

**Distribution-Free, Uniformly-Tighter
Linear Approximations for
Chance-Constrained Programming[†]**

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Distribution-Free, Uniformly-Tighter Linear Approximations[†]
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Abstract

We propose a class of deterministic linear approximations to mathematical programs with individual chance constraints. In these approximations, we replace each chance constraint by a set of linear inequalities. The linear inequalities approximate the chance constraint as opposed to methods that approximate the variance of the chance constraint. This allows us to provide near-optimal solutions of chance-constrained problems that have random variables with arbitrary distributions; the random variables can be dependent.

If the feasible region defined by a chance constraint is convex, we show that the linear approximation is uniformly tighter than the constraint it replaces: any solution to the approximation is feasible to the original constraint. At the cost of doing more work, the number of linear inequalities can be infinitely increased to make the approximation exact. As examples, we present three simple methods from this class of linear approximations. The computation results show that these approximations work well when the number of random variables is small. The methods can potentially be applied to production problems where planning is done on a rolling horizon basis.

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1. Introduction

In this paper, we present a class of linear approximations to chance-constrained problems where the random variables have arbitrary distributions. We focus on chance-constrained linear programs (LP) with stochastic technology coefficients--coefficients on the left-hand-side of a constraint. However, the method can also be applied to mathematical programs that have chance-constrained nonlinear inequalities with stochastic or deterministic resource parameters, the right-hand-side parameter of a constraint. These problems appear in service constrained applications that have uncertainties in the yield or demand. The broad classification of these applications are problems (a) with carry-over resources (inter-period constraints), (b) with portfolio selection (intra-period constraints), and (c) with both carry-over resources and portfolio selection.

Applications of the first type can be found in the areas of production planning and inventory control; facility location planning; project planning (PERT); financial investment planning; cash management; cost-volume-profit analysis; and environmental, public services, and utilities (hospital staffing, reservoir capacity, rail-road system, solid waste system) planning. Portfolio-selection applications include problems in investment portfolio management, activity analysis and technology planning, capital budgeting, animal and human dietary planning, and material composition selection. The lists are not intended to be comprehensive but illustrates the variety of applications. For more

applications and detail of specific problems, see [Hogan, Morris, and Thompson 1981].

In general, chance-constrained programs with random technical coefficients are hard to solve. To make the problem tractable, it is typical to assume that the random coefficients are normally distributed. We propose, in this paper, a new alternative approach. The method relaxes the assumptions needed for the probability distributions of the random coefficients. Our main goal is to derive an approach that is intuitive and allows easy extraction of a problem's structural properties for building simple effective heuristics.

In this paper, we present three simple methods from our class of approximations. These methods are particularly suited to problems with a small number of random coefficient in a chance constraint. However, in some problems, the structural properties of the problem may permit large n . For example, in production/inventory problems, the service constraints for the planning horizon have a block triangular structure. The chance constraints with few variables are the service constraints for the more immediate periods; the periods further into the horizon have more variables. Thus when the planning is done on a rolling horizon basis, the large errors for the periods further away are not important as long as the nearer periods are well approximated. Examples of applying the methods appear in our earlier work: [Bitran and Leong 1989a], [--- 1989b], [--- 1989c].

The paper is organized as follows. Section 2 reviews the work of other researchers. Section 3 describes the general principle of our approach and illustrates it with some examples. In section 4, we recapitulate the equations used in the major alternative approximations and test our method against them. We report the results in section 5 with

comments on extensions and future research. In the last section, we conclude with a short summary.

2. Literature Review and Important Results

(For a quick tutorial on stochastic programming problems and chance-constrained programs, we refer the reader to the introductory comments in [Hillier 1967] and the references mentioned there. Greater details on the subject can be found in stochastic programming texts like [Sengupta 1972], [Vadja 1972], [Kall 1976], and [Dempster 1980].)

Under the condition that the joint distribution function of the random variables is continuous, Symonds [1967] proved that the feasible region of a chance-constrained program can be replaced by a deterministic equivalent. He also provided results for linear chance-constrained problems where only the resource vector is random. Problems with random resource are quite well covered in the literature. In fact, most of the research on chance-constrained problems has centered on the cases where only the resource is random. The reader is directed to see [Charnes, Cooper, and Symonds 1958] and [Charnes and Cooper 1963] for more details. In a LP with stochastic technology coefficients, a chance constraint comprises a linear combination of random variables. The weights used for combining the random variables are the decision variables of the program. Hence in general, before solving the program, the probability of violating the resource capacity is difficult to evaluate.

To simplify the discussion, without loss of generality, we assume that the resource parameter is deterministic. Problems with stochastic resource parameters in the chance constraints can be transformed by multiplying the parameters with dummy decision variables, thus converting them into technology coefficients. Adding new constraints to the problem,

we set the value of these dummy variables to one. Now consider the chance-constrained linear inequality

$$\text{Prob}(\sum_{i=1}^n a_i x_i \leq b) \geq \alpha. \quad (1)$$

a_i , $i=1, \dots, n$, are the random technology coefficient variables with continuous joint distribution; x_i , $i=1, \dots, n$, are the decision variables; b is the deterministic resource parameter; and $\alpha \in [0,1]$, the service performance target, is the probability that $\sum_{i=1}^n a_i x_i \leq b$ is satisfied. The random variables a_i , $i=1, \dots, n$, are assumed to have finite mean $E[a_i]$ and finite variance $V[a_i] = \sigma_i^2$.

By Symonds' theorem, constraint (1) has a deterministic equivalent

$$g(x) \leq b, \quad (2)$$

with vector $x \equiv (x_1, \dots, x_n)$. In general, $g(x)$ is a nonlinear function and can be written as

$$g(x) = E[\sum_{i=1}^n a_i x_i] + z_\alpha (V[\sum_{i=1}^n a_i x_i])^{1/2}, \quad (3)$$

where $z_\alpha = z_\alpha(x)$, the safety factor, is a function of the service target α as well as the decision vector x . When a_i , $i=1, \dots, n$, are independent and normally distributed, (3) becomes

$$g(x) = \sum_{i=1}^n E[a_i] x_i + Z_\alpha [\sum_{i=1}^n \sigma_i^2 x_i^2]^{1/2}. \quad (4)$$

Here Z_α is the "one-tail" normal variate for α and depends on α only. This value can be obtained easily from the tables in most basic statistics texts. (4) is true because a linear combination of normal random variables is also normally distributed.

The representation in (4) is also applicable to the class of stable probability distributions (see [Allen, Braswell, and Rao 1974]). Stable distributions are distributions, completely specified by their means and

standard deviations, such that a linear combination of random variables with a common stable distributional form has the same distributional form. It is not necessary for the random variables to be identically distributed. Their means and variances may be different but they must be from a common distributional form. The class of stable distributions include the normal, Poisson, Chi-square, and binomial distributions. Though not as readily available as the normal variate, the safety factor for the other stable distributions can be computed or found in published tables. Again, the safety factor for stable distributions is independent of the decision variables: the safety factor is a function of α only.

CONDITION C1: Random variables a_i , $i=1, \dots, n$, share a common stable distributional form. ■

When condition C1 is true and the random variables are dependent, z_α is still a function of α only. However, in general, since z_α may be dependent on x , $g(x)$ is difficult to evaluate and usually cannot be expressed in closed form.

THEOREM 1 [Kataoka 1963]: A chance-constrained linear inequality under C1 and with $z_\alpha \geq 0$ is convex. ■

Theorem 1 is the main published result on the convexity of a chance-constrained linear inequality. Little is known to date about the convexity of (1) when the random variables are of other distributions. Hillier [1967] mentioned that for arbitrary distributions, under fairly weak conditions, the central limit theorem may permit the normal approximation. Charnes, Cooper, and Thompson [1963] suggested that a mixture of normal distributions can be used to approximate distributions of fairly arbitrary shapes. Under this approximation, the term $\sum_{i=1}^n a_i x_i$ in (1) is again a normal random variable. Z_α , the safety factor for the normal distribution, is non-negative when $\alpha \geq 0.5$. Therefore, using theorem 1, we can argue that

for $\alpha \geq 0.5$, (1) is usually convex. Our simulation experiments, on common distributions, indicate that for α close to 1, (1) is convex. We did not find any general theoretical result on the necessary conditions for convexity of (1). Some work in this area has been done by Prekopa [1971,1974]. This remains an interesting research question for further study.

ASSUMPTION A1: The feasible region of (1) is convex. ■

We will assume A1 to be true for the rest of this paper. This condition has been assumed to be true in almost all the work we came across. Hence the assumption we make is no more restrictive than those that have been made (see for example, [Hillier 1967] and [Seppälä 1971]).

The usual approach taken by previous studies, after assuming C1 and convexity (A1), is to solve the chance-constrained problem with nonlinear programming methods. The nonlinear programming methods usually linearize the problem and search along subgradients. An example of this is Kelly's [1960] cutting plane method. This method solves chance-constrained programs, in multiple passes, as linear programs. A linear program is first solved without the chance constraints. At each subsequent iteration, a hyperplane tangent to each chance constraint is defined using the preceding iteration's solution and its partial derivatives. These are introduced into the program as additional linear constraints. Prekopa [1988] summarizes the numerical approaches available for solving chance-constrained problems. These nonlinear programming methods tend to be complicated: they require partial derivatives, multiple-pass techniques, and non-standard computer codes. Moreover, they are usually restricted to cases under condition C1.

Allen, Braswell, and Rao [1974] developed methods for approximating a chance-constrained set using information derived from sample data only. This approach is based on Wilks's [1963] work on the use of statistically

equivalent blocks to construct 100% tolerance regions. The level of confidence of satisfying the chance constraints can be determined from the size of the sample. In addition to the type-one error information implicit in the chance-constraints, this approach gives the decision-maker type-two error information. The paper compared the percentages of empirical constraint satisfaction of the actual feasibility region by (a) assuming stable distribution, (b) using a safety factor derived from Chebyshev's inequality, (c) using a simple linear approximation, and (d) using a hyperspherical approximation. Charnes, Kirby, and Raike [1970] studied this further and developed the "acceptance region theory".

Allen, Braswell, and Rao's approximations are not uniformly tighter than the original constraint; the solutions to their approximations are not necessarily feasible to the original problem. However, since the uniformly-tighter characteristic is important in many practical applications of chance-constrained programs, Hillier [1967] and later Seppälä [1971, 1972] devised approaches that satisfy that condition. Both methods require condition C1. In this section, we give a brief sketch of these two methods. We provide their technical detail in section 4.

Hillier restricted his study to cases where the decision variables are (a) 0-1, (b) 0 or 1, or (c) bounded. He approximated the variance term in (3) with a separable nonlinear function. The separability property of the approximation makes it easier to apply nonlinear programming techniques. The nonlinear approximation can also be further approximated with a piecewise-linear function using a standard approach in separable convex programs ([Bradley, Hax, and Magnanti 1977]). The piecewise-linear approximation is uniformly tighter than the nonlinear approximation; in turn, the nonlinear approximation is uniformly tighter than the chance constraint.

Despite the set-back of not getting the exact optimal solution, Hillier expounded on the value of linear approximations. The advantages he listed are (a) the relatively high efficiency of solving linear programs, (b) the ability to do sensitivity analysis, and (c) the availability of linear duality theory for analyzing the solutions for managerial implications. The service levels in the chance constraints may be initially selected by managerial policy. After solving the problem, the service levels should be re-evaluated against their corresponding optimal dual variable values. These give a measure of the costs of maintaining the corresponding service levels and hence provide guidance for revising them.

Seppälä [1971] relaxed the restrictions, on the decision variables, required by Hillier. Focussing, as in Hillier's approach, on the variance term of the deterministic equivalent (3), Seppälä introduced new variables to break this nonlinear variance term into simpler nonlinear functions. Each nonlinear function is a function of two variables only. He then approximated each of these by piecewise-linear segments. The resulting linear approximation is uniformly tighter than the chance constraint. In this, as well as Hillier's approach, the decision-maker can choose the number of linear constraints to use to replace the chance constraint. As the number of linear constraints increases, the error of Seppälä's solution from the optimum approaches zero. For Hillier's method, the error will decrease with more linear constraints but may not go to zero.

In a recent paper, Olson and Swenseth [1987] suggested the simple approximation of replacing each random coefficient a_i , $i=1, \dots, n$, in (1) by the sum of its expected value and its standard deviation multiplied by a safety factor, shown below:

$$g(\mathbf{x}) = \sum_{i=1}^n (E[a_i] + z_\alpha \sigma_i) x_i . \quad (5)$$

Assuming C1 and with the means and the variances given, this approximation is the same as Allen, Braswell, and Rao's [1974] linear approximation. As an illustration, Olson and Swenseth solve Van de Panne and Popp's [1963] cattle-feed mix problem--assuming independent normal distributions--and showed that the errors are small.

The details of this approximation is given later. Briefly, this linear approximation consists of only one inner-linearization hyperplane. For the convex case (A1), the hyperplane defines a half space that guarantees feasibility to (1). Since the feasible region defined by a chance constraint is nonlinear, in the worst case the gap from optimality can be very large. This is particularly so when the magnitudes of the variances relative to the means of the random variables are huge. The coefficients of variation (COV) in the cattle-feed mix problem are extremely small--less than 0.01--and hence the excellent results.

In cases where condition C1 is not true, both Hillier and Seppälä suggest that the Chebyshev's inequality be used to obtain the safety factor. However, the Chebyshev's inequality is known to give safety factors with magnitudes much larger than they need to be and thus the approximations tend to constrict excessively the feasible region (e.g. see Allen, Braswell, and Rao [1974]). They are particularly bad when the random variables have finite supports (for example, $a_i \geq 0$ or $a_i \in [0,1]$). In the literature we have encountered, most test cases have small COV's. However, in some applications the COV's can be large: Albin and Friedman [1989] reported that the distributions of defects in integrated circuit fabrication have COV's larger than 1.

3. Description of the method

Unlike Hillier and Seppälä who linearize the variance term in (3), we

linearize $g(x)$ in (2). This way, we do not have to deal with z_α explicitly. Our method constructs hyperplanes, each formed by connecting selected points on the boundary of (2). The hyperplanes approximate the feasible region of (2). The main principle in our approach is to make some "guesses" about the relative magnitudes of the decision variables. For each guess, $g(x)$ becomes a function with one random coefficient and one decision variable. Specifically, we "guess" that $x_i = s_i w$, $i = 1, \dots, n$, for a selected deterministic vector $s \equiv (s_1, \dots, s_n)$ and decision variable w . In this case, (1) becomes

$$\text{Prob}(w \sum_{i=1}^n a_i s_i \leq b) \geq \alpha. \quad (6)$$

Constraint (6) can be re-written as linear inequality

$$\phi(s) w \leq b \quad (7)$$

where fractile $\phi(s) = F^{-1}(\sum_{i=1}^n a_i s_i; \alpha)$ is a deterministic coefficient, $F(u;v)$ is the cumulative density function of random variable u evaluated at v , and $F^{-1}(u;.)$ is the inverse function of $F(u;v)$.

Vector s corresponds to a ray from the origin. With each vector s , the evaluation of (7) gives us a point on the boundary surface of (2). We repeat the process for a set of selected rays to get a set of points on the boundary of (2). We then connect adjacent points to form hyperplanes. The hyperplanes are introduced into the problem as linear inequalities (linear constraints). These linear inequalities replace the chance constraint (1) in the problem.

Geometrically, the linear inequalities form a polyhedron that has extreme points touching the boundary of the feasible region. Consequently, if the feasible region formed by (2) is convex, it "contains" the polyhedron. Therefore when (2) is convex, the set of linear inequalities is uniformly tighter than (2). The extreme points are the same points at which the selected rays from the origin intersect the boundary of the feasible

region. Figure 1 illustrates the polyhedron formed by the extreme points in a convex chance constraint in R^2 .

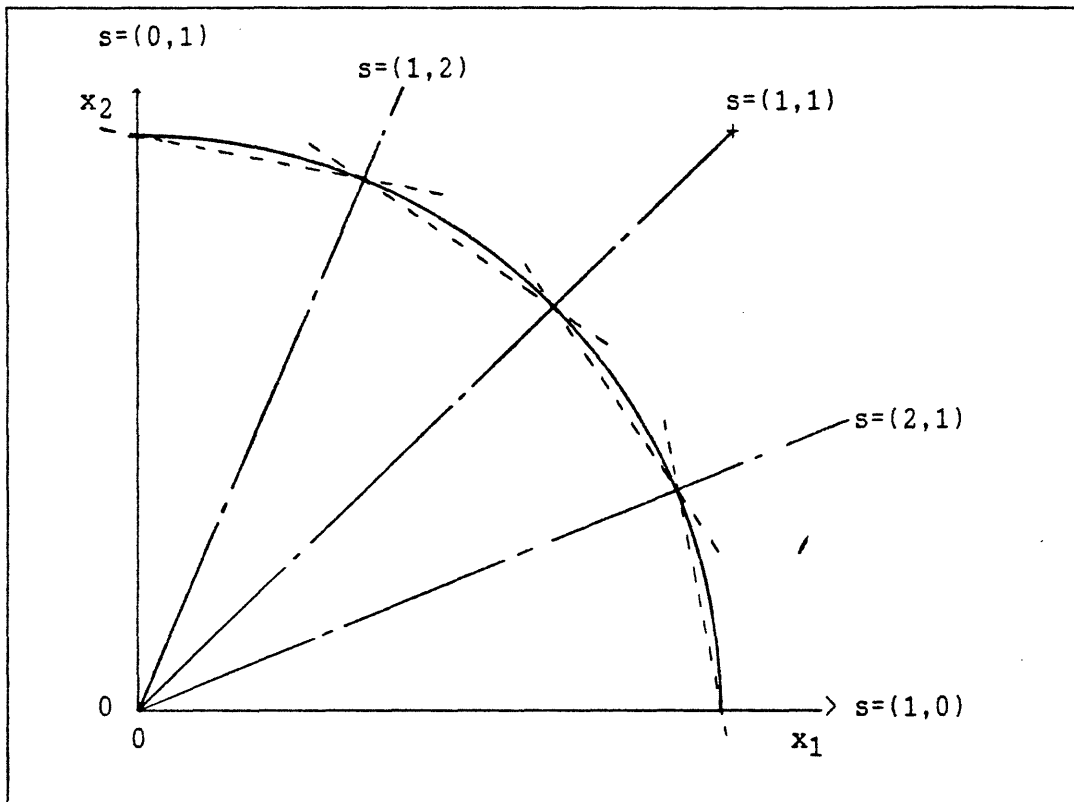


Figure 1. Linear Approximation of a Chance Constraint.

With an infinite number of rays, the linear inequalities reproduce the chance constraint (1). In practice, only a few "well-chosen" rays are needed to give solutions with small relative errors from the optimal value.

EXAMPLES

All our linear approximations have the form

$$\sum_{i=1}^n \Omega_{ik} x_i \leq b, \quad k=1, \dots, K, \quad (8)$$

where Ω_{ik} , $i=1, \dots, n$, are deterministic coefficients. For each k , $\sum_{i=1}^n \Omega_{ik} x_i = b$ defines a hyperplane and Ω_{ik} , $i=1, \dots, n$, are obtained by solving a system of equations; each equation corresponds to an extreme point (of the polyhedron) that rests on the hyperplane. This effort is done once only for each approximation; they are not solved each time the approximation is used. (Details are presented in the appendix.)

In this study we provide, as examples of the general approach, three approximations: (RAY1), (RAY2), (RAY3). The rays used in these approximations are the decision variable axes and the centroid rays, rays in the center of the cone formed by subsets of the axes. Allen, Braswell, and Rao [1974] and Olson and Swenseth's [1987] methods are special cases of (RAY1); the rays used in (RAY1) are the decision variable axes only.

We define the unit vector $e_i = (s_1, \dots, s_n)$ where $s_j = 1$ for $j = i$ and $s_j = 0$ otherwise. Below, we provide the approximations.

(RAY1)

$$\begin{aligned} \mathcal{E}_{\text{RAY1}}(x) &= \sum_{i=1}^n \Omega_{ik} x_i, \quad k=1, \dots, K=1, \\ \text{where } \Omega_{ik} &= \phi(e_i), \quad k=1, \dots, K=1. \end{aligned}$$

(RAY2)

$$\begin{aligned} \mathcal{E}_{\text{RAY2}}(x) &= \sum_{i=1}^n \Omega_{ik} x_i, \quad k=1, \dots, K=n, \\ \text{where } \Omega_{ik} &= \phi(\sum_{j=1}^n e_j) - \sum_{j=1}^n \phi(e_j) + \phi(e_i) \text{ for } i = k \\ \text{and } \Omega_{ik} &= \phi(e_i) \text{ otherwise.} \end{aligned}$$

For $n = 3$, for example, matrix $\Omega \equiv [\Omega_{ik}]$

$$= \begin{bmatrix} \phi(1,1,1) - \phi(0,1,0) - \phi(0,0,1) & \phi(0,1,0) & \phi(0,0,1) \\ \phi(1,0,0) & \phi(1,1,1) - \phi(1,0,0) - \phi(0,0,1) & \phi(0,0,1) \\ \phi(1,0,0) & \phi(0,1,0) & \phi(1,1,1) - \phi(1,0,0) - \phi(0,1,0) \end{bmatrix}$$

For (RAY3), we construct unique sets $\{\pi(i,k), i=1, \dots, n\}, k=1, \dots, K$. By itself, each $\pi(i,k)$ is also a set such that $\pi(i-1,k) \subset \pi(i,k)$, $i=1, \dots, n$, with $\pi(n,k) = \{1, \dots, n\}$ and $\pi(0,k) = \{\}$ for all k . Note that there can be $n!$ unique sets of $\{\pi(i,k), i=1, \dots, n\}$ and hence $K = n!$.

(RAY3)

$$g_{RAY3}(x) = \sum_{i=1}^n \Omega_{ik} x_i, \quad k=1, \dots, K=n!,$$

where $\Omega_{ik} = \phi(\sum_{r \in \pi(j,k)} s_r) - \phi(\sum_{r \in \pi(j-1,k)} s_r)$
with j such that $i = \pi(j,k) \setminus \pi(j-1,k)$. (\setminus is the set subtraction operation.)

Now for $n = 3$,

$$\Omega = \begin{bmatrix} \phi(1,1,1) - \phi(0,1,1) & \phi(0,1,1) - \phi(0,0,1) & \phi(0,0,1) \\ \phi(1,1,1) - \phi(0,1,1) & \phi(0,1,0) & \phi(0,1,1) - \phi(0,1,0) \\ \phi(1,0,1) - \phi(0,0,1) & \phi(1,1,1) - \phi(1,0,1) & \phi(0,0,1) \\ \phi(1,0,0) & \phi(1,1,1) - \phi(1,0,1) & \phi(1,0,1) - \phi(1,0,0) \\ \phi(1,1,0) - \phi(0,1,0) & \phi(0,1,0) & \phi(1,1,1) - \phi(1,1,0) \\ \phi(1,0,0) & \phi(1,1,0) - \phi(1,0,0) & \phi(1,1,1) - \phi(1,1,0) \end{bmatrix}$$

(RAY1), (RAY2), and (RAY3) are by no means the only approximations possible using our methodology. There can be numerous variations by using different rays and constructing different hyperplanes from them. Moreover, an iterative multi-pass approach may be devised to generate the rays based on previous iterations solution. We leave this challenge for future research.

FRACTILES

When the joint probability distribution of the random variables are known, the fractiles $\phi(s)$ may sometimes be obtained as closed-form expressions. Alternatively, at a level of confidence that corresponds to their sample sizes, the fractiles can be extracted from Monte-Carlo simulation (Levy [1967]) or sample data. If the form of the distribution is known but not the exact distribution, the fractiles may be estimated using statistical approximation methods (Bache [1979], Cornish and Fisher [1937], Fisher and Cornish [1960]). These methods can make use of the sampling

information beyond the first two moments. When the sample data only is available, distribution-free or non-parametric methods (Allen, Braswell, and Rao [1974], Wilks [1963]) may be used.

4. Comparative Experiments

We compare our method against those by Hillier, Seppälä, and Olson and Swenseth. Before proceeding, we sketch how these methods approximate $g(x)$ for the test conditions we are using. Following that, we describe the test conditions and then state the comparison criterion.

$$g(x) \equiv F^{-1}(\sum_{i=1}^n a_i x_i; \alpha)$$

Normal distribution:

$$g(x) = \sum_{i=1}^n E[a_i] x_i + Z_\alpha [\sum_{i=1}^n \sigma_i^2 x_i^2]^{1/2}.$$

Uniform distribution and $x_i \in \{0,1\}$, $i=1, \dots, n$:

$$g(x) = m - (m!(1-\alpha))^{1/m} \quad \text{where } m = \sum_{i=1}^n x_i.$$

For $x_i \in \{0,1\}$, $g(x)$ for the uniform distribution case can be derived using geometry.

(HILL)--Hillier

Normal distribution and $x_i \in [0,1]$, $i=1, \dots, n$:

$$g_{HILL}(x) = \sum_{i=1}^n E[a_i] x_i + Z_\alpha [\sum_{i=1}^n (\theta^2 - \sigma_i^2 + \sigma_i^2 x_i^2)]^{1/2} - (n-1)\theta$$

Uniform distribution and $x_i \in \{0,1\}$, $i=1, \dots, n$:

$$g_{HILL}(x) = \sum_{i=1}^n E[a_i] x_i + (\alpha/(1-\alpha))^{1/2} [\sum_{i=1}^n (\theta - (\theta^2 - \sigma_i^2)^{1/2}) x_i + \sum_{i=1}^n (\theta^2 - \sigma_i^2)^{1/2} - (n-1)\theta]$$

where $\theta = (\sum_{i=1}^n \sigma_i^2)^{1/2}$.

$(\alpha/(1-\alpha))^{1/2}$ is the one-sided Chebyshev's inequality safety factor.

Approximation $g_{HILL}(x)$ given above is only the nonlinear approximation of

Hillier's method. For the comparison tests, we did not linearize it. The reader will realize that, for our comparison criterion, the result using $g_{HILL}(x)$ will be an upper bound on the linearized version.

(SPLK)--Seppälä

Normal distribution and $x_i \geq 0$, $i=1, \dots, n$:

$$g_{SPLK}(x) = \sum_{i=1}^n E[a_i] x_i + Z_\alpha y_n$$

where $y_i = \text{Max} \{r_{ik} y_{i-1} + s_{ik} x_i, k=1, \dots, K\}$ with $y_0 = 0$,

$$r_{ik} = \frac{t_{ik} (1 + \sigma_i^2 t_{i,k-1}^2)^{1/2} - t_{i,k-1} (1 + \sigma_i^2 t_{ik}^2)^{1/2}}{t_i - t_{i,k-1}},$$

$$s_{ik} = \frac{(1 + \sigma_i^2 t_{i,k}^2)^{1/2} - (1 + \sigma_i^2 t_{i,k-1}^2)^{1/2}}{t_i - t_{i,k-1}},$$

and $t_{ik} = \text{TANGENT}[(k/K)(\pi/2)]/i^{1/2}$, $i=1, \dots, n$, $k=1, \dots, K$.

Uniform distribution and $x_i \geq 0$, $i=1, \dots, n$:

Same as above except replace Z_α by $(\alpha/(1-\alpha))^{1/2}$.

We consider the instances where $K = 3$ and 6 which we label (SPL3) and (SPL6) respectively. Seppälä's approach replaces each chance constraint with $O(nK)$ linear inequalities.

(OLSW)--Olson and Swenseth

Normal distribution and $x_i \geq 0$, $i=1, \dots, n$:

$$g_{OLSW}(x) = \sum_{i=1}^n (E[a_i] + Z_\alpha \sigma_i) x_i.$$

Uniform distribution and $x_i \geq 0$, $i=1, \dots, n$:

$$g_{OLSW}(x) = \sum_{i=1}^n (E[a_i] + (\alpha/(1-\alpha))^{1/2} \sigma_i) x_i.$$

(OLSW) is the same as (RAY1) under condition C1 and if z_α is available.

(RAY1, RAY2, and RAY3)

The approach we propose in this study leads to a class of approximations.

For the purpose of comparison, we use only (RAY1), (RAY2), and (RAY3).

Under the normal distribution, $\phi(s) = \sum_{i=1}^n E[a_i] s_i + Z_\alpha [\sum_{i=1}^n \sigma_i^2 s_i^2]^{1/2}$; and for the uniform distribution case with $s_i \in \{0,1\}$, $\phi(s) = m - (m!(1-\alpha))^{1/m}$ where $m = \sum_{i=1}^n s_i$. With the expression for $\phi(s)$ given, approximations (RAY1), (RAY2), and (RAY3) are completely defined.

For simplicity of conducting the experiments, we test only cases where the random variables are independent identically distributed. (All the methods tested can be used for dependent and non-identically distributed random variables. In the future, we hope to perform extensive experiments on these cases and report our findings.) First, we evaluate cases where a_i , $i=1, \dots, n$, are normally distributed with $x_i \in [0,1]$, $i=1, \dots, n$. Here, we examine how well our methods compare against the other methods in the conditions specified for those methods. Since the conditions are those that the other methods were derived under, we can expect our methods to do no better than these methods. Then we compare the methods when a_i , $i=1, \dots, n$, are uniformly distributed with $x_i \in \{0,1\}$, $i=1, \dots, n$. Here, we test for the situation of a non-stable distribution for which the exact analytical form of $g(x)$ is known. We would expect the other methods to do poorly under non-stable distributions since they were intended for such conditions and have to use the Chebyshev's inequality.

In general, the conditions are specified to keep the tests simple or to replicate the conditions that were originally intended for the other methods. The test input data are generated as follows:

(a) For the normal distribution cases, we let $E[a_i] = 0.5$, $i=1, \dots, n$, and pick variance $\sigma_i^2 = \sigma^2$, $i=1, \dots, n$, from a uniform distribution in $[0, VA]$. We examine cases where $VA = 10^{-\beta}$, $\beta = 0, \dots, 4$. The maximum coefficient of variation (COV), in each case of β , is $\sqrt{VA}/0.5$. Overall, the COV ranges from 0 to 2. The values of decision variables x_i , $i=1, \dots, n$, are sampled

from a uniform distribution in $[0,1]$. We tested cases where $n = 2, 4,$ and 6 .

(b) For the uniform distribution cases, we sample $a_i, i=1,\dots,n$, from a uniform distribution in $[0,1]$. Hence, $E[a_i] = 0.5, i=1,\dots,n$, and variance $\sigma_i^2 = \sigma^2 = 1/12$ and $COV = (1/12)/0.5 = 0.58$. To obtain $x_i \in \{0,1\}$, we sample from a uniform distribution in $[0,1]$. We then let $x_i = 0$ when the sampled value is less than 0.5 ; and $x_i = 1$ otherwise. We evaluated the cases where $n = 2, 4, 6, 8, 16,$ and 32 .

The criteria of comparison is the relative error that the values of the approximation deviate from $g(x)$: relative error of method $h \equiv (g_h(x) - g(x))/g(x)$. Since all the methods to be compared are uniformly tighter than (1), $g(x) \leq g_h(x) \leq b, h \in \{HILL, SPL3, SPL6, OLSW, RAY1, RAY2, RAY3\}$. Therefore, the relative error is non-negative; it is a measure of how constricted approximation h is when (1) is binding and the sampled value of x is the optimum solution. We replicated each set of test conditions 15 times and compute their average and standard deviation.

5. Results and Comments

First, we note that our three approximations have different computation requirements. (RAY1) replaces each chance constraint with one linear inequality; (RAY2) uses n linear inequalities and (RAY3) uses $n!$ linear inequalities. (RAY1) and (RAY3) represent examples of the extreme types of our class of approximations. (RAY3), especially, will be difficult to implement for n larger than 6 ; it took about 50 seconds to evaluate one chance-constrained approximation on a IBM-PC compatible 80286 class machine when $n = 6$. (RAY1), (RAY2), and other methods took a negligible amount of time--typically less than 1 second per evaluation of $g(x)$. Second, the result of (RAY3) dominates that of (RAY2) and the result of (RAY2)

dominates that of (RAY1). This is because (RAY3) is uniformly tighter than (RAY2) and (RAY2) is uniformly tighter than (RAY1). We can therefore use the results of one approximation as a bound to another.

The comparisons outcome is tabulated in figures 2a through 3b. Figure 2a tabulates the average and standard deviation of the relative errors for the normal distribution cases. These results are graphed in figures 2b through 2d, for $n = 2, 4, \text{ and } 6$ respectively. For maximum-COV less than 0.5, (RAY2) does as well as (HILL) and (SPL6), but it gets progressively worse as n becomes larger. The average relative errors in such cases are about 5% or less. Since (SPL6) uses 6 times more constraints than (RAY2) and (HILL) is only the nonlinear approximation part of Hillier's method, the comparable results among the three methods suggest that our methods perform well for small n .

[INSERT FIGURES 2a THROUGH 2d HERE]

For the uniform distribution cases, figure 3a tabulates the results while figure 3b presents it graphically. In these cases for n up to 6, (RAY3) is exact. We did not test (RAY3) beyond $n = 6$ since the computation effort is too much. (RAY1) and (RAY2) performed equally well and are about an order of magnitude better than (HILL) and (SPL6) when n is less than 6. They dominate the other methods up to $n = 16$. The methods, except for (OLSW), have about the same size of errors when n is larger than 16. (OLSW), on the whole, demonstrated to be an inferior method.

[INSERT FIGURES 3a AND 3b HERE]

The results of both the normal and uniform distribution cases show that the average relative errors become worse with larger n . As such, we do not recommend using any of the methods, ours included, for COV's larger than 0.5 when n gets beyond 12; the average relative errors then goes up to more than 50%. As mentioned in the introduction of this paper, large n may be permitted in production/inventory problems because of its block triangular structure. When the planning is done on a rolling horizon basis, the large errors for the periods further away are not important as long as the nearer periods are well approximated. This is the case when a method like (RAY2) is applied. In fact, (RAY1), (RAY2), and (RAY3) are exact for $n = 1$, the first period problem.

The number of random coefficient, n , may also be large when the capacity constraints limit the number of non-zero decision variables, of those in the chance constraint. (An example of this is where there are many alternative processes of producing a product and only a few are permitted.) In these cases, the "effective" number of random coefficients in the chance constraint will be small. Again, (RAY2) should perform very well there.

EXTENSIONS AND FUTURE RESEARCH

We have assumed zero-order decision rules (see Charnes and Cooper [1963]); all the decision variables to be determined before the value of any random variables are known. The linear formulation of our method can be adapted easily to give solutions for linear decision rules. A linear decision rule for a decision variable is a linear combination of the random variables that will be realized before that the decision variable needs to be determined. Therefore for problems with sufficient stationarity, we can solve the problem once, replacing all decision variables by the decision rules, to obtain the parameters for the decision rules. As the value of

random variables become known, we apply the decision rules to get the values of the decision variables, without resolving the linear program.

The chance constraints we have examined are for linear inequalities-- that is, the term inside of Prob(.) in (1) is a linear function. Our approach does not restrict us to linear inequalities; we can have nonlinear inequalities or situations where $a_i = a_i(x)$, $i=1, \dots, n$, are functions of x . In production problems with stochastic yield, the last situation corresponds to the case where the yields are not independent of the lot size; a problem that has chance constraint like constraint (1) corresponds to the case where the yields are independent of the lot size.

For future research, we suggest to consider the following: (a) examine how the relative errors can be parametrically bounded, (b) provide a multi-pass ϵ -optimal iterative approach, (c) consider problems with joint chance constraints, (d) provide theoretical results on the convexity of the feasible region.

6. Summary and Conclusions

We provided a class of linear approximations for problems with individual chance constraints that have random variables with arbitrary distributions. In our method, we linearized the nonlinear deterministic equivalents of chance constraints when the deterministic equivalents may not have closed-form expressions. The linear approximation is uniformly tighter than the chance constraint when the feasible region defined by the latter is convex. Therefore under convexity, the solutions generated by our approximations will satisfy or do better than the service target specified by the chance constraint.

Our method gives linear inequalities that retain the original decision variables. This make it easy to extract heuristics or apply higher

order decision rules. The resulting programs are linear programs which permit the use of standard LP codes, perform sensitivity analysis, and have both primal and dual solutions. The coefficients in the approximating linear constraints can be extracted from assumed distributions or sample data. We do not restrict the distributional form of the random variables, disallow their dependencies on each other or require their partial derivatives.

The user has the option of trading-off the amount of computation effort against the accuracy of the solution by selecting the number of linear inequalities. In the simulation tests, when the number of random variables are small, applications of our approach with very few inequalities compare very well against the existing methods for both the normal and the uniform distributions.

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APPENDIX

Notations

- Prob(.): Probability of the event argument.
- E[.]: Expectation function.
- V[.]: Variance function.
- F(u;v): Cumulative density function of any random variable u evaluated at v, and $F^{-1}(u;.)$ is its inverse function.
- n: Number of decision variables in the chance constraint.
- a_i : Random technology coefficients with finite mean $E[a_i]$ and finite variance σ_i^2 , $i=1, \dots, n$.
- b: Resource parameter.
- x_i : Decision variables, $i=1, \dots, n$.
- α : Service performance target; probability target for satisfying the chance constraint. (Typically, $\alpha \in [0,1]$ is close to 1.)
- $\phi(.): \phi(s_1, \dots, s_n) = F^{-1}(\sum_{i=1}^n a_i s_i; \alpha)$ where $s_i \geq 0$, $i=1, \dots, n$.
- z_α : Safety factor with service target α .
- Z_α : Safety factor for normal distributions: one-tail normal variate for α .

System of Equations to solve to obtain matrix $\Omega \equiv [\Omega_{ik}]$

$$\text{Prob}(\sum_{i=1}^n a_i x_i \leq \sum_{i=1}^n \Omega_{ik} x_i, k=1, \dots, K) \geq \alpha \quad (\text{a1})$$

$$\text{and } \sum_{i=1}^n \Omega_{ik} x_i \leq b, k=1, \dots, K \quad (\text{a2})$$

$$\Rightarrow \text{Prob}(\sum_{i=1}^n a_i x_i \leq b) \geq \alpha. \quad (\text{a3})$$

Since (a2) replaces chance constraint (a3) in the linear program approximation, to approximate (a3) we need only focus on (a1). We determine Ω_{ik} , $i=1, \dots, n$ and $k=1, \dots, K$ by solving a system of equations, one for each k .

Note that in inequality (a1), the absolute value of vector x is not important; we need only to know the relative values among its components x_i , $i=1, \dots, n$. $\sum_{i=1}^n \Omega_{ik} x_i = b$, $k=1, \dots, K$, are a set of hyperplanes. We let $\{S_{1k}, \dots, S_{nk}\}$ be the set of rays such that hyperplane k , $k=1, \dots, K$, is formed by the points at which these rays intersect the boundary of (a3). Each S_{ik} , $i=1, \dots, n$ and $k=1, \dots, K$, is a vector of dimension n .

The extreme points of the polyhedron that form hyperplane k , $k=1, \dots, K$, correspond to the fractiles $\phi(S_{ik})$, $i=1, \dots, n$. (Recall that for any vector (s_1, \dots, s_n) , $\phi(s_1, \dots, s_n) = F^{-1}(\sum_{i=1}^n a_i s_i; \alpha)$, $i=1, \dots, n$.) Therefore, $\text{Prob}(\sum_{i=1}^n a_i w_i \leq \sum_{i=1}^n \Omega_{ik} w_i) \geq \alpha$ for all $w \equiv (w_1, \dots, w_n) = \sum_{j=1}^n \mu_j S_{jk}$, $k=1, \dots, K$, where $\sum_{j=1}^n \mu_j = 1$ and $0 \leq \mu_j \leq 1$, $j=1, \dots, n$. Hence, the system of equation to solve to obtain Ω is

$$\begin{bmatrix} S_{1k} \\ \vdots \\ S_{nk} \end{bmatrix} \begin{bmatrix} \Omega_{1k} \\ \vdots \\ \Omega_{nk} \end{bmatrix} = \begin{bmatrix} \phi(S_{1k}) \\ \vdots \\ \phi(S_{nk}) \end{bmatrix}, \quad k=1, \dots, K. \quad (\text{a4})$$

For $n=3$, we present the following examples:

(RAY2)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Omega_{11} \\ \Omega_{21} \\ \Omega_{31} \end{bmatrix} = \begin{bmatrix} \phi(1 \ 1 \ 1) \\ \phi(0 \ 1 \ 0) \\ \phi(0 \ 0 \ 1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Omega_{12} \\ \Omega_{22} \\ \Omega_{32} \end{bmatrix} = \begin{bmatrix} \phi(1 \ 0 \ 0) \\ \phi(1 \ 1 \ 1) \\ \phi(0 \ 0 \ 1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \Omega_{13} \\ \Omega_{23} \\ \Omega_{33} \end{bmatrix} = \begin{bmatrix} \phi(1 \ 0 \ 0) \\ \phi(0 \ 1 \ 0) \\ \phi(1 \ 1 \ 1) \end{bmatrix}$$

(RAY3)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Omega_{11} \\ \Omega_{21} \\ \Omega_{31} \end{bmatrix} = \begin{bmatrix} \phi(1 \ 1 \ 1) \\ \phi(0 \ 1 \ 1) \\ \phi(0 \ 0 \ 1) \end{bmatrix} ,$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \Omega_{12} \\ \Omega_{22} \\ \Omega_{32} \end{bmatrix} = \begin{bmatrix} \phi(1 \ 1 \ 1) \\ \phi(0 \ 1 \ 0) \\ \phi(0 \ 1 \ 1) \end{bmatrix} ,$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Omega_{13} \\ \Omega_{23} \\ \Omega_{33} \end{bmatrix} = \begin{bmatrix} \phi(1 \ 0 \ 1) \\ \phi(1 \ 1 \ 1) \\ \phi(0 \ 0 \ 1) \end{bmatrix} ,$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Omega_{14} \\ \Omega_{24} \\ \Omega_{34} \end{bmatrix} = \begin{bmatrix} \phi(1 \ 0 \ 0) \\ \phi(1 \ 1 \ 1) \\ \phi(1 \ 0 \ 1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \Omega_{15} \\ \Omega_{25} \\ \Omega_{35} \end{bmatrix} = \begin{bmatrix} \phi(1 \ 1 \ 0) \\ \phi(0 \ 1 \ 0) \\ \phi(1 \ 1 \ 1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Omega_{16} \\ \Omega_{26} \\ \Omega_{36} \end{bmatrix} = \begin{bmatrix} \phi(1 \ 0 \ 0) \\ \phi(1 \ 1 \ 0) \\ \phi(0 \ 0 \ 1) \end{bmatrix} .$$

RELATIVE ERROR'S
Averages for the Normal distribution

n	Max.-COV	HILL	SPL3	SPL6	OLSW	RAY1	RAY2	RAY3
2	0.02	1.43	0.22	0.21	0.51	0.51	0.08	0.08
	0.06	0.75	1.19	1.10	1.56	1.56	0.29	0.29
	0.20	3.57	2.22	1.57	3.17	3.17	0.80	0.80
	0.63	5.23	5.58	3.17	9.71	9.71	1.68	1.68
	2.00	17.66	3.49	0.96	19.44	19.44	2.98	2.98
4	0.02	0.18	0.53	0.52	0.89	0.89	0.51	0.11
	0.06	0.64	2.09	1.93	2.63	2.63	1.76	0.37
	0.20	2.55	5.37	4.19	7.20	7.20	4.44	0.99
	0.63	6.67	9.06	3.80	20.55	20.55	11.65	2.70
	2.00	13.85	5.86	1.37	36.75	36.75	22.06	5.72
6	0.02	0.21	0.67	0.64	1.08	1.08	0.80	0.11
	0.06	0.76	2.39	2.16	3.04	3.04	2.18	0.38
	0.20	1.93	7.51	5.67	10.21	10.21	7.12	1.11
	0.63	3.13	12.00	5.39	25.98	25.98	16.27	2.48
	2.00	9.44	5.33	1.42	60.96	60.96	45.72	6.34

RELATIVE ERROR'S
Standard Deviations for the Normal distribution

n	Max.-COV	HILL	SPL3	SPL6	OLSW	RAY1	RAY2	RAY3
2	0.02	3.76	0.21	0.21	0.22	0.22	0.05	0.05
	0.06	0.97	0.64	0.60	0.67	0.67	0.17	0.17
	0.20	6.45	1.47	1.22	1.82	1.82	0.40	0.40
	0.63	10.93	3.41	2.03	4.87	4.87	0.95	0.95
	2.00	27.43	2.16	0.90	7.38	7.38	1.42	1.42
4	0.02	0.25	0.27	0.26	0.31	0.31	0.28	0.07
	0.06	0.58	0.91	0.80	1.02	1.02	1.00	0.17
	0.20	3.09	2.36	1.71	3.23	3.23	2.34	0.45
	0.63	6.49	2.69	1.95	6.68	6.68	5.81	0.99
	2.00	11.72	4.25	1.09	10.07	10.07	11.00	2.10
6	0.02	0.23	0.44	0.42	0.50	0.50	0.37	0.07
	0.06	1.00	1.20	1.07	1.34	1.34	0.92	0.22
	0.20	1.55	2.57	1.73	3.69	3.69	2.78	0.58
	0.63	4.13	3.49	1.96	10.76	10.74	6.49	0.97
	2.00	7.93	3.46	1.95	14.55	14.55	15.79	2.57

Figure 2a. Results of Tests under Normal Distribution.

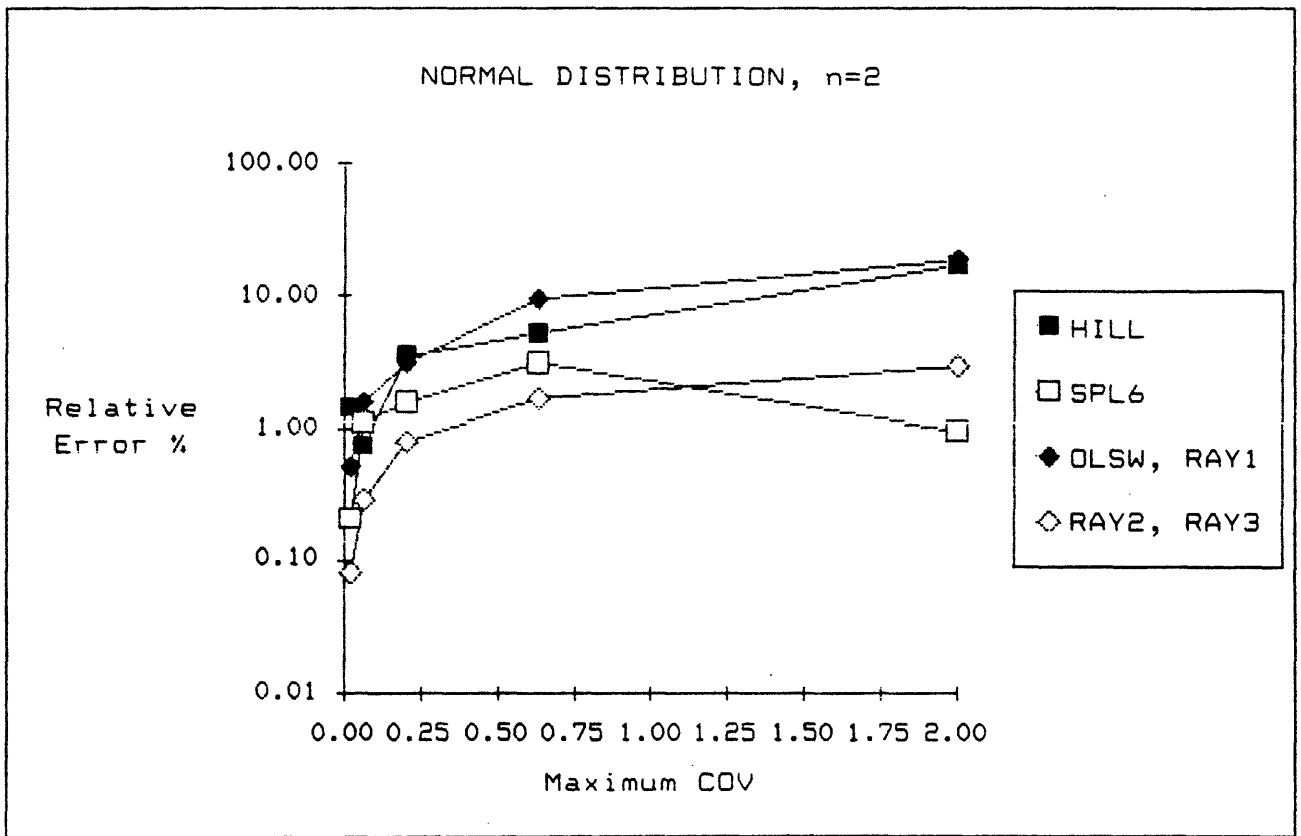


Figure 2b. Results of Tests under Normal Distribution.

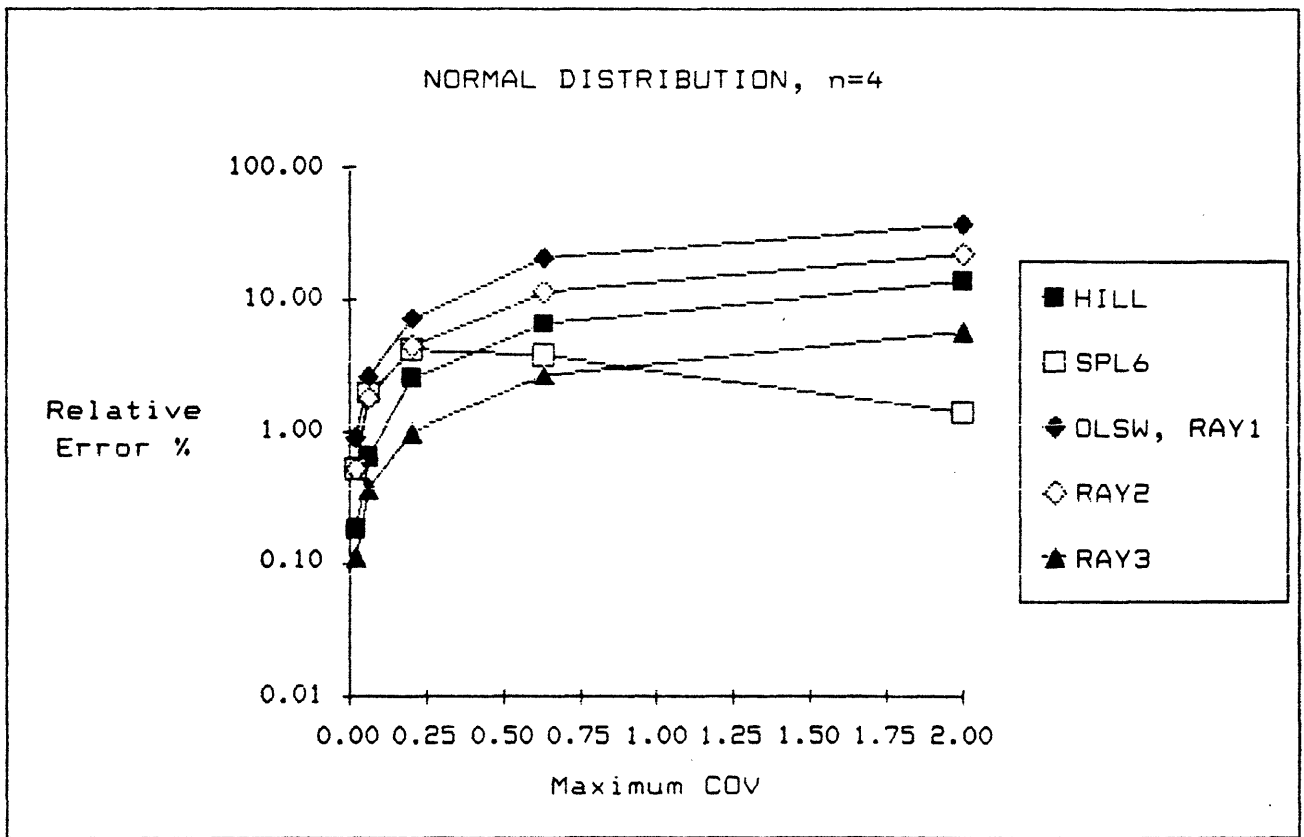


Figure 2c. Results of Tests under Normal Distribution.

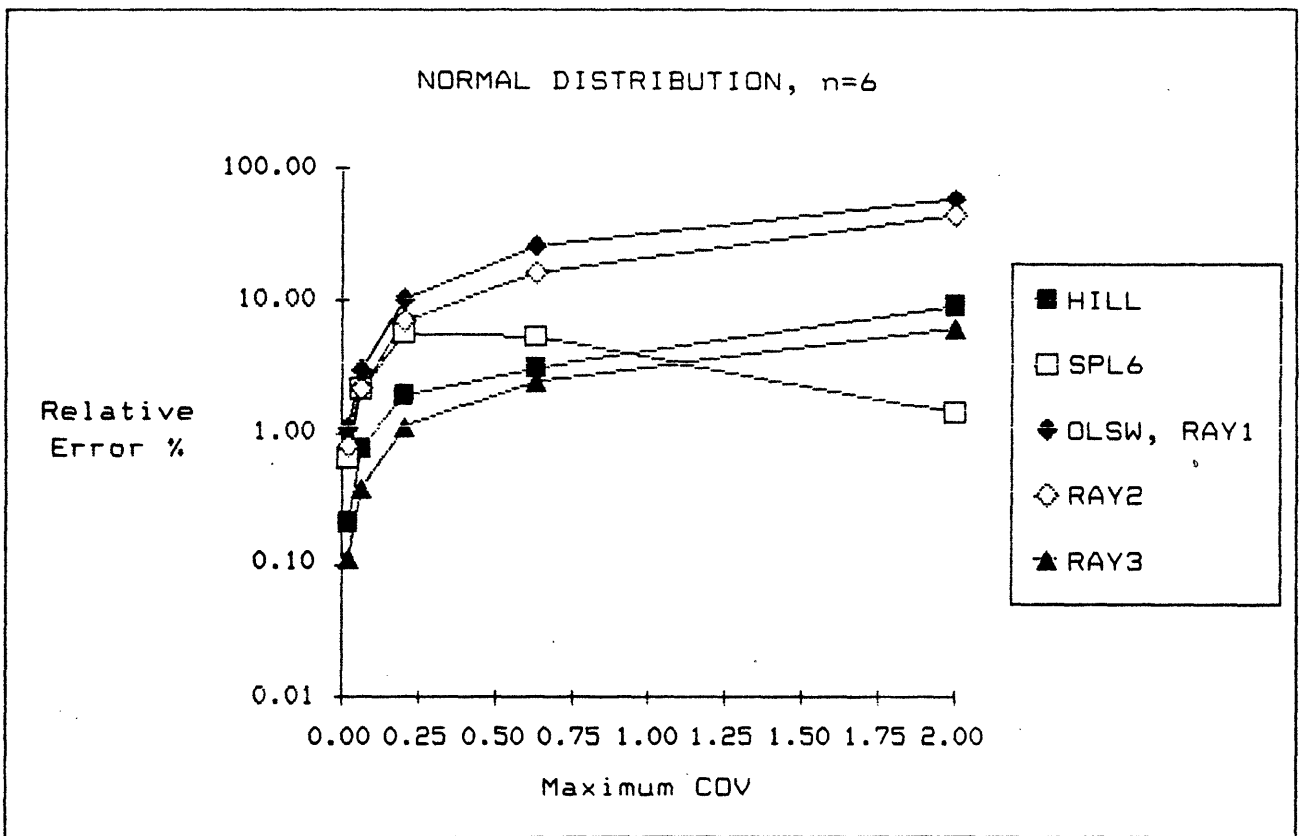


Figure 2d. Results of Tests under Normal Distribution.

RELATIVE ERROR'S
Averages for the Uniform distribution

n	HILL	SPL3	SPL6	OLSW	RAY1	RAY2	RAY3
2	79.75	85.30	81.20	91.42	3.42	0.00	0.00
4	81.41	87.85	75.54	106.75	11.70	7.88	0.00
6	71.89	92.30	72.90	121.80	19.84	17.42	0.00
8	63.55	94.86	70.50	131.94	25.32	25.32	----
16	44.46	86.65	52.26	158.95	39.91	39.91	----
32	28.49	72.46	36.13	171.52	46.70	46.70	----

RELATIVE ERROR'S
Standard Deviations for the Uniform distribution

n	HILL	SPL3	SPL6	OLSW	RAY1	RAY2	RAY3
2	9.16	0.91	6.29	10.88	5.88	0.00	0.00
4	25.63	3.28	9.83	20.63	11.14	9.33	0.00
6	27.21	5.67	10.48	18.23	9.85	9.98	0.00
8	12.23	3.78	8.06	12.06	6.52	6.52	----
16	11.82	11.05	13.10	8.47	4.58	4.58	----
32	5.09	8.56	7.54	1.97	1.06	1.06	----

Figure 3a. Results of Tests under Uniform Distribution.

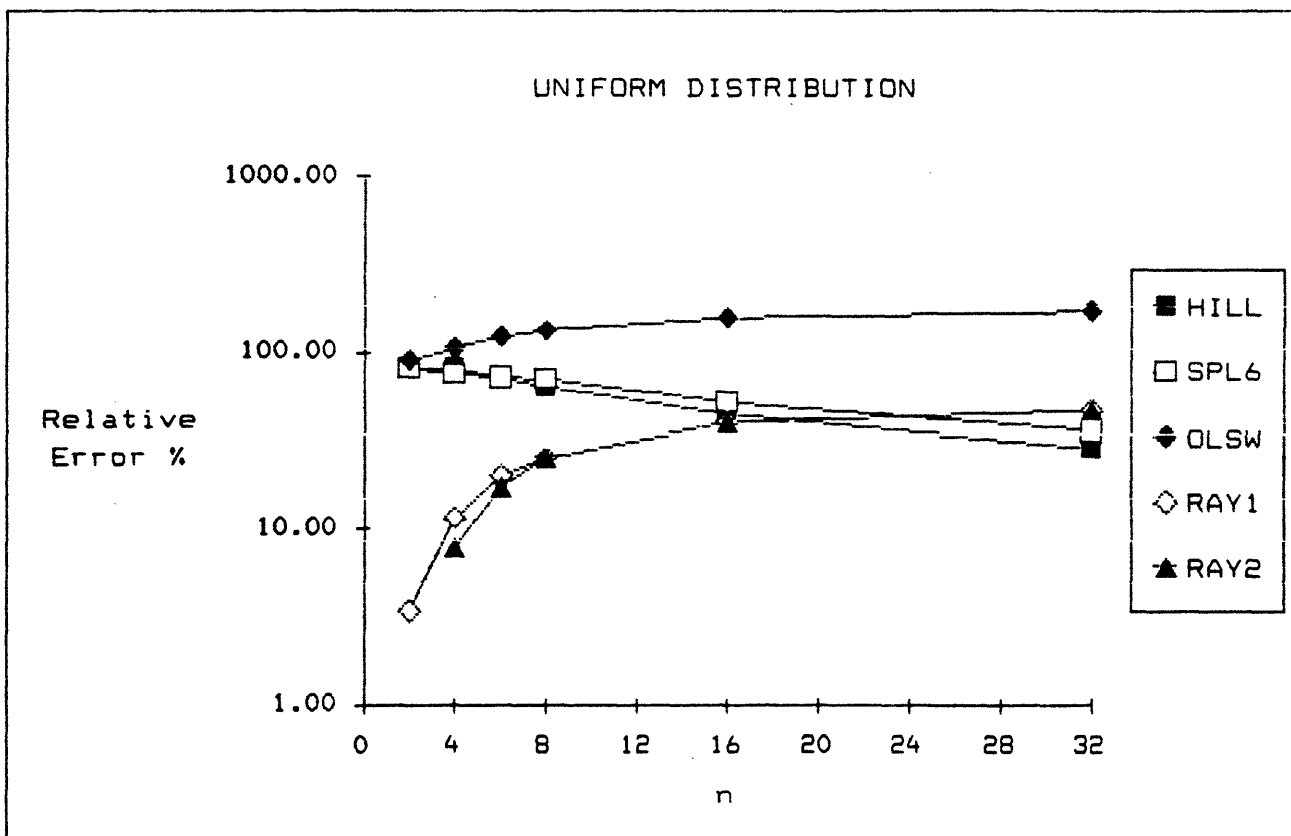


Figure 3b. Results of Tests under Uniform Distribution.