# Nonlinear formulations and improved randomized approximation algorithms for multiway and multicut problems

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**Abstract**

We introduce nonlinear formulations of the multiway cut and multicut problems. By simple linearizations of these formulations we derive several well known formulations and valid inequalities as well as several new ones. Through these formulations we establish a connection between the multiway cut and the maximum weighted independent set problem that leads to the study of the tightness of several LP formulations for the multiway cut problem through the theory of perfect graphs. We also introduce a new randomized rounding argument to study the worst case bound of these formulations, obtaining a new bound of $2\alpha(H)(1-\frac{1}{k})$ for the multicut problem, where $\alpha(H)$ is the size of a maximum independent set in the demand graph $H$.

## 1 Introduction

Given a graph $G = (V, E)$ with edge weight $c_e$ for each $e \in E$, and a set of terminal vertices $T = \{v_1, v_2, \ldots, v_k\}$, a *multiway cut* is a set of edges whose removal disconnects every pair of terminal vertices. The problem of finding the multiway cut of minimum total weight is called the multiway cut problem. When $T$ consists of only two terminals ($k = 2$) the problem reduces to the well known minimum cut problem. For $k \geq 3$, it has been shown by Dalhaus et. al. [7] that the problem is $NP-$hard even on planar graphs.

The case $k = 2$ is not the only polynomially solvable instance of the multiway cut problem. Lovász [12] and Cherkasskij [3] show that when $c_e = 1 \; \forall e \in E$ and $G$ is Eulerian, then the multiway cut problem is polynomially solvable. Erdos and Székely [8] have shown that a generalization of the multiway cut problem is polynomially solvable when the underlying graph $G$ is a tree. Dalhaus et. al. [7] have shown the problem to be polynomial solvable for *fixed* $k$ on planar graphs.

Chopra and Rao [5] and Cunningham [6] have investigated the multiway cut problem using a polyhedral approach. They derive a number valid inequalities and facets. For one particular formulation of the problem, Cunningham [6] shows that the value of the minimum multiway cut is

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at most twice the value of its linear programming relaxation. Chopra and Owen [4] proposed an extended formulation of the problem which was shown to be tighter than all previously proposed. In addition, when the underlying graph is a tree, they show that the extended formulation is integral. They also report computational results that show that their formulation consistently yields high quality solutions to the multiway cut problem. Regarding approximation algorithms, Dalhaus et. al. [7] proposed a $2(1-\frac{1}{k})$ approximation algorithm.

A more general problem that we also consider in the present paper is the *multicut problem*. Given a graph $G=(V,E)$ with edge weights $c_e$ for each $e \in E$, and a demand graph $H=(V(H),E(H))$, find a minimum weight set of edges whose removal disconnects each node $s \in V(H)$ from $t \in V(H)$ if $(s,t) \in E(H)$. If $V(H)$ is a complete graph on $k$ vertices, the multicut problem reduces to the multiway cut problem. Regarding approximation algorithms Garg et. al. [9] propose an algorithm that produces a multicut whose weight is within $O(\log(|V(H)|))$ from the optimal solution.

Our contributions and the structure of the present paper are as follows:

1. In Section 2, we express the multiway cut problem as a *continuous nonlinear* program and establish its integrality through the probabilistic method (Alon and Spencer [1]). The formulation provides a framework for the study of extended linear formulations. Many of the known standard and extended formulations and valid inequalities, as well as new ones, can be derived from simple linearizations of the nonlinear constraints. This provides a systematic way to construct improved extended formulations for the multiway cut problem. In particular we derive the extended formulation of Chopra and Owen [4].

2. In Section 3, we establish a connection between the multiway cut problem and the stable set problem. This allows us to derive relaxations for the multiway cut problem that are stronger than previously known. In addition, we use the theory of perfect graphs to prove the integrality of some extended formulations of the multiway cut problem that have special structure. In this way we identify new polynomially solvable cases. Moreover, we obtain the integrality result of Chopra and Owen [4] mentioned above (when the underlying graph is a tree) and the result of Erdos and Székely [8] as a corollary.

3. In Section 4, we propose a new randomized approximation algorithm for the multiway cut problem based on probabilistically rounding the optimal fractional solution of an associated linear program. Compared with traditional randomized rounding, our algorithm introduces dependencies in the rounding process. If $Z_{LP}$ denotes the optimal objective function value of the linear relaxation and $Z_{IP}$ the value of the optimal integer solution, we show that $Z_{IP} \leq 2(1-\frac{1}{k})Z_{LP}$. It offers a slight improvement over the Dalhaus et. al. [7] approximation bound. As a by product we get a new proof of the max-flow-min-cut theorem based on the probabilitsic method.

4. In Section 5, we introduce a new formulation for the multicut problem and apply the randomized rounding technique of Section 4 to obtain a new $2\alpha(H)(1-\frac{1}{k})$ approximation algorithm for the problem, where $\alpha(H)$ is the size of a maximum independent set in the demand graph $H$. Notice that the bound is a direct generalization of the bound for the multiway cut problem. Our bound is stronger than the bound of $O(\log(|V(H)|))$ derived by Garg et. al. [9] for dense demand graphs $H$, but weaker for sparse graphs.

In Section 6 we include some concluding remarks.

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2. In Section 3, we establish a connection between the multiway cut problem and the stable set problem. This allows us to derive relaxations for the multiway cut problem that are stronger than previously known. In addition, we use the theory of perfect graphs to prove the integrality of some extended formulations of the multiway cut problem that have special structure. In this way we identify new polynomially solvable cases. Moreover, we obtain the integrality result of Chopra and Owen [4] mentioned above (when the underlying graph is a tree) and the result of Erdos and Székely [8] as a corollary.

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In Section 6 we include some concluding remarks.

## 2 A nonlinear formulation for the multiway cut problem and its linearizations

In this section we present a continuous nonlinear formulation for the multiway cut problem, prove its validity through randomization and linearize it to obtain tight linear relaxations.

Let $T = \{v_1, v_2, \ldots, v_k\}$ denote the set of terminal vertices. Let $y^j(u)$ denote the probability that node $u$ belongs to the same component as $v_j$ in a multiway cut. Clearly

$$P\{\text{edge } (u,v) \text{ in the multiway cut}\} = 1 - \sum_{j=1}^{k} y^j(u) y^j(v). \tag{1}$$

This motivates the following formulation :

$$\begin{aligned}
(NLF)\ Z_N &= \min \sum_{(u,v)\in E} c(u,v)x(u,v) \\
\text{subject to} & \\
x(u,v) &= 1 - \sum_{j=1}^{k} y^j(u)y^j(v); \quad (u,v) \in E \\
\sum_{j=1}^{k} y^j(u) &= 1; \quad u \in V \\
y^j(v_j) &= 1; \quad \forall j \\
y^j(v_l) &= 0; \quad \forall l \neq j \\
0 \leq\ & y^j(u) \leq 1, \\
0 \leq\ & x(u,v) \leq 1.
\end{aligned}$$

Let $IZ_{MC}$ denote the value of a minimum multiway cut.

**Theorem 1** $IZ_{MC} = Z_N$.

**Proof :** Let $(x_N, y_N)$ be an optimal solution to Problem $(NLF)$. Vertex $u$ is assigned to the component of $v_j$ with probability $y_N^j(u)$. Let $x$ be the incidence vector of the multiway cut obtained. From equation (1),

$$E[x(u,v)] = P\{x(u,v) = 1\} = 1 - \sum_{j=1}^{k} y_N^j(u) y_N^j(v) = x_N(u,v),$$

so $E[\sum_{(u,v)\in E} c(u,v)x(u,v)] = E[\sum_{(u,v)\in E} c(u,v)x_N(u,v)] = Z_N$. The random process always produces a multiway cut solution, so its expected value cannot be smaller than the minimum. Hence $IZ_{MC} \leq E[\sum_{(u,v)\in E} c(u,v)x(u,v)] = Z_N$. Since all multiway cuts are feasible in $(NLF)$, $Z_N \leq IZ_{MC}$. Therefore, $IZ_{MC} = Z_N$. □

Linearizing the previous formulation we immediately obtain the extended formulation proposed by Chopra and Owen [4] via combinatorial arguments. In particular, by using the following linearization trick:

$$\sum_{j=1}^{k} y^j(u)y^j(v) \leq \sum_{j\in S} y^j(u) + \sum_{j\notin S} y^j(v), \tag{2}$$

we obtain the following extended formulation (relaxation) of the multiway cut problem :

$$\begin{aligned}
(EF1) \quad Z_{EF1} &= \min \sum_{(u,v)\in E} c(u,v)x(u,v) \\
\text{subject to} \\
x(u,v) + \sum_{j\in S} y^j(u) + \sum_{j\notin S} y^j(v) &\geq 1 \\
\sum_j y^j(u) &= 1; \quad u \in V \\
y^j(v_j) &= 1; \quad \forall j \\
y^j(v_l) &= 0; \quad \forall l \neq j \\
0 \leq y^j(u) &\leq 1, \\
0 \leq x(u,v) &\leq 1.
\end{aligned}$$

In the above formulation the edge variables, $x(u,v)$, can be considered as the "natural" variables, while the node variables, $y^j(u)$, can be viewed as the auxiliary variables.

Even though $(EF1)$ has an exponential number of constraints, its linear relaxation can be solved in polynomial time. This is because the associated separation problem is polynomial (see Chopra and Owen [4]). If we represent the product terms in $(NLF)$ by a variable:

$$z^j(u,v) = y^j(u)y^j(v)$$

we obtain a second extended formulation:

$$\begin{aligned}
(EF2) \quad Z_{EF2} &= \min \sum_{(u,v)\in E} c(u,v)x(u,v) \\
\text{subject to} \\
x(u,v) + \sum_j z^j(u,v) &= 1; \quad (u,v) \in E \\
z^j(u,v) &\leq y^j(u), \\
z^j(u,v) &\leq y^j(v), \\
z^j(u,v) &\geq y^j(u) + y^j(v) - 1 \\
\sum_j y^j(u) &= 1; \quad u \in V \\
y^j(v_j) &= 1; \quad \forall j \\
y^j(v_l) &= 0; \quad \forall l \neq j \\
0 \leq \ y^j(u) &\leq 1, \\
0 \leq \ x(u,v) &\leq 1.
\end{aligned}$$

Under the condition that the weight function $c$ is nonnegative, it is easy to see that the constraints $z^j(u,v) \geq y^j(u) + y^j(v) - 1$ are redundant. Chopra and Owen [4] prove that

**Theorem 2** *(Chopra and Owen [4]) When the cost function $c$ is nonnegative, $Z_{EF1} = Z_{EF2}$.*

In contrast to $(EF1)$ the formulation $(EF2)$ involves only polynomially many variables and constraints. In the sequel we will assume that $c_e \geq 0$ for all $e \in E$, so that we will not distinguish between $(EF1)$ and $(EF2)$.

We next derive a third extended formulation, equivalent to $(EF2)$, not previously considered. Since $\sum_{j=1}^k y^j(u) = 1$ in $(NLF)$ we can write

$$x(u,v) = \sum_{j=1}^{k} y^j(u) - \sum_{j=1}^{k} y^j(u)y^j(v).$$

Replacing $u$ by $v$ (note that $x(u,v) = x(v,u)$) we get

$$x(u,v) = \sum_{j=1}^{k} y^j(v) - \sum_{j=1}^{k} y^j(u)y^j(v).$$

Adding these two equations together yields:

$$2x(u,v) = \sum_{j=1}^{k} [y^j(u) + y^j(v) - 2y^j(u)y^j(v)].$$

Now,

$$y^j(u) + y^j(v) - 2y^j(u)y^j(v) \geq |y^j(u) - y^j(v)|$$

as long as $0 \leq y^j(u), y^j(v) \leq 1$. Thus we get the following *convex* programming formulation:

$$\begin{array}{rrcl}
(EF3) & Z_{EF3} & = & \min \sum_{(u,v)\in E} c(u,v)x(u,v) \\
& \text{subject to} & & \\
& 2x(u,v) & \geq & \sum_{j=1}^{k} |y^j(u) - y^j(v)| \\
& \sum_j y^j(u) & = & 1; \quad u \in V \\
& y^j(v_j) & = & 1; \quad \forall j \\
& y^j(v_l) & = & 0; \quad \forall l \neq j \\
& 0 \leq y^j(u) & \leq & 1, \\
& 0 \leq x(u,v) & \leq & 1.
\end{array}$$

$(EF3)$ can be turned into a linear program using the usual trick of introducing extra variables.

**Theorem 3** *When the cost function $c$ is nonegative, $Z_{EF2} = Z_{EF3}$.*

**Proof :** Let $(x_0, y_0, z_0)$ be any optimal solution to $(EF2)$. Notice that $z_0^j(u,v) = \min\{y_0^j(u), y_0^j(v)\}$. Let $A(u,v) = \{j : y_0^j(u) \geq y_0^j(v)\}$ and $A^c(u,v)$ be the complement. Now, $(x_0, y_0, z_0)$ must satisfy

$$x_0(u,v) + \sum_j z_0^j(u,v) = 1 = \sum_j y_0^j(u)$$

and

$$x_0(u,v) + \sum_j z_0^j(u,v) = 1 = \sum_j y_0^j(v).$$

Adding them together we get

$$2x_0(u,v) + 2\sum_j z_0^j(u,v) = \sum_j y_0^j(u) + \sum_j y_0^j(v).$$

Hence

$$2x_0(u,v) = \sum_j [y_0^j(u) - z_0^j(u,v)] + \sum_j [y_0^j(v) - z_0^j(u,v)].$$

Now the right hand side of the above can be rewritten as

$$\sum_{j \in A(u,v)} [y_0^j(u) - z_0^j(u,v)] + \sum_{j \in A^c(u,v)} [y_0^j(v) - z_0^j(u,v)] = \sum_j |y_0^j(u) - y_0^j(v)|.$$

Hence, any optimal solution to $(EF2)$ is a feasible solution to $(EF3)$.

Now suppose that $(x_0, y_0)$ is an optimal solution to $(EF3)$. Define $z_0^j(u,v) = \min\{y_0^j(u), y_0^j(v)\}$. We show that $(x_0, y_0, z_0)$ is feasible for $(EF2)$.

Let $A(u,v) = \{j : y_0^j(u) \geq y_0^j(v)\}$ and $A^c(u,v)$ be the complement. Then

$$\sum_j |y_0^j(u) - y_0^j(v)| = \sum_{j \in A(u,v)} [y_0^j(u) - y_0^j(v)] + \sum_{j \in A^c(u,v)} [y_0^j(v) - y_0^j(u)].$$

Also

$$2\sum_j z_0^j(u,v) = 2\sum_{j \in A(u,v)} y_0^j(v) + 2\sum_{j \in A^c(u,v)} y_0^j(u).$$

Since $x_0(u,v) = \sum_j |y_0^j(u) - y_0^j(v)|$ it follows that

$$2x_0(u,v) + 2\sum_j z_0^j(u,v) = \sum_{j \in A(u,v)} [y_0^j(u) - y_0^j(v) + 2y_0^j(v)] + \sum_{j \in A^c(u,v)} [y_0^j(v) - y_0^j(u) + 2y_0^j(u)] = 2.$$

So, an optimal solution to $(EF3)$ is feasible for $(EF2)$. □

By projecting out the auxiliary variables in these extended formulations we can derive the standard formulations involving edge variables alone as well as several (facet defining) valid inequalities for the multiway cut. This is described in the next theorem.

**Theorem 4** *Let $L$ denote a subgraph of $G$ which contains some demand nodes $v \in V(H)$, which we label as $v_1, ..., v_p$, and at least a non-demand node $w \in V \setminus V(H)$. Suppose the edges of $L$ can be oriented in such a way that there are exactly $q$ internally vertex disjoint paths from each $v_j$ to*

*$w$, i.e., for all $j = 1, \ldots, p$ all the paths from $v_j$ to $w$ are node-disjoint except for the nodes $v_j$ and $w$. Then,*

$$\sum_{e \in E(H)} x(e) \geq q(p-1)$$

*is a valid inequality for the multiway cut problem.*

**Proof :** Consider an orientation of the edges of $H$ so that there are $q$ (fixed) internally disjoint directed paths from each demand node $v_j$ to $w$. For each edge $(u, v)$ oriented from $u$ to $v$, let $S(u, v)$ denote the set of demand nodes which use this edge along one of their $q$ paths to $w$. We have, from $(EF1)$

$$x(u,v) \geq 1 - \sum_{j \in S(u,v)} y^j(v) - \sum_{j \notin S(u,v)} y^j(u).$$

Note that if $v \neq w$, then $S(u_1, v), S(u_2, v)$ are disjoint, since the $q$ paths from each demand node to $w$ are internally disjoint. Let $N^+(v), N^-(v)$ denote respectively the set of in-neighbors and out-neighbors of $v$ under the orientation. Let $u \in N^-(v)$, for each term

$$1 - \sum_{j \notin S(v,u)} y^j(v),$$

note that the set $\{j : j \in S(v, u)\}$ is contained in $\cup\{j : j \in S(r, v), r \in N^+(v)\}$. Since $\sum_j y^j(v) = 1$, and

$$\cup_{\, u \in N^-(v)} S(v,u) = \cup_{\, r \in N^+(v)} S(r,v),$$

we have

$$-\sum_{u \in N^+(v)} \sum_{j \in S(u,v)} y^j(v) + \sum_{u \in N^-(v)} \{1 - \sum_{j \notin S(v,u)} y^j(v)\} \geq 0. \quad (3)$$

On the other hand, each demand node has degree at least $q$, and for each neighbor $u$ of a demand node,

$$x(v_j, u) \geq 1 - y^j(u).$$

So

$$\sum_{u \in N^-(v_j)} x(v_j, u) \geq q - \sum_{u \in N^-(v)} y^j(u).$$

For the node $w$, since $w$ has no out-neighbor, and each demand has exactly $q$ internally disjoint paths to $w$,

$$\sum_{u \in N^+(w)} \sum_{j \in S(u,w)} y^j(w) = q \sum_j y^j(w) = q.$$

By (3),

$$\sum_{e=(u,v) \in E(H)} x(u,v) \geq \sum_{e=(u,v) \in E(H)} \{1 - \sum_{j \in S(u,v)} y^j(v) - \sum_{j \notin S(u,v)} y^j(u)\} \geq qp - q.$$

Hence the result follows . □

As an example, pick any two vertices $v_r$ and $v_s$ in $T$ and let $L$ be any path between them. The theorem implies:

$$\sum_{e \in L} x_e \geq 1.$$

If we apply the theorem to every path between every pair of terminal vertices we get the *path formulation* of the multiway cut problem. To describe this formulation let $P(i,j)$ be the set of paths between terminals $v_i$ and $v_j$.

$$
\begin{aligned}
(PF) \quad Z_{PF} &= \min \sum_{e \in E} c_e x_e \\
\text{subject to} & \\
\sum_{e \in p} x_e &\geq 1; \quad \forall p \in P(i,j)\ \forall v_i, v_j \in T \\
0 \leq x_e &\leq 1.
\end{aligned}
$$

As another example pick a tree $S$ on $G$ all of whose leaves are in $T$, and no terminal vertex is a non-leaf vertex. Call such a tree a $T$-*tree.* Then, by the theorem we get

$$\sum_{e \in S} x_e \geq |S \cap T| - 1,$$

which are called the tree inequalities. Generating all tree inequalities gives us another formulation called the *tree formulation* that was considered by Chopra and Rao [5] and Cunningham [6]. To describe this formulation let $\mathcal{T}$ be the set of all $T$-trees.

$$
\begin{aligned}
(TF) \quad Z_{TF} &= \min \sum_{e \in E} c_e x_e \\
\text{subject to} & \\
\sum_{e \in S} x_e &\geq |S \cap T| - 1; \quad S \in \mathcal{T} \\
0 \leq x_e &\leq 1.
\end{aligned}
$$

In the same way one can derive the odd-wheel inequalities, and bipartite inequalities. These are known to be facets of the tree formulation (see Chopra and Rao [5]).

### 2.1 Fractional extreme points

The examples above illustrate that the previous extended formulations are rather powerful as they lead to a large collection of facet-defining inequalities. The following example, taken from Cunningham [6], shows, however, that there are fractional extreme points in $(EF1)$ and also $(EF2)$.

For the above example, nodes 1, 3 and 5 are the demand nodes. Using the convention that node 0 means node 6, we have the following fractional extreme point:

$$y^i(i) = 1, y^i(i+1) = y^i(i-1) = z^i(i,i+1) = z^i(i,i-1) = z^i(i-1,i+1) = 1/2, \quad i = 1,3,5;$$

and $y, z = 0$ otherwise. By this choice of $y$, $z$, $x(e) = \frac{1}{2}$ for all edges in the graph. Assuming $c(u,v) = 2$ if the edge $(u,v)$ is incident to a terminal, $c(u,v) = 1$ otherwise, then $Z_{EF1} = Z_{EF2} \leq 7.5$, while it can be easily seen the cardinality of any multiway cut is at least 8, i.e., $IZ_{MC} \geq 16/15 Z_{MC}$.

The computational results of Chopra and Owen [4] show that the formulation $(EF1)$ and $(EF2)$ consistently yield high quality bound to the multiway cut problems. The preceeding example shows

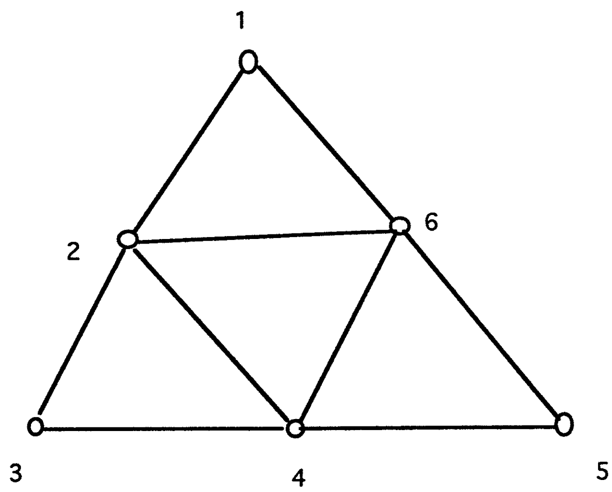


Figure 1: $G$ is the above graph and $H = \{1, 3, 5\}$.

that the gap can be as large as 16/15. We next generalize the construction of this example to give an example whose gap is asymtotically close to 10/9.

Define the graph $G$ with nodes denoted $v_1, v_2, \ldots, v_k$ (terminals) and $u_{i,j}$ where $i \neq j$, $1 \leq i, j \leq k$. The edge set of $G$ consists of edges $\{v_i, u_{i,j}\}$, $j = 1, \ldots, k$, and $\{u_{i,j}, u_{i,r}\}$ for $r, j = 1, \ldots, k$. Let $c(u,v) = k-1$ if $(u,v)$ is incident to a terminal, otherwise $c(u,v) = 1$. When $k = 3$, the construction reduces to the preceeding example.

Let $y^i(u_{i,j}) = 1/2$, $z^i(u_{i,j}, u_{i,k}) = 1/2$, $z^i(u_{i,j}, v_i) = 1/2$ for each $i = 1, \ldots, k$. Thus $x(u,v) = 1/2$ for all edges in $G$. This yields a fractional LP solution with cost $\frac{k(k-1)^2}{2} + \frac{k(k-1)(k-2)}{4} = \frac{3k^3}{4} + o(k^3)$. On the other hand, consider an optimum multiway cut solution. Let $T_i$ denote the set of vertices in the same componenet as the terminal $v_i$. Let $A_i$ denote the number of vertices in $T_i$ of the type $u_{i,j}$ for some $j$, $B_i = |T_i| - A_i$. Then there are exactly $k(k-1) - \sum_{i=1}^k A_i$ edges with cost $k-1$ in the cut. Furthermore, there are at least $(k-2-A_i)A_i$ edges of the type $\{u_{i,j}, u_{i,k}\}$ in the cut. For each $u_{i,j}$, there are $(k-2)$ other neighbours of the type $u_{j,k}$. Hence there are at least $(k-2)\sum_{i=1}^k A_i - 2\sum_{i=1}^k B_i$ edges of the type $\{u_{i,j}, u_{j,k}\}$ in the cut. The last term arising because there are at most $2B_i$ edges between the nodes enumerated by $A_i$ and $B_i$, which do not belong to the cut. By eliminating duplication, we have

$$IZ_{MC} \geq (k-1)(k(k-1) - \sum_{i=1}^{k} A_i) + \frac{1}{2}\{\sum_{i=1}^{k}(k-2-A_i)(A_i) + (k-2)\sum_{i=1}^{k} A_i - 2\sum_{i=1}^{k} B_i\}$$

Since $\sum_{i=1}^k T_i = \frac{k(k-1)}{2}$, we have

$$IZ_{MC} \geq k^3 - \frac{\sum_i A_i^2}{2} + o(k^3).$$

It is easy to see that $\sum_i A_i^2$ is maximized when

- the vertex $u_{i,j}$ belongs to either $T_i$ or $T_j$.

- $A_i \neq A_j$ if $i \neq j$.

Hence

$$\sum_i A_i^2 \leq 1^2 + 2^2 + \ldots + (k-1)^2 = \frac{k^3}{3} + o(k^3).$$

Using this bound, we have $IZ_{MC} \geq \frac{5k^3}{6} + o(k^3)$. Thus $IZ_{MC}/Z_{MC} \geq 10/9$ for large $k$.

### 2.2 Connection with quadratic zero-one programming

In order to strengthen the formulations further, we can consider stronger linearizations of the quadratic terms $y^j(u)y^j(v)$. The problem of linearizing quadratic terms of this type has been addressed within the context of unconstrained quadratic zero-one programming problems, leading to the Boolean Quadric polytope $(BQP)$ (see Padberg [13] for a comprehensive treatment of the subject). The polyhedron $(BQP)$ is also called the correlation polytope by Laurent and Poljak [11]. In this way, all valid inequalities known for the $(BQP)$ can easily be converted to valid inequalities for the multiway cut problem. For instance, we can add the following valid inequalities

- $z^j(u,u) + z^j(v,w) \geq z^j(u,v) + z^j(u,w)$,
- $z^j(u,u) + z^j(v,v) + z^j(w,w) \leq 1 + z^j(u,v) + z^j(v,w) + z^j(u,w)$.

Unfortunately these valid inequalities do not cutoff the fractional extreme point of Figure 1.

## 3 Relation between the multiway cut and the independent set problem

In this section, we establish that the multiway cut problem on $G$ can be solved as an independent set problem on a related graph $I(G)$. We then use this connection to establish a new stronger extended formulation for the multiway cut problem.

In $(EF2)$ we can use the equation

$$x(u,v) + \sum_j z^j(u,v) = 1$$

to eliminate the variable $x(u,v)$ from the formulation. This yields the following version of $(EF2)$:

$$\begin{array}{rcl}
(EF2) \quad Z_{EF2} & = & \min \sum_{(u,v)\in E} c(u,v)[1 - \sum_{j=1}^{k} z^j(u,v)] \\
\text{subject to} & & \\
\sum_j z^j(u,v) & \leq & 1;\ (u,v) \in E \\
z^j(u,v) & \leq & y^j(u), \\
z^j(u,v) & \leq & y^j(v), \\
\sum_j y^j(u) & = & 1;\ u \in V
\end{array}$$

$$\begin{array}{rcl} y^j(v_j) & = & 1;\ \forall j \\ y^j(v_l) & = & 0;\ \forall l \neq j \\ 0 \leq & y^j(u) & \leq 1, \\ 0 \leq & z^j(u,v) & \leq 1. \end{array}$$

Now the constraints $z^j(u,v) \leq y^j(u)$ and $\sum_j y^j(u) = 1$ make the constraint $\sum_j z^j(u,v) \leq 1$ redundant. Eliminating the constant term in the objective function we deduce that $(EF2)$ is equivalent to

$$\begin{array}{rcl} (EF) \quad Z_{EF} & = & \max \sum_{(u,v)\in E} \sum_{j=1}^{k} c(u,v) z^j(u,v) \\ \text{subject to} & & \\ z^j(u,v) & \leq & y^j(u), \\ z^j(u,v) & \leq & y^j(v), \\ \sum_j y^j(u) & = & 1;\ u \in V \\ y^j(v_j) & = & 1;\ \forall j \\ y^j(v_l) & = & 0;\ \forall l \neq j \\ 0 \leq & y^j(u) & \leq 1, \\ 0 \leq & z^j(u,v) & \leq 1. \end{array}$$

To summarize, $(EF)$ and $(EF2)$ have identical feasible regions and will have the same optimal solution. We focus on $(EF)$.
From the constraints of $(EF)$ we deduce that for $i \neq j$,

$$z^j(u,v) + z^i(u,v) \leq y^j(u) + y^i(u) \leq \sum_{j=1}^{k} y^j(u) = 1.$$

Furthermore, for any two neighbors $v$ and $w$ of $u$ and $i \neq j$ we have

$$z^j(u,v) + z^i(u,w) \leq y^j(u) + y^i(u) \leq 1.$$

Using the two inequalities just derived we obtain the following relaxation of $(EF)$:

$$\begin{array}{rcl} (REF) \quad Z_{REF} & = & \max \sum_{(u,v)\in E} \sum_{j=1}^{k} c(u,v) z^j(u,v) \\ \text{subject to} & & \\ z^i(u,v) + z^j(u,v) & \leq & 1,\ i \neq j \\ z^i(u,v) + z^j(u,w) & \leq & 1,\ i \neq j,\ (u,v),(u,w) \in E \\ 0 \leq & z^j(u,v) & \leq 1. \end{array}$$

Although $(REF)$ is a relaxation of $(EF)$, there is an one to one correspondance between *integer* solutions to $(EF)$ and $(REF)$. If we associate a vertex with each variable $z^i(u,v)$ in $(REF)$ we see that $(REF)$ can be interpreted as the linear relaxation of the problem of finding a maximum weight independent set.

Formally, given $G$, let $I(G)$ denote the graph with vertex set

$$\{(u,v,j):\ (u,v)\in E(G), j=1,2,\ldots,k\}\setminus\{(v_j,u,i):\ i\neq j,(v_j,u)\in E(G)\};$$

and edge set

$$\{((u_1,w_1,i),(u_2,w_2,j)):\ i\neq j; |\{u_1,w_1\}\cap\{u_2,w_2\}|\geq 1\}.$$

Consider a maximal independent (stable) set $I$ in $I(G)$. Let $F_j=\{(u,v)\in E(G):\ (u,v,j)\in I\}$. The edge induced subgraphs $G[F_j]$ are node disjoint in the graph $G$, since $I$ is a stable set. Moreover, by maximality of $I$, each non-demand node must be a vertex in one of the subgraphs $G[F_j]$. This partition induces a solution to the multiway cut problem and vice versa. For each vertex $(u,v,j)$ in $I(G)$ we assign costs $c(u,v)$. Then the cost of the multiway cut in $G$ and the cost of the maximum independent set in $I(G)$ are related as follows:

$$IZ_{MC}=\sum_{(u,v)\in E(G)} c(u,v)-\max\{c(I): I \text{ stable set in } I(G)\}.$$

The problem of finding a maximum weight stable set in $I(G)$ can be formulated as:

$$\begin{aligned}
(IREF)\quad Z_{IREF} &= \max\sum_{(u,v)\in E}\sum_{j=1}^{k} c(u,v)z^j(u,v)\\
\text{subject to}&\\
z^i(u,v)+z^j(u,v) &\leq 1,\ i\neq j\\
z^i(u,v)+z^j(u,w) &\leq 1,\ i\neq j,\ (u,v),(u,w)\in E\\
z^j(u,v)=0,1.&
\end{aligned}$$

Notice that $(REF)$ is just the linear relaxation of $(IREF)$.

This correspondence has many interesting consequences. As an example, since the multiway cut for fixed $k\geq 3$ is NP-hard, we obtain as a direct corollary that the maximum independent set on $k$-partite graph ( $k\geq 3$) is also NP-hard. In addition, many classes of facet-defining inequalities for the stable set problem can be interpreted as valid inequalities for the multiway cut problem.

A natural way to strengthen $(REF)$ is to include the the maximal clique inequalities for the stable set problem. The maximal cliques in $I(G)$ are of the form (type I)

$$\{(u,w_1,1),\ldots,(u,w_k,k)\};$$

where $(u,w_i)\in E(G)$ or type II

$$\{(e_1,1),\ldots,(e_k,k)\}$$

where $e_i\in\{(u,w),(u,v),(v,w)\}$ are edges of a triangle in $G$. We will show that in relaxing $(EF)$ to $(REF)$, the inequalities thrown out are those that correspond to all maximal cliques of type I in $I(G)$. Consider the formulation, $(RIND)$ obtained from $(REF)$ by adding all clique constraints corresponding to cliques of type I:

$$Z_{RIND} = \max\sum_{e\in E}\sum_{j} c_e z^j(e)$$

subject to

$$\sum_{j=1}^{k} z^j(u, w_j) \leq 1; \; \forall u, \; w_j \text{ adjacent to } u$$

$$0 \leq z^j(e) \leq 1.$$

These clique constraints are implied by the constraints in $(EF)$, so $(RIND)$ is a relaxation of $(EF)$. So, $Z_{REF} \geq Z_{RIND} \geq Z_{EF}$.

**Theorem 5**

$$Z_{EF} = Z_{RIND}.$$

**Proof :** Since $(RIND)$ is a relaxation of $(EF)$, it suffices to show that $Z_{RIND} \leq Z_{EF}$. Let $z$ be an optimal solution to $(RIND)$. Let $y^j(u) = \max_e z^j(e)$ where the maximum is taken over all edges $e$ incident to $u$. By optimality of $z$, for each $u$, there exists a set of edges $e_j$ (possibly repeat), $j = 1, ..., k$, incident to $u$, such that

$$\sum_j z^j(e_j) = 1.$$

So by definition of $y$,

$$\sum_j y^j(u) = 1 \text{ for each node } u.$$

At the demand node $v_j$, by construction, $y^i(v_j) = 0$ if $i \neq j$. Hence $y^j(v_j) = 1$. Therefore, the solution $(z, y)$ is feasible in $(EF)$, and hence the result follows. □

The previous theorem implies that the feasible regions of $(EF)$ and $(RIND)$ coincide, i.e., $(RIND) = (EF)$. Hence, any facet of $(RIND)$ is automatically a facet of $(EF)$. In particular, the clique constraints corresponding to maximal cliques of type II in $I(G)$ are facets.

If we add them to $(RIND)$ we obtain:

$$(IND) \quad Z_{IND} = \min \sum_{(u,v) \in E(G)} c(u,v)(1 - \sum_j z^j(u,v)) \tag{4}$$

subject to (5)

$$\sum_{j=1}^{k} z^j(u, w_j) \leq 1; \; \forall u, \; w_j \text{ adjacent to } u \tag{6}$$

$$\sum_{j=1}^{k} z^j(u_j, w_j) \leq 1; \; \forall \text{ triangles } \Delta \; (u_j, w_j) \text{ are edges of } \Delta \tag{7}$$

$$0 \leq z^j(e) \leq 1. \tag{8}$$

Formulation $(IND)$ is strictly stronger than $(EF2)$ , as inequalities (7) cut off the fractional extreme points in the example of Figure 1 . Recall that

$$y^i(i) = 1, y^i(i+1) = y^i(i-1) = z^1(i, i+1) = z^1(i, i-1) = z^1(i-1, i+1) = 1/2, \quad i = 1, 3, 5.$$

Hence the solution violates the triangle inequality

$$z^1(2,6) + z^3(2,4) + z^5(4,6) \leq 1.$$

Other valid inequalities for the multiway cut problem can be constructed from facets for the independent set problem. For example, the well known odd cycle inequality for the stable set problem translates to :

$$\sum_{i=1}^{l} z^{j(i)}(e_i) \leq \lfloor |C|/2 \rfloor; \tag{9}$$

where $C = \{(e_i, j(i))\}_{i=1}^{l}$ is an odd cycle in $I(G)$. It is well known (see [10]) that the separation problem for the odd cycle inequalities can be solved in polynomial time.

In general the clique constraints for the independent set problem cannot be separated in polynomial time. In this case, because of the specific nature of $I(G)$, we can separate the clique constraints in $I(G)$ in polynomial time. Hence $(IND)$ is solvable in polynomial time via the ellipsoid algorithm.

**Theorem 6** *$(IND)$ is solvable in polynomial time.*

**Proof :** For each node $u$ in $G$, the inequalities

$$\sum_j z^j(u, w_j) \leq 1 \text{ , } \forall \; w_j \text{ adjacent to } u;$$

are satisfied if and only if the inequality is satisfied by a single choice of $w_j$. In particular, choose

$$w_j = \text{argmax}\{z^j(u,w) : w \text{ adjacent to } u\}.$$

Similarly, the inequalities corresponding to the triangles can be checked by verifying only for the case $e_j = argmax\{z^j(e), z^j(f), z^j(g)\}$. □

It follows directly from the theory of perfect graphs that

**Theorem 7** *If $I(G)$ is a perfect graph, formulation $(IND)$ for the multiway cut problem is integral.*

Note that for $k = 2$, $I(G)$ is bipartite and is therefore a perfect graph. Thus $(IND)$ is always integral in this case.

## 3.1 Polynomially solvable cases of the multiway cut problem

Chopra and Owen [4] showed that $(EF2)$ is integral when the underlying graph G is a tree. The proof of this result follows easily from the theory of perfect graphs.

**Theorem 8 ([4])** *Formulation $(EF2)$ is integral when $G$ is a tree.*

**Proof :** Since $G$ does not contain any triangles, $(EF2)$ coincides with the formulation $(IND)$. It suffices, then, to show that $I(G)$ is perfect. If $G$ corresponds to a star on $n+1$ nodes, then $I(G)$ is a complete $n$-partite graph and therefore perfect. Since all trees $G$ are formed by "gluing" star graphs on cut-edges, $I(G)$ is formed by clique-gluing of complete multipartite graphs. Since clique-gluing operations preserve perfectness, $I(G)$ is perfect, when $G$ is a tree. Therefore $(IND)$

and hence $(EF2)$ is integral in this instance. $\square$

This result is interesting, as the multiway cut problem over trees has important generalizations in the study of evolutionary trees. The generalized multiway cut problem introduced in Erdos and Szekely [8] is as follows: Given a graph $G = (V, E)$ and a partial $k$-coloring of the vertices, i.e., a subset $V' \subseteq V$ and a function $f : V' \to \{1, ..., k\}$, find an extension of $f$ to $V$ such that the total weight of edges with different colored endpoints is minimized. Erdos and Szekely [8] contains a nice illustration of how this problem arises naturally in the study of evolutionary trees. They have also constructed a polynomial time dynamic programming algorithm for the generalized multiway cut problem on trees. In $(EF2)$, this amounts to setting

$$y^{f(u)}(u) = 1 \quad , y^j(u) = 0 \quad \text{otherwise} ,$$

for each $u$ in $V'$. This has the effect of deleting nodes in $I(G)$ that corresponds to $(u, v, i)$ if $i \neq f(u)$. Since subgraphs of perfect graphs are also perfect, the above argument yields directly that the generalized multiway cut problem is solved by $(EF2)$ when $G$ is a tree. Using this notion of generalized multiway cut, it follows directly (see Erdos and Szekely [8]) that if the set of demands $V(H)$ in $G$ intersects every cycle in $G$, then by splitting the demands into multiple demands with the same coloring, we can transform $G$ to a forest. Thus the multiway cut problem in this case can be transformed to a generalized multiway cut problem on a forest. In fact, the result can be improved as follows.

**Theorem 9** *If the demands $V(H)$ in $G$ intersect every cycle of length greater or equal to 4, then the multiway cut problem is solvable in polynomial time.*

**Proof :** Consider the following class of graphs (called *triangular cactus*) obtained from node gluing of triangles and edges. By splitting demands as in Erdos and Szekely [8], the multiway cut problem in the above theorem can be transformed to a generalized multiway cut problem on triangular cactus. Hence the theorem follows immediately from the following property:

**Lemma 10** *When $G$ is a triangular cactus, $I(G)$ is perfect.*

**Proof :** We next sketch the main idea of the proof of the lemma. Let $M$ be a node-induced subgraph of $I(G)$. $H(M)$ corresponds to a set of colored edges in $G$. Let $q(M)$ denote the size of a maximum clique in $M$. We only need to show that the chromatic number of $M$ is at most $q(M)$. We proceed by induction, using the fact that $G$ is built up with node gluing of edges and triangles. $\square$

## 4 A new randomized rounding technique and its applications to the multiway cut problem

In this section we describe a new randomized rounding heuristic for the multiway cut problem and use it to analyze the worst case bound of the extended formulation $(EF3)$ ($(EF1)$, and $(EF2)$) relative to the optimum multiway cut solution.

**Dependent randomized rounding heuristic**

1. Solve the relaxation $(EF3)$ obtaining an optimal solution $(x_0, y_0)$.

2. Generate a random number $U$, uniformly between 0 and 1.

3. For each vertex $u$ and $j$, set $y^j(u)$ to 1 if $U \leq y_0^j(u)$ and 0 otherwise.
In this way sets $S_j = \{u : y^j(u) = 1\}$ (not necessarily disjoint) are generated, $j = 1, \ldots, k$ such that $S_j$ contains demand node $v_j$ but not $v_i$ for $i \neq j$ (recall $y^j(v_i) = 0$ if $i \neq j$). Compute $c(\delta(S_j))$. Let $S_{max} = \text{argmax } c(\delta(S_j))$.

4. The proposed solution is the set of edges in

$$D = \cup_{j:S_j \neq S_{\max}} \delta(S_j).$$

Notice that unlike the usual independent randomized rounding method which rounds each $y_0^j(u)$ independently, we correlate the rounding process by generating a single random variable $U$. Clearly, the solution is feasible for the multiway cut problem. Let $Z_H$ be the value of the heuristic.

**Theorem 11** *For $c \geq 0$, $IZ_{MC} \leq E[Z_H] \leq 2(1 - \frac{1}{k})Z_{EF3}$.*

**Proof :** Clearly, the value of the multicut $D$ is

$$c(D) \leq (1 - \frac{1}{k}) \sum_j c(\delta(S_j)).$$

Since

$$P\{(u,v) \in \delta(S_j)\} = P\{(y^j(u) = 1, y^j(v) = 0) \cup (y^j(u) = 0, y^j(v) = 1)\} =$$
$$P\{\min(y_0^j(u), y_0^j(v)) \leq U \leq \max(y_0^j(u), y_0^j(v))\} = |y_0^j(u) - y_0^j(v)|,$$

we have

$$E[\sum_j c(\delta(S_j))] = \sum_{j=1}^k \sum_{(u,v)\in E} c(u,v) P\{(u,v) \in \delta(S_j)\} = \sum_{j=1}^k \sum_{(u,v)\in E} c(u,v) |y_0^j(u) - y_0^j(v)|.$$

From $(EF3)$ we know that

$$\sum_{j=1}^k |y_0^j(u) - y_0^j(v)| \leq 2x_0(u,v).$$

Hence

$$E[\sum_j c(\delta(S_j))] \leq 2 \sum_{(u,v)\in E} c(u,v) x_0(u,v).$$

Therefore,

$$E[Z_H] = E[c(D)] \leq (1 - \frac{1}{k}) E[\sum_j c(\delta(S_j))] \leq 2(1 - \frac{1}{k}) Z_{EF3}.$$

□

**Remark:** When $k = 2$, the bound is exact. In this case, since (EF1) is equivalent to the dual of a max-flow problem, we have obtained a randomized proof of the max-flow-min-cut theorem.

Using the conditioning method (see Alon and Spencer [1]) we can make step 3 of the dependent randomized heuristic deterministic. So, we obtain an approximation algorithm that delivers a multiway cut at most twice the optimal. Another easier way to obtain a deterministic 2-approximation algorithm is to find, for each $j$, a minimum cut containing the terminals $v_j$ but not the other terminals, this time among the sets $\{v_j, u_1, u_2, u_3, ...\}$ with the vertices ordered (in non-increasing order, breaking ties arbitrarily) according to the value of $y^j(u_i)$. Dalhaus et. al. [7] constructed directly a combinatorial algorithm to approximate the multiway cut problem, also within a bound of $2(1-\frac{1}{k})$. However, our result is a little stronger in that the bound on the heuristic solution is in terms of the linear programming relaxation and not the integer optimal.

## 4.1 The case k = 4

Dalhaus et. al. [7] showed that the approximation bound 3/2 can be improved further to 4/3 for the 4-demand cut problems. By a randomized version of their heuristic, we propose next a linear relaxation attaining the same worst case bound of 4/3 for the four-demand cut problem.

$$
\begin{array}{rrcl}
(4T) & Z_{4T} & = & \min \sum_{(u,v)\in E} c(u,v)x(u,v) \\
\text{subject to} & & & \\
& 2x(u,v) & \geq & \sum_{i=2}^{4} |y^1(u)+y^i(u)-y^1(v)-y^i(v)|; \quad (u,v)\in E, \\
& y^j(v_j) & = & 1; \forall j \\
& y^j(v_l) & = & 0; \forall l \neq j \\
& 0 \leq & y^j(u) & \leq 1, \\
& 0 \leq & x(u,v) & \leq 1.
\end{array}
$$

Note that the above convex programming problem is essentially a linear program. We keep this form as it makes the following analysis more transparent. Let $IZ_{4T}$ be the corresponding optimal integer programming value.

**Theorem 12** $IZ_{4T} \leq \frac{4}{3} Z_{4T}$.

**Proof :** Let $(x, y)$ be an optimal solution to $Z_{4T}$. We generate randomly cuts of the form $F(1, i)$, which separates demands $v_1, v_i$ from the other two demands, in the following way:

- $F(1,i) = \emptyset$. Generate $U$ randomly on [0,1].
- If $y^1(u) + y^i(u) \geq U$ then $F(1,i) \leftarrow F(1,i) \cup \{u\}$. Repeat for all $u$.

Note that

$$E[c(\delta(F(1,i)))] = \sum_{(u,v)} c(u,v)|y^1(u)+y^i(u)-y^1(v)-y^i(v)|.$$

Now, since union of any 2 of the 3 cuts generated is a valid 4-demand cut, by taking the minimum $Z_H$ of the 3 feasible solutions, we have

$$Z_H \leq \frac{2}{3}\sum_{i=2}^{4} E(c(\delta(F(1,i))) \leq \frac{4}{3} Z_{4T}.$$

□

Similarly, the combinatorial approximation algorithm for the 8-way-cut problem [7] can be turned into a LP formulation with equivalent bound to the 8-way-cut problem.

# 5 A new approximation algorithm for the multicut problem

In this section, we introduce a nonlinear formulation for the multicut problem, and a linear formulation to which we apply the randomized rounding technique of the previous section to obtain an approximation algorithm within $2\alpha(H)(1-\frac{1}{k})$ of the LP relaxation, where $\alpha(H)$ is the independent set number of the demand graph $H$.

## 5.1 An exact nonlinear formulation

Let $E(H) = \{(s_j, t_j),\ j = 1, \ldots, m\}$. We assign node $u$ of $G$ in the same component with $s_j$ with probability $y^j(u)$ ($y^j(t_j) = 0$). This creates a set $S_j$ of nodes that are in the component of node $s_j$. Then the multicut solution is $D = \cup_j \delta(S_j)$. Then

$$P\{(u,v) \in D\} = 1 - P\{(u,v) \notin D\} = 1 - P\{\cap_j[(u \in S_j, v \in S_j) \cup (u \notin S_j, v \notin S_j)]\} =$$

$$1 - \prod_{j=1}^{m}(P\{u \in S_j, v \in S_j\} + P\{u \notin S_j, v \notin S_j\}) = 1 - \prod_{j=1}^{m}[y^j(u)y^j(v) + (1-y^j(u))(1-y^j(v))] =$$

$$1 - \prod_{j=1}^{m}(1 - y^j(u) - y^j(v) + 2z^j(u,v)),$$

with $z^j(u,v) = y^j(u)y^j(v)$. With this motivation we consider the following nonlinear formulation.

$$\begin{array}{rcl} Z_{NM} & = & \min \sum_{(u,v)\in E} c(u,v)[1 - \prod_{j=1}^{m}(1 - y^j(u) - y^j(v) + 2z^j(u,v))] \\ \text{subject to} & & \\ z^j(u,v) & = & y^j(u)y^j(v);\ \forall j,\ \forall (u,v) \in E \\ y^j(s_j) & = & 1;\ \forall j \\ y^j(t_j) & = & 0;\ \forall j \\ 0 \leq & y^j(u) & \leq 1, \\ 0 \leq & x(u,v) & \leq 1. \end{array}$$

Let $IZ_M$ denote the value of an optimum multicut. Using a similar idea as in Theorem 1, we have

**Theorem 13** $IZ_M = Z_{NM}$.

By linearizing the above formulation, we obtain next a classical formulation of the multicut problem (see Garg et.al. [9]).
Since $z^j(u,v) \leq \min(y^j(u), y^j(v))$ and

$$\prod_j (1 - y^j(u) - y^j(v) + 2z^j(u,v)) \leq \min_j(1 - y^j(u) - y^j(v) + 2z^j(u,v)),$$

we have

$$1-\prod_j (1-y^j(u)-y^j(v)+2z^j(u,v)) \geq |y^j(u)-y^j(v)| \quad \forall\, j.$$

We obtain immediately the following formulation for the multicut problem :

$$\begin{array}{rcl}
(M1) \quad Z_{M1} & = & \min \sum_{(u,v)\in E} c(u,v)x(u,v) \\
\text{subject to} & & \\
x(u,v) & \geq & y^j(u)-y^j(v); \quad (u,v)\in E \\
x(u,v) & \geq & y^j(v)-y^j(u); \quad (u,v)\in E \\
y^j(s_j) & = & 1; \quad \forall j \\
y^j(t_j) & = & 0; \quad \forall j \\
0 \leq & y^j(u) & \leq 1, \\
0 \leq & x(u,v) & \leq 1.
\end{array}$$

The above formulation is usually obtained by considering the LP dual to a multicommodity flow problem. The route through the nonlinear characterization offers additional insights into the above formulation. Moreover, it indicates how the above formulation can be strengthened via better linearizations of the nonlinear term.

## 5.2 A new relaxation and a randomized approximation algorithm

Garg et.al. [9] propose a primal-dual approximation algorithm $A$ with cost $Z_A$ based on formulation (M1) that leads to the bound $Z_A \leq O(\log(k))Z_{M1}$, for an arbitrary demand graph $H$. This bound, however, is quite weak for the multiway cut problem. In this section, we generalize the formulation for the multiway cut problems to yield a stronger formulation for the multicut problem. The new relaxation uses variables $y^j(u)$ where $j$ is indexed on the set of terminals. Note that the formulation in (M1) uses variables $y^j(u)$ where $j$ is indexed on the edges of the demand graph.

Let $\alpha(H)$ the independent set number of $H$ with node set $V(H)$ and edge set $E(H)$.

In order to improve the bound we consider the following formulation

$$\begin{array}{rcl}
(M2) \quad Z_{M2} & = & \min \sum_{(u,v)\in E} c(u,v)x(u,v) \\
\text{subject to} & & \\
2\alpha(H)x(u,v)+\sum_j z^j(u,v) & \geq & \sum_{j\in S} y^j(u)+\sum_{j\notin S} y^j(v); (u,v)\in E, S\subset V(H) \\
z^j(u,v) & \leq & y^j(u), \\
z^j(u,v) & \leq & y^j(v), \\
\sum_j y^j(u) & \leq & \alpha(H); u\in V \\
x(u,v) & \geq & y^j(u)-y^j(v), \\
x(u,v) & \geq & y^j(v)-y^j(u), \\
y^j(s_j) & = & 1; \forall j
\end{array}$$

$$
\begin{array}{rcl}
y^j(v) & = & 0; \forall l \text{ such that } (v, s_j) \in E(H) \\
0 \leq & y^j(u) & \leq 1, \\
0 \leq & x(u,v) & \leq 1.
\end{array}
$$

**Theorem 14** *$Z_{M2} \leq IZ_M$. $Z_{M2}$ can be computed in polynomial time.*

**Proof :** Consider the incidence vector $x$ of any multicut solution. Let $y^j(u) = 1$ if node $u$ and $v_j \in V(H)$ lie in the same component in the multicut. Let $z^j(u,v) = y^j(u)y^j(v)$. If $u, v$ lie in the same component $C$, then

$$\sum_{j \in S} y^j(u) + \sum_{j \notin S} y^j(v) = |V(H) \cap C| = \sum_{j \in V(H)} z^j(u,v).$$

If $u, v$ lie in two different components $C_1$ and $C_2$, then $\forall\ S \subset V(H)$

$$\sum_{j \in S} y^j(u) + \sum_{j \notin S} y^j(v) \leq 2\alpha(H).$$

Hence

$$2\alpha(H)x(u,v) + \sum_j z^j(u,v) \geq \sum_{j \in S} y^j(u) + \sum_{j \notin S} y^j(v)$$

is a valid inequality. Clearly, $\sum_j y^j(u) \leq \alpha(H)$ is also a valid inequality. Moreover, $x(u,v) \geq |y^j(u) - y^j(v)|$ follows directly from definition. Since

$$\max\{\sum_{j \in S} y^j(u) + \sum_{j \notin S} y^j(v) : S \subset A\}$$

is solvable in polynomial time (see [4]), we can solve (M2) in polynomial time by the ellipsoid method. □

Let $(x, y, z)$ be a feasible solution to LP relaxation (M2). Consider demand edges in the form $\{v_i, v_j\}$. Define new variables $y_1^{v_i,v_j}(u) = y^i(u)$ for all $u$ in $G$. Clealry $x(u,v) \geq |y_1^{v_i,v_j}(u) - y_1^{v_i,v_j}(v)|$. Hence $(x, y_1)$ is feasible to (M1). It follows immediately that (M2) is a tighter relaxation to the multicut problem.

We next apply the dependent randomized rounding heuristic $H$ to the optimal solution of (M2). Using the same notation we obtain

**Theorem 15** *$IZ_M \leq E[Z_H] \leq 2\alpha(H)(1 - \frac{1}{|V(H)|})$.*

**Proof :** Let $(x_0, y_0, z_0)$ be an optimal solution to (M2). Then

$$
\begin{aligned}
E[\sum_j c(\delta(S_j))] &= \sum_j \sum_{(u,v) \in E(G)} c(u,v)|y_0^j(u) - y_0^j(v)| \\
&\leq \sum_{(u,v) \in E(G)} c(u,v)\{\sum_{j \in T(u,v)} (y_0^j(u) - z_0^j(u,v)) + \sum_{j \notin T(u,v)} (y_0^j(v) - z_0^j(u,v))\} \\
&\leq 2\alpha(H) \sum_{(u,v) \in E(G)} c(u,v)x_0(u,v) = 2\alpha(H)Z_{M2}.
\end{aligned}
$$

By disregarding the set $S_{max}$ with the maximum cut we obtain a multicut solution $D'$ such that

$$E[Z_H] = E[c(D')] \leq (1 - \frac{1}{k})E[\sum_j c(\delta(S_j))] \leq 2\alpha(H)(1 - \frac{1}{|V(H)|})Z_{M2}.$$

□

## 6 Concluding remarks

In this paper, we proposed several extended formulations for the multiway and multicut problems. We showed that these formulations can be constructed from linearizations of exact nonlinear formulations of the underlying problems. By reducing the multiway cut problem to the maximum independent set problem, we utilized the tools of perfect graph theory to study several instances of the multiway cut problems. Finally, we used a new randomized rounding argument to analyze the worst case behaviour of the formulations, resulting in a new approximation bound for the multicut problem. The rounding argument differs from the traditional randomized rounding method by imposing certain dependency structure on the rounding.

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