

**"Ordering Policies in an Environment of
Stochastic Yields and Substitutable Demands"**

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Abstract

In this paper we model production problems where yields are stochastic, demands are substitutable, and several items are jointly produced. We have formulated this problem as a profit maximizing convex program, and have studied two solution procedures. The first method solves finite horizon stochastic programs on a rolling horizon basis. We have developed a decomposition algorithm for solving the finite horizon problems. These are linear programs and our algorithm utilizes the "network-like" structure. The second method is a heuristic procedure that is based on the structure of the optimal policy for a two period problem. It parallels the decision rules used by managers in practice. The computational results suggest that the performance of this heuristic is comparable to that of the rolling horizon approach.

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1. INTRODUCTION

In this paper we are concerned with situations where there is a hierarchy in the grades of outputs. We assume that a higher grade product can be substituted for a product lower in the hierarchy. Under these circumstances, the manufacturer may occasionally downgrade a product rather than backorder the demand for a lower grade item. This type of action may be motivated by a variety of reasons, for example - to prevent customer dissatisfaction, or to reduce inventory costs.

We are interested in identifying appropriate ordering policies in an environment of stochastic yields and substitutable demands. Our interest in this problem resulted from a specific application in the semiconductor industry. For the sake of concreteness we will describe this application in greater detail. Although the discussion is motivated by a specific application, a wide variety of inventory -production problems have a similar structure. This claim is particularly applicable to the semiconductor industry (Leachman 86).

In the semiconductor industry, products and manufacturing technologies change rapidly. The manufacturing process is complex and often not very well understood. Consequently yields vary significantly. Decision aids that facilitate inventory management in this uncertain environment can play an important role in improving profitability.

Semiconductor chips are produced in wafers. Each wafer contains several chips. The number of chips produced per wafer depends on the complexity of the circuit on the chip, and can vary from 10 to 100,000. The production of wafers is essentially a flow process involving several steps. To start, disks (wafers) of either Silicon or Gallium Arsenide are cut from ingots. Next, several layers of semiconducting material are deposited on top of the wafer. At the end of the fabrication process wafers are cut into

individual chips, also called dice, and delivered to the assembly stage. The chip is then assembled into a package. For a more detailed description of the production process the reader is referred to Kothari (84), and Bitran and Tirupati (88).

The company where this research took place manufactures devices for both military and commercial applications. The particular facility that motivated this study produces diodes (an electronic valve) and is one of the largest facility of its kind in the world. Each wafer yields approximately 5000 diodes. Typically, diodes from the same wafer exhibit different electrical properties. The application (end-use) that a diode finds depends on its electrical properties. The number of diodes of each type obtained from a wafer is a random variable. The distribution of these random variables depends on the process employed to produce the wafer. A diode that does not meet its intended specification can often be useful for some other application.

In this facility diodes are produced for approximately 100 different applications. These applications can be grouped into 12 families. Within each family there is a hierarchy. If the electrical performance of a diode meets the requirements of an application then it satisfies the requirements of all applications lower in the hierarchy. Thus the demands are substitutable in this hierarchical sense.

In the general context considered in this paper we have (i) processes, (ii) items and (iii) applications or customers. A process corresponds to the production process for a wafer, and an item corresponds to a diode. Associated with each process is a set of items that can be produced using that process. The actual yields are random variables. If we process n units (chips), then we obtain np_i units of item i , where p_i - the fraction of type i items - is a random variable. The distribution of these random variables is determined by the production process employed.

Customers can be grouped into families. Within each family there is a hierarchy such that an item that is suitable for a member in the family is also suitable for all members lower in the hierarchy. The relationship between production process, items and customers is illustrated by Fig. 1.

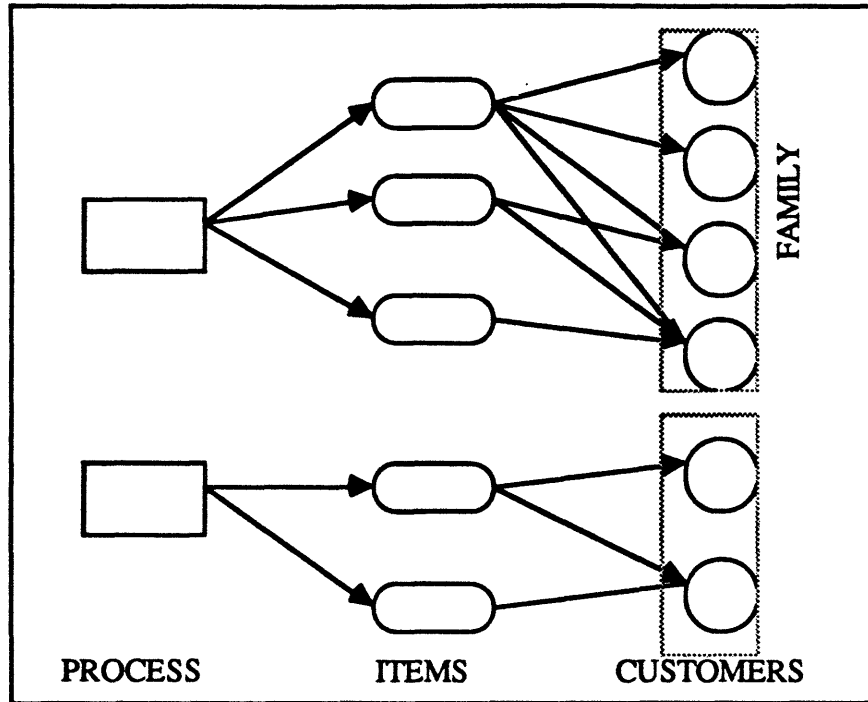


Figure 1. Relationship between Processes, Items and Customers

Our objective in this paper, is to develop a model to assist managers in deciding how many units to produce, in each period, using each process . Once the yields are known the model must assist managers in deciding how to allocate the items to customers. One of our tasks is to develop an understanding of how to manage the item inventories. We have formulated the problem as a profit maximizing, convex, stochastic program, and have developed approximation procedures for solving it. Based on the structure of a two period problem we propose heuristics for allocating items to customers in a multiperiod setting.

Section 2 contains a brief survey of the related literature. In section 3 the main assumptions on which our model is based, are described in greater detail. We present the formulation in section 4 and show that it is a convex program. In section 5 we examine the stochastic linear program and identify embedded networks. Section 5 also contains approximation procedures for solving the stochastic program. In section 6 we derive the structure of the optimal inventory allocation policy for a two period problem. Based on this result we propose and test heuristics for multiperiod problems.

2. LITERATURE REVIEW

There is a vast body of literature dealing with inventory and production problems. Veinott(60) provides an excellent review of early literature on basic lot-sizing models. An implicit assumption in most of the inventory models is that the yield is 100% or is deterministic and known. There are relatively few papers addressing variability in yields. Over the last decade there has been considerable increase in interest in modelling production processes with variable yields.

Silver(76) derives the economic order quantity when the quantity received from the supplier does not match the quantity ordered. Under the assumption of constant deterministic demand, the economic order quantity is shown to be dependent upon only the mean and standard deviation of the quantity received.

Kairo and Gohil(82) extend Silver's model to the case where the demand during the stock-out period is either partially or completely backordered. In both cases the results obtained are shown to be extensions of the well known results of the lot sizing problem with backordering. Shih(80) analyzed a single period inventory model with random demand, variable yield and no ordering cost.

Mazzola, McCoy, and Wagner(87) consider a multiperiod problem. They derive an EOQ model when the production yield follows a binomial distribution and backlogging of demand is permitted. They test several heuristic applications of this lot sizing results, to discrete time problems. The most promising heuristics are based on adjusting deterministic lot-sizing policies, as calculated either optimally using the Wagner-Whitin algorithm or approximately using the Silver-Meal heuristic. Moinzadeh and Lee (87) have analyzed a continuous review model, when the quantity received contains defectives and the demands are uncertain.

Lee and Yano(88) have modelled a multi-stage single period single product problem. They assume that the proportion of defective pieces produced at each stage is a random variable. The decision variables are the quantities to be produced at each stage. They show that the cost incurred at each stage is a convex function of the quantity produced at that stage.

The by-product problem has been studied by Pierskalla and Deuermeyer(78). They consider the control of a production system that consists of two processes which produce two products. Process A produces both products in some fixed (deterministic) proportion while process B produces only one product. The demands are random, but are not substitutable. The authors formulate the problem as a convex program and derive properties of the optimal policy. They show that the decision space (inventory levels) can be divided into 4 regions depending on whether or not a production process is used.

The substitutable demand problem has also been studied (Sadowski 59, Wolfson 65, Pentico 88). This literature is not concerned with production process. The focus is on allocating items to customers so as to reduce ordering, inventory, and backorder costs. The yields are assumed to be deterministic. The work by McGillivray and Silver (78), Parlar and Goyal (84), Sadowski (59), Wolfson(65), Tryfos (85), and Pentico (88) is representative of this work.

We model production processes where demands are substitutable, yields are stochastic, and several items are jointly produced. In a sense our model generalizes the notion of 'defect' by permitting a number of grades of quality. We restrict the analysis to a particular substitution structure that is common in practice.

There is a large body of literature on computational procedures for solving stochastic programs. One of the earliest works in this field is due to Dantzig(55). He posed a multi-time period stochastic program. Since then, several strategies have been proposed for solving these problems. In general it is very difficult to obtain exact solutions, hence the emphasis has been on developing good approximations. In principle, if the random variables are discrete, the problem of maximizing the expected value of the objective function reduces to one of solving a large linear-program. If the random variables have a continuous distribution then a popular strategy is to approximate the continuous problem by a sequence of discrete problems. Olsen(76) proved that the sequence of solutions of the discrete approximations would converge to the solution of the continuous problem under relatively mild conditions. Birge and Wets(86) provide a very good review of approximation schemes for stochastic optimization problems.

3. MODELLING ASSUMPTIONS

In order to develop a mathematical model we have made a few assumptions regarding demand patterns, the nature of variability of yields, and lead times. In this section we state and provide justification for our assumptions

The customers for the diodes, are manufacturers of electronic goods who specify their requirements over a horizon of 4 to 5 months. The delivery schedule specifies the quantity to be dispatched each week. Accordingly, we assume that the demands are dynamic and deterministic.

In our facility the yield varies from lot to lot, and the distribution of the fraction of each item obtained in the output is not significantly influenced by the lot size. Accordingly, we assume that the production of n units yields np_i units of item i ; P_i is a discrete random variable and the sum of the p_i s is less than or equal to one. We also assume that the number of yield outcomes is finite.

The production activities in this facility are split into two groups - the wafer fabrication facility and the assembly operations. The assembly operations in turn, are organized into several sub-groups on the basis of product families (lines). The wafer fabrication shop is a central facility, that acts like a supplier to the assembly units. It is the responsibility of each assembly unit to determine the number of wafers it requires. This is done on a weekly basis. Typically, each week the assembly unit sends in a requisition for a set of wafers. The capacity issue is solved at the wafer fabrication facility through mechanisms such as working additional shifts, working on week-ends, and subcontracting wafers that involve technologies that are neither proprietary nor very sophisticated. The changes in the load at the wafer fabrication facility have a bearing on the delivery lead time. However, these lead times are stable over horizons of few months. Since the planning model developed in this paper aids decisions made on a weekly basis, we have ignored the lead time variability. To simplify the presentation of our results, much of the analysis is based on the assumption that lead times are zero. We indicate how the model is to be modified to incorporate longer lead times.

Another assumption made in this paper, relates to the order quantity. The order quantity is stated in terms of chips, instead of wafers. This is equivalent to ignoring the

integrality constraint on the number of wafers ordered. This assumption is based on the demand levels. Typically, the lot size exceeds 10 wafers.

The objective function of our model is to maximize expected profits. Customers higher in the hierarchy pay a higher price. The costs include production, holding, and backorder costs. There are no set-up costs.

4. PROBLEM FORMULATION

For ease of presentation we first formulate and analyze a multiperiod problem with zero lead time. In section 5 we indicate how to incorporate longer lead times. In the absence of capacity constraints the problem separates by families. Hence the formulation given below is for a single family. The members of the family are numbered in descending order - the highest member of the family is #1. The number of an item designates the highest member in the family that it is suitable for. Without loss of generality, in the remainder of this paper we assume that the number of items equals the number of customers.

In the absence of lead times, production is initiated at the beginning of the day and is completed by evening, and then, the yields are known. In the evening, items are allocated to customers. In this framework, in the morning we must decide the number of units to produce, and in the evening we must allocate the items optimally.

NOTATION:

A_{ij}	: The number of 'i' items allocated to customer 'j'.
$B_{t,j}$: Backorders of customer 'j' carried into period t
C_i	: Unit salvage value of item i
h_i	: Unit holding cost of item i
$D_{t,j}$: Demand of customer j in period t
$I_{t,i}$: Inventory of item 'i' carried into period t
N_t	: Number of units produced in period t
B_j	: Unit backorder cost for customer 'j'
K	: Production cost per unit
S_j	: Price per unit paid by customer j
p_m^i	: Fraction of item i in the m th outcome of the random variables

E_p : Expectation with respect to the yields.

Bold : Bold letters denote vectors (except for the expectation operator)

The periods are indexed backwards. The index indicates the number of periods remaining in the horizon. In this notation period 1 is the last period in the horizon, and period 2 is the penultimate period. The formulation given below is for a two customer family. It can be extended in a straight forward manner to multi-customer families.

MORNING PROBLEM

$F_t(I_t, B_t) =$

$$(1) \quad \begin{array}{l} \text{Max } E_p G_t(p_m, N_t, I_t, B_t) - KN_t \\ \text{S.T:} \quad N_t \geq 0 \end{array}$$

EVENING PROBLEM

$G_t(p_m, N_t, I_t, B_t) =$

$$\text{Max } S_1 A_{11} + S_2(A_{12} + A_{22}) + \sum_{i=1}^2 h_i I_{t-1,i} - \sum_{j=1}^2 \beta_j B_{t-1,j} + F_{t-1}(I_{t-1}, B_{t-1})$$

S.T.

$$(2) \quad A_{11} + A_{12} + I_{t-1,1} = N_t p_m^1 + I_{t,1}$$

$$(3) \quad A_{22} + I_{t-1,2} = N_t p_m^2 + I_{t,2}$$

$$(4) \quad A_{11} + B_{t-1,1} = D_{t,1} + B_{t,1}$$

$$(5) \quad A_{12} + A_{22} + B_{t-1,2} = D_{t,2} + B_{t,2}$$

$$(6) \quad A_{11}, A_{12}, A_{22}, B_{t-1,1}, B_{t-1,2}, I_{t-1,1}, I_{t-1,2} \geq 0$$

CONSTRAINTS

Constraints (2) and (3) are inventory balance constraints. The right hand side is the inventory of each item available at the end of the day. Constraints (4) and (5) account for the customers' demands. The right side is the net demand and the left hand side shows how it is satisfied (or backordered).

CONVEXITY

We assume that $F_0(I_0, B_0)$ equals $C I_0 + B_0 B_0$. C is the vector of salvage prices. The following propositions are readily derived.

Proposition 1: For any outcome p_m , $G_1(p_m, \cdot)$ is a concave function of N_1, I_1, B_1

and D_1 .

Proof : $G_1(p_m, ..)$ is the right hand side parametric of a maximization problem with a concave objective function and linear constraints, hence $G_1(p_m, ..)$ is concave.

Proposition 2: $F_1(I_1, B_1)$ is a concave function.

Proof: Please see Lemma 1 in appendix 1.

The propositions can also be extended to the multiperiod problem.

To prove that $F_T(I_T, B_T)$ and $G_T(p_m, N_T, I_T, B_T)$ are concave functions for all T , we use an inductive argument. Assume that $F_t(I_t, B_t)$ is concave, then by proposition 1 $G_{t+1}(p_m, N_{t+1}, I_{t+1}, B_{t+1})$ is also concave. If $G_{t+1}(p_m, N_{t+1}, I_{t+1}, B_{t+1})$ is concave, then by Lemma 1 $F_{t+1}(I_{t+1}, B_{t+1})$ will be concave. Since $F_1(I_1, B_1)$ is concave Propositions 1, and 2 are valid for a multiperiod problem.

The problem on hand is an infinite horizon problem. Unfortunately, determining the solution of this problem is computationally intractable, and approximation procedures are needed. In the remainder of the paper we examine two approaches for solving the problem. In the first approach, we solve the production planning problem on a rolling horizon basis. This is a popular method, that has been employed by several researchers (eg. Morton 79, Modigliani and Hohn 55). This method entails solving finite horizon stochastic programs. In Section 5, we develop an efficient algorithm for solving such programs. The second approach we propose, is a heuristic method. In section 6, we study the optimal allocation policy in a two period problem. Based on the structure of this policy, we develop and test a heuristic for solving multiperiod problems.

5. ROLLING HORIZON APPROACH

In the rolling horizon approach, at the beginning of each period, we solve a stochastic program that determines the quantity to be produced in the current period, and the allocation under each yield outcome. There are several approaches that can be adopted. One strategy is to solve a single period problem. Although the single period problem can be solved fairly easily, the quality of the solutions provided by this

approach were found to be very poor. A reasonable alternative is to assume that yields are stochastic in the first few periods, and are deterministic (equal to their means) in subsequent periods. As the length of the horizon increases, the quality of the solution improves. On the other hand, as the length of the horizon increases, the computational burden grows very rapidly. The appropriate trade-off between the computational burden and the quality of the solution should be determined on the basis of the level of uncertainty in the yields, and the level of the inventory, backorder, and production costs. In the next sub-section we develop a decomposition procedure for finite horizon problems. The finite horizon problem is a linear program. The decomposition procedure exploits the "network" like structure of the coefficient matrix of the L.P.

5.1. Computational Procedures

We begin with a finite horizon problem with stochastic yields in the first period (FHFP) and deterministic yields in the rest. We then extend the procedure to problem with stochastic yields in multiple periods (FHMP).

The stochastic program is decomposed into two components - a master problem and a subproblem. The master problem transmits to the subproblem the production level (or production levels - in a multiperiod setting). For Example, in a two-period problem, with two yield outcomes, we have to determine N_2 , and N_1 quantities to be produced in the first and second period of the horizon, respectively. In the second period, we have to specify the quantity to be produced under each yield outcome in the first period. Hence N_1 is a vector. Given the production quantities, the subproblem determines the optimal allocations, and thereby computes the maximum expected revenues and the marginal value of altering the production level. This information is used by the master program to revise the production quantities, thereby setting up an iterative procedure. In this section we focus primarily on developing efficient procedures for solving the subproblems, and do not discuss extensively procedures for solving the master problem. Nevertheless, for the sake of completeness, we describe below one approach for solving the master problem. Note that the expected revenues are a concave function of the production level.

Let

N^k : Production decision transmitted to the subproblem at the kth iteration
(a vector for multiperiod problems)

In a T period problem $N^k = (N^k_T, N^k_{T-1}, N^k_{T-2}, \dots, N^k_2, N^k_1)$. If yields are uncertain N^k_j $j \neq T$ are vectors. In the two period example described above, N^k_1 is a vector consisting of two elements $(N^k_{1,m}; m=1,2)$, where m is the index for the yield outcomes.

$\phi(N^k)$: Optimal expected revenues when production level is N^k . For example, in a single period problem $\phi(N^k) = E_p G_1(p_m, N^k, I_1, B_1) - KN^k$

$\phi'(N^k)$: Marginal value of altering the production level from N^k ($\phi'(N^k)$ is a vector in multiperiod problems).

The hyper-plane $[\phi'(N^k)(X - N^k) + \phi(N^k)]$ is tangential to $\phi(\cdot)$ at N^k . Since $\phi(\cdot)$ is concave, for all X $\phi(X) \leq \phi'(N^k)(X - N^k) + \phi(N^k)$ (Shapiro 79). The master problem (MP) constructs a piece-wise linear envelope for the function $\phi(\cdot)$. At the kth iteration MP solves the following linear program:

$$\text{Max } (W \mid \phi'(N^q)(N^k - N^q) + \phi(N^q) \geq W ; q = 1,2,3, \dots, k-1; N^k \geq 0)$$

The decision variables in this L.P. are N^k and W . N^k is then passed on to the subproblem to determine $\phi(N^k)$ and $\phi'(N^k)$.

There are several advantages to this approach of decomposing the stochastic program. The subproblems have network like structures and can be solved efficiently, and in parallel. In the presence of positive lead times, the production decisions for the first few periods are fixed, and the production levels for the subsequent periods are to be determined. The algorithms we develop here are directly applicable to this situation.

We now describe an algorithm for solving the subproblems arising in FHFP. The multiperiod problem can be reformulated as:

$$\text{Max } \phi(N_T, N_{T-1}, N_{T-2}, \dots, N_2, N_1)$$

$$\text{S.T. } N_T, N_{T-1}, N_{T-2}, \dots, N_2, N_1 \geq 0$$

$\phi(N_T, N_{T-1}, N_{T-2}, \dots, N_2, N_1)$ is the expected maximum revenues, given that we

produce $N_T, N_{T-1}, N_{T-2}, \dots, N_2, N_1$ in periods T, T-1, T-2, ... 2,1. N_T , the production level for the first period of the horizon is a scalar, and N_t ($t \neq T$) is a

vector. In FHFP the dimension of N_t ($t \neq T$) equals the number of yield outcomes in period T; i.e., $N_t = (N_{t,1}, N_{t,2}, N_{t,3}, \dots, N_{t,M})$. Once again $\phi(\cdot)$ is a concave function.

Given the production quantities in each of the T periods, the allocation problem for each yield outcome in period T can be reformulated as a network flow problem. Figure 2

illustrates the network for a 2 period, 3 product problem. There is one such network

for each outcome in period T. In FHFP, the subproblem can be solved efficiently by algorithms developed for minimum cost flow problems (Tardos 85). However, the subproblems arising in FHMP are not network flow problems. This motivated us to develop algorithms that exploit the special structure of the subproblem, and can also solve subproblems arising in FHMP. We begin by describing an algorithm for subproblems in FHFP, and show how it can be employed in FHMP.

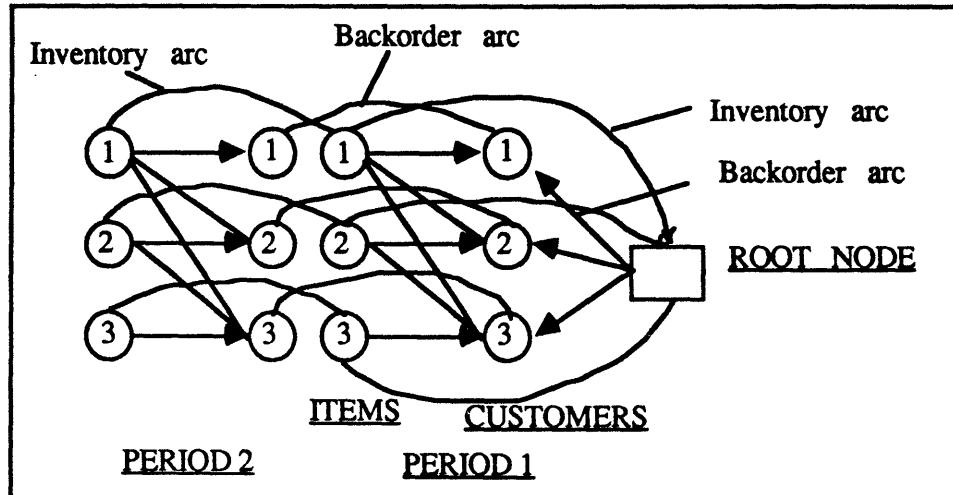


Figure 2. Allocation network for a two period problem

The network flow problem arising in FHFP is solved by algorithm A1 described below by applying it for each yield outcome in period T. This algorithm initially assumes that downgrading is not permitted, and allocates item i to customer i in periods T through 1. This assumption is gradually relaxed. First, downgrading is allowed in period 1, and sequentially downgrading is permitted in all T periods. While downgrading, in each period we begin with customer 2 and work our way down to the lowest member of the hierarchy. Appendix 1 illustrates the algorithm.

Algorithm for FHFP (A1)

Notation:

M	Number of outcomes
η	Number of customers
r^m	Probability of outcome m
$D_{t,i}$	Demand of customer i in period t
$Y_{t,i}$	Output of item i in period t
$B(i,t)$	Backorder arc for customer i from period $t-1$ to t
$C(i,t)$	Node corresponding to customer i in period t

$H(i,t)$ Inventory arc for item i from period t to period $t-1$
 $S(i,t)$ Node corresponding to item i in period t
 $J(i,t)$ Set of all downgrading arcs to $C(i,t)$ from $S(j,t)$ $j < i$

Initialization: Starting from period T , allocate item i to customer i , backorder any shortages and carry forward any inventories; i.e., solve the T period problem assuming that no downgrading is permitted.

Through out this algorithm we restrict the set of arcs on which the flows can be altered. We call these arcs admissible arcs, and denote the set of admissible arcs by ϑ . Initially $\vartheta = \phi$, (Null set).

$t \leftarrow 1, i \leftarrow 0$

Step 1: $i \leftarrow i+1$, if $i > \eta$ then $t \leftarrow t+1, i \leftarrow 1$

$\vartheta \leftarrow \vartheta \cup B(i,t) \cup H(i,t) \cup J(i,t)$

Step 2: $R(i,t) = \sum_{j=t}^T (D_{j,i} - Y_{j,i})$. If $R(i,t) \leq 0$ then go to Step 1

Step 3: We identify a set of arcs that are to be considered for inclusion; i.e., arcs that are candidates for pivoting into the basis. The candidate arcs have the following properties :

- (1) Belong to the set ϑ
- (2) Have zero flows - i.e, are not in the current basis
- (3) Increasing flow along these directed arcs, will result in an oriented cycle that causes the backorders of $C(i,t)$ to decrease.

Step 4: From the candidate arcs, pick the arc with the highest reduced cost. If this reduced cost is non-positive go to Step 1.

Step 5: Introduce the arc selected in step 4, and pivot it in - i.e., circulate flow along the cycle created. If this results in the backorders of $C(i,t)$ to go to zero, return to Step 1; else go to Step 3.

Step 1 augments the set of admissible arcs. This step is analogous to introducing new columns into the simplex tableau (Dantzig 63). Steps 3 and 4 pick the next column to be pivoted in. Proof of the optimality of this algorithm and an illustration of the algorithm are given in appendix 1. In appendix 1 (proposition 4a) we show that all arcs in ϑ with positive reduced costs are candidate arcs. This greatly simplifies steps 3 and 4. The arc that is pivoted in, is the arc with the highest positive reduced cost in the

set ϑ . A nice feature of this algorithm is that at each step, we pivot in only arcs that decrease the backorders of $C(i,t)$. In appendix 1 (proposition 3a) we show that at any step of the algorithm we have an optimal solution, provided that we are not permitted to alter flows on the arcs $\vartheta^C + B(i,t)$. ϑ^C is the set of arcs not included in ϑ . In particular, consider the situation where we have completed the downgrading process to meet the demands for all customers in period t and are about to explore downgrading options for customer 2 in period $t + 1$. If we terminate the algorithm at this step, then the allocations proposed by the algorithm are optimal, provided that we are not allowed to downgrade in periods T through $t + 1$.

For any production vector, the optimal allocations are used to determine : (i) the expected revenues, and (ii) the marginal value of altering any production quantity. Let $\pi_{i,t,m}(N^k)$ be the node potential of item i , period t , in the network corresponding to outcome m . The potential of a node is the shadow price of the constraint associated with that node (Ford and Fulkerson 62). To compute the node potentials, the potential of the root node is set at zero. $\pi_{i,t,m}(N^k)$ is the marginal value of an additional unit of item i , in period t , given that we produce N^k units, and that we observe outcome m . Let $p(i,m, t)$ be the proportion of item i obtained under outcome m in period t ($t \neq T$), then the marginal value of altering $N_{t,m}$ is $r^m[\sum_i p(i,t,m)\pi_{i,t,m}(N^k) - K]$. The marginal value of altering N_T is

$$\sum_m \{r^m[\sum_i p(i,T,m)\pi_{i,T,m}(N^k) - K]\}$$

5.1.1. Multiperiod Stochastic Yields

Let us now consider the case when yields are uncertain in more than one period. If we fix the production levels, then the problem separates into subproblems based on the yield outcome in the first period of the horizon. In the ensuing discussion, we describe the structure of these sub-problems. The coefficient matrix for the subproblem has a network structure, except for columns corresponding to inventory and backorder variables. Hence, the subproblems are not network flow problems.

Algorithm A1 can be viewed as a primal simplex method that dictates the sequence in which columns are considered for inclusion into the basis. In that sense, algorithm A1 can be directly applied to subproblems arising in FHMP. However, we have to make some modifications in the procedures employed to compute (a) reduced costs of non-

basic variables , (b) shadow price associated with the rows , and (c) the updated basis after a pivot step. For this purpose we describe the structure of the sub-problems arising in FHMP. At the end of this subsection we indicate how to change algorithm A1.

For ease of exposition, and without loss of generality , assume that we have a two period problem with two possible yield outcomes in each period. To solve this problem we must specify three production levels - one for the first period of the horizon and two in the last period (one for each outcome in the first period). We must also determine the allocations in each period under each outcome. Since there are two outcomes in the first period of the horizon, there are two subproblems. Given below is a formulation of one of the subproblems:

$$\begin{aligned} \text{Max } & S_1 A_{11} + S_2(A_{12} + A_{22}) + \sum_{i=1}^2 h_i l_{1,i} - \sum_{j=1}^2 \beta_j B_{1,j} \\ & + \sum_{m=1}^2 r^m \{ S_1 A_{11}^m + S_2(A_{12}^m + A_{22}^m) + \sum_{i=1}^2 C_i l_{0,i}^m + \sum_{j=1}^2 \beta_{0,j} B_{0,j}^m \} \end{aligned}$$

S.T.

$$\begin{aligned} (1') \quad & A_{11} + A_{12} + l_{1,1} & = & N_2 p^1_1 \\ (2') \quad & A_{22} + l_{1,2} & = & N_2 p^2_1 \\ (3') \quad & -A_{11} - B_{1,1} & = & -D_{2,1} \\ (4') \quad & -A_{12} - A_{22} - B_{1,2} & = & -D_{2,2} \\ (5') \quad & A_{11}^m + A_{12}^m + l_{0,1}^m - l_{1,1} & = & N^1_1 p^1_m \quad m = 1,2 \\ (6') \quad & A_{22}^m + l_{0,2}^m - l_{1,2} & = & N^1_1 p^2_m \quad m = 1,2 \\ (7') \quad & -A_{11}^m - B_{0,1}^m + B_{1,1} & = & -D_{1,1} \quad m = 1,2 \\ (8') \quad & -A_{12}^m - A_{22}^m - B_{0,2}^m + B_{1,2} & = & -D_{1,2} \quad m = 1,2 \\ (9') \quad & - \sum_{i=1}^2 l_{0,i}^m + \sum_{j=1}^2 B_{0,j}^m & = & 2 \sum_{t=1}^2 \sum_{i=1}^2 D_{t,i} - N^1_1 \sum_{m=1}^2 (p^1_m + p^2_m) - 2 N_2 (p^1_1 + p^2_1) \end{aligned}$$

Nonnegativity Constraints

The last constraint is a redundant constraint (dummy). The shadow price of this constraint is set to zero. Observe that this formulation has a network structure, except for the columns corresponding to $l_{1,1}$, $l_{1,2}$, $B_{1,1}$, and $B_{1,2}$. The elements in the inverse of every basis for this problem are either 0, -1 or +1. Because of this structure of the coefficient matrix, the pivot operations, and the computation of the reduced costs of non-basic variables are considerably simplified. In particular, the reduced cost of a non-basic variable can be computed quite simply from the shadow

prices of the rows that the variable is associated with. For example, the reduced cost of variable A_{12} is $\pi_4 + S_2 - \pi_1$. π_i is the shadow price of constraint (i). It is convenient to visualize this problem as a network flow problem (with a few distortions) by associating directed arcs with variables and nodes with rows. With this abuse of notation, algorithm A1 can be used to solve FHMP.

Although the multiperiod problem can be solved as a system of 'network-like' problems, the real difficulty arises from the rapid growth in the size of the subproblems.

6. DERIVATION OF DECISION RULE

Until now we have concentrated on solving stochastic programs, either exactly or approximately. Our initial reaction was to solve the stochastic programs. However we found that the managers were using simple intuitive rules. This motivated us to search for decision rules that managers would feel comfortable with and which would perform similarly to finite horizon problems. In this section we examine the allocation process and derive the structure of the optimal allocation policy for a two period, two product problem. Based on this structure, we propose and test simple heuristics for determining production quantities and managing the allocation process in multiperiod problems.

6.1 Item Allocation

Consider a two product, one period problem. In a single period setting, the allocation process is straightforward. Let $S_1 + \beta_1 \geq S_2 + \beta_2$, and $C_1 \geq C_2$. The first inequality, ensures that the cost of foregoing a sale to customer 1, is greater than that for customer 2. The second inequality requires the salvage price of item 1 to be higher than that for item 2. We also require $S_j + \beta_j \geq C_j$. Consequently it is suboptimal to backorder demand for customer j and salvage item j . Under these conditions it is optimal to allocate item #1 to customer 2 if, (i) item #1 yield exceeds customer 1's demand, and item #2 yield is less than customer 2's demand; and (ii) $S_2 + \beta_2 \geq C_1$. Henceforth, we assume that $S_2 + \beta_2 \geq C_1$.

Next, consider the allocation process in the penultimate period of a two period

problem. In our model, costs and sale prices do not change from period to period. Hence we do not backorder demand from customer j and hold inventories of item j . Again, the only case that is interesting is when we have an excess of item 1 and a shortage of item 2. In a single period problem if it is optimal to downgrade item #1, then we downgrade until either customer 2's demand is fully satisfied, or we run out of item 1. In general, this is not the case in a two period problem. We may downgrade only some of the excess, and start the final period with item 1 inventory and backorders of customer 2.

In the penultimate period let the yields for item 1 and 2, be x and y , respectively. Assume $x > D_{2,1}$ and $y < D_{2,2}$. Define $\epsilon = x - D_{2,1}$, and $U = D_{2,2} - y$. After we allocate item 1 to customer 1, and item 2 to customer 2, we are left with ϵ units of item 1 and U units of unsatisfied customer 2 demand. At this point, every unit of item 1 allocated to customer 2 has the following consequences:

- (1) Increase period 2 profits by $S_2 + \beta_2 + h_1$
- (2) Inventory of item 1 will decrease by 1 unit
- (3) Backorders of customer 2 will decrease by 1 unit.
- (4) As a result of (2) and (3), period 1 profit will change from $F_1(\epsilon, 0, 0, U)$ to $F_1(\epsilon-1, 0, 0, U-1)$.

Since $F_1(\dots)$ is a concave function it is optimal to downgrade so long as the marginal increase in the current period's profits exceeds the marginal decrease in the final period's profits. If we are to say anything more about the down-grading process it is necessary to understand the structure of $F_1(\cdot, \cdot)$. Fortunately, it has an intuitively appealing structure. In the next sub-section we describe the structure of $F_1(\cdot, \cdot)$.

6.2 Structure of $F_1(L, B)$

Throughout this sub-section we assume:

- (1) $I_{1,j} \cdot B_{1,j} = 0$ for $j = 1, 2$. This is consistent with the observation that in the penultimate period it is not optimal to backorder demand for customer j and carry item j inventory.
- (2) $I_{1,j} \leq D_{1,j}$ for $j = 1, 2$. The other cases are neither interesting nor useful to investigate.

We also make a minor modification in the notation by including the final periods' demand into the arguments of the function $F_1(\cdot, \cdot)$. The definition of the function

otherwise remains the same as in section 4.

We first show how to transform a problem with initial inventories and backorders to one without any inventories or backorders (Proposition 3). This permits us to restrict our attention to problems without any starting inventories or backorders. Next, we examine $F_1(D_{1,1}, D_{1,2}, 0, 0)$ over the domain $D_{1,1} \geq 0$ and

$D_{1,2} \geq 0$. In Proposition 4 we establish that doubling demand causes $F_1(\cdot, \cdot, 0, 0)$ to double. This shows that in the $(D_{1,1}, D_{1,2})$ space, along any ray emanating from the origin, $F_1(D_{1,1}, D_{1,2}, 0, 0)$ is linear. The structure of $F_1(D_{1,1}, D_{1,2}, 0, 0)$ is derived in Proposition 5.

Proposition 3:

If $I_{1,j} \leq D_{1,j}$ for $j = 1, 2$ then:

$$F_1(D_{1,1}, D_{1,2}, I_{1,1}, I_{1,2}, B_{1,1}, B_{1,2}) = S_1 I_{1,1} + S_2 I_{1,2} + F_1(D_{1,1} - I_{1,1} + B_{1,1}, D_{1,2} - I_{1,2} + B_{1,2}, 0, 0)$$

Proof: Appendix 2.

Proposition 4

Let $\mu \geq 0$, then $F_1(\mu D_{1,1}, \mu D_{1,2}, 0, 0) = \mu F_1(D_{1,1}, D_{1,2}, 0, 0)$

Proof: Appendix 2

We know that $F_1(D_{1,1}, D_{1,2}, 0, 0)$ is concave. Since the number of yield outcomes is finite, the morning problem can be formulated as a finite linear program. Thus $F_1(D_{1,1}, D_{1,2}, 0, 0)$ is piece-wise linear concave. The number of linear regions is finite. Each region corresponds to a basis. An immediate consequence of Proposition 4 is that, in the $(D_{1,1}, D_{1,2})$ space, along any ray emanating from the origin, $F_1(D_{1,1}, D_{1,2}, 0, 0)$ is linear. Therefore, the boundary of each linear region of $F_1(D_{1,1}, D_{1,2}, 0, 0)$ must be a ray emanating from the origin. Thus we have shown:

Proposition 5:

There exists a sequence of numbers $0 \leq \pi_1 \leq \pi_2 \leq \pi_3 \leq \dots \leq \pi_n \leq \infty$ such that for $\pi_j \leq D_{1,1} / D_{1,2} \leq \pi_{j+1}$, $F_1(D_{1,1}, D_{1,2}, 0, 0)$ is linear.

Figure 3 illustrates the structure of $F_1(D_{1,1}, D_{1,2}, 0, 0)$. The domain is the non-negative orthant of $(D_{1,1}, D_{1,2})$. Inside each region $F_1(D_{1,1}, D_{1,2}, 0, 0)$ is linear. Over the entire domain it is piece-wise linear concave.

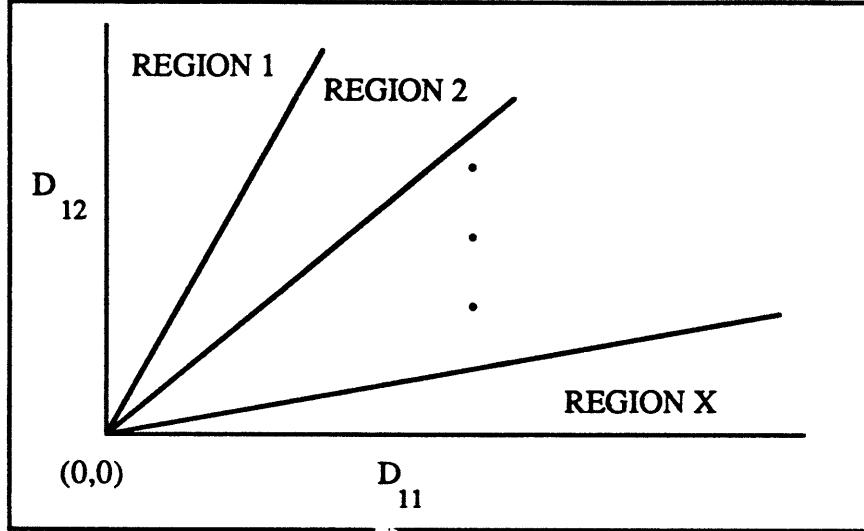


Figure 3. Structure of $F_1(D_{1,1}, D_{1,2}, 0, 0)$

6.3 Item Allocation in the Penultimate Period

Recall that after we allocate item 1 to customer 1, and item 2 to customer 2, we are left with ϵ units of item 1 and U units of unsatisfied customer 2 demand. Let Δ denote the net change in objective value resulting from down-grading one unit of item 1. Then

$$\Delta = S_2 + \beta_2 + h_1 + F_1(D_{1,1}, D_{1,2}, \epsilon, 0, 0, U) - F_1(D_{1,1}, D_{1,2}, \epsilon-1, 0, 0, U-1)$$

On the basis of Proposition 3

$$\Delta = S_2 + \beta_2 + h_1 + S_1 + F_1(D_{1,1} - \epsilon, D_{1,2} + U, 0, 0) - F_1(D_{1,1} - \epsilon + 1, D_{1,2} + U - 1, 0, 0)$$

Let $D^*_{1,1}$ and $D^*_{1,2}$ be the net final period demand; i.e., $D^*_{1,1} = D_{1,1} - \epsilon + \Omega$ and $D^*_{1,2} = D_{1,2} + U - \Omega$. Ω is the number of item 1s downgraded in the penultimate period. By down-grading we are increasing the ratio $D^*_{1,1} / D^*_{1,2}$. Clearly, it is optimal to downgrade so long as $\Delta > 0$. The magnitude of Δ will change whenever the down-grading process moves the net demands from one linear region of the function $F_1(D^*_{1,1}, D^*_{1,2}, 0, 0)$ to the next. By proposition 5, this happens every time $D^*_{1,1} / D^*_{1,2}$ exceeds π_j , for some j . If Δ becomes negative then we stop down-

grading. This leads us to the following result:

Proposition 6:

There exists a non-negative number π^* (perhaps = 1) such that it is optimal to continue to downgrade if and only if the following conditions are satisfied:

- (1) $D^*_{1,1} / (D^*_{1,2} + D^*_{1,1}) \leq \pi^*$;
- (2) $\varepsilon - \Omega \geq 0$; and
- (3) $U - \Omega \geq 0$.

In this problem the mix of the output is random. A priori, we do not know the proportion of each item, that we will get in the final product. The likelihood of observing any specific proportion is independent of the quantity produced. Therefore, it is not surprising that in a single period problem, doubling of the demand results in the optimal solution being doubled. Given the nature of yield uncertainty, Proposition 6 is intuitive. Down-grading alters the proportions of the net demand by increasing the proportion of item 1's that are required in the final period. Thus, it is reasonable to stop down-grading if the proportion of item 1's that will be needed increases beyond some critical value π^* . Although the nature of the optimal down-grading policy is intuitive, π^* is difficult to determine. It depends on the cost parameters and the yield distribution. Nevertheless, Proposition 6 is insightful and is the basis for the heuristic procedures we propose.

We have determined the nature of the optimal down-grading policy for a two-product two-period problem. The results extend to a multiproduct problem. Unfortunately, this policy is not necessarily optimal if the number of periods exceeds 2. The result extends to a multiperiod problem only if the yields are deterministic except for period T. With stochastic yields the down-grading process is complex and can not be easily characterized.

In view of the difficulty in describing the optimal down-grading policy for multiple periods we propose an heuristic .

6.4 Heuristic

The heuristic consists of two modules. It first determines the quantity to be

produced, and on the basis of the actual output, it allocates the items to the customers. The production level is determined so as to ensure that the demands are met with some probability; i.e., meet a service level constraint. The allocation process ensures that in the next period, the demands for the items being downgraded do not exceed a pre-determined proportion of the total demand.

Let

y_i : Fraction of item i in the output ; a random variable

$F_i(x)$: Probability ($\sum_{j=1}^i y_j \leq x$)

f_i : Max ($x \mid F_i(x) \leq \alpha_i$)

$D^*_{T,j}$: Net demand in period T for item j (could be negative)

Production Level: The service level constraint for customer i is given by

$$\text{Probability } (N(\sum_{j=1}^i y_j) \geq \sum_{j=1}^i D^*_{T,j}) \geq 1 - \alpha_i \text{ for } i = 1 \text{ to } \eta$$

α_i is a parameter of the heuristic. This definition of the service level implicitly assumes that all surpluses are downgraded. Several heuristics can be designed, by changing the definition of the service level. The parameter α_i , should depend on the holding and backorder costs. One simple approach is to set α_i equal to $h_i / (\beta_i + h_i)$ (as in a newsboy problem).

In order to satisfy all the service constraints, the production quantity N must satisfy:

$$N \geq (\sum_{j=1}^i D^*_{T,j}) / f_i \text{ for } i = 1 \text{ to } \eta$$

Therefore the production level is set at $\text{Max}_i ((\sum_{j=1}^i D^*_{T,j}) / f_i)$.

Allocation : The output is allocated to the customers in the following manner. First, to customer "i" we allocate item "i". Then, starting from customer "2", we see if downgrading is required. While exploring downgrading options for customer "i", we begin with item "i-1", and go up in the hierarchy. In period T , we continue to allocate excess of item k to customer $k+i$ so long as,

$$\sum_{j=1}^w D^*_{T-1,j} / D^*_{T-1} \leq f_w \quad \text{for } w = k \text{ to } k+i-1.$$

In this expression $D^*_{T-1,j}$ is the effective or net demand in period T-1. $D^*_{T-1,j} = D_{T-1,j} - I_{T-1,j} + B_{T-1,j}$. Note that $D^*_{T-1,j}$ depends on the allocations made in period T. D^*_{T-1} is the total net demand in period T-1, after allocating item i to customer i in

period T but before downgrading. $D^*_{T-1} = \max_{k=1, \dots, n} \left\{ \sum_{j=1}^k (D_{T-1,j} - I^*_{T,j} + B^*_{T,j}) \right\}$, where $I^*_{T,j}$ and $B^*_{T,j}$ are the inventory and backorder, respectively in period T for item j, after allocating item i to customer i but, prior to downgrading. If $D^*_{T-1} = 0$ then no downgrading is to be considered. The allocation policy is consistent with the production rule. It ensures that the cumulative requirements of customers 1 through i, as a fraction of the total demand, do not exceed f_i .

We have tested this heuristic on a set of problems that were randomly generated. Appendix 3 contains the problem descriptions. In order to assess the performance of this simple rule, we implemented the heuristic, and the stochastic program on a rolling horizon basis. Each period we computed the decision variables using: (i) the heuristic and (ii) the stochastic program. The stochastic program had a horizon of five periods. The yield outcome was randomly generated. We simulated these rules over a 5000 period (approximate) horizon and compared the average revenues (per period) obtained under these two decision rules. In our preliminary tests the heuristic dominated the stochastic program with stochastic yields only in the first period of the horizon. Hence, in the test results reported here, the stochastic program has uncertain yields in the first and the second period of the horizon. In 4 of the 10 problems, the heuristic provided better solutions than the stochastic program. On an average, the performance of the stochastic program was 2.7% better than that of the heuristic. The percentage difference ranged between - 7.2% and 10.8%. In view of the simplicity of the heuristic, these results are encouraging.

TABLE 2
COMPARISON OF THE HEURISTIC AND THE STOCHASTIC PROGRAM

	<u>Average Revenue Per Period</u>		
	Heuristic	Stochastic Program	% Difference
1)	3097.21	3158.58	1.9
2)	2005.25	2239.09	10.5
3)	2803.44	2913.92	3.8
4)	1551.65	1532.59	-1.2
5)	2775.64	2762.50	- 0.5
6)	3728.07	3478.50	- 7.2
7)	3268.72	3.170.40	- 3.1
8)	3374.41	3609.43	6.5
9)	3357.28	3623.99	7.4
10)	318.92	349.70	8.8

For any problem instance, the performance of the heuristic can be improved by altering the parameters (f_i). This can be done by testing the performance of the policy (via simulation) for different values of f_i . Several variants of this heuristic can be constructed by altering the definition of the service levels, and the horizon over which we try to keep the demands in balance. In the heuristic stated above, we are only concerned with the net demands in the next period.

7. Summary and Conclusions

In this paper we have modelled a production planning problem arising in the semiconductor industry. In this environment, yields vary significantly and demands are substitutable. In addition, from each production lot we get several different items. We have formulated the problem as a convex stochastic program, and showed that the coefficient matrix of the linear program describing this problem has a special block diagonal structure. Each of these blocks can be reformulated as "network-flow" problems. Based on this observation, we proposed a decomposition procedure for solving the finite horizon problem and developed algorithms for solving the embedded "networks".

The computational burden grows very rapidly as the length of the horizon grows. Hence we proposed two approximation procedures . The first approach, is a rolling horizon procedure. In each period a finite horizon problem is solved. Further, we approximate the finite horizon problem by assuming that yields, in periods other than the first few periods of the horizon, are deterministic.

The second approach is a heuristic. On the basis of the structure of a two period problem, we have identified a class of heuristics for allocating items to customers. The heuristics allocate items to customers in a manner that keeps the net demand for the items in balance. The objective of the allocation heuristic, is not to allow the net demand for any item to exceed some (predetermined) fraction of the total demand for all the items. This simple heuristic approach compared well with the approach of solving stochastic programs on a rolling horizon basis.

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APPENDIX 1

Proposition 2 claims $F_1(l_1, B_1)$ to be a concave function.

We prove the result for a more general setting.

Lemma 1. Let $h(\mu) = \text{Max} \{L(\mu, N) - cN \mid N \geq 0\}$. If $L(\dots)$ is a concave function then $h(\cdot)$ is also a concave function.

Proof: Let $\mu^* = \beta\mu' + (1-\beta)\mu''$. Where $0 < \beta < 1$. Let N^*, N', N'' be optimal for the problem with $\mu^*, \mu',$ and μ'' respectively.

$$h(\mu^*) = L(\mu^*, N^*) - cN^* \geq L(\mu^*, \beta N' + (1-\beta)N'') - c(\beta N' + (1-\beta)N'').$$

By concavity of $L(\dots)$ we have

$$\begin{aligned} & L(\mu^*, \beta N' + (1-\beta)N'') - c(\beta N' + (1-\beta)N'') \\ \geq & \beta \{L(\mu', N') - cN'\} + (1-\beta) \{L(\mu'', N'') - cN''\} \\ = & \beta h(\mu') + (1-\beta) h(\mu''). \end{aligned}$$

Hence

$$h(\beta\mu' + (1-\beta)\mu'') = h(\mu^*) \geq \beta h(\mu') + (1-\beta)h(\mu''). \quad \bullet$$

ALGORITHMS A1

We begin by illustrating A1. Consider a 2 period, 3 product problem with the following data for items 1, 2, and 3 respectively. Selling price (10, 6, 5); Holding cost (0.5, 0.5, 0.5); Backorder cost (1, 0.6, 0.5); Salvage values (3, 2.5, 2); Stationary Demand (500, 300, 350), Output in 1st period (500, 350, 200), Output in 2nd period (450, 351, 250). The flows after this initial step are shown in figure 1.a

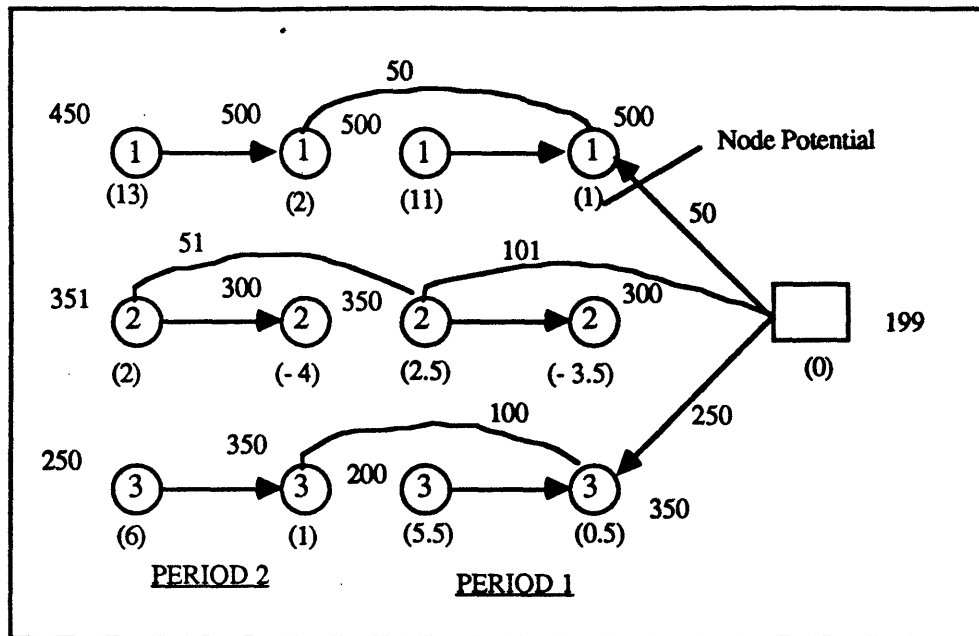


Figure 1a. Flows after the initial step

The algorithm then explores downgrading options starting from customer 2 in period 1. The first opportunity arises with customer 3 in period 1. At this point the set of arcs on which flows can be altered: δ = all arcs that are not connected to a node in the second period. The candidate arcs are S(1,1) to C(3,1) and S(2,1) to C(3,1). The reduced cost of arc S(2,1) to C(3,1) is the highest and equals +3. The network, after pivoting in the new arc, is given in figure 2.a .

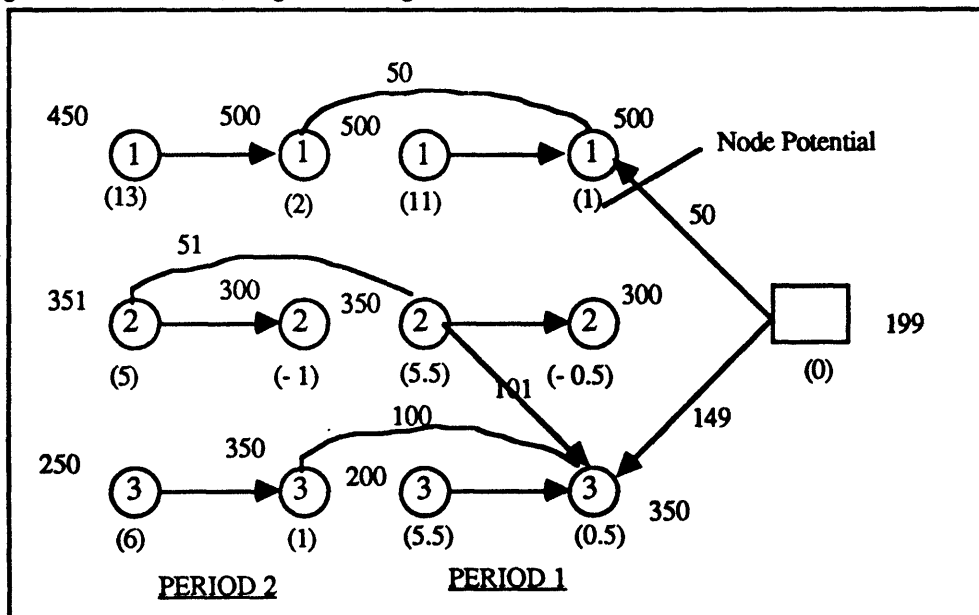


Figure 2a. Allocations after the first pivot

In the updated graph the only candidate arcs are: (i) root node to C(2,1) (backordering 2), and (ii) S(1,1) to C(3,1). The reduced cost of both these arcs is negative. Therefore, the algorithm proceeds to period 2. The allocations at the termination of A1 are given in figure 3.a.

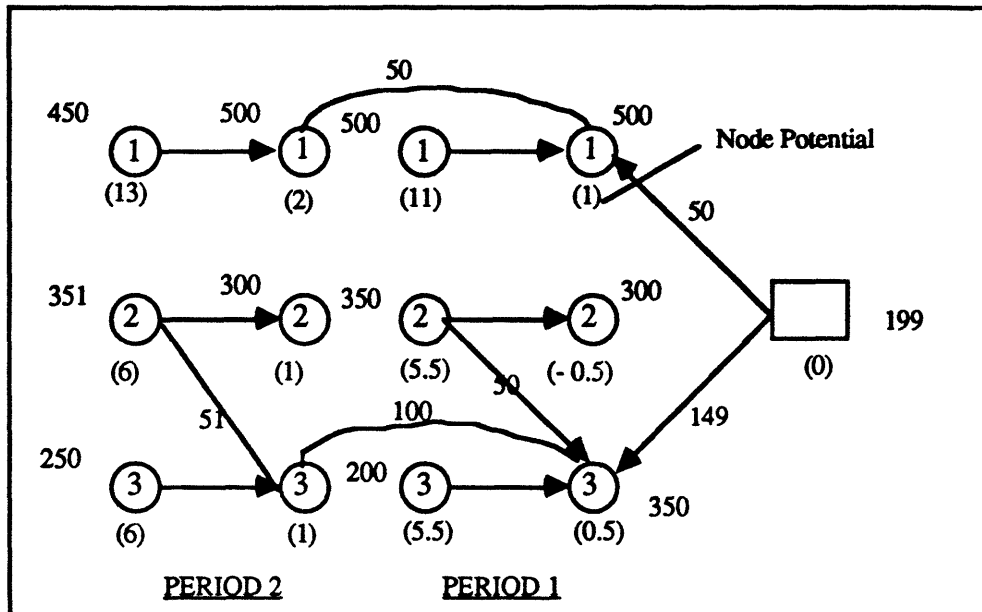


Figure 3a. Optimal allocations

The optimality of the algorithm is established by an inductive argument. We show that at any stage of the algorithm all positive cycles involving the arcs in \mathcal{A} , will include the downgrading arc of the customer being considered at that step. Next we show that the orientation of a positive cycle, should one exist, is such that it results in a decrease in the backorders of the customer being considered. Since the algorithm terminates with the lowest member of the hierarchy in period T, and at that point the set \mathcal{A} includes all arcs, A1 must find an optimal solution. The following proposition facilitate the task on hand.

Proposition 1A: Let A be a subset of the arcs of the allocation network. Suppose we fix the flows on the arcs in A at some value X. Let $G(A,X)$ denote the allocation problem so modified, and $Z(A,X)$ the optimal value (objective value) of $G(A,X)$. Then $Z(A,X)$ is a concave function of X.

Proof: $G(A,X)$ is a linear program. Altering X corresponds to altering the right hand

side of this linear program. For instance, if we change the flow on one arc in A by d , then the right hand side of two constraints in $G(A,X)$ will change. One will increase by d , and the other will decrease by d . Since the objective value of a linear program is a concave function of its right hand side (assuming that it is a maximization program), $Z(A,X)$ is a concave function of X .

Proposition 2A: Assume that the initial step of allocating item "i" to customer "i" in periods T through 1 is carried out. Consider the immediate next step in which we are trying to meet the demands of customer 2 in period 1. At this point ϑ includes all arcs between the following nodes $C(1,1)$, $C(2,1)$, $S(1,1)$, $S(2,1)$, root node. Figure 4a. depicts the corresponding subgraph. If there exists a positive cycle involving only arcs in ϑ , then the cycle will have the following properties:

Property 1: It will include the backorder arc of $C(2,1)$,

Property 2: The orientation of this cycle will be such that it results in the decrease in the backorders of $C(2,1)$.

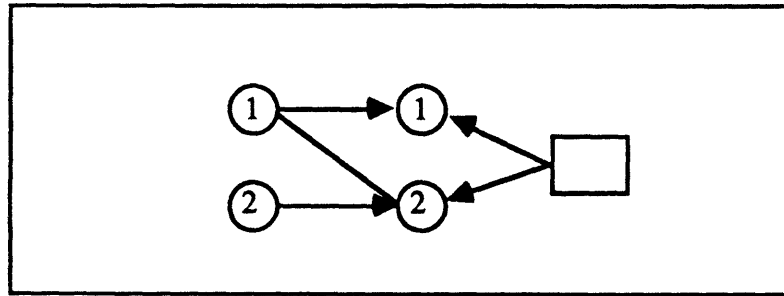


Figure 4a. Illustration of the set ϑ

Proof: The set ϑ restricts us to a two product, single period problem. If customer 2's demand is met then there is no positive cycle involving only the arcs in ϑ . If the demand of customer 2 is backordered, we only need to check if item 1 is to be allocated to customer 2. This proves the proposition and establishes the basis for an inductive argument.

Proposition 3A: Assume that

(i) the algorithm is in period t , and is attempting to reduce the backorders of customer j .

(ii) there exists candidate arcs with positive reduced costs, and that we pivot in the arc with highest reduced cost, and

(iii) prior to the pivot step properties 1 and 2 are not violated. (Property 1 and 2 are as stated above with $C(2,1)$ replaced by $C(j,t)$).

Then, after the pivot step properties 1 and 2 continue to be true.

Proof: Let c' be the arc that is pivoted in, and let the corresponding cycle be E . Consider the situation after we pivot in c' . Assume that there is an arc c'' belonging to ϑ , with positive reduced cost, whose inclusion into the graph results in a positive cycle D completely contained in ϑ . Further assume that in violation of property 1, D does not include $B(j,t)$. Then D must include c' , and the orientation of D should be such that it decreases the flow on c' . If this is not the case then D would have been feasible prior to the pivot step. This violates the assumption that property 1 is valid prior to the pivot step. Hence if D does not include $B(j,t)$, then it must reduce the flow on c' . We now show that if D does not include $B(j,t)$, but includes c' , then c' was not the candidate arc with the highest reduced cost.

Let us briefly return to the situation prior to the inclusion of c' . Consider the oriented cycle $(D \setminus c') \cup (E \setminus c')$. This is a positive cost cycle that results in a decrease of flow on the arc $B(j,t)$. Therefore, c'' was a candidate arc. Since D is a positive cost cycle that reduces the flow on c' , the value of cycle $(D \setminus c') \cup (E \setminus c')$ must have exceeded that of E . This in turn implies that the reduced cost of c'' must have exceeded that of c' . But c' was chosen as the candidate arc with the highest reduced cost. Hence, cycle D must include $B(j,t)$. This proves that property 1 is retained by the graph after we pivot in c' .

Property 1 implies that the allocations are optimal for $L(\vartheta^C + B(j,t), X)$. Hence, before and after we pivot in c' , we have an optimal solution for $L(\vartheta^C + B(j,t), X)$. Inclusion of c' , alters X . By proposition 1A, we know that $Z(\vartheta^C + B(j,t), X)$ is a piece-wise linear concave function of the flow on $B(j,t)$. If prior to the inclusion of c' , reducing the flow on $B(j,t)$ caused $Z(\vartheta^C + B(j,t), X)$ to increase, then after a single pivot, an increase in the flow on $B(j,t)$ can not cause $Z(\vartheta^C + B(j,t), X)$ to increase. The orientation of D must be such that it results in a decrease in the backorders of customer j in period t . This completes the proof of the proposition.

Combining propositions 2a and 3a, and noting that when the algorithm terminates ϑ^C is empty, shows that algorithm A1 is optimal.

Let c be an arc that belongs to ϑ , but is not in the current basis. The structure

of our graph is such that inclusion of c will result in a cycle that is entirely contained in \mathcal{C} . Combining this fact with the proposition 4a we conclude :

Proposition 4a: All arcs in \mathcal{C} with positive reduced costs are candidate arcs.

This greatly simplifies the task of selecting an arc for inclusion into the graph. We have to pick that arc in \mathcal{C} which has the highest reduced cost (provided its reduced cost is positive).

APPENDIX 2

Proposition 3:

If $l_{1,j} \leq D_{1,j}$ for $j = 1, 2$ then:

$$F_1(D_{1,1}, D_{1,2}, l_{1,1}, l_{1,2}, B_{1,1}, B_{1,2}) = S_1 l_{1,1} + S_2 l_{1,2} + F_1(D_{1,1} - l_{1,1} + B_{1,1}, D_{1,2} - l_{1,2} + B_{1,2}, 0, 0)$$

Proof:

Let :

$$D^*_{1,1} = D_{1,1} - l_{1,1} + B_{1,1}$$

$$D^*_{1,2} = D_{1,2} - l_{1,2} + B_{1,2}$$

$$A^*_{11} = A_{11} - l_{1,1}$$

$$A^*_{22} = A_{22} - l_{1,2}$$

Since $l_{1,j} \leq D_{1,j}$ for $j = 1, 2$; $N_1 p^1_m \geq 0$ and $N_1 p^2_m \geq 0$; in the optimal solution to the evening problem $A_{11} \geq l_{1,1}$ and $A_{22} \geq l_{1,2}$. As a result $A^*_{11} \geq 0$ and $A^*_{22} \geq 0$.

Therefore, we can reformulate the evening problem as follows:

$$G_1(p_m, N_1, D_{1,1}, D_{1,2}, l_1, B_1) =$$

$$\text{Max } S_1(A^*_{11} + l_{1,1}) + S_2(A_{12} + A^*_{22} + l_{1,2}) + \sum_{i=1}^2 C_i l_{0,i} - \sum_{j=1}^2 \beta_j B_{0,j}$$

S.T.

$$(2') \quad A^*_{11} + A_{12} + l_{0,1} = N_1 p^1_m$$

$$(3') \quad A^*_{22} + l_{0,2} = N_1 p^2_m$$

$$(4') \quad A^*_{11} + B_{0,1} = D^*_{1,1}$$

$$(5') \quad A_{12} + A^*_{22} + B_{0,2} = D^*_{1,2}$$

$$(6') \quad A^*_{11}, A_{12}, A^*_{22}, B_{0,1}, B_{0,2}, l_{0,1}, l_{0,2} \geq 0 \quad \text{NON-NEGATIVITY}$$

Thus:

$$G_1(p_m, N_1, D_{1,1}, D_{1,2}, l_1, B_1) = S_1 l_{1,1} + S_2 l_{1,2} + G_1(p_m, N_1, D^*_{1,1}, D^*_{1,2}, 0, 0)$$

Consequently:

$$F_1(D_{1,1}, D_{1,2}, l_{1,1}, l_{1,2}, B_{1,1}, B_{1,2}) = S_1 l_{1,1} + S_2 l_{1,2} + F_1(D_{1,1} - l_{1,1} + B_{1,1}, D_{1,2} - l_{1,2} + B_{1,2}, 0, 0)$$

Proposition 4

Let $\mu \geq 0$, then $F_1(\mu D_{1,1}, \mu D_{1,2}, 0, 0) = \mu F_1(D_{1,1}, D_{1,2}, 0, 0)$

Proof: Let $\mu A^*_{ij} = A_{ij}$, $\mu l^*_{0j} = l_{ij}$, $\mu B^*_{0j} = B_{0j}$, $\mu N^*_1 = N_1$.

By following the steps employed in Proposition 3 we can show that :

$$G_1(p_m, \mu N_1, \mu D_{1,1}, \mu D_{1,2}, 0, 0) = \mu G_1(p_m, N_1, D_{1,1}, D_{1,2}, 0, 0).$$

Therefore:

$$F_1(\mu D_{1,1}, \mu D_{1,2}, 0, 0) = \mu F_1(D_{1,1}, D_{1,2}, 0, 0) \quad \cdot$$

APPENDIX 3

TEST PROBLEMS FOR THE HEURISTIC

There are ten problem sets. In all the test problems the number of yield outcomes is 5 and the number of items is 3. Associated with each outcome are 4 numbers, the first number is the probability of observing that outcome, and the other three numbers are the yields of items 1, 2 and 3 respectively. The demand was assumed to be stationary. The three numbers listed as demands are the demands of items 1,2, and 3 respectively. The selling price, holding cost, and the backorder cost are also listed in that manner.

- # 1 Yield Outcomes : (.143, .132, .330, .072) ; (.498, .336, .124, .305) ; (.202, .31, .445, .199)
(.029, .245, .216, .150) ; (.128, .332, .058, .449)
Demand (200, 275, 320) ; Selling Price (15, 12, 10) ; Backorder cost (3, 2.4, 2) ; Holding cost (1,1,1) ; Production cost 6
- # 2 Yield Outcomes : (.050, .376, .403, .015) ; (.058, .277, .264, .270) ; (.275, .003, .158, .628)
(.332, .184, .174, .614) ; (.285, .516, .023, .179)
Demand (200, 150, 490) ; Selling Price (15,12,10) ; Backorder cost (3, 2.4, 2) ; Holding cost (1,1,1) ; Production cost 6
- # 3 Yield Outcomes : (.273, .295, .266, .181) ; (.218, .049, .330, .226) ; (.288, .193, .235, .242)
(.107, .147, .510, .053) ; (.114, .478, .030, .113)
Demand (175, 300, 250) ; Selling Price (12,10, 9) ; Backorder cost (1.2, 1, .9) ; Holding cost (.6, .6, .6) ; Production cost 4
- # 4 Yield Outcomes : (.017, .236, .086, .611) ; (.071, .209, .209, .411) ; (.304, .059, .203, .440)
(.441, .137, .320, .234) ; (.167, .249, .124, .152)
Demand (110, 240, 350) ; Selling Price (12,10, 9) ; Backorder cost (1.2, 1, .9) ; Holding cost (.6, .6, .6) ; Production cost 5
- # 5 Yield Outcomes :(.159, .540, .277, .029) ; (.215, .399, .255, .271) ; (.286, .261, .378, .344)
(.164, .191, .386, .150) ; (.175, .471, .160, .249)
Demand (300, 300, 230) ; Selling Price (10, 8, 6) ; Backorder cost (1, .8, .6) ; Holding cost (.5, 5, .5) ; Production cost 4
- # 6 Yield Outcome : (.082, .414, .140, .066) ; (.130, .372, .336, .087) ; (.231, .380, .159, .315)
(.236, .258, .303, .268) ; (.321, .370, .339, .244)
Demand (350, 300, 300) ; Selling Price (15,10, 8) ; Backorder cost (3, 2, 1.6) ; Holding cost (.5, .5, .5) ; Production cost 6
- # 7 Yield Outcome : (.260, .367, .230, .330) ; (.174, .239, .338, .171) ; (.215, .137, .024, .384)
(.193, .098, .129, .429) ; (.158, .162, .407, .430)
Demand (200, 250, 400) ; Selling Price (15, 12,10) ; Backorder cost (3, 2.4, 2) ; Holding cost (1, 1, 1) ; Production cost 5
- # 8 Yield Outcome : (.138, .082, .311, .327) ; (.374, .309, .026, .244) ; (.041, .133, .359, .368)
(.123, .137, .132, .435) ; (.324, .160, .212, .345)
Demand (200, 200, 385) ; Selling Price (20, 15, 10) ; Backorder cost (4, 3, 2) ; Holding cost (1.5, 1.5, 1.5) ; Production cost 5.5
- # 9 Yield Outcome : (.200, .441, .010, .385) ; (.011, .128, .394, .342) ; (.248, .053, .314, .395)
(.229, .263, .080, .246) ; (.312, .178, .360, .396)
Demand (200, 215, 450) ; Selling Price (20, 15, 12) ; Backorder cost (2, 1.5, 1.2) ; Holding cost

(1, 1, 1); Production cost 7.5

10 Yield Outcome : (.177, .234, .174, .265); (.145, .087, .580, .235); (.360, .276, .353, .326)
(.237, .398, .098, .171); (.081, .250, .304, .196)
Demand (250, 350, 300); Selling Price (15,12,10); Backorder cost (3, 2.4, 2); Holding cost
(1.5, 1.5, 1.5); Production cost 9