OPTIMAL CONSUMPTION AND PORTFOLIO POLICIES WHEN ASSET PRICES FOLLOW A DIFFUSION PROCESS

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Table of Contents

1. Introduction

Optimal intertemporal consumption and portfolio policies in continuous time under uncertainty have traditionally been characterized by stochastic dynamic programming. Merton (1971) is the pioneering paper in this regard. To show the existence of a solution to the consumption-portfolio problem using dynamic programming, there are two approaches. The first is through application of the existence theorems in the theory of stochastic control. These existence theorems often require an admissible control to take its values in a compact set. However, if we are modeling a frictionless financial market, any compactness assumption on the values of controls is arbitrary and unsatisfactory. Moreover, many of the results are limited to cases where the controls affect only the drift term of the controlled processes. This, unfortunately, rules out the portfolio problem under consideration.

The second approach is through construction: construct a control, usually by solving a nonlinear partial differential equation, and then use the *verification theorem* in dynamic programming to verify that it indeed is a solution. Merton's paper uses this second approach. It is in general very difficult, however, to construct a solution. Moreover, when there are constraints on controls, such as the nonnegative constraints on consumption and on the wealth, this approach becomes even more difficult.

Recently, a martingale representation technology has been used in place of the theory of stochastic control to show the existence of optimal consumption and portfolio policies without the requirement of compactness of the values of admissible controls; see Cox and Huang (1986) and Pliska (1986). Notably, Cox and Huang show that, for a quite general class of utility functions, it suffices to check, for the existence of optimal controls, whether the sufficient conditions for the existence and uniqueness of a system of stochastic differential equations, derived completely from the price system, are satisfied. The sufficient conditions for the existence and the uniqueness of a solution to a system of stochastic differential equations have been well studied.

The focus of this paper is on explicit construction of optimal controls while taking into account the nonnegativity constraints on consumption and on final wealth by using a martingale technique. We provide two characterization theorems of optimal policies (Theorems 2.1 and 2.2) and a verification theorem (Theorem 2.3), which is a counterpart of the verification theorem in dynamic programming. One advantage of our approach is that we need only to solve a linear partial differential equation in constructing solutions unlike a nonlinear partial differential equation in the case of dynamic programming. In many specific situations, optimal controls can even be directly computed without solving any partial differential equation.

The rest of this paper is organized as follows. Section 2 contains our general theory. We formulate a dynamic consumption-portfolio problem for an agent in continuous time with general diffusion price processes in Section 2.1. The agent's problem is to dynamically manage a portfolio of securities and withdraw funds out of it in order to maximize his expected utility of consumption over time and of the final wealth, while facing nonnegativity constraints on consumption as well as on final wealth. Section 2.2 contains the main results of Section 2. In Theorems 2.1 and 2.2, we give characterizations of optimal consumption and portfolio policies. We show in Theorem 2.3 how candidates for optimal policies can be constructed by solving a *linear* partial differential equation. We also show ways to verify that a candidate is indeed optimal. The relationship between our approach and dynamic programming is discussed in Section 2.3. We also demonstrate the connection between a solution with nonnegativity constraints and a solution without the constraints.

In Section 3, we specialize the general model of Section 2 to a model considered originally by Merton (1971). Risky securities price processes follow a geometric Brownian motion. In this case, optimal consumption and portfolio policies can be computed directly without solving any partial differential equation. Several examples of utility functions are considered. In particular, we solve the consumption and portfolio problem for the family of HARA utility functions. In the unconstrained case given in Merton (1971), the optimal policies for HARA utility functions are linear in wealth; when nonnegativity constraints are included, this is no longer true. We also obtain some characterization of optimal policies that are of independent interest.

Section 4 contains the concluding remarks.

2. The General Case

In this section, a model of securities markets in continuous time with diffusion price processes will be formulated. We will consider the optimal consumption-portfolio policies of an agent. The agent's problem is to dynamically manage a portfolio of securities and withdraw funds out of the portfolio in order to maximize his expected utility of consumption over time and of final wealth, while facing a nonnegativity constraint on consumption and on final wealth. The connection between our approach and dynamic programming will be demonstrated and the advantages of our approach will be pointed out. We will also discuss the relationship between a solution to the agent's problem with the constraint and a solution to his problem without the constraint.

2.1. The Formulation

Taken as primitive is a complete probability space $(\Omega, \mathcal{F}, P)$ and a time span $[0, T]$, where T is a strictly positive real number. Let there be an N dimensional standard Brownian motion defined on the probability space, denoted by $w = \{w_n(t); t \in [0, T], n = 1, 2, \ldots, N\}$. Let $\mathbf{F} = \{\mathcal{F}_t; t \in [0, T]\}$ be the *filtration* generated by w. (A filtration is an increasing family of sub sigma-fields of $\mathcal{F}$.)

We assume that $\mathbf{F}$ is complete in that $\mathcal{F}_0$ contains all the P null sets and that $\mathcal{F}_T = \mathcal{F}$. Since for an N dimensional standard Brownian motion, $w(0) = 0$ *a.s.*, $\mathcal{F}_0$ is almost trivial.

We use $\mathcal{O}$ to denote the $\mathbf{F}$-optional sigma-field and ν to denote the product measure on $\Omega \times [0, T]$ generated by P and the Lebesgue measure. (The $\mathbf{F}$ optional sigma-field is the sigma-field on $\Omega \times [0, T]$ generated by $\mathbf{F}$-adapted right-continuous processes; see, e.g., Chung and Williams (1983).) The consumption space for an agent is

$$L^2(\nu) \times L^2(P) \equiv L^2(\Omega \times [0, T], \mathcal{O}, \nu) \times L^2(\Omega, \mathcal{F}, P),$$

where $L^2(\nu)$ is the space of consumption rate processes and $L^2(P)$ is the space of final wealth. Note that all element of $L^2(\nu)$ are $\mathbf{F}$-adapted processes (see, e.g., Chung and Williams (1983)). All the processes to appear will be adapted to $\mathbf{F}$.

Consider frictionless securities markets with $N + 1$ long-lived securities traded, indexed by $n = 0, 1, 2, \ldots, N$. Security $n \neq 0$ is risky and pays dividends at rate $\iota_n(t)$ and sells for $S_n(t)$, at time t. We will henceforth use $S(t)$ to denote $(S_1(t), \ldots, S_N(t))^\top$. Assume that $\iota_n(t)$ can be written as $\iota_n(S(t), t)$ with $\iota_n(x, t) : \Re^N \times [0, T] \mapsto \Re$ Borel measurable. Security 0 is (locally) riskless, pays no dividends, and sells for $B(t) = \exp\{\int_0^t r(s)ds\}$ at time t. Assume further that $r(t) = r(S(t), t)$ with $r(x, t) : \Re^N \times [0, T] \to \Re_+$ continuous.

We shall use the following notation: If σ is a matrix, then $|\sigma|^2$ denotes $\mathrm{tr}(\sigma\sigma^\top)$ and tr denotes *trace*.

The price process for risky securities S follows an Itô process satisfying

$$S(t) + \int_0^t \iota(S(s), s)ds = S(0) + \int_0^t \varsigma(S(s), s)ds + \int_0^t \sigma(S(s), s)dw(s) \quad \forall t \in [0, T], \quad a.s., \quad (2.1)$$

where ι is an N-vector of ι_n's, $\varsigma(x,t) : \Re^N \times [0,T] \mapsto \Re^N$ and $\sigma(x,t) : \Re^N \times [0,T] \mapsto \Re^{N\times N}$ are continuous in x and t, and $\sigma(x,t)$ is nonsingular for all x and t.

Defining $S^*(t) \equiv S(t)/B(t)$ and $\iota^*(t) \equiv \iota(t)/B(t)$, Itô's lemma implies that

$$\begin{aligned} S^*(t) + \int_0^t \iota^*(s)ds &= S^*(0) + \int_0^t \frac{1}{B(s)}[\varsigma(S(s),s) - r(S(s),s)S(s)]\,dt \\ &\quad + \int_0^t \frac{\sigma(S(s),s)}{B(s)} dw(s) \\ &\equiv S^*(0) + \int_0^t \varsigma^*(S(s),B(s),s)ds \\ &\quad + \int_0^t \sigma^*(S(s),B(s),s)dw(s) \quad \forall t \in [0,T]. \quad a.s. \end{aligned} \tag{2.2}$$

Next we define

$$G(t) = S(t) + \int_0^t \iota(s)ds$$

and

$$G^*(t) = S^*(t) + \int_0^t \iota^*(s)ds;$$

the former is the N vector of gains processes and the latter is the N vector of gains processes in units of the 0–th security, both for the risky securities.

A *martingale measure* is a probability measure Q equivalent to P such that

$$E[(dQ/dP)^2] < \infty$$

and under which $G^*(t)$ is a martingale. We assume that there exists a unique martingale measure, denoted by Q. The existence of a martingale measure ensures that there be no arbitrage opportunities for *simple strategies*;[1] see Harrison and Kreps (1979). It follows also from Harrison and Kreps (1979, Theorem 3) that

$$dQ/dP = \exp\left\{\int_0^T \kappa(S(s),s)^\top dw(s) - \frac{1}{2}\int_0^T |\kappa(S(s),s)|^2 ds\right\},$$

where

$$\kappa(S(t),t) \equiv -\sigma(S(t),t)^{-1}(\varsigma(S(t),t) - r(S(t),t)S(t)). \tag{2.3}$$

For future use, we define a square-integrable martingale under P

$$\begin{aligned} \eta(t) &= E[dQ/dP|\mathcal{F}_t] \quad a.s. \\ &= \exp\left\{\int_0^t \kappa(S(s),s)^\top dw(s) - \frac{1}{2}\int_0^t |\kappa(S(s),s)|^2 ds\right\} \quad a.s. \end{aligned} \tag{2.4}$$

We will use $E^*[\cdot]$ to denote the expectation under Q.

The following lemma will be useful later.

[1] A strategy is said to be simple if it is bounded and changes its values at a finite number of nonstochastic time points. Formally, (α,θ) is a simple trading strategy if there exist time points $0 = t_0 < t_1 < \ldots < t_N = 1$ and bounded random variables x_n, y_{jn}, $n = 0,\ldots,N-1$ and $j = 1,\ldots,M$, such that x_n and y_{jn} are measurable with respect to $\mathcal{F}_{t_n}$ and $\alpha(t) = x_n$ and $\theta_j(t) = y_{jn}$ if $t \in [t_n, t_{n+1})$.

Lemma 2.1. *Under* Q,

$$w^*(t) \equiv w(t) - \int_0^t \kappa(S(s), s)ds$$

is a Standard Brownian motion and

$$G^*(t) = S^*(0) + \int_0^t \sigma(S(s), s)/B(s)dw^*(s) \quad a.s.,$$

a square-integrable martingale.

Proof. The first assertion follows from the Girsanov Theorem: see, e.g., Liptser and Shiryayev (1977, Chapter 6). The second assertion follows from substitution of w^* into (2.2).

∎

Remark 2.1. Since P and Q are equivalent and thus have the same probability zero sets, the *almost surely* statements above and henceforth will be with respect to either.

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Remark 2.2. For sufficient conditions for the existence and uniqueness of a martingale measure see, for example, Proposition 3.1 and Theorem 3.1 of Cox and Huang (1986).

∎

A trading strategy is an $N+1$-vector process $(\alpha, \theta) = \{\alpha(t), \theta_n(t); n = 1, 2, \ldots, N\}$, where $\alpha(t)$ and $\theta_n(t)$ are the numbers of shares of the 0-th security and the n-th security, respectively, held at time t.

A trading strategy (α, θ) is admissible if

$$E^* \left[\int_0^T \theta(t)^\top \sigma(t)\sigma(t)^\top \theta(t)dt \right] < \infty. \tag{2.5}$$

if the stochastic integral

$$\int_0^t \left(\theta(s)^\top dG(s) + \alpha(s)dB(s) \right)$$

is well-defined, and if there exists a *consumption plan* $(c, W) \in L^2(\nu) \times L^2(P)$ such that

$$\begin{aligned} & \alpha(t)B(t) + \theta(t)^\top S(t) + \int_0^t c(s)ds \\ = \quad & \alpha(0)B(0) + \theta(0)^\top S(0) + \int_0^t \left(\alpha(s)dB(s) + \theta(s)^\top dG(s) \right) \qquad \forall t \ \ a.s., \end{aligned} \tag{2.6}$$

and

$$W = \alpha(T)B(T) + \theta(T)^\top S(T) \quad a.s. \tag{2.7}$$

The consumption plan (c, W) of (2.6) and (2.7) is said to be *financed* by (α, θ) and the quadruple (α, θ, c, W) is said to be a *self-financing strategy*. Let H denote the space of self financing strategies. By the linearity of the stochastic integral and the Cauchy Schwartz inequality, it is easily verified that H is a linear space. We record a well known fact about a self financing strategy and a mathematical result:

Lemma 2.2. *Let $(\alpha, \theta, c, W) \in H$. For all $t \in [0, T]$.*

$$
\begin{aligned}
&E^*\left[\int_0^T c^*(s)ds + W^*|\mathcal{F}_t\right] \\
=&\alpha(0) + \theta(0)^\top S^*(0) + \int_0^t \theta(s)^\top dG^*(s) \\
=&\alpha(t) + \theta(t)^\top S^*(t) + \int_0^t c^*(s)ds \quad a.s.,
\end{aligned}
$$

where $W^* \equiv W/B(T)$ and $c^*(t) \equiv c(t)/B(t)$, the normalized final wealth and time t consumption. Hence the value of (c, W) at time t is

$$
\alpha(t)B(t) + \theta(t)^\top S(t) = B(t)E^*[W^* + \int_t^T c^*(s)ds|\mathcal{F}_t] \quad a.s.
$$

Moreover, $\int_0^T c^*(s)ds + W^*$ is an element of $L^2(Q)$, the space of square integrable random variables on $(\Omega, \mathcal{F}, Q)$.

Proof. See, e.g., Cox and Huang (1986, Proposition 3.2).

∎

Lemma 2.3. *Let $g \in L^2(\nu)$. Then*

$$
E^*\left[\int_t^T g(s)ds|\mathcal{F}_t\right] = E\left[\int_t^T g(s)\eta(s)ds|\mathcal{F}_t\right]/\eta(t) \quad a.s.
$$

Proof. See Dellacherie and Meyer (1982, VI.57).

∎

Consider an agent with a time-additive utility function for consumption $u : \Re_+ \times [0, T] \mapsto \Re \bigcup \{-\infty\}$ and a utility function $V : \Re_+ \mapsto \Re \bigcup \{-\infty\}$ for final wealth. His problem is to choose a self-financing strategy to maximize his expected utility:

$$
\begin{aligned}
&\sup_{(\hat{\alpha},\hat{\theta},\hat{c},\hat{W})\in H} \quad E\left[\int_0^T u(\hat{c}(t), t)dt + V(\hat{W})\right]. \\
\text{s.t.} \quad &\hat{\alpha}(0)B(0) + \hat{\theta}(0)^\top S(0) \leq W(0), \\
&\hat{c} \geq 0 \ \nu - a.e. \text{ and } \hat{W} \geq 0 \ a.s.,
\end{aligned}
\tag{2.8}
$$

where $W(0) > 0$ is the agent's initial wealth. Note that the utility functions are allowed to take the value $-\infty$.

We assume that $u(y, t)$ and $V(y)$ are continuous, increasing, and strictly concave in y. For future reference, we cite several properties of a concave utility function. At every interior point of the domain of a concave function, the right-hand derivative and the left-hand derivative exist. At the left boundary of its domain, the right-hand derivative exists. The right-hand derivatives and the left-hand derivatives are decreasing functions and are equal to each other except possibly at most a countable number of points. That is, a concave function is differentiable except possibly at most a countable number of points and thus continuously differentiable except possibly at most a countable number of points. (Note that for strictly concave functions, the relation above such as *decreasing* becomes a strict relation.) The right-hand derivative is a right-continuous function. Moreover, at every point, the left-hand derivative is greater than the right-hand derivative.

Now let $u_{y+}(y,t)$, $V'_+(y)$ and $u_{y-}(y,t)$, $V'_-(y)$ denote the right-hand derivatives and left-hand derivatives, respectively, for $u(y,t)$ with respect to y and for $V(y)$. Let $y < y'$, then $V'_-(y) \geq V'_+(y) > V'_-(y')$; and similarly for $u(y,t)$. We assume that

$$\lim_{y\to\infty} u_{y+}(y,t) = 0$$

and

$$\lim_{y\to\infty} V'_+(y) = 0.$$

Define *inverse functions* $\hat{f}(x^{-1},t) = \inf\{y \in \Re_+ : u_{y+}(y,t) \leq x^{-1}\}$ and $V'^{-1}_+(x^{-1}) = \inf\{y \in \Re_+ : V'_+(y) \leq x^{-1}\}$. By the right-continuity of the right-hand derivatives, the infima are equal to minima.

Remark 2.3. Note that the assumption that utility functions are increasing and strictly concave implies that they are strictly increasing.

∎

Now we will state two assumptions that will be used in the next subsection. First define a process

$$Z(t) = Z(0) + \int_0^t (r(S(s),s) + |\kappa(S(s),s)|^2)Z(s)ds - \int_0^t \kappa(S(s),s)^\top Z(s)dw(s) \quad a.s. \tag{2.9}$$

for some constant $Z(0) > 0$. Using Itô's lemma, it is easily verified that

$$Z(t) = Z(0)B(t)/\eta(t) \quad a.s. \tag{2.10}$$

Note that $(\log Z(T) - \log Z(0))/T$ is the realized continuously compounded growth rate from time 0 to time T of the growth-optimal portfolio — the portfolio that maximizes the expected continuously compounded growth rate.

We adopt the following notation:

$$D_y^m = \frac{\partial^m}{\partial y^m} = \frac{\partial^{m_1+m_2+\ldots m_N}}{\partial y_1^{m_1}\ldots\partial y_N^{m_N}}; \quad m = m_1 + \ldots + m_N$$

for positive integers $m_1, m_2, \ldots, m_N$. If $g : \Re^N \times \mathbf{T} \mapsto \Re$ has partial derivatives with respect to its first N arguments, the vector $(\partial g/\partial y_1, \ldots, \partial g/\partial y_N)^\top$ is denoted by $D_y g$ or g_y.

Assumption 2.1. *Write (2.1) and (2.9) compactly as follows:*

$$\begin{pmatrix} S(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} S(0) \\ Z(0) \end{pmatrix} + \int_0^t \hat{\varsigma}(S(s), Z(s), s)ds + \int_0^t \hat{\sigma}(S(s), Z(s), s)dw(s) \quad \forall t \in [0, T], \ a.s.$$

Suppose that there exist strictly positive constants K_1, K_2, K_3, and γ such that for all t

$$\begin{aligned} |\hat{\varsigma}(x,t)| &\leq K_1(1+|x|), \qquad |\hat{\sigma}(x,t)| \leq K_1(1+|x|) \quad \forall x \in \Re^{N+1} \\ |\hat{\varsigma}(x,t) - \hat{\varsigma}(y,t)| &\leq K_2|x-y|, \qquad |\hat{\sigma}(x,t) - \hat{\sigma}(y,t)| \leq K_2|x-y| \quad \forall x, y \in \Re^{N+1}, \end{aligned} \tag{2.11}$$

that $D_y^m \hat{\varsigma}(y,t)$ and $D_y^m \hat{\sigma}(y,t)$ exist for $m = 1, 2$, and are continuous in y and t, and that

$$|D_y^m \hat{\varsigma}(y,t)| + |D_y^m \hat{\sigma}(y,t)| \leq K_3(1 + |y|^\gamma) \tag{2.12}$$

for all $y \in \Re^{N+1}$.

Assumption 2.2. *$f(x,t) \equiv x^{-1}\hat{f}(x^{-1}, t)$ and $g(x) \equiv x^{-1}V_+'^{-1}(x^{-1})$ are such that for $m \leq 2$, $D_x^m f(x,t)$ and $D_x^m g(x)$ are continuous,*

$$|D_x^m f(x,t)| \leq K(1 + |x|^\gamma),$$

and

$$|D_x^m g(x)| \leq K(1 + |x|^\gamma),$$

for some strictly positive constants K and γ.

The purpose of (2.11) is to guarantee the existence and the uniqueness of a solution of (2.1) and (2.9). The purpose of (2.12) and Assumption 2.2 is to ensure that certain functionals of S and Z have two continuous derivatives. To have a feel of the restrictiveness of Assumption 2.2, we note that if $V(y) = \frac{1}{1-b}x^{1-b}$, then $b \leq 1/3$. In many specific situations, differentiability will obtain under much weaker conditions. For example, in Section 3 of this paper, we consider a special case of the general model developed here. All the HARA utility functions give rise to the desired differentiability conditions.

2.2. Main Results

We will give explicit characterizations of an optimal consumption portfolio policy under the following assumption that the optimal consumption plan is also a solution to a corresponding static maximization problem. Assumption 2.3 is valid under quite mild regularity conditions, to which we refer readers to Sections 2-4 of Cox and Huang [1986].

Assumption 2.3. *There exists a solution to (2.8), denoted by (α, θ, c, W), if and only if (c, W) is a solution to*

$$\begin{aligned} &\sup_{(\hat{c},\hat{W})\in L^2_+(\nu)\times L^2_+(P)} && E\left[\int_0^T u(\hat{c},t)dt + V(\hat{W})\right] \\ &\text{s.t.} && E\left[\int_0^T \hat{c}(t)\eta(t)/B(t)dt + \hat{W}\eta(T)/B(T)\right] \leq W(0), \end{aligned} \tag{2.13}$$

where $L^2_+(\nu)$ denotes the positive orthant of $L^2(\nu)$ and likewise for $L^2_+(P)$.

Remark 2.4. The idea behind Assumption 2.3 is as follows. The assumption that the martingale measure is unique together with the square-integrable restriction on the trading strategies of (2.5) implies that any element of $L^2(\nu^*) \times L^2(Q)$ is attainable by an admissible trading strategy, or is *marketed*, where ν^* is the product measure generated by Q and the Lebesgue measure. As long as the solution to (2.13) lies in $L^2(\nu^*) \times L^2(Q)$, Assumption 2.3 will be valid. In this case, any consumption plan that is a candidate for the optimal solution for some initial wealth is marketed and thus markets can be said to be *dynamically complete*.

∎

It follows from the Lagrangian theory (see, e.g., Holmes (1975) and Rockafellar (1975)) that if (c, W) is a solution to (2.13), there exists a strictly positive real number λ such that

$$\begin{aligned} u_{c+}(c(\omega,t),t) &\begin{cases} \leq \lambda\eta(\omega,t)/B(\omega,t) \leq u_{c-}(c(\omega,t),t) & \text{for } \nu \text{ a.e. } (\omega,t) \text{ such that } c(\omega,t) > 0, \\ \leq \lambda\eta(\omega,t)/B(\omega,t) & \text{for } \nu \text{ a.e. } (\omega,t) \text{ such that } c(\omega,t) = 0; \end{cases} \\ V'_+(W(\omega)) &\begin{cases} \leq \lambda\eta(\omega,T)/B(\omega,T) \leq V'_-(W(\omega)) & \text{for } P \text{ a.e } \omega \text{ such that } W(\omega) > 0, \\ \leq \lambda\eta(\omega,T)/B(\omega,T) & \text{for } P \text{ a.e } \omega \text{ such that } W(\omega) = 0. \end{cases} \end{aligned} \tag{2.14}$$

Now let $\{Z(t); t \in [0,T]\}$ be defined as in (2.9) with the initial condition $Z(0) = 1/\lambda$. Then the above first order conditions become

$$\begin{aligned} u_{c+}(c(\omega,t),t) &\begin{cases} \leq Z(\omega,t)^{-1} \leq u_{c-}(c(\omega,t),t) & \text{for } \nu \text{ a.e. } (\omega,t) \text{ such that } c(\omega,t) > 0, \\ \leq Z(\omega,t)^{-1} & \text{for } \nu \text{ a.e. } (\omega,t) \text{ such that } c(\omega,t) = 0; \end{cases} \\ V'_+(W(\omega)) &\begin{cases} \leq Z(\omega,T)^{-1} \leq V'_-(W(\omega)) & \text{for } P \text{ a.e } \omega \text{ such that } W(\omega) > 0, \\ \leq Z(\omega,T)^{-1} & \text{for } P \text{ a.e } \omega \text{ such that } W(\omega) = 0. \end{cases} \end{aligned} \tag{2.15}$$

where we have used (2.9). Thus we have

$$\begin{aligned} c(t) &= \hat{f}(Z(t)^{-1}, t) \quad \nu - a.e. \\ W &= V'^{-1}_+(Z(T)^{-1}) \quad a.s. \end{aligned} \tag{2.16}$$

Here is our first main result:

Theorem 2.1. *Suppose that Assumptions 2.1 and 2.2 are satisfied and that there exists a solution to (2.13). Let* $Z(0) = 1/\lambda$ *and let the function* F *be defined by*

$$F(Z(t), S(t), t) = Z(t)E\left[\int_t^T Z(s)^{-1}\hat{f}(Z(s)^{-1}, s)ds + V_+^{\prime -1}(Z(T)^{-1})Z(T)^{-1}|Z(t), S(t)\right],$$

where we have fixed a right-continuous version of the conditional expectation. Then $D_y^m F(y,t)$ *and* $F_t(y,t)$ *exist and are continuous for* $m \leq 2$, *and together with* F *satisfy the following linear partial differential equation with a boundary condition:*

$$\mathcal{L}F + F_t = F_Z Z\kappa^\top \kappa - F_S^\top \sigma\kappa + rF - \hat{f}, \tag{2.17}$$

$$F(Z, S, T) = V_+^{\prime -1}(Z^{-1}), \tag{2.18}$$

where $\mathcal{L}$ *is the differential generator of* (Z, S). *The optimal portfolio policy is*

$$\begin{aligned}\theta(t) =& \Big[F_S(Z(t), S(t), t) + (\sigma(S(t), t)\sigma(S(t), t)^\top)^{-1} \\ & (\varsigma(S(t), t) - r(S(t), t)S(t))Z(t)F_Z(Z(t), S(t), t)\Big], \quad \nu - a.e. \\ \alpha(t) =& (F(Z(t), S(t), t) - \theta(t)^\top S(t))/B(t) \quad \nu - a.e.,\end{aligned} \tag{2.19}$$

and the optimal consumption plan is specified in (2.16).

Proof. By Lemma 2.2, the value of (c, W) at time t is

$$\begin{aligned}& B(t)E^*\left[\int_t^T \hat{f}(Z(s)^{-1}, s)/B(s)ds + V'^{-1}(Z(T)^{-1})/B(T)|\mathcal{F}_t\right] \\ =& \eta(t)^{-1}B(t)E\left[\int_t^T \hat{f}(Z(s)^{-1}, s)\eta(s)/B(s)ds + V_+^{\prime -1}(Z(T)^{-1})\eta(T)/B(T)|\mathcal{F}_t\right] \\ =& Z(t)E\left[\int_t^T Z(s)^{-1}\hat{f}(Z(s)^{-1}, s)ds + V_+^{\prime -1}(Z(T)^{-1})Z(T)^{-1}|Z(t), S(t)\right],\end{aligned}$$

where the first equality follows from Lemma 2.3 and the second equality follows from (2.9) and the fact that under Assumption 2.1, (Z, S) is a diffusion process and thus possesses the Strong Markov property.

Putting

$$\hat{F}(Z(t), S(t), t) \equiv E\left[\int_t^T Z(s)^{-1}\hat{f}(Z(s)^{-1}, s)ds + V_+^{\prime -1}(Z(T)^{-1})Z(T)^{-1}|Z(t), S(t)\right],$$

Assumption 2.2 and a multidimensional version of Remark 11.3 of Gihman and Skorohod (1972, p.77) shows that $D_y^m \hat{F}(y,t)$ and $\hat{F}_t(y,t)$ exist and continuous for $m \leq 2$, and

$$\mathcal{L}\hat{F} + \hat{F}_t + Z^{-1}\hat{f} = 0.$$

where $\mathcal{L}$ is the differential generator of (Z, S). In addition, $\hat{F}(Z(T), S(T), T) = WZ(T)^{-1}$ a.s. This implies that

$$\mathcal{L}F + F_t = F_Z Z\kappa^\top\kappa - F_S^\top\sigma\kappa + rF - \hat{f},$$

and

$$F(Z, S, T) = V_+^{\prime -1}(Z^{-1}),$$

which are (2.17) and (2.18). Itô's formula implies that

$$\begin{aligned}
&F(Z(t), S(t), t)/B(t) + \int_0^t \hat{f}(Z(s)^{-1}, s)/B(s)ds \\
&= F(Z(0), S(0), 0) + \int_0^t (\mathcal{L}F(s) + F_s(s))/B(s)ds + \int_0^t F_S(Z(s), S(s), s)^\top\sigma^*(S(s), s)dw(s) \\
&- \int_0^t F_Z(Z(s), S(s), s)Z(s)\kappa(Z(s), S(s), s)^\top/B(s)dw(s) - \int_0^t r(s)F(Z(s), S(s), s)/B(s)ds \\
&= F(Z(0), S(0), 0) + \int_0^t F_S(Z(s), S(s), s)^\top dG^*(s) \\
&\qquad - \int_0^t F_Z(Z(s), S(s), s)Z(s)\kappa(Z(s), S(s), s)^\top\sigma(Z(s), S(s), s)^{-1}dG^*(s),
\end{aligned} \tag{2.20}$$

where the second equality follows from (2.17) and Lemma 2.1. Evaluating (2.20) at $t = T$, we get

$$\begin{aligned}
&W/B(T) + \int_0^T \hat{f}(Z(s)^{-1}, s)/B(s)ds = F(Z(0), S(0), 0) \\
&+ \int_0^t \left(F_S(Z(s), S(s), s)^\top + F_Z(Z(s), S(s), s)Z(s)\kappa(Z(s), S(s), s)^\top\sigma(Z(s), S(s), s)^{-1}\right) dG^*(s).
\end{aligned} \tag{2.21}$$

From Lemma 2.2 we know

$$\begin{aligned}
&W/B(T) + \int_0^T \hat{f}(Z(s)^{-1}, s)/B(s)ds \\
=&F(Z(0), S(0), 0) + \int_0^T \theta(s)^\top dG^*(s).
\end{aligned} \tag{2.22}$$

Lemma 2.2 also says that the left-hand side of (2.22) lies in $L^2(Q)$. We can thus subtract (2.22) from (2.21), and then use Jacod (1979, Proposition 2.48) to take expectation under Q of its inner product:

$$E^*\left[\int_0^T (\theta(t) - (F_S + (\sigma\sigma^\top)^{-1}(\varsigma - rS)ZF_Z))^\top\sigma^*\sigma^{*\top}(\theta(t) - (F_S + (\sigma\sigma^\top)^{-1}(\varsigma - rS)ZF_Z))dt\right] = 0.$$

By the hypothesis, $\sigma^*\sigma^{*\top}$ is positive definitive. Therefore, we must have

$$\begin{aligned}
\theta(t) =&\Big[F_S(Z(t), S(t), t) + (\sigma(S(t), t)\sigma(S(t), t)^\top)^{-1} \\
&(\varsigma(S(t), t) - r(S(t), t)S(t))Z(t)F_Z(Z(t), S(t), t)\Big], \quad \nu - a.e.
\end{aligned}$$

The rest of the assertion is easy.

∎

Theorem 2.1 makes enough assumptions to ensure that F has desired derivatives and (2.17) and (2.18) are satisfied. Sometimes, we can explicitly compute the function F and it may have the property that $D_y^m F(y,t)$ and F_t exist and are continuous for $m \leq 2$. The theorem below gives a way to verify the existence of a solution and to compute optimal consumption and portfolio policies.

Theorem 2.2. *Suppose that F defined in Theorem 2.1 is such that $D_y^m F(y,t)$ and $F_t(y,t)$ exist and are continuous for $m \leq 2$. Suppose further that there exists Z_0 such that $F(Z_0, S(0), 0) = W(0)$ and that (2.16) lies in $L^2(\nu) \times L^2(P)$ when $Z(0) = Z_0$. Then there exists a solution to (2.8) with optimal policies specified in (2.16) and (2.19). Moreover, F satisfies (2.17) and (2.18).*

Proof. Define Z by (2.9) with $Z(0) = Z_0$. Then $F(Z_0, S(0), 0)$ is the present value of (2.16) and by hypothesis is equal to $W(0)$. Also, (2.16) lies in $L^2(\nu) \times L^2(P)$. Thus the first order conditions for (2.13) are satisfied. Since (2.13) is a concave program, the first order conditions are sufficient. Hence there exists a solution to (2.13) and, by Assumption 2.3, to (2.8).

Since $D_y^m F(y,t)$ and $F_t(y,t)$ exist and are continuous for $m \leq 2$, Itô's lemma implies that

$$
\begin{aligned}
& F(Z(t), S(t), t)/B(t) + \int_0^t \hat{f}(Z(s)^{-1}, s)/B(s)ds \\
=& F(Z(0), S(0), 0) + \int_0^t F_S(Z(s), S(s), s)^\top \sigma(S(s), s)/B(s)dw^*(s) \\
& - \int_0^t F_Z(Z(s), S(s), s)Z(s)\kappa(Z(s), S(s), s)^\top/B(s)dw^*(s). \\
& + \int_0^t \left(\mathcal{L}F(s) + F_S(s) - F_Z(s)Z(s)\kappa(s)^\top\kappa(s) + F_S(s)^\top\sigma(s)\kappa(s) - r(s)F(s) + \hat{f}(s)\right)/B(s)ds.
\end{aligned}
$$

By Lemma 2.2, the left–hand side of the above relation is a square integrable martingale under Q. Now note that the integrands of the two stochastic integrals on the right hand side are continuous functions of $Z(t)$ and $S(t)$ and thus are bounded on bounded sets. Let

$$T_n = \inf\{t \in [0,T] : \int_0^t |F_S(s)^\top \sigma^*(s) - F_Z(s)\kappa(s)^\top|^2/B^2(s) \geq n\}.$$

It is clear that $T_n \to T$ *a.s.* On the stochastic interval $[0, T_n]$, the two stochastic integrals on the right–side of the above equation are square–integrable martingales under Q. Hence the integrand of the Lebesgue integral on the right–hand side must be zero on $[0, T_n]$, since any martingale having continuous and bounded variation paths must be zero or be a constant; cf. Fisk (1965). Since $T_n \to T$ with probability one, we have (2.17). Finally, (2.18) follows directly from the definition of F.

∎

When F can not be explicitly computed, we can utilize the following theorem, which is a counterpart of the verification theorem in dynamic programming.

Theorem 2.3. *Let $u(y,t)$ and $V(y)$ be such that*

$$|x^{-1}\hat{f}(x^{-1},t)| + |x^{-1}V_+^{\prime\,-1}(x^{-1})| \leq K(1+|x|)^{\gamma}$$

for some K and γ and let (2.11) be satisfied. Suppose $F : \Re^{N+1} \times [0,T] \mapsto \Re$ with

$$|F(y,t)| \leq K(1+|y|)^{\gamma}$$

for some constants K and γ, is a solution to the partial differential equation of (2.17) with a boundary condition (2.18). Suppose also that there exists $Z_0 > 0$ such that $F(Z_0, S(0), 0) = W(0)$ and that $(\{\hat{f}(Z(t)^{-1},t)\}, V_+^{\prime\,-1}(Z(T)^{-1})) \in L^2(\nu) \times L^2(P)$ for Z with $Z(0) = Z_0$. Then there exists a solution to (2.8) with optimal policies described in (2.16) and (2.19) and with $Z(0) = Z_0$.

Proof. First we note that (2.11) implies that for all positive integers m there exist constants L_m such that

$$E[|Z(t)|^{2m}] \leq (1+|Z(0)|^{2m})\exp\{L_m t\};$$

see, e.g., Theorem 5.2.3 of Friedman (1975). Therefore,

$$E\left[\int_0^T Z(t)^{-1}\hat{f}(Z(t)^{-1},t)dt\right] < \infty,$$

for every $Z(0) > 0$. This, (2.17), (2.18), and Theorem V.5.2 of Fleming and Rishel (1975) then imply that

$$\begin{aligned} F(Z(0),S(0),0) &= Z(0)^{-1}E\left[\int_0^T Z(t)^{-1}\hat{f}(Z(t)^{-1},t)dt + V_+^{\prime\,-1}(Z(T)^{-1})\right] \\ &= E^*\left[\int_0^T \hat{f}(Z(t)^{-1},t)/B(t)dt + V_+^{\prime\,-1}(Z(T)^{-1})/B(T)\right]. \end{aligned}$$

In particular, we can take $Z(0) = Z_0$ in the above relation. This is simply the value at time zero of $(\{\hat{f}(Z(t)^{-1},t)\}, V_+^{\prime\,-1}(Z(T)^{-1}))$, which lies in $L^2(\nu) \times L^2(P)$ by hypothesis. Also by the hypothesis that there exists Z_0 such that $F(Z_0,S(0),0) = W(0)$. Hence $(\{\hat{f}(Z(t)^{-1},t)\}, V_+^{\prime\,-1}(Z(T)^{-1}))$ satisfies the first order condition for an optimum for the program (2.13). Therefore, it is a solution to (2.13). The rest of the assertion then follows from Assumption 2.3 and Theorem 2.2.

∎

Unlike the verification theorem in dynamic programming, the verification procedure in Theorem 2.3 involves a linear partial differential equation.

For the rest of this section, we will assume that there exists a solution to (2.13) and that Assumptions 2.1 and 2.2 are satisfied.

We use $\{W(t); t \in [0, T]\}$ to denote the process of the optimally invested wealth:

$$W(t) = F(Z(t), S(t), t).$$

It is clear that $W(T) = W$ *a.s.*. The following proposition shows that after the optimally invested wealth reaches zero, the optimal consumption and portfolio policies are zeros. The following notation will be needed.

Define an optional time $\mathcal{T} = \inf\{t \in [0, T) : W(t) \leq 0\}$, the first time the optimally invested wealth reaches zero. As a convention, when the infimum does not exist, it is set to be T.

Proposition 2.1. *On the stochastic interval* $[\mathcal{T}, T]$,

$$\begin{aligned} \theta(Z(t), S(t), t) &= 0 \quad \nu - a.e. \\ \alpha(Z(t), S(t), t) &= 0 \quad \nu - a.e. \\ c(t) &= 0 \quad \nu - a.e. \\ W &= 0 \quad a.s. \end{aligned}$$

Proof. From the definition of F, it is clear that it is equal to zero at $\mathcal{T}$ if and only if on $[\mathcal{T}, T]$ the optimal consumption and final wealth are zeros. Arguments similar to the last half of the proof of Theorem 2.1 starting from (2.21) prove the rest of the assertion.

∎

Note that if we consider the agent's problem in the context of the theory of stochastic control, given the set up of the securities markets, we would like the optimal controls such as (α, θ, c, W) to be feedback controls. That is, the optimal controls at each time t depend only upon time t, the values of $S(t)$, and the agent's optimally invested wealth at that time. In the above theorem, the optimal controls are functions of $S(t)$, $Z(t)$, and t. However, Z is determined in part by the agent's initial wealth through the initial condition $Z(0) = 1/\lambda$. The following proposition shows that given $S(t)$ and t, the agent's optimally invested wealth at time t is an invertible function of Z if $u(y, t)$ and $V(y)$ are differentiable in y. Hence, the optimal controls are indeed feedback controls.

Proposition 2.2. $F_Z \geq 0$. *Suppose that* $u(y, t)$ *and* $V(y)$ *are differentiable in* y. *Then* $F_Z > 0$ *if* $F > 0$. *Thus there exists a function* $F^{-1}(W(t), S(t), t) = Z(t)$ *if* $W(t) > 0$. *In addition*, F^{-1}_{WW},

F_W^{-1}, F_{SS}^{-1}, F_S^{-1}, and F_t^{-1} exist and are continuous. We can write,

$$
\begin{aligned}
\theta(Z(t),S(t),t) &= \begin{cases} \theta(F^{-1}(W(t),S(t),t),S(t),t) \quad \nu-a.e. & \textit{if } W(t)>0; \\ 0 & \textit{if } W(t)=0; \end{cases} \\
\alpha(Z(t),S(t),t) &= \begin{cases} \theta(F^{-1}(W(t),S(t),t),S(t),t) \quad \nu-a.e. & \textit{if } W(t)>0; \\ 0 & \textit{if } W(t)=0; \end{cases} \\
c(t) &= \begin{cases} \hat{f}(1/F^{-1}(W(t),S(t),t),t) \quad \nu-a.e. & \textit{if } W(t)>0; \\ 0 & \textit{if } W(t)=0; \end{cases} \\
W &= W(T) \quad a.s.
\end{aligned}
\tag{2.23}
$$

Proof. It follows from Friedman (1975, Theorem 5.5.5) and the fact that $\partial Z(s)/\partial Z(t) = Z(s)/Z(t)$ if $s \geq t$ we have

$$
F_Z(Z(t),S(t),t) = -E\left[\int_t^T \frac{\hat{f}'(Z(s)^{-1},s)}{Z^2(s)}ds + \frac{V_+'^{-1\prime}(Z(T)^{-1})}{Z^2(T)} | Z(t),S(t)\right],
$$

where $\hat{f}'$ denotes the derivative of $\hat{f}$ with respect to its first argument, and where $V_+'^{-1\prime}$ denotes the derivative of $V_+'^{-1}$. Thus $F_Z \geq 0$, since $\hat{f}(y,t)$ and $V_+'^{-1}(y)$ are decreasing in y. Note that if $u(y,t)$ is differentiable in y then $\hat{f}$ is strictly decreasing when $\hat{f}(y,t) > 0$; and similarly $V_+'^{-1}$ is strictly decreasing if V is differentiable and if $V_+'^{-1}(y) > 0$. If $F_Z = 0$, it must be that $F(Z(t),S(t),t) = 0$ and Proposition 2.1 gives the optimal consumption and portfolio policies. If $F(Z(t),S(t),t) > 0$ then $F_Z(Z(t),S(t),t) > 0$. Therefore, given $S(t)$ and t, $Z(t)$ is an invertible function of $W(t)$ if $W(t) > 0$. Let this function be denoted by $F^{-1}(W(t),S(t),t)$. The differentiability of F^{-1} follows from the implicit function theorem; see, e.g., Hestenes (1975, p.172). The rest of the assertion then follows from Theorem 2.1 and substitution.

∎

Remark 2.5. For $F_Z > 0$ when $F > 0$, it is certainly not necessary that $u(y,t)$ and $V(y)$ be differentiable in y. In the special case of our current general model to be dealt with in Section 3, many utility functions that are concave and nonlinear yields $F_Z > 0$ for $F > 0$.

∎

When utility functions have a finite marginal utility at zero, the optimal consumption policy may involve zero consumption. The following proposition identifies the circumstances in which optimal consumption is zero.

Proposition 2.3. *Suppose that* $u_{c+}(0,t) < \infty$. *Consumption at time* t *is zero only if* $W(t) \leq F(u_{c+}(0,t),S(t),t)$. *Suppose in addition that* $u(y,t)$ *and* $V(y)$ *are differentiable in* y. *Then an optimal policy has the property that consumption will be zero if and only if wealth is less than a stochastic boundary* $F(u_c(0,t)^{-1},S(t),t)$.

Proof. From (2.15) we know that $c(t) \leq 0$ if and only if $u_c(0,t) \leq Z(t)^{-1}$. It then follows from Proposition 2.2 that $u_c(0,t) \leq Z(t)^{-1}$ only if

$$F(u_c(0,t)^{-1}, S(t), t) \geq F(Z(t), S(t), t) = W(t).$$

This is the first assertion. Next suppose that both $u(y,t)$ and $V(y)$ are differentiable. We want to show that if $W(t) \leq F(u_c(0,t)^{-1}, S(t), t)$, then $c(t) = 0$. We take two cases. Case 1: $W(t) = 0$. Then Proposition 2.1 shows that $c(t) = 0$. Case 2: $W(t) > 0$. Proposition 2.1 also shows that when $W(t) > 0$, $F_Z > 0$. Thus $u_c(0,t) \leq Z(t)^{-1}$ if and only if

$$F(u_c(0,t)^{-1}, S(t), t) \geq F(Z(t), S(t), t) = W(t).$$

∎

By inspection of (2.19) and (2.23), we easily see that, when $F_Z > 0$, the *feedback controls* are differentiable functions of $W(t)$ and $S(t)$. In particular, the optimal consumption policy is twice continuously differentiable in $W(t)$ and $S(t)$, which follows directly from the assumption that $\hat{f}(y,t)$ is two times continuously differentiable with respect to y (see Assumption 2.2).

The following proposition gives a complete characterization of utility functions such that $\hat{f}(y,t)$ is twice continuously differentiable with respect to y, given that $u(y,t)$ is differentiable in y.

Proposition 2.4. *Suppose that $u(y,t)$ is differentiable with respect to y. $D_y^m \hat{f}(y,t)$ exists and is continuous for $m \leq 2$ if and only if $D_y^m u(y,t)$ exists and is continuous for $m \leq 3$, and for $u_{y+}(0,t) < \infty$,*

$$\lim_{y \downarrow 0} -\frac{u_y(y,t)}{u_{yy}(y,t)} = 0; \tag{2.24}$$

and

$$\lim_{y \downarrow 0} -\frac{u_{yyy}(y,t)}{u_{yy}(y,t)} \left(\frac{u_y(y,t)}{u_{yy}(y,t)} \right)^2 = 0. \tag{2.25}$$

Similar conclusions also hold for $V(y)$.

Proof. On the interval $(0, u_y(0,t))$, u_y is continuous and strictly decreasing. Hence $D_y^m \hat{f}(y,t)$ exists and is continuous for $m \leq 2$ if and only if $D_y^m u(y,t)$ exists and is continuous for $m \leq 3$. When $u_y(0,t) < \infty$, on $(u_y(0,t), \infty)$, $D_y^m \hat{f}(y,t)$ is equal to zero for $m \leq 2$. This implies (2.24) and (2.25).

The proof for $V(y)$ is identical.

∎

2.3. Relation to dynamic Programming

Traditionally, the agent's optimal consumption portfolio policy is computed by stochastic dynamic programming; see, e.g., Merton (1971). We will demonstrate the connection between our approach and the stochastic dynamic programming.

The usual formulation of the consumption–portfolio problem uses a consumption policy and a vector of dollar amounts invested in risky assets to be the controls. The former is denoted by $c(W(t), S(t), t)$ and the latter will be called an *investment policy* and be denoted by $A(W(t), S(t), t)$. Given a pair of controls (c, A), dynamic behavior of the wealth is

$$W(t) = W(0) + \int_0^t \Big(W(s)r(s) - c(s) + A(s)I_{S^{-1}}(t)\left(\varsigma(s) - r(s)S(s)\right)\Big)ds + \int_0^t A(s)I_{S^{-1}}(s)\sigma(s)dw(s) \quad \forall t \in [0, T] \text{ a.s.}$$

where $I_{S^{-1}}(t)$ is a diagonal matrix with diagonal elements $S_n(t)^{-1}$. Define

$$J(W(t), S(t), t) = \sup_{c \geq 0, A} E\left[\int_t^T u(c(s), s)ds + V(W(T)))|W(t), S(t)\right]$$

subject to the constraints that the wealth follows the above dynamics, that consumption cannot be negative, and that

$$J(0, S(t), t) = \int_t^T u(0, s)ds + V(0). \tag{2.26}$$

The last constraint is basically a nonnegative wealth constraint that rules out arbitrage opportunities.

The existence of a pair of optimal controls is a nontrivial problem. We will refer readers to, for example, Krylov (1980) for an extensive treatment using the theory of stochastic controls. For a much easier approach specific to the consumption–portfolio problem, we refer readers to Cox and Huang (1986) and the references given there.

We assume that there exists a pair of optimal controls (c, A) and that J has two continuous derivatives with respect to its first two arguments and a continuous derivative with respect to t. The Bellman equation is

$$0 = \max_{\hat{c}(t), \hat{A}(t)} \Big\{u(\hat{c}(t), t) + \mathcal{L}J(W(t), S(t), t) + J_t(W(t), S(t), t)\Big\}. \tag{2.27}$$

where $\mathcal{L}$ is the differential generator of (W, S). The optimal controls satisfy the first order necessary conditions:

$$\begin{aligned} u_{c+}(c(t), t) &\begin{cases} \leq J_W(t) \leq u_{c-}(c(t), t) & \text{if } c(t) > 0; \\ \leq J_W(t) & \text{if } c(t) = 0. \end{cases} \\ V'_+(W) &\begin{cases} \leq J_W(T) \leq V'_-(W) & \text{if } W > 0; \\ \leq J_W(T) & \text{if } W = 0; \end{cases} \end{aligned} \tag{2.28}$$

$$A(t) = I_S(t)\Big[\Big(-\frac{J_{WS}(t)}{J_{WW}(t)}\Big) + (\sigma(t)\sigma(t)^\top)^{-1}(\varsigma(t) - r(t)S(t))\Big(-\frac{J_W(t)}{J_{WW}(t)}\Big)\Big]. \tag{2.29}$$

Substituting (2.28) and (2.29) into (2.27), we have a nonlinear partial differential equation of J. To compute the optimal controls, we need to solve this nonlinear partial differential equation with two boundary conditions: (2.26) and $J(W,S,T) = V(W)$. Once we solve this partial differential equation, the optimal controls can be gotten by simply substituting the solution into (2.28) and (2.29). Note that in solving the nonlinear partial differential equation, the nonnegativity constraint on consumption usually makes this nontrivial problem even more difficult; see, e.g., Karatzas, Lehoczky, Sethi, and Shreve (1986) in a special case of our general model.

To see that dynamic programming is consistent with our approach, note that at each time t, dynamic strategy corresponds to the allocation that would be chosen in a newly initiated static problem of the form of (2.13) and that $Z(t)^{-1}$ is the marginal utility of wealth. Hence

$$\begin{aligned} u_{c+}(c(\omega,t),t) &\begin{cases} \leq Z(\omega,t)^{-1} \leq u_{c-}(c(\omega,t),t) & \text{for } \nu\text{-a.e. } (\omega,t) \text{ such that } c(\omega,t) > 0, \\ \leq Z(\omega,t)^{-1} & \text{for } \nu\text{-a.e. } (\omega,t) \text{ such that } c(\omega,t) = 0; \end{cases} \\ V'_{+}(W(\omega)) &\begin{cases} \leq Z(\omega,t)^{-1} \leq V'_{-}(W(\omega)) & \text{for } P\text{-a.e } \omega \text{ such that } W(\omega) > 0, \\ \leq Z(\omega,t)^{-1} & \text{for } P\text{-a.e } \omega \text{ such that } W(\omega) = 0. \end{cases} \end{aligned} \tag{2.30}$$

$$J_W(t) = Z(t)^{-1} = 1/F^{-1}(W(t),S(t),t) \quad \nu - a.e. \tag{2.31}$$

and

$$J_{WW}(t) = -F_W^{-1}(t)/(F^{-1}(t))^2 \quad \text{and} \quad J_{WS}(t) = F_S^{-1}(t)/(F^{-1}(t))^2. \tag{2.32}$$

Recall that

$$Z(t) = F^{-1}(W(t),S(t),t) \quad \nu - a.e.$$

Therefore,

$$F_Z F_W^{-1} = 1 \tag{2.33}$$

and

$$F_Z F_S^{-1} + F_S = 0. \tag{2.34}$$

Relations (2.33) and (2.34) imply that

$$-\frac{J_{WS}}{J_{WW}} = F_S$$

and

$$-\frac{J_W}{J_{WW}} = F^{-1}/F_W^{-1} = F_Z Z.$$

Hence, it follows from (2.29) that

$$A(t) = I_S(t)\left(F_S(t) + (\sigma(t)\sigma(t)^\top)^{-1}(\varsigma(t) - r(t)S(t))F_Z(t)Z(t)\right). \quad \nu - a.e. \tag{2.35}$$

Relations (2.30) and (2.35) are consistent with (2.15) and (2.19).

Although our approach and stochastic dynamic programming are essentially consistent, there are several advantages to our approach.

First, as mentioned above, the problem of the existence of optimal consumption–portfolio policies can be dealt with with much ease, in the context of our model, using our approach. This issue has been extensively discussed in Cox and Huang (1986), to which we refer interested readers.

Second, in the verification theorem of dynamic programming, one needs to solve a nonlinear partial equation. On the other hand, a linear partial differential equation needs to be solved in Theorem 2.2.

Third, our approach yields optimal policies without the knowledge of the indirect utility function. The indirect utility function will be a by–product of our analysis even when it does not have desired derivatives to satisfy the Bellman's equation. To see this, we put

$$\hat{J}(Z(t), S(t), t) \equiv E\left[\int_t^T u(\hat{f}(Z(s)^{-1}, s)ds + V(V_+'^{-1}(Z(T)^{-1})|Z(t), S(t)\right].$$

Once we have $F_Z > 0$, the indirect utility function is

$$J(W(t), S(t), t) = \begin{cases} \hat{J}(1/F^{-1}(W(t), S(t), t), S(t), t) & \text{if } W(t) > 0; \\ \int_t^T u(0, s)ds + V(0) & \text{if } W(t) = 0. \end{cases}$$

The indirect utility function J may not be twice continuously differentiable in $W(t)$ and $S(t)$ and continuously differentiable in t. In such event, the optimal policies cannot even be computed by solving a nonlinear partial differential equation.

2.4. The Relationship Between the Constrained and the Unconstrained Solutions

The optimization problem of (2.13) has nonnegativity constraints on the consumption as well as on final wealth. For utility functions that exhibit infinite marginal utilities at zero consumption and at zero wealth, the nonnegativity constraints are not binding at the optimal solution. For problems for which the nonnegativity constraints are binding, it is sometimes difficult to compute an optimal solution. In this subsection, we will consider utility functions that are defined on the whole of the real line. If the consumption–portfolio problems for these utility functions have optimal solutions without the nonnegativity constraint, it is possible to obtain the optimal constrained solutions in a simple and direct way. In effect, the market informs an agent that he or she can follow an unconstrained consumption–portfolio policy only if he or she simultaneously buys an insurance package that will pay off the negative consumption and wealth as they are incurred. An optimal constrained policy will be one that allocates the initial wealth between an unconstrained policy and the insurance package on the unconstrained policy and exhausts all the initial wealth.

Formally, consider an agent with an utility function for consumption $u : \Re \times [0, T] \mapsto \Re$ and a utility function for final wealth $V : \Re \times [0, T] \mapsto \Re$. Assume that $u(y, t)$ and $V(y)$ are increasing and strictly concave in y. Consider the following program:

$$
\begin{aligned}
&\sup_{(\hat{c},\hat{W})\in L^2(\nu)\times L^2(P)} E\left[\int_0^T u(\hat{c},t)dt + V(\hat{W})\right] \\
\text{s.t.} \quad & E\left[\int_0^T \hat{c}(t)\eta(t)/B(t)dt + \hat{W}\eta(T)/B(T)\right] = W_\lambda(0).
\end{aligned}
\tag{2.36}
$$

Note that there is no nonnegativity constraint on consumption and on final wealth in (2.36). If there exists a solution to (2.36), by the strict concavity of utility functions, the solution is unique and is denoted by $(\hat{c}_\lambda, \hat{W}_\lambda)$. By the Lagrangian theory, there exists a unique $\lambda > 0$ such that

$$
\begin{aligned}
u_{c+}(c_\lambda(t),t) \le \lambda\eta(t)/B(t) \le u_{c-}(c_\lambda(t),t) \qquad & \nu - a.e. \\
V'_+(W_\lambda) \le \lambda\eta(T)/B(T) \le V'_-(W_\lambda) \qquad & a.s.
\end{aligned}
\tag{2.37}
$$

We will use the following notation. Let $(\hat{c},\hat{W}) \in L^2(\nu)\times L^2(P)$. Then $\hat{c}^+ \equiv \{\max[\hat{c}(t),0]; t\in[0,T]\}$ and $\hat{W}^+ \equiv \max[\hat{W},0]$. Similarly, $\hat{c}^- \equiv \{\max[-c(t),0]; t\in[0,T]\}$ and $W^- \equiv \max[-\hat{W},0]$. By definition, we have $\hat{c} = \hat{c}^+ - \hat{c}^-$ and $\hat{W} = \hat{W}^+ - \hat{W}^-$. Moreover, by the fact that $L^2(\nu)$ and $L^2(P)$ are lattices, we know $\hat{c}^+$, $\hat{c}^-$ are elements of $L^2(\nu)$ and $\hat{W}^+$ and $\hat{W}^-$ are elements of $L^2(P)$.

The following is the main result of this subsection:

Theorem 2.4. *Suppose that (c_λ, W_λ) is the solution to (2.36) with an initial wealth $W_\lambda(0) \in (0, W(0)]$ and that*

$$
E\left[\int_0^T c_\lambda^-(t)\eta(t)/B(t)dt + W_\lambda^-\eta(T)/B(T)\right] = W(0) - W_\lambda(0).
\tag{2.38}
$$

Then $(c_\lambda^+, W_\lambda^+)$ is the solution to (2.36) with additional nonnegativity constraints that $\hat{c} \ge 0$ and $\hat{W} \ge 0$ and with an initial wealth $W(0) > 0$. Conversely, suppose that there exists a solution to (2.36) with the additional nonnegativity constraints on consumption and on final wealth. Denote this solution by (c, W). Let λ be the Lagrangian multiplier associated with (c, W). Suppose that there exists $(c_\lambda, W_\lambda) \in L^2(\nu)\times L^2(P)$ such that (2.37) holds. Then there exists $W_\lambda(0) \in (0, W(0)]$ such that (c_λ, W_λ) is a solution to (2.36) with $(c_\lambda^+, W_\lambda^+) = (c, W)$ and (2.38).

Proof. By concavity of utility functions and (2.37) we have

$$
\begin{aligned}
u_{c+}(c_\lambda^+(\omega,t),t) &\begin{cases} \le \lambda\eta(\omega,t)/B(\omega,t) \le u_{c-}(c_\lambda^+(\omega,t),t) & \text{for } \nu \text{ a.e. } (\omega,t) \text{ such that } c_\lambda^+(\omega,t) > 0; \\ \le \lambda\eta(\omega,t)/B(\omega,t) & \text{for } \nu \text{ a.e. } (\omega,t) \text{ such that } c_\lambda^+(\omega,t) = 0. \end{cases} \\
V'_+(W_\lambda^+(\omega)) &\begin{cases} \le \lambda\eta(\omega,T)/B(\omega,T) \le V'_-(W) & \text{for } P \text{ a.e. } \omega \text{ such that } W_\lambda^+(\omega) > 0; \\ \le \lambda\eta(\omega,T)/B(\omega,T) & \text{for } P \text{ a.e. } \omega \text{ such that } W_\lambda^+(\omega) = 0. \end{cases}
\end{aligned}
\tag{2.39}
$$

Next we claim that $(c_\lambda^+, W_\lambda^+)$ has an initial value $W(0)$. To see this, we recall that $c_\lambda = c_\lambda^+ - c_\lambda^-$ and $W_\lambda = W_\lambda^+ - W_\lambda^-$. Therefore,

$$\begin{aligned} & E\left[\int_0^T c_\lambda^+(t)\eta(t)/B(t)dt + W_\lambda^+\eta(T)/B(T)\right] \\ = \ & E\left[\int_0^T (c_\lambda(t) + c_\lambda^-(t))\eta(t)/B(t)dt + (W_\lambda + W_\lambda^-)\eta(T)/B(T)\right] \\ = \ & W_\lambda(0) + W(0) - W_\lambda(0) = W(0), \end{aligned} \tag{2.40}$$

where the second equality follows from (2.38). Finally, $(c_\lambda^+, W_\lambda^+) \in L_+^2(\nu) \times L_+^2(P)$, the concavity of utility functions, (2.39), and (2.40) imply that $(c_\lambda^+, W_\lambda^+)$ is the solution to (2.36) with the nonnegativity constraints and with an initial wealth $W(0)$.

Conversely, let (c, W) be the solution to (2.36) with the additional nonnegativity constraints and let $\lambda > 0$ be the Lagrangian multiplier associated with it. By the hypothesis, there exists $(c_\lambda, W_\lambda) \in L^2(\nu) \times L^2(P)$ such that (2.37) holds. By the definition of (c_λ, W_λ), it is obvious that $(c_\lambda^+, W_\lambda^+) = (c, W)$. Now define

$$W_\lambda(0) \equiv E\left[\int_0^T c_\lambda(t)\eta(t)/B(t)dt + W_\lambda\eta(T)/B(T)\right].$$

The rest of the assertion then follows from direct verification.

∎

The agent invests $W_\lambda(0)$ in the unconstrained policy and then spends $W(0) - W_\lambda(0)$ on an insurance package that pays $(c_\lambda^-, W_\lambda^-)$. The combination of the unconstrained policy and the insurance package gives precisely the constrained policy. Note that the insurance package can be thought of as consisting of a continuum of put options with zero exercise price. To see this, we observe that $c_\lambda^-(t) = \max[c_\lambda(t), 0]$ is the payoff of an European put option written on the unconstrained consumption policy at time t with a zero exercise price and $W_\lambda^- = \max[W, 0]$ is the payoff of an European put option written on the unconstrained policy for final wealth with an exercise price zero. The price at time 0 for the former is $E[c_\lambda^-(t)\eta(t)/B(t)]$ and for the latter is $E[W_\lambda^-\eta(T)/B(T)]$. Consider buying a continuum of these put options on consumption according to the Lebesgue measure on $[0, T]$ and the put option on the final wealth. The payoff of this package is just $(c_\lambda^-, W_\lambda^-)$ and its price at time 0 is

$$\begin{aligned} & \int_0^T E[c_\lambda^-(t)\eta(t)/B(t)]dt + E[W_\lambda^-\eta(T)/B(T)] \\ = \ & E\left[\int_0^T c_\lambda^-(t)\eta(t)/B(t)dt + W_\lambda^-\eta(T)/B(T)\right] = W(0) - W_\lambda(0), \end{aligned}$$

where the first equality follows from the Fubini Theorem.

Once we solve the static problem, then we can use the methodology developed in Section 2.2 to compute the optimal portfolio strategy. In many specific situations, the optimal consumption and portfolio policies for the unconstrained problem are well known. We can thus simply find the optimal allocation of the initial wealth between the unconstrained policy and its associated insurance package, and then compute the portfolio strategy for the insurance package. The optimal consumption policy is then the positive part of the unconstrained consumption policy and the optimal portfolio policy is the sum of the known portfolio policy for the unconstrained problem and the portfolio policy for the insurance policy. This procedure will be demonstrated in the next section in the context of the model of Merton (1971).

3. A Special Case

We now specialize our general model of uncertainty developed in Section 2 to the model considered by Merton (1971) and revisited recently by Karatzas, Lehoczky, Sethi, and Shreve (1986). We will employ the general method developed in the previous section in place of dynamic programming used in Karatzas and et al and Merton. The optimal consumption-portfolio policies for a class of utility functions will be explicitly computed. For many of the HARA utility functions for which the nonnegativity constraints are binding, the optimal policies fail to be linear policies.

3.1. Formulation

We take the model of uncertainty of Section 2 with the following specialization. Assume that risky security gain processes follow a multiplicative geometric Brownian motion:

$$S(t) + \int_0^t \iota(S(s), s)ds = S(0) + \int_0^t I_S(s)\mu ds + \int_0^t I_S(s)\hat{\sigma}dw(s) \quad \forall t \in [0, T], \ a.s.,$$

where μ is an $N \times 1$ vector of constants and $\hat{\sigma}$ is an $N \times N$ nonsingular matrix of constants. Assume further that $r(t) = r$ is a constant.

Given $Z(0) > 0$, the process Z becomes

$$\begin{aligned} Z(t) &= Z(0) + \int_0^t \left(r + (\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}(\mu - r\mathbf{1})\right) Z(s)ds + \int_0^t (\mu - r\mathbf{1})^\top \hat{\sigma}^{-1\top} Z(s)dw(s) \\ &= Z(0)\exp\left\{(\mu - r\mathbf{1})^\top\hat{\sigma}^{-1\top}w(t) + (r + \frac{1}{2}(\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}(\mu - r\mathbf{1})t\right\}, \\ &= Z(0)\exp\left\{\left(r - \frac{1}{2}(\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}(\mu - r\mathbf{1})\right)t - (\mu - r\mathbf{1})^\top\hat{\sigma}^{-1\top}w^*(t)\right\} \end{aligned}$$

where $\mathbf{1}$ is an N vector of ones. Thus, $\ln Z(t)$ is normally distributed with mean

$$\ln Z(0) + (r + \frac{1}{2}(\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}(\mu - r\mathbf{1})t$$

and variance

$$(\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}(\mu - r\mathbf{1})t$$

under P and is normally distributed with mean

$$\ln Z(0) + \left(r - \frac{1}{2}(\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}(\mu - r\mathbf{1})\right)t$$

and variance

$$(\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}(\mu - r\mathbf{1})t$$

under Q. To simplify notation, we note that

$$\frac{(\hat{\sigma}^\top\hat{\sigma})^{-1}(\mu - r\mathbf{1})}{(\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}\mathbf{1}}$$

is an N vector of constants that sum to one and therefore can be thought of as a vector of portfolio weights on the N risky securities. The mean m and the variance σ^2 of the rate of return of this portfolio are

$$\hat{\mu} = \frac{(\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}\mu}{(\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}\mathbf{1}}$$

$$\sigma^2 = \frac{(\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}(\mu - r\mathbf{1})}{[(\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}\mathbf{1}]^2}.$$

To avoid the degenerate case, we assume that $\hat{\mu} \neq r$. Now note that

$$(\mu - r\mathbf{1})^\top(\hat{\sigma}\hat{\sigma}^\top)^{-1}(\mu - r\mathbf{1}) = \frac{(\hat{\mu} - r)^2}{\sigma^2} \equiv \varrho^2$$

and we can write, when satiation does not occur, the mean and variance of $\ln Z(t)$ under P as $\ln Z(0) + \left(r + \frac{1}{2}\varrho^2\right)t$ and $\varrho^2 t$; and under Q as $\ln Z(0) + \left(r - \frac{1}{2}\varrho^2\right)t$ and $\varrho^2 t$.

For this special case of uncertainty, we will be able to consider a class of utility functions that is larger than that specified in Assumption 2.2. We assume that utility functions for consumption are continuous, increasing, and concave. They are either defined on the positive real line with a value at zero level of consumption possibly equal to minus infinity or defined on the whole of the real line. The utility function for the final wealth has the similar characteristics. As in Section 2, we use $u(y,t)$ and $V(y)$ to denote utility functions for consumption at time t and the utility function for final wealth. We also assume that either $u(y,t)$ or $V(y)$ is nontrivial, and when they are nontrivial, they are nonlinear. We still maintain that

$$\lim_{y\to\infty} u_{y+}(y,t) = 0$$

and

$$\lim_{y\to\infty} V'_+(y) = 0;$$

and define $\hat{f}(y,t)$ and $V_+^{\prime -1}(y)$ as in Section 2. We futher assume that $u_t(y,t)$ exists and is continuous in t.

3.2. Explicit Formulas for Optimal Consumption and Portfolio Policies

We will continue to impose Assumption 2.3. Note that since $u(y,t)$ and $V(y)$ may not be strictly increasing in y, satiation may be attained. In such event, investing completely in the riskless security while withdrawing minimum satiation levels of consumption over time is an optimal consumption–portfolio policy. Note that whenever

$$W(0) \geq \int_0^T e^{-rt}\hat{f}(0,t)dt + e^{-rT}V_+^{\prime -1}(0)$$

satiation occurs.

When satiation does not occur, define

$$\begin{aligned}F(Z(t),t) &= E^*\left[\int_t^T e^{-r(s-t)}\hat{f}(Z(s)^{-1},s)ds + e^{-r(T-t)}V_+^{\prime -1}(Z(T)^{-1})|Z(t)\right]\\ &= \int_0^{T-t} e^{-rs}\frac{1}{\varrho\sqrt{s}}\int_{-\infty}^{+\infty}\hat{f}(e^{-x},t+s)\,n\left(\frac{x-\ln Z(t)-(r-\frac{1}{2}\varrho^2)s}{\varrho\sqrt{s}}\right)dxds\\ &\quad + e^{-r(T-t)}\frac{1}{\varrho\sqrt{T-t}}\int_{-\infty}^{+\infty}V_+^{\prime -1}(e^{-x})\,n\left(\frac{x-\ln Z(t)-(r-\frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right)dx,\end{aligned} \tag{3.1}$$

where

$$n(y) = \frac{1}{\sqrt{2\pi}}\exp\{-\frac{y^2}{2}\}$$

is the standard normal density function. This function is just the F defined in Theorem 2.1. In our present set up, F is independent of $S(t)$.

For future reference, we will use $N(\cdot)$ to denote the distribution function for a standard normal random variable.

The following proposition shows that the optimally invested wealth will never become zero before time T.

Proposition 3.1. *Suppose that there exists a solution to (2.13). The optimally invested wealth will never reach zero before time T.*

Proof. We take cases. Case 1: the agent reaches satiation. The assertion is obvious. Case 2: satiation does not occur. Define Z by taking $Z(0)$ to be $1/\lambda$. Since either $u(y,t)$ or $V(y)$ is nontrivial, nonlinear, and concave, and since the support of a normally distributed random variable is the whole real line, the right-hand side of (3.1) is strictly positive for all $Z(t)$ and all $t \in [0,T)$. When there is no satiation, $F(Z(t),t)$ is equal to the optimally invested wealth at time t, and the assertion follows. ∎

We also have

Proposition 3.2. *Suppose that there exists a solution to (2.13). When satiation does not occur, $F(Z(t),t)$ is strictly increasing in $Z(t)$ and thus $Z(t) = F^{-1}(W(t),t)$ ν a.e.*

Proof. When satiation dose not occur, an increase in $Z(t)$ implies an increase in the mean for $Z(s)$, $s > t$, while the variance stays the same. The assertion then follows from the hypothesis that either $u(y,t)$ or $V(y)$ is nontrivial, nonlinear, and concave.

∎

The following proposition is a specialization of Proposition 2.3.

Proposition 3.3. *Suppose that $u(y,t)$ is nontrivial, that $u_{c+}(0,t) < \infty$, and that satiation does not occur. An optimal consumption policy has the property that consumption will be zero if and only if wealth is less than the nonstochastic time dependent boundary given by*

$$\underline{W}(t) = \int_0^{T-t} \frac{e^{-rs}}{\varrho\sqrt{s}} \int_{-\infty}^{+\infty} \hat{f}(e^{-x}, t+s)\, n\left(\frac{x + \ln u'_+(0,t) - (r - \frac{1}{2}\varrho^2)s}{\varrho\sqrt{s}}\right) dx ds$$
$$+ \frac{e^{-r(T-t)}}{\varrho\sqrt{T-t}} \int_{-\infty}^{+\infty} V_+^{\prime -1}(e^{-x})\, n\left(\frac{x + \ln u_{c+}(0,t) - (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right) dx.$$

Proof. Note that $c(t) = 0$ if and only if $u'_+(0,t) \leq Z(t)^{-1}$. The assertion then follows from Proposition 3.2.

∎

The following proposition gives a set of sufficient conditions for $D_y^m F(y,t)$ and $F_t(y,t)$ to exist and to be continuous.

Proposition 3.4. *Suppose that (3.1) is finite for all $Z(t)$. Suppose further that for every subinterval $[a,b]$ of $\Re$ and for every subinterval $[a',b')$ of $[0,T)$ there exist functions $G^m(x,s)$, $m = 1,2$, such that*

$$\frac{1}{\varrho\sqrt{s}} \hat{f}(e^{-x}, s+t) \left| \frac{\partial^m}{\partial y^m} n\left(\frac{x - y - (r - \frac{1}{2}\varrho^2)s}{\varrho\sqrt{s}}\right)\right| \leq G^m(x,s) \quad \forall y \in (a,b)$$

and

$$\int_t^T \int_{-\infty}^{+\infty} G^m(x,s) dx ds < \infty.$$

and $\forall t \in [0,T)$ and for every subinterval $[a,b]$ of $\Re$ there exists function $H(x,s)$ such that $\forall s \in (0, T-t)$, $y \in (a,b)$, and x

$$\left| \frac{1}{\varrho\sqrt{s}} \hat{f}_t(e^{-x}, t+s)\, n\left(\frac{x - y - (r - \frac{1}{2}\varrho^2)s}{\varrho\sqrt{s}}\right)\right| \leq H(x,s)$$

and

$$\int_{-\infty}^{+\infty} \hat{H}(x)dx < \infty.$$

Then $D_y^m F(y,t)$ $m \leq 2$ *and* $F_t(y,t)$ *exist and are continuous. In particular,*

$$F_Z(Z(t),t) = \int_0^{T-t} \frac{e^{-rs}}{\varrho\sqrt{s}} \int_{-\infty}^{+\infty} \hat{f}(e^{-x}, t+s) \frac{\partial}{\partial Z(t)} n\left(\frac{x - \ln Z(t) - (r - \frac{1}{2}\varrho^2)s}{\varrho\sqrt{s}}\right) dx ds$$
$$+ \frac{e^{-r(T-t)}}{\varrho\sqrt{T-t}} \int_{-\infty}^{+\infty} V_+^{\prime -1}(e^{-x}) \frac{\partial}{\partial Z(t)} n\left(\frac{x - \ln Z(t) - (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right) dx.$$

Proof. The assertions follow from repeated application of the Lebesgue convergence theorem, the fact that $u_{y+}(y,t)$ is continuous in t for every y, and the fact that the normal distribution density function is an exponential function; see, e.g., Theorems 10.38 and 10.39 of Apostol (1974).

∎

Note that the conditions in Proposition 3.4 do not involve the the differentiability of $\hat{f}(e^{-x}, t)$ with respect to x, in contrast to Theorem 2.1. They do involve differentiability with respect to t, however. The proposition below is a direct consequence of Theorem 2.2 with a difference that now there exists a possibility of satiation.

Proposition 3.5. *Suppose that* $D^m F(y,t)$ *and* F_t *exist and are continuous for* $m \leq 2$, *and that differentiation of* F *can be carried out under the integral sign. If* $W(0) < \int_0^T e^{-rt}\hat{f}(0,t)dt + e^{-rT}V_+^{\prime -1}(0)$ *and if there exists* $Z_0 > 0$ *such that* $F(Z_0, 0) = W(0)$, *a solution to (2.13) exists. Defining* Z *by taking* $Z(0) = Z_0$, *an optimal consumption portfolio policy and its corresponding indirect utility function are*

$$c(W(t),t) = \hat{f}(e^{-\ln F^{-1}(W(t),t)}, t)$$

$$A(W(t),t) = (\hat{\sigma}^\top\hat{\sigma})^{-1}(\mu - r\mathbf{1})\Bigg[\int_0^{T-t} \frac{e^{-rs}}{(\varrho^2 s)^{\frac{3}{2}}} \times$$
$$\left(\int_{-\infty}^{+\infty} \hat{f}(e^{-x}, t+s)\left(x - \ln F^{-1}(W(t),t) - (r - \frac{1}{2}\varrho^2)s\right) n\left(\frac{x - \ln F^{-1}(W(t),t) - (r - \frac{1}{2}\varrho^2)s}{\varrho\sqrt{s}}\right) dx\right) ds$$
$$+ \frac{e^{-r(T-t)}}{(\varrho^2(T-t))^{\frac{3}{2}}} \int_{-\infty}^{+\infty} V_+^{\prime -1}(e^{-x})\left(x - \ln F^{-1}(W(t),t) - (r - \frac{1}{2}\varrho^2)(T-t)\right) \times$$
$$n\left(\frac{x - \ln F^{-1}(W(t),t) - (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right) dx\Bigg]$$

$$J(W(t),t) = \int_0^{T-t} \frac{1}{\varrho\sqrt{s}} \int_{-\infty}^{+\infty} u(\hat{f}(e^{-1}, t+s), t+s)\, n\left(\frac{x - \ln F^{-1}(W(t),t) - (r + \frac{1}{2}\varrho)s}{\varrho\sqrt{s}}\right) dx ds$$
$$+ \frac{1}{\varrho\sqrt{T-t}} \int_{-\infty}^{+\infty} V(V_+^{\prime -1}(e^{-x}))\, n\left(\frac{x - \ln F^{-1}(W(t),t) - (r + \frac{1}{2}\varrho)(T-t)}{\varrho\sqrt{T-t}}\right) dx.$$

When $W(0) \geq \int_0^T e^{-rt}\hat{f}(0,t)dt + e^{-rT}V_+'^{-1}(0)$, there is satiation and therefore investing completely in the riskless security and consuming $c(t) = \hat{f}(0,t)$ at time t is an optimal strategy.

Proof. The first assertion is a consequence of Theorem 2.2. The second assertion is obvious.

∎

Note that with exponential discounting, the utility function has the form $u(y,t) = e^{-\rho t}u(y)$. For this important special case, $\hat{f}(e^{-x},t) = u_+'^{-1}(e^{-x+\rho t})$.

Now we will pause for a moment to present several examples. Using Proposition 3.5 or Cox and Huang (1986), one can verify that Assumption 2.3 is valid and there exists an optimal consumption-portfolio policy for all the examples. We will demonstrate our proposed method by computing explicit optimal consumption-portfolio policies. In particular, Example 3.4 solves the optimal consumption–portfolio problem for the complete family of HARA utility functions while taking into account the nonnegativity constraints on consumption and on the final wealth.

Example 3.1. Let $u(y,t) = 0$ and

$$V(y) = \begin{cases} y & \text{for } 0 \leq y < \bar{y}, \\ \bar{y} & \text{for } y \geq \bar{y}. \end{cases}$$

In this case, $V_+'(y)$ equals 1 for $0 \leq y < \bar{y}$ and equals 0 for $y \geq \bar{y}$. Hence, $V_+'(e^{-x}) = \bar{y}$ for $x > 0$ and $V_+'(e^{-x}) = 0$ for $x \leq 0$. Computation yields

$$F(Z(t),t) = \bar{y}e^{-r(T-t)}\, N\left(\frac{\ln Z(t) + (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right).$$

Note that if $W(0) > e^{-rT}\bar{y}$, there is no $Z(0) < \infty$ such that $F(Z(0),0) = W(0)$. This is so, because by investing $W(0)$ completely in the riskless asset, the agent will reach satiation at time T with probability one and this riskless strategy is an optimal strategy. For $W(0) < e^{-rT}\bar{y}$, an optimal investment strategy and the its corresponding indirect utility function are

$$A(W(t),t) = \bar{y}e^{-r(T-t)}(\hat{\sigma}^\top\hat{\sigma})^{-1}(\mu - r\mathbf{1})\frac{1}{\varrho\sqrt{T-t}}\, n\left(\frac{\ln F^{-1}(W(t),t) + (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right)$$

$$J(W(t),t) = \bar{y}\, N\left(\frac{\ln F^{-1}(W(t),t) + (r + \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right).$$

Note that for any given time t, the optimal amounts invested in the risky assets are the largest when $F^{-1}(W(t),t) = e^{-(r-\frac{1}{2}\varrho^2)(T-t)}$, which occurs when $W(t) = \frac{1}{2}\bar{y}e^{-r(T-t)}$.

∎

Example 3.2. Let $u(y,t) = 0$ and $V(y) = -\frac{1}{a}e^{-ay}$, where $a > 0$ is the constant absolute risk aversion. Then $V'_+(y) = e^{-ay}$ and $V'^{-1}_+(e^{-x}) = [x/a]^+$. Hence we have

$$F(Z(t),t) = e^{-r(T-t)} \frac{1}{a\varrho\sqrt{T-t}} \int_0^\infty x\, n\left(\frac{x - \ln Z(t) - (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right) dx$$

$$= e^{-r(T-t)} \frac{\varrho\sqrt{T-t}}{a}\left[\left(\frac{\ln Z(t) + (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right) N\left(\frac{\ln Z(t) + (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right) + n\left(\frac{\ln Z(t) + (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right)\right].$$

$$A(W(t),t) = e^{-r(T-t)} \frac{1}{a}(\hat{\sigma}^\top\hat{\sigma})^{-1}(\mu - r\mathbf{1})\, N\left(\frac{\ln F^{-1}(W(t),t) + (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right).$$

Note that the optimal amount invested in the risky assets is not independent of the wealth level. This is a consequence of the nonnegativity constraint. However, note the following

$$\lim_{W(t)\to\infty} A(W(t),t) = e^{-r(T-t)} \frac{1}{a}(\hat{\sigma}^\top\hat{\sigma})^{-1}(\mu - r\mathbf{1}),$$

which is a constant policy.

∎

Recall from Section 2.4 that there exists a relationship between constrained solutions and unconstrained solutions. The following example illustrates this connection.

Example 3.3. Consider the utility function for wealth of Example 3.3. Without the nonnegativity constraint. Then $V'^{-1}_+(e^{-x}) = x/a$. Since x is normally distributed, x/a lies in $L^2(P)$. Let

$$\hat{F}(Z_\lambda(t),t) \equiv e^{-r(T-t)} \frac{1}{a\varrho\sqrt{T-t}} \int_{-\infty}^{+\infty} x\, n\left(\frac{x - \ln Z_\lambda(t) - (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right) dx$$

$$= e^{-r(T-t)} \frac{1}{a\varrho\sqrt{T-t}} \ln Z_\lambda(t) + (r - \frac{1}{2}\varrho^2)(T-t),$$

where Z_λ denotes the process Z with $Z(0) = 1/\lambda$. $\hat{F}$ is the value, at time t, of the optimally invested wealth given that the initial wealth $W_\lambda(0)$ gives rise to the Lagrangian multiplier λ. Independent of the initial wealth, the optimal amounts invested in risky assets are

$$\hat{A}(W_\lambda(t),t) = e^{-r(T-t)} \frac{1}{a}(\hat{\sigma}^\top\hat{\sigma})(\mu - r\mathbf{1}).$$

Following this strategy, the final wealth will be

$$W_\lambda(T) = \ln Z_\lambda(0) + \left\{(r - \frac{1}{2}\varrho^2)T - (\mu - r\mathbf{1})^\top\hat{\sigma}^{-1\top}w^*(T)\right\}.$$

The value of the European put option written on $W_\lambda(T)$ is

$$\hat{\hat{W}}_\lambda(t) = p(Z_\lambda(t), t) = e^{-r(T-t)} \frac{1}{a\varrho\sqrt{T-t}} \int_{-\infty}^{0} x\, n\left(\frac{x - \ln Z_\lambda(t) - (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right) dx$$

$$= e^{-r(T-t)} \frac{\varrho\sqrt{T-t}}{a} n\left(\frac{-\ln Z_\lambda(t) - (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right)$$

$$- \frac{\ln Z_\lambda(t) + (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}} N\left(\frac{-\ln Z_\lambda(t) - (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right).$$

The investment strategy in the risky assets that replicates this put option is

$$\hat{\hat{A}}(\hat{\hat{W}}_\lambda(t), t) = -e^{-r(T-t)} \frac{1}{a} (\hat{\sigma}\hat{\sigma}^\top)^{-1}(u - r\mathbf{1})\, N\left(-\frac{p^{-1}(\hat{\hat{W}}_\lambda(t), t) + (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right).$$

Now we want to find $Z_\lambda(0)$ so that

$$\hat{F}(Z_\lambda(0), 0) + p(Z_\lambda(0), 0) = W(0).$$

Note that $\hat{F}(t) + p(t) = F(t)$, where $F(t)$ is the value of the constrained policy at time t. Hence

$$1/\lambda = Z_\lambda(0) = F^{-1}(W(0), t),$$

which is what we anticipated. Now the process Z_λ is well defined and the optimal investment strategy for the constrained problem is

$$A(W(t), t) = \hat{A}(W_\lambda(t), t) + \hat{\hat{A}}(\hat{\hat{W}}_\lambda(t), t)$$

$$= e^{-r(T-t)} \frac{1}{a} (\hat{\sigma}^\top \hat{\sigma})^{-1}(\mu - r\mathbf{1})\, N\left(\frac{\ln F^{-1}(W(t), t) + (r - \frac{1}{2}\varrho^2)(T-t)}{\varrho\sqrt{T-t}}\right),$$

which is identical to that of Example 3.2.

∎

Among many other results in a pioneering paper, Merton (1971) derived optimal consumption and portfolio rules for hyperbolic absolute risk aversion (HARA) utility functions when securities prices follow a geometric Brownian motion and the interest rate is constant. However, as Merton noted, the solution given for some members of the HARA family are not completely appropriate, since they allow the agent to incur negative wealth and may require negative consumption. One might hope that this difficulty could be easily remedied by setting consumption equal to zero whenever negative consumption would have been required and by following the designated rules only as long as wealth remains positive. Unfortunately, this is not the case. The optimal solution with nonnegativity constraints on consumption and wealth will have a completely different form as evidenced already by Example 3.2. In the following example, we will derive explicit solutions that satisfy these constraints.

Example 3.4. Let

$$u(y,t) = e^{-\rho t}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\beta y}{1-\gamma}+\xi\right)^{\gamma}$$

$$V(y) = u(y,T).$$

with $\beta > 0$, $\gamma \neq 0$ or 1. It is understood that if $\gamma > 1$, then $u(y,t) = 0$ for all $y \geq (\gamma - 1)\xi/\beta$. With $\gamma < 1$ and $\xi < 0$, the agent's problem is not completely specified because the utility function does not state the consequence of consuming less than $|\xi|(1-\gamma)/\beta$. Furthermore, for sufficiently low initial wealth,

$$W(0) < |\xi|(1-\gamma)\left(1-e^{-rT}\right)/\beta r,$$

there is no policy that can guarantee $c(t) \geq |\xi|(1-\gamma)/\beta$ for all t with probability one. Consequently, we only consider the case $\xi \geq 0$.

By evaluating the integrals of (3.1), we obtain the following results for the HARA functions:

$$\begin{aligned}
F(Z(t),t) = {} & \left(\frac{1-\gamma}{\beta}\right)\int_0^{T-t}\Bigg((Z(t)\beta)^{\frac{1}{1-\gamma}}e^{-\delta s}\,N\Big(\frac{\ln(\beta Z(t))^{\frac{1}{1-\gamma}} - \ln\xi + (r-\delta+\frac{1}{2}\bar{\sigma}^2)s}{\operatorname{sgn}(1-\gamma)\bar{\sigma}\sqrt{s}}\Big) \\
& - \xi e^{-rs}\,N\Big(\frac{\ln(\beta Z(t))^{\frac{1}{1-\gamma}} - \ln\xi + (r-\delta-\frac{1}{2}\bar{\sigma}^2)s}{\operatorname{sgn}(1-\gamma)\bar{\sigma}\sqrt{s}}\Big)\Bigg)ds \\
& + \left(\frac{1-\gamma}{\beta}\right)\Bigg[(Z(t)\beta)^{\frac{1}{1-\gamma}}e^{-\delta(T-t)}\,N\Big(\frac{\ln(\beta Z(t))^{\frac{1}{1-\gamma}} - \ln\xi + (r-\delta+\frac{1}{2}\bar{\sigma}^2)(T-t)}{\operatorname{sgn}(1-\gamma)\bar{\sigma}\sqrt{T-t}}\Big) \\
& - \xi e^{-r(T-t)}\,N\Big(\frac{\ln(\beta Z(t))^{\frac{1}{1-\gamma}} - \ln\xi + (r-\delta-\frac{1}{2}\bar{\sigma}^2)(T-t)}{\operatorname{sgn}(1-\gamma)\bar{\sigma}\sqrt{s}}\Big)\Bigg]
\end{aligned}$$

where

$$\delta = \left(\frac{1}{1-\gamma}\right)\left(\rho - \gamma\left(r + \frac{\varrho^2}{2(1-\gamma)}\right)\right)$$

and

$$\bar{\sigma}^2 = \left(\frac{1}{1-\gamma}\right)^2\varrho^2.$$

Using the properties of $n(\cdot)$ and $N(\cdot)$, it can be verified that $D_y^m F(y,t)$, $m \leq 2$, and $F_t(y,t)$ exist and are continuous. In particular, $F_Z(Z(t),t)$ can be computed by differentiating under the integral sign. When $\gamma > 1$, satiation occurs if

$$W(0) \geq \frac{1-(1-r)e^{-rT}}{r}\frac{(\gamma-1)\xi}{\beta}.$$

When satiation does not occur, an optimal policy and its corresponding indirect utility function are

$$c(W(t),t) = \left[\left(\frac{1-\gamma}{\beta}\right)\left((F^{-1}(W(t),t)\beta)^{\frac{1}{1-\gamma}} - \xi\right)\right]^+$$

$$A(W(t),t) = (\hat{\sigma}\hat{\sigma}^\top)^{-1}(u - r\mathbf{1})\left(\frac{(F^{-1}(W(t),t)\beta)^{\frac{1}{1-\gamma}}}{\beta}\right) \times$$

$$\left[\int_0^{T-t} e^{-\rho s} N\Big(\frac{\ln(\beta F^{-1}(W(t),t))^{\frac{1}{1-\gamma}} - \ln\xi + (r - \delta - \frac{1}{2}\bar{\sigma}^2)s}{\operatorname{sgn}(1-\gamma)\bar{\sigma}\sqrt{s}}\Big)ds\right.$$

$$\left. + e^{-\rho(T-t)} N\Big(\frac{\ln(\beta F^{-1}(W(t),t))^{\frac{1}{1-\gamma}} - \ln\xi + (r - \delta - \frac{1}{2}\bar{\sigma}^2)(T-t)}{\operatorname{sgn}(1-\gamma)\bar{\sigma}\sqrt{T-t}}\Big)\right]$$

$$J(W(t),t) = \left(\frac{1-\gamma}{\gamma}\right)e^{-\rho t}\left[\int_0^{T-t}\left((\beta F^{-1}(W(t),t))^{\frac{\gamma}{1-\gamma}} e^{-\delta s} N\Big(\frac{\ln(\beta F^{-1}(W(t),t))^{\frac{\gamma}{1-\gamma}} - \ln\xi^\gamma + (\rho - \delta + \frac{1}{2}\gamma^2\bar{\sigma}^2)s}{\operatorname{sgn}(1-\gamma)\,\gamma\bar{\sigma}s}\Big)\right.\right.$$

$$\left. + \xi^\gamma e^{-\rho s} N\Big(\frac{-\ln(\beta F^{-1}(W(t),t))^{\frac{\gamma}{1-\gamma}} + \ln\xi^\gamma - (\rho - \delta - \frac{1}{2}\gamma^2\bar{\sigma}^2)s}{\operatorname{sgn}(1-\gamma)\,\gamma\bar{\sigma}s}\Big)\right)ds$$

$$+ (\beta F^{-1}(W(t),t))^{\frac{\gamma}{1-\gamma}} e^{-\delta(T-t)} N\Big(\frac{\ln(\beta F^{-1}(W(t),t))^{\frac{\gamma}{1-\gamma}} - \ln\xi^\gamma + (\rho - \delta + \frac{1}{2}\gamma^2\bar{\sigma}^2)(T-t)}{\operatorname{sgn}(1-\gamma)\,\gamma\bar{\sigma}(T-t)}\Big)$$

$$\left. + \xi^\gamma e^{-\rho(T-t)} N\Big(\frac{-\ln(\beta F^{-1}(W(t),t))^{\frac{\gamma}{1-\gamma}} + \ln\xi^\gamma - (\rho - \delta - \frac{1}{2}\gamma^2\bar{\sigma}^2)(T-t)}{\operatorname{sgn}(1-\gamma)\,\gamma\bar{\sigma}(T-t)}\Big)\right].$$

As $W(t)$ becomes large, the optimal consumption and investment policies approach linear functions of wealth given in Merton (1971).

∎

Remark 3.1. Example 3.4 can easily be generalized to allow the utility function for final wealth to be a HARA function having different coefficients than those of the utility functions for consumption. No substantial changes need to be made of the solution except for some cosmetic changes in notation.

∎

For many utility functions, the optimal consumption policy will not be differentiable and may not even by continuous in wealth. A specific example is given below.

Example 3.5. Let

$$u(y,t) = \begin{cases} y & \text{for } 0 \le y < \bar{y}, \\ \bar{y} & \text{for } y \ge \bar{y}, \end{cases}$$

and let $V(y) = 0$. Suppose that satiation does not occur. We know that

$$\hat{f}(e^{-x},t) = \begin{cases} \bar{y} & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

Direct computation yields

$$F(Z(t),t) = \bar{y} \int_0^{T-t} e^{-rs} N\left(\frac{\ln Z(t) + (r - \frac{1}{2}\varrho^2)s}{\varrho\sqrt{s}}\right) ds.$$

The optimal time t consumption is zero if and only if $Z(t) < 1$. By the strict monotonicity of $F(y,t)$ in y, we know $Z(t) < 1$ if and only if $F(Z(t),t) < F(1,t)$. Thus

$$c(t) = \begin{cases} 0 & \text{if } W(t) < F(1,0); \\ \bar{y} & \text{if } W(t) \geq F(1,0). \end{cases}$$

The optimal consumption is not a continuous function of the wealth and fails to be differentiable at a single point.

∎

We conclude this section by giving, in the two propositions below, necessary and sufficient conditions for the consumption policy prescribed by $\hat{f}$ to have certain derivatives.

Proposition 3.6. *Suppose that utility functions for consumption have possibly time dependent satiation level $\bar{c}(t)$ and yield an F such that $D_y^m F(y,t)$ and F_t exist and are continuous. Suppose also that satiation does not occur. Let y' be a point of discontinuity of $u'_{c+}(c,t)$. A necessary and sufficient condition for $c(W(t),t)$ to be a differentiable function of $W(t)$ and t is that for all $t \in [0,T]$ and for all y'*

(i) $u(y,t)$ *is strictly concave for all* $y < \bar{c}(t)$;

(ii) $u(y,t)$ *is twice differentiable with respect to* y *for all* $y < \bar{c}(t)$ *except at* y';

(iii)

$$\lim_{y \to y'} -\frac{u_y(y,t)}{u_{yy}(y,t)} = 0;$$

(iv) for $u_{y+}(0,t) < \infty$,

$$\lim_{y \downarrow 0} -\frac{u_y(y,t)}{u_{yy}(y,t)} = 0;$$

(v) for $\bar{c}(t) < \infty$,

$$\lim_{y \uparrow \bar{c}(t)} -\frac{u_y(y,t)}{u_{yy}(y,t)} = 0.$$

$c(W(t),t)$ is a continuously differentiable function of $W(t)$ and t if and only if, in addition, $u(y,t)$ is twice continuously differentiable with respect to y for all $y < \bar{c}(t)$ except at y' and continuously differentiable with respect to t.

Proof. Suppose first that $\bar{c}(t) = \infty$. $c(W(t),t)$ is differentiable in $W(t)$ if and only if $\hat{f}(y,t)$ is differentiable in y. For every subinterval (a,b) on which $u_{y+}(y,t)$ is continuous, $\hat{f}(y,t)$ is differentiable if and only if $u_{y+}(y,t)$ is strictly decreasing and differentiable in y, which is (i) and (ii). On the interval $(u_{y+}(y',t), u_{y-}(y',t))$, $\hat{f}$ is flat. Hence $\hat{f}(y,t)$ is differentiable in y at y' if and only if (iii). When $u_{y+}(0,t) < \infty$, $\hat{f}$ is flat on the interval $(u_{y+}(0,t), \infty)$. Thus $\hat{f}(y,t)$ is differentiable at 0 if and only if (iv). Similar arguments proves (v).

∎

The following proposition gives circumstances in which c_{WW} exists and is continuous.

Proposition 3.7. *Suppose that utility functions for consumption have possibly time dependent satiation level $\bar{c}(t)$ and yield an F such that $D_y^m F(y,t)$ and F_t exist and are continuous. Suppose also that satiation does not occur. Let y' be a point of discontinuity of $u'_{c+}(c,t)$ and suppose that $u(y,t)$ is three times differentiable with respect to y except at y'. A necessary and sufficient condition for $c(W(t),t)$ to be twice differentiable with respect to $W(t)$ is that for all t and for all y', (i)-(v) of Proposition 3.4* are satisfied and

(vi)

$$\lim_{y \to y'} \left(\frac{u_y(y,t)}{u_{yy}(y,t)} \right)^2 \left(\frac{u_{yyy}(y,t)}{u_{yy}(y,t)} \right) = 0;$$

(vii) for $u_{y+}(0,t) < \infty$,

$$\lim_{y \downarrow 0} \left(\frac{u_y(y,t)}{u_{yy}(y,t)} \right)^2 \left(\frac{u_{yyy}(y,t)}{u_{yy}(y,t)} \right) = 0;$$

(viii) for $\bar{c}(t) < \infty$,

$$\lim_{y \uparrow \bar{c}(t)} \left(\frac{u_y(y,t)}{u_{yy}(y,t)} \right)^2 \left(\frac{u_{yyy}(y,t)}{u_{yy}(y,t)} \right) = 0.$$

$c(W(t),t)$ is a twice continuously differentiable function of $W(t)$ if and only if, in addition, $u(y,t)$ is three times continuously differentiable with respect to y for all $y < \bar{c}(t)$ except at y'.

Proof. Arguments are similar to those of Propositions 2.3 and 3.6, so we omit them.

∎

4. Concluding Remarks

This paper is a companion paper of Cox and Huang (1986). In that paper, we tackled the general existence question. The focus of this paper in on characterization and computation of optimal policies. Both of these papers depend critically upon the assumption that the number of risky securities is equal to the number of the underlying independent Brownian motions that describe the uncertain environment. How our technique can be useful when that assumption is not met is an important open question.

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