

Some Structural Properties  
of the  
Seasonal Product Pricing Problem

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# **Some Structural Properties of the Seasonal Product Pricing Problem**

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## **Abstract**

We define the seasonal product pricing problem as the problem of determining the dynamic optimal pricing policy for a retail product under a limited sale period and a fixed stock of product. In this paper, we propose and investigate four conjectures about the structure of this problem. We consider both deterministic and stochastic demand versions of the problem. The first three conjectures state that the optimal price is non-increasing in the level of inventory and non-decreasing in the level of demand, and the marginal revenue function is non-increasing in the level of inventory. The fourth conjecture states that the retailer does not benefit from reserving some stock for future periods if the demand for the product is known to decrease over time. We show through a series of counter-intuitive examples that these conjectures are not true in general. For some cases of the seasonal product pricing problem, we derive a set of sufficient conditions under which the conjectures can be guaranteed to hold and present examples of some demand functions and distributions which satisfy these conditions. In describing our results, we also attempt to provide some economic insights on optimal retail pricing behavior under different demand and inventory conditions. We conclude by highlighting some open analytical issues related to our work.

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# 1. INTRODUCTION

## Seasonal product pricing

Retailers who sell seasonal products, such as air conditioners and seasonal fashion apparel, often have to place orders for these items much in advance of the sale season. Once the initial order is placed, they often have no flexibility in placing reorders. In addition, such merchandise may only be in demand during the season. The retailer's pricing policy is therefore governed by the objective of maximizing the yield (i.e., the revenues) from a *fixed stock* of product within a *limited sale period*. We define the *seasonal product pricing problem* to be that of determining the dynamic optimal pricing policy for retail products in contexts that include the two factors identified above: a fixed stock of product and a limited sale period. Examples of applicable business contexts abound: fashion apparel, seasonal sports and home equipment, vacation cruises, and airline, hotel and car reservations. In this paper, we formulate the seasonal product pricing problem analytically and discuss some of its structural properties. Our focus in this paper is not on modeling specific aspects of any particular seasonal product environment, but on analyzing certain important general characteristics of the problem.

In certain seasonal product contexts, the retailer may be in a position to place one or more reorders for the product during the season. While our focus in this paper is mainly on situations where no reorders are possible (which is often the case with seasonal products), we will comment in Section 6 on the extension of our work to contexts where reordering is possible.

A third factor that is often present in such seasonal product environments is a high level of *demand uncertainty*. Goods that have a high fashion content have highly unpredictable demand, and even for non-fashion seasonal items such as air conditioners, it is often not feasible to predict demand accurately at the beginning of the season. The product's price therefore needs to be adjusted dynamically to incorporate new demand information and to balance supply with demand over the sale period. We will consider both deterministic and stochastic demand conditions in our analysis of the problem.

Some researchers (Gallego and Van Ryzin, 1994, Bitran and Mondschein, 1995) have recently developed models of the seasonal product pricing problem and devised solution approaches for these models. Our focus in this paper, however, is different - we attempt to provide some general insights into the nature of this pricing problem through a series of structural conjectures, counterexamples and sufficient conditions, touching upon issues that have either been ignored by past researchers or been studied under very restrictive conditions. We provide an overview of our work in the remainder of this section.

## Overview of research

The seasonal product pricing problem will be formulated in formal analytical terms in Section 3. It may be described as follows: A retailer has a certain initial level of inventory of a product in stock at the beginning of a season, which consists of a certain fixed number of periods. Demand in each period is either a deterministic function of price (the *deterministic demand case*) or a random variable whose distribution depends on price (the *probabilistic demand case*). It may also

depend on the period in question - that is, the demand distribution (or demand function) at any fixed price  $p$  may be non-stationary. The retailer initially decides on the price to charge for the product in period 1. After each period, the retailer checks the current inventory level and determines the price to be charged in the next period. The pricing problem is to determine a pricing strategy that maximizes the expected revenues over the sales season. We do not consider inventory holding costs or discount rates in this problem. These additional factors could be included without affecting the key results in this paper, and we have chosen to leave them out in order to keep the exposition simple. We assume that demand is a smooth and decreasing function of price. (Throughout this paper, we will use the term 'smooth' to denote a function that has a continuous second derivative with respect to the variable being considered.) The notion of demand being decreasing in price has a clear meaning in the deterministic demand context. For the probabilistic demand context, we will use a *stochastic dominance*-based extension of this condition that will be defined in Section 3.

Although the literature has examined both continuous-time and discrete-time models, we have considered only the discrete time case for the following key reason: the continuous-time model, while representing an interesting idealized case, provides pricing strategies that are not practical, since price revisions are allowed at all points in time. In addition, properties that hold in the continuous time case do not directly transfer over to the more realistic discrete time case. For instance, the sales limit issue we consider in Conjecture 4 (discussed later in this section) will not arise in a continuous time model for in that context a retailer would never benefit from imposing such a limit.

In framing the pricing problem in the above fashion, we have chosen to focus on certain key factors in the retailer's environment while ignoring other issues. We have chosen a fairly basic version of the problem for our analysis since it allows us to study the interactions of a few fundamental elements of the problem - such as the price, the level of inventory, and the level of demand - and to derive a number of interesting analytical and economic insights about them.

In this paper, we investigate the following conjectures related to the above problem. Conjectures 1-3 below are stated with reference to the following quantities in any period  $t$ : the inventory level at the beginning of period  $t$ , the optimal price for period  $t$ , the level of demand in period  $t$ , and the marginal revenue in period  $t$ . The *level of demand* is meant to capture factors that determine the demand for the product, such as the size of the market, the availability of competitive products, and the attractiveness of the product. By an increase in the level of demand, we mean that the demand at each price point has shifted upward. This concept will be defined more rigorously in Section 3. The *marginal revenue* at any given inventory level  $I$  is defined as the incremental expected revenue that the retailer will receive from having an additional unit of inventory.

1. The optimal price is non-increasing in the level of inventory.
2. The optimal price is non-decreasing in the level of demand.
3. The marginal revenue is non-increasing in the level of beginning inventory. (This is equivalent to stating that the optimal revenue function is concave in the initial level of inventory.)

4. The retailer does not benefit from reserving some stock of the product for future periods (and thereby limiting sales in early periods) if the level of demand is decreasing over time.

As we discuss in Section 3, these conjectures seem quite intuitive. However, we will show through a series of counterexamples that they are not true. While some of these counterexamples have arguably limited practical relevance, they serve an important analytical purpose by demonstrating that the intuitive properties defined in the conjectures above cannot be derived under general conditions. Thus, we need to impose further conditions on the problem in order to guarantee these results. We derive a set of sufficient conditions in this paper and present examples of demand functions and distributions that satisfy these conditions for certain cases of the seasonal product pricing problems. We also highlight certain open analytical issues for further research.

Some researchers (e.g., Thowsen, 1975, Bitran and Mondschein, 1993, Gallego and Van Ryzin, 1995) have studied the properties in Conjectures 1 and 3 in the past for certain special cases of the seasonal product pricing problem we study in this paper. Each of these special cases arise from assuming a specific form for the price-dependent demand distribution. In contrast, in this paper we study these conjectures under very general demand conditions. The main contributions of this paper are the identification of certain structural properties of the seasonal product pricing problem, the demonstration through counterexamples that these properties do not hold in general, and the derivation of conditions under which the properties can in fact be guaranteed to hold. The paper also attempts to provide some interesting economic insights on optimal retail pricing behavior under different demand and inventory conditions.

The remaining sections of this paper are as follows. Section 2 provides a review of the related literature. In Section 3, we provide analytical formulations for the seasonal product pricing problem and the conjectures mentioned above. Section 4 discusses a set of counterexamples that motivate the search for conditions under which the conjectures will hold true. In Section 5, we derive some of these conditions and present examples of demand functions and distributions that satisfy these conditions. Section 6 provides some concluding remarks, including prospective directions for further research.

## **2. LITERATURE REVIEW**

Our study of Conjectures 1 and 3 is related to some earlier research in the area of joint ordering-pricing models and to recent work by Bitran and Mondschein (1993) and Gallego and Van Ryzin (1995). To our knowledge, our study of the other two conjectures is quite new and unrelated to past research. An informative review of ordering-pricing models can be found in the survey of joint production-marketing models by Eliashberg and Steinberg (1991). In our review, we only mention papers that have studied the structural properties of interest to us in this paper.

Thowsen (1975) considered an demand model with additive uncertainty of the form

$\mathcal{D}(p) = D(p) + \mathcal{X}$ , where  $\mathcal{D}(p)$  is a random variable representing demand at price  $p$  in any given period,  $D(p)$  is a deterministic function of price and  $\mathcal{X}$  is a random variable with  $E[\mathcal{X}] = 0$ .

The author derived a list of sufficient conditions under which Conjectures 1 and 3 would hold for such a demand model. One situation where these conditions are satisfied is when  $D(p)$  is linear and  $\mathcal{X}$  is in the class  $PF_2$ . The  $PF_2$  class contains many common distributions, such as the exponential, uniform, normal and truncated normal.

Polatoglu (1991) considered a single-period version of our seasonal product pricing model and showed that the optimal expected revenue function was unimodal in the initial level of inventory when the demand model had one of the following forms:

$\mathcal{D}(p) = D(p) + \mathcal{X}$ , where  $D(p)$  is a linear function of price and  $\mathcal{X}$  is a uniform random variable with  $E[\mathcal{X}] = 0$ .

$\mathcal{D}(p) = D(p)\mathcal{X}$ , where  $D(p)$  is a linear function of price and  $\mathcal{X}$  is an exponential random variable.

This result is related to Conjecture 3 in this paper, since unimodality is a relaxation of the concavity property stated for the optimal expected revenue function in that conjecture. Zabel (1970) also proved the same result for the second demand model given above.

Thomas (1973) studied a multiperiod joint ordering-pricing model with linear ordering costs plus a fixed order placement cost in each period. Demand was taken to be a random variable with a distribution that depended on price and which satisfied the stochastic monotonicity property described in Condition 3.3 in this paper. The author noted that an "(s,S,p)" ordering policy was optimal for this problem for a range of numerically tested problems, and suggested that such a policy may be optimal for this problem under 'fairly general conditions'. Using our counterexample to Conjecture 3 (described in section 4), it can be shown that even for the deterministic demand case such a result would not hold without the imposition of more conditions.

Bitran and Mondschein (1993) and Gallego and Van Ryzin (1995) studied continuous time versions of the seasonal product pricing problem. They showed that under certain assumptions, Conjectures 1 and 3 were satisfied in the continuous time model when demand (for each price level) was given by a Poisson point process.

## Limitations

As mentioned earlier, researchers in the past have not, in our knowledge, addressed the issues encapsulated in Conjectures 2 and 4. The other two conjectures have been analyzed only under fairly restrictive conditions. For instance, for the additive demand model, the variance of demand is always independent of its expected value, while for the multiplicative demand model, this variance is always proportional to the square of the expected value. Our analysis of the conjectures employs a very general form of a demand model, and we derive sufficient conditions that guarantee the conjectures for some cases of the seasonal product pricing problem. Finally, we believe that the counterexamples we provide to each of the conjectures are also new

contributions in this area that, through their counterintuitive nature, provide some interesting insights into the problem.

### 3. ANALYTICAL FORMULATIONS

In this section, we formulate the seasonal product pricing problem and the conjectures mentioned in Section 1 in analytical terms. We assume that the retailer has  $T$  periods in which to sell the product, and that the salvage value for the product at the end of  $T$  periods is zero. The results presented in the paper remains valid for the case of a linear salvage value function also.

#### 3.1 Demand

In the deterministic demand case, demand at price  $p$  in period  $t$  is a deterministic quantity, denoted by  $D_t(p)$ . The following conditions are assumed regarding  $D_t(p)$ :

Condition 3.1:  $D_t(\cdot)$  is a decreasing function of  $p$  for all  $t$ , i.e.,  $p_1 > p_2$  implies  $D_t(p_1) < D_t(p_2)$  for all  $p_1, p_2 > 0$

Condition 3.2:  $D_t(p)$  is a smooth function of  $p$  for all  $t$ .

Note that Condition 3.1 implies that  $D_t(\cdot)$  is invertible.

In the probabilistic demand case, demand at price  $p$  in period  $t$  is a random variable, denoted by  $\mathcal{D}_t(p)$ . We assume that  $\mathcal{D}_t(p)$  is a continuous random variable for all  $t$ , and denote its distribution and density functions by  $F_t(p)$  and  $f_t(p)$  respectively. For both the deterministic and probabilistic demand cases, we assume that demand is continuous and that the inventory comes in continuous amounts. With the exception of Theorem 5.5, the results of this paper can, with appropriate reformulations of the conditions discussed, be extended to the case of probabilistic demand with discrete demand distributions and discrete inventory levels.

The following conditions will be assumed regarding  $\mathcal{D}_t(p)$ :

Condition 3.3: For any  $t$ , and any prices  $p_1 > p_2 > 0$ ,  $\mathcal{D}_t(p_1)$  is stochastically dominated by  $\mathcal{D}_t(p_2)$ , i.e.,  $F_t[x|p_1] \geq F_t[x|p_2]$  for all  $x \geq 0$

(Throughout this paper, we will use the term *stochastic dominance* to refer to *first order stochastic dominance*)

Condition 3.4:  $F_t(x|p)$  is a smooth function of  $x$  and  $p$  for all  $t$

Condition 3.3 is a way of translating the inverse demand-price relationship of the deterministic case (Condition 3.1) to the probabilistic case. It may be described in words as follows: *If, in any period, the retailer raises the price, then the probability that the demand in that period will exceed some given level  $X$  will 'decrease', and this holds true for all levels  $X$ .* (We actually mean 'not increase' instead of 'decrease'. The formal definitions, conjectures and theorems in the paper will be stated using the correct terminology. However, we will use the terms 'decreasing' for

'non-increasing', 'increasing' for 'non-decreasing', etc., when we are focusing on intuitive explanations, and will indicate this abuse of language by using quotation marks.)

Conditions 3.2 and 3.4 are assumed purely for analytical convenience, and are common assumptions in economic models of demand.

Note that the above conditions imply the following:

$$\frac{\partial D_t(p)}{\partial p} \leq 0 \quad \text{for all } t, p \quad (\text{for the deterministic case})$$

$$\frac{\partial F_t(x|p)}{\partial p} \geq 0 \quad \text{for all } x > 0, t, p \quad (\text{for the probabilistic case})$$

### 3.2 Pricing model

The seasonal product pricing problem for the probabilistic demand case is a stochastic dynamic program as described below. We model the time index  $t$  as increasing with time (so that  $t = 1$  is the first period and  $t = T$  is the last) and define the following functions:

- $V_t(I)$  = The maximum expected revenue from period  $t$  onwards, given that the retailer has  $I$  units of product in stock at the beginning of period  $t$ .
- $W_t(I, p)$  = The maximum expected revenue from period  $t$  onwards, given that the retailer has  $I$  units of product in stock at the beginning of period  $t$  and charges a price  $p$  in period  $t$ .

#### Seasonal Product Pricing Problem

*Backward recursion equation*

$$V_t(I) = \max_{p \geq 0} W_t(I, p), \text{ where}$$

$$W_t(I, p) = E[p \min\{\mathcal{D}_t(p), I\} + V_{t+1}(I - \min\{\mathcal{D}_t(p), I\})]$$

*Boundary conditions*

$$V_T(I) = 0 \quad \text{for all } I$$

$$V_t(0) = 0 \quad \text{for all } t$$

One special case of this model, mentioned in Bitran and Mondschein (1993, 1995), is where  $\mathcal{D}_t(p)$  is a Poisson random variable. A continuous time version of the Poisson demand case has also been analyzed by Gallego and Van Ryzin (1994) and Bitran and Mondschein (1994). We will consider this Poisson demand model later when we present examples of demand models.

The seasonal product pricing problem under deterministic demand can be expressed as a special case of the above dynamic program. We replace the random variable  $\mathcal{D}_t(p)$  by  $D_t(p)$  and the backward recursion equation involving  $W_t(I, p)$  now reads as follows:

$$W_t(I, p) = p \min\{D_t(p), I\} + V_{t+1}(I - \min\{D_t(p), I\})$$



### 3.3 Model with sales limit

We describe below an extension to the dynamic programming model formulated above that considers the issue of a *sales limit* discussed in Section 1. In this model, the retailer is allowed to limit the sales in any period to an amount less than the inventory available at the beginning of that period. The retailer therefore has to make two decisions at the beginning of each period: what price to charge in that period, and what limit to impose on sales in that period. We define the two functions  $V_t(I)$  and  $W_t(I,p,S)$  as follows:

- $V_t(I)$  = The maximum expected revenue from period  $t$  onwards, given that the retailer has  $I$  units of product in stock at the beginning of period  $t$ .
- $W_t(I,p,S)$  = The maximum expected revenue from period  $t$  onwards, given that the retailer has  $I$  units of product in stock at the beginning of period  $t$ , charges a price  $p$  in period  $t$ , and imposes a sales limit of  $S$  on period  $t$ . It is assumed that  $S \leq I$ .

#### Seasonal Product Pricing Problem with Sales Limit

*Backward recursion equation*

$$V_t(I) = \max_{p \geq 0, I \geq S \geq 0} W_t(I,p,S), \text{ where}$$

$$W_t(I,p,S) = E[p \min\{D_t(p), S\} + V_{t+1}(I - \min\{D_t(p), S\})]$$

*Boundary conditions*

$$V_T(I) = 0 \quad \text{for all } I$$

$$V_t(0) = 0 \quad \text{for all } t$$

### 3.4 Level of demand

We describe below how the notion of a *level of demand* is expressed in our analytical framework. Consider first the deterministic demand case. We model the demand function  $D_t(p)$  as a function of a non-negative parameter,  $\alpha$ , which we call the *demand parameter*. When relevant, we will therefore write the demand function as  $D_t(p|\alpha)$ . We now wish to define the following property: *The level of demand at each price is 'increasing' in  $\alpha$* . This is done by assuming the following:

Condition 3.5:  $\alpha_1 > \alpha_2 > 0$  implies that  $D_t(p|\alpha_1) \geq D_t(p|\alpha_2)$  for all  $t, p$ .

Thus, a higher value for  $\alpha$  implies a higher (strictly, at least as high) level of demand at each price  $p$ . For instance, suppose the demand function is linear:

$$D_t(p) = A_t - b_t p, \quad p \leq A_t/b_t$$

It is straightforward to observe that a higher value for  $A_t$  should lead to a higher level of demand at each price, and so the above condition is satisfied if we take  $\alpha = A_t$ . For analytical convenience, we will assume the following condition:

Condition 3.6:  $D_t(p|\alpha)$  is a smooth function of  $\alpha$  for all  $t, p$ .

The extension of this idea to the probabilistic case is analogous to the extension of Condition 3.1 to Condition 3.3. We again assume that the random variable  $D_t(p)$  is associated with a demand

parameter  $\alpha$ . Where relevant, we will denote this random variable by  $\mathcal{D}_t(p|\alpha)$ . Analogous to conditions 3.5 and 3.6, we will assume the following:

Condition 3.7:  $\alpha_1 > \alpha_2 > 0$  implies that  $\mathcal{D}_t(p|\alpha_1)$  stochastically dominates  $\mathcal{D}_t(p|\alpha_2)$  for all  $t, p$ , i.e.,  $F[x|p, \alpha_1] \leq F[x|p, \alpha_2]$  for all  $x \geq 0$ .

Condition 3.8:  $F_t(x|p, \alpha)$  and  $f_t(x|p, \alpha)$  are smooth functions of  $x, p$  and  $\alpha$  for all  $t$ .

The use of  $\alpha$  to parametrically capture the notion of the *level of demand* will be further illustrated in the examples that follow below.

### 3.5 Change in level of demand over time

We model increasing or decreasing trends in demand over time in the following manner. Consider the deterministic demand case where we have  $T$  demand functions  $\{D_t(p): t = 1, \dots, T\}$  over the  $T$  periods of the season. To model decreasing demand levels over time, we will assume the following:

Condition 3.9:  $D_t(p) \geq D_{t+1}(p)$  for all  $t, p$ .

Hence, in words, demand is decreasing over time if, at all price levels  $p$ , the demand  $D_t(p)$  is decreasing in  $t$ . If the demand functions  $D_t(p)$  were all of the form:

$$D_t(p) = D(p|\alpha_t) \text{ for all } t,$$

where  $\alpha_t$  is a demand parameter, then, given Condition 3.5, we would have Condition 3.9 being equivalent to the following condition:  $\alpha_{t+1} \leq \alpha_t$  for all  $t$ .

This approach can be extended to the probabilistic demand case in a straightforward manner, yielding the condition below:

Condition 3.10:  $\mathcal{D}_t(p)$  stochastically dominates  $\mathcal{D}_{t+1}(p)$  for all  $t, p$ , i.e.,  $F_t[x|p] \leq F_{t+1}[x|p]$  for all  $x \geq 0$ , and all  $t, p$ .

As for the deterministic case, if the demand random variables  $\mathcal{D}_t(p)$  were all of the form:

$$\mathcal{D}_t(p) = \mathcal{D}(p|\alpha_t) \text{ for all } t,$$

where  $\alpha_t$  is a demand parameter, then, given Condition 3.7, we would have Condition 3.10 being equivalent to the following condition:  $\alpha_{t+1} \leq \alpha_t$  for all  $t$ .

Increasing trends in demand could be modeled in a similar manner.

### 3.6 Examples

We provide below a number of examples of demand models (demand functions  $D(p)$  and random variables  $\mathcal{D}(p)$ ) that we will use in this paper to illustrate the application of the sufficient conditions to be derived in Section 5.

*Deterministic case*

3.6.1. Linear:  $D(p) = A - bp$  where  $A > 0, b > 0$  and  $p \leq A/b$

The parameter  $A$  may represent the size of the customer population, while  $b$  may signify their price sensitivity to the product. Either  $A$  or  $1/b$  could be taken as the demand parameter  $\alpha$  – in the former case, an increase in  $\alpha$  may signify an increase in the overall customer population, and in the latter case, it may signify a decrease in customers' price sensitivity to the product

3.6.2. Weibull:  $D(p) = A e^{-b^a p^a}$  where  $A > 0, b > 0$  and  $a > 0$

Here too,  $A$  may represent the size of the customer population, while  $b$  may signify their price sensitivity to the product. Again, either  $A$  or  $1/b$  could be taken as the demand parameter  $\alpha$ . When  $a = 1$ , we get the exponential demand function.

(We term this demand function as a Weibull function since it can be derived by assuming that the customer population has a Weibull distribution of reservation prices. A customer is said to have a *reservation price* of  $r$  if he or she will purchase the product for any price  $\leq r$ . The cumulative distribution function for a Weibull is given by:  $F(r) = 1 - e^{-b^a r^a}$ .

This allows us to derive the Weibull-based demand function as follows:

$$D(p) = (\text{Total population of customers}) \times (\text{Fraction of customers who will purchase at price } p) = A(1-F(r)) = A e^{-b^a p^a}$$

Both the linear and Weibull demand functions satisfy conditions 3.1, 3.2, and they both satisfy conditions 3.5 and 3.6 for  $\alpha = A$  and for  $\alpha = 1/b$ . For reasons of space, we do not provide any proofs in this paper for our claims about the demand models satisfying the conditions in Section 3. These results are fairly straightforward to derive.

*Probabilistic case*

3.6.3. Multiplicative Noise:  $\mathcal{D}(p) = D(p)\mathcal{X}$ , where  $D(p)$  is a deterministic demand function that satisfies conditions 3.1 and 3.2, and  $\mathcal{X}$  is a continuous random variable with expected value 1.

This model satisfies conditions 3.3 and 3.4. A special case of this example is when  $\mathcal{D}(p)$  of the following 'linear-exponential' form:

$$\mathcal{D}(p) = (A - bp)\mathcal{X}, \quad \text{where } \mathcal{X} \text{ is an exponential random variable, and } A > 0, b > 0, p \leq A/b.$$

This model was studied by Zabel (1970) and Polatoglu (1991). In this case, also, conditions 3.7 and 3.8 are satisfied when the demand parameter is taken to be  $A$  or  $1/b$ .

3.6.4. Exponential-Exponential:  $P[\mathcal{D}(p) \geq x] = A e^{-b_p x}$  where  $1/b_p = e^{-bp}$ , and  $A > 0, b > 0$

In this model,  $\mathcal{D}(p)$  is an exponential random variable (scaled by a certain constant  $A$ ) with a mean  $1/b_p$  that is an exponential function of price.  $\mathcal{D}(p)$  satisfies conditions 3.3 and 3.4. We can again consider either one of  $A$  and  $1/b$  as the demand parameter for this model, and in both cases, conditions 3.7 and 3.8 are satisfied.

3.6.5 Poisson-Weibull:  $\mathcal{D}(p)$  = Poisson r.v. with arrival rate  $\lambda(p)$ , where

$$\lambda(p) = A e^{-b \cdot p^a}, \text{ where } A > 0, b > 0, a > 0.$$

In this model,  $\mathcal{D}(p)$  is a Poisson random variable with an arrival rate  $\lambda(p)$  that is a Weibull function of price. This demand model was employed by Bitran and Mondschein (1993, 1995). The case of Poisson distribution with an exponential arrival rate function (derived by setting  $a = 1$ ) was also employed by Gallego and Van Ryzin (1995) in a continuous time context. This model satisfies conditions 3.3 and 3.4. It also satisfies conditions 3.7 and 3.8 for both  $\alpha = A$  and  $\alpha = 1/b$ .

As discussed in Section 2, the multiplicative demand model is very restrictive in form. However, due in part to the analytical convenience that this model and the additive demand model offer, researchers in the past have almost exclusively focussed on these demand models when analyzing structural properties of the type considered in conjectures 1 and 3 in this paper (two exceptions are Bitran and Mondschein, 1993, and Gallego and Van Ryzin, 1994). The models in examples 3.5 and 3.6 do not fit into either of these forms, and they will be used to demonstrate the broader applicability of the conditions we derive in Section 5.

### 3.7 Conjectures

We now state the conjectures mentioned in Section 1 in formal analytical terms. We also provide some intuitive justification for these conjectures.

Consider the seasonal product pricing problem defined in Section 3.2, and suppose we are at the beginning of any period  $t$ . We define  $p_{it}$  to be the optimal price in period  $t$  when the inventory at the beginning of period  $t$  is  $I$ , and  $p_{t\alpha}$  to be the optimal price in period  $t$  when the demand parameter in period  $t$  is  $\alpha$  (for a fixed level of period  $t$  inventory). We will assume in this paper that this optimal price  $p_{it}$  (or  $p_{t\alpha}$ ) is unique. We do so to keep the exposition simple - the results in this paper can be generalized to the case where the optimal price is not unique.

The first conjecture relates the optimal price in period  $t$  to the inventory level at the beginning of that period. It is intuitive to expect that as one increases the initial inventory level from low to high levels, the optimal initial price ‘decreases’ - as the retailer wants to induce the market to purchase a larger quantity - and then plateaus at some level beyond which it is not beneficial to the retailer to lower the price further. This leads to our first conjecture:

Conjecture 1: *Consider the seasonal product pricing problem defined in Section 3.2 from any period  $t$  onwards. Suppose for each period  $r = t, \dots, T$ , the demand random variable  $\mathcal{D}_r(p)$  (or, in the deterministic demand case, the demand function  $D_r(p)$ ) satisfies conditions 3.3 and 3.4 (conditions 3.1 and 3.2 in the deterministic demand case). Then the optimal price  $p_{it}$  is non-increasing in  $I$ .*

In Conjecture 2, we wish to describe the influence of the level of demand in period  $t$  on the price in that period. Suppose that the retailer revises his or her estimate of the level of demand in period  $t$  upwards. Hence, at each price level, the demand in period  $t$  has increased. One would

intuitively expect that this should lead to an ‘increase’ in the optimal initial price, as expressed below:

**Conjecture 2:** *Consider the seasonal product pricing problem defined in Section 3.2 from any period  $t$  onwards. Suppose the demand random variable  $D_t(p|\alpha)$  (or, in the deterministic demand case, the demand function  $D_t(p|\alpha)$ ) satisfies conditions 3.3, 3.4, 3.7 and 3.8, and for each period  $r = t+1, \dots, T$ , the demand random variable  $D_r(p)$  (or, in the deterministic demand case, the demand function  $D_r(p)$ ) satisfies conditions 3.3 and 3.4 (conditions 3.1 and 3.2 in the deterministic demand case). Then the optimal price  $p_{t\alpha}$  is non-decreasing in  $\alpha$ .*

A different version of Conjecture 2 could be one where the retailer revises the demand information simultaneously for all the periods, i.e., the demand functions  $D_r(\cdot)$  ( $r = t, \dots, T$ ) are all functions of the same demand parameter  $\alpha$ , and the change in  $\alpha$  affects them all simultaneously. This version of the conjecture is not considered in this paper.

The marginal revenue function in period  $t$  is given by  $dV_t(I)/dI$ . This will always be non-negative, since the retailer can choose to ignore the additional unit of inventory (also recall that we are not considering ordering costs or inventory holding costs). One may also, however, expect this marginal value to be ‘decreasing’ with the inventory level  $I$  - as  $I$  increases, the retailer would need to lower the initial price in order to sell the additional unit of product, and this would lead to lower incremental revenues from the additional unit. This leads to Conjecture3:

**Conjecture 3:** *Consider the seasonal product pricing problem defined in Section 3.2 from any period  $t$  onwards. Suppose for each period  $r = t, \dots, T$ , the demand random variable  $D_r(p)$  (or, in the deterministic demand case, the demand function  $D_r(p)$ ) satisfies conditions 3.3 and 3.4 (conditions 3.1 and 3.2 in the deterministic demand case). Then the marginal revenue function  $dV_t(I)/dI$  is non-increasing in  $I$ .*

Finally, consider Conjecture4. It is quite intuitive to expect that in certain cases, a retailer may benefit by reserving some of the inventory for future periods, as allowed by the model with sales limit. For instance, in the airline industry, booking limits are imposed on advance purchase fares in order to keep some inventory of seats reserved for later arrivals. (In this illustrative context, the *product* is a seat on a particular flight, the initial *inventory* is the total number of seats on the flight, and the sales *season* is the time period in which the flight's seats are made available for sales, ending on the day the flight departs.) These later arrivals typically constitute the 'high-end' segment of business travelers, while early arrivals typically constitute the 'low-end' segment of leisure travelers. Since the 'high end' travelers are willing to pay more for the 'product' (a seat on a certain flight), and since some of the inventory still needs to be sold at a lower fare to the low-end segment (as there is not enough demand by the high-end segment to sell all the seats on the aircraft), an airline benefits from making the product available at lower fares early in the season while imposing a booking limit to reserve a certain number of seats for sale in future periods. Similar ‘yield management’ practices are also prevalent in the hotel and car rental industries.

As illustrated by the airline example, the retailer may benefit from imposing a sales limit in the early part of the sale period for products that can be sold at higher prices later in the sale period. In contexts where the product's value decreases over the course of the season, however, one would not expect the retailer to benefit from reserving product for future periods in this manner. This decrease may come about both as a result of the product losing value to customers over time (as the useful life of the product decreases over the course of the season), as well as by the exit of high-end customers (who are willing to pay higher prices) and the entry of low-end customers (who are willing to pay only lower prices). Examples of such contexts are seasonal apparel and sportswear. This leads to the following conjecture:

**Conjecture 4:** *Consider the probabilistic case of the seasonal product pricing problem with sales limit defined in Section 3.3. Suppose that for all  $r = t, \dots, T$ , the demand random variables  $\mathcal{D}_r(p)$  satisfy conditions 3.3 and 3.10. Then  $W_t(I, p, S) \leq W_t(I, p, I)$  for all  $I \geq 0$ ,  $P > 0$  and  $S \leq I$ . Hence, the retailer does not benefit from imposing a sales limit in period  $t$ .*

Note that unlike conjectures 1-3, Conjecture 4 is only of interest in the probabilistic demand case. It is easy to see that a sales limit will never be useful in a deterministic demand context.

## 4. COUNTEREXAMPLES

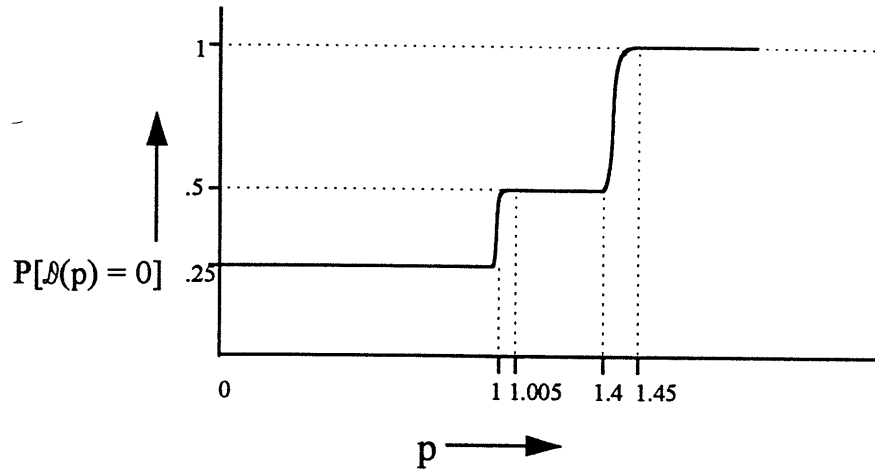
In this section, we will discuss examples that contradict Conjectures 1-4. For Conjectures 1, 2 and 3, our examples are based on *single period* situations - that is, there is only one sale period and a single price set by the retailer. Also, for Conjectures 2 and 3, our examples are based on *deterministic demand* conditions. It is interesting that we can get counter-intuitive results even in the simple context of a single period and deterministic demand. By confounding our intuition, the results of this section provide us with fresh insights into the nature of the seasonal product pricing problem and motivate the search for conditions under which the conjectures may be guaranteed to hold true. This will lead us to the next section, where such conditions will be identified.

Proofs for all the results in this section are provided in the appendix.

### 4.1 Counterexample to Conjecture 1

Consider the following seasonal product pricing problem. There is a single sale period ( $T = 1$ ), and the demand is uncertain, with the demand random variable  $\mathcal{D}(p)$  having a discrete distribution for all  $p$ . The probability mass function for  $\mathcal{D}(p)$  is defined as follows.

For any price  $p$ , the random variable  $\mathcal{D}(p)$  has zero mass at all points except possibly at the points 0, 2 and 3.  $P[\mathcal{D}(p) = 0]$  is given by the curve in Figure 4.1a.



**Figure 4.1a:  $P[D(p) = 0]$  as a function of  $p$**

This curve is flat over the intervals  $[0, 1]$ ,  $[1.005, 1.4]$  and  $[1.45, \infty)$ . In the two intervals  $(1, 1.005)$  and  $(1.4, 1.45)$ ,  $P[D(p) = 0]$  is defined such that the curve is smooth and increasing in  $p$  over  $[0, \infty)$ . We do not define the curve analytically in these two intervals since the particular values that  $P[D(p) = 0]$  takes in these intervals is not of consequence to our analysis.

$P[D(p) = 2]$  and  $P[D(p) = 3]$  are given by:

$$\begin{aligned}
 P[D(p) = 2] &= 0.5 - P[D(p) = 0] && \text{for } p \text{ in } [0, 1.4] \\
 &= 0 && \text{for } p \text{ in } (1.4, \infty) \\
 P[D(p) = 3] &= 0.5 && \text{for } p \text{ in } [0, 1.4] \\
 &= 1 - P[D(p) = 0] && \text{for } p \text{ in } (1.4, \infty)
 \end{aligned}$$

$P[D(p) = 2]$  and  $P[D(p) = 3]$  have been defined so that the sum of  $P[D(p) = 0]$ ,  $P[D(p) = 2]$ ,  $P[D(p) = 3]$  is 1 for all  $p$ , as desired.

We note the following additional characteristics of this family of probability mass functions:

- At each price  $p$ , the demand distribution is discrete, and  $P[D(p) = n]$  is a smooth function of  $p$  for all  $n$ . Thus,  $D(p)$  satisfies (the discrete distribution analog of) Condition 3.3.
- For  $p > q > 0$ ,  $D(p)$  is stochastically dominated by  $D(q)$ . Thus,  $D(p)$  satisfies (the discrete distribution analog of) Condition 3.4.

The following series of lemmas prove that the optimal price at inventory level 2 is less than the optimal price at inventory level 3. Thus, we would have a case where the optimal price *increases* when the initial inventory is increased, contradicting Conjecture 1.

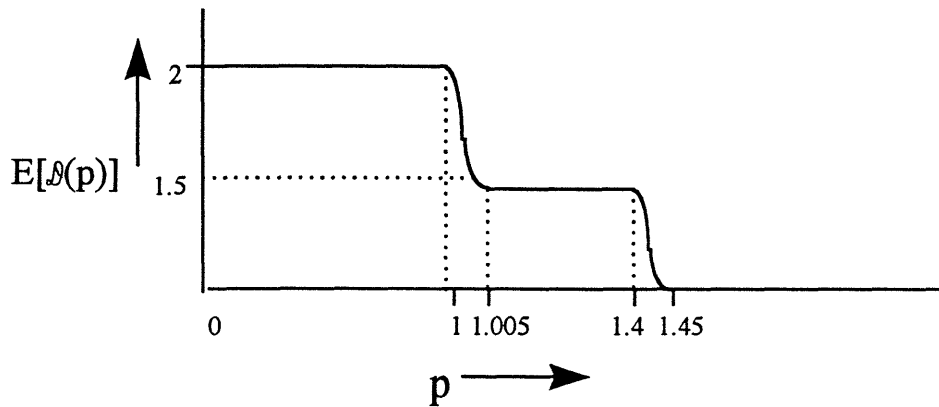
**Lemma 4.1:** *For all inventory levels  $I$ , the optimal price  $p_1$  for the above problem lies in the region  $[1, 1.005) \cup [1.4, 1.45)$ .*

**Lemma 4.2:** *The optimal price  $p_2$  for the above problem when the inventory level is 2 lies in the interval  $[1, 1.005)$ .*

**Lemma 4.3:** *The optimal price  $p_3$  for the above problem when the inventory level is 3 lies in the interval  $[1.4, 1.45)$ .*

Lemmas 4.2 and 4.3 establish the desired result.

Figure 4.1b shows the expected value function  $E[\mathcal{J}(p)]$  for this problem. This curve has the same ‘flat-drop’ form as the demand function considered later in counterexample 3.

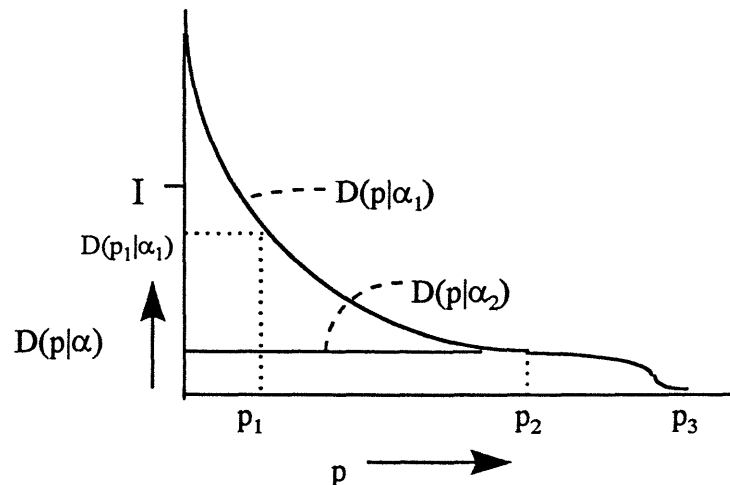


**Figure 4.1b: Expected value function  $E[\mathcal{J}(p)]$**

The above illustration employed a probabilistic demand context. For the seasonal product pricing problem with deterministic demand, Conjecture 1 is true - conditions 3.1 and 3.2 suffice in guaranteeing that the optimal initial price, in the general multiperiod case, is a non-increasing function of the initial inventory level. This result is formally stated in the next section, and proved in the Appendix.

## 4.2 Counterexample to Conjecture 2

This example is based on a single period, deterministic demand problem. The demand function  $D(p|\alpha)$  at two values  $\alpha_1$  and  $\alpha_2$  of the demand parameter  $\alpha$ , with  $\alpha_1 > \alpha_2 > 0$ , are as shown in Figure 4.2.



**Figure 4.2: Demand functions for  $\alpha_1$  and  $\alpha_2$**



The demand functions and the price  $p_1$  are chosen such that  $p_1 D(p_1|\alpha_1) > p_2 D(p_2|\alpha_1)$ . This condition will easily be satisfied by choosing a high-enough value for the point  $D(p_1|\alpha_1)$ . Note that  $D(p|\alpha_1) = D(p|\alpha_2)$  for all  $p \geq p_2$ , and that  $D(p|\alpha_1) > D(p|\alpha_2)$  for  $p < p_2$ , and so Condition 3.5 is satisfied for  $\alpha_1$  and  $\alpha_2$ . For  $\alpha_1 < \alpha < \alpha_2$ ,  $D(p|\alpha)$  is defined in any manner that satisfies conditions 3.1, 3.2, 3.5 and 3.6. It is easy to see that it will be feasible to do so, and we therefore do not furnish a specific analytical form for  $D(p|\alpha)$  over this interval. The initial inventory  $I$  is taken to be larger than  $D(p_1|\alpha_1)$ .

Lemma 4.4 shows that the optimal price decreases as  $\alpha$  increases from  $\alpha_2$  to  $\alpha_1$ . Here,  $p_\alpha$  stands for the optimal price when the demand parameter is  $\alpha$ .

Lemma 4.4: *For the pricing problem described above,  $p_{\alpha_1} < p_{\alpha_2}$ .*

We offer the following explanation for why Conjecture 2 is not true in general. Consider the unconstrained optimal pricing problem:

$$\max_{p \geq 0} pD(p|\alpha)$$

where we are maximizing revenues over a single period without an inventory constraint. The optimal solution  $\bar{p}_\alpha$  can be shown, by the first order optimality condition, to satisfy

$$\bar{p}_\alpha = - \frac{D(\bar{p}_\alpha|\alpha)}{\left( \frac{\partial D(\bar{p}_\alpha|\alpha)}{\partial p} \right)} \quad (4.1)$$

We observe that the optimal price depends both on the level of demand,  $D(p|\alpha)$ , and the rate of change of demand with price,  $\left( \frac{\partial D(p|\alpha)}{\partial p} \right)$  (which we loosely term as the price sensitivity). An

increase in the demand parameter will effect not only the level of demand, but potentially also the price sensitivity of demand at various points on the demand curve. The overall effect of the increase in the demand parameter  $\alpha$  on  $\bar{p}_\alpha$  will depend on the change in both of these quantities over all price levels. Note that the example assumes there was enough inventory to meet demand at the relevant prices, and so it is not necessary to assume a constrained inventory situation to derive this behavior.

The above situation could occur in practice when a retailer experiences a surge in demand caused by the entry of a number of predominantly low-end, price sensitive customers into the market. (This may be caused by recent advertising or promotional activity by the retailer that generates a disproportionately greater response from the low-end customers.) This may cause an overall *increase* in demand at each price level, but may still make it attractive for the retailer to *lower* the price since he or she may be able to sell a much higher amount at a lower price than was possible earlier.

### 4.3 Counterexample to Conjecture 3

This example is also based on a single period, deterministic demand problem. The demand function  $D(p)$ , shown by the curve in Figure 4.3, is defined as:

$$\begin{aligned} D(p) &= 4 & p \leq 60 \\ D(p) &= 2 & 65 \leq p \leq 100 \end{aligned}$$

In the intervals  $(60, 65)$  and  $(100, \infty)$ ,  $D(p)$  is defined in a way that allows the curve to be smooth and decreasing in  $p$  in the range  $(0, \infty)$ . We do not define the curve analytically in these two intervals since the particular values that  $D(p)$  takes in these intervals is not of consequence to our analysis. In addition, we require that  $D(p)$  decrease very dramatically as  $p$  goes beyond \$100, thus ensuring that the optimal price is \$100 when the inventory level is 2.

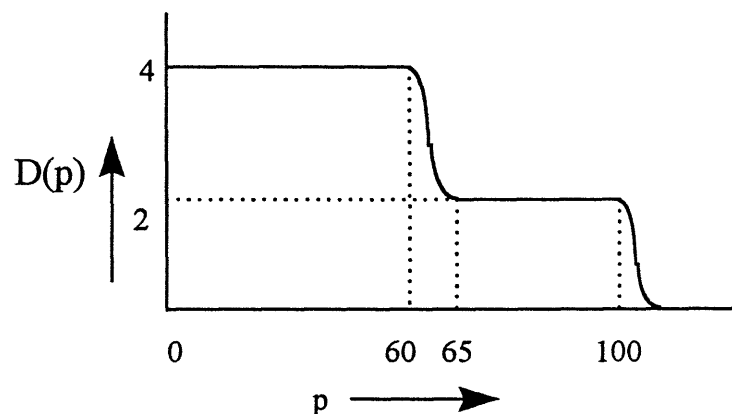


Figure 4.3: Demand function

The optimal prices at initial inventory levels of  $I = 2, 3$ , and 4 are given by:

$$\begin{aligned} p_I &= \$100 & \text{at } I = 2 \\ &= \$100 & \text{at } I = 3 \\ &= \$60 & \text{at } I = 4 \end{aligned}$$

$$\begin{aligned} \text{This yields } V(I) &= \$200 & \text{at } I = 2 \\ &= \$200 & \text{at } I = 3 \\ &= \$240 & \text{at } I = 4 \end{aligned}$$

Hence  $V(3) - V(2) = 0 < 40 = V(4) - V(3)$ , and so the rate at which  $V(I)$  increases has increased in going from 2 to 3, contradicting Conjecture 3.

This counterexample hinges on the sudden drop in demand after the price level of \$60 followed by the flat demand between \$65 and \$100. This demand pattern makes the optimal price move in a very irregular fashion as a function of the initial inventory. This in turn causes  $V(I)$  to increase as a function of  $I$  in an irregular manner itself, leading to the non-concave behavior shown above. Note that though we have studied the values of  $V(I)$  at the integral points  $I = 2, 3$  and 4, the example is applicable to a context where  $I$  is a continuous variable.

Can the above situation arise in practice? One context where this might happen is when there are, say, two distinct segments of customers in the market - a high-end segment, willing to pay a high price (say,  $\geq \$100$ ), and a low-end segment, willing to pay only a much lower price (say,  $\approx \$60$ ) for the product. Under tight inventory conditions, the retailer will then focus on the high-end segment and charge a high price ( $\geq \$100$ ) for the product. When the inventory level exceeds the level that the high-end segment can buy, the retailer may still desist from lowering the price to clear the inventory if the size of the additional inventory is small, since the price would need to be lowered substantially in order to attract the other customers (from the low-end segment), and so the additional inventory will not be of any additional value to the retailer. When the inventory level is much higher, though, the retailer will be able to make higher revenues by lowering the price to attract the low-end segment and achieving much higher sales. At this point, the additional inventory will again be of positive value to the retailer. Note that we are implicitly assuming that the retailer cannot price discriminate by charging different prices to the two different customer segments. A counter-intuitive outcome of this behavior is that the retailer would be willing to pay the supplier a *higher* per-unit price when the supplier provides a high level of additional stock (over and above a small initial order level) than when the supplier provides lower levels of additional stock.

#### 4.4 Counterexample to Conjecture 4

Finally, we consider the sales-limit issue introduced in Conjecture 4. Consider a two period problem where the demand in each period is given as follows:

$$\begin{aligned} \text{In period 1, } P[D(p) = 0] &= 1 && \text{for all } p \text{ in } (50, \infty] \\ P[D(p) = 5] &= 1 && \text{for all } p \text{ in } (25, 50] \\ P[D(p) = 30] &= .5 && \text{for all } p \text{ in } [0, 25] \\ P[D(p) = 50] &= .5 && \text{for all } p \text{ in } [0, 25]. \end{aligned}$$

$$\begin{aligned} \text{In period 2, } P[D(p) = 0] &= 1 && \text{for all } p \text{ in } (40, \infty] \\ P[D(p) = 5] &= 1 && \text{for all } p \text{ in } (20, 40] \\ P[D(p) = 20] &= 1 && \text{for all } p \text{ in } [0, 20] \end{aligned}$$

Figure 4.4 shows the demand model for periods 1 and 2.

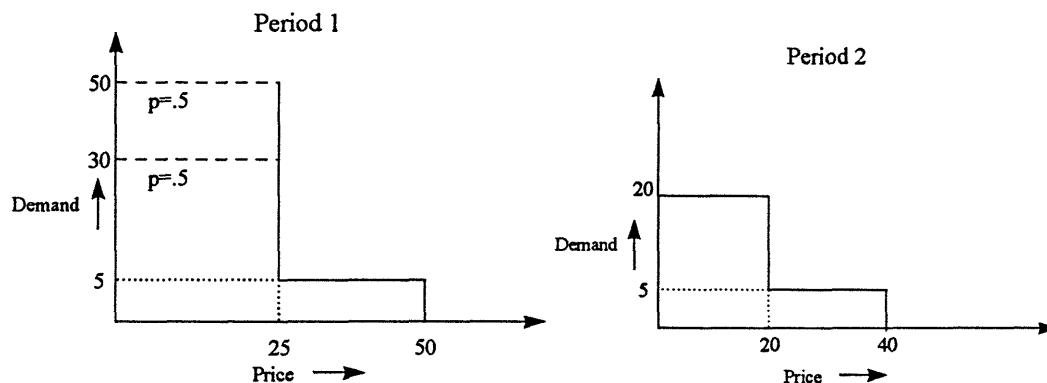


Figure 4.4: Demand-price relationships for periods 1 and 2

It is straightforward to check that the demand model represented above satisfies conditions 3.3 and 3.10. In particular, the level of demand decreases from period 1 to period 2. Suppose we start with 50 units of initial inventory. The optimal pricing strategy under the no-sales limit case would then be to charge \$25 in period 1. Then, if the period 1 demand was 50, the inventory would be exhausted and the period 1 revenue would be \$1,250. If the period 1 demand was 30, 20 units would be passed on to period 2 and the period 1 revenue would be \$750. In that case, the optimal period 2 price would be \$20 and the period 2 revenue would be \$400. Hence the optimal expected revenue under no sales limit is  $.5(\$1,150 + \$1,250) = \$1,200$ .

In the sales limit case, the optimal period 1 strategy would be to reserve 5 units of stock for period 2 and again charge \$25. In this case, the period 1 revenues would be \$1,125 under the high demand scenario and \$750 under the low demand scenario. In the high demand scenario, period 1 sales would be limited to 45, and 5 units would be passed on to period 2. These would then be priced at \$40 in period 2, fetching revenues of \$200. In the low demand scenario, 20 units would be passed on to period 2, and these would be priced at \$20 in that period, fetching revenues of \$400. Hence the optimal expected revenues under a sales limit would be  $.5(1,325+1,150) = \$1,237.50$ . This is larger than the same figure for the model with no sales limit.

The sales limit was useful in the above examples because it enabled the retailer to 'price discriminate': the retailer was able to reserve some inventory to be sold at a higher price to a price-insensitive segment in period 2 and sell the rest at a lower price to a price sensitive segment in period 1. An alternative use of the sales limit, not illustrated in the above example, may be the following. At the beginning of each period, the retailer has to commit to a price for that period. This price is based on the available level of inventory and the retailer's expectation of current and future demand. If an unexpectedly large number of customers arrive in the current period to purchase the product, the retailer will be caught by surprise, and may then regret having sold so much product at the period 1 price instead of keeping some for selling at a higher price over the subsequent periods. A sales limit can therefore provide the retailer with the capability to insure against lost future revenues under unexpectedly high early demand conditions.

In certain retail environments, the retailer cannot raise the price over time - that is, the retailer can mark down prices but cannot mark them up. In such cases, the sales limit is not necessary, and we will prove this in the next section. Another context where a sales limit is not necessary is where the retailer can revise the price in a continuous manner, and this is also further discussed in the next section.

The sales limit may play a very critical role in certain situations - notably, one where the retailer does not know how customers value the product beforehand but learns of this over the course of the season. In this situation, the retailer may benefit substantially from insuring against an underestimation in demand by capping the level of sales allowed in the initial period.

## 5. SUFFICIENT CONDITIONS

The examples in the previous section have demonstrated that the conjectures in Section 1 are not true in general. We turn now to the task of identifying conditions under which they can be guaranteed to hold.

We first derive a series of sufficient conditions under which Conjectures 1-3 can be guaranteed. This is done in three stages - we first consider the case of *deterministic demand*, then the case of *uncertain demand* in a *single period* context, and finally, the most general case of *uncertain demand* in a *multi-period* context. We employ this incremental approach in order to highlight the different conditions required under each of these contexts, and because we have limited results at this stage for the most general case of *uncertain demand* in a *multi-period* context. In each case, we provide examples of demand models (from among those discussed in Section 3) that satisfy the identified conditions. (For reasons of space, we will not provide proofs for these claims - that certain demand models satisfy certain conditions in this section - in this paper.) Finally, we prove that in a common retail context - where the price is not allowed to increase over time - Conjecture 4 can be also guaranteed to hold. We also comment on another context - where the price can be revised on a continuous basis - in which Conjecture 4 can be guaranteed to hold.

As in Section 3, we will use the term  $p_t$  to represent the optimal price in period  $t$  when the inventory at the start of period  $t$  is  $I$ , and the term  $p_{t\alpha}$  to represent the optimal price in period  $t$  when the demand parameter for period  $t$  is  $\alpha$  (for a fixed level of inventory at the beginning of period  $t$ ).

For reasons of space, we only provide proofs for Corollary 5.1 and Theorems 5.1, 5.2 and 5.6 in the appendix. Proofs for the other results are given in Wadhwa (1996), and can be obtained from the authors.

### 5.1 Deterministic demand

The counterexample to Conjecture 1 in the previous section was based on a probabilistic demand context. Theorem 5.1 states that Conjecture 1 is true for the single period deterministic demand problem.

**Theorem 5.1:** *Consider the single period seasonal product pricing problem under deterministic demand defined in Section 3. Suppose  $D_t(p)$  satisfies conditions 3.1 and 3.2 for all  $p$ . Then the optimal price  $p_{1I}$  is non-increasing in  $I$ .*

The extension of this result to the multi-period case, however, requires that the value function  $V_t(I)$  be concave in  $I$ . This links Conjecture 1 with Conjecture 3, and so we turn first to defining sufficient conditions for Conjecture 3 to be true.

**Theorem 5.2:** *Consider the seasonal product pricing problem under deterministic demand. Suppose, for all  $t$ , the demand function  $D_t(\cdot)$  satisfies conditions 3.1 and 3.2. Also, suppose it satisfies the following condition:*

$$2(D_t'(p))^2 - D_t(p)D_t''(p) \geq 0 \text{ for all } p > 0 \quad (5.1)$$

Then the value function  $V_t(I)$  is concave in  $I$  for all time periods  $t$ .

**Corrolary 5.1:** *Under the conditions defined in Theorem 5.2, the optimal price  $p_t$  is non-increasing in  $I$  for all  $t$ .*

We have used the following notation above:  $D'(p) = \frac{\partial D(p)}{\partial p}$ , and  $D''(p) = \frac{\partial^2 D(p)}{\partial p^2}$ . Condition 5.1

will hold if  $D_t(\cdot)$  is concave (though this is not necessary for Condition 5.1), since then  $D_t''(p) < 0$  for all  $p$ . Another way to state Condition 5.1 is by requiring that the term  $[p + D(p)/D'(p)]$  be increasing in  $p$ .

The linear demand function (example 3.6.1) satisfies Condition (5.1) for  $A > 0$ ,  $b > 0$  and  $p \geq A/b$ . The Weibull demand function (example 3.6.2) satisfies Condition (5.1) for  $A > 0$ ,  $b > 0$  and  $a \geq 1$ .

We now turn our attention to Conjecture 2. We write the demand function for period  $t$  as  $D_t(\cdot|\alpha)$ . Theorem 5.3 specifies conditions under which Conjecture 2 holds for this deterministic demand case.

**Theorem 5.3:** *Consider the seasonal product pricing problem under deterministic demand. For any period  $t$ , let the inventory level at the beginning of period  $t$  be fixed, and let  $p_{t\alpha}$  be the optimal period  $t$  price when the demand function for period  $t$  is  $D_t(\cdot|\alpha)$ . Suppose the demand functions  $D_t(\cdot|\alpha)$ ,  $D_r(\cdot)$  ( $r = t+1, \dots, T$ ) satisfy conditions (3.1), (3.2) and (5.1). Also, suppose  $D_t(\cdot|\alpha)$  satisfies conditions (3.5), (3.6) and the following condition:*

$$D_t(p|\alpha) \left( \frac{\partial^2 D_t(p|\alpha)}{\partial \alpha \partial p} \right) - \left( \frac{\partial D_t(p|\alpha)}{\partial p} \right) \left( \frac{\partial D_t(p|\alpha)}{\partial \alpha} \right) \geq 0 \text{ for all } p, \alpha > 0 \quad (5.2)$$

Then  $p_{t\alpha}$  is non-decreasing in  $\alpha$ .

The linear demand function (example 3.6.1) satisfies Condition (5.2) for  $A > 0$ ,  $b > 0$  and  $p \geq A/b$ , for the case of  $\alpha = A$  as well as for  $\alpha = 1/b$ . The Weibull demand function (example 3.6.2) satisfies Condition (5.2) for  $A > 0$ ,  $b > 0$  and  $a \geq 1$ , for the case of  $\alpha = A$  as well as for  $\alpha = 1/b$ .

It is worthwhile to pause at this stage and gain some intuition about conditions 5.1 and 5.2. Since the demand function  $D_t(\cdot)$  is invertible, we can write the revenue function  $pD_t(p)$  as a function of the demand  $d = D_t(p)$  as follows:

$$R_t(d) = D_t^{-1}(d) \cdot d$$

One can show that Condition 5.1 is equivalent to requiring that  $R_t(d)$  be concave in  $d$ . This is formally shown in the proof of Theorem 5.2 in the Appendix. A concave revenue function  $R_t(d)$  is unimodal in demand, and this unimodality prevents the kind of behavior we observed in the value function  $V(I)$  in Counterexample 3.

Condition 5.2 can be connected to equation 4.1. It is easy to see that Condition 5.2 is equivalent to requiring that the derivative (with respect to  $\alpha$ ) of the right hand side of the equation be non-negative.

## 5.2 Uncertain demand, single period

As demonstrated by counterexample 1 in the previous section, Conjecture 1 is not true in the case of uncertain demand. Theorem 5.4 provides the additional conditions required to guarantee this result.

**Theorem 5.4:** *Consider the single period seasonal product pricing problem under probabilistic demand. Suppose  $A(p)$  satisfies conditions 3.3 and 3.4 for all  $p$ , and, additionally, it satisfies the following condition for all  $p$  and  $I$ :*

$$(1 - F_1(I|p)) \int_0^I F'_1(x|p) dx - F'_1(I|p) \int_0^I (1 - F_1(x|p)) dx \leq 0 \quad (5.3)$$

*Then the optimal price  $p_{II}$  is non-increasing in  $I$ .*

We have used the following notation above:  $F'(x|p) = \frac{\partial F(x|p)}{\partial p}$ , and  $F''(x|p) = \frac{\partial^2 F(x|p)}{\partial p^2}$ .

Examples of models for which Condition 5.3 is satisfied are the multiplicative demand model (example 3.6.3) when  $x$  follows an exponential distribution, the exponential-exponential demand model (example 3.6.4), and the Poisson-Weibull demand model (example 3.6.5). Except for the case of the Poisson-Weibull demand model, we have analytical proofs that these demand models satisfy the relevant conditions in this section as claimed. For the Poisson-Weibull case, we have checked the relevant conditions by running a set of numerical tests to investigate this issue. We used a grid of points ranging over different values of  $p$ ,  $I$ , and the demand model parameters  $A$ ,  $a$  and  $b$  (with all these variables non-negative, and with  $a \geq 1$ ). All the points checked satisfied the relevant conditions.

Some comments about the nature of Condition 5.3 are in order. First consider the following lemma.

**Lemma 5.1:** *For any period  $t$ ,  $E[\min\{I, A(p)\}] = \int_0^I (1 - F_t(x|p)) dx$*

(We suppress the time period subscript 1 in the discussion below.)

Lemma 5.1 shows that, for the single period model mentioned in Theorem 5.4:

$$\text{Expected sales} = \int_0^I (1 - F(x|p)) dx$$

$$\text{Also, Probability of having a shortage} = \text{Probability that demand exceeds inventory} = 1 - F(I|p)$$

As price is increased, both of the above quantities will decrease. Condition 5.3 can be shown to be equivalent to requiring that

$$\frac{\partial \left( \frac{\int_0^I (1 - F(x|p)) dx}{(1 - F(I|p))} \right)}{\partial p} \geq 0,$$

which implies that as price increases, the consequent decrease in expected sales is at a slower rate than the decrease in the probability of having a shortage.

The additional condition required for Conjecture 2 to hold is similar to Condition 5.3, as described below.

**Theorem 5.5:** Consider the single-period seasonal product pricing problem under probabilistic demand. For a fixed level  $I$  of initial inventory, suppose  $\mathcal{A}(p)$  satisfies conditions 3.1, 3.2, 3.5, and 3.6 for all  $p, \alpha$  and, additionally, also satisfies the following condition for all  $p, \alpha$ :

$$-\left( \int_0^I (1 - F_1(x|p, \alpha)) dx \right) \int_0^I \left( \frac{\partial^2 F_1(x|p, \alpha)}{\partial \alpha \partial p} \right) dx - \int_0^I \left( \frac{\partial F_1(x|p, \alpha)}{\partial \alpha} \right) dx \int_0^I \left( \frac{\partial F_1(x|p, \alpha)}{\partial p} \right) dx \geq 0 \quad (5.4)$$

Then  $p_{1\alpha}$  is non-decreasing in  $\alpha$ .

We have checked Condition 5.4 for the Poisson-Weibull demand model using  $\alpha = 1/b$ , and for the exponential-exponential model, using both  $\alpha = A$  and  $\alpha = 1/b$ .

Condition 5.4 is similar to Condition 5.2 for the deterministic demand case, with the expected demand  $\int_0^I (1 - F_1(x|p, \alpha)) dx$  replacing the demand  $D_1(p|\alpha)$ .

Conjecture 3 requires a fairly complicated condition defined in the following theorem.

**Theorem 5.6:** Consider the single period seasonal product pricing problem under probabilistic demand. Suppose  $\mathcal{A}(p)$  satisfies conditions 3.3 and 3.4 for all  $p$ , and, additionally, it satisfies the following condition for all  $p$  and  $I$ :

$$-\left( (1 - F_1(I|p_I)) \int_0^I F'_1(x|p_I) dx - F'_1(I|p_I) \int_0^I (1 - F_1(x|p_I)) dx \right)^2 + \left( 2 \left( \int_0^I F'_1(x|p_I) dx \right)^2 + \int_0^I F''_1(x|p_I) dx \int_0^I (1 - F_1(x|p_I)) dx \right) \int_0^I (1 - F_1(x|p_I)) dx \cdot f_1(I|p_I) \geq 0 \quad (5.5)$$

Then  $V_1(I)$  is concave in  $I$ .

An example of a model for which Condition 5.5 is satisfied is the exponential-exponential demand model (example 3.6.4).



### 5.3 Uncertain demand, multiple periods

We now examine the most general case of the season product pricing problem - with uncertain demand and multiple periods. In the discussion below, we show how Theorems 5.4 and 5.5 can be extended to the multiple period context. Both these results will assume that the value function  $V_t(\cdot)$  is concave, which is what Conjecture 3 asserts. (It is fairly easy to construct examples where the conjectures are violated when the value function  $V_t(I)$  is *not* concave in  $I$ , even though  $V_t(I)$  may be increasing in  $I$ .) We do not, at present, have a satisfactory set of sufficient conditions that guarantee Conjecture 3 in this multi-period context. However, as we will discuss later, we have strong computational evidence that  $V_t(\cdot)$  is in fact concave (for all  $t$ ) when the random variable  $D_t(p)$  is of the Poisson-Weibull form.

**Theorem 5.7:** *Consider the seasonal product pricing problem under probabilistic demand. Consider any period  $t$ . Suppose  $V_{t+1}(I)$  is concave in  $I$ . Further, suppose  $\mathcal{A}(p)$  satisfies conditions 3.3, 3.4, and 5.3, and, in addition, satisfies the following condition for all  $p$ :*

$$\frac{\frac{\partial F_t(J|p)}{\partial p}}{\int_0^J \frac{\partial F_t(x|p)}{\partial p} dx} \text{ is decreasing in } J \text{ for all } J \quad (5.6)$$

*Then  $p_{tJ}$  is non-increasing in  $I$ .*

**Theorem 5.8:** *Consider the seasonal product pricing problem under probabilistic demand. Consider any period  $t$  and a fixed level  $I$  of inventory at the beginning of period  $t$ . Suppose  $V_{t+1}(J)$  is concave in  $J$ . Assume further that  $\mathcal{A}(p)$  satisfies conditions 3.3, 3.4, 3.7, 3.8 and 5.4 for all  $p, \alpha$  and, in addition, satisfies the following condition for all  $p, \alpha$ :*

$$\frac{\int_0^J \left( \frac{\partial^2 F_t(x|p, \alpha)}{\partial \alpha \partial p} \right) dx}{\int_0^J \left( \frac{\partial F_t(x|p, \alpha)}{\partial p} \right) dx} \text{ is increasing in } J \text{ for all } J. \quad (5.7)$$

*Then  $p_{t\alpha}$  is non-decreasing in  $\alpha$ .*

Examples of models for which conditions 5.6 and 5.7 are satisfied are:

- Exponential-exponential demand model (example 3.6.4)
- Poisson-Weibull demand model (example 3.6.5)

Note that conditions 5.6 and 5.7 are similar in the following sense. The term in Condition 5.6 can

be written as  $\frac{\partial \left( \int_0^J \frac{\partial F_t(x|p)}{\partial p} dx \right)}{\partial J}$ , while the term in Condition 5.7 can be written as  $\frac{\partial \left( \int_0^J \frac{\partial F_t(x|p, \alpha)}{\partial p} dx \right)}{\partial \alpha}$ .

We have check for the concavity of  $V_t(I)$  for the seasonal product pricing problem under the Poisson-Weibull demand model by making a number of runs of the stochastic dynamic program

for this problem. (The backward recursion solution approach for this dynamic program has been described in Bitran and Mondschein, 1993.) These runs have involved a range of settings for the initial inventory, the number of time periods, and the demand model parameters  $A$ ,  $b$  and  $a$ . In all of these computational tests, the function  $V_t(I)$  has been found to be concave in  $I$  for all period  $t = 1, \dots, T$ . We therefore believe there is numerical evidence that Conjecture 3 is indeed true for the multi-period, Poisson-Weibull demand model. As we have shown above, Conjecture 3 implies Conjecture 1 and 2, and thus the computational evidence points towards these being true as well for this demand model.

#### 5.4 Sales limit

We now turn to Conjecture 4. We show below that the retailer will in fact not benefit from limiting sales in the early periods if he or she follows a non-increasing price policy - that is, if price is not allowed to increase over time. This situation is commonly observed in many retail settings, such as department stores, and may be motivated, for example, by concern about customers' negative reaction to price increases. We also comment briefly on another context in which a sales limit is not necessary - when the retailer is allowed to adjust the price on a continuous basis. Retailers may often undertake promotional discounting to promote sales or awareness of a product, or to stimulate higher store traffic. In such cases, prices are temporarily reduced, and then raised to their original levels after the promotional period. Our discussion focusses on permanent markups or markdowns, and so the increase in price caused by the elimination of the temporary, promotional price, is not considered.

Consider the dynamic programming model of the seasonal product pricing problem under sales limit, formulated in Section 3. We modify the model below to impose a non-increasing price constraint in every period. This is done by adding an additional variable, price, to the state space of the dynamic program. The value function is now defined as:

$V_t(I, P)$  = The maximum expected revenue from period  $t$  onwards, given that the retailer has  $I$  units of product in stock at the beginning of period  $t$  and the price charged in period  $t-1$  was  $P$ .

The dynamic program can then be formally defined as:

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*Backward recursion equation*

$$V_t(I, P) = \max_{P \geq p \geq 0, I \geq S \geq 0} W_t(I, p, S), \text{ where}$$

$$W_t(I, p, S) = E[p \min\{D_t(p), S\} + V_{t+1}(I - \min\{D_t(p), S\}, p)]$$

*Boundary conditions*

$$V_T(I, p) = 0 \quad \text{for all } I$$

$$V_t(0, p) = 0 \quad \text{for all } t$$

Suppose we are in period  $t$  with the price in the last period being  $P$ . Then the maximum amount of per-unit revenues that we could make, given the non-increasing price constraint, should be  $P$ . This is formally proved by the following lemma.

Lemma 5.2:  $V_t(I,P) \leq V_t(S,P) + (I-S)P$  for all  $t=1,\dots,T$ ,  $I \geq 0$ ,  $P > 0$  and  $S \leq I$ .

This result suggests that, under a non-increasing price constraint, the retailer should not lose an opportunity to sell at the current price, since the per-unit future value of the product can never be higher than the current price. Thus, the retailer should not impose an artificial sales limit ( $S < I$ ) to the current period sales. This is the essence of the proof for the claim in Theorem 5.8. This theorem shows that Conjecture 4 is true in cases where prices are not allowed to increase over time.

Theorem 5.8:  $W_t(I,p,S) \leq W_t(I,p,I)$  for all  $t=1,\dots,T$ ,  $I \geq 0$ ,  $P > 0$  and  $S \leq I$ .

A second situation in which a sales limit would not be necessary is one where the retailer can adjust the price on a continuous basis. In such cases, the price could always be adjusted upwards to an appropriate level whenever an unexpectedly high level of sales occurred over any period of time. In fact, the optimal price path for such a continuous time model involves a jump in price immediately after each sale (for examples of price paths, see the papers by Bitran and Mondschein, or Gallego and Van Ryzin). Thus, sales-limit extensions of the continuous time models of Bitran and Mondschein (1993, 1995) and Gallego and Van Ryzin (1994) are not necessary. We have assumed here that at any price  $p$  the probability of observing more than one unit of demand in any period of time of length  $\Delta t$  approaches zero as  $\Delta t$  approaches zero. In situations where this was not the case (for instance, where demand at any price  $p$  is a non-homogeneous Poisson point process), a sales limit may still be necessary. An example of this would be group bookings in the airline or hotel industries.

## 6. CONCLUSIONS

In this paper, we have proposed and analyzed a number of conjectures regarding the structure of the seasonal product pricing problem. We have discussed the intuitive nature of the conjectures, and have demonstrated through a series of counterexamples that in fact these conjectures are not in general true for the problem. Further, we have developed a number of sufficient conditions that guarantee the conjectures in many situations. These conditions are satisfied not only by certain commonly used demand models (such as the multiplicative uncertainty model under an exponential distribution), but also by other demand models that do not fit into their restrictive structure.

A number of issues arising out of our analysis could be investigated further. These include the derivation of sufficient conditions that guarantee Conjecture 3 for the probabilistic, multi-period problem, and the analysis of other demand models with respect to the sufficient conditions that have been derived. One may also seek to determine if alternative sufficient conditions may be found (to those derived in this paper) that may facilitate the analysis of larger classes of demand models.

In certain seasonal product contexts, retailers can reorder the product (usually only once or twice) during the season. This leads to an extension of the seasonal product pricing model that will incorporate the reordering decision. While a number of researchers have developed and analyzed joint ordering-pricing models in the past, they have used demand models that are specific in form (such as the additive and multiplicative uncertainty demand models described in section 3). See, for instance, Kunreuther and Richard (1969, 1971), Kunreuther and Schrage (1973) and Thomas (1970) for deterministic demand models, and Karlin and Carr (1962), Ernst (1970), Zabel (1972), Thomas (1974), Thowsen (1975), and Polatuglu (1991) for probabilistic demand models. A review of this area can be found in the survey of joint production-marketing models by Eliashberg and Steinberg (1991). Some of the results in this paper can be used to demonstrate certain structural properties for the joint ordering-pricing problem. For instance, consider a two-period seasonal product pricing model with a single reorder decision at the beginning of period 2, zero lead time, and a linear ordering cost. Under the conditions specified in Theorem 5.6, it can be shown that the optimal period 2 ordering and pricing policy will be of the following (S,p) form: If the inventory I at the end of period 1 is less than S, order S - I and charge price p in period 2. If  $I \geq S$ , order nothing (and set the same price as you would for the seasonal product pricing model without reordering).

Finally, one may pose and analyze other structural properties of interest relating to the pricing problem considered in this paper, or to extensions of this problem such as the one discussed above. These may, for instance, examine the relationship between the optimal price and the variance in the demand distribution (at each price p) in any period, or that between the optimal initial order quantity and the level of demand.

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## APPENDIX

### Proof of Lemma 4.1

First note that for all  $p$  in  $[1.005, 1.4)$  and for all  $I$ ,

$$V(I, p) = pE[\min\{\mathcal{D}(p), I\}] < 1.4E[\min\{\mathcal{D}(1.4), I\}] = V(I, 1.4).$$

Similarly, for all  $p$  in  $[0, 1)$  and all  $I$ ,  $V(I, p) < V(I, 1)$ .

Also, for all  $p$  in  $[1.45, \infty)$ , and all  $I$ ,  $V(I, p) = 0$  since  $P[\mathcal{D}(p) = 0] = 1$ .

The above results imply that the optimal price lies in  $[1, 1.005) \cup [1.4, 1.45)$  for all  $I$ . □

### Proof of Lemma 4.2

$$V(2, 1) = 1(1.P[\mathcal{D}(1) = 1] + 2.P[\mathcal{D}(1) > 1]) = 1(0 + 2(0.75)) = 1.5 \quad (A1)$$

Also, for any  $p$  in  $[1.4, 1.45)$ ,  $V(2, p) \leq p(2(0.5)) = p \leq 1.45$  (A2)

Lemma 4.1 along with equations (A1) and (A2) imply that  $p_2$  is in  $[1, 1.005)$ . □

### Proof of Lemma 4.3

$$\begin{aligned} V(3, 1.4) &= 1.4 (1.P[\mathcal{D}(1.4) = 1] + 2.[\mathcal{D}(1.4) = 2] + 3.P[\mathcal{D}(1.4) > 2]) \\ &= 1.4 (0 + 3 (.5)) = 2.1 \end{aligned} \quad (A3)$$

Also, for any  $p$  in  $[1, 1.005)$ ,

$$V(3, p) \leq p.E[\min\{\mathcal{D}(1), 3\}] = p (2(.25) + 3(.5)) = 2p \leq 2.01 < 2.1 \quad (A4)$$

Lemma 4.1 along with equations (A4) and (A5) imply that  $p_3$  is in  $[1.4, 1.45)$  □

### Proof of Lemma 4.4

Define  $V(I, p|\alpha) = p.\min\{D(p|\alpha), I\}$

Since  $D(p)$  is flat in the price range  $[0, p_2]$ , it follows that for all  $p < p_2$ ,  $V(I, p|\alpha_2) < V(I, p_2|\alpha_2)$ .

Hence, 
$$p_{\alpha_2} \geq p_2 \quad (A5)$$

Also, for all  $p$  in  $[p_2, p_3]$ ,

$$V(I, p|\alpha_1) \leq p_3.D(p|\alpha_1) \leq p_3.D(p_2) < p_1D(p_1) = p_1\min\{D(p_1), I\} = V(I, p_1|\alpha_1)$$

and for  $p$  in  $[p_3, \infty)$ ,  $V(I, p_3|\alpha_1) = 0$ .

Hence,  $p_{\alpha_1} < p_2$  (A6)

(A5) and (A6) imply that  $p_{\alpha_1} < p_{\alpha_2}$ .

□

Proof of Theorem 5.1

We prove this result in a more general setting than that addressed in the theorem. We assume that there exists a salvage value for the unsold product at the end of the period, with the salvage value function  $S(\cdot)$  being concave. The original problem in the Theorem is a special case of this problem where the salvage value is zero (i.e.,  $S(y) = 0$  for all  $y$ ). The more general formulation will be used later in deriving Corollary 5.1.

We define the value function  $V^s(\cdot)$  as:

$$\begin{aligned} V^s(I) &= \max_{p \geq 0} W^s(I, p), \text{ where} \\ W^s(I, p) &= p \min\{D(p), I\} + S(I - \min\{D(p), I\}). \end{aligned}$$

Let  $I, \bar{I}$  be any two inventory levels with  $\bar{I} > I$ . We will first show that for the optimal price  $p_I$  at inventory level  $I$ ,  $\min\{D(p_I), I\} = D(p_I)$ . Suppose this were not true, i.e., suppose  $D(p_I) > I$ . Let us increase  $p$  from  $p_I$  to  $D^{-1}(I)$ . Now

$$\begin{aligned} W^s(I, D^{-1}(I)) &= D^{-1}(I) \min\{D(D^{-1}(I)), I\} + S(I - \min\{D(D^{-1}(I)), I\}) \\ &= D^{-1}(I)I > p_I I = V(I, p_I), \end{aligned}$$

which contradicts that fact that  $p_I$  is an optimal price for the deterministic problem with inventory level  $I$ . Hence we have the following result:

Lemma A.1: For the optimal price  $p_I$  at inventory level  $I$ ,

$$\min\{D(p_I), I\} = D(p_I) \quad (A7)$$

Since  $\bar{I} > I$ , (A7) implies that

$$\min\{D(p_I), \bar{I}\} = D(p_I). \quad (A8)$$

Since  $D(p)$  is decreasing in  $p$ , (A7) and (A8) also imply that for all  $p > p_I$

$$\min\{D(p), I\} = D(p) \quad (A9)$$

and  $\min\{D(p), \bar{I}\} = D(p)$ . (A10)

Now consider any price  $p > p_I$ . We will show that  $W^s(\bar{I}, p) < W^s(\bar{I}, p_I)$ . This will show that the optimal price  $p_{\bar{I}}$  at inventory level  $\bar{I}$  satisfies  $p_{\bar{I}} \leq p_I$ .

$$\begin{aligned} &W^s(\bar{I}, p) - W^s(\bar{I}, p_I) \\ &= p \min\{D(p), \bar{I}\} + S(\bar{I} - \min\{D(p), \bar{I}\}) \\ &\quad - p_I \min\{D(p_I), \bar{I}\} - S(\bar{I} - \min\{D(p_I), \bar{I}\}) \\ &= p D(p) - p_I D(p_I) + S(\bar{I} - D(p)) - S(\bar{I} - D(p_I)) && \text{by (A8) and (A10)} \\ &= p D(p) - p_I D(p_I) + S(I - D(p) + \delta) - S(I - D(p_I) + \delta) && \text{where } \delta = \bar{I} - I > 0 \\ &\leq p D(p) - p_I D(p_I) + S(I - D(p)) - S(I - D(p_I)) && \text{since } S(\cdot) \text{ is concave} \\ &= W^s(I, p) - W^s(I, p_I) && \text{by (A7) and (A9)} \\ &< 0 && \text{by definition of } p_I. \end{aligned}$$

Hence  $W^s(\bar{I}, p) < W^s(\bar{I}, p_I)$ , and so the optimal price  $p_I$  at inventory level  $\bar{I}$  must satisfy  $p_I \leq p$ .  $\square$

**Proof of Theorem 5.2:**

We will first show that, for the problem with a concave salvage value function that was discussed in the proof of Theorem 5.1, the value function  $V^s(I)$  is concave in  $I$  if the demand function  $D(p)$  satisfies Condition (5.1). This result will then be used to prove Theorem 5.2 by induction on  $t$ .

We had shown in the proof of Theorem 5.1 that for the optimal solution  $p_I$  at inventory level  $I$ ,  $D(p_I) \leq I$ . Since  $D(p)$  is decreasing in  $p$ , this also means that  $p_I \geq D^{-1}(I)$ . We can therefore write  $V^s(I)$  as follows:

$$V^s(I) = \max_{p \geq D^{-1}(I)} p D(p) + S(I - D(p)).$$

Re-writing this in terms of the demand variable  $d = D(p)$ , we have

$$V^s(I) = \max_{0 \leq d \leq I} d D^{-1}(d) + S(I - d).$$

We now show that this problem has a concave objective function. Then, since the feasible region for the problem is  $\{d: 0 \leq d \leq I\}$ , it will follow that  $V^s(I)$  is concave in  $I$ . The objective function is the sum of the two functions  $d D^{-1}(d)$  and  $S(I - d)$ . Since  $S(I - d)$  is concave in  $d$ , so the objective function will be concave in  $d$  as long as  $d D^{-1}(d)$  is concave in  $d$ . This is proved below.

Let  $F(d) = d D^{-1}(d)$

$$\text{Then } \frac{\partial F(d)}{\partial d} = D^{-1}(d) + d \frac{\partial D^{-1}(d)}{\partial d}, \text{ and so } \frac{\partial^2 F(d)}{\partial d^2} = d \frac{\partial^2 D^{-1}(d)}{\partial d^2} + 2 \frac{\partial D^{-1}(d)}{\partial d}$$

Now  $D^{-1}(d) = p$  and so

$$\frac{\partial D^{-1}(d)}{\partial d} = \frac{\partial p}{\partial d} = \left( \frac{\partial d}{\partial p} \right)^{-1} = \left( \frac{\partial D(p)}{\partial p} \right)^{-1}, \text{ and } \frac{\partial^2 D^{-1}(d)}{\partial d^2} = \frac{\partial \left( \frac{\partial D(p)}{\partial p} \right)^{-1}}{\partial d} = \frac{\partial \left( \frac{\partial D(p)}{\partial p} \right)^{-1}}{\partial p} \frac{\partial p}{\partial d} = - \frac{\frac{\partial^2 D(p)}{\partial p^2}}{\left( \frac{\partial D(p)}{\partial p} \right)^3}$$

$$\text{So } d \frac{\partial^2 D^{-1}(d)}{\partial d^2} + 2 \frac{\partial D^{-1}(d)}{\partial d} = \frac{2 \left( \frac{\partial D(p)}{\partial p} \right)^2 - D(p) \frac{\partial^2 D(p)}{\partial p^2}}{\left( \frac{\partial D(p)}{\partial p} \right)^3}$$

Since  $D(p)$  is decreasing in  $p$ , the denominator in the last expression is negative. The numerator is non-negative since  $D(p)$  satisfies Condition (5.1). Hence  $\frac{\partial^2 F(d)}{\partial d^2} \leq 0$ , and so  $F(d)$  is concave in  $d$ . Hence, as reasoned above,  $V^s(I)$  is concave in  $I$ .

Now consider the multiperiod problem in the statement of Theorem 5.2. For  $t = T$ , we have  $V_T(I) = \max_{p \geq 0} p \min\{D_T(p), I\}$ , which can be considered as the value function for a single period problem with zero salvage value, a special case of the single period problem considered above.  $V_T(I)$  is therefore concave in  $I$ . Now suppose for some  $t$ ,  $V_{t+1}(I)$  is concave in  $I$ . Consider  $V_t(I) = \max_{p \geq 0} p \min\{D_t(p), I\} + V_{t+1}(I - \min\{D_t(p), I\})$ . This can now be considered as the value

function for a single period problem with a concave salvage value function  $V_{t+1}(\cdot)$ . This allows us to conclude that  $V_t(\cdot)$  is concave, and so, by induction, Theorem 5.2 is proved.  $\square$

Proof of Corollary 5.1

$V_t(I) = \max_{p \geq 0} p \min\{D_t(p), I\} + V_{t+1}(I - \min\{D_t(p)\})$ , where, for notational convenience, we define  $V_{T+1}(\cdot)$  as  $V_{T+1}(J) = 0$  for all  $J$ . By Theorem 5.2,  $V_{t+1}(\cdot)$  is concave (this holds trivially for  $t = T$ ). Hence,  $V_t(I)$  can be considered as the value function of a single-period problem with a concave salvage value function given by  $V_{t+1}(\cdot)$ . The result now follows from the proof of Theorem 5.1.  $\square$

Proof of Theorem 5.6

(We have suppressed the time subscript 1 below.)

$$\begin{aligned} V(I) &= \max_{p \geq 0} V(I, p), \text{ where} \\ V(I, p) &= E[p \min\{D(p), I\}] \\ &= p \int_0^I (1 - F(x|p)) dx \quad (\text{by Lemma 5.1}) \end{aligned}$$

$$\text{So } \frac{\partial V(I, p)}{\partial p} = -p \int_0^I F'(x|p) dx + \int_0^I (1 - F(x|p)) dx \quad (\text{A11})$$

The first order optimality condition implies that  $\frac{\partial V(I, p)}{\partial p} = 0$  for  $p = p_I$ . Hence, by equation

$$\text{(A11), } -p_I \int_0^I F'(x|p_I) dx + \int_0^I (1 - F(x|p_I)) dx = 0 \quad (\text{A12})$$

where we have again used the notation  $F'(x|p) = \frac{\partial F(I, p)}{\partial p}$ .

$$\text{This yields } p_I = \frac{\int_0^I (1 - F(x|p_I)) dx}{\int_0^I F'(x|p_I) dx} \quad (\text{A13})$$

Note that since  $F'(x|p) \geq 0$  by Condition 3.4, and since  $F'(x|p)$  is continuous in  $x$  by Condition

3.3,  $\int_0^I F'(x|p) dx = 0$  implies that  $F'(x|p) = 0$  for all  $x$  in  $[0, I]$ . Then, given the continuity of  $F(x|p)$

in  $x$  (by Condition 3.3) and the fact that  $F(0|p) = 0$ , we must have  $F(x|p) = 0$  for all  $x$  in  $[0, I]$ . We assume that for any inventory level  $I$  of interest and for all  $p$ , there exists some  $x$  in the range  $[0, I]$  for which the density function  $f(x|p)$  is positive. Under this assumption, and conditions 3.3 and

3.4, it is straightforward to then see that  $\int_0^I F'(x|p) dx > 0$ . We will continue to make this

assumption throughout our analysis in this paper. This assumption is satisfied by the multiplicative demand model (example 3.6.3) with exponential uncertainty, the exponential-exponential demand model (example 3.6.4) as well as the Poisson-Weibull demand model (example 3.6.5), for any positive value of the inventory level  $I$ .



Differentiating equation (A13) with respect to I, we get:

$$p'_I = \frac{\int_0^I F'(x|p_I) dx \left[ 1 - F(I|p) - \int_0^I F'(x|p_I) p'_I dx \right] - \int_0^I (1 - F(x|p_I)) dx \left[ F'(I|p_I) + \int_0^I F''(x|p_I) p'_I dx \right]}{\left( \int_0^I F'(x|p_I) dx \right)^2}$$

where  $p'_I = \frac{\partial p_I}{\partial I}$ ,  $F''(x|p) = \frac{\partial^2 F(I, p)}{\partial p^2}$ , and this implies that

$$p'_I = \frac{(1 - F(I|p_I)) \int_0^I F'(x|p_I) dx - F'(I|p_I) \int_0^I (1 - F(x|p_I)) dx}{2 \left( \int_0^I F'(x|p_I) dx \right)^2 + \int_0^I F''(x|p_I) dx \int_0^I (1 - F(x|p_I)) dx} \quad (\text{A14})$$

Now,  $V(I)$  can also be written as  $V(I) = p_I \int_0^I (1 - F(x|p_I)) dx$

Differentiating with respect to I, we get

$$V'(I) = p'_I \int_0^I (1 - F(x|p_I)) dx + p_I \left( (1 - F(I|p_I)) - \int_0^I F'(x|p_I) p'_I dx \right)$$

Using equation (A12), this becomes  $V'(I) = p_I (1 - F(I|p_I))$

Differentiating again with respect to I, we get  $V''(I) = p'_I (1 - F(I|p_I)) + p_I (-F'(I|p_I) p'_I - f(I|p_I))$

$$= p'_I \left( \frac{(1 - F(I|p_I)) \int_0^I F'(x|p_I) dx - F'(I|p_I) \int_0^I (1 - F(x|p_I)) dx}{\int_0^I F'(x|p_I) dx} \right) - \frac{\int_0^I (1 - F(x|p_I)) dx \cdot f(I|p_I)}{\int_0^I F'(x|p_I) dx} \quad (\text{using equation A13})$$

$$= \frac{\left( (1 - F(I|p_I)) \int_0^I F'(x|p_I) dx - F'(I|p_I) \int_0^I (1 - F(x|p_I)) dx \right)^2 - \left( 2 \left( \int_0^I F'(x|p_I) dx \right)^2 + \int_0^I F''(x|p_I) dx \int_0^I (1 - F(x|p_I)) dx \right) \int_0^I (1 - F(x|p_I)) dx \cdot f(I|p_I)}{\left( \int_0^I F'(x|p_I) dx \right)^2}$$

(using equation A14)

$F'(x|p_I) \geq 0$  for all  $x > 0$  since  $F(\cdot|p)$  is non-decreasing in  $p$ . So  $\int_0^I F'(x|p_I) dx \geq 0$ . Hence the

denominator in the previous expression is non-negative. Also, by Condition (5.5), the numerator in this expression is non-positive. Hence  $V''(I) \leq 0$ , and so  $V(I)$  is concave in I.  $\square$