

Pricing Perishable Products: An
Application to the Retail Industry

by
Gabriel R. Bitran
Susana V. Mondschein

WP #3592-93

July 1993

PRICING PERISHABLE PRODUCTS: AN APPLICATION TO THE RETAIL INDUSTRY

GABRIEL R. BITRAN

Sloan School of Management, MIT

SUSANA V. MONDSCHHEIN

Operations Research Center, MIT

July, 1993

Abstract

This paper studies intertemporal pricing strategies when selling perishable products in retail stores. Initially we present a continuous time model where a seller faces a stochastic arrival of customers with different valuations of the product. For this model, we characterize the optimal pricing policies as functions of time and inventory. We find necessary and sufficient conditions for the optimal pricing strategy, which provide an efficient algorithm to compute the optimal pricing policies. This model is extended to consider a more realistic framework with periodic pricing reviews. We show that the structure of the optimal pricing policies is consistent with the procedures observed in practice; retail stores successively discount the product during the season and promote a liquidation sale at the end of the planning horizon. We also show that the loss experienced when implementing periodic pricing review policies instead of continuous time policies is small when the appropriate number of reviews is chosen.

Finally, we generalize the model to the case of a company that has multiple retail stores oriented to different market segments. For this case, we develop a heuristic to solve real size problems; this heuristic has a satisfactory performance with respect to an upper bound that we derive for the optimization problem. The paper also discusses the broader applicability of the models to other industries that have similar features.

1 INTRODUCTION

Pricing a product is one of the most important decisions a seller has to make. As quoted in Monroe (1990) “more and more, today’s pricing environment demands better, faster, and more frequent pricing decisions than ever before. It is also forcing companies to take a new look at pricing and its role in an increasingly complex marketing climate.”

In this paper we study optimal pricing strategies for perishable products in retail stores. However, the same framework can be used to analyze other applications that present the same structure. Section 4 presents examples of applications in the hotel, airline and other industries.

Initially we model the situation of a store that must sell a product within a desired period of time. This is a fairly common situation, in practice, with seasonal products as for example fashion clothing. Usually, if there is an advertisement campaign during the selling season, retail stores promote discount ranges for a family of products without necessarily specifying individual prices. In some cases a sample of representative items are used to promote the sale and customers do not know all the individual prices. Therefore, the arrival rate of potential customers to the store is a response to the regular purchasing patterns during the selling season (which can be affected by an advertisement campaign) rather than a function of individual prices. Another instance that fits in our framework is the case of stores that are known to operate with a strategy of periodic pricing reviews until the products are sold. These stores use this strategy even though the products that they sell are not necessarily perishable. As in the previous case, the arrival of customers to the store is a function of the normal selling practices rather than a reaction to individual prices.

The population of potential customers, those who come to the store, is characterized by a distribution of *reservation prices*, i.e., the maximum price that they are willing to pay for the product. Thus, customers only buy the product if their reservation prices are higher than or equal to the product’s price. The reservation price reflects the value that customers assign to the product. Hence, in general, it has a continuous distribution over the population of potential customers.

The goal of the seller is to determine the pricing policy during the planning horizon that maximizes the total expected profit, considering the heterogeneity of the population of

customers in their willingness to pay for the product.

We first present a model where the price is updated, for all practical purposes, continuously during the planning horizon. We characterize the optimal pricing strategy as a function of the inventory and time left in the planning horizon, considering that all price paths are allowed. We formulate a necessary condition for the optimal solution which is also sufficient for a large family of problems. In those cases, the optimal pricing strategy has a close form solution that can be efficiently computed. The solution to this model is not implementable in practice because prices react to every sale destroying the notion of value of the product.

Later, we extend the basic model to incorporate periodic pricing reviews where prices are allowed to change only at specific instants in time. This type of strategy simplifies the implementation of price changes and is more acceptable to consumers. We show that by choosing the appropriate number of reviews, the loss experienced by the seller when implementing periodic pricing review policies instead of continuous pricing policies is very small. We also show that the optimal pricing policies mimic the procedure commonly observed in practice where retail stores successively discount the product over the planning horizon and finally promote a liquidation sale.

Finally, we extend the models above to consider a company that has multiple retail stores oriented to different market segments. Every store has its own inventory but a centralized management system allows movement of merchandise from one store to another at the end of each period taking into account the associated transportation costs. Hence, the goal is to find the pricing strategies for the stores and the level of inventory that maximize the company's total expected profit. Optimal solutions are difficult to obtain in this case, specially when solving real size problems. Therefore, we develop an heuristic that has a satisfactory performance with respect to an upper bound for this problem.

Traditional pricing strategies based on applying a mark-up to the cost of the article, see Gabor and Granger (1964) and Monroe (1990), are not well suited for our problem. The main factors that determine the pricing policy in our paper are the finite planning horizon, the perishability of the products and the fact that after deciding the initial inventory, the cost of the goods is a "sunk cost".

There are several papers that study intertemporal pricing models, where a monopolistic seller faces a market of consumers with heterogeneous valuations for the product. These

models assume perfect information, i.e., potential consumers always know the current price of the product. Thus, for example, Kalish (1983) assumes that for nondurable goods, customers buy as soon as the price falls below their reservation prices. Besanko and Winston (1990) extend this model to incorporate the assumption that consumers are intertemporal utility maximizers. These papers determine the optimal pricing policies assuming a known deterministic distribution for the reservation prices. They also consider the uncapacitated problem where the demand is always satisfied. The problem that we study in section 2 differs from those described above in that it considers that customers do not have perfect information about the product's price before shopping at the store. Hence, sellers are not able to do a perfect price skimming. We also assume that the reservation price is stochastic, and sellers only know its probability distribution. Finally, an important factor in our model is the limited number of units in inventory that must be managed efficiently in order to maximize the total expected yield.

Gallego and van Ryzin (1993) present a model that is somehow related to the problem that we study in subsection 2.1. Although the mathematical formulations are similar they reflect different customers' behavior. In the case of their paper the arrival of customers to the store responds continuously to changes in price of the specific product under consideration. One possible interpretation of their formulation is that customers are continuously informed about specific prices. As we explained earlier, in our case the arrival of customers to the store is a function of the way the store conducts business rather than a function of a specific price. In our framework the demand is also a function of the price through the distribution of reservation prices. In section 2.2 we present comparative computational results for instances where both approaches can be applied.

To the best of our knowledge the multiple store problem has not been addressed in the literature in a formal way as we do it in section 3.

The remainder of this paper is organized as follows. In Section 2 we present the single outlet model and characterize the optimal pricing policy. We also extend the formulation to include periodic pricing reviews. Section 3 formulates the multiple outlet model and develops a heuristic to solve the mathematical formulation. We provide an upper bound for the problem that allows us to measure the performance of the heuristic. Finally, in Section 4 we present conclusions and extensions.

2 THE SINGLE OUTLET MODEL

In this section we study the case where a single retail store must sell a product within a preestablished time frame using price adjustments to influence the demand. For example, fashionable clothing and food products for special holidays usually fall in this category. We assume that the arrival process of potential customers to the store is described by a Poisson process with an arrival rate that is a function of the general purchasing patterns rather than a function of the specific price of the product under consideration. This behavior is realistic even in the cases where the store conducts an advertisement campaign during the selling season because usually a discount range is announced for a family of products instead of individual prices. In the models that we present in this section we consider a constant arrival rate during the selling season. However, the mathematical formulations and algorithms to find the optimal solutions are also applicable to the case of a time dependent arrival rate, which could represent the effect of a specific promotion during the season.

The population of potential customers is characterized by the *reservation prices*, i.e., the maximum price that they are willing to pay for the product. Hence, if the product's price is lower than their reservation prices, customers buy the product. We consider that the seller only knows the probability distribution of the customers' reservation prices. Thus, she faces the trade-off of losing a customer due to a high price and losing the *consumer's surplus*¹ due to a low price. Hence, at the beginning of every period the seller has to decide the product's price that maximizes the total expected profit during the planning horizon, given the current inventory and the probability distributions for the arrival process and reservation prices. Since the interval of time covered for a selling season tends, in general, not to exceed two or three months inventory reorders are difficult. Thus, we consider that the seller orders the product at the beginning of the planning horizon and reorders are not allowed.

We first present a model that determines the optimal pricing policy for perishable goods where all price paths are allowed, i.e., prices can decrease or increase arbitrarily during the planning horizon. In the second model we incorporate periodic pricing reviews where the seller can modify the price on a periodic basis, as for example once a day, a week or a month.

¹The consumer surplus is equal to the difference between the reservation price and the product's price.

Before presenting the models, we introduce the following notation:

λ = customers' arrival rate per unit of time.

C = inventory at the beginning of the planning horizon.

L = length of the planning horizon.

$f(x)$ = probability density function for the reservation price.

$F(x)$ = cumulative distribution function for the reservation price.

2.1 THE CONTINUOUS TIME MODEL

In this model we consider that the price is updated, for all practical purposes, continuously during the planning horizon. For this purpose we divide the planning horizon in intervals of time small enough so that at most one arrival occurs in each time period. In the limit when these time intervals go to zero we obtain the continuous time formulation. In the remainder of this paper we refer to this model as the continuous time formulation even though it is only an approximation to it.

We define the function $V_t(c)$ as the maximum expected profit if the store starts with c units at the beginning of period t . Since the planning horizon is usually short (one or two months), the implicit discount factor is equal to one. The objective function at the beginning of period t is given by the immediate expected revenue of selling a product at period t plus the expected revenue from period $t-1$ onwards (we count the periods in the planning horizon backwards, i.e., 1 is the last period). The probability of selling a unit at price p in period t is equal to the probability of an arrival times the probability that the arrival's reservation price is higher than or equal to the current price p . For simplicity, we assume that the salvage value of the products is zero. We define Δt as the time interval where at most one arrival occurs (the value of Δt is determined by the customers' arrival rate). Hence, the total number of periods during the planning horizon is equal to $T = L/\Delta t$. The mathematical model is given by the following stochastic and dynamic programming formulation:

$$V_t(c) = \max_{p \geq 0} \{ \lambda \Delta t (1 - F(p))(p + V_{t-1}(c-1)) + (1 - \lambda \Delta t (1 - F(p))) V_{t-1}(c) \}. \quad (1)$$

Boundary conditions:

$$V_t(0) = 0 \quad \forall t,$$

and,

$$V_0(c) = 0 \quad \forall c.$$

The following two propositions characterize the optimal pricing policy. The first proposition shows that for a given period in time, the larger the inventory, the smaller the optimal price. In our model, the only mechanism that the seller has to affect the demand is through the pricing policy. Hence, the seller has to reduce the price to increase the demand when the inventory increases. The second proposition shows that as long as the inventory remains constant, the optimal price is decreasing in time; as time goes by the seller has less possibilities of selling the products. Therefore, combining these two propositions, the optimal pricing policies resulting from the model described above, are non monotonic during the planning horizon, i.e., for a particular outcome of the arrival process and the reservation prices, the optimal price is decreasing in time with jumps that correspond to the instances where the product is sold. Furthermore, as we will show in section 2.2, the expected price, obtained by taking the expectation over all the possible outcomes for the arrival process and for the reservation prices, is not necessarily a decreasing function of time. We define $p_{t,c}$ as the optimal price at period t if the total number of units in inventory is equal to c .

Proposition 1 *For a given period of time, the optimal price is a non-increasing function of the inventory.*

$$p_{t,c} \geq p_{t,c+1} \quad \forall t, c$$

PROOF: See Appendix 1 . ■

Proposition 2 *For a given inventory, the optimal price is a non-increasing function of time.*

$$p_{t,c} \geq p_{t-1,c} \quad \forall t, c$$

PROOF: See Appendix 1 . ■

Next we derive useful first order conditions for the stochastic dynamic formulation described above. We first consider the single-period case, where the seller must price one good,

there will be at most one buyer and the good perishes after one period. A necessary condition for p to be the optimal price is that the seller have no incentive to modify this price. The additional revenue obtained by increasing the price by a small amount dp comes from being able to sell the good at a higher price. The expected benefit this generates is equal to the probability that a customer's reservation price is greater than p times the price change: $(1 - F(p))dp$. Yet this additional revenue comes at a cost since a fraction of customers who were willing to buy the good at the old price are no longer willing to buy it. This fraction is equal to $f(p)dp$, thus the seller expects to lose $pf(p)dp$ because of them. For p to be the optimal price it must be the case that the gain and loss from a small change in price be the same. It follows that:

$$pf(p) = 1 - F(p). \quad (2)$$

In the multi-period case the benefit associated to increasing the price (and therefore the right hand side of equation (2)) remains unchanged. Yet the loss associated by not selling the product is partly offset by the possibility of selling it in the future. Since the expected benefit of selling the product in the future is given by $f(p)dp(V_{t-1}(c) - V_{t-1}(c - 1))$ it follows that the first order condition in the general case is given by:

$$[p - (V_{t-1}(c) - V_{t-1}(c - 1))]f(p) = 1 - F(p). \quad (3)$$

The optimal pricing policy can be found by solving the non linear equation (3) backwards in time. Examples can be easily constructed to show that there may be values of p satisfying equation (3) that do not correspond to the optimal price. A sufficient condition for a price satisfying (3) to be optimal is given by requiring that the function $(1 - F(p))^2/f(p)$ be decreasing in p . We summarize the results above in the following proposition:

Proposition 3 *Assuming that the reservation price can be described by a differentiable cumulative distribution function supported by the positive real line, a necessary condition for the price p to be optimal at time t given an inventory equal to c corresponds to:*

$$p = \frac{1 - F(p)}{f(p)} + V_{t-1}(c) - V_{t-1}(c - 1) \quad (4)$$

If the function $(1 - F(p))^2/f(p)$ is decreasing in p , then the first order condition described in (4) has a unique solution and it corresponds to the optimal price.

PROOF: See Appendix 1 . ■

There are several probability density functions for which the sufficient condition of Proposition 3 is satisfied. For example, the exponential and Weibull distributions ($k \geq 1$).² There are other probability density functions with bounded, convex support for which the first order condition leads to a feasible optimal solution. For example, the uniform distribution in $[0, b]$ has the property mentioned above. In general, when the density has a bounded support, the first order condition is given by the corresponding Kuhn Tucker system of equations.

A stronger sufficient condition for a price satisfying the necessary condition to be optimal is given by requiring that the hazard function associated with the reservation price distribution be increasing, where the hazard function $H(p)$ is defined as $H(p) = f(p)/(1 - F(p))$. This function also has an interesting interpretation: given that the product's price is equal to p and that the reservation price associated to the current request is larger than or equal to p , $H(p)dp$ is approximately equal to the probability that the current customer's reservation price is in the range $[p, p + dp]$. Hence the larger the hazard function, the more likely it is that the seller obtains all the consumer's surplus, *conditional on making the sale*.

The model has the property that the optimal pricing policy is constant over the planning horizon when the capacity is large enough. If the planning horizon is divided in T periods, then the total number of requests is bounded by T (considering that in a period of time at most one arrival occurs). Therefore, for any initial inventory larger than or equal to T , the optimal solution can be found by solving the single period problem given by:

$$\max_{p \geq 0} \{ \lambda \Delta t (1 - F(p)) p \}.$$

In what follows we present a numerical example of the optimal price path given by the model described above. We use a Poisson process to represent the customers' arrival process to the store and a Weibull distribution to represent the probability density function for the reservation prices³. When using a unimodal distribution for the reservation price implicitly

²The Weibull distribution with parameters k and τ is equal to $f(p) = k\tau(\tau p)^{k-1} \exp(-(\tau p)^k) \quad \forall p > 0, \quad \forall k > 0, \tau > 0$.

³The Weibull distribution allows us to obtain a large variety of behavior for the reservation prices: symmetry with respect to the mean, a heavier right tail, a heavier left tail, etc. For example, a form of the Weibull distribution with shape parameter of 3.25 is almost identical with the unit normal distribution (Johnson and Kotz (1970))

we are assuming that the store faces a single market segment. This assumption is close to reality, specially for stores located in local malls. The parameters considered for the Weibull distribution correspond to $r = 0.01$ and $k = 1.5$. The shape of this distribution can be seen in figure 4 in Appendix 1. We consider a planning horizon of one month and an arrival rate of 200 customers per month. In this example, the length of the period of time, Δt , is such that the probability of more than one arrival is less than 99.8%. Figure 1 shows the price path for a particular outcome of the arrival process and the reservation prices. We observe that the optimal price changes continuously during the planning horizon. In practice, this solution is unrealistic because of the coordination and management costs associated to this type of strategy and the confusing information that customers receive about the product's value. The jumps in the price curve correspond to the instants in time where the product is sold. The difficulty to implement in practice the optimal policies given by this model motivates the formulation of a model that allows only periodic pricing reviews. We present this model in the next subsection.

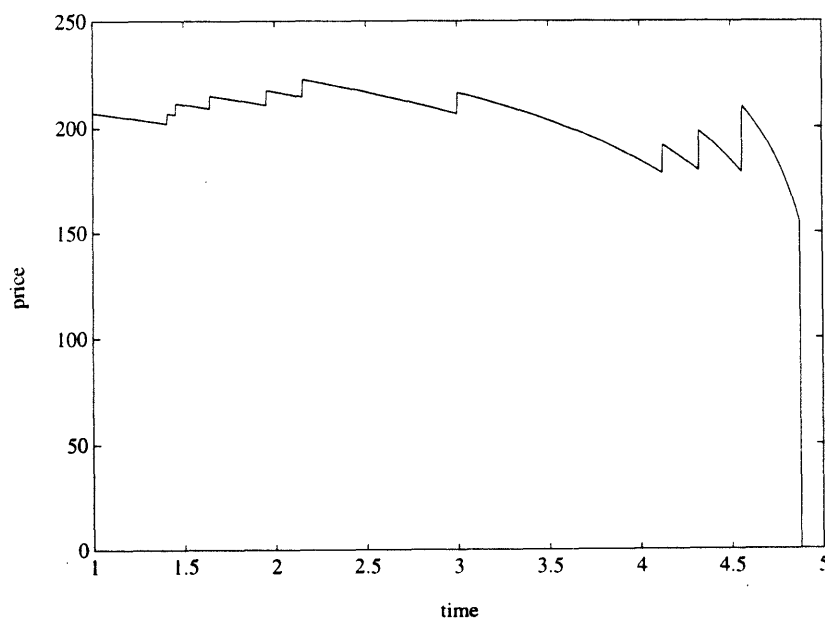


Figure 1: Price path for initial capacity equal to 50 units

2.2 PERIODIC PRICING REVIEWS

In this section we extend the basic model to incorporate periodic pricing reviews where prices are modified at discrete intervals of time, as for example once a day, week or month, reducing management costs and coordination problems. We define K as the number of times the price can be modified during the planning horizon and ΔT_k as the length of the k^{th} time interval, and therefore, $\sum_{k=1}^K \Delta T_k = L$. This division of the planning horizon gives the seller the flexibility of revising the price more frequently towards the end of the season. The new formulation allows multiple arrivals during an interval of time.

We define the function $VP_k(c)$ as the maximum expected profit from period k onwards if the initial inventory is c . Hence, the mathematical formulation is given by:

$$VP_k(c) = \max_{p \geq 0} \left\{ \sum_{j=0}^{\infty} [\min(c, j)p + VP_{k-1}(c - \min(c, j))] \Pr\{j_k(p) = j\} \right\}. \quad (5)$$

Boundary conditions:

$$VP_k(0) = 0 \quad \forall k$$

and,

$$VP_0(c) = 0 \quad \forall c.$$

In equation (5), $j_k(p)$ denotes the random variable that represents the number of potential sales in period k if the price is p . Its probability mass function is given by:

$$\Pr\{j_k(p) = j\} = \sum_{n=j}^{\infty} \left(\frac{n!}{j!(n-j)!} \right) (1 - F(p))^j (F(p))^{n-j} \frac{\exp^{-\lambda \Delta T_k} (\lambda \Delta T_k)^n}{n!}.$$

Observing that the arrival process of *actual buyers* (customers that show up in the store and whose reservation prices are higher than or equal to the current price) can also be seen as a non-homogeneous Poisson process with arrival rate equal to $\lambda(p) = \lambda(1 - F(p))$ and doing some algebraic manipulations the model above can be rewritten as follows:

$$VP_k(c) = \max_{p \geq 0} p \exp^{-\lambda(p)\Delta T_k} \sum_{j=1}^c (\lambda(p)\Delta T_k)^j \frac{(j-c)}{j!} + pc(1 - \exp^{\lambda(p)\Delta T_k}) + \sum_{j=0}^c VP_{k-1}(c-j) \exp^{-\lambda(p)\Delta T_k} \frac{(\lambda(p)\Delta T_k)^j}{j!} \quad (6)$$

This dynamic programming formulation is solved backwards in time. For each stage and initial capacity it is necessary to solve a unidimensional non-linear optimization problem. In all the computational experiments we use the Fibonacci algorithm to solve the non-linear problem.

This model also has the property of a constant pricing policy when the capacity is large enough. To prove this property, we observe that the constant pricing policy obtained for formulation (1) is also feasible for formulation (6), therefore it is also optimal for (6).

In what follows we present a set of computational experiments that show the expected profits given by periodic pricing reviews in comparison to continuous time policies. We consider a single store that faces an average arrival rate of 50 customers per week for the product under consideration. The planning horizon is 4 weeks and the parameters for the Weibull distribution are $r = 0.01$ and $k = 1.5$. Table 1 summarizes the numerical results. The first column contains the initial inventory in number of units. The second, third, fourth and fifth columns correspond to the maximum expected profit for the periodic pricing review problems with 1, 2, 4, and 6 periods respectively, relative to the expected profit for the continuous time problem. In table 1 we use VP^K to denote the maximum expected profit for the periodic pricing review problem with K periods and V^* to denote the maximum expected profit for the continuous time problem described in equation 1. We observe from Table 1 that the expected profits increase significantly when prices are allowed to change during the planning horizon. Comparing the expected profits given by the periodic pricing review problems with one period and six periods, we observe an improvement in the expected profits in the range of 2.5% to 5.3%, which can be crucial to survive in the retail industry. We also observe that the loss experienced by the seller when implementing periodic pricing reviews instead of the continuous pricing strategies is very small when selecting the appropriate number of reviews. In this set of experiments, the losses are less than 1.7% when making three reviews and approximately 1.0% when making five reviews. Figure 2 shows the expected price during the planning horizon for the continuous pricing policy and the periodic pricing review policies with 1,2, and 4 periods. We observe that the expected price for the case of 4 periods follows the continuous curve very closely during the first three weeks and the difference is only significant in the last period.

Figure 3 shows the expected price during the planning horizon for the case of periodic pricing reviews with four periods. The curves correspond to initial inventories of 4, 8 and

Table 1: The one store problem

Initial Inventory	$(VP^1/V^*) * 100$	$(VP^2/V^*) * 100$	$(VP^4/V^*) * 100$	$(VP^6/V^*) * 100$
1	94.1%	97.5%	98.9%	99.4%
2	94.4%	97.1%	98.6%	99.1%
3	94.6%	97.0%	98.4%	98.9%
4	94.8%	97.0%	98.4%	98.9%
5	94.9%	97.0%	98.3%	98.9%
6	95.0%	97.0%	98.3%	98.9%
8	95.2%	97.1%	98.3%	98.8%
10	95.4%	97.2%	98.4%	98.8%
12	95.5%	97.2%	98.4%	98.9%
18	95.8%	97.4%	98.5%	98.9%
25	96.1%	97.6%	98.5%	98.9%
30	96.3%	97.7%	98.6%	99.0%
35	96.5%	97.8%	98.7%	99.0%
40	96.6%	97.9%	98.7%	99.1%

12 units respectively. Additionally to the curves shown in this figure we have computed the expected prices for a range of initial inventories between 1 and 40 units. We have observed that for small initial inventories, relative to the length of the planning horizon and the arrival rate of customers, most of the price paths are decreasing as a function of time, and therefore, the expected price is also monotonically decreasing. An example of this behavior can be observed in figure 3 when the initial inventory is equal to 4. A different behavior is observed for initial inventories that are in the middle of the inventory range considered in this problem. In these cases there are several price paths for which the prices increase after one or two periods in the planning horizon. Hence, the expected prices are not necessarily monotonically decreasing as a function of time. The dynamics of the system in these cases can be explained as follows: due to the fact that the initial inventory is large in comparison to the planning horizon, the initial optimal price is relatively low. Thus, on average the seller is able to get rid off an important fraction of the inventory in the first period. Then, in the

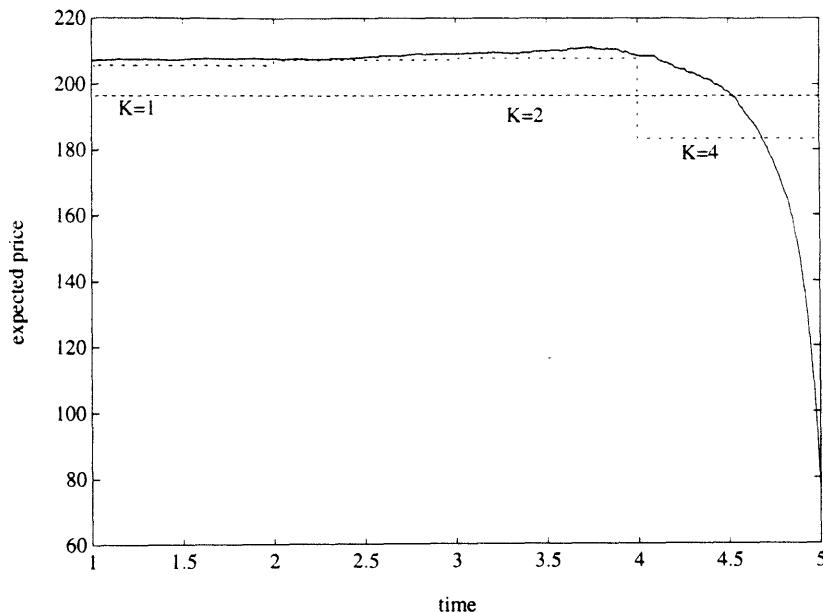


Figure 2: Continuous time vs. Periodic review policies

next periods the optimal expected price is higher because the remaining inventory is “well balanced” with respect to the time left until the end of the planning horizon. We also point out that for the cases where the expected price goes up, the increment that we have observed in the computational experiments has been less than 1.2%. When the inventory is large the price paths tend to be constant during the planning horizon. As we showed earlier, in the limit when the inventory goes to infinity, the optimal price is constant over the planning horizon.

Summarizing the results shown in figure 3, we observe that the expected price is approximately constant during the first three periods with a big sale in the last week. These solutions mimic the procedures commonly observed in practice where retail stores successively discount the product over the planning horizon and finally promote a liquidation sale.

2.3 OPTIMAL INITIAL INVENTORY

The models presented in the previous sections also allow to determine the optimal initial inventory that the seller has to carry to maximize the total expected profit. The optimal initial inventory, C^* , can be found solving the problem:

$$Z = \max_{C \in \mathbb{Z}^+} \{V_T(C) - gC\},$$

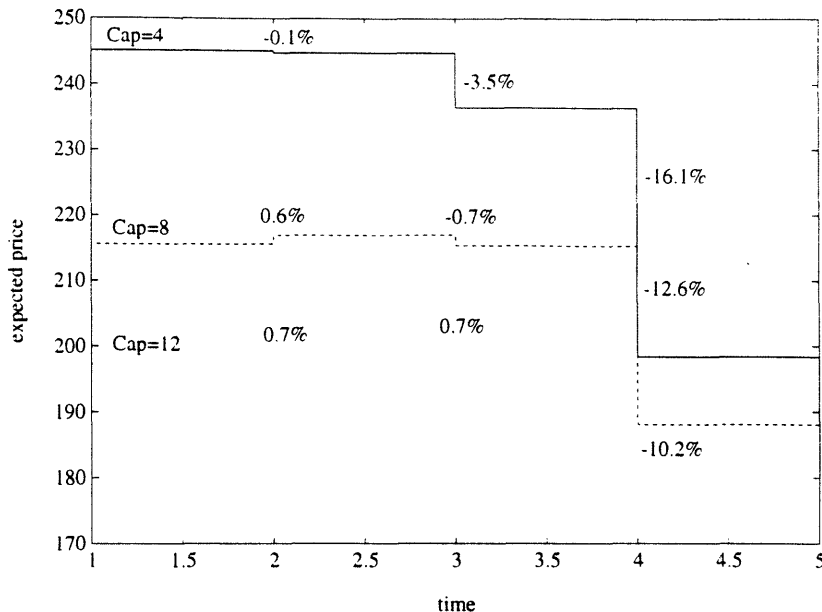


Figure 3: Expected price for different initial inventories

where g is the unit cost of the goods. By lemma 2 in Appendix 1, we have that the function $V_T(C)$ is concave as a function of the capacity. Hence, the optimal initial inventory is determined by the inventory such that the marginal increment in the function $V_T(C)$ is equal to the unit good's cost. The mathematical condition is given by:

$$V_T(C^*) - V_T(C^* - 1) \geq g \geq V_T(C^* + 1) - V_T(C^*).$$

Hence, when solving the dynamic programming formulation to find the optimal pricing policies, we can also compute the optimal initial inventory with little additional work.

3 THE MULTIPLE OUTLET CASE

In this section we study the case of a company that has multiple outlets, selling the product to different market segments. We consider the case where the company can make price discrimination, i.e., the same product in different outlets can be sold at different prices. This situation usually happens with companies that have several retail stores with different names that are oriented to different classes of customers. We assume that each store manages its own inventory, however, a global allocation allows to move merchandise from one store to

another at the end of each period of time (day, week or month). The goal is to determine the pricing policies for the stores and the inventory management that maximize the company's total expected profit. For simplicity in the presentation, in what follows we study the case of a company with two retail stores. However, all the results can be directly extended to the case of multiple stores. We introduce the following additional notation:

λ_i = arrival rate of customers at store i , $i = 1, 2$.

(c_k^1, c_k^2) = inventory at the end of period k in stores 1 and 2 respectively.

I_k^i = inventory at the beginning of period k in store i after adjusting the inventories, $i = 1, 2$.

v = moving cost per unit.

z_k = number of units moved from one store to another in period k .

$j_{ik}(p)$ = random variable that represents the number of potential buyers in store i if the price is equal to p during period k .

$F_i(p)$ = cumulative density function for the reservation price in store i , $i = 1, 2$.

$VP_k(c_{k+1}^1, c_{k+1}^2)$ = maximum expected profit from period k onwards if the company starts with an inventory equal to (c_{k+1}^1, c_{k+1}^2) .

The objective function is given by the immediate expected profit in period k plus the expected profit from period $k - 1$ onwards. Given the limited number of units in inventory, the expected number of sales is the minimum between the inventory and the expected number of buyers. Thus, the model has the following mathematical formulation:

$$VP_k(c_{k+1}^1, c_{k+1}^2) = \max_{p_1, p_2, I_k^1, I_k^2} \left\{ p_1 \sum_{i=0}^{\infty} \min\{i, I_k^1\} \Pr\{j_{1k}(p_1) = i\} + p_2 \sum_{i=0}^{\infty} \min\{i, I_k^2\} \Pr\{j_{2k}(p_2) = i\} + \right. \\ \left. -v z_k + E_{j_{1k}(p_1), j_{2k}(p_2)} [VP_{k-1}(c_k^1, c_k^2)] \right\}$$

s.t.

$$I_k^1 + I_k^2 = c_{k+1}^1 + c_{k+1}^2 \quad (7)$$

$$z_k \geq I_k^1 - c_{k+1}^1 \quad (8)$$

$$z_k \geq I_k^2 - c_{k+1}^2 \quad (9)$$

$$c_k^1 = I_k^1 - \min(I_k^1, j_{1k}(p1)) \quad (10)$$

$$c_k^2 = I_k^2 - \min(I_k^2, j_{2k}(p2)) \quad (11)$$

$$p_1 \geq 0, p_2 \geq 0, I_k^1 \in Z^+ \text{ and } I_k^2 \in Z^+,$$

where the probability distribution for $j_{ik}(p)$ is given by:

$$\Pr\{j_{ik}(p) = j\} = \sum_{n=j}^{\infty} \frac{(1 - F_i(p))^j (F_i(p))^{n-j}}{j!(n-j)!} \exp^{-\lambda_i \Delta T_k} (\lambda_i \Delta T_k)^n \quad i = 1, 2.$$

The first constraint corresponds to the balance equation for the inventory; the total initial inventory must be equal to the total inventory after moving merchandise from one store to another. Constraints (8) and (9) define the total number of units moved from one store to another at the beginning of period k . Finally, constraints (10) and (11) update the inventory at the end of period k .

This mathematical formulation is difficult to solve specially when solving real size problems. Thus, in what follows we present a heuristic developed to find a pricing policy for the two stores model. The heuristic evaluates all possible inventory adjustments at the beginning of each period and chooses the one that maximizes the total expected profit assuming that the prices must be held constant until the end of the planning horizon.

Description of the heuristic: HEUR1

The following heuristic determines the pricing policy at the beginning of period k if the initial inventories are equal to c_1 and c_2 in stores 1 and 2 respectively.

Step 0: Initialization

$$prof = -\infty,$$

$$c = c_1 + c_2$$

$$Inv_1 = 0$$

$$Inv_2 = c$$

Step 1: Solve the non-linear programming problem below to find the optimal prices in period k if the inventories at stores 1 and 2 are Inv_1 and Inv_2 respectively.

$$D(Inv_1, Inv_2) = \max_{p^1, p^2} \{p^1 \min[N_{\lambda(p^1)}, Inv_1] + p^2 \min[N_{\lambda(p^2)}, Inv_2] - v|Inv_1 - c_1|\}$$

s.t.

$$p^1 \geq 0, p^2 \geq 0.$$

where $\Delta T = \sum_{n=1}^k \Delta T_n$, and $N_{\lambda(p^i)}$ is a Poisson random variable with arrival rate $\lambda(p^i) = \lambda_i \Delta T (1 - F_i(p^i))$

Step 2: Check if the current solution leads to an improvement in the objective function.

If($D(Inv_1, Inv_2) > prof$)then

$$I_1 = Inv_1, I_2 = c - Inv_1$$

$$p_k^1 = p^1, p_k^2 = p^2$$

Endif

$$Inv_1 = Inv_1 + 1, Inv_2 = Inv_2 - 1$$

If($Inv_1 \leq c$) Goto Step1

Step 3: STOP: the prices in period k in stores 1 and 2 are p_k^1 and p_k^2 respectively and the initial inventories are equal to I_1 and I_2 .

In what follows we present a model which solution corresponds to an upper bound for the model presented above. In this upper bound both stores share the inventory permanently during the planning horizon and prices can be updated after each Δt units of time, where Δt is small enough so that in total, considering both stores, at most one arrival occurs in every time period. Hence, the following model considers a total of $L/\Delta t$ periods.

Proposition 4 *The solution of the following problem is an upper bound to the optimization problem described in this section.*

$$V_t(c_{t+1}) = \max_{p_1 \geq 0, p_2 \geq 0} \{ \lambda_1 \Delta t (1 - F_1(p_1)) [p_1 + V_{t-1}(c_{t+1} - 1)] + \lambda_2 \Delta t (1 - F_2(p_2)) [p_2 + V_{t-1}(c_{t+1} - 1)] \\ + [1 - \lambda_1 \Delta t (1 - F_1(p_1)) - \lambda_2 \Delta t (1 - F_2(p_2))] V_{t-1}(c_{t+1}) \}$$

where $c_{t+1} = c_{t+1}^1 + c_{t+1}^2$.

PROOF: The proof is straightforward if we note that any feasible outcome for the selling strategy in the optimization problem can be reproduced for the upper bound formulation, without the costs of moving merchandise from one store to another. Thus, for example, suppose that at period k , the optimal prices are p_1 and p_2 and the optimal initial inventories (after the adjustments) are I^1 and I^2 for stores 1 and 2, respectively. Then, the upper bound

formulation can reproduce the same solution taking p_1 and p_2 as the optimal prices; if store i sells I^i units during period k then p_i is set equal to a large number (“infinity”) for any additional unit. This avoids selling more units than those available at store i . Hence, any selling strategy for the optimization problem is also feasible for the upper bound formulation to a lower cost. ■

The upper bound formulation is solved using the same approach used for the one store model: we solve the first order condition for the optimal pricing strategy backwards in time. This condition is given by:

$$p_{c,t}^i = H_i^{-1}(p_{c,t}^i) + V_{t-1}(c) - V_{t-1}(c - 1). \quad i = 1, 2$$

We observe that the optimal price in store i consists of the sum of two terms. The first term, $H_i^{-1}(p)$, depends only on the probability distribution for the reservation prices of the i -th store’s customers. The second term, $V_{t-1}(c) - V_{t-1}(c - 1)$, takes into account the interaction between both stores. The optimal expected profit, $V_{t-1}(c)$, depends on the optimal pricing policies for both stores from period $t - 1$ onwards, for any initial capacity.

In what follows we present computational experiments that show the performance of this heuristic with respect to the upper bound. We use Monte Carlo simulations to estimate the expected profits given by the heuristic. At every period in time, we determine the pricing policy defined by the heuristic considering the available inventory and the remaining time in the planning horizon. Then, we simulate the arrival process during the next period in the planning horizon with the corresponding reservation prices. Finally, using this outcome, we update the inventory at the end of the period and apply the pricing policy again. Taking the average of the profits given by repeated simulations, we estimate the expected profit. The simulations stop when the standard deviation is less than or equal to 0.1% of the expected profit. The upper bounds are computed solving the corresponding dynamic programming formulations.

Table 2 shows the performance of the heuristic for two problems that have the same planning horizon divided in 2 and 4 periods where prices can be changed. We refer to these problems as P2 and P4 respectively. The planning horizon is one month for both problems. We consider arrival rates, for the product under consideration, of 200 and 150 customers per month to stores 1 and 2, respectively. We use a Weibull distribution for the reservation prices with parameters $k_1 = 5, r_1 = .010$, and $\epsilon_1 = 0$ for store 1 and $k_2 = 4, r_2 = .007$, and

$\epsilon_2 = 0$ for store 2. The shape of these two distributions can be seen in figure 4 in Appendix 1.

Table 2: The two stores problem

Initial Inventory	HUER1: P2 w/r BOUND1	HUER1: P4 w/r BOUND1
2	94.2%	94.4%
4	93.7%	94.4%
6	94.0%	94.8%
8	94.8%	95.0%
10	95.2%	95.7%
15	95.5%	96.5%
18	95.9%	96.6%
25	96.4%	97.1%
30	96.7%	97.5%
40	97.1%	97.7%
60	97.4%	98.0%
80	97.5 %	98.3%

The first column is the initial inventory. The second and third columns contain the performance of the heuristic with respect to the upper bound for problems P1 and P2 respectively. We observe that the heuristic's performance improves as long as the inventory increases. Similarly to the single outlet case, when the inventory is large enough the optimal pricing policy is constant over the planning horizon for each store and is equal to the pricing policy given by the heuristic. In this case no merchandise is reallocated from one outlet to another. The overall performance of the heuristic is satisfactory with results in the range of 94.4% to 98.3% relative to the upper bound for problem P4. It is important to notice that the upper bound tends not to be very tight with respect to the optimal solution. This is due to the fact that in the upper bound formulation the inventory is permanently shared by both outlets. Therefore, the problem can be seen as a single store that faces a joint demand distribution and the inventory can be optimally allocated between these two different market

segments as customers show up. To illustrate that the upper bound is not necessarily tight we present the following two examples. In the first case we consider a problem with one period and an initial inventory of one unit. The upper bound for this case is equal to \$266.0. Furthermore, it is straightforward to prove that, for this set of parameters, the heuristic leads to the optimal solution equal to \$247.5 which corresponds to 93.0% of the upper bound (even though it is the optimal solution). In the second case we consider a problem where the planning horizon is divided in four periods and the initial inventory is equal to one unit. For this small example it is possible to compute the optimal expected profit which is 97.7% of the upper bound. We observe that the performance of the heuristic is 94.4% relative to the upper bound. However it improves to 96.4% when we compare it with respect to the optimal solution.

4 CONCLUSIONS AND EXTENSIONS

This paper has studied optimal pricing strategies for perishable products in retail stores. We have considered a continuous time problem where a seller faces a stochastic arrival of customers with heterogeneous valuations of the product. For this model we have characterized the optimal pricing policies as follows: for every outcome of the arrival process with the corresponding reservation prices, the optimal price is a decreasing function of time with jumps when goods are sold. A necessary and sufficient condition for the optimal price is given, which is satisfied for a large group of distributions for the reservation price. For these distributions, the optimal pricing strategy can be easily computed solving the first order condition backwards in time. We have also extended this model to incorporate periodic pricing reviews which is a feature usually desired in practical applications. Computational experiments have shown that the loss of profit when including the appropriate number of reviews is small. We have also shown that the optimal pricing policies reproduce what is usually observed in practice: retail stores promote successive price discounts with a liquidation sale at the end of the season.

Finally, we have generalized the basic model to consider a company that has multiple retail stores in different markets. We have developed an efficient heuristic to find approximations to the optimal pricing policies for this case, which are particularly useful when solving real size problems.

The models developed in this paper can be extended to incorporate reservation prices that evolve over time. There are examples in practice where people are willing to pay less for the same item as time goes by. For example, winter clothings have less value for customers as the spring approaches. Another interesting extension is to consider bayesian updating of the parameters in the reservation price distribution functions. We leave the formal study of these two topics for future research.

OTHER APPLICATIONS

The models studied in this paper can also be applied to other industries that sell perishable products or services (in the sense that their residual values are eventually equal to zero). In what follows we cite three other potential applications.

1. Tickets for special events as for example sport games, theater performances and concerts.
2. Hotel rooms is another example that fits in this framework. Usually the reservation process starts several weeks before the actual target date and managers develop pricing strategies to maximize the total expected yield. The models developed in this paper can be applied to this case without major changes when cancellations are not an important issue.
3. Finally, the models can be used to determine the pricing strategies in the airline industry. As in the hotel industry case, the results derived in this paper can be applied to selling seats for a particular flight when cancellations are not significant.

Appendix 1

Lemma 1 *The function $V_t(c)$ is non-increasing as a function of time.*

$$V_t(c) \geq V_{t-1}(c)$$

PROOF: For $t = 1$ the inequality holds trivially. We assume that it holds for t , and prove it for $t + 1$.

$$V_{t+1}(c) = \max_{p \geq 0} \{ \lambda \Delta t (1 - F(p))(p + V_t(c - 1)) + (1 - \lambda \Delta t (1 - F(p))) V_t(c) \},$$

using that the proposition is true for t , we obtain:

$$V_{t+1}(c) \geq \max_{p \geq 0} \{ \lambda \Delta t (1 - F(p))(p + V_{t-1}(c-1)) + (1 - \lambda \Delta t (1 - F(p))) V_{t-1}(c) \},$$

which is equal to:

$$V_{t+1}(c) \geq V_t(c). \quad \blacksquare$$

Lemma 2 *The function $V_t(c)$ is a concave function of the capacity,*

$$V_t(c+1) - V_t(c) \geq V_t(c+2) - V_t(c+1) \quad \forall t, c$$

and the additional profit given by an extra unit increases as long as the remaining time until the end of the planning horizon increases,

$$V_{t+1}(c+1) - V_{t+1}(c) \geq V_t(c+1) - V_t(c) \quad \forall t, c$$

PROOF: We define the following inequalities:

$$I1(c, t) : V_{t+1}(c+1) - V_{t+1}(c) \geq V_t(c+1) - V_t(c) \quad \forall t, c$$

$$I2(c, t) : V_{t+1}(c) - V_t(c) \geq V_{t+2}(c) - V_{t+1}(c) \quad \forall t, c$$

$$I3(c, t) : V_t(c+1) - V_t(c) \geq V_t(c+2) - V_t(c+1) \quad \forall t, c$$

The proof is done by induction in $k = t + c$. The inequalities $I1(c, t)$, $I2(c, t)$, and $I3(c, t)$ hold trivially for $k = 0$. We assume that the three inequalities are satisfied for all $t + c < k$ and we prove that they hold for $t + c = k$.

i) We prove that $I1(c, t)$ holds when $t + c = k$. For $c = 0$, using lemma 1 we obtain that $I1(0, t)$ is true for all t . Suppose $c > 0$, hence:

for some \bar{p} , we have,

$$V_{t+1}(c) = \lambda \Delta t (1 - F(\bar{p})) \bar{p} + \lambda \Delta t (1 - F(\bar{p})) V_t(c-1) + (1 - \lambda \Delta t (1 - F(\bar{p}))) V_t(c),$$

subtracting $V_t(c)$ from both sides of the equation above, we get,

$$V_{t+1}(c) - V_t(c) = \lambda \Delta t (1 - F(\bar{p})) \bar{p} + \lambda \Delta t (1 - F(\bar{p})) (V_t(c-1) - V_t(c)) \quad (12)$$

Because \bar{p} is feasible for $V_{t+1}(c+1)$ we have,

$$V_{t+1}(c+1) \geq \lambda \Delta t (1 - F(\bar{p})) \bar{p} + \lambda \Delta t (1 - F(\bar{p})) V_t(c) + (1 - \lambda \Delta t (1 - F(\bar{p}))) V_t(c+1),$$

subtracting $V_t(c+1)$ from both sides of the previous inequality we get,

$$V_{t+1}(c+1) - V_t(c+1) \geq \lambda\Delta t(1 - F(\bar{p}))\bar{p} + \lambda\Delta t(1 - F(\bar{p}))(V_t(c) - V_t(c+1)) \quad (13)$$

By $I3(c-1, t)$ we know that the following inequality holds,

$$V_t(c) - V_t(c+1) \geq V_t(c-1) - V_t(c)$$

hence, replacing it in (13) we obtain:

$$V_{t+1}(c+1) - V_t(c+1) \geq \lambda\Delta t(1 - F(\bar{p}))\bar{p} + \lambda\Delta t(1 - F(\bar{p}))(V_t(c-1) - V_t(c)) \quad (14)$$

Finally, (12) and (14) together lead to,

$$I1(c, t) : V_{t+1}(c+1) - V_t(c+1) \geq V_{t+1}(c) - V_t(c).$$

ii) We prove that $I2(c, t)$ holds for $t + c = k$.

For some \bar{p} we have,

$$V_{t+2}(c) = \lambda\Delta t(1 - F(\bar{p}))\bar{p} + \lambda\Delta t(1 - F(\bar{p}))V_{t+1}(c-1) + (1 - \lambda\Delta t(1 - F(\bar{p})))V_{t+1}(c),$$

subtracting $V_{t+1}(c)$ from both sides of the equation above, we get,

$$V_{t+2}(c) - V_{t+1}(c) = \lambda\Delta t(1 - F(\bar{p}))\bar{p} + \lambda\Delta t(1 - F(\bar{p}))(V_{t+1}(c-1) - V_{t+1}(c)) \quad (15)$$

Because \bar{p} is feasible for $V_{t+1}(c)$ we have,

$$V_{t+1}(c) \geq \lambda\Delta t(1 - F(\bar{p}))\bar{p} + \lambda\Delta t(1 - F(\bar{p}))V_t(c-1) + (1 - \lambda\Delta t(1 - F(\bar{p})))V_t(c),$$

subtracting $V_t(c)$ from both sides we get,

$$V_{t+1}(c) - V_t(c) \geq \lambda\Delta t(1 - F(\bar{p}))\bar{p} + \lambda\Delta t(1 - F(\bar{p}))(V_t(c-1) - V_t(c)) \quad (16)$$

By $I1(c-1, t)$ we know that,

$$V_t(c-1) - V_t(c) \geq V_{t+1}(c-1) - V_{t+1}(c)$$

Replacing this inequality in (16) we obtain,

$$V_{t+1}(c) - V_t(c) \geq \lambda\Delta t(1 - F(\bar{p}))\bar{p} + \lambda\Delta t(1 - F(\bar{p}))(V_{t+1}(c-1) - V_{t+1}(c)) \quad (17)$$

Finally, (15) and (17) lead to:

$$I2(c, t) : V_{t+1}(c) - V_t(c) \geq V_{t+2}(c) - V_{t+1}(c).$$

iii) Finally, we prove that the inequality $I3(c, t)$ holds for $k = c + t$.

Similarly to the previous cases, for some \bar{p} , we get the following two inequalities:

$$V_t(c+2) - V_{t-1}(c+1) = \lambda\Delta t(1 - F(\bar{p}))\bar{p} + (1 - \lambda\Delta t(1 - F(\bar{p}))) (V_{t-1}(c+2) - V_{t-1}(c+1)) \quad (18)$$

and,

$$V_{t+1}(c+1) - V_t(c) \geq \lambda\Delta t(1 - F(\bar{p}))\bar{p} + (1 - \lambda\Delta t(1 - F(\bar{p}))) (V_t(c+1) - V_t(c)) \quad (19)$$

Using $I1(c, t-1)$ and $I3(c, t-1)$ we have the following inequalities,

$$V_t(c+1) - V_t(c) \geq V_{t-1}(c+1) - V_{t-1}(c)$$

and,

$$V_{t-1}(c+1) - V_{t-1}(c) \geq V_{t-1}(c+2) - V_{t-1}(c+1)$$

hence,

$$V_{t+1}(c+1) - V_t(c) \geq \lambda\Delta t(1 - F(\bar{p}))\bar{p} + (1 - \lambda\Delta t(1 - F(\bar{p}))) (V_{t-1}(c+2) - V_{t-1}(c+1)) \quad (20)$$

(18) and (20) lead to,

$$V_{t+1}(c+1) - V_t(c) \geq V_t(c+2) - V_{t-1}(c+1) \quad (21)$$

Additionally, by $I2(c+1, t-1)$ we have,

$$V_t(c+1) - V_{t-1}(c+1) \geq V_{t+1}(c+1) - V_t(c+1)$$

or equivalently,

$$2V_t(c+1) \geq V_{t-1}(c+1) + V_{t+1}(c+1) \quad (22)$$

(21) and (22) together lead to the desire inequality,

$$I3(c, t) : V_t(c+1) - V_t(c) \geq V_t(c+2) - V_t(c+1). \quad \blacksquare$$

Proof of Proposition 1

Defining the function $h_t(p, c)$ equal to:

$$h_t(p, c) = \lambda\Delta t(1 - F(p))p + \lambda\Delta t(1 - F(p))V_{t-1}(c - 1) + (1 - \lambda\Delta t(1 - F(p)))V_{t-1}(c),$$

the maximization problem is equivalent to,

$$V_t(c) = \max_{p \geq 0} \{h_t(p, c)\}$$

Let $p_{t,c}$ be the optimal price at the beginning of period t when the initial inventory is c . Hence, the following inequality holds for all p ,

$$h_t(p_{c,t}, c) \geq h_t(p, c) \quad \forall p.$$

Because $p_{c,t}$ is feasible for the maximization problem starting with an inventory of $c + 1$ units, a sufficient condition for $p_{t,c+1}$ to be smaller than or equal to $p_{t,c}$ is:

$$h_t(p_{c,t}, c + 1) \geq h_t(p, c + 1) \quad \forall p \geq p_{t,c}.$$

Thus, a stronger sufficient condition is given by,

$$h_t(p_{c,t}, c + 1) - h_t(p, c + 1) \geq h_t(p_{c,t}, c) - h_t(p, c) \quad \forall p \geq p_{t,c}.$$

Replacing the function $h_t(p, c)$ by its corresponding value, we get that the sufficient condition is equivalent to,

$$V_{t-1}(c) - V_{t-1}(c - 1) \geq V_{t-1}(c + 1) - V_{t-1}(c)$$

which is true by lemma 2. Therefore, the optimal price is a non increasing function of the capacity. ■

Proof of Proposition 2

Using the same notation as in the previous proof, we have:

$$h_t(p_{c,t}, c) \geq h_t(p, c) \quad \forall p.$$

Because $p_{c,t}$ is feasible for the maximization problem starting at period $t - 1$, a sufficient condition for $p_{t-1,c}$ to be smaller than or equal to $p_{t,c}$ is:

$$h_{t-1}(p_{c,t}, c) \geq h_{t-1}(p, c) \quad \forall p \geq p_{t,c}.$$

Hence, a stronger sufficient condition is given by:

$$h_{t-1}(p_{c,t}, c) - h_{t-1}(p, c) \geq h_t(p_{c,t}, c) - h_t(p, c) \quad \forall p \geq p_{t,c}.$$

Thus, replacing $h_t(p, c)$ by its corresponding expression we obtain the following sufficient condition:

$$V_{t-1}(c-1) - V_{t-2}(c-1) \leq V_{t-1}(c) - V_{t-2}(c)$$

which is true by Lemma 2. Hence, the optimal price is non increasing as a function of time.

■

Proof of Proposition 3

The first order condition for the optimal price given by

$$p = \frac{1 - F(p)}{f(p)} + V_{t-1}(c) - V_{t-1}(c-1),$$

is obtained by setting the derivative of the objective function equal to zero. This equation has a unique solution if the function $G(p)$ is increasing as a function of p , where $G(p)$ corresponds to:

$$G(p) = p - \frac{1 - F(p)}{f(p)}.$$

After some algebraic manipulations we can show that an equivalent condition for $G(p)$ to be an increasing function of p is given by:

$$2\partial[\log(1 - F(p))] \leq \partial[\log(f(p))] \quad \forall p,$$

or equivalently,

$$\frac{[1 - F(p_2)]^2}{f(p_2)} \leq \frac{[1 - F(p_1)]^2}{f(p_1)} \quad \forall p_1 \leq p_2$$

Therefore, the function $G(p)$ is increasing in p if and only if the function $(1 - F(p))^2/f(p)$ is decreasing in p . Hence, the first order condition has a unique solution if $(1 - F(p))^2/f(p)$ is a decreasing function of p . Finally, assuming that the probability density function for the reservation price is bounded, this unique solution must be the optimal solution. ■

Figure 4 shows the Weibull density function for the parameters used in the computational experiments.

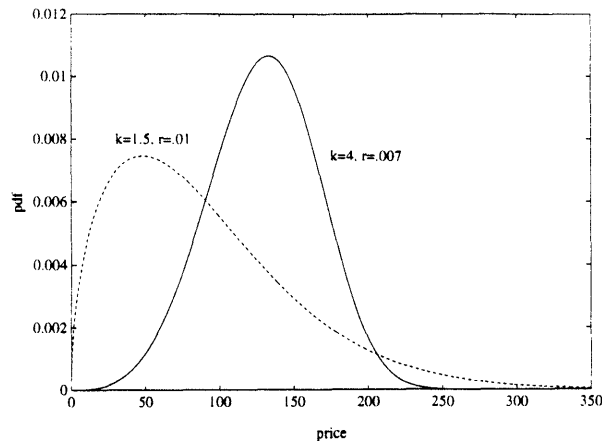


Figure 4: Weibull p.d.f.

REFERENCES

1. BESANKO, D. and WINSTON, W. (1990), "Optimal Price Skimming by a Monopolist Facing Rational Consumers", *Management Science*, Vol.36, No. 5, 555-567.
2. GABOR, A. and GRANGER, C.W.J. (1964), "Price Sensitivity of the Customers", *Journal of Advertising Research*, Vol.4, (Dec.), 40-44.
3. GALLEGO, G. and VAN RYZIN, G. (1993), "Optimal Dynamic Pricing of Inventories with Stochastic Demand Over Finite Horizons", Columbia University.
4. JOHNSON, N.L. and KOTZ, S. (1970), *Continuous Univariate Distributions - 1*, Houghton Mifflin Company, Boston.
5. KALISH, S. (1983), "Monopolistic Pricing with Dynamic Demand and Production Cost," *Marketing Science*, 2, 135-159.
6. MONROE, K.B. (1990), *Pricing: Making Profitable Decisions* McGraw-Hill, Series in Marketing.
7. PASTERNAK, B.A. (1985), "Optimal Pricing and Return Policies for Perishable Commodities", *Marketing Science*, Vol.4, No. 2, 166-176.