APPLICATIONS OF A GENERALIZATION OF A SET INTERSECTION THEOREM OF VON NEUMANN

by

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Abstract

In his paper on the expanding economic model, von Neumann developed a set intersection theorem that asserts that the intersection of two special sets in the cross-product of two convex sets cannot be empty. This theorem is generalized in two ways, first by expanding the number of sets involved beyond two, and then by relaxing the restriction of cross-products. The generalization of the theorem yields a direct proof of the existence of an equilibrium in an n-person noncooperative game, and a direct proof of the existence of a ϕ-flexible solution to a system of linear inequalities.

Keywords

set intersection, fixed point, game theory, flexible solutions, noncooperative game.

I. Introduction

A cell is defined to be a nonempty closed and bounded convex set. Let $S \in \mathbb{R}^{m_1}$ and $T \in \mathbb{R}^{m_2}$ be cells and let $R = S \times T$. Let U and V be subsets of R such that for each $x \in S$, $U(x) = \{y \in \mathbb{R}^{m_2} | (x, y) \in U\}$ is a cell, and for each $y \in T$, $\{x \in \mathbb{R}^{m_1} | (x, y) \in V\}$ is a cell. von Neumann's set intersection theorem [2] states that under the above conditions, $U \cap V \neq 0$. This theorem was used to prove the existence of equilibria in von Neumann's expanding economic model [2], and can be used to demonstrate the existence of an equilibrium in a two-person noncooperative game. In this note, von Neumann's theorem is generalized in two ways, first by expanding the number of sets involved beyond two, and then by relaxing the condition that R be a cross-product of cells. This generalization yields a direct proof of the existence of an equilibrium in an n-person noncooperative game. It also yields a proof of the existence of a ϕ-flexible solutions to systems of linear inequalities [4].

II. The Generalized Set Intersection Theorem

Let $R \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \ldots \times \mathbb{R}^{m_n}$ be a cell. For each $k = 1,\ldots, n$ let P^k denote the projection of R onto the coordinate space $\mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_{k-1}} \times \mathbb{R}^{m_{k+1}} \times \ldots \times \mathbb{R}^{m_n}$, i.e.
$P^k = \{(y^1, \ldots, y^{k-1}, y^{k+1}, \ldots, y^n) |$ there exists $x^k \in \mathbb{R}^{m_k}$ with $(y^1, \ldots, y^{k-1}, x^k, y^{k+1}, \ldots, y^n) \in R\}$. Let $S^1, \ldots, S^n$ each be a closed subset of R, with the property that for each $k = 1,\ldots, n$, and each $y \in P^k$, the set $S^k(y) = \{x^k | (y^1, \ldots, y^{k-1}, x^k, y^{k+1}, \ldots, y^n) \in S^k\}$ is a cell. The generalized set intersection theorem is:

<u>Theorem</u> Under the above conditions, $\bigcap_{k=1}^{n} S^k \neq \phi$.

Note that von Neumann's theorem is a special case of the above, where $n = 2$, $S \subset \mathbb{R}^{m_1}$, $T \subset \mathbb{R}^{m_2}$, $R = S \times T$, $U = S^2$, and $V = S^1$.

<u>Proof of the Theorem</u>: The proof follows from Kakutani's fixed point theorem, see [1]. For each $k = 1,\ldots, n$, define Q^k to be the projection of R onto the coordinate space $\mathbb{R}^{m_k}$, i.e. $Q^k = \{x^k \in \mathbb{R}^{m_k} \mid \text{there exists } y \in P^k$ with $(y^1, \ldots, y^{k-1}, x^k, y^{k+1}, \ldots, y^n) \in R\}$. Let $Q = Q^1 \times Q^2 \times \ldots \times Q^n$, i.e. Q is the smallest cross-product (of sets in $\mathbb{R}^{m_k}$) that contains R. For each $x = (x^1, \ldots, x^k, \ldots, x^n) \in Q$, define the point-to-set mapping $F(x) = (F^1(x), \ldots, F^n(x)) : Q \to Q^*$, where Q^* is the set of cells contained in Q, by $F^k(x) = S^k(x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^n)$ if $x \in R$. If $x \notin R$, define $F(x) = F(\bar{x})$, where $\bar{x} \in R$ is the unique point in R which minimizes the Euclidean distance $\| \bar{x} - x \|_2$ over all points $\bar{x}$ in R. Because each set S^k is closed, and Q is bounded, $F(\cdot)$ is upper semi-continuous and maps elements of Q to cells that are subsets of Q. Thus, by Kakutani's fixed-point theorem, there exists $\tilde{x} \in Q$ such that $\tilde{x} \in F(\tilde{x})$. If $\tilde{x} \in R$, then $\tilde{x} \in F(\tilde{x})$ means $\tilde{x}^k \in S^k(\tilde{x}^1, \ldots, \tilde{x}^{k-1}, \tilde{x}^{k+1}, \ldots, \tilde{x}^n)$ for all $k = 1, \ldots, n$, i.e. $\tilde{x} \in S^k$ for $k = 1, \ldots, n$, proving the theorem. It only remains to show that $\tilde{x}$ must be an element of R.

Suppose $\tilde{x}$ is not an element of R. Let $\bar{x}$ be the unique point in R whose distance $\| \bar{x} - \tilde{x} \|_2$ to $\tilde{x}$ is minimized over all points in R. Then $\tilde{x} \in F(\tilde{x}) = F(\bar{x})$. By a separating hyperplane thoerem, $x(\tilde{x} - \bar{x}) \leq \bar{x}(\tilde{x} - \bar{x})$ for any $x \in R$, and $\tilde{x}(\tilde{x} - \bar{x}) > \bar{x}(\tilde{x} - \bar{x})$. Since $\tilde{x} \in F(\bar{x})$, $\tilde{x}^k \in S^k(\bar{x}^1, \ldots, \bar{x}^{k-1}, \bar{x}^{k+1}, \ldots, \bar{x}^n)$, which means that $(\bar{x}^1, \ldots, \bar{x}^{k-1}, \tilde{x}^k, \bar{x}^{k+1}, \ldots, \bar{x}^n) \in S^k \subset R$, for all $k = 1,\ldots, n$.

Therefore, $(\bar{x}^1, \ldots, \bar{x}^{k-1}, \tilde{x}^k, \bar{x}^{k+1}, \ldots, \bar{x}^n)(\tilde{x} - \bar{x}) \le \bar{x}(\tilde{x} - \bar{x})$, whereby $\tilde{x}^k(\tilde{x}^k - \bar{x}^k) \le \bar{x}^k(\tilde{x}^k - \bar{x}^k)$ for all $k = 1, \ldots, n$. Summing this last inequality over k, we obtain $\tilde{x}(\tilde{x} - \bar{x}) \le \bar{x}(\tilde{x} - \bar{x})$, which is a contradiction. Therefore $\tilde{x} \in R$, and so the theorem is proved. ⊠

III. Applications of the Generalized Set Intersection Theorem

A. N-Person Noncooperative Game Theory

An n-person noncooperative game is characterized by n players, each of which has m_k pure strategies, $k = 1, \ldots, n$, and a loss function $a(j; i_1, \ldots, i_n)$ defined as the loss incurred by player j if the k^{th} player chooses pure strategy i_k, where $i_k \in \{1, \ldots, m_k\}$, $k = 1, \ldots, n$. A mixed strategy for player k is a vector $x^k \in T^k = \{x^k \in \mathbb{R}^{m_k} \mid \sum_{j=1}^{m_k} x_j^k = 1, x^k \geq 0\}$, which is the standard simplex in $\mathbb{R}^{m_k}$, $k = 1, \ldots, n$. Let $T = T^{m_1} \times T^{m_2} \times \ldots \times T^{m_k}$, and for any strategy vector $x \in T$, the loss to player k is given by the function

$$f_k(x) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \ldots \sum_{i_n=1}^{m_n} a(k; i_1, \ldots, i_n) \prod_{j=1}^{n} x_{i_j}^j .$$

A strategy $\tilde{x} \in T$ is said to constitute a (Nash) equilibrium if

$$f_k(\tilde{x}) \le f_k(\tilde{x}^1, \ldots, \tilde{x}^{k-1}, x^k, \tilde{x}^{k+1}, \ldots, \tilde{x}^n) \quad \text{for any } x^k \in T^{m_k} \text{, for}$$

$k = 1, \ldots, n$.

Most proofs of the existence of an equilibrium rely directly on Brouwer's or Kakutani's fixed-point theorem. Below, the result is derived as a consequence of the generalized set intersection theorem, rendering more of a geometric interpretation to the existence of an equilibrium.

<u>Proof of the Existence of an Equilibrium</u>. Let $R = T$, and let $S^k = \{\bar{x} \in R \mid f_k(\bar{x}) \le f_k(\bar{x}^1, \ldots, \bar{x}^{k-1}, x^k, \bar{x}^{k+1}, \ldots, \bar{x}^n) \text{ for any } x^k \in T^{m_k}\}$. R is a cell, and S^k is closed, and $S^k(x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^n)$ is a

subsimplex of T^{m_k}, and so is a cell. Thus the hypotheses of the set intersection theorem are satisfied, whereby there exists $\tilde{x} \in R$ such that $\tilde{x} \in S^k$ for $k = 1, \ldots, n$. But then

$$f(\tilde{x}) \le f(\tilde{x}^1, \ldots, \tilde{x}^{k-1}, x^k, \tilde{x}^{k+1}, \ldots, \tilde{x}^n) \quad \text{for any} \quad x^k \in T^{m_k},$$

and so $\tilde{x}$ is an equilibrium.

B. Flexible Solutions to Linear Inequalities

Let $R = \{x \in \mathbb{R}^n \mid Ax \le b\}$ for a given matrix and vector (A, b), and assume that R is bounded, whereby R is a cell. For a given $x \in R$, define for $k = 1, \ldots, n$,

$$s_+^k(x) = \min_{a_{ik}>0} \left\{\frac{b_i - A_{i.}x}{a_{ik}}\right\} \text{ and } \quad s_-^k(x) = \min_{a_{ik}<0} \left\{\frac{b_i - A_{i.}x}{-a_{ik}}\right\},$$

where $A_{i.}$ is the i^{th} row of A. $s_+^k(x)$ and $s_-^k(x)$ are the distances from x to the boundary of R along the k^{th} coordinate in the positive and negative directions, respectively. For a given vector ϕ with $0 \le \phi \le (1, \ldots, 1)$, $x \in R$ is said to be ϕ_k-flexible if $(1 - \phi_k) s_+^k(x) = \phi_k\, s_-^k(x)$, for $k = 1, \ldots, n$, see van der Vet [4]. This corresponds to $\dfrac{s_+^k(x)}{s_+^k(x) + s_-^k(x)} = \phi_k$ in the case where $s_+^k(x) + s_-^k(x) \ne 0$. If x is simultaneously ϕ_k-flexible for all $k = 1, \ldots, n$, x is said to be a ϕ-flexible solution.

In [4], van der Vet uses Brouwer's fixed-point theorem to prove that for any ϕ with $0 \le \phi \le (1, \ldots, 1)$, a ϕ-flexible solution exists. Herein, this existence result is proved as a consequence of the generalized set intersection theorem.

Proof of the Existence of a ϕ-Flexible Solution: Let R be given as above, and let $m_1 = m_2 = \ldots = m_n = 1$. Define P^k as in section II for $k = 1, \ldots, n$. For a given $y \in P^k$, define

$$f^k(y) = (1 - \phi_k) \min \{x_k \mid (y_1, \ldots, y_{k-1}, x_k, y_{k+1}, \ldots, y_n) \in R\}$$
$$+ \phi_k \max \{x_k \mid (y_1, \ldots, y_{k-1}, x_k, y_{k+1}, \ldots, y_n) \in R\} .$$

$f^k(y)$ is the value of x_k such that $(y_1, \ldots, y_{k-1}, f^k(y), y_{k+1}, \ldots, y_n)$ is ϕ_k-flexible. Define $S^k = \{(y_1, \ldots, y_{k-1}, f^k(y), y_{k+1}, \ldots, y_n) \mid y \in P^k\}$, i.e., S^k is the graph of the function $f^k(\cdot)$, $k = 1, \ldots, n$. Because each $S^k \subset R$, and $S^k(y)$ is a single point for $y \in P^k$, R and $S^1, \ldots, S^n$ satisfy the hypotheses of the set intersection theorem. Thus there exists $\tilde{x} \in R$ with $\tilde{x} \in S^k$, $k = 1, \ldots, n$. This in turn means that $\tilde{x}$ is ϕ_k-flexible for $k = 1, \ldots, n$, whereby $\tilde{x}$ is a ϕ-flexible solution.

Remark

The geometry of the sets S^k in the flexible solution problem is similar to that of von Neumann's set intersection theorem for the case when $n = 2$. It was this similarity that motivated the development of a generalization of von Neumann's theorem applicable to the problem of ϕ-flexible solutions.

References

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