A Variational Problem Arising in Financial Economics

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Last Revised October 1988

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December 1985
Revised: November 1987
Last Revised: October 1988

Abstract

We provide sufficient conditions for a dynamic consumption-portfolio problem in continuous time to have a solution for a class of utility functions, when the price system follows a diffusion process and when the space of admissible policies is a linear space. Besides a regularity condition, it suffices to check whether a uniform growth and a local Lipschitz condition are satisfied by the parameters of a system of stochastic differential equations, which is completely derived from the price system. The class of utility functions includes concave functions that are, roughly, either bounded from below or strictly concave, and whose coefficients of relative risk aversion have nonzero limit infima as consumption/wealth goes to infinity.

*The authors would like to acknowledge helpful conversations with Sergiu Hart, Andreu Mas Colell, and Ho–Mou Wu. The expositions of this paper were improved by comments from two anonymous referees. Various comments were made by seminar participants at Brown University, Cornell University, Massachusetts Institute of Technology, Northwestern University, University of California at Berkeley, University of Chicago, University of Minnesota, University of Pennsylvania, the Western Finance Association Meeting in 1987, for which the authors are thankful. This paper was completed when the second author was on a fellowship provided by Batterymarch Financial Management, whose generous support is greatly appreciated. Remaining errors are of course the authors'.

1 Introduction

Optimal intertemporal consumption and portfolio policies in continuous time under uncertainty are traditionally characterized by stochastic dynamic programming; cf. Merton (1971). To show the existence of a solution to the consumption–portfolio problem using dynamic programming, there are two approaches. The first is through some existence theorems in the theory of stochastic control. Those existence theorems, however, often require an admissible control to take its values in a compact set. This is unsatisfactory. If we are modeling a frictionless market, any compactness assumption on the values of controls is arbitrary. Moreover, many existence theorems allow controls to affect only the drift term of the controlled process (see Fleming and Rishel (1975)). This, unfortunately, rules out the portfolio problem under consideration.

The second approach is through construction: construct a control and then use the *verification theorem* in dynamic programming to verify that it indeed is a solution. Merton's paper uses this second approach. It is very difficult, however, to construct a solution in general.

Recently, a martingale representation technology has been used in place of dynamic programming: notably, Cox and Huang (1987) and Pliska (1986) in portfolio theory (purely microeconomics), and Chamberlain (1987) and Huang (1987) in characterizing equilibrium portfolio rules. This technology allows the space of admissible policies to be a linear space and is made available by the connection between an arbitrage free price system and martingales demonstrated by Harrison and Kreps (1979) and developed further by Harrison and Pliska (1981). The possibility of this new technology is first vaguely foreshadowed in Kreps (1979).

Pliska (1986) provides joint conditions on an agent's utility function and on a price system that are sufficient for the optimal policies to exist in a very general stochastic environment where asset prices are semimartingales[1] and when consumption and final wealth are allowed to become strictly negative. Those conditions, however, are restrictive and usually hard to verify. Cox and Huang (1987) is a companion paper of this one and is concerned with computation of optimal policies when they exist and verification of candidate policies to be optimal. They show, among others things that, a *linear* partial differential equation need be solved to construct optimal policies. They also solve, in closed form, the optimal consumption and portfolio policies for the HARA class utility functions, when asset prices follow a geometric Brownian motion and when a nonnegativity constraint on consumption is present.

The purpose of this paper is to provide a set of easily verifiable sufficient conditions for existence when asset prices follow a diffusion process, when the space of admissible *controls* or *policies* is a linear

[1] A semimartingale is a continuous time stochastic process that can be decomposed into the sum of a process of bounded variation and a local martingale; see Dellacherie and Meyer (1982).

space, and when there is a nonnegativity constraint on consumption as well as on final wealth. We restrict our attention to a class of concave utility functions that are either bounded from below or strictly concave, and whose whose coefficients of relative risk aversion have nonzero limit infima as the consumption/wealth goes to infinity. For this class of utility functions, for the existence of an optimal policy, it is sufficient to check whether a uniform growth condition and a local Lipschitz condition are satisfied by the parameters of a system of stochastic differential equations completely derived from the price system itself.

The rest of this paper is organized as follows: Section 2 formulates the dynamic consumption–portfolio problem when asset prices follow a diffusion process. It is shown there that, under certain conditions, there exists a correspondence between the dynamic problem and an Arrow–Debreu type static problem in that the solution to the static problem is a solution to the dynamic problem. The Arrow–Debreu prices in the static problem are derived from the price processes in the dynamic problem and will be termed the implicit Arrow–Debreu prices. It follows that the existence of a solution to the dynamic problem can be ensured if there exists a solution to the static problem. We then turn our attention in Section 3 to study a class of Arrow–Debreu style static maximization problems. For the class of utility functions mentioned in the above paragraph, for existence, it suffices to check whether the the "inverse" of the implicit Arrow–Debreu price system, per unit of probability, has a certain finite moment.

We come back in Section 4 to the dynamic problem formulated in Section 2. From Section 3, we know that for the existence of a solution to the dynamic problem for a class of utility functions, it suffices to check whether the inverse of the implicit Arrow–Debreu prices has a finite certain moment. It turns out that this condition is ensured if parameters of a system of stochastic differential equations completely derived from the price processes satisfy a local Lipschitz and a uniform growth condition.

In Section 5 we demonstrate the technique developed in earlier sections by showing that there exists an optimal consumption–portfolio policy in the geometric Brownian motion model originally considered by Merton (1971) except that now we impose a nonnegativity constraint on consumption and on final wealth. Section 6 contains some generalizations and concluding remarks.

2 The Dynamic Consumption–Portfolio Problem

In this section, we will formulate a consumption–portfolio problem for an individual in continuous time. Our final purpose is to provide sufficient conditions for the existence of an optimal consumption–portfolio policy for a class of utility functions. This, however, will come much later in Section 4. We will focus our attention here on characterizing certain properties of an optimal policy when one exists.

The essential idea to be elaborated is the observation that there exists a correspondence between the dynamic problem under study and a particular static problem of the Arrow–Debreu type where elementary time–state contingent securities are traded. This observation is not new and can be found in many places in the literature; see, for example, Harrison and Kreps (1979) and Huang (1985). We shall, however, pay special attention to the characterization of the budget feasible set for the class of trading strategies allowed here, which includes as a subset those considered by Harrison and Kreps (1979) and Huang (1985).

2.1 Formulation

We fix a complete probability space $(\Omega, \mathcal{F}, P)$ and a time span $[0, T]$, where T is a strictly positive real number.[2] An element of Ω, denoted by ω, is a state of nature, which is a complete description of the exogenous uncertain environment from time 0 to time T. The sigma–field $\mathcal{F}$ is the collection of events distinguishable at time T and P is a probability measure representing an individual's beliefs about the likelihood of distinguishable events.

There is defined on the probability space an N–dimensional standard Brownian motion denoted by $w = \{w_n(t); t \in [0,T], n = 1, 2, \ldots, N\}$. Let $\mathcal{F}_t$ be the smallest sigma–field containing all the P–measure zero sets with respect to which $\{w(s); 0 \leq s \leq t\}$ is measurable, or simply the *completed* sigma–field *generated* by $\{w(s); 0 \leq s \leq t\}$. Since the Brownian motion w is defined on $(\Omega, \mathcal{F}, P)$, $\mathcal{F}_t$ is a sub–sigma–field of $\mathcal{F}$. The increasing family of sub–sigma–fields $\mathbf{F} \equiv \{\mathcal{F}_t; t \in [0,T]\}$ is usually termed the *filtration* generated by w. A filtration is an abstract way of representing information arrival over time. We assume that $\mathcal{F}_T = \mathcal{F}$, that is, the true state of nature will be revealed at time T by observing w from time 0 to time T. Since a standard Brownian motion starts at zero $P-a.s.$, $\mathcal{F}_0$ contains only sets of probability zero or one.

A process $X = \{X(t); t \in [0,T]\}$ is said to be *adapted* to $\mathbf{F}$ if $X(t)$ is measurable with respect to $\mathcal{F}_t$ $\forall t \in [0,T]$, that is, the value of X at time t cannot depend on the realizations of the Brownian motion strictly after time t.

Consider a frictionless security market with $N+1$ long–lived traded securities indexed by $n = 0, 1, 2, \ldots, N$. A long–lived security is a security available for trading all the time from time 0 to time T. Security $n \neq 0$ is risky and, at time t, pays dividend at rate $f_n(t)$ and sells for $S_n(t)$. We will henceforth denote $(S_1(t), \ldots, S_N(t))^\top$ and $(f_1(t), \ldots, f_N(t))^\top$ by $S(t)$ and $f(t)$, respectively, where $^\top$ denotes "transpose." Assume that $f_n(t)$ can be written as $f_n(S(t), t)$ with $f_n(z, t) : \Re^N \times [0,T] \mapsto \Re$ continuous in z and t. Security 0 is *locally riskless*, pays no dividends, and sells for $B(t) = B(0)\exp\{\int_0^t r(s)ds\}$ at time t, where $B(0)$ is a strictly positive real number and where $r(t)$ is the *instantaneous riskless rate* at

[2] A probability space $(\Omega, \mathcal{F}, P)$ is said to be complete if $A \in \mathcal{F}$ and $P(A) = 0$ imply $A' \in \mathcal{F}$ for any $A' \subset A$.

time t. We assume that $r(t)$ is nonnegative and can be written as $r(S(t),t)$, with $r(z,t) : \Re^N \times [0,T] \mapsto \Re_+$ continuous in x and t. Henceforth the 0–th security will be termed the bond.

Assume that S is an Itô process adapted to $\mathbf{F}$ satisfying

$$S(t) + \int_0^t f(S(s),s)ds = S(0) + \int_0^t \varsigma(S(s),s)ds + \int_0^t \sigma(S(s),s)dw(s) \quad \forall t \in [0,T], \quad P-a.s., \tag{1}$$

where $\varsigma(z,t) : \Re^N \times [0,T] \mapsto \Re^N$ and $\sigma(z,t) : \Re^N \times [0,T] \mapsto \Re^{N\times N}$ are continuous in z and t and $\sigma(z,t)$ is nonsingular for all z and t.

An individual's objects of choice are pairs of consumption rate process and final wealth, denoted generically by (c,W), where $c(t)$ denotes the consumption rate at time t. We require that a consumption rate process be adapted to $\mathbf{F}$ and a final wealth be a random variable defined on $(\Omega, \mathcal{F}, P)$. These are natural informational constraints. For tractability, we will now further impose some conditions on (c,W). Before proceeding, a definition is needed.

Note that a process can always be viewed as a mapping from $\Omega \times [0,T]$ to the real line, $\Re$. The smallest sigma–field of subsets of $\Omega \times [0,T]$ with respect to which all the processes adapted to $\mathbf{F}$ having right–continuous sample paths are measurable as mappings from $\Omega \times [0,T]$ to $\Re$ is termed the *optional–*sigma field, denoted by $\mathcal{O}$. It is known that any process measurable with respect to $\mathcal{O}$ is adapted to $\mathbf{F}$; see Chung and Williams (1983, p.56).

We will say that a consumption–final wealth pair (c,W) is *admissible* if

$$(c,W) \in L^p_+(\nu) \times L^p_+(P) \equiv L^p_+(\Omega \times [0,T], \mathcal{O}, \nu) \times L^p_+(\Omega, \mathcal{F}, P),$$

where ν is the product measure generated by P and Lebesgue measure, where $1 < p < \infty$, and where $L^p_+(\nu)$ and $L^p_+(P)$ are positive orthants of $L^p(\nu) \equiv L^p(\Omega \times [0,T], \mathcal{O}, \nu)$ and $L^p(P) \equiv L^p(\Omega, \mathcal{F}, P)$, respectively. In the terminology of general equilibrium theory, we have taken the commodity space of the economy to be $L^p(\nu) \times L^p(P)$ and the consumption set of an individual to be $L^p_+(\nu) \times L^p_+(P)$. Note that our choice of the commodity space makes easy the representation of a consumption rate process (adapted to $\mathbf{F}$) – it is simply a measurable function defined on the measurable space $(\Omega \times [0,T], \mathcal{O})$. The reader will find out later that this representation allows us to solve the consumption–portfolio problem in a manner different from and much simpler than the stochastic control theory.

Remark 2.1 *Our results do not hold for $p = 1$, since Lemma 2.4 fails and thus Proposition 2.8 does not hold for this case.*

We will use the following notation: If g is a matrix, $|g|^2$ denotes $\mathrm{tr}(gg^\top)$ and $|g|$ denotes $\sqrt{\mathrm{tr}(gg^\top)}$, where "tr" denotes trace.

Henceforth, all the processes will be adapted to $\mathbf{F}$ unless otherwise specified.

A trading strategy is an $(N+1)$–vector of processes, denoted generically by

$$\{\alpha(t), \theta(t) \equiv (\theta_1(t), \ldots, \theta_N(t))^\top; t \in [0,T]\},$$

where $\alpha(t)$ and $\theta_n(t)$ are the number of shares of the 0–th and the n–th security held at time t, respectively. We will specify the set of admissible trading strategies more fully later. For now, an admissible trading strategy must satisfy the following conditions:

1. $$\int_0^T |\alpha(t)B(t)r(t) + \theta(t)^\top \varsigma(t)|dt < \infty \quad P-a.s., \tag{2}$$

2. $$\int_0^T |\theta(t)^\top \sigma(t)|^2 dt < \infty \quad P-a.s., \tag{3}$$

3. there exists consumption–final wealth pair $(c, W) \in L^p_+(\nu) \times L^p_+(P)$ such that, P–a.s.,

$$\begin{aligned} &\alpha(t)B(t) + \theta(t)^\top S(t) + \textstyle\int_0^t c(s)ds \\ &= \alpha(0)B(0) + \theta(0)^\top S(0) + \textstyle\int_0^t (\alpha(s)B(s)r(s) + \theta(s)^\top \varsigma(s))ds + \int_0^t \theta(s)^\top \sigma(s)dw(s) \quad \forall t \in [0,T], \end{aligned} \tag{4}$$

4. and

$$\alpha(T)B(T) + \theta(T)^\top S(T) = W \quad a.s.. \tag{5}$$

Relations (2) and (3) ensure that the integrals of (4) are well–defined; see Liptser and Shiryayev (1977, Chapter 4). The left–hand side of (4) is the value of the portfolio at time t plus the accumulated withdrawals for consumption from time 0 to time t; while the right–hand side is equal to the initial value of the portfolio plus accumulated capital gains (losses) and dividends from time 0 to time t. That the left–hand side is equal to the right–hand side is a natural budget constraint. Relation (5) simply says that the final wealth is equal to the final value of the portfolio. Note that since $W \in L^p_+(P)$ and thus $W \geq 0$ P–a.s., (5) also ensures that borrowing to consume without paying back is not admissible. The consumption–final wealth pair (c, W) of (4) and (5) will be said to be *financed* by the trading strategy (α, θ).

Remark 2.2 *Trading strategies satisfying (2)–(3) include all the simple trading strategies, those that change portfolio compositions only at finitely many predetermined nonstochastic time points. Simple trading strategies satisfying (4) and (5) are among the strategies that can actually be implemented in real world. Their inclusion in the set of admissible strategies is a necessity for our model to be reasonable.*

Now we will turn our attention momentarily to the price processes before completing our specification of the set of admissible trading strategies. Thus far, we have not put any restriction on the price processes other than certain continuity and nonsingularity conditions on their parameters. For our consumption–portfolio problem to be well–posed, we certainly do not want the price processes to allow something to be created from nothing, when reasonable strategies are employed. Formally, a *free lunch* is a consumption–final wealth pair $(c, W) \in L^p_+(\nu) \times L^p_+(P)$ financed by an admissible trading strategy (α, θ) such that $\alpha(0)B(0) + \theta(0)^\top S(0) = 0$ and either $c > 0$ with a strictly positive ν–measure or $W > 0$ with a strictly positive P–measure. In other words, a free lunch is a consumption–final wealth pair that is nonnegative and nonzero and is financed by an admissible trading strategy with zero initial cost. Harrison and Kreps (1979) and Huang (1985) have shown that for free lunches not to be available for *simple strategies* it suffices that S is related to martingales after a change of unit and a change of probability, or equivalently, there exists an *equivalent martingale measure*. An equivalent martingale measure Q is a probability measure on $(\Omega, \mathcal{F})$ equivalent to P so that the Radon–Nikodym derivative dQ/dP lies in $L^q(\Omega, \mathcal{F}, P)$ with $1/p + 1/q = 1$ and

$$G^*(t) \equiv S(t)/B(t) - S(0)/B(0) + \int_0^t f(s)/B(s)ds,$$

the accumulated capital gains plus accumulated dividends, in units of the bond, is a martingale under Q. The existence of an equivalent martingale measure can be ensured by some regularity conditions on the parameters of the price processes. This is the subject to which we now turn.

Remark 2.3 *Probability measure Q is said to be equivalent to P if they have the same measure zero sets. This definition is symmetric and thus we say P and Q are equivalent to each other. A necessary and sufficient condition for this is that the Radon–Nikodym derivative dQ/dP is strictly positive. If dQ/dP is merely positive, we say that Q is absolutely continuously with respect to P.*

Remark 2.4 *Harrison and Kreps (1979) and Kreps (1981) show that the existence of an equivalent martingale measure is not only a sufficient but also a necessary condition for free lunches not to be available for simple strategies in the limit. Interested readers should consult their work for details. We should also note that, in the setup of Harrison and Kreps (1979), securities do not pay dividends and an individual maximizes his preferences only for final wealth. Our model here is more general and uses results of Huang (1985).*

Itô's lemma implies that

$$G^*(t) = \int_0^t \frac{1}{B(s)} \left[\varsigma(S(s),s) - r(S(s),s)S(s)\right] ds + \int_0^t \frac{\sigma(S(s),s)}{B(s)} dw(s).$$

Now put

$$\kappa(S(t),t) \equiv -\sigma(S(t),t)^{-1}(\varsigma(S(t),t) - r(S(t),t)S(t))$$

and

$$\eta(t) \equiv \exp\left\{\int_0^t \kappa(S(s),s)^\top dw(s) - \frac{1}{2}\int_0^t |\kappa(S(s),s)|^2 ds\right\}. \tag{6}$$

Note that $\sigma(z,t)^{-1}$ is continuous in z and t, since $\sigma(z,t)$ is. Then we have

$$\int_0^T |\kappa(S(t),t)|^2 dt < \infty \qquad a.s., \tag{7}$$

by the continuity of $\varsigma(z,t)$ and $r(z,t)$. It follows that the integrals on the right–hand side of (6) are well–defined; see Liptser and Shiryayev (1977, Chapter 4).

We will assume throughout this paper that $E[\eta(T)] = 1$, where $E[\cdot]$ is the expectation under P. (A well-known sufficient condition for $E[\eta(T)] = 1$ is that

$$E\left[\exp\left\{\frac{1}{2}\int_0^T |\kappa(S(s),s)|^2 ds\right\}\right] < \infty; \tag{8}$$

see Liptser and Shiryayev (1977, Theorem 6.1).) One can verify that $\{\eta(t)\}$ is a martingale under P.

We will now use $\eta(T)$ to define a probability Q and show in Proposition 2.1 that Q is an equivalent martingale measure under certain conditions. Putting

$$Q(A) \equiv \int_A \eta(T)P(d\omega) \qquad \forall A \in \mathcal{F}, \tag{9}$$

one can easily check that Q is a probability measure on $(\Omega, \mathcal{F})$ absolutely continuous with respect to P since $E[\eta(T)] = 1$ and since $\eta(T) \geq 0$ $P-a.s.$

Proposition 2.1 *Suppose that $E[|\eta(T)|^q] < \infty$ with $1/p + 1/q = 1$ and*

$$E^*\left[\left(\int_0^T |\sigma(S(t),t)|^2 dt\right)\right] < \infty, \tag{10}$$

where $E^*[\cdot]$ is the expectation under Q. Then Q is the unique equivalent martingale measure. Moreover,

$$w^*(t) \equiv w(t) - \int_0^t \kappa(s)ds \quad t \in [0,T]$$

is a standard Brownian motion under Q and we can write

$$G^*(t) = \int_0^t \frac{\sigma(s)}{B(s)} dw^*(s) \quad t \in [0,T]\, P-a.s.$$

PROOF. See Appendix. ∎

We henceforth assume that the conditions of Proposition 2.1 are valid until further notice and use ν^* to denote the product measure generated by the unique martingale measure, Q, and Lebesgue measure. The meaning of $L^p(\nu^*)$ and $L^p_+(\nu^*)$ should be evident. In addition, since P and Q are equivalent and thus have the same probability zero sets, we shall use *a.s.* to denote *almost surely* with respect to both from now on.

The existence of an equivalent martingale measure ensures that there be no free lunches for simple strategies. However, free lunches financed by trading strategies satisfying (2)–(5) still exist. For example, a doubling strategy, named after the strategy of doubling one's bet each time one loses at a roulette, produces a free lunch, as was pointed out by Harrison and Kreps (1979). Therefore, either we allow only simple strategies or we impose conditions in addition to (2)–(5) to rule out free lunches for non–simple strategies. The former solution proves to be mathematically intractable since the set of consumption–final wealth pairs financed by simple strategies is not closed in $L^p(\nu) \times L^p(P)$ and optimization problems in this case are not well–posed. Moreover, the set of simple strategies is not "rich enough" in that it does not include strategies of practical interest such as those that produce call options on securities.

The latter solution can be implemented in two ways. Both are motivated by the observation that for a doubling strategy to be implementable, it is necessary that an individual can borrow without bound and that there is no limit on the number of shares of risky securities held over time. The first approach is to put a nonnegative wealth constraint on trading strategies. Such a constraint certainly rules out doubling strategies, since it limits the amount of borrowing that one can make. Harrison and Kreps (1979) conjectured that this constraint would also rule out all the free lunches. This conjecture was verified by Dybvig (1980) in the model of Black and Scholes (1973) and by Dybvig and Huang (1987) in a model like ours. The second approach is to put a constraint on the θ. Note that a bound on θ also constrains α through the budget constraint (5). It turns out that a bound across states of nature is too strong – θ can be allowed to grow unbounded on sets of small Q–probability. Formally, the appropriate constraint is the following integrability condition on θ:

$$E^*\left[\left(\int_0^T |\theta(t)^\top \sigma(t)/B(t)|^2 dt\right)^{\frac{p}{2}}\right] < \infty. \tag{11}$$

Duffie and Huang (1985) used this kind of integrability constraint in their general equilibrium model. These two approaches are, however, shown to be equivalent for individuals with strictly increasing preferences by Dybvig and Huang (1987). They showed that any trading strategy that satisfies (2), (3), (4), (5), and (11) must satisfy the nonnegative wealth constraint. The strategies satisfying (2)–(5) and the nonnegative wealth constraint but not (11) are suicidal strategies – strategies that essentially run a free lunch in reverse and throw money away. Any individual with strictly increasing preferences will never employ a suicidal strategy! Before proceeding, the following lemma shows that (11) is sufficient for (3).

Lemma 2.1 *Let θ satisfy (11). Then θ satisfies (3).*

PROOF. Let θ satisfy (11). Then it is necessary that

$$\int_0^T |\theta(t)\sigma(t)/B(t)|^2 dt < \infty \quad a.s.$$

Since $B(t)$ is a continuous process, a sample path is bounded on $[0,T]$ almost surely. Thus

$$\int_0^T |\theta(t)\sigma(t)|^2 dt < \infty \quad a.s.,$$

which is (3). ∎

Now we are ready to complete the specification of admissible trading strategies. A trading strategy (α,θ) is admissible if it satisfies (2), (4), (5) and (11). We will use $H(Q)$ to denote the space of admissible trading strategies, where Q signifies that the expectation of (11) is taken with respect to the unique equivalent martingale measure Q. One can verify that $H(Q)$ is a linear space.

Now consider an agent with a time-additive utility function for consumption, $u(c(t),t)$, a utility function for final wealth, $V(W)$, and an initial wealth $W_0 > 0$. He wants to solve the following problem:

$$\begin{aligned} \sup_{(c,W)\in L^p_+(\nu)\times L^p(P)} \quad & E\left[\int_0^T u(c(t),t)dt + V(W)\right] \\ \text{s.t. } (c,W) \quad & \text{is financed by some } (\alpha,\theta)\in H(Q) \\ \text{with} \quad & \alpha(0)B(0)+\theta(0)^\top S(0) = W_0. \end{aligned} \tag{12}$$

Our task is to provide a set of easily verifiable conditions to ensure that (12) has a solution in that the supremum is finite and is attained. Note that the consumption–portfolio problem of (12) has infinitely many budget constraints as specified in (4) and (5).

2.2 The correspondence between a dynamic problem and a static problem

In this subsection, we will show the connection between the dynamic problem of (12) with infinitely many budget constraints with a static problem with a single budget constraint. We begin by defining the set of consumption–final wealth pairs (c,W) financed by admissible trading strategies:

$$F \equiv \{(c,W)\in L^p_+(\nu)\times L^p_+(P) : (c,W) \text{ is financed by some } (\alpha,\theta)\in H(Q)\}.$$

Propositions 2.2 and 2.3 to follow will characterize properties of the set F. We first give a definition and record a technical lemma. A martingale $\{X(t); t \in [0,T]\}$ is a $L^p(Q)$–martingale if it is martingale under Q and

$$E[|X(t)|^p] < \infty \quad \forall t \in [0,T].$$

Lemma 2.2 *Let θ satisfy (11). Then*

$$\int_0^t \theta(s)^\top \sigma(s)/B(s) dw^*(s)$$

is a $L^p(Q)$–martingale.

PROOF. See Jacod (1979, Chapter IV). ∎

The following proposition shows that the cost over time of any $(c,W) \in F$ can be computed by taking a conditional expectation with respect to the martingale measure Q.

Proposition 2.2 *Let $(c,W) \in F$ be financed by (α,θ). The initial cost of (c,W), $\alpha(0)B(0)+\theta(0)^\top S(0)$, is*

$$\alpha(0)B(0) + \theta(0)^\top S(0) = B(0)E^* \left[\int_0^T c(t)/B(t)dt + W/B(T)\right]. \tag{13}$$

More generally, the cost of $\{c(s); s \in [t,T]\}$ and W at time t is

$$\alpha(t)B(t) + \theta(t)^\top S(t) = B(t)E^* \left[\int_t^T c(s)/B(s)ds + W/B(T)|\mathcal{F}_t\right].$$

PROOF. Let $(c,W) \in F$ be financed by $(\alpha,\theta) \in H(Q)$. Itô's lemma implies that

$$\begin{aligned} & \alpha(t) + \theta(t)^\top S(t)/B(t) + \textstyle\int_0^t c(s)/B(s)ds \\ = \quad & \alpha(0) + \theta(0)^\top S(0)/B(0) + \textstyle\int_0^t \theta(s)^\top \sigma(s)/B(s)dw^*(s) \quad \forall t \in [0,T] \ a.s., \end{aligned} \tag{14}$$

where we have used (4). Lemma 2.2 implies that the stochastic integral on the right–hand side of (14) is an $L^p(Q)$–martingale. Evaluating (14) at $t = T$, using (5) and (14) we have

$$\begin{aligned} & W/B(T) + \int_t^T c(s)/B(s)ds \\ = \quad & \alpha(t) + \theta(t)^\top S(t)/B(t) + \int_t^T \theta(s)^\top \sigma(s)/B(s)dw^*(s) \quad \forall t \in [0,T] \ a.s. \end{aligned}$$

Taking expectation conditional on $\mathcal{F}_t$ under Q of both sides of the above relation we get

$$E^* \left[\int_t^T c(s)/B(s)dt + W/B(T)|\mathcal{F}_t\right] = \alpha(t) + \theta(t)^\top S(t)/B(t).$$

Multiplying both sides of the above relation by $B(t)$ we have the second assertion. Evaluating the second assertion at $t = 0$, we get (13). ∎

Proposition 2.2 says that the initial cost of a consumption–final wealth pair financed by an admissible strategy is equal to $B(0)$ times the expectation, under the martingale measure Q, of the "sum" of discounted future consumption and final wealth. With the aid of the following lemma, (13) has a very intuitive interpretation in the context of an Arrow–Debreu economy.

Lemma 2.3 *Let g be an adapted process, then*

$$E^*\left[\int_0^T g(t)dt\right] = E\left[\int_0^T g(t)\eta(t)dt\right],$$

whenever the integrals are well-defined. Thus for any $(c, W) \in L^p(\nu) \times L^p(P)$ we can write

$$B(0)E^*\left[\int_0^T c(t)/B(t)dt + W/B(T)\right] = B(0)E\left[\int_0^T c(t)\eta(t)/B(t)dt + W\eta(T)/B(T)\right] \le W_0. \quad (15)$$

PROOF. The first assertion follows from Dellacherie and Meyer (1982, VI.57). Using the first assertion and the definition of Q we have the second assertion, since $c(t)$ and $B(t)$ are adapted processes. ∎

From (15), we can interpret $B(0)\eta(\omega, t)P(d\omega)/B(\omega, t)$ to be the time 0 Arrow–Debreu price of a security that pays one unit of consumption in state ω at time t and nothing otherwise. In other words, there exists an implicit system of Arrow–Debreu prices in the dynamic economy that value any $(c, W) \in F$ by (15). Note that since the equivalent martingale measure is unique and, by (9), is defined by $\eta(T)$, the system of Arrow–Debreu prices is unique (recall that $\{\eta(t)\}$ is a martingale under P and thus $\eta(t) = E[\eta(T)|\mathcal{F}_t]$ *a.s.*).

The next proposition shows that F contains the intersection of $L^p_+(\nu) \times L^p_+(P)$ and $L^p_+(\nu^*) \times L^p_+(Q)$ when $\kappa(t)$ satisfies an integrability condition. The following technical lemma is instrumental for the proof of Proposition 2.3.

Lemma 2.4 *Suppose that*

$$E^*\left[\int_0^T |\kappa(t)|^2 dt\right] < \infty. \quad (16)$$

Then for any $x \in L^p(Q)$, there exists an N-dimensional process $\{\phi(t); t \in [0, T]\}$ satisfying

$$E^*\left[\left(\int_0^T |\phi(t)|^2 dt\right)^{\frac{p}{2}}\right] < \infty \quad (17)$$

such that

$$E^*[x|\mathcal{F}_t] = E^*[x] + \int_0^t \phi(s)^\top dw^*(s) \quad \forall t \in [0, T]\, a.s.$$

PROOF. This is a consequence of Jacod (1976, Chapter IV) and Fujisaki, Kallianpur, and Kunita (1972, Theorem 3.1). ∎

Lemma 2.4 is a *martingale representation theorem*. Note that for any $x \in L^p(Q)$, $\{E^*[x|\mathcal{F}_t]; t \in [0,T]\}$ defines a $L^p(Q)$–martingale. Conversely, any $L^p(Q)$ martingale $\{x(t); t \in [0,T]\}$ can be represented by the random variable $x(T) \in L^p(Q)$ in that the whole process $\{x(t); t \in [0,T]\}$ can be "recovered" by taking conditional expectation of $x(T)$ with respect to Q. Lemma 2.4 says that any $L^p(Q)$ martingale can be represented by an Itô integral with respect to the N–dimensional Brownian motion w^* under Q defined in Proposition 2.1, if (16) is satisfied.

Proposition 2.3 *Suppose that (16) holds. Then* $L^p_+(\nu) \times L^p_+(P) \bigcap L^p_+(\nu^*) \times L^p_+(Q) \subset F$.

PROOF. Let $(c, W) \in L^p_+(\nu) \times L^p_+(P) \bigcap L^p_+(\nu^*) \times L^p_+(Q)$. First note that, by Jensen's inequality and by the fact that $B(t)$ is bounded below away from zero,

$$\int_0^T c(t)/B(t)dt \in L^p_+(Q) \quad \text{and} \quad W(T)/B(T) \in L^p_+(Q).$$

Hence

$$\int_0^T c(t)/B(t)dt + W/B(T) \in L^p(Q).$$

By Lemma 2.4, there exists an N–dimensional process ϕ satisfying (17) such that

$$\begin{aligned} & E^*\left[\textstyle\int_0^T c(t)/B(t)dt + W/B(T)|\mathcal{F}_t\right] \\ = \quad & E^*\left[\textstyle\int_0^T c(t)/B(t)dt + W/B(T)\right] + \textstyle\int_0^t \phi(s)^\top dw^*(s) \quad \forall t \in [0,T]\ a.s. \end{aligned} \tag{18}$$

Now define

$$\begin{aligned} \theta(t) \quad & \equiv \phi(t)\sigma(t)^{-1}B(t), \\ \alpha(t) \quad & \equiv E^*\left[\textstyle\int_t^T c(s)/B(s)ds + W/B(T)|\mathcal{F}_t\right] - \theta(t)^\top S(t)/B(t). \end{aligned} \tag{19}$$

We claim that (α, θ) of (19) lies in $H(Q)$ and finances (c, W).

We note that $(\alpha(t), \theta(t))$ of (19) are finite a.s. and θ satisfies (11). Moreover, evaluating (19) at $t = T$ gives (5). We need to verify (2) and (4). Relations (18) and (19) and Proposition 2.2 imply that

$$\begin{aligned} & \alpha(t) + \theta(t)^\top S(t)/B(t) + \textstyle\int_0^t c(s)/B(s)ds \\ = \quad & \alpha(0) + \theta(0)^\top S(0)/B(0) + \textstyle\int_0^t \theta(s)^\top \sigma(s)/B(s)dw^*(s). \end{aligned} \tag{20}$$

Using Itô's lemma and (20) we get

$$\begin{aligned} & \alpha(t)B(t) + \theta(t)S(t) + \int_0^t c(s)ds \\ = \quad & \alpha(0)B(0) + \theta(0)^\top S(0) + \int_0^t \theta(s)^\top \sigma(s)dw(s) + \int_0^t \left(\alpha(s)B(s)r(s) + \theta(s)^\top \varsigma(s)\right) ds. \end{aligned}$$

The integral on the right-hand-side is well-defined since the left-hand-side is. This is just (4). Moreover, since the second integral on the right–hand side is well–defined, we have (2). Hence any element of $L^p(\nu) \times L^p_+(P) \bigcap L^p_+(\nu^*) \times L^p_+(Q)$ is in F, and the last assertion follows. ∎

Now we are ready to show the correspondence between the dynamic problem of (12) with infinitely many budget constraints and a static problem with a single budget constraint. Consider the following static variational problem:

$$\begin{aligned} \sup_{(c,W)\in L^p_+(\nu)\times L^p_+(P)} \quad & E\left[\int_0^T u(c(t),t)dt + V(W)\right] \\ \text{s.t. } B(0) \quad & E\left[\int_0^T c(t)\eta(t)/B(t)dt + W\eta(T)/B(T)\right] = W_0. \end{aligned} \tag{21}$$

Basically, in this static problem, we take the Arrow–Debreu prices implicit in the second assertion of Lemma 2.3 to formulate a single budget constraint. The interpretation of the single budget constraint of (21) is that for a price $B(0)\eta(\omega,t)P(d\omega)/B(t)$, an individual can purchase at time 0 an Arrow–Debreu security that pays one unit of consumption at time t in state ω. We will show that if there exists a solution to (21) and it lies in F, then it must also be a solution to the dynamic problem of (12). The argument goes as follows: Suppose this is not the case. Then there must be $(c,W)\in F$ which the agent strictly prefers to the solution to (21). But (c,W) is budget feasible in (21) by Lemma 2.3 and Proposition 2.2, which is a contradiction. The following proposition formalizes this idea:

Proposition 2.4 *Suppose that (c,W) is a solution to the static problem of (21) and lies in F. Then it is a solution to the dynamic problem. Conversely, if every solution to (21) lies in F, then if there exists a solution to the dynamic problem, it is also a solution to (21).*

PROOF. Suppose that there exists $(\hat{c},\hat{W})\in F$ financed by (α,θ) with $\alpha(0)B(0)+\theta(t)^\top S(0) = W_0$ such that

$$E\left[\int_0^T u(\hat{c}(t),t))dt + V(\hat{W})\right] > E\left[\int_0^T u(c(t),t))dt + V(W)\right].$$

Proposition 2.2 and Lemma 2.3 imply that

$$B(0)E\left[\int_0^T \hat{c}(t)\eta(t)/B(t)dt + \hat{W}\eta(T)/B(T)\right] = \alpha(0)B(0)+\theta(0)^\top S(0) \le W_0.$$

Thus $(\hat{c},\hat{W})$ is budget feasible in the static problem of (21). This contradicts the hypothesis that (c,W) is a solution to (21).

The second assertion is obvious. ∎

Given Proposition 2.4, if we can show that there exists a solution to (21) and that it lies in F, then there exists a solution to the dynamic problem of (12). Recall from Proposition 2.3 that

$L_+^p(\nu) \times L_+^p(P) \bigcap L_+^p(\nu^*) \times L_+^p(Q) \subset F$, provided that conditions of Proposition 2.1 and (16) are satisfied. Thus it suffices to look for conditions under which there exists a solution to (21) that lies in $L_+^p(\nu^*) \times L_+^p(Q)$ and the conditions of Proposition 2.1 and (16) are satisfied. This is the subject to which we now turn in the following two sections.

3 A Static Variational Problem

In this section we will study a class of static variational problems of the kind described in (21). For expositional purpose, we will first analyze a problem in detail without the time dimension. Later, we will generalize the results in this simpler case to the case with the time dimension. In both cases, the sufficient condition for existence involves whether the "inverse" of the implicit Arrow–Debreu price system has a certain finite moment. We will see in Section 4 that the implicit Arrow–Debreu price system in the context of Section 2 is a solution to a stochastic differential equation. The moments of a solution to a stochastic differential equation turn out to be easily estimated and thus we are able to provide easily verifiable sufficient conditions for existence.

3.1 The problem

Fix throughout this section a probability space $(\Omega, \mathcal{F}, P)$. We are interested in finding the solution to the following problem:

$$\begin{aligned} &\sup_{x \in L_+^p(P)} \quad \int_\Omega V(x(\omega))P(d\omega) \\ &\text{s.t. } \phi(x) \equiv \quad \int_\Omega x(\omega)\xi(\omega)P(d\omega) = K_0, \end{aligned} \qquad (A_1)$$

where $L^p(P)$ denotes $L^p(\Omega, \mathcal{F}, P)$ with $1 \le p < \infty$, $\xi \in L_+^q(P)$ with $1/p + 1/q = 1$ and $\xi > 0$ P–a.e., and K_0 is a strictly positive constant. As usual, we shall say that there exists a solution to (A_1) if the supremum is attained by some $x \in L_+^p(P)$. To begin, we will consider the case where $V : \Re_+ \mapsto \Re_+$ is a continuous, concave, and increasing function. Note that by concavity,

$$\int_\Omega V(x(\omega))P(d\omega) < \infty \quad \forall x \in L^p(P).$$

Thus if the supremum is attained it must be finite. Therefore, the problem of the existence of a solution is equivalent to whether the supremum is attained. As will be shown in the following two examples, however, when $L_+^p(P)$ is infinite dimensional, (A_1) may not have a solution. Consider the following two examples.

Take Ω to be $[0,1]$, F to be the Borel σ-field of $[0,1]$, and P to be Lebesgue measure on $[0,1]$. Suppose that $u(z) = z^{\frac{2}{3}}$, $p = 1$, and

$$\phi(x) = \int_0^1 2\omega x(\omega)d\omega \quad \forall x \in L^1(\Omega, \mathcal{F}, P).$$

Consider the sequence

$$x_n(\omega) = K_0 n^2 1_{[0,\frac{1}{n}]}(\omega) \quad n = 1, 2, \ldots$$

We first show that $x_n \in L^1(P)\ \forall n$:

$$\int_0^1 \mid x_n(\omega) \mid d\omega = K_0 n < \infty.$$

Next we note that

$$\begin{aligned} \phi(x_n) &= 2K_0 \int_0^1 \omega n^2 1_{[0,\frac{1}{n}]}(\omega) d\omega \\ &= 2n^2 K_0 \int_0^{\frac{1}{n}} \omega d\omega \\ &= K_0 \quad \forall n. \end{aligned}$$

Thus every $x_n \in L^1(P)$ is budget-feasible. However,

$$\begin{aligned} \int_0^1 u(x_n(\omega)) d\omega &= K_0^{\frac{2}{3}} n^{\frac{1}{3}} \\ &\to \infty \quad \text{as } n \to \infty. \end{aligned}$$

Note that in the above example, the utility function is strictly concave, increasing, and continuous. It has a zero derivative at infinity and an infinite right–hand derivative at zero – all the nice properties that you would like a utility function to have. The problem arises because the prices for commodities, as captured by 2ω, are not bounded below away from zero, and the utility function grows too fast asymptotically. The commodities close to $\omega = 0$ are worth almost nothing. The agent would like to put all his money in the commodity indexed by $\omega = 0$, but his expected utility will be zero since the event $\{\omega = 0\}$ is of zero measure. Thus he tries to purchase commodities as close to $\omega = 0$ as possible. He achieves this by going along the sequence $(K_0 n^2 1_{[0,\frac{1}{n}]}(\omega))_{n=1}^{\infty}$. However, the slope of the utility function does not approach zero fast enough, thus the expected utility explodes. Recall that the expected utility of any $x \in L^1_+(P)$ is finite by concavity. Thus the supremum of (A_1) is not attained.

The following example, which is adapted from Aumann and Perles (1965), shows that the supremum may not be attained even if it is finite. Take Ω to be $[0, 1]$, $\mathcal{F}$ to be the Borel sigma–field of $[0, 1]$, and

$$P(A) = \int_A 2\omega d\omega, \quad \forall A \in \mathcal{F}.$$

In addition, let $V(z) = z$ and

$$\phi(x) = \int_0^1 x(\omega) \frac{1}{2\omega} P(d\omega) = \int_0^1 x(\omega) d\omega.$$

In this case, it is easily verified that $\int_0^1 V(x(\omega))P(d\omega) < 2K_0$ but its supremum over all budget feasible $x \in L^1_+(P)$ is equal to $2K_0$, which is not attained. In this case, the prices as captured by $1/2\omega$ are bounded from below; but the utility function is linear and grows too fast to infinity. Hence the agent chooses to concentrate his wealth buying inexpensive commodities close to $\omega = 1$.

Aumann and Perles (1965) studied a class of problems very similar to (A_1). Briefly, they considered conditions under which there exists a solution to the following program:

$$\begin{array}{ll} \sup_{x \in L^1_+([1,0],\mathcal{B}[0,1],\lambda)} & \int_0^1 f(x(\omega),\omega)d\omega \\ \text{s.t.} & \int_0^1 x(\omega)d\omega = K_0, \end{array} \tag{A_2}$$

where $f(z,\omega)$ is increasing, concave, and continuous in z, for P–almost every ω. The program (A_2) differs from (A_1) in that f is state dependent and the prices of commodities are unity. We will show in what follows that after a change of unit, the same conditions that ensure a solution to (A_2) guarantee a solution to (A_1). This change of unit can be carried out by a change of measure. Lemma 3.1 and Theorems 3.1 and 3.2 make the connection between (A_2) and (A_1).

We first give a definition which is just a generalization of Aumann and Perles'.

Definition 3.1 *Let $f : \Re_+ \times \Omega \mapsto \Re_+$ be measurable with respect to the product σ-field $\mathcal{B}(\Re_+) \times \mathcal{F}$, where $\mathcal{B}(\Re_+)$ denotes the Borel sigma–field of $\Re_+$. Then $f(z,\omega) = o(z)$ as $z \to \infty$, $L^p(P)$–integrably in ω, if for each $\epsilon > 0$ there exists $y \in L^p_+(P)$ such that $f(z,\omega) \le \epsilon z$ for P–almost every $\omega \in \Omega$ whenever $z \ge y(\omega)$.*

We will need the following lemma:

Lemma 3.1 *Suppose that $f : \Re_+ \times \Omega \mapsto \Re_+$ is such that $f(z,\omega) = o(z)$ as $z \to \infty$, $L^p(P)$-integrably in ω. Suppose futher that P^* is a measure on $(\Omega,\mathcal{F})$ absolutely continuous with respect to P such that the Radon-Nikodym derivative of P^* with respect to P belongs to $L^q(P)$, where $1/q + 1/p = 1$. Then $f(z,\omega) = o(z)$ as $z \to \infty$, $L^1(P^*)$–integrably in ω.*

PROOF. Let $y \in L^p_+(P)$ be such that $f(z,\omega) \le \epsilon z$ for P–almost every $\omega \in \Omega$ whenever $z \ge y(\omega)$. Once we show that $y \in L^1(P^*)$ we are done. Note that

$$\int_\Omega y(\omega)P^*(d\omega) = \int_\Omega y(\omega)\frac{dP^*}{dP}(\omega)P(d\omega) < \infty,$$

where dP^*/dP is the Radon–Nikodym derivative of P^* with respect to P and lies in $L^q_+(P)$ by assumption, and where the inequality follows from the Hölder's inequality (cf. Royden (1968), p. 113). This was to be shown. ∎

Now define a finite measure P^* on $(\Omega,\mathcal{F})$ by

$$P^*(A) = \int_A \xi(\omega)P(\omega) \quad \forall A \in \mathcal{F},$$

where we recall that ξ is the "price" in (A_1). It follows from the Radon–Nikodym theorem that P^* is equivalent to P, since by the hypothesis $\xi > 0$ P–a.s. For brevity of notation, throughout the rest of this section, we will simply use *a.s.* to denote *almost surely* under P and *a.e.* to denote *almost everywhere* under P^*. This distinction is needed since P^* may not be a probability measure.

Here is our first main result, which is a straightforward generalization of the theorem of Aumann and Perles (1965):

Theorem 3.1 *Suppose that $V : \Re_+ \rightarrow \Re_+$ is continuous, concave, and increasing. Suppose further that $V(z)\xi(\omega)^{-1} = o(z)$ as $z \rightarrow \infty$, $L^p(P)$–integrably in ω. Then there exists a solution to the following program:*

$$\begin{array}{ll} \sup_{x \in L^1_+(P^*)} & \int_\Omega V(x(\omega))P(d\omega) \\ s.t. & \int_\Omega x(\omega)\xi(\omega)P(d\omega) = K_0. \end{array} \tag{A_1^*}$$

PROOF. The above program can be rewritten as follows:

$$\begin{array}{ll} \sup\limits_{x \in L^1_+(P^*)} & \displaystyle\int_\Omega V(x(\omega))\xi(\omega)^{-1}P^*(d\omega) \\ \text{s.t.} & \displaystyle\int_\Omega x(\omega)P^*(d\omega) = K_0. \end{array}$$

Then the assertion follows from generalizations of Lemma 2.1 and Proposition 2.2 of Aumann and Perles (1965). ∎

Note that the consumption set in (A_1^*) is $L^1_+(P^*)$. Therefore, a solution to (A_1^*) may not be a solution to (A_1), since $L^p_+(P) \subset L^1_+(P^*)$ and the inclusion may be strict. It turns out that under the same set of conditions of the above theorem, every solution of (A_1^*) is a solution of (A_1), as long as $V(z)$ is strictly increasing.

Theorem 3.2 *Under the same set of conditions of Theorem 3.1 and the assumption that $V(z)$ is strictly increasing, there exists a solution to (A_1). Indeed, in such event, every solution to (A_1^*) is a solution to (A_1).*

PROOF. Let $x^* \in L^1_+(P^*)$ be a solution to (A_1^*). Since the Slater's condition (cf. Holmes (1975)) is obviously satisfied, it follows from the Saddle-Point Theorem and Rockafellar (1975) that there exists a strictly positive constant λ such that for all $x \in L^1_+(P^*)$:

$$u(x^*(\omega))\xi(\omega)^{-1} - u(x(\omega))\xi(\omega)^{-1} \geq \lambda\,(x^*(\omega) - x(\omega)) \quad a.e.$$

Without loss of generality we can assume that $V(0) = 0$. Now take $x(\omega) = 0$ $\forall\omega \in \Omega$ in the above relation. We get $u(x^*(\omega))\xi(\omega)^{-1} \geq \lambda x^*(\omega)$ *a.e.* Since $u(z)\xi(\omega)^{-1} = o(z)$, $L^p(P)$–integrably in ω, and since P and P^* are equivalent, there must exists an $y \in L^p_+(P)$ such that

$$x^*(\omega) \leq y(\omega) \quad a.s.$$

This implies that $x^* \in L^p_+(P)$, which was to be proved. ∎

Note that the class of utility functions considered above maps $\Re_+$ into $\Re_+$. Since solutions to (A_1) are invariant under strictly positive affine transformations of the utility function, Theorems 3.1 and 3.2 can be applied to utility functions that are not necessarily nonnegative but are bounded from below. We simply transform the utility function to be nonnegative.

3.2 Moment conditions for a class of utility functions

The integrability condition of Theorem 3.1 is a joint condition on the utility function and the prices of commodities and is hard to verify in general in applications. For a class of utility functions, however, the integrability condition reduces to whether a certain moment of the inverse of the price system is finite. This condition amounts to restricting how fast the prices for commodities asymptote to zero.

Before we proceed, we first consider the class of constant relative risk aversion utility functions:

$$\begin{aligned} V(z) &= \frac{z^{1-b}-1}{1-b} \qquad z>0, \quad b>0; \\ &= 0 \quad \text{if } z=0 \text{ and if } b<1; \\ &= \quad \text{not defined if } z=0 \text{ and } b \geq 1; \end{aligned}$$

where $V(z) = \ln z$ when $b=1$ by the l'Hopital's rule. Note that when $b \geq 1$, $V(z)$ is unbounded from below, and thus is not covered by Theorem 3.1 or Theorem 3.2. For this class of utility functions, we have a simple necessary and sufficient condition for the existence of a solution to (A_1).

Theorem 3.3 *Suppose that $V(z)$ is of constant relative risk aversion with a coefficient of relative risk aversion $b>0$. There exists a solution to (A_1) if and only if $\xi^{-1} \in L^{\frac{p}{b}}(P)$.*

PROOF. We first prove necessity. Suppose that $x \in L^p_+(P)$ is a solution to (A_1). Since $V(z)$ is strictly increasing and since $\lim_{z\to 0} V'(z) = \infty$, there must exist a strictly positive constant λ such that

$$x(\omega)^{-b} = \lambda\xi(\omega) \quad a.s.;$$

cf. Rockafellar (1975). Equivalently,

$$x(\omega) = \lambda^{\frac{-1}{b}}\xi(\omega)^{\frac{-1}{b}} \quad a.s..$$

Thus $x \in L^p(P)$ *only if* $\xi^{-1} \in L^{\frac{p}{b}}(P)$.

Next suppose that $\xi^{-1} \in L^{\frac{p}{b}}(P)$. Let λ be any strictly positive real number. Define

$$x_\lambda(\omega) = \lambda^{\frac{-1}{b}} \xi(\omega)^{\frac{-1}{b}}.$$

It follows from the hypothesis that $x_\lambda \in L^p(P)$. Now define

$$B(\lambda) = E(x_\lambda \xi) = \lambda^{\frac{-1}{b}} \int_\Omega \xi(\omega)^{1-\frac{1}{b}} < \infty,$$

where the inequality follows from the fact that $\xi \in L^q(P)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and the Hölder's inequality. It is clear that there exists a unique $\lambda > 0$ such that $B(\lambda) = K_0$. Then we have

$$x_\lambda^{-b}(\omega) = \lambda \xi(\omega) \qquad a.s.$$

and

$$\int_\Omega x_\lambda(\omega)\xi(\omega)P(d\omega) = K_0.$$

The above two conditions constitute sufficient conditions for x_λ to be a solution to (A_1). ∎

Thus for constant relative risk aversion utility functions the condition for the existence of a solution to (A_1) reduces to whether a certain moment of ξ^{-1} is finite. The higher the relative risk aversion, the less the requirement of ξ^{-1}, or, equivalently, the faster the prices are allowed to asymptote to zero. Intuition suggests that this kind of moment condition should at least extend to the class of utility functions that hehave like power functions asymptotically, since it is what happens at "infinity" that causes problem for existence. The reader will soon find out that we can do better than the class of utility functions that behave like power functions asymptotically.

The following theorem gives a sufficient condition for existence.

Theorem 3.4 *Suppose that $V : \Re_+ \mapsto \Re_+$ is continuous, concave, and strictly increasing. Suppose further that there exists $b \in (0,1)$, $B > 0$, $z^* > 0$ such that*

$$V(z) \le Bz^{1-b} \qquad \forall z \ge z^*.$$

For there to exist a solution to (A_1), it is sufficient that $\xi^{-1} \in L^{\frac{p}{b}}(P)$. Moreover, if $\xi^{-1} \in L^{\frac{p'}{b}}(P)$ for some $p' > p$, then every solution to (A_1) is an element of $L^{p'}(P)$.

PROOF. Let $\xi^{-1} \in L^{\frac{p}{b}}(P)$. By the hypothesis, we know there exist $B > 0$ and $z^* > 0$ such that

$$V(z) \le g(z) \equiv Bz^{1-b} \qquad \forall z \ge z^*.$$

We claim that given $\epsilon > 0$, there exists $y \in L^p_+(P)$ such that

$$g(z) \le \epsilon z \xi(\omega) \quad \forall z \ge y(\omega),$$

for almost every $\omega \in \Omega$. Note that $g(z)/z$ is of the same order of magnitude of z^{-b} as $z \to \infty$ and is decreasing. Thus there exists constants $K > 0$ and $\hat{z} > 0$ such that

$$\frac{g(z)}{z} \leq K z^{-b} \quad \forall z \geq \hat{z}.$$

Now putting $z^{\circ} \equiv \max\{z^*, \hat{z}\}$ and

$$\Lambda = \{\omega \in \Omega : \frac{g(z^{\circ})}{z^{\circ}} \leq \epsilon\xi(\omega)\},$$

we define

$$y(\omega) = \begin{cases} z^{\circ} & \text{if } \omega \in \Lambda; \\ \frac{\epsilon}{K}^{\frac{-1}{b}} \xi(\omega)^{\frac{-1}{b}} & \text{if } \omega \notin \Lambda. \end{cases}$$

It is clear that $y \geq 0$ *a.s.* and $y \in L^p(P)$ since $\xi^{-1} \in L^{\frac{p}{b}}(P)$. Note also that $y \geq z^{\circ}$ *a.s.* Next note the following. On Λ, since $g(z)/z$ is decreasing,

$$\frac{g(z)}{z} \leq \frac{g(z^{\circ})}{z^{\circ}} \leq \epsilon\xi(\omega) \qquad \forall z \geq y(\omega).$$

On $\Omega \setminus \Lambda$, we know $z \geq y(\omega)$ implies that

$$\frac{g(z)}{z} \leq K z^{-b} \leq K y(\omega)^{-b} = \epsilon\xi(\omega).$$

Finally, since $V(z) \leq g(z)$ for all $z \geq z^{\circ}$ and since $y(\omega) \geq z^{\circ}$ *a.s.* by construction, we know

$$V(z) \leq \epsilon z \xi(\omega) \quad \forall z \geq y(\omega).$$

Hence $V(z)\xi(\omega)^{-1} = o(z)$ $L^p(P)$–integrably in ω and there exists a solution to (A_1) by Theorem 3.2.

The proof for the second assertion is similar to that for Theorem 3.2. ∎

Remark 3.1 *A sufficient condition for there to exist $b \in (0,1)$, $z^* > 0$, and $B > 0$ such that $V(z) \leq Bz^{1-b}$ $\forall z \geq z^*$ is that V is asymptotically twice differentiable with* $\liminf_{z\to\infty} \frac{-u''(z)z}{u'(z)} = b \in (0,1]$. *That is, the Arrow-Pratt measure of relative risk aversion exists asymptotically and has a limit infimum* $b \in (0,1]$.

Remark 3.2 *Note that bounded from below utility functions whose coefficients of relative risk aversion have limit infima strictly greater than one are bounded from above and thus will be covered by Theorem 3.5. Also, utility functions that are concave and not strictly increasing must be bounded from above and will also be covered by Theorem 3.5.*

The proof for the above theorem reveals the following.

Theorem 3.5 *Suppose that $V : \Re_+ \mapsto \Re_+$ is continuous, concave, strictly increasing, and is bounded from above. If $\xi^{-1} \in L^p(P)$, there exists a solution to (A_1). In addition, if $\xi^{-1} \in L^{p'}(P)$ for some $p' > p$, any solution to (A_1) is an element of $L^{p'}(P)$.*

PROOF. We note that if $V(z)$ is bounded from above, then $V(z)/z$ is of the same order of magnitude as $1/z$ as $z \to \infty$. Along the same line of arguments as in the previous theorem, the assertion can be proved easily. ∎

The above two theorems include, for example, the class of HARA utility functions that are strictly increasing, concave, and bounded from below.

3.3 Existence of a solution for a class of utility functions unbounded from below

Except for the class of constant relative risk aversion utility functions, the previous two subsections give sufficient conditions for the existence of a solution to (A_1) when utility functions are continuous, increasing, concave, and bounded from below. When utility functions are strictly concave, a set of sufficient conditions can be given that covers utility functions unbounded from below.

Throughout this section we consider utility function V that may be unbounded from below at zero in that $V : \Re_+ \setminus \{0\} \mapsto \Re$ with $\lim_{z \to 0} V(z) = -\infty$. We assume that $V(\cdot)$ is nontrivial, increasing, concave, and strictly concave on any subset of its domain where $V(\cdot)$ is strictly increasing. In the case that $V(\cdot)$ is strictly increasing, we further suppose that there exists $b > 0$, $z^* > 0$, and $K > 0$ such that

$$V'_+(z) \leq K z^{-b} \qquad \forall z \geq z^*,$$

where $V'_+(\cdot)$ denotes the right–hand derivative of V. (Here we remark that the right–hand derivative of a concave function is always well-defined.) Note that the purpose of this last assumption is to make sure that the "inverse" of $V'_+(\cdot)$, appropriately defined, is bounded from above asymptotically by a power function with a negative exponent. Thus when the inverse of the Arrow-Debreu price ξ has a certain finite moment, the supremum in (A_1) is finite. This, however, does not guarantee that the supremum is attained and therefore the assumption about the strict cocavity of V is needed.

Remark 3.3 *A sufficient condition for there to exist $z^* > 0$, $K > 0$, and $b > 0$ such that $V'_+(z) \leq K z^{-b}$ is that $V(\cdot)$ is asymptotically twice differentiable with*

$$\liminf_{z \to \infty} -\frac{zV''(z)}{V'(z)} = b > 0.$$

Note that since V is continuous, concave and nontrivial, if it is not strictly increasing, there must exists $z^\circ > 0$ such that $V'_+(z) = 0 \; \forall z \geq z^\circ$. We shall put

$$z^\circ = \inf\{z \in \Re_+ : V'_+(z) = 0\}. \tag{22}$$

It is easily seen that the infimum is actually attained.

Now define the inverse of V'_+, $g : \Re_+ \mapsto \Re_+$, by

$$g(y) = \inf\{z \in \Re_+ : V'_+(z) \leq y\},$$

where we recall the convention that if the infimum does not exist, we assign $+\infty$ to g.

Proposition 3.1 *g is decreasing, continuous, and $\lim_{y\to\infty} g(y) = 0$. When $V(\cdot)$ is strictly increasing, we have $\lim_{y\to 0} g(y) = \infty$. When $V(\cdot)$ is not strictly increasing, we have $\lim_{y\to 0} g(y) = z^\circ$.*

PROOF. The fact that g is continuous and decreasing follows from the hypothesis that $V(\cdot)$ is strictly concave when $V(\cdot)$ is strictly increasing. The assertion that $g(y) \to 0$ as $y \to \infty$ follows from concavity.

Now we take cases. Suppose $V(\cdot)$ is strictly increasing. By the hypothesis that

$$V'_+(z) \leq Kz^{-b} \qquad \forall z \geq z^*,$$

for some $b > 0$, $K > 0$, and $z^* > 0$, $V'_+(z) \to 0$ as $z \to \infty$. Hence $g(y) \to \infty$ as $y \to 0$.

Finally, suppose that $V(\cdot)$ is not strictly increasing. Then it is easily seen that $g(0) = z^\circ$, where z° is defined in (22). ∎

For any $\lambda > 0$, put

$$x_\lambda(\omega) = g(\lambda\xi(\omega)).$$

By the definition of g we know that

$$V'_+(x_\lambda(\omega)) \begin{cases} \leq \lambda\xi(\omega) \leq V'_-(x_\lambda(\omega)) & \text{for } a.e.\, \omega \text{ s.t. } x_\lambda(\omega) > 0; \\ \leq \lambda\xi(\omega) & fora.e.\, \omega \text{ s.t. } x_\lambda(\omega) = 0; \end{cases}$$

where $V'_-(z)$ denotes the left–hand derivative of $V(z)$. If we can show that $x_\lambda \in L^p(P)$, then by the Saddle-Point theorem, x_λ is a solution to (A_1) with an initial wealth

$$K(\lambda) \equiv \int_\Omega x_\lambda(\omega)\xi(\omega)P(d\omega).$$

If we further demonstrate that there exists $\lambda > 0$ such that $K(\lambda) = K_0$ the we are done. These are the steps on which we now take.

Proposition 3.2 *Suppose $V(\cdot)$ is strictly increasing, then $x_\lambda \in L^p_+(P) \; \forall \lambda > 0$ if $\xi^{-1} \in L^{\frac{p}{b}}(P)$. When $V(\cdot)$ is not strictly increasing, $x_\lambda \in L^p_+(P) \; \forall\lambda > 0$.*

PROOF. See Appendix. ∎

Proposition 3.2 says that given any strictly positive "Lagrangian multiplier" λ, the random consumption/wealth x_λ that is consistent with λ and $V(\cdot)$ lies in $L^p(P)$ when the inverse of the Arrow–Debreu price system has a finite certain moment. Our task now is to show that there exists a Lagrangian multiplier $\lambda^* > 0$ so that x_{λ^*} exhausts the initial wealth. To accomplish this, we first characterize some properties of the function $K(\cdot)$ in the following proposition, whose proof is in the Appendix.

Proposition 3.3 *Suppose that $\xi^{-1} \in L^{\frac{p}{b}}(P)$ whenever $V(\cdot)$ is strictly increasing. Then $K(\lambda)$ is finite for all $\lambda \in (0, \infty)$, is a continuous function of λ, and $\lim_{\lambda\to\infty} K(\lambda) = 0$. In addition, if $V(\cdot)$ is strictly increasing, we have*

$$\lim_{\lambda\to 0} K(\lambda) = \infty;$$

on the other hand if $V(\cdot)$ is not strictly increasing, we have

$$\lim_{\lambda\to 0} K(\lambda) = z^\circ \int_\Omega \xi(\omega)P(d\omega),$$

which is finite and strictly positive.

Here is the main theorem:

Theorem 3.6 *Suppose that $\xi^{-1} \in L^{\frac{p}{b}}(P)$ whenever $V(\cdot)$ is strictly increasing. There exists a solution to (A_1).*

PROOF. We take cases. Case 1: $V(\cdot)$ is strictly increasing. By Proposition 3.3 there exists $\lambda > 0$ such that $K(\lambda) = K_0$. Hence $x_\lambda \in L^p_+(P)$ satisfies

$$\int_\Omega x_\lambda(\omega)\xi(\omega)P(d\omega) = K_0,$$

and thus is budget-feasible. Also, by definition of x_λ we know

$$V'_+(x_\lambda(\omega)) \begin{cases} \leq \lambda\xi(\omega) \leq V'_-(x_\lambda(\omega)) & \text{for } P-a.e.\,\omega \text{ s.t. } x_\lambda(\omega) > 0; \\ \leq \lambda\xi(\omega) & \text{for } P-a.e.\,\omega \text{ s.t. } x_\lambda(\omega) = 0. \end{cases}$$

It then follows from the Saddle-Point theorem that x_λ is a solution to (A_1).

Case 2: $V(\cdot)$ is not strictly increasing. Put

$$K^0 \equiv z^\circ \int_\Omega \xi(\omega)P(d\omega).$$

If $K_0 \leq K^\circ$, there exists a solution along the same line of arguments of the previous paragraph. If $K_0 > K^\circ$, any $x \in L^p_+(P)$ such that $x \geq z^\circ$ *a.s.* and that

$$\int_\Omega x(\omega)\xi(\omega)P(d\omega) = K_0$$

is a solution. ∎

3.4 Generalization

In this subsection, we will give sufficient conditions for there to exist a solution to the Arrow–Debreu style variational problem of (21) with a time dimension. Proofs of the following theorems are straightforward generalizations of their counterparts in Sections 3.2 and 3.3. We will prove Theorem 3.8 in the Appendix as a demonstration and omit the proofs for other theorems.

We will first rewrite (21) in a form that can be analyzed more conveniently. Let $\hat{\lambda}$ denote Lebesgue measure plus a unit mass at T and let γ denote the product measure on $\Omega \times [0,T]$ generated by P and $\hat{\lambda}$. Also, recall from Section 2 that ν denotes the product measure generated by P and Lebesgue measure. Letting $u(z,T) = V(z)$, (21) is equivalent to

$$\begin{aligned} \sup_{c \in L^p_+(\gamma)} \quad & E\left[\int_0^T u(c(t),t)\hat{\lambda}(dt)\right] \\ \text{s.t.} \quad & E\left[\int_0^T c(t)\xi(t)\hat{\lambda}(dt)\right] = W_0, \end{aligned} \tag{TA}$$

where $L^p(\gamma) \equiv L^p(\Omega \times [0,T], \mathcal{O}, \gamma) = L^p(\nu) \times L^p(P)$, and where, for convenience, we have put $\xi(t) \equiv \eta(t)B(0)/B(t)$. Note that redefining the utility function for time T consumption to be the utility function for final wealth has no effect on the solution to (21) since we only alter utility function of consumption on a Lebesgue measure zero set. Note also that any $c \in L^p_+(\gamma)$ represents a consumption–final wealth pair $(c,W) \equiv (c, c(T)) \in L^p_+(\nu) \times L^p_+(P)$.

The following definition is a generalization of Definition 3.1:

Definition 3.2 *Let $f : \Re_+ \times \Omega \times [0,T] \mapsto \Re_+$ be measurable with respect to the product σ-field $\mathcal{B}(\Re_+) \times \mathcal{F} \times \mathcal{B}([0,T])$. Then $f(z,\omega,t) = o(z)$ as $z \to \infty$, $L^p(\gamma)$-integrably in (ω,t), if for each $\epsilon > 0$ there exists $y \in L^p_+(\gamma)$ such that $f(z,\omega,t) \le \epsilon z$ for γ-almost every $(\omega,t) \in \Omega \times [0,T]$ whenever $z \ge y(\omega,t)$.*

Here is the generalization of Theorem 3.2:

Theorem 3.7 *Suppose that*

1. *$u(z,t) : \Re_+ \times [0,T] \mapsto \Re_+$ is Borel measurable and is continuous, strictly increasing, and concave in the first argument, for $\hat{\lambda}$-a.e. $t \in [0,T]$; and*
2. *$u(z,t)\xi(\omega,t)^{-1} = o(z)$, $L^p(\gamma)$-integrably in (ω,t).*

Then there exists a solution to (TA).

Theorem 3.4 can be generalized as follows:

Theorem 3.8 *Suppose that $u(z,t) : \Re_+ \times [0,T]$ is Borel measurable, and is continuous, concave, and strictly increasing in z for $\hat{\lambda}$ almost every $t \in [0,T]$. Suppose further that there exist $b \in (0,1)$, $z^* > 0$, $B > 0$ such that*

$$u(z,0) \leq Bz^{1-b} \quad \text{and} \quad u(z,T) \leq Bz^{1-b} \quad \forall z \geq z^*.$$

For there to exist a solution to (TA), it is sufficient that, $u(z,t)$ exhibits discounting in that

$$u(z,t) \leq u(z,0) \quad \text{for } \hat{\lambda} - a.e.\ t \in [0,T),$$

and $\xi^{-1} \in L^{\frac{p}{b}}(\gamma)$. Moreover, if $\xi^{-1} \in L^{\frac{p'}{b}}(\gamma)$ for some $p' > p$, then every solution to (TA) is an element of $L^{p'}(\gamma)$.

PROOF. See Appendix. ∎

The following theorem generalizes Theorem 3.5, whose proof is omitted.

Theorem 3.9 *Suppose that $u(z,t) : \Re_+ \times [0,T] \to \Re_+$ is Borel measurable, and is continuous, concave, and strictly increasing in z for $\hat{\lambda}$-a.e. $t \in [0,T]$. Suppose further that $u(z,t)$ exhibits discounting in that*

$$u(z,t) \leq u(z,0) \quad \text{for } \hat{\lambda} - a.e. t \in [0,T),$$

and $u(z,0)$ and $u(z,T)$ is bounded from above for all z. If $\xi^{-1} \in L^p(\gamma)$, there exists a solution to (TA). In addition, if $\xi^{-1} \in L^{p'}(\gamma)$ for some $p' > p$, any solution to (TA) is an element of $L^{p'}(\gamma)$.

Under a different discounting condition, the following theorem generalizes results in Section 3.3.

Theorem 3.10 *Suppose that $u(z,t) : \Re_+ \times [0,T] \to \Re \bigcup \{-\infty\}$ is Borel measurable and can be unbounded from below at $z = 0$. Suppose also that, for $\hat{\lambda}$-a.e. $t \in [0,T]$, $u(z,t)$ is nontrivial, increasing, and concave, and is strictly concave on any subset of z where $u(z,t)$ is strictly increasing in z. In the case that $u(z,t)$ is strictly increasing in z for a strictly positive $\hat{\lambda}$-measure of t, we further suppose that*

1. *there exists $b > 0$, $z^* > 0$ and $K > 0$ such that*

$$u_{z+}(z,0) \leq Kz^{-b} \quad \forall z \geq z^*,$$

if $u(z,T)$ is not strictly increasing, where $u_{z+}(z,0)$ denotes the right-hand partial derivative of $u(z,0)$ with respect to z;

2. there exist $b > 0$, z^*, and $K > 0$ such that

$$u_{z+}(z,T) \leq Kz^{-b} \quad \forall z \geq z^*,$$

if $u(z,T)$ is strictly increasing in z; moreover,

$$u_{z+}(z,0) \leq Kz^{-b} \quad \forall z \geq z^*$$

if there exists a strictly positive Lebesgue measure of t such that $u(z,t)$ is strictly increasing in z.

If $u(z,t)$ exhibits discounting in that

$$u_{z+}(z,t) \leq u_{z+}(z,0) \quad \text{for } \hat{\lambda} - a.e.\ t \in [0,T),$$

then there exists a solution to (TA), provided that

1. *$\xi^{-1} \in L^{\frac{2}{b}}(\gamma)$ when $u(z,t)$ is strictly increasing in z for a strictly positive $\hat{\lambda}$-measure of t; or*
2. *$u(z,t)$ is not strictly increasing in z for $\hat{\lambda}$-almost every t.*

Remark 3.4 *The discounting conditions appear in Theorems 3.9–3.10 are satisfied by time-separable utility functions $u(z,t) = h(t)g(z)$ with $h(t)$ a decreasing function of time.*

Remark 3.5 *In Theorems 3.8–3.10, utility functions of consumption and the utility function of final wealth have similar characteristics. For example, $u(z,0)$ and $V(z)$ of Theorem 3.8 both satisfy conditions of Theorem 3.4. We can also allow $u(z,0)$ and $V(z)$ to have different characteristics. But we feel that this rarely happens in applications and thus leave it for the interested reader.*

4 Sufficient Conditions for the Existence of a Solution to the Dynamic Problem

We will give in this section easily verifiable sufficient conditions on the price processes for the existence of a solution to the dynamic consumption–portfolio problem of (12) for various classes of utility functions discussed in Section 3. Before doing that, we will first collect in Propositions 4.1, 4.2, and 4.3 three sets of sufficient conditions for existence of a solution to (12). Throughout, we will use the notation of $\hat{\lambda}$ and γ as defined in Section 3.4. Also, $\xi(t) = \eta(t)B(0)/B(t)$. The product measure generated by $\hat{\lambda}$ and Q will be denoted by γ^*. An element $c \in L^p_+(\gamma)$ represents a consumption–final wealth pair $(c, c(T)) \in L^p_+(\nu) \times L^p_+(P)$.

Consider the following three conditions:

(a) $E[\eta(T)^q] < \infty$ with $1/p + 1/q = 1$;

(b)

$$E^*\left[\int_0^T |\sigma(S(t),t)|^2 dt\right] < \infty;$$

(c) and

$$E^*\left[\int_0^T |\kappa(t)|^2 dt\right] < \infty.$$

Note that (a) and (b) are sufficient for Q to be the unique equivalent martingale measure by Proposition 2.1, and (c) is (16), which is needed for the martingale representation theorem of Lemma 2.4.

Proposition 4.1 *Let $u(z,T) = V(z)$. Suppose that $u(z,t)$ satisfies conditions in Theorem 3.8 with $b \in (0,1)$. Then there exists a solution to (12) if (a), (b), and (c) hold and $\xi^{-1} \in L^{\frac{p^p}{b}}(\gamma)$.*

PROOF. By Proposition 2.1, (a) and (b) ensure that there exists a unique martingale measure Q. From Theorem 3.8 and (d) we know that there exists a solution to (21) that lies in $L^{p^p}(\gamma)$. Denote this solution by c^*. We claim that $c^* \in L^p(\gamma^*)$. Next, by the Jensen's inequality

$$\left(\int_0^T |c^*(\omega,t)|^p \hat{\lambda}(dt)\right)^p \leq (T+1)^{p-1} \int_0^T |c^*(\omega,t)|^{p^p} \hat{\lambda}(dt) \quad \forall \omega.$$

Thus

$$E\left[\left(\int_0^T |c^*(t)|^p \hat{\lambda}(dt)\right)^p\right] \leq (T+1)^{p-1} E\left[\int_0^T |c^*(t)|^{p^p} \hat{\lambda}(dt)\right] < \infty,$$

where the last inequality follows since $c^* \in L^{p^p}(\gamma)$. That is, $\int_0^T |c^*(t)|^p \hat{\lambda}(dt) \in L^p(P)$. It then follows from Hölder's inequality that

$$E^*\left[\int_0^T |c^*(t)|^p \hat{\lambda}(dt)\right] = E\left[\eta(T) \int_0^T |c^*(t)|^p \hat{\lambda}(dt)\right] < \infty,$$

where we have used (a). That is, $c^* \in L^p(\gamma^*)$. Hence $c^* \in L^p(\gamma) \bigcap L^p(\gamma^*)$, or, equivalently,

$$(c^*, c^*(T)) \in L^p(\nu) \times L^p(P) \bigcap L^p(\nu^*) \times L^p(Q).$$

Proposition 2.3 and (c) imply that

$$(c^*, c^*(T)) \in L^p(\nu) \times L^p(P) \bigcap L^p(\nu^*) \times L^p(Q) \subset F.$$

This and Proposition 2.4 show that there exists a solution to (12), which was to be shown.

$\blacksquare$

The reader may find condition on ξ^{-1} of Proposition 4.1 a bit unusual. Its purpose is to ensure that a solution to (21) exists that lies in F. Note that since $p > 1$, $p^p > p$.

The following two propositions utilize Theorems 3.9 and 3.10, whose proofs are similar to that of Proposition 4.1 and are omitted.

Proposition 4.2 *Let $u(z,T) = V(z)$. Suppose that $u(z,t)$ satisfies conditions in Theorem 3.9. Then there exists a solution to (12) if (a), (b), and (c) are satisfied and $\xi^{-1} \in L^{p^p}(\gamma)$.*

Proposition 4.3 *Let $u(z,T) = V(z)$. Suppose that $u(z,t)$ satisfies conditions in Theorem 3.10 with $b > 0$ when $u(z,t)$ is strictly increasing in z for a strictly positive $\hat{\lambda}$ measure of $t \in [0,T]$. Then there exists a solution to (12) provided that (a), (b), and (c) are satisfied and $\xi^{-1} \in L^{\frac{p^p}{b}}(\gamma)$ when $u(z,t)$ is strictly increasing in z for a strictly positive $\hat{\lambda}$-measure of $t \in [0,T]$.*

Now we are ready to provide easily verifiable sufficient conditions for the requirements of Propositions 4.1, 4.2, and 4.3. Some preliminaries are in order.

Putting $\phi(t) \equiv B(t)/\eta(t)$, Itô's lemma implies that

$$\eta(t) = 1 + \int_0^t \eta(s)\kappa(S(s),s)^\top dw(s) \qquad \forall t \in [0,T],\ a.s. \tag{23}$$

and

$$\phi(t) = \phi(0) + \int_0^t \phi(s)r(S(t),t)\kappa(S(s),s)^\top \kappa(S(s),s)ds - \int_0^t \phi(s)\kappa(S(s),s)^\top dw(s) \qquad \forall t \in [0,T]\ a.s. \tag{24}$$

Here is the main theorem:

Theorem 4.1 *Consider the system of stochastic integral equations (1), (23) , and (24), compactly written as:*

$$\begin{pmatrix} S(t) \\ \eta(t) \\ \phi(t) \end{pmatrix} = \begin{pmatrix} S(0) \\ 1 \\ \phi(0) \end{pmatrix} + \int_0^t \widehat{\varsigma}(S(s),\eta(s),\phi(s),s)ds + \int_0^t \widehat{\sigma}(S(s),\eta(s),\phi(s),s)dw(s) \quad \forall t \in [0,T],\ a.s.$$

Suppose that there exists a constant K_1 such that

$$|\widehat{\varsigma}(z,t)| \le K_1(1+|z|), \qquad |\widehat{\sigma}(z,t)| \le K_1(1+|z|) \tag{25}$$

for all $z \in \Re^{N+2}$ and for all $t \in [0,T]$, and that for any $M > 0$ there is a constant K_M such that for all $y, z \in \Re^{N+2}$ with $|y| \leq M$ and $|z| \leq M$ and $t \in [0,T]$

$$|\hat{\varsigma}(y,t) - \hat{\varsigma}(z,t)| \leq K_M|y - z|, \qquad |\hat{\sigma}(y,t) - \hat{\sigma}(z,t)| \leq K_M|y - z|, \tag{26}$$

that is, $\hat{\varsigma}(z,t)$ and $\hat{\sigma}(z,t)$ satisfy a local Lipschitz and a uniform growth condition. Then there exists a solution to the dynamic problem of (12) if the conditions on the utility function of Theorem 3.8, Theorem 3.9, or Theorem 3.10 are satisfied.

PROOF. Under the hypothesis, Theorem 5.2.3 of Friedman (1975) implies that, for all positive integers m, there exist constants L_m such that

$$E\left[|\eta(t)|^{2m}\right] \leq 2\exp\{L_m t\}, \tag{27}$$

$$E\left[|B(t)\eta(t)^{-1}|^{2m}\right] \leq (1 + |B(0)\eta(0)^{-1}|^{2m})\exp\{L_m t\}, \tag{28}$$

and

$$E\left[|S(t)|^{2m}\right] \leq (1 + |S(0)|^{2m})\exp\{L_m t\}, \tag{29}$$

for all $t \in [0,T]$.

Note first that Relation (27) implies that $E[\eta(T)^q] < \infty$ with $1/p + 1/q = 1$. This is (a). Relation (28) and Fubini theorem implies that

$$E\left[\int_0^T (B(t)\eta(t)^{-1})^{p'}\hat{\lambda}(dt)\right] = \int_0^T E\left[B(t)\eta(t)^{-1}\right]^{p'}\hat{\lambda}(dt) < \infty \tag{30}$$

for any $p' \geq 2$. Thus the conditions on $\xi(t) = B(0)\eta(t)/B(t)$ of Propositions 4.1–4.3 are satisfied. Next note that, by again the Fubini's theorem and Lemma 2.3, we have

$$E^*\left[\int_0^T |\sigma(S(t),t)|^2 dt\right] = \int_0^T E[\eta(t)|\sigma(S(t),t)|]dt$$

and

$$E^*\left[\int_0^T \eta(t)|\kappa(S(t),t)|^2 dt\right] = \int_0^T E[\eta(t)|\kappa(S(t),t)|^2]dt.$$

The growth condition of (25), relations (27) and (29), and Hölder's inequality imply that

$$E[\eta(t)|\sigma(S(t),t)|^2] < K\exp\{Lt\}, \quad \forall t,$$

for some constants K and L. Hence

$$\int_0^T E[\eta(t)|\sigma(S(t),t)|^2] < \infty.$$

This is condition (b).

Finally, note that, by the definition of $\phi(t)$,

$$\begin{aligned} E[\eta(t)|\kappa(S(t),t)|^2] &= E\left[\frac{\eta^2(t)}{B(t)}\phi(t)|\kappa(S(t),t)|^2\right] \\ &\leq \frac{1}{B(0)}E\left[\eta^2(t)\phi(t)|\kappa(S(t),t)|^2\right], \end{aligned}$$

where the inequality follows from the assumption that $r(t)$ is positive. It is then easily verified that

$$E\left[\eta^2(t)\phi(t)|\kappa(S(t),t)|^2\right] < K\exp\{Lt\} \quad \forall t \in [0,T]$$

for some constants K and L, by the growth condition of (25), relations (27) and (29), and Hölder's inequality. Then (c) follows from Fubini's theorem and Lemma 2.3.

The rest of the assertion then follows from Propositions 4.1–4.3. ∎

For the three classes of utility functions of Propositions 4.1–4.3, for the existence of a solution to (12), it suffices to check whether the parameters of a system of stochastic differential/integral equations, completely derived from the price system, satisfy a local Lipschitz and a uniform growth condition. Thus we have provided easily verifiable conditions for the existence of a solution to (12), where the space of admissible strategies form a linear space. Before we leave this section, the reader is cautioned to note that throughout our analysis it is assumed that $E[\eta(T)] = 1$, for which a sufficient condition is (8). Thus, for existence, besides the local Lipschitz and the uniform growth conditions of Theorem 4.1, one still needs to verify, for example, the regularity condition of (8).

5 An Example

In this section, we will briefly discuss a simple example of the general securities market analyzed above. Assume that $\varsigma(S(t),t) = I_S(t)\mu$, $\sigma(S(t),t) = I_S(t)\sigma$, and $r(S(t),t) = r$, where μ is an $N \times 1$ vector of constants, σ is an $N \times N$ matrix of constants, $I_S(t)$ is a diagonal matrix with elements $S_n(t)$, and r is a constant. This is the geometric Brownian motion model considered by Merton (1971). In this case,

$$\kappa(S(t),t) = -\sigma^{-1}(\mu - r\mathbf{1}),$$

which is a constant, where $\mathbf{1}$ denotes an N–vector of 1's. Then

$$\eta(t) = \exp\left\{(\mu - r\mathbf{1})^\top\sigma^{-1\top}w(t) - \frac{1}{2}(\mu - r\mathbf{1})^\top(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1})t\right\}$$

is lognormally distributed with a mean equal to one. Since a lognormal distribution has finite moments of any order, $E[\eta(T)^q] < \infty$ for any $1 < q < \infty$. It follows that η defines a unique equivalent martingale measure. It can also be easily verified that (30) holds.

Now consider the class of HARA utility functions

$$\begin{aligned} u(y,t) &= e^{-\rho t}\left(\frac{b}{1-b}\right)\left(\frac{\beta y}{b}+\xi\right)^{1-b} \\ V(y) &= u(y,T), \end{aligned} \tag{31}$$

with $\beta > 0$, $b \neq 0$ or 1. It is understood that if $b < 0$, then $u(y,t) = 0$ for all $y \geq (-b)\xi/\beta$. With $b > 0$ and $\xi < 0$, the agent's problem is not completely specified because the utility function does not state the consequence of consuming less than $|\xi|b/\beta$. Furthermore, for sufficiently low initial wealth,

$$K_0 < |\xi|(b)\left(1-e^{-rT}\right)/\beta r,$$

there is no policy that can guarantee $c(t) \geq |\xi|(b)/\beta$ for all t with probability one. Consequently, we only consider the case $\xi \geq 0$.

Merton (1971) has shown that, for this class of utility functions, there exists a solution to (12) without the constraint of nonnegative consumption and nonnegative final wealth when asset prices follow geometric Brownian motion and when the instantaneous riskless rate is a constant. His method of proof is by solving a partial differential equation and then using the verification theorem in dynamic programming. We will now use the results of previous sections to show that there also exists a solution to (12) when the nonnegativity constraint is present. Regardless whether the utility functions of (31) are bounded from below or not, they are strictly concave when they are strictly increasing and thus fall into the class of utility functions of Section 3.3. Note that since

$$\lim_{y\to\infty}\frac{-u_{yy}(y,t)y}{u_y(y,t)} = b$$

when utility functions are strictly increasing, Remark 3.3 and Theorem 4.3 show that there exists a solution to (12).

Optimal consumption–portfolio policies for HARA utility functions can in fact be explicitly calculated. They are no longer linear in wealth in contrast to the optimal policies calculated by Merton (1971). We refer the interested reader to Cox and Huang (1987) for complete details.

6 Generalizations and Concluding Remarks

Our analysis can be extended in several directions. First, we have assumed that the instantaneous interest rate $r(t)$ depends only on $S(t)$ and t. This is not necessary. It can be allowed to depend on the bond price process, for example, in that $r(t) = r(S(t), B(t), t)$. In this case, we simply add one more equation

$$B(t) = B(0) + \int_0^t B(s)r(S(s), B(s), s)ds$$

to the system of stochastic integral equations in Theorem 4.1.

Second, the parameters of the price processes at any time do not have to depend only on the realizations of the prices at that time. For example, assume that

$$S(t)+\int_0^t f(Y(s),S(s),s)ds = S(0)+\int_0^t \varsigma(Y(s),S(s),s)ds+\int_0^t \sigma(Y(s),S(s),s)dw(s)$$

where Y is an M–vector of Itô processes:

$$Y(t)=Y(0)+\int_0^t \mu_y(Y(s),s)ds+\int_0^t \sigma_y(Y(s),s)dw(s)$$

with $M \leq N$. Then Theorem 4.1 goes through when the system of stochastic integral equations is expanded to include Y.

Third, the parameters of the price processes at any time can be allowed to depend on their own historical realizations. In that case, the bounds of the moments of certain processes as described in (27)–(29) are still valid under a generalized Lipschitz and a generalized growth condition; see Theorem 4.6 of Liptser and Shiryayev (1977). The conclusion of Theorem 4.1 naturally follows. Moreover, a generalization toward a combination of this and the previous case is also straightforward.

Throughout our analysis, we have assumed that the economy is of finite horizon. Our results can, however, be extended to an infinite horizon economy in the following manner. Consider price processes that admit a unique equivalent martingale measure. Roughly, use Chapter 8 of Friedman (1975) to find conditions on the parameters of the price processes so that (30) holds with T replaced by ∞ and $\hat{\lambda}$ replaced by Lebesgue measure. This approach is not satisfactory, however. For there to exist a unique equivalent martingale measure, we want condition (b) of Section 4 or, equivalently, relation (10) to be satisfied when T is replaced by ∞. This entails that $\sigma(S(t),t)$ goes to zero when t approaches ∞ and rules out the case where the price processes are an N–dimensional geometric Brownian motion, which happens to be the most prevalent case in applications. The question of how our approach could be extended to the infinite horizon case while including the geometric Brownian motion as a special case needs be resolved. This is a high priority issue.

The class of utility functions covered by Theorem 4.1 is quite general. It includes the HARA class functions that are increasing and concave, continuous and concave functions that are bounded from above and from below, and asymptotically non-risk-neutral utility functions that are increasing and strictly concave, for example. It does not, however, include utility functions that are unbounded from below and are not strictly concave when they are strictly increasing. The existence result needs to be extended for this latter class of functions.

The results in Sections 2 and 4 depend largely upon the uniqueness of the martingale measure, which in turn implies that the markets are dynamically complete as described in Proposition 2.3. How the martingale technique can be useful when the markets are not dynamically complete is an open question.

Another weakness of the results reported is the lack of necessary condition for existence. This, unfortunately, is a feature of the theory of stochastic differential equations. For example, not much is known about the necessary conditions for the existence and uniqueness of a solution to a system of stochastic differential equations.

7 References

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Appendix

Proof of Proposition 2.1.

PROOF. We first show that Q is equivalent to P, that is, $\eta(T) = dQ/dP$ is strictly positive P–a.s. It suffices to demonstrate that

$$\int_0^T \kappa(S(t),t)^\top dw(t) - \frac{1}{2}\int_0^T |\kappa(S(t),t)|^2 dt > -\infty \qquad P - a.s.$$

We note that (7) implies

$$\left|\int_0^T \kappa(S(t),t)^\top dw(t)\right| < \infty;$$

see Liptser and Shiryayev (1977, Theorem 7.1). This, together with (7), ensures that $\eta(T) > 0$ P-a.s.

Next, we want to show that $G^*(t)$ is a martingale under Q. By the Girsanov theorem (see Liptser and Shiryayev (1977, Chapter 6)), we know

$$w^*(t) \equiv w(t) - \int_0^t \kappa(S(s),s)ds \qquad \forall t \in [0,T]$$

is an N-dimensional standard Brownian motion under Q. Thus we can write

$$G^*(t) = S(0)/B(0) + \int_0^t \frac{\sigma(S(s),s)}{B(s)} dw^*(s) \qquad \forall t \in [0,T]. \tag{A.1}$$

Given (10) and the fact that $B(t)$ is bounded below away from zero, the right–hand side of (A.1) is a (square–integrable) martingale under Q; see Liptser and Shiryayev (1977, (4.48)). The first assertion then follows from the assumption that $E[\eta(T)^q] < \infty$.

The uniqueness of Q follows from similar arguments of Theorem 3 of Harrison and Kreps (1979). ∎

Proof or proposition 3.2:

PROOF. We take cases.

Case 1: $V(\cdot)$ is strictly increasing. From the hypothesis and the definition of g we know that there exists $y^* > 0$ such that

$$g(y) \le K^{\frac{1}{b}} y^{\frac{-1}{b}} \qquad \forall y \le y^*.$$

Let $\lambda > 0$ be given. Put

$$\Lambda = \{\omega \in \Omega : \lambda\xi(\omega) \le y^*\}.$$

On Λ, we know

$$g(\lambda\xi(\omega)) \le K^{\frac{1}{b}} (\lambda\xi(\omega))^{\frac{-1}{b}}.$$

On $\Omega \setminus \Lambda$, however, we have

$$g(\lambda\xi(\omega)) \le g(y^*).$$

Thus

$$\begin{aligned}\int_\Omega |g(\lambda\xi(\omega))|^p P(d\omega) &= \int_\Lambda |g(\lambda\xi(\omega))|^p P(d\omega) + \int_{\Omega\setminus\Lambda} |g(\lambda\xi(\omega))|^p P(d\omega) \\ &\leq K^{\frac{p}{b}} \int_\Lambda (\lambda\xi(\omega))^{\frac{-p}{b}} P(d\omega) + g(y^*) < \infty,\end{aligned}$$

which was to be shown.

Case 2: $V(\cdot)$ is not strictly increasing. Since $g(0) = z^\circ$, $g(\cdot)$ is bounded from above. Thus the assertion follows. ∎

Proof of Proposition 3.3:

PROOF. We first note that $K(\lambda)$ is always finite for all $\lambda \in (0, \infty)$ since by Proposition 3.2 $x_\lambda \in L^p(P)$ and by assumption $\xi \in L^q(P)$ with $1/p + 1/q = 1$.

Next let $\lambda_n \downarrow \lambda$. We note that

$$\lambda_n \xi \to \lambda \xi \qquad P-a.e.$$

By the continuity and monotonicity of g (see Proposition 3.1), it follows that

$$g(\lambda_n\xi)\xi \uparrow g(\lambda\xi)\xi \qquad P-a.e.$$

By the Monotone Convergence Theorem, we know

$$\lim_{n\to\infty} \int_\Omega g(\lambda_n\xi(\omega))\xi(\omega)P(d\omega) = \int_\Omega g(\lambda\xi(\omega))\xi(\omega)P(d\omega). \tag{A.6}$$

Equivalently, $\lim_{n\to\infty} B(\lambda_n) = K(\lambda)$.

Next consider $\lambda_n \uparrow \lambda$. Using similar arguments while applying Lebesgue Convergence Theorem, we get

$$\lim_{n\to\infty} B(\lambda_n) = K(\lambda). \tag{A.7}$$

Relations (A.6) and (A.7) imply that $B(\cdot)$ is a continuous function.

Next note that from Proposition 3.1 we know

$$\lim_{\lambda\to\infty} g(\lambda\xi)\xi = 0 \qquad P-a.e.$$

Since g is decreasing, by Lebesgue Convergence Theorem we have

$$\lim_{\lambda\to\infty} K(\lambda) = 0.$$

Now we take cases. Case 1: V is strictly increasing. Then

$$\lim_{\lambda\to 0} g(\lambda\xi)\xi = \infty \qquad P-a.e.$$

By Fatou's lemma, we get

$$\lim_{\lambda\to 0} K(\lambda) \geq \infty.$$

Thus

$$\lim_{\lambda\to 0} K(\lambda) = \infty.$$

Case 2: V is not strictly increasing. Then

$$\lim_{\lambda \to 0} g(\lambda\xi)\xi = z^\circ \xi \qquad P-a.e.$$

By the dominated convergence theorem we have

$$\lim_{\lambda \to 0} K(\lambda) = z^\circ \int_\Omega \xi(\omega) P(d\omega),$$

which was to be shown. ∎

Proof of Theorem 3.8:

PROOF.

Let $\xi^{-1} \in L^{\frac{p}{b}}(\gamma)$. By the hypothesis, we know there exist $B > 0$ and $z^* > 0$ such that

$$u(z,0) \leq g(z) \equiv Bz^{1-b} \qquad \forall z \geq z^*$$

and that

$$u(z,T) \leq \hat{g}(z) \equiv Bz^{1-b} \qquad \forall z \geq z^*$$

The discounting hypothesis implies that

$$u(z,t) \leq u(z,0) \leq g(z) \quad \forall z \geq z^*, \hat{\lambda} - a.e.\ t \in [0,T).$$

We claim that given $\epsilon > 0$, there exists $y \in L^p_+(\gamma)$ such that

$$g(z) \leq \epsilon z \xi(\omega,t) \quad \forall z \geq y(\omega,t),$$

for ν–a.e. $(\omega,t) \in \Omega \times [0,T]$ and

$$\hat{g}(z) \leq \epsilon z \xi(\omega,T) \quad \forall z \geq y(\omega,T),$$

for P–a.e. $\omega \in \Omega$.

Note that $g(z)/z$ and $\hat{g}(z)/z$ are of the same order of magnitude of z^{-b} as $z \to \infty$ and is decreasing. Thus there exists constants $K > 0$ and $\hat{z} > 0$ such that

$$\frac{g(z)}{z} \leq Kz^{-b} \quad \forall z \geq \hat{z}$$

and

$$\frac{\hat{g}(z)}{z} \leq Kz^{-b} \quad \forall z \geq \hat{z}.$$

Now putting $z^\circ \equiv \max\{z^*, \hat{z}\}$,

$$\Lambda = \{(\omega,t) \in \Omega \times [0,T) : \frac{g(z^\circ)}{z^\circ} \leq \epsilon \xi(\omega,t)\},$$

and

$$\hat{\Lambda} = \{\omega \in \Omega : \frac{\hat{g}(z^\circ)}{z^\circ} \leq \epsilon \xi(\omega,T)\},$$

we define

$$y(\omega,t)=\begin{cases} z^\circ & \text{if } (\omega,t)\in\Lambda; \\ z^\circ & \text{if } \omega\in\hat{\Lambda} \text{ and } t=T; \\ \frac{\epsilon}{K}^{\frac{-1}{b}}\xi(\omega,t)^{\frac{-1}{b}} & \text{if } (\omega,t)\in\Omega\times[0,T)\setminus\Lambda; \\ \frac{\epsilon}{K}^{\frac{-1}{b}}\xi(\omega,T)^{\frac{-1}{b}} & \text{if } \omega\in\Omega\setminus\hat{\Lambda} \text{ and } t=T. \end{cases}$$

It is clear that $y\geq 0$ γ–a.e. and $y\in L^p(\gamma)$ since $\xi^{-1}\in L^{\frac{p}{b}}(\gamma)$. Note also that $y\geq z^\circ$ γ–a.e.

Next note the following: On Λ, since $g(z)/z$ is decreasing,

$$\frac{g(z)}{z}\leq\frac{g(z^\circ)}{z^\circ}\leq\epsilon\xi(\omega,t)\qquad\forall z\geq y(\omega,t).$$

On $\Omega\times[0,T)\setminus\Lambda$, we know $z\geq y(\omega,t)$ implies that

$$\frac{g(z)}{z}\leq Kz^{-b}\leq Ky(\omega,t)^{-b}=\epsilon\xi(\omega,t).$$

Similarly, on $\hat{\Lambda}$,

$$\frac{\hat{g}(z)}{z}\leq\frac{\hat{g}(z^\circ)}{z^\circ}\leq\epsilon\xi(\omega,T)\qquad\forall z\geq y(\omega,T).$$

On $\Omega\setminus\hat{\Lambda}$,

$$\frac{\hat{g}(z)}{z}\leq Kz^{-b}\leq Ky(\omega,T)^{-b}=\epsilon\xi(\omega,t).$$

Finally, since $u(z,t)\leq g(z)$ for all $z\geq z^\circ$ and for $\hat{\lambda}$–a.e. $t\in[0,T)$, and since $y\geq z^\circ$ γ–a.e. by construction, we know

$$u(z,t)\leq\epsilon z\xi(\omega,t)\quad\forall z\geq y(\omega,t)\quad\text{for }\gamma-a.e.\,(\omega,t).$$

Hence $u(z,t)\xi(\omega,t)^{-1}=o(z)$ $L^p(\gamma)$–integrably in (ω,t) and there exists a solution to (TA).

The proof for the second assertion is similar to that for Theorem 3.2. ∎