

OPTIMAL ORDERING REGIONS UNDER
JOINT REPLENISHMENT

by

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1. Introduction

The inventory theory literature contains extensive treatment of single item policies, as well as treatment of joint replenishment (i.e. ordering frequency) under deterministic demand. What is not treated very thoroughly is the problem of joint replenishment, that is, a single set-up cost for two or more items, under stochastic demand. Ignall [2] notes that this problem is difficult, and that optimal policies may be complex. Johnson [3] and Kalin [4] identify (σ, s) policies under certain conditions where σ is an optimal ordering region. Silver [6] hypothesizes can order regions under the conditions of major and minor set-up costs. There are no definitive results, however, about the nature of the optimal ordering region σ .

This paper deals with the most basic type of issue in joint replenishment: when to order two items under a common set-up charge and independent stochastic demands. Both the theoretical and practical implications are self-evident. Indeed, practitioners are commonly faced with this problem. Typically items are ordered when an order point is reached for either item. Some practitioners, however, will also set order points when the weighted sum of stock for the two items reaches a given order level. This approach has some theoretical basis, as the development in the remainder of the paper shows. The development demonstrates the nature of the optimal order surfaces for a single period. While the analogous results are not shown for multiple periods or the infinite horizon, the single-period results should indicate the nature of the solution.

Based on results from single-item theory, under linear penalty and holding costs, when items are ordered, there are unique levels to order up to. Hence with two independent items, when items are ordered, there is a unique vector level (s_1, s_2) to order up to. The issue is the determination of the surface on the cartesian space (if it exists) from which it is optimal to order.

2. Analysis

Consider a family of two items with a joint order cost. Thus the cost of ordering is

$$c(x,y) = k + c_1x + c_2y$$

if either $x > 0$ or $y > 0$

$$= 0 \quad \text{otherwise}$$

where

$$x = \text{order for item 1}$$

$$y = \text{order for item 2}$$

$$k = \text{setup cost}$$

The issue becomes the costs of alternative policies for a given inventory position (x_0, y_0) . For linear holding and penalty costs for the two items of h_1 and h_2 and p_1 and p_2 respectively, the total expected costs given an initial inventory level (x_0, y_0) and an order-up-to level (x, y) is

$$k + c_1(x - x_0) + c_2(y - y_0)$$

$$+ \int_x^\infty p_1(\xi - x)\phi_1(\xi)d\xi + \int_{-\infty}^x h_1(x - \xi)\phi_1(\xi)d\xi$$

$$+ \int_y^\infty p_2(\xi - y)\phi_2(\xi)d\xi + \int_{-\infty}^y h_2(y - \xi)\phi_2(\xi)d\xi$$

where ϕ_1 and ϕ_2 are the two demand densities. If $x=x_0$ and $y=y_0$, then the same expression with $k=0$ holds.

Let

$$G_i(y) = \int_y^{\infty} (\xi - y)\phi_i(\xi)d\xi = \int_y^{\infty} (1 - F_i(\xi))d\xi$$

where F_i is the cumulative distribution for ϕ_i . For a normal distribution, for example,

$$G_i(y) = \sigma_i G_0\left(\frac{y - u_i}{\sigma_i}\right)$$

where

$$u_i = \text{mean of } \phi_i$$

$$\sigma_i = \text{standard deviation of } \phi_i$$

$$G_0(y) = \int_y^{\infty} (\xi - y)\phi(\xi)d\xi$$

where

ϕ is a standard normal density function.

Additionally,

$$\int_{-\infty}^y (y - \xi)\phi_1(\xi)d\xi = y - u_1 + \int_y^{\infty} (\xi - y)\phi_1(\xi)d\xi$$

Given these expressions, we see that if an order is made to bring inventories up to the levels x and y , the cost is

$$\begin{aligned} k - c_1 x_0 - c_2 y_0 - h_1 u_1 - h_2 u_2 \\ + \{(c_1 + h_1)x + (c_2 + h_2)y \\ + (h_1 + p_1)G_1(x) + (h_2 + p_2)G_2(y)\} \end{aligned} \quad (1)$$

Denoting the function in braces as $F(x,y)$, and noting that $F(x,y)$ is convex with a unique minimum $F(x^*, y^*)$ we see that the optimal policy is

- 1) order up to (x^*, y^*) if $x_0 < x^*, y_0 < y^*$
and $F(x_0, y_0) > F(x^*, y^*) + k$
- 2) order up to (x_0, y^*) if $x_0 > x^*, y_0 < y^*$
and $F(x_0, y_0) > F(x_0, y^*) + k$
- 3) order up to (x^*, y_0) if $x_0 < x^*, y_0 > y^*$
and $F(x_0, y_0) > F(x^*, y_0) + k$
- 4) Do not order otherwise

Conditions 2 and 3 yield the usual type of order points in an (s, S) policy in y^* and x^* respectively, as the function in braces is separable. The interesting case is condition 1, which defines the optimal regions to order when both items are below their optimal levels. The optimal surface in the region $x_0 < x^*, y_0 < y^*$ is

$$F(x_0, y_0) = F(x^*, y^*) + k \quad (2)$$

With a continuous function of two variables x and y , isocontours of the form (2) can be found by the equating the directional derivative of F along $y(x)$ to be zero.

So

$$\frac{\partial F}{\partial y} \frac{dy}{dx} + \frac{\partial F}{\partial x} = 0$$

or

$$\frac{dy}{dx} = - \frac{F_x}{F_y}$$

From (1), the partials are of the form

$$c_i + h_i + (p_i + h_i)G_i$$

Furthermore, since

$$G_1(z) = \int_z^{\infty} (\xi - z)\phi_1(\xi)d\xi = \int_z^{\infty} (1 - F_1(\xi))d\xi$$

$$G_1(z) = F_1(z) - 1$$

Hence the differential equation for the optimal ordering contour is

$$\frac{dy}{dx} = - \frac{(p_1 + h_1)F_1(x) - (p_1 - c_1)}{(p_2 + h_2)F_2(y) - (p_2 - c_2)}$$

which yields

$$G_2(x)(p_2 + h_2) + G_1(x)(p_1 + h_1) + (c_2 + h_2)y + (c_1 + h_1)x = C \quad (3)$$

where C is a constant.

The interpretation of (3) is as follows: The optimal ordering surface is obtained when the sum of functions of x and y are constant. Each function represents the relative "vulnerability" of the inventory for one of the two products. (The functions are minimized upon ordering). When the total vulnerability reaches a threshold, an order is placed.

3. Discussion

This type of relationship typically yields a curve of the form shown in Figure 1. When x is near its optimal order point and y is near its optimal order up to level (the point a in Figure 1) the function $R_2(y) = G_2(y)(p_2 + h_2) + (c_2 + h_2)y$ is close to its minimum (that defines the order-up-to level) and is changing very slowly. On the other hand, $R_1(x) = G_1(x)(p_1 + h_1) + (c_1 + h_1)x$ is changing rapidly, as x is near its order point and not near its minimum. Thus, in order for (3) to hold, small changes in x must be accompanied by large changes in y. The opposite holds at point b. At point c, both functions $R_1(y)$ and $R_2(x)$ are changing at about the

same rate (when scaled) and hence the optimum curve is pitched at 45° in order to maintain (3).

In practice, since the use of (3) might be computationally difficult, it might make sense to approximate the curve by the union of two rays and one line segment arising from the three curves

$$x = \text{constant} = x^* - Q_1 \quad (4a)$$

$$y = \text{constant} = y^* - Q_2 \quad (4b)$$

and

$$Q_1x + Q_2y = \text{constant} \quad (4c)$$

where Q_1 and Q_2 are the lot sizes (order-up-to-level minus single item order point) for x and y respectively. x^* and y^* are the optimal order-up to levels. The constants in 4a and 4b hence are the individual order points if joint replenishment did not exist. The ordering region is the union of three regions below or to the left of the curves 4a through 4c.

In other words, there should be separate trigger points for x , y and a weighted sum of x and y . This allows more accuracy than a system with order points for x and y only. While the curve of (3) has not been demonstrated to hold for the infinite horizon or average cost cases, there is a theoretical basis for trigger levels for aggregate sums of stock.

As a practical matter, the weighting on the weighted sum curve should be based on the maximum replenishment quantities for each item (i.e. the lot sizes Q_1 and Q_2). If, for example, there is a one-hundred unit replenishment amount for y and a ten-unit amount for x , then y should be more heavily weighted in determining the weighted sum reorder level. This type of approach examines the sum of proportional proximity to the individual item order points. The

constant for the weighted sum constraint is the same as C in relationship (3) and is hence the left side of (3) evaluated at either $(x^*, y^* - Q_2)$ or $(x^* - Q, y^*)$.

Sometimes, when both items are ordered, there is an additional smaller set-up for the second item. This leads to the desirability of what is known as can-order points. The same type of arguments can be used to show the optimality of the type of policy shown in Figure 2. The curve of the joint order surface between the unconditional order and can-order points is based on the same type of constant directional derivative argument.

References

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Optimal y_1
order up to
level for y

y

Optimal y^*
order point
for y alone

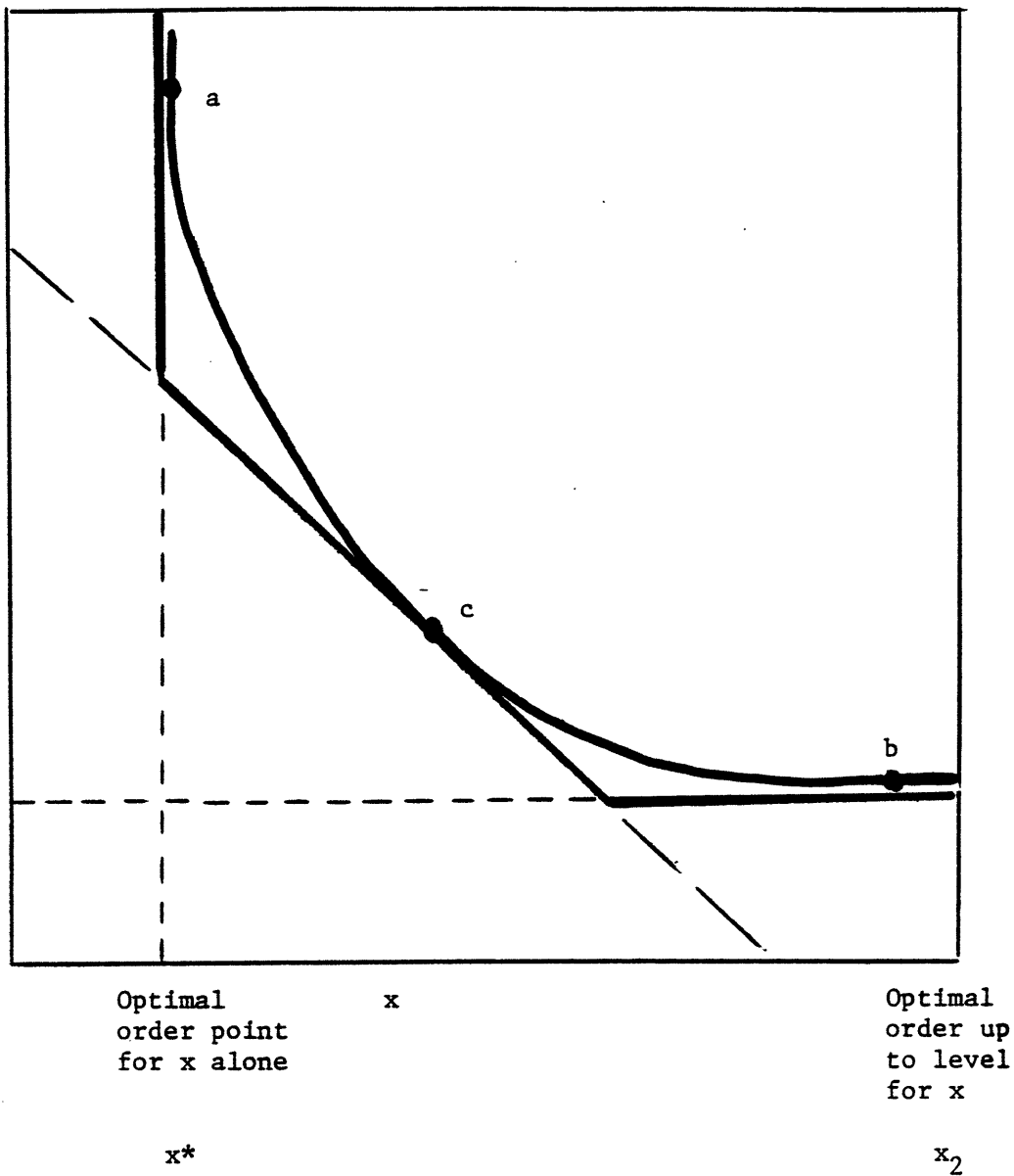
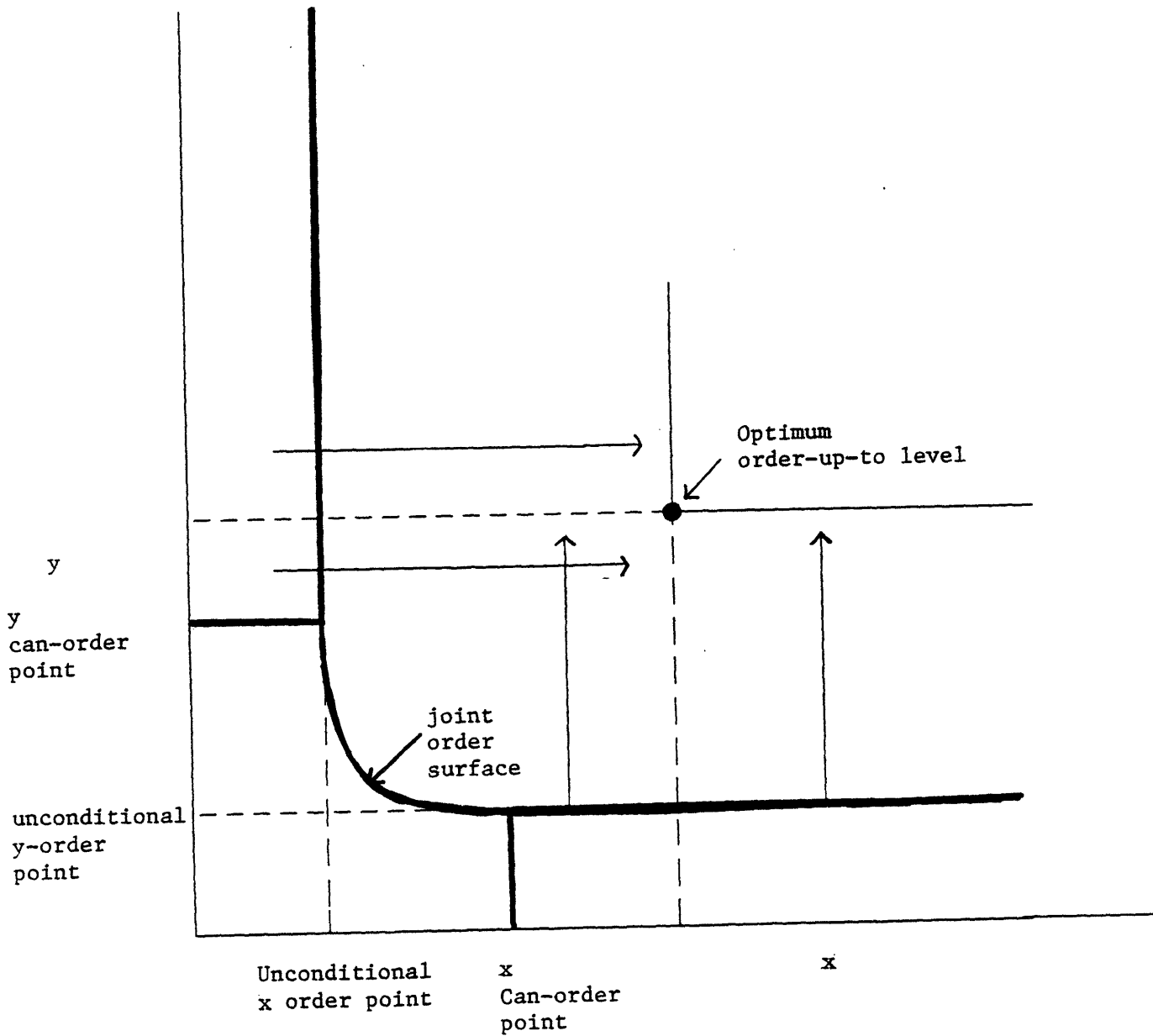


Figure 1. Optimal Ordering Surface



Arrows Represent Optimal Orders

Figure 2. Optimal ordering with can-order points.