COMPUTATIONAL COMPLEXITY OF THE CAPACITATED LOT SIZE PROBLEM

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ABSTRACT

In this paper we study the computational complexity of the capacitated lot size problem with a particular cost structure that is likely to be used in practical settings. For the single item case new properties are introduced, classes of problems solvable by polynomial time algorithms are identified, and efficient solution procedures are given. We show that special classes are NP-hard, and that the problem with two items and independent setups is NP-hard under conditions similar to those where the single item problem is easy. Topics for further research are discussed in the last section.

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1. INTRODUCTION

We first consider the problem of determining an optimal production plan for a single product under capacity constraints. In section four we extend the discussion to the multiple item case.

The single product can be seen as an aggregate product representing a family of items. This problem, although simplistic in nature, is useful in particular settings and its analysis is likely to provide a better understanding of more complex production planning models. The single product capacitated problem can be written as follows:

$$(P) \qquad v(P) = \min \sum_{t=1}^{T} [s_t \delta(X_t) + p_t(X_t) + h_t(I_t)]$$

$$\text{s.t.} \quad I_{t-1} + X_t - I_t = d_t \qquad t=1,2,\ldots,T$$

$$X_t \leq C_t \qquad t=1,2,\ldots,T$$

$$\delta(X_t) = \begin{cases} 1 \text{ if } X_t > 0 \\ 0 \text{ otherwise} \end{cases} \qquad t=1,2,\ldots,T$$

$$I_t, X_t \geq 0 \qquad t=1,2,\ldots,T.$$

X_t, I_t, C_t, d_t, and s_t denote respectively for period t, the production quantity, the ending inventory, the capacity available, the demand, and the setup cost. The first two quantities are decision variables and the last three are given parameters. I_o is the initial inventory. The functions $p_t(X_t)$ and $h_t(I_t)$ represent the continuous component of the production cost and the holding cost incurred in period t.

The computational complexity of problem (P) has attracted the attention of several researchers in recent years. In an insightful paper Florian, Lenstra and Rinnooy Kan [3] have shown that problem (P) is NP-hard for quite general objective functions. They have also provided a brief introduction to computational complexity theory. The terminology that we use in this paper conforms to their introduction. The readers are referred to Garey and Johnson [4] for a comprehensive discussion of computational complexity theory. Baker, Dixon, Magazine, and Silver [1] devised an $O(2^T)$ algorithm to solve problem (P) for the case where the functions $p_t(\cdot)$ and $h_t(\cdot)$ are linear and do not depend on t. The computational results they provide suggest that the algorithm is quite effective. Florian and Klein [2] have shown that when the cost function is concave and the capacities are constant over time, problem (P) can be solved by a polynomial algorithm of $O(T^4)$. Love [6] provided an $O(T^3)$ algorithm to solve the single product lot size problem with piecewise concave cost and upper bounds on inventories rather than on production. Pseudopolynomial algorithms can be obtained by dynamic programming as discussed in [3]. The uncapacitated version of problem (P) has been extensively analyzed under a variety of conditions by several authors including Zangwill [11], Veinott [9], and Wagner and Whitin [10]. In particular, Wagner and Whitin provided an $O(T^2)$ algorithm. We do not expand on these references since they have been extensively discussed in the literature [3], [8], [5], and [7].

In this paper we address issues of computational complexity of the capacitated lot size problem for the cases where the continuous components of the production and holding costs are linear. The resulting cost functions are concave. These cost structures represent satisfactorily the cost functions encountered in many practical settings and are frequently adopted in the literature. In particular, the non-increasing cost that we often assume, arises in practice due to discount factor effects.

For future reference we indicate the assumptions and notation used throughout the paper.

Assumptions

- $p_t(X_t) = v_t X_t \qquad t=1,2,\ldots,T$

  $h_t(I_t) = h_t I_t \qquad t=1,2,\ldots,T$

  v_t and h_t are given nonnegative parameters.
- The demands d_t and capacities C_t, $t=1,2,\ldots,T$, are non-negative integers.
- Without loss of generality, we assume that the initial inventory I_0 is equal to zero.

Notation

Let F denote the feasible set of problem (P) and let (X,I) denote a 2T-vector (X_t,I_t), $t=1,2,\ldots,T$, of production and inventory levels for a generic feasible solution. In order to classify the special families or classes of problem (P), we introduce the following notation $\alpha/\beta/\gamma/\delta$, where α, β, γ, and δ specify respectively a special structure for the setup costs, holding costs, production costs, and capacities. α, β, γ, δ will be taken equal to the following letters: G, C, ND, NI, Z if the parameter under consideration is assumed over time to follow no prespecified pattern, be constant, non-decreasing, non-increasing, and have value zero. For example, the notation NI/ND/C/G indicates the family of problems (P) where, over time, the setup sequence s_t is non-increasing, the unit holding costs h_t are non-decreasing, the unit production costs v_t are constant, and the set of capacities C_t are not restricted to any prespecified pattern. It is useful to point out that if a family is NP-hard, its subfamilies need not be NP-hard. However, if a family is solvable by a polynomial algorithm, its subfamilies are also solvable by a polynomial algorithm. Whenever possible we have avoided, for the benefit of clarity, the use of excessive algebra in the proofs. We believe that the reader will have no difficulty in filling the possible gaps in the mathematical development.

The plan of this paper is as follows. In section two we introduce new properties of problem (P) and summarize known results that are used in subsequent developments. In particular, we show that in the absence of a satisfactory forecast, a problem in the class NI/G/NI/ND can be partitioned in two subprograms that when solved independently have a total cost that differs from the optimal cost, by a value not larger than the maximum setup cost. In section three we discuss classes of problem (P) that are polynomial and provide an $O(T^4)$ algorithm to solve the class NI/G/NI/ND. When specialized for the family NI/G/NI/C it reduces to an $O(T^3)$ algorithm. In section four, we identify NP-hard classes and show that the two item problem with independent setups is NP-hard under conditions similar to those where the single item problem is polynomial. In the section on conclusions and topics for further research, we discuss classes that have not yet been classified as either polynomial or NP-hard.

2. SOME PROPERTIES OF THE CAPACITATED LOT SIZE PROBLEM

In this section we present new results for problem (P) and summarize known properties that will be used in the subsequent development.

Baker, Dixon, Magazine, and Silver [1] provided a property of optimal solutions of the family G/C/C/G. We present in the next proposition an extension of their result. It will be used in Proposition 2.4.

Proposition 2.1: For feasible problems of the family G/G/NI/G there is an optimal solution (X,I) satisfying the conditions

$$I_{t-1}X_t(C_t-X_t) = 0 \qquad t=1,2,\ldots,T$$

Another important property that plays a central role in the development of algorithms to solve problem (P) when the objective function is concave is given by Florian and Klein [2]. It characterizes the extreme points of F.

Proposition 2.2 ([2]): A plan (X,I) is an extreme point of F if and only if it can be partitioned into a sequence of subplans with the following properties:

i) The ending inventory is strictly positive in every period, except the last where it is zero, and
ii) The production is either zero or at full capacity in every period except in at most one.

In section three we exploit Proposition 2.2 to derive a polynomial algorithm for the class NI/G/NI/ND.

The following equivalent representation of problem (P) will simplify our development. Although we prove the result for the cost structure assumed in this paper, it holds for more general objective functions.

Proposition 2.3: If problem (P) is feasible, it can be rewritten as an equivalent capacitated lot size problem where in each period the demand is not greater than the capacity.

Proof: For every (X,I) feasible in F, define (X_t', I_t'), $t=1,2,\dots,T$, as

$$I_{t-1}' = I_{t-1} - \max_{\tau=0,\dots,T-t} \{0, \sum_{\ell=t}^{t+\tau} (d_\ell - C_\ell)\} \text{ and}$$

$$X_t' = X_t.$$

Since (X,I) is feasible in F,

$$I_{t-1} \geqq \max_{\tau=0,\dots,T-t} \{0, \sum_{\ell=t}^{t+\tau} (d_\ell - C_\ell)\}$$

and consequently,

$$I_{t-1}' \geqq 0 \text{ and } X_t' \geqq 0 \qquad t=1,2,\dots,T.$$

Consider,

$$\begin{aligned} I_{t-1}' + X_t' - I_t' &= d_t + \max_{\tau=1,\dots,T-t} \{0, \sum_{\ell=t+1}^{t+\tau} (d_\ell - C_\ell)\} - \\ &\quad - \max_{\tau=0,\dots,T-t} \{0, \sum_{\ell=t}^{t+\tau} (d_\ell - C_\ell)\} = \\ &= C_t + \max_{\tau=1,\dots,T-t} \{0, \sum_{\ell=t+1}^{t+\tau} (d_\ell - C_\ell)\} - \\ &\quad - \max_{\tau=1,\dots,T-t} \{C_t - d_t, \sum_{\ell=t+1}^{t+\tau} (d_\ell - C_\ell)\} \\ &= d_t' \end{aligned} \tag{2.1}$$

From (2.1), $d_t' \geqq 0$ and $d_t' \leqq C_t$. Substituting X_t, I_t as a function of X_t', I_t' in the objective function we obtain

$$\sum_{t=1}^{T} \{s_t\delta(X_t') + p_t(X_t') + h_t\left(I_t' + \max_{\tau=1,\dots,T-t} [0, \sum_{\ell=t+1}^{t+\tau}(d_\ell - C_\ell)]\right)\}$$

$$= \sum_{t=1}^{T} \{s_t\delta(X_t') + v_tX_t' + h_tI_t'\} + \sum_{t=1}^{T} h_t \max_{\tau=1,\dots,T-t} [0, \sum_{\ell=t+1}^{t+\tau}(d_\ell - C_\ell)]$$

The new objective function differs from the original one by a constant. ▨

The transformation of (P) as a problem having demands not larger than capacities, in every period, can be made in O(T) operations. In what follows, we assume, without loss of generality, that every problem (P) has this property. We also point out that the original problem is infeasible if and only if the transformed version has a negative initial inventory.

<u>Proposition 2.4</u>: For problems in NI/G/NI/ND there is at least one optimal solution (X_t, I_t), $t=1,2,\dots,T$, satisfying the property:

$$I_{t-1} < d_t \text{ for every } t \text{ such that } I_{t-1}X_t > 0 \quad . \tag{2.2}$$

<u>Proof</u>: Let (X,I) be an optimal solution satisfying Proposition 2.1. Assume that for some t

$$I_{t-1}X_t > 0 \text{ and } I_{t-1} \geq d_t.$$

Let t^* be the first period after t with no production. Since $d_t \leq C_t$, $t=1,2,\dots,T$, such period exists because otherwise I_T would be strictly positive. In this case, we obtain a solution not worse than (X,I) by producing in period t^* instead of period t. Consequently any optimal extreme point solution of NI/G/NI/ND can be transformed into an optimal solution satisfying the proposition. ▨

<u>Corollary 2.1</u>: For problems in NI/G/NI/ND there is an optimal solution with

the property that there is no production in periods of zero demand.

Proof: Parallels the proof of Proposition 2.4.

In many practical situations, it may not be possible to solve problems NI/G/NI/ND. The reason is that the demand data may not be available, or even forecastable with reasonable accuracy, for the whole planning horizon T. In these instances we may want to partition the problem into smaller horizon problems. First solve one with T_1 periods and at time T_1+1, solve a second problem with $T-T_1$ periods. This procedure implicitly assumes that the second problem is feasible with zero initial inventory. Although there will probably be a loss in optimality, the next proposition shows that it might not be severe.

Proposition 2.5: Let $(P) \varepsilon$ NI/G/NI/ND. Let (P1) and (P2) be a partition of (P) where (P1) corresponds to the first T_1 periods and (P2) to the last $T_2 = T-T_1$ periods. Assume that (P2) is feasible with zero initial inventory. Then, the optimal values v(P), v(P1), and v(P2) relate as follows

$$v(P) \leq v(P1) + v(P2) \leq v(P) + s_{T_1+1}$$

The proof of this result is given in the Appendix.

3. POLYNOMIAL TIME ALGORITHMS

In this section we derive algorithms that run in polynomial time for special cases of the single product capacitated production problem. We show that the class of problems NI/G/NI/ND, NI/G/NI/C, C/Z/C/G, and ND/Z/ND/NI can be solved by algorithms that run respectively in $O(T^4)$, $O(T^3)$, $O(T \log T)$ and $O(T)$.

Consider problem

$$
\begin{array}{llll}
(P_{uv}) & E_{uv} = \min \sum_{t=u+1}^{v} [s_t\delta(X_t) + p_t(X_t) + h_t(I_t)] & & \\
& I_u = I_v = 0 & & \\
& I_t > 0 & t=u+1,\ldots,v-1 & \\
& I_{t-1} + X_t - I_t = d_t & t=u+1,\ldots,v & \\
& X_t = C_t \text{ or } X_t = 0 & t=u+2,\ldots,v & \\
& 0 \leq X_{u+1} \leq C_{u+1} & & \\
& (X,I) \text{ satisfies (2.2)} & & (3.1)
\end{array}
$$

Problem (P) can be solved by using the recursions:

$$
\begin{array}{lll}
f_o = 0 & & \\
& & (3.2) \\
f_v = \min_{0 \leq u < v} \{f_u + E_{uv}\} & v=1,2,\ldots,T &
\end{array}
$$

where f_v is the optimal value of problem (P) with T = v. There are $\frac{1}{2}T(T+1)$ subproblems (P_{uv}) and the recursions (3.2) can be carried out in $O(T^2)$ time.

Problems (P_{uv}) are inspired by the characterization of extreme points given in Proposition 2.2 and the existence of an optimal solution for the class NI/G/NI/ND with the property expressed in (2.2). Note that for problems in G/G/NI/G, by Proposition 2.1, it is sufficient to consider extreme points with subplans as given in Proposition 2.2 with ii) stated as "The production is either zero or at full capacity in every period except the first." Florian and Klein [2], Florian, Lenstra, and Rinnooy Kan [3], and Love [6] have explored a related problem to (P_{uv}) to derive algorithms for the classes of problems they have studied.

Let $[P_{uv}(q)]$ be the problem derived from (P_{uv}) by letting $X_{u+1} = d_{u+1} + q \leq C_{u+1}$ for some $q > 0$, $q \varepsilon R$.

Proposition 3.1: If $[P_{uv}(q)]$ is feasible, it has a unique optimal solution.

The proof of the proposition follows from the fact that $[P_{uv}(q)]$ may have at most one feasible solution.

The solution to $[P_{uv}(q)]$, if it exists, can be constructed by producing at full capacity at every period $t > u+1$ where the incoming inventory is smaller than the demand. This procedure is an $0(v-u)$ algorithm.

An $0(T^4)$ Algorithm to solve NI/G/NI/ND

The algorithm we propose is based on the recursions (3.2). A critical step is the solution of problems (P_{uv}).

For a given problem (P_{uv}), only the production of the first period can be positive and smaller than the capacity (without loss of generality we assume $d_{u+1} > 0$ and we concentrate on integer solutions). Therefore, we must consider implicitly or explicitly all values $d_{u+1} + q$ for $q=1,2,\ldots,C_{u+1}-d_{u+1}$. As we show below, the number of different values of q that need to be examined is of $0[(v-u-1)^2]$.

The algorithm is as follows. Start with $q=1$, i.e., $X_{u+1} = d_{u+1} + 1$ and construct a solution satisfying Proposition 2.4, that is, produce as late as possible. Whenever production occurs at a period $t > u+1$, produce at capacity. For every q considered we shall generate a plan for problem (P_{uv}) according to the rule just described. Although these plans may be infeasible (if $[P_{uv}(q)]$ is infeasible) we compute them for reference purposes. Assume that we should produce in period t_1. If I_{t_1-1} were larger, we could avoid setting up in period t_1. The amount needed to avoid this set up is $\Delta_{t_1} = d_{t_1} - I_{t_1-1}$. At each period where a set up is incurred, we compute the quantity Δ_t which is the smallest increase in q that would push the set up to a later period. If $\Delta(q)$ denotes the minimum of the Δ_t's, the next production to be considered at period u+1 is

$$X_{u+1} = d_{u+1} + q + \Delta(q) \tag{3.3}$$

i.e., the new value of q is $q + \Delta(q)$. Plans having a production quantity in period u+1 between $d_{u+1}+q$ and $d_{u+1}+q+\Delta(q)$ need not be examined because they will not alter the periods where production occurs, and will only increase the holding cost and in particular the value of I_v.

With the new value of X_{u+1} given by (3.3) we repeat the process. After having considered all possible values of q we select among the feasible plans, if any, the one with minimum cost. This plan solves (P_{uv}).

In order to obtain an upper bound on the number of possible values of q that the algorithm may compute for each (P_{uv}) we observe that for each new increment $\Delta(q)$ considered, at least one set up is shifted to a later period. Thus, in the worst case, each set up is moved to a later period at most v-u-1 times. Therefore, the number of plans that we need to compute is bounded by $\sum_{i=1}^{v-u-1} i$, i.e. $O[(v-u)^2]$. The algorithm described to solve a given problem (P_{uv}) is of $O[(v-u)^2] \times O(v-u) = O[(v-u)^3]$ since it takes a run time of $O(v-u)$ to compute the plan for each q.

This algorithm combined with the recursions (3.2) runs in $O(T^5)$ time, there are $O(T^2)$ problems (P_{uv}), and the recursions run in time $O(T^2)$. However, we can reduce the run time to $O(T^4)$ if we observe that when we solve (P_{uv}) we are at the same time solving $(P_{uv'})$ for all $u < v' \leq v$. Thus, we should compute (P_{1T}), then $(P_{2T}),\ldots,(P_{T-1,T})$. Therefore all (P_{uv})'s can be solved in $O[T^3 + (T-1)^3 + \ldots + 1] \equiv O(T^4)$ time. Since the dynamic recursion runs in time $O(T^2)$ the algorithm to solve NI/G/NI/ND will run in time $O(T^4) + O(T^2) = O(T^4)$.

Therefore

<u>Proposition 3.2</u>: There is an $O(T^4)$ algorithm to solve the class NI/G/NI/ND.

<u>Corollary 3.1</u>: There is an $O(T^3)$ algorithm to solve the class NI/G/NI/C.

<u>Proof</u>: In the previous algorithm there is a unique value of q that needs to be considered for each (P_{uv}). More specifically, q is the remainder of the division of the demands in problem (P_{uv}) and the constant capacity C.

We conclude this section by providing polynomial algorithms for two other classes of problem (P).

<u>Proposition 3.3</u>: There is an $O(T \log T)$ algorithm to solve the class C/Z/C/G.

<u>Proof</u>: Since the production costs are constant we can eliminate them from the problem. Therefore, we are left with the set up costs only. An $O(T \log T)$ algorithm can be devised by producing at the periods of largest capacity. At every period t, if the entering inventory is smaller than the demand d_t, we have to consider for production periods $1,2,\ldots,t$ and select the one with largest capacity not yet scheduled for production.

Proposition 3.4: There is an O(T) algorithm to solve the class ND/Z/ND/NI.

Proof: The reader will have no difficulty in verifying that an O(T) algorithm can be obtained by producing in the first periods as much as possible to satisfy the future demand. ▨

4. NP-HARD CASES

Florian, Lenstra, and Rinnooy Kan [3] have shown that several general families of problem (P) are NP-hard. In what follows we first show that some less general families, i.e., families with particular cost structures, also are NP-hard. We conclude this section proving that for the multiple item problem with independent setups the polynomial cases discussed for problem (P), are NP-hard.

Proposition 4.1: The following classes are NP-hard:

a) C/Z/NI/NI

b) C/Z/ND/ND

c) ND/Z/Z/ND

d) NI/Z/Z/NI

e) C/G/Z/NI

f) C/C/ND/NI

Proof: The proof of cases a) to d) parallel the proof of Proposition 1 in [3] by conveniently ordering the elements in the knapsack and/or taking the setup costs equal to capacities in every period. Period zero of Proposition 1 in [3] must be deleted and the demand pattern redefined.

The proof of e) is obtained by showing that each problem in the class C/Z/NI/NI can be written as an equivalent problem belonging to the class C/G/Z/NI. Consider an instance of problem (P) in the class C/Z/NI/NI, with parameters $s_t = s$, $h_t = 0$, v_t, C_t, and d_t, $t=1,2,\dots,T$. The objective function of problem (P) expressed in terms of the inventory variables is

$$\sum_{t=1}^{T} s\delta(d_t+I_t-I_{t-1}) + \sum_{t=1}^{T-1} (v_t-v_{t+1})I_t + v_T I_T + \sum_{t=1}^{T} v_t d_t - v_1 I_o \qquad (4.1)$$

By leeting $h'_t = v_t - v_{t+1}$ and $h'_T = v_T$ in (4.1), we obtain the objective function of an equivalent problem in C/G/Z/NI. Since the h'_t, $t=1,\ldots,T$ are non-negative, by the assumption that the v_t are nonincreasing, it follows that the class C/G/Z/NI is NP-hard.

We prove case f) by showing that any instance of C/Z/NI/NI can be transformed into an equivalent instance of C/C/ND/NI. We first observe that substituting $\sum_{\tau=1}^{t} (X_\tau - d_\tau) + I_o$ for I_t in the objective function shows that any instance of C/C/ND/NI, with parameters v'_t and $h_t = h'$, $t=1,\ldots,T$ is equivalent to a problem (P') with zero holding costs, the same setup costs, and production costs given by

$$v^*_t \;=\; v'_t + (T-t+1)h' \qquad t=1,\ldots,T \qquad (4.2)$$

Thus it will be sufficient to show that any generic instance of problem (P) in the class C/Z/NI/NI with parameters $s_t=s$, $h_t=0$, v_t, C_t, and d_t, $t=1,2,\ldots,T$, can be reduced to a problem of the form (P'). That is, we show that we can define parameters v'_t and h' of an instance of C/C/ND/NI, with setup costs s, capacities C_t, and demands d_t, that satisfies (4.2). Let $v^*_t = v_t$. Then, we wish to show that the system with unkowns h' and v'_t, $t=1,2,\ldots,T$:

$$\left.\begin{array}{lll} v_t = v'_t + (T-t+1)h' & & t=1,2,\ldots,T \\ v'_t \le v'_{t+1} & & t=1,2,\ldots,T-1 \\ h' > 0 & & \\ v'_1 \ge 0 & & \end{array}\right\} \qquad (4.3)$$

is feasible. For example, take the solution given by:

$$0 \le v'_1 < \min_{t=2,\ldots,T} \{v_1,\ v_1 - T(v_{t-1} - v_t)\} + K \qquad (4.4)$$

$$h' = (v_1 + K - v'_1)/T, \text{ and}$$

$$v' = v - (T-t+1)h' \qquad t=2 \qquad T$$

where K is chosen so that the right most expression in (4.4) is positive.

With this choice of v'_t and h' we have an instance in the class C/C/ND/NI that is equivalent to the instance of problem (P) in C/Z/NI/NI.

Multiple Items with Independent Setups

The multiple items capacitated lot size problem with independent setups is written as follows

$$\text{(PMI)} \quad \min \sum_{i=1}^{N} \sum_{t=1}^{T} [s_{it}\delta(X_{it}) + p_{it}(X_{it}) + h_{it}(I_{it})]$$

$$\text{s.t.} \quad I_{it-1} + X_{it} - I_{it} = d_{it} \qquad i=1,\ldots,N;\ t=1,\ldots,T$$

$$\sum_{i=1}^{N} X_{it} \leq C_t \qquad t=1,\ldots,T$$

$$\delta(X_{it}) = \begin{cases} 1 & \text{if } X_{it} > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \begin{matrix} i=1,\ldots,N \\ t=1,\ldots,T \end{matrix}$$

$$I_{it},\ X_{it} \geq 0 \qquad i=1,\ldots,N;\ t=1,\ldots,T.$$

It is convenient to extend the notation $\alpha/\beta/\gamma/\delta$ to $1/\alpha/\beta/\gamma/\delta$, $2/\alpha/\beta/\gamma/\delta$, $N/\alpha/\beta/\gamma/\delta$ for problems (PMI) with one, two, or a variable number of items. The assumptions made in section one are assumed to hold for each item and parameters in (PMI). For example, the symbol 2/G/Z/ND/NI indicates the class of problems, with two items, where the setup costs of the items do not necessarily follow a prespecified pattern; the holding and production unit costs of both items are respectively equal to zero and non-decreasing, and the capacities are non-increasing over time.

In section two we presented four families of problems (P) that can be solved by polynomial time algorithms. However, as the next two propositions show, the corresponding classes in (PMI) are NP-hard.

Proposition 4.2: The following classes of problems (PMI) are NP-hard

a) 2/C/Z/Z/ND

b) 2/C/Z/Z/NI

Proof: Consider the partition problem: "Given $\{A_t\}_{t=1}^{T}$ find $S \subseteq \{1,2,\ldots,T\}$ such that $\sum_{t\in S} A_t = \sum_{t\in\{1,\ldots,T\}-S} A_t$". This problem is NP-Complete [4]. We prove the proposition by showing that the classes in a) and b) above are at least as hard to solve as the Partition Problem.

Assume that $\{A_t\}_{t=1}^{T}$ is a non-decreasing sequence of positive integer numbers. Consider the following instance of the class 2/C/Z/Z/ND with two items and parameters:

$$d_{1t} = d_{2t} = 0 \qquad t=1,2,\ldots,T-1;$$

$$d_{1T} = d_{2T} = \frac{1}{2}\sum_{t=1}^{T} C_t;$$

$$s_{1t} = s_{2t} = 1 \qquad t=1,2,\ldots,T;$$

$$v_{1t} = v_{2t} = 0 \qquad t=1,2,\ldots,T;$$

$$h_{1t} = h_{2t} = 0 \qquad t=1,2,\ldots,T;$$

and $$C_t = A_t \qquad t=1,2,\ldots,T.$$

Since the (PMI) problem constructed has $d_{1T} + d_{2T} = \sum_{t=1}^{T} C_t$, the available capacity C_t in every period will have to be used. Hence, its optimal value is not smaller than T and will be equal to T if and only if the sequence $\{C_t\}_{t=1}^{T}$ or equivalently $\{A_t\}_{t=1}^{T}$, can be partitioned such that

$$\sum_{t\in S} C_t = \sum_{t\in\{1,\ldots,T\}-S} C_t = \frac{1}{2}\sum_{t=1}^{T} C_t$$

that is, if and only if the partition problem has solution. Consequently the class 2/C/Z/Z/ND is HP-hard. The proof of case b) parallels the one given above and is omitted. ▨

The same proof does not apply to the family 2/NI/G/NI/C since the partition problem would have all A_t's equal. The corresponding result is stated below.

Proposition 4.3: The class 2/C/Z/NI/C of problems (PMI) is NP-hard.

Proof: Consider an instance of 1/C/Z/NI/NI, with parameters s_1, v_{1t}, d_{1t}, C_t, $t=1,\ldots,T$. This problem is computationally equivalent to a two items problem with capacities $C'_t = C = \max_{q=1,\ldots,T} \{C_q\}$, $t=1,\ldots,T$, and where item two has parameters

$$s_2 = s_1$$

$$v_{2T} = Ts_1 + \left(\sum_{t=1}^{T} v_{1t}\right) \sum_{t=1}^{T} d_{1t},$$

$$v_{2t} = (T-t+1)v_{2T} \qquad t=1,\ldots,T-1,$$

$$d_{2t} = C'_t - C_t \qquad t=1,\ldots,T.$$

Because the unit production costs of item two are very high and decreasing, and the magnitude of the difference between two consecutive periods is also high, item number two must be produced in every period. Moreover, the optimal schedule of item one remains unchanged. Therefore, we have shown that the class 1/C/Z/NI/NI can be expanded to a subclass of 2/C/Z/NI/C with equivalent computational complexity. Hence, the result follows by Proposition 4.1. ▨

In addition to the families in Propositions 4.2 and 4.3, the corresponding classes in problem (PMI), of those presented in Propostion 4.1 for problem (P), are also NP-hard. These results show that even under simplified cost assumptions, frequently made in practice, the capacitated lot size problem with as little as two items is hard to solve.

The next propositions permit the identification of a few more families of problems (PMI) that are also NP-hard.

Proposition 4.4: Every class $2/\alpha/\beta/\gamma/C$ of problems (PMI), with $\beta \neq Z$ is at least as hard as the corresponding class $\alpha/\beta/\gamma/G$ of problems (P).

Proof: Can be derived using similar arguments as in Proposition 4.3 and by choosing the second item with unit holding costs such that it will be produced in every period. ▨

Note that the computational complexity of some classes $\alpha/\beta/\gamma/G$ of problems (P) is still undetermined. Therefore, the computational complexity of the corresponding classes $2/\alpha/\beta/\gamma/C$ of problems (PMI) cannot be established by Proposition 4.4. The following proposition provides stronger results.

Corollary 4.1: The class 3/C/C/Z/C of problems (PMI) is NP-hard.

Proof: The proof is similar to the one given in Proposition 4.4. Select an instance of 2/C/C/Z/G, which is an NP-hard class by Proposition 4.2. Then, define an instance of the class 3/C/C/Z/C by selecting one item such that in any optimal solution it must be produced in every period. ▨

We point out that using similar arguments, the same results stated in Propoistion 4.4 and Corollary 4.1 apply when $\beta = Z$ and $\gamma = NI$.

5. CONCLUSIONS AND TOPICS FOR FURTHER RESEARCH

In this paper we have identified special classes of problem (P) that can be solved by polynomial time algorithms and classes that are NP-hard. In section two we provided several properties of the problem considered. In particular, we have shown for the family NI/G/NI/ND, that if a forecast with the desired accuracy is not available, the problem can be partitioned at a cost not higher than the maximum setup cost. Still, for this family we proved that there is an optimal solution with the property that in every period where production occurs the demand is larger than the entering inventory. This property was instrumental in devising an $O(T^4)$ algorithm. There are several families of problems that can be classified as solvable by polynomial algorithms or as NP-hard as a direct consequence of the propositions obtained in this paper. For example, the families C/G/NI/ND, NI/G/C/ND, C/Z/ND/NI, and ND/Z/C/NI are polynomial while the families G/Z/Z/G, ND/ND/ND/NI, G/Z/Z/ND, and NI/Z/G/ND are NP-hard. These results can be summarized in a network form. For space reasons we only present, in Figure 1, a few networks for NP-hard families. The origin vertices are families shown to be NP-hard. The descendent of each vertex is at least as hard as its predecessors.

It is also not difficult to prove that for the conditions under which problem (P) is NP-hard, the lot size problem with multiple items and independent setups is NP-hard. In fact we have shown, in Propositions 4.2 and 4.3, that this last problem is NP-hard in several cases where (P) is polynomial. However, the same extensions for the conditions where (P) is polynomial, to the problem with multiple items sharing the same setup, are not obvious and remain to be explored.

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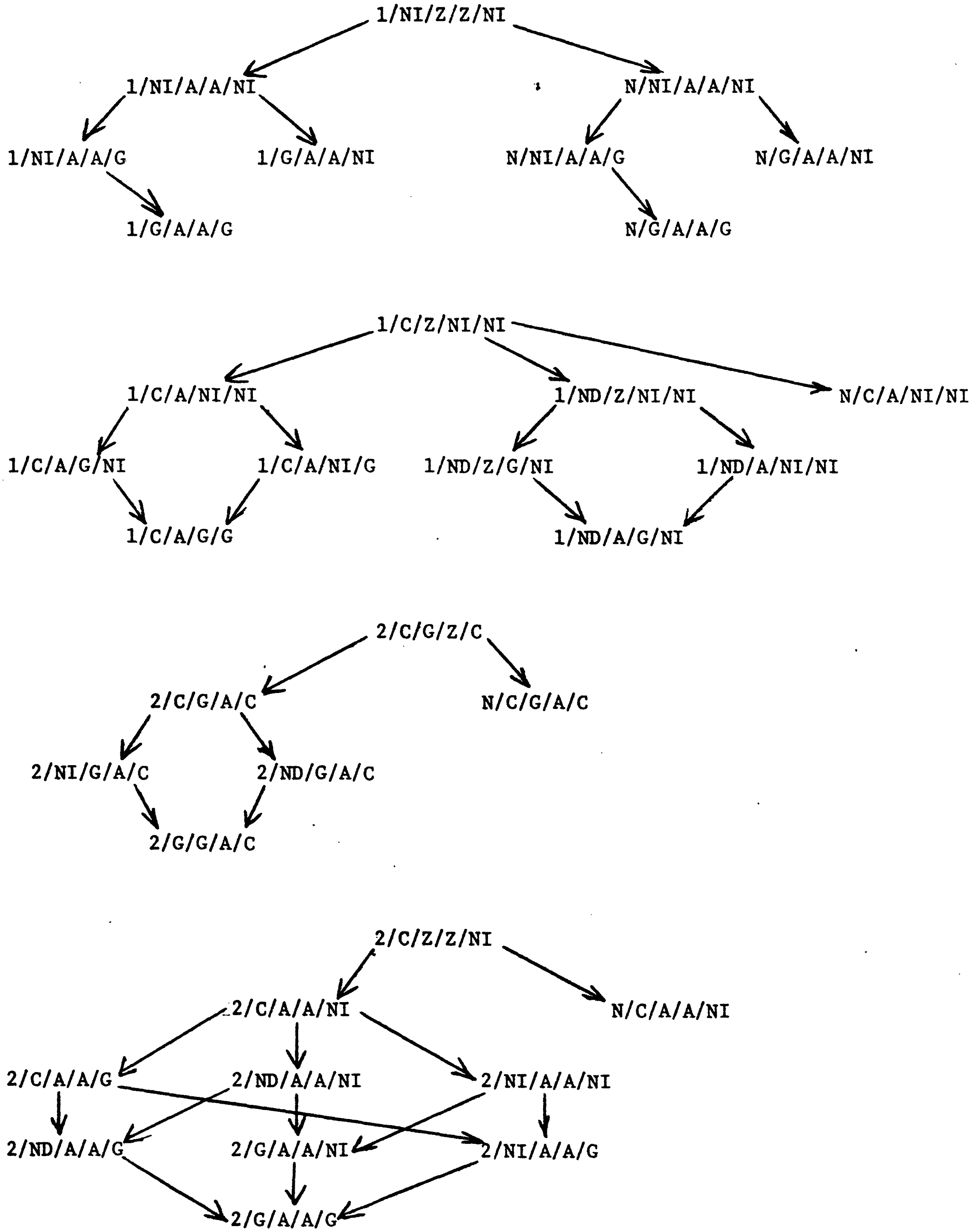

Figure 1: Some NP-hard families. The letter A for either β, γ indicates the union of all cases, that is $C/A/G/NI = \bigcup_{\beta=G,C,NI,ND,Z} \{C/\beta/G/NI\}$.

Appendix: PROOF OF PROPOSITION 2.5

Proof: Let (X^*, I^*) be an optimal solution of (P) satisfying Propositions 2.1 and 2.4, and let $v(P) = v1+v2$ where v1 and v2 are the costs associated with the first T_1 and the last $(T-T_1)$ periods. Since v(P1) represents the optimal value for the first T_1 periods it follows that

$$v(P1) \leqq v1.$$

We next show that $v(P2) \leqq v2 + s_{T_1+1}$ by considering three cases:

i) $I^*_{T_1} = 0$. The result is trivial.

ii) $I^*_{T_1} X^*_{T_1+1} > 0$. In this case $I^*_{T_1} < d_{T_1+1}$ by Proposition 2.4. Moreover, $I^*_{T_1} < C_{T_1+1}$. We can obtain a feasible solution for (P2) by following the same schedule as in (P) for periods T_1+1 on and schedule production in the first period after T_1 where $X^*_t = 0$. Since by assumption the capacities are nondecreasing we can produce in that period the quantity $I^*_{T_1}$. Therefore, $v(P2) \leqq v2+s_\tau$ for some $\tau > T_1+1$. From $(P) \varepsilon$ NI/G/NI/ND it follows that $v(P2) \leqq v2+s_{T_1+1}$.

iii) $I^*_{T_1} > 0$ and $X^*_{T_1+1} = 0$. Let $\tau \leqq T_1$ be the last period prior to T_1+1 where a setup occurred in (X^*, I^*). Then, by Proposition 2.4 and the assumption of non-decreasing capacities it follows that

$$I^*_{\tau-1} \leqq d_\tau \leqq C_\tau \leqq C_{T_1+1} \qquad \text{and}$$

$$I^*_{T_1} \leqq I^*_\tau \leqq I^*_{\tau-1} + C_\tau - d_\tau \leqq C_{T_1+1}.$$

Hence, we can obtain a feasible solution for (P2) by following the same production as in (X^*, I^*) for periods T_1+2 on and by producing $I^*_{T_1}$ in period T_1+1. Therefore,

$$v(P2) \leqq v2 + s_{T_1+1} \quad .$$

Finally, since to solve (P1) and (P2) is equivalent to impose the additional constraint $I_{T_1} = 0$ to (P) we have that

$$v(P) \leqq v(P1) + v(P2).$$ ▨

References

1. Baker, K. R., Dixon, P. S., Magazine, M. J., Silver, E. A., "An Algorithm for the Dynamic Lot-Size Problem with Time-Varying Production Capacity Constraints", Management Science, Vol. 24, No. 16, December 1978, pp. 1710-1720.

2. Florian M; Klein, M., "Deterministic Production Planning with Concave Costs and Capacity Constraints", Management Science, Vol. 18, No. 1, September 1971, pp. 12-20.

3. Florian, M; Lenstra, J. K.; Rinnooy Kan, A. H. G., "Deterministic Production Planning: Algorithms and Complexity", Management Science, Vol. 26, No. 7, July 1980, pp. 669-679.

4. Garey, M. R.,; Johnson, D. S., "Computers and Intractability: A Guide to the Theory of NP-Completeness", Freeman, San Francisco, 1979.

5. Johnson, L. A.; Montgomery, D. C., "Operations Research in Production Planning, Scheduling, and Inventory Control", Wiley, New York, 1974.

6. Love, S. F., "Bounded Production and Inventory Models with Piecewise Concave Costs", Management Science, Vol. 20, No. 3, November 1973, pp. 313-318.

7. McClain, J. O.; Thomas, L. J., "Operations Management", Prentice Hall, Englewood Cliffs, NJ, 1980.

8. Peterson, R.; Silver, E. A., "Decision Systems for Inventory Management and Production Planning", Wiley, New York, 1979.

9. Veinott, Jr., A. F., "Extreme Points of Leontief Substitution Systems", Linear Algebra and Its Application 1, 181-194 (1968).

10. Wagner, H. M.; Whitin, T. M., "Dynamic Version of the Economic Lot Size Model", Management Science, Vol. 5, October 1958, pp. 89-96.

11. Zangwill, W. I., "A Deterministic Multi-Period Production Scheduling Model with Backlogging", Management Science, Vol. 13, 1966, pp. 105-119.