SOME VERY EASY KNAPSACK/PARTITION PROBLEMS

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Abstract

Consider the problem of partitioning a group of b indistinguishable objects into subgroups each of size at least ℓ and at most u. The objective is to minimize the additive separable cost of the partition, where the cost associated with a subgroup of size j is c(j). In the case that $c(\cdot)$ is convex, we show how to solve the problem in O(log u) steps. In the case that $c(\cdot)$ is concave, we solve the problem in $O(\min(\ell, b/u, (b/\ell)-(b/u), u-\ell))$ steps.

Consider the problem of partitioning a group of b objects into subgroups each of size at least ℓ and at most u. The objective is to minimize an additive cost $\sum_{j=\ell}^{u} c(j)x_j$ where $c(\cdot)$ is some real-valued function and x_j is the number of subgroups of size j. This problem may be expressed as the knapsack problem P below.

$$\begin{aligned} \text{Minimize} \quad & \sum_{j=\ell}^{u} c(j)x_j \\ & \sum_{j=\ell}^{u} jx_j = b \\ & x_j \geq 0 \text{ integer for } j=\ell,\ldots, u. \end{aligned} \qquad \text{(P)}$$

It is well known that the problem P may be solved in o(ub) steps via dynamic programming recursion. Moreover, if $b \geq u^2$, then P may be solved in $o(u^2)$ steps because the optimal solution to the associated group problem is feasible for P. (See Garfinkel and Nemhauser [2] for further details.)

The purpose of this note is to provide very efficient algorithms for the case that $c(\cdot)$ is either concave or convex. In particular, we show that we can solve the case of P in which $c(\cdot)$ is convex in O(log u) steps. This algorithm extends a previous algorithm by Chand [1] for a variant of the discrete time EOQ model, as mentioned below.

In the case that $c(\cdot)$ is concave, we show how to solve the knapsack/partition problem in $O(\min(\ell, b/u, (b/\ell) - (b/u), u-\ell))$ steps. It is an open question as to whether the concave case can be solved in a number of steps that is polynomially bounded in log b.

In both the case that $c(\cdot)$ is convex and the case that $c(\cdot)$ is concave, if the number of subgroups in the partition is specified, then the resulting problem is solvable in O(1) steps.

As an example of the knapsack/partition problem consider the problem of subdividing a group of b people into subcommittees, each consisting of between ℓ and u members. Suppose further that the "value" of a committee with j members is c(j), where the function c(·) is concave, reflecting decreasing marginal returns. Here the objective is to maximize the value of the partition of people into subcommittees.

As another example, consider the problem of aggregating large amounts of data. Here the cost of aggregating j pieces of data is c(j), reflecting a loss in accuracy but a gain in computational convenience. The optimum solution serves as a rough estimate of how much aggregation is desirable.

A third example is an application to a finite-horizon discrete-time variant of the EOQ model as described and solved by Chand. Here b is the number of time periods in a finite horizon, and x_j represents the number of order intervals of j time periods.

The methodology developed below for the case that c is convex is an extension of Chand's work for the EOQ problem. In fact, his proof technique, which used interchange arguments, is similar to the technique used to prove Theorem 1 below.

The Convex Case

We first consider the case in which c(·) is strictly convex. (If c(·) were convex but not strictly convex we could replace c(·) by $c'(j) = c(j) + \varepsilon^j$ for a suitably small ε.)

Henceforth, we let c_j denote c(j). As a preliminary we define the parametric linear programming problem LP(M).

$$\begin{aligned} \text{Minimize} \quad & \sum_{j=\ell}^{u} c_j x_j \\ \text{Subject to} \quad & \sum_{j=\ell}^{u} j x_j = b \qquad \qquad \text{LP(M)} \\ & \sum_{j=\ell}^{u} x_j = M \\ & x_j \geq 0 \quad \text{for } j = \ell, \ldots, u. \end{aligned}$$

We denote each instance of P as a quaduple $<c, \ell, u, b>$, and we denote each instance of LP(M) as a quintuple $<c, \ell, u, b, M>$

LEMMA 1. Let $<c, \ell, u, b, M>$ be an instance of LP(M) such that c is strictly convex and such that M is an integer with $b/u \le M \le b/\ell$. Let $t = \lfloor b/M \rfloor$. Then the unique optimal solution for $<c, \ell, u, b, M>$ is x^* defined as follows:

$$x_j^* = \begin{cases} (t + 1)M - b & \text{for } j = t \\ b - tM & \text{for } j = t + 1 \\ 0 & \text{otherwise} \end{cases} .$$

PROOF. First, it is clear that the linear program is feasible and bounded and thus there is some optimal basic solution x^* in which x_r^* and x_s^* are the basic variables. If only one of these variables is non-zero (say x_r^*) then $x_r^* = M$ and the theorem is true with $r = t$. Otherwise, assume that $r < s$. We now claim that $s = r + 1$. To see this suppose that $s > r + 1$, and let x' be obtained from x^* by (1) decreasing x_s^* and x_r^* by ε and (2) increasing x_{s-1} and x_{r+1} by ε, where $\varepsilon > 0$ is sufficiently small. Then x' is feasible for LP(M). Moreover, since $c(\cdot)$ is strictly convex it follows that

$$c_{r+1} + c_{s-1} < c_r + c_s,$$

and thus x' is a strict improvement over x^*, contrary to assumption.

By the above, x^* is a non-degenerate optimal basic solution in which x_r^* and x_{r+1}^* are basic. It follows that $x_r^* = (r + 1)M - b$ and $x_{r+1}^* = b - rM$. The lemma is true for $r = t$.□

In the following, we let $x(M)$ denote the unique optimal solution for instance $<c, \ell, u, b, M>$ of LP(M). We let $z(M)$ denote its objective value.

COROLLARY. Suppose that M^* is an integer chosen so that $z(M^*) = \min (z(M) : M \text{ integer})$. Then $x(M^*)$ is optimal for P.

<u>PROOF</u>. Suppose that x' is optimal for P, and let z' denote its objective value. Then $z' \le z^*(M)$ for all M. Suppose now that $M' = \sum_{j=\ell}^{u} x_j'$. Then x' is

feasible for LP(M') and thus $z^*(M') \le cx' = z'$. Thus $M^* = M'$ and the proof is complete.□

LEMMA 2. Let r be an index chosen so that $c_r/r = \min(c_j/j : \ell \le j \le u)$ for problem P. Let $M' = b/r$. Then the integer value M^* that minimizes z(M) is $\lfloor M' \rfloor$ or $\lceil M' \rceil$.

PROOF. Let LP denote the continuous relaxation of the Knapsack problem P. Then an optimum solution for LP is as follows: $x_r' = M'$, $x_j' = 0$ for $j \ne r$. Then x' is also optimal for LP(M') and thus z(M) is minimized at M = M'. Finally, z(M) is convex in M since it is the optimal value function of a parametric linear program, and thus the integer value of M that minimizes z(M) is either $\lfloor M' \rfloor$ or $\lceil M' \rceil$.□

We combine Lemma 1 and its corollary and Lemma 2 into the following theorem.

THEOREM 1. Suppose that $\langle c, \ell, u, b\rangle$ is an instance of problem P, and suppose that c is a convex function. Let r be chosen so that $c(r)/r = \min\{c(j)/j : \ell \le j \le u\}$, and let $M' = \lfloor b/r \rfloor$. Then an optimum solution for problem P (if one exists) is one of the following

(i)
$$x_j = \begin{cases} (b+1)M' - b & \text{if } j = t \\ b - tM' & \text{if } j = t+1 \\ 0 & \text{otherwise} \end{cases}$$

where $t = \lfloor b/M' \rfloor$,

or

(ii)
$$x_j = \begin{cases} (t+1)(M'+1) - b & \text{if } j = t \\ b - t(M'+1) & \text{if } j = t+1 \\ 0 & \text{otherwise} \end{cases}$$

where $t = \lfloor b/(M'+1) \rfloor$. □

We note that we have dropped our assumption of strict convexity. As before, if c is convex then we may perturb c to c' where c' is strictly convex. Moreover, an optimal solution for $\langle c', \ell, u, b\rangle$ is also optimal for $\langle c, \ell, u, b\rangle$.

We note that although c(·) is convex, it does not follow that the function

$c(x)/x$ is convex. For example, consider the function $c(x) = x - 1$. Nevertheless, we can compute the minimum ratio in $\log(u - \ell)$ iterations, as is implicit in the following Lemma.

LEMMA 3. Suppose that c is a convex function. If $c(r)/r \le c(r+1)/(r+1)$, then $c(r)/r \le c(j)/j$ for all $j \ge r+1$. If $c(r)/r \ge c(r+1)/(r+1)$, then $c(r+1)/(r+1) \le c(j)/j$ for all $j \le r$.

PROOF. Since c is convex, for $j \ge r+1$,

$$(j-r)\ c(r+1)/j \le (j-r-1)\ c(r)/j + c(j)/j. \qquad (1)$$

Moreover, if we assume that $c(r)/r \le c(r+1)/(r+1)$ then

$$(j-r)(r+1)\ c(r)/(rj) \le (j-r)\ c(r+1)/j. \qquad (2)$$

Combining (1) and (2) yields that $c(r)/r \le c(j)/j$.

Since c is convex, it is also true that for $j \le r$

$$(r-j+1)\ c(r)/j \le (r-j)\ c(r+1)/j + c(j)/j. \qquad (3)$$

If we assume that $c(r)/r \ge c(r+1)/(r+1)$ then

$$r(r-j+1)\ c(r+1)/((r+1)j) \le (r-j+1)\ c(r)/j. \qquad (4)$$

Combining (3) and (4) yields that $c(r+1)/(r+1) \le c(j)/j$ for $j \le r$, completing the proof.□

By Lemma 3, we can determine the minimum ratio of $c(j)/j$ in $O(\log(u-\ell))$ steps by using binary search to locate the minimum index r for which $c(r)/r \le c(r+1)/(r+1)$.

The Concave Case

Below we solve the case in which $c(\cdot)$ is strictly concave. We offer two different algorithms, one of which takes $O(\min(\ell, b/u))$ steps and the other of which takes $O(\min((b/\ell) - (b/u), u-\ell))$ steps. Unfortunately, the author does not know of any algorithm that runs in time polynomial in log b. We do note that for $\ell = 1$ or $u = b$, the first algorithm runs in $O(1)$ steps.

As a preliminary, we define the parametric integer program P(M) similarly to the problem LP(M) for the convex case

$$\begin{aligned} &\text{Minimize} \quad \sum_{j=\ell}^{u} c_j x_j \\ &\sum_{j=\ell}^{u} j x_j = b \\ &\sum_{j=\ell}^{u} x_j = M \\ &x_j \geq 0 \ \text{ integer } \ \ell \leq j \leq u. \end{aligned} \qquad P(M)$$

LEMMA 4. Suppose that <c, ℓ, u, b> is an instance of the Knapsack problem P and that c is strictly concave. Suppose further that M is an integer such that $b/u \leq M \leq b/\ell$. Then there is a unique optimal solution x^* for P(M) defined as follows:

(i) $x_u^* = \lfloor (b - \ell M)/(u - \ell) \rfloor$,

(ii) $x_\ell^* = \lfloor (uM - b)/(u - \ell) \rfloor$,

(iii) If $r \equiv (b - \ell M) \bmod (u - \ell)$ and $1 \leq r \leq u - \ell - 1$ then $x_{\ell+r}^* = 1$,

(iv) $x_j^* = 0$ otherwise.

PROOF. Let $k = x_{\ell+1}^* + \ldots + x_{u-1}^*$. We first show that $k \leq 1$. Suppose otherwise that $k \geq 2$. Choose s, t so that $\ell + 1 \leq s, t \leq u - 1$ and either (1) $x_s, x_t \geq 1$ and $s \neq t$ or else (2) $s = t$ and $x_s \geq 2$. Let x' be obtained from x^* by decrementing both x_s and x_t by 1 (i.e., if s = t we increment x_s by 2), and incrementing x_{s-1} and x_{t+1} by 1. Let z^* and z' be the objective values for x^* and x' respectively. Then x' is feasible for P(M). Moreover,

$$z^* - z' = c(s) + c(t) - c(s - 1) - c(t - 1),$$

and $z^* - z' > 0$ by the concavity of $c(\cdot)$, contradicting the optimality of x^*. Thus we have proved that $k \leq 1$.

If k = 0, then x_ℓ^* and x_u^* are determined uniquely by the equations "$\ell x_\ell^* + u x_u^* = b$", and "$x_\ell^* + x_u^* = M$". Thus $x_\ell^* = (uM - b)/(u - \ell)$ and

$x_u^* = (b - \ell M)/(u - \ell)$. Since x_u^* is integral it follows that $(u - \ell)$ is a divisor of $b - \ell M$. Thus (i) - (iv) all hold.

If $k = 1$, let r be the index such that $1 \le r \le u - \ell - 1$ and $x_{\ell+r}^* = 1$. Then $\ell x_\ell^* + u x_u^* = b - \ell - r$ and $x_\ell^* + x_u^* = M - 1$. Solving for x_ℓ^* and x_u^* we get that $x_\ell^* = (Mu - b - (u - \ell - r))/(u - \ell)$ and $x_u^* = (b - M\ell - r)/(u - \ell)$. Since $1 \le r \le u - \ell - 1$, it follows that (i) and (ii) hold. Since x_u^* is integral it follows that (iii) hold, and (iv) also holds.□

LEMMA 5. Suppose that <c, ℓ, u, b> is an instance of problem P and that c is strictly concave. Suppose further that x^* is optimal for P. Then at least one of (i), (ii), (iii) and (iv) is true.

(i) $x_\ell^* = 0$,

(ii) $x_\ell^* = (b - uX_u^*)/\ell$,

(iii) $x_\ell^* = \lceil (b - ux_u^* - u + 1)/\ell \rceil$,

or (iv) $x_\ell^* = \lfloor (b - ux_u^* - \ell - 1)/\ell \rfloor$.

PROOF. Let $k = x_{\ell+1}^* + \ldots + x_{u-1}^*$. By Lemma 4 we know that $k = 0$ or 1. If $k = 0$ then (ii) holds. Henceforth we consider the case that $k = 1$ and that $x_{\ell+r}^* = 1$ for $1 \le r \le u - \ell - 1$. Let $b' = b - ux_u^*$.

Then $x_\ell^* = (b' - \ell - r)/\ell$. If $1 \le r \le \ell - 1$, then (iv) holds. If $u - 2\ell \le r \le u - \ell - 1$, then (iii) holds. Let us now assume that $\ell \le r \le u - 2\ell - 1$ and that $x_\ell^* \ge 1$ and we will derive a contradiction.

Let x' be obtained from x^* by incrementing x_ℓ by 1, incrementing x_r by 1, and decrementing $x_{r+\ell}$ by 1. Let x'' be obtained from x^* by decrementing x_ℓ and $x_{r+\ell}$ by 1 and incrementing $x_{r+2\ell}$ by 1. Let z', z'' and z^* be the objective values for x', x'' and x^* respectively. Then

$$2z^* - z' - z'' = 2c(r + \ell) - c(r) - c(r + 2\ell) ,$$

and thus

$$(z^* - z') + (z^* - z'') > 0$$

by the strict concavity of $c(\cdot)$. It follows that $z' < z^*$ or $z'' < z^*$, contradicting the optimality of z^*. Thus the lemma is true.□

Lemma 4 suggests the following method for solving for problem P : solve P(M) for all integral M such that $b/\ell \le M \le b/u$ and choose the best of these solutions. Lemma 5 suggests the following method for solving P : for each integral value s with $0 \le s \le b/u$ let x^s be the best of the solutions (i), (ii), (iii) and (iv) of Lemma 5 with $x_u^s = s$. Then choose the best of the solutions x^s.

In order to improve the computational bounds of these two procedures, we show that the range of values for M and s can be limited further.

LEMMA 6. Suppose that $\langle c, \ell, u, b \rangle$ is an instance of problem P and that $c(\cdot)$ is strictly concave. Suppose further that x^* is an optimal solution and that $M = x_\ell^* + \ldots + x_u^*$. Then

(i) If $c_\ell/\ell < c_u/u$, then $0 \le x_u^* \le \ell - 1$ and $-(u - \ell) + (b/\ell) \le M \le b/\ell$.

(ii) If $c_\ell/\ell > c_u/u$, then $(b/u) - \ell - 1 \le x_u^* \le b/u$ and $(b/u) \le M \le (u-\ell) + (b/u)$.

PROOF. We note first that in any feasible solution $(b/u) \le M \le (b/\ell)$ and $0 \le x_u^* \le b/u$. If $c_\ell/\ell < c_u/u$ and $x_u^* \ge \ell$, then we can find an improved solution x' by decrementing x_u^* by ℓ and incrementing x_ℓ^* by u, contradicting the optimality of x^* . If $x_u^* \le \ell - 1$, then by (i) of Lemma 4 it follows that $M \ge -(u - \ell) + (b/\ell)$.

If $c_\ell/\ell > c_u/u$ and $x_\ell^* \ge u$, we can find an improved solution x' by decrementing x_ℓ^* by u and incrementing x_u^* by ℓ. Thus $x_\ell^* \le u - 1$ and by (ii) of Lemma 4 it follows that $M \le (u - \ell) + (b/u)$, and thus by (i) of Lemma 4 it follows that $x_u^* \ge (-\ell - 1) + (b/u)$.□

We observe that if $c_\ell/\ell = c_u/u$ then there may be multiple optimum. In this case, there is an optimum solution so that the conclusion (i) of Lemma 6 holds and "another" optimum solution so that the conclusion (ii) of Lemma 6 holds. The proof of this fact follows from the same "interchange" argument as in the proof of Lemma 6.

THEOREM 2. Suppose that $\langle c, \ell, u, b\rangle$ is an instance of problem P and that $c(\cdot)$ is strictly concave. Then we may solve P in $O(\min(\ell, b/u, (b/\ell) - (b/u), u - \ell))$ steps.

PROOF. The first method is to solve $P(M)$ for $b/u \le M \le b/\ell$ and choose the best of these solutions. Moreover, by Lemma 6 we may further restrict our search to a range of at most $u - \ell + 1$ consecutive integers. Thus this procedure is $O(\min((b/\ell) - (b/u), u - \ell))$ steps.

The second method is to consider the four solutions determined upon setting $x_u = s$ as provided by Lemma 5. The best of these x^s calculated in $O(1)$ steps. Moreover, we can restrict our search to at most $\min(\ell, b/u)$ values of x_u by Lemma 6. Thus determining the best solution x^s for this range of the parameter s takes $O(\min(\ell, b/u))$ steps, completing the proof.□

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