FUNDAMENTAL SOLUTIONS OF INVARIANT DIFFERENTIAL OPERATORS ON SYMMETRIC SPACES

BY S. HELGASON[1]

Communicated by I. M. Singer, June 27, 1963

1. Introduction and notation. Let S be a Riemannian globally symmetric space, G the largest connected group of isometries of S in the compact open topology. We assume that S is of the noncompact type, that is, G is semisimple and has no compact normal subgroup $\neq \{e\}$. Let o be any point in S, K the isotropy subgroup of G at o, $\mathfrak{k}$ and $\mathfrak{g}$ their respective Lie algebras, and $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form B of $\mathfrak{g}$. Let $\mathfrak{a}$ be any maximal abelian subspace of $\mathfrak{p}$ and let $A=\exp(\mathfrak{a})$. For each λ in the dual space of $\mathfrak{a}$ (which we identify with $\mathfrak{a}$, via B) let $\mathfrak{g}_\lambda=\{X\in\mathfrak{g} \mid [H,X]=\lambda(H)X$ for all $H\in\mathfrak{a}\}$. Let $d_\lambda=\dim(\mathfrak{g}_\lambda)$. Choose some order on $\mathfrak{a}$ and let

$$\mathfrak{n} = \sum_{\lambda>0} \mathfrak{g}_\lambda, \qquad \rho = \sum_{\lambda>0} d_\lambda \lambda, \qquad \pi' = \prod_{\lambda>0} \lambda^{d_\lambda}$$

and let π denote the product of the distinct prime factors in π'. Then we have the Iwasawa decompositions $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$, $G=KAN$ where N is the nilpotent group $\exp(\mathfrak{n})$. Given $g\in G$, let $H(g)$ denote the unique element in $\mathfrak{a}$ for which $g\in K \exp H(g)N$. Let W denote the Weyl group M'/M where M and M', respectively, denote the centralizer and normalizer of $\mathfrak{a}$ in K.

For each $\lambda\in\mathfrak{a}$ consider the spherical function

$$\phi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))}dk \qquad (x \in G)$$

dk being the normalized Haar measure on K. Let $\mathbf{c}(\lambda)$ denote Harish-Chandra's function on $\mathfrak{a}$ which occurs in the leading term of the asymptotic expansion of ϕ_λ [2, p. 283], i.e.,

$$\phi_\lambda(\exp H) \sim \sum_{s\in W} \mathbf{c}(s\lambda)e^{(is\lambda-\rho)(H)}$$

where λ and H are suitably restricted in $\mathfrak{a}$.

Each $x\in G$ can be written uniquely in the form $x=k \exp X$ ($k\in K$, $X\in\mathfrak{p}$). We put $|X|=(B(X, X))^{1/2}$ and $\omega(X)=\{\det (\sinh ad X/ad X)_{\mathfrak{p}}\}^{1/2}$ where the subscript $\mathfrak{p}$ indicates restriction to $\mathfrak{p}$ of the linear

[1] Work supported in part by the National Science Foundation NSF GP-149.

transformation of $\mathfrak{g}$ given by

$$\sinh ad\, X/ad\, X = \sum_{n \geq 0} (ad\, X)^{2n}/(2n+1)!.$$

Let $D(G)$ and $D(S)$ denote the set of left invariant (resp. G-invariant) differential operators on G (resp. S). Let $S(\mathfrak{a})$ denote the symmetric algebra over $\mathfrak{a}$, $\mathcal{S}(\mathfrak{a})$ the space of C^∞ functions on $\mathfrak{a}$ which are rapidly decreasing together with all their derivatives. Let $I(\mathfrak{a})$ and $\mathcal{I}(\mathfrak{a})$ denote the set of W-invariants in $S(\mathfrak{a})$ and $\mathcal{S}(\mathfrak{a})$, respectively; $\mathcal{S}(\mathfrak{a})$ is taken with its usual locally convex topology [7, p. 90] and then $\mathcal{I}(\mathfrak{a})$ is a closed subspace. According to a theorem of Harish-Chandra (cf. [2, Theorem 1, p. 260], also [3, p. 432]) there exists an isomorphism Γ of the algebra $D(S)$ onto $I(\mathfrak{a})$. Let $I(G)$ denote the set of C^∞ functions f on G which are bi-invariant under K and for each integer $q \geqq 0$ and each $D \in D(G)$ satisfy

$$\tau_{D,q}(f) = \sup_{H \in \mathfrak{a}} (1 + |H|)^q \omega(H) |(Df)(\exp H)| < \infty.$$

Let $I_0(G)$ denote the space of functions of the form

$$\phi_a(x) = \int_{\mathfrak{a}} \pi(\lambda) a(\lambda) \phi_\lambda(x) d\lambda \qquad (x \in G)$$

where $a \in \mathcal{S}(\mathfrak{a})$ and $d\lambda$ is the Euclidean measure on $\mathfrak{a}$. Then, by [2, p. 586], $I_0(G) \subset I(G)$ (it can be shown using [1] that $\pi = \pi_0$). The seminorms $\tau_{D,q}$ turn $I_0(G)$ and $I(G)$ into locally convex spaces.

LEMMA 1. *$I_0(G)$ is an algebra under convolution on G.*

Under the restriction from G to A, $I_0(G)$ and $I(G)$ are mapped isomorphically onto spaces $I_0(A)$ and $I(A)$ of W-invariant C^∞ functions on A. We carry the algebraic and topological structure of $I_0(G)$ over on $I_0(A)$ by means of this mapping. The space $\mathcal{I}(\mathfrak{a})$ is an algebra under convolution on $\mathfrak{a}$.

2. **Transmutation operators.** A function f on the space $S = G/K$ is called a *radial function* if $f(k \cdot p) = f(p)$ for all $k \in K$, $p \in S$. The set of continuous (resp. C^∞) radial functions on G/K is in one-to-one correspondence $f \rightarrow \bar{f}$ with the set of all continuous (resp. C^∞) W-invariant functions on A. Here $f(aK) = \bar{f}(a)$ for $a \in A$. Let $D \in D(S)$; then by [2, p. 265] there exists a unique differential operator $\delta'(D)$ on A' (the set of regular elements in A) such that $(Df)^- = \delta'(D)\bar{f}$ for all C^∞ radial functions f. The operator $\delta'(D)$ is called the radial part of D in analogy with the radial part $D_r^2 + (n-1)/r\ D_r$ of the Laplacian on $\mathbf{R}^n$, $(D_r = d/dr)$. It is known [5] that there exists an isomorphism X

("transmutation operator") of the vector space of even C^∞ functions on $\boldsymbol{R}$, onto itself, under which the singular operator $D_r^2+(n-1)/r\ D_r$ corresponds to D_r^2. The operators $\delta'(D)$ $(D\in\boldsymbol{D}(S))$ are singular when considered as differential operators on A but Theorem 1 shows that they have a simultaneous transmutation operator X under which they correspond to differential operators on the Euclidean space $\mathfrak{a}$ with constant coefficients.

Given a W-invariant function ϕ on A, let $\tilde{\phi}$ denote the corresponding radial function on S. For $\phi\in I(A)$, put

$$(X\phi)(H) = e^{\rho(H)}\int_N \tilde{\phi}((\exp H)n\cdot o)dn \qquad (H\in\mathfrak{a})$$

where dn is a suitably normalized invariant measure on N. As proved by Harish-Chandra [11, p. 595], X is a continuous mapping of $I(A)$ into $\mathcal{I}(\mathfrak{a})$.

THEOREM 1. *The mapping X is a topological isomorphism of the algebra $I_0(A)$ onto the algebra $\mathcal{I}(\mathfrak{a})$. Moreover, if $D\in\boldsymbol{D}(S)$ then*

$$X\delta'(D)\phi = \Gamma(D)X\phi, \qquad \phi\in I_0(A).$$

Here $\Gamma(D)$ is considered as a differential operator on $\mathfrak{a}$.

The proof is based on the Plancherel formula for functions in $I_0(G)$, proved by Harish-Chandra [2]. It also uses the recent result of Gindikin and Karpelevič [1] according to which the function $c(\lambda)$ above can be expressed in terms of Gamma functions.

REMARKS. At the end of [2], Harish-Chandra states the following two conjectures which would imply that $I_0(A)$ contains all the W-invariant C^∞ functions on A with compact support.

(I) There exists a polynomial $p\in S(\mathfrak{a})$ such that $|c(\lambda)\pi(\lambda)p(\lambda)|\geqq 1$ for all $\lambda\in\mathfrak{a}$. (Here we have used the fact that $\pi=\pi_0$.)

(II) The mapping X is one-to-one on $I(A)$.

Now (I) can be verified on the basis of the mentioned result of Gindikin and Karpelevič. Theorem 1 shows that (II) is equivalent to $I_0(G)=I(G)$. On the other hand, (II) is easily implied by the Plancherel formula for the functions in $I(G)$. This formula is not proved in [2] but I understand that Harish-Chandra has proved it in recent, as yet unpublished, work. In the next section we shall therefore assume that $I_0(G)=I(G)$.

3. **Fundamental solutions.** Let $C_c^\infty(S)$ denote the space of C^∞ functions on S with compact support. Let δ denote the distribution on S

given by $\delta(f)=f(o)$ where $f\in C_c^\infty(S)$. Let $D\in \boldsymbol{D}(S)$. A distribution T on S is called a *fundamental solution* of D if $DT=\delta$. If $f\in C_c^\infty(S)$, then a fundamental solution T of D gives a solution of the equation $Du=f$ by putting $u=f*T$ where $*$ is the operation on distributions on G/K induced by the convolution product of distributions on G.

THEOREM 2. *Each invariant differential operator $D\in \boldsymbol{D}(S)$ $(D\neq 0)$ on the symmetric space S has a fundamental solution.*

This is a consequence of Theorem 1 and the fact that a nonzero differential operator on $\boldsymbol{R}^n$ with constant coefficients always has a tempered fundamental solution [**4**; **6**].

Added in proof. In the case when G is complex the following formula (which is a simple consequence of Lemma 55 in [**2**]) gives a simpler proof of Theorem 2.

$$(DF)\circ \mathrm{Exp} = \frac{1}{\omega}\lambda(D)(\omega(F\circ \mathrm{Exp})) \qquad (D\in \boldsymbol{D}(S)).$$

Here Exp is the usual Exponential mapping of $\mathfrak{p}$ onto S, λ is a certain isomorphism of $\boldsymbol{D}(S)$ onto the algebra of Ad(K)-invariant polynomials on $\mathfrak{p}$ and F is any radial function on S.

REFERENCES

1. S. G. Gindikin and F. I. Karpelevič, *Plancherel measure of Riemannian symmetric spaces of nonpositive curvature*, Dokl. Akad. Nauk. SSSR **145** (1962), 252–255= Soviet Math. Dokl. **3** (1962), 962–965.
2. Harish-Chandra, *Spherical functions on a semisimple Lie group*. I, II, Amer. J. Math. **80** (1958), 241–310, 553–613.
3. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
4. L. Hörmander, *On the division of distributions by polynomials*, Ark. Mat. **3** (1958), 555–568.
5. J. L. Lions, *Équations différentielles opérationelles*, Springer, Berlin, 1961.
6. S. Lojasiewicz, *Division d'une distribution par une fonction analytique de variables réelles*. C. R. Acad. Sci. Paris **246** (1958), 683–686.
7. L. Schwartz, *Théorie des distributions*. II, Hermann, Paris, 1951.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY