

SALVAGING OF EXCESS INVENTORY

by

Donald B. Rosenfield  
Visiting Associate Professor  
M.I.T. Sloan School of Management

Senior Consultant  
Arthur D. Little, Inc.

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## 1. Introduction

A critical problem in the management of many businesses; manufacturing, distributing, and retailing; is the management of inventory that is slow-moving or obsolete. For many reasons, businesses will find themselves in an overstock situation. Favorable buying opportunities followed by a reduction in activity is a typical cause. In other situations, the management of large numbers of items, coupled with inadequate information and forecasting systems, may lead to overestimates of demand that are not reviewed at proper intervals. Finally, substitution of products or parts can significantly and instantaneously reduce demand for specific items.

Unfortunately, there are no real guidelines for the disposal of excess inventory. While items have value in that they may ultimately sell, albeit far in the future, they can incur space and other holding costs in the interim. Furthermore they may have some salvage value, and, before they sell, the items could perish or become completely obsolete. While the space costs and holding costs are not as substantial as the usual capital costs of inventory (which should not be considered in the analysis of slow-moving inventory, as this capital is sunk), they can amount to several percent of item value.

While the problem of salvaging of excess inventory has been a problem, of interest for many years, [2], [3], [5], no previous treatment addressed the probabilistic nature of demand. In fact, the slow-moving stock is subject to uncertain demand and the expected value of potential sales needs to be considered. Typically, when demand events are infrequent, a Poisson distribution for demand, and hence exponential times between sales, may be appropriate. We assume a Poisson distribution of demand in this paper. We also assume, without loss of generality, a batch size (number of units per

sale event) of one. With a larger batch size the average sale and salvage values increase proportionally.

The real question in the treatment of excess inventory is not necessarily whether to salvage the lot, but what number of units (or batches) to keep. When items are ordered, each successive item is worth less, since it is expected to sell at a later time. Hence higher numbered items should be salvaged. An item is worth saving if its salvage value exceeds its expected discounted sales value less expected space and holding costs up until the time of sale. There is also a question of perishability or complete obsolescence. When this is likely, the future sales value is further reduced. In any case, there will be an item number above which salvage (e.g. disposal) is optimal, and this paper presents a derivation of the threshold number of items worth saving.

The next section presents the basic formula and derivation for the correct number of items to save and an expression for the savings obtained from following the optimal salvaging policy. Section 3 presents the effects of perishability, which can manifest itself in different ways. Section 4 presents an example based on an actual case study for a distributor faced with substantial excess inventory.

## 2. The Basic Problem

To solve the problem we make the following definitions:

- $\lambda$  = Number of demands per item per unit time
- T = Expected time until next demand ( $T = 1/\lambda$  if demand is Poisson)
- A = Average ultimate sale or disposal value as a percentage of current value

$r$  = Cost of space and other non-capital holding costs as a percent of current value

$i$  = discount rate

$S$  = Salvage value at present time as a percent of current value

We assume that demand is Poisson, and the issue is to determine how many units (or sets of units if ordered in fixed batches) to dispose and what the resulting savings will be.

Suppose a specific item sells at time  $W$ . Then the total value is the selling value less storage costs:

$$\begin{aligned} \text{Value} &= Ae^{-iW} - \int_0^W re^{-iW} dt \\ &= e^{-iW}(A + r/i) - r/i \end{aligned} \quad (1)$$

Now assume there are  $m$  units in stock. Each one is sold in turn at times  $T_1, T_2, \dots, T_m$ . The issue is the expected value of  $e^{-iT_j}$ . If the time of each successive sale is exponential, then the  $j$ th unit is gamma with parameters  $\lambda$  and  $j$ .

Hence

$$\begin{aligned} E(e^{-iT_j}) &= \int_0^{\infty} \frac{1}{\Gamma(j)} \lambda^j (T_j)^{j-1} e^{-\lambda T_j} e^{-iT_j} dT_j \\ &= \frac{\lambda^j}{\Gamma(j)} \int_0^{\infty} T_j^{j-1} e^{-(\lambda+i)T_j} dT_j \\ &= \frac{\lambda^j}{\Gamma(j)} \frac{\Gamma(j)}{(\lambda+i)^j} = \left(\frac{\lambda}{\lambda+i}\right)^j \end{aligned}$$

Thus the value of the  $j$ th unit is

$$\left(\frac{\lambda}{\lambda + i}\right)^j (A + r/i) - r/i$$

Note that this is monotonic in  $j$ , and hence by equating it to the salvage value  $S$ , we obtain that the optimal number of units to be maintained is

$$m^* = \text{greatest integer less than } \frac{\log \left(\frac{S + r/i}{A + r/i}\right)}{\log \left(\lambda/(\lambda + i)\right)}$$

Alternatively, if  $T = \text{mean time until usage} = 1/\lambda$

$$m^* = \text{greatest integer less than } \frac{\log \left(\frac{A + r/i}{S + r/i}\right)}{\log (1 + iT)}$$

The next issue becomes the potential discounted savings in salvaging units in excess of the level  $m^*$ . Let  $\Delta = \text{number of units disposed so the direct savings (in percentage of unit value is)}$

$$S\Delta$$

By (1) the cumulative value "lost" is

$$\sum_{j=m^*+1}^{m^*+\Delta} \{e^{-iTj}(A + r/i) - r/i\} = (A + r/i) \left( \sum_{j=m^*+1}^{m^*+\Delta} e^{-iTj} \right) - \frac{\Delta r}{i}$$

and the expected value of this is

$$(A+r/i) \left( \sum_{m^*+1}^{m^*+\Delta} \left(\frac{\lambda}{\lambda + i}\right)^j \right) - \frac{\Delta r}{i} = (A+r/i) \frac{\left(\frac{\lambda}{\lambda + i}\right)^{m^*+1} - \left(\frac{\lambda}{\lambda + i}\right)^{m^*+\Delta+1}}{1 - (\lambda/(\lambda+i))} - \frac{\Delta r}{i}$$

Hence, the net savings is the difference between  $S\Delta$  and this, which reduces to

$$\Delta S + \Delta r/i - (A + r/i)(1 - (1 + iT)^{-\Delta})/(iT(1 + iT)^{m^*})$$

### 3. The Case of Perishability

The decision problem becomes somewhat more complex if stocks can perish or become obsolescent. There are actually a number of different assumptions about perishability that can be made:

- (a) All items perish or become obsolete together at a random time.
- (b) All items perish or become obsolete together at a known time.
- (c) Each item can perish at a random time.

Cases (a) and (b) might represent situations where items become obsolete due to product substitution. Case (c) might represent real perishing. Substitution might affect all products and perishing affects each one separately.

Note that these approaches cover the situations where all items can be treated identically. If, for example, there are known perishing dates that vary, then it makes sense that the salvage values will vary as well. When products have different ages, the optimum depletion problem is a separate problem area by itself (eg. [1])

a) If all items perish together at a random time then an items value, analogous to relationship (1), is

$$Ae^{-iT_j}(1 - F(T_j)) - \int_0^{T_j} re^{-it}(1 - F(t))dt \quad (2)$$

where  $T_j$  now represents the items hypothetical sell date (when it sells if it hasn't perished) and where  $F(t)$  = distribution function of perishing time. For a constant hazard rate,  $F(t)$  is exponential, and we obtain a value of

$$\begin{aligned}
 & A e^{-iT_j} e^{-hT_j} - \int_0^{T_j} r e^{-it} e^{-ht} dt \quad (3) \\
 & = A e^{-(i+h)T_j} - \frac{r}{i+h} (1 - e^{-(i+h)T_j}) \\
 & = e^{-(i+h)T_j} \left( A + \frac{r}{i+h} \right) - \frac{r}{i+h}
 \end{aligned}$$

where  $h =$  hazard rate (i.e.  $F(t) = 1 - e^{-ht}$ ).  $T_j$  is still a  $j$ th-order gamma. The identical relationships hold with  $i$  replaced by  $i + h$ . Hence

$$m^* = \text{greatest integer less than } \frac{\log \left( \frac{A + r/(i+h)}{S + r/(i+h)} \right)}{\log (1 + (i+h)T)}$$

with an analogous relationship for savings.

b) If all items perish at a known date  $p$ , then the items value, given a sale date  $T_j$ , is

$$A e^{-iT_j} - \int_0^{T_j} r e^{-it} dt = e^{-iT_j} (A + r/i) - r/i \quad \text{if } T_j < p$$

and

$$- \int_0^p r e^{-it} dt = - \frac{r}{i} (1 - e^{-ip}) \quad \text{if } T_j \geq p$$

which is monotonic in  $T_j$  and hence  $j$ .

Hence the expected value of the  $j$ th item is

$$\begin{aligned}
 & (A+r/i) \int_0^p \frac{1}{\Gamma(j)} \lambda^j (T_j)^{j-1} e^{-(\lambda+i)T_j} dT_j + \frac{r}{i} \text{Prob} (T_j \geq p) e^{-ip} - \frac{r}{i} \\
 & = (A+r/i) \left( \frac{\lambda}{\lambda+i} \right)^j \int_0^p \frac{1}{\Gamma(j)} (\lambda+i)^j T_j^{j-1} e^{-(\lambda+i)T_j} dT_j + \frac{r}{i} \text{Prob} (T_j > p) e^{-ip} - \frac{r}{i}
 \end{aligned}$$

The integral is the probability that a gamma function with parameters  $j$  and  $\lambda+i$  will be less than or equal to  $p$ . This is the same as a poisson

variate with parameter  $(\lambda+i)p$  will be at least  $j$ . Similarly  $\text{Prob}(T_j \geq p)$  is the probability that a gamma with parameters  $j$  and  $\lambda$  will be greater than  $p$ , or equivalently, the probability that a poisson variate with parameter  $(\lambda p)$  is less than  $j$ . So the value of the  $j$ th item is

$$(A + r/i) \left( \frac{\lambda}{\lambda + i} \right)^j \sum_{k=j}^{\infty} \frac{[(\lambda+i)p]^k e^{-(\lambda+i)p}}{k!} - \frac{r}{i} + \frac{r}{i} e^{-ip} \sum_{k=0}^{j-1} \frac{(\lambda p)^k e^{-\lambda p}}{k!}$$

Since the marginal value is monotonic in  $j$ ,  $m^*$  is determined by equating this marginal value to  $S$ . So  $m^* = \max j$  such that

$$S < (A + r/i)(1 + iT)^{-j} \sum_{k=j}^{\infty} \frac{(pi + p/T)^k e^{-(pi+p/T)}}{k!} - r/i + r/i \sum_{k=0}^{j-1} \frac{(p/T)^k e^{-p/T}}{k!}$$

c) The most complex case is when each item can perish randomly. Again we assume constant hazard rates. In addition each item can perish separately. Suppose we keep  $m^*$  items and we order them so that we try to sell item  $j+1$  only after item  $j$  sells or perishes.

Lemma: Each successive item has a decreasing marginal value.

Proof: The time that an item  $j$  sells or is disposed  $T_j$  is the minimum of its perishing time  $P_j$  or sale date  $S_j$ . (We define sales date as the time a units "turn" comes plus an exponential variate with parameter  $\lambda$ ).

Also note that  $P_j$  is independent of  $S_j$ . If  $j > k$  then  $S_j > S_k$  and since  $P_j$  has the same distribution as  $P_k$ ,  $T_j$  stochastically dominates  $T_k$ . (In particular  $\text{Prob}(T_j \leq Y) = \text{Prob}((S_j \leq Y) \cup (P_j \leq Y)) = 1 - \text{Prob}(S_j > Y) \text{Prob}(P_j > Y) < 1 - \text{Prob}(S_k > Y) \text{Prob}(P_k > Y) = \text{Prob}(T_k < Y)$ ). Since marginal value is a monotonic function of



$T_j$  (storage costs increase and sales value decreases as  $T_j$  increases), and the expected value of a monotonic function  $G(x)$  is

$$\int_0^{\infty} G(x)f(x)dx = G(0) + \int_0^{\infty} G'(x)[1-F(x)]dx$$

by the definition of stochastic dominance, marginal value decreases. ■

The last item (which is tagged or specified as such) gets its turn when all of the other items have been sold or have perished. Suppose this occurs at time  $T_0$ . Then the value of the last item is

$$\begin{aligned} Ae^{-iT_{m^*}} - \int_0^{T_{m^*}} re^{-it} dt &= Ae^{-iT_{m^*}} - \frac{r}{i} (1 - e^{-iT_{m^*}}) \\ &= \left(A + \frac{r}{i}\right)e^{-iT_{m^*}} - r/i \end{aligned}$$

if it sells at a future time  $T_{m^*}$ . It is

$$- \int_0^{T_{m^*}} re^{-it} dt = - \frac{r}{i} (1 - e^{-iT_{m^*}})$$

if it perishes at a future time  $T_{m^*}$ . Finally it is

$$- \int_0^{\bar{T}} re^{-it} dt = - \frac{r}{i} (1 - e^{-i\bar{T}})$$

if perished at a prior time  $\bar{T}$ .

The likelihood that it already perished is  $(1 - e^{-hT_0})$  and the conditional distribution of  $\bar{T}$  is  $he^{-h\bar{T}} / (1 - e^{-hT_0})$ . The conditional distribution of the time interval after  $T_0$  that the item sells or perishes is  $(\lambda + h)e^{-(\lambda + h)t}$  and the likelihood that it sells before it perishes given that it perishes or sells in the future is

$$\frac{\lambda}{\lambda + h}$$

The latter relationships follow since the minimum of two exponentials is exponential with parameter equal to the sum of parameters. Furthermore, the conditional likelihoods of the respective parts being the minimum are proportional to the parameters. Putting this together, the value of the last item is

$$\begin{aligned}
 & \int_0^{T_0} h e^{-h\bar{T}} \left( -\frac{r}{i} (1 - e^{-i\bar{T}}) \right) \\
 & + e^{-hT_0} \frac{h}{\lambda+h} \int_0^{\infty} -\frac{r}{i} (1 - e^{-i(T_0+t)}) (\lambda+h) e^{-(\lambda+h)t} dt \\
 & + e^{-hT_0} \frac{\lambda}{\lambda+h} \int_0^{\infty} \left( (A + \frac{r}{i}) e^{-i(T_0+t)} - \frac{r}{i} \right) (\lambda+h) e^{-(\lambda+h)t} dt \\
 & = -\frac{r}{i} + \frac{r}{i} \cdot \frac{h}{h+i} (1 - e^{-(i+h)T_0}) \\
 & \quad + \frac{r}{i} \frac{h}{\lambda+h+i} e^{-(i+h)T_0} \\
 & \quad + (A + \frac{r}{i}) \left( \frac{\lambda}{\lambda+h+i} \right) e^{-(i+h)T_0} \\
 & = -\frac{r}{i+h} + e^{-(i+h)T_0} \left[ \frac{\lambda r}{(h+i)(\lambda+h+i)} + \frac{A\lambda}{\lambda+h+i} \right]
 \end{aligned}$$

So then the issue is the distribution of  $T_0$ . This is the expected sum of  $m^*-1$  independent exponential variates  $x_i$  in succession. (We now are essentially reordering the first  $m^*-1$  events). The first variate is the minimum of an exponential sale and  $m^*-1$  exponential perish events. This random variable is exponential with parameter  $\lambda+(m^*-1)h$  since again the minimum of a set of exponentials is exponential with parameter equal to the sum of parameters. The second is exponential with parameter  $\lambda+(m^*-2)h$ , and so forth. It follows inductively that

$$E(e^{-(i+h)T_0}) = E(e^{-\sum_{i=1}^{m^*-1} x_i})$$

$$= \prod_{k=1}^{m^*-1} \frac{\lambda+kh}{\lambda+i+(k+1)h}$$

It follows that  $m^*$  is

$$m^* = \max j \text{ such that } \prod_{k=1}^{j-1} \frac{1/T+kh}{1/T+i+(k+1)h} > \frac{S+r/(i+h)}{\left[ \frac{r/T}{(h+i)(h+i+1/T)} + \frac{A/T}{(h+i+1/T)} \right]}$$

or equivalently,

$$m^* = \max j \text{ such that } \prod_{k=1}^j \left( \frac{1/T+(k-1)h}{1/T+i+kh} \right) < \frac{S+r/(i+h)}{A+r/(i+h)}$$

To show the left side indeed goes to zero and that  $m^*$  is defined, we note that

$$\prod_{k=1}^j \frac{1/T+(k-1)h}{1/T+i+kh}$$

$$= \frac{1}{T} \cdot \frac{1/T+h}{1/T+i+h} \cdot \dots \cdot \frac{1/T+(j-1)h}{1/T+i+(j-1)h} \cdot \frac{1}{1/T+i+jh}$$

$$\leq \frac{1/T}{1/T+i+jh} \text{ which goes to zero as } j \rightarrow \infty.$$

#### 4. Applications

The relative simplicity of the basic problem and the simultaneous perishing case enables relatively direct determination of optimal salvaging values. The author was recently involved in a study for a distributor of durable goods. Based on the sales activities for a large number of items, there was a significant amount of excess stocks. In fact, 30% of the inventory did not move in the previous nine months and items for which there was only one transaction accounted for an additional 10% of inventory. Using time since last sale as a proxy for expected time until sale, no perishing, a capital charge of .12, a space cost  $r$  of .025 and three different salvage values (0, .25, and .50 percent), the excess stocks were carefully reviewed for a sample of 1000 items. This sample had known previous sales dates between 6 and 12 months prior. (This stock moved faster than stock that did not move at all, but for the latter, there were no known sales dates to analyse.)

We discovered, for example, that hundreds of years of supply existed for some items. The aggregate results obtained by applying the formula to each item were as follows:

	Salvage Value		
	0	75%	50%
Percentage of Investment Retained	51%	43%	36%
Savings as a Percentage of Inventory Investment	1%	15%	33%

The company was able to substantially reduce its inventory investment through the application of these methods.

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