RADON-FOURIER TRANSFORMS ON SYMMETRIC SPACES AND RELATED GROUP REPRESENTATIONS[1]

BY S. HELGASON

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In §2 we announce some results in continuation of [10], connected with the Radon transform. §1 deals with tools which also apply to more general questions and §§2–3 contain some applications to group representations. A more detailed exposition of §2 appears in Proceedings of the U. S.-Japan Seminar in Differential Geometry, Kyoto, June, 1965.

1. **Radial components of differential operators.** Let V be a manifold, v a point in V and V_v the tangent space to V at v. Let G be a Lie transformation group of V. A C^∞ function f on an open subset of V is called locally invariant if $Xf=0$ for each vector field X on V induced by the action of G.

Suppose now W is a submanifold of V satisfying the following transversality condition:

$$\text{(T)} \qquad \text{For each } w \in W,\ V_w = W_w + (G\cdot w)_w \qquad \text{(direct sum)}.$$

If f is a function on a subset of V its restriction to W will be denoted $\bar{f}$.

LEMMA 1.1. *Let D be a differential operator on V. Then there exists a unique differential operator $\Delta(D)$ on W such that*

$$(Df)^- = \Delta(D)\bar{f}$$

for each locally invariant f.

The operator $\Delta(D)$ is called the *radial component* of D. Many special cases have been considered (see e.g. [1, §2], [4, §5], [5, §3], [7, §7], [8, Chapter IV, §§3–5]).

Suppose now dv (resp. dw) is a positive measure on V (resp. W) which on any coordinate neighborhood is a nonzero multiple of the Lebesgue measure. Assume dg is a bi-invariant Haar measure on G. Given $u \in C_c^\infty(G \times W)$ there exists [7, Theorem 1] a unique $f_u \in C_c^\infty(G\cdot W)$ such that

$$\int_{G\times W} F(g\cdot w)u(g, w)\, dg dw = \int_V F(v)f_u(v)\, dv \qquad (F \in C_c^\infty(G\cdot W)).$$

Let $\phi_u \in C_c^\infty(W)$ denote the function $w \to \int u(g, w)dg$.

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THEOREM 1.2. *Suppose G leaves dv invariant. Let T be a G-invariant distribution on $G\cdot W$. Then there exists a unique distribution $\overline{T}$ on W such that*

$$\overline{T}(\phi_u) = T(f_u), \qquad u \in C_c^\infty(G \times W).$$

If D is a G-invariant differential operator on V then

$$(DT)^- = \Delta(D)\overline{T}.$$

The proof is partly suggested by the special case considered in [7, §9]. See also [12, §4].

2. **The Radon transform and conical distributions.** Let G be a connected semisimple Lie group, assumed imbedded in its simply connected complexification. Let K be a maximal compact subgroup of G and X the symmetric space G/K. Let $G=KAN$ be an Iwasawa decomposition of G (A abelian, N nilpotent) and let M and M', respectively, denote the centralizer and normalizer of A in K. The space Ξ of all horocycles ξ in X can be identified with G/MN [10, §3]. Let $\boldsymbol{D}(X)$ and $\boldsymbol{D}(\Xi)$ denote the algebras of G-invariant differential operators on X and Ξ, respectively; let $S(A)$ denote the symmetric algebra over the vector space A and $I(A)$ the set of elements in $S(A)$ which are invariant under the Weyl group $W=M'/M$. There are isomorphisms Γ of $\boldsymbol{D}(X)$ onto $I(A)$ [6, p. 260], [9, p. 432] and $\hat{\Gamma}$ of $\boldsymbol{D}(\Xi)$ onto $S(A)$ [10, p. 676].

The Radon transform $f\to\hat{f}$ ($f\in C_c^\infty(X)$) and its dual $\phi\to\check{\phi}$ ($\phi\in C^\infty(\Xi)$) are defined by

$$\hat{f}(\xi) = \int_\xi f(x)dm(x), \qquad \check{\phi}(x) = \int_{\check{x}} \phi(\xi)d\mu(\xi) \qquad (x \in X,\ \xi \in \Xi)$$

where dm is the measure on ξ induced by the canonical Riemannian structure of X, $\check{x}$ is the set of horocycles passing through x and $d\mu$ is the measure on $\check{x}$ invariant under the isotropy subgroup of G at x, satisfying $\mu(\check{x})=1$. The easily proved relation

$$(1) \qquad \int_X f(x)\check{\phi}(x)dx = \int_\Xi \hat{f}(\xi)\phi(\xi)d\xi \qquad (f \in C_c^\infty(X),\ \phi \in C_c^\infty(\Xi))$$

dx and $d\xi$ being G-invariant measures on X and Ξ, respectively, suggests immediately how to extend the integral transforms above to distributions.

Let $\mathfrak{G}$ and $\mathfrak{A}$ be the Lie algebras of G and A, respectively, and $\mathfrak{A}^*$

the dual space of $\mathfrak{A}$. Let $\lambda \rightarrow c(\lambda)$ be the function on $\mathfrak{A}^*$ giving the Plancherel measure $|c(\lambda)|^{-2}d\lambda$ for the K-invariant functions on X (Harish-Chandra [6, p. 612]). Let j be the operator on rapidly decreasing functions on A which under the Fourier transform on A corresponds to multiplication by c^{-1}. Let ρ denote the sum (with multiplicity) of the restricted roots on $\mathfrak{A}$ which are positive in the ordering given by N. Let e^ρ denote the function on Ξ defined by $e^\rho(kaMN) = \exp[\rho(\log a)]$ $(k \in K,\ a \in A)$. Viewing Ξ as a fibre bundle with base K/M, fibre A [10, p. 675] we define the operator Λ on suitable functions ϕ on Ξ by $(e^\rho \Lambda\phi)|F = j((e^\rho\phi)|F)$, where $|F$ denotes restriction to any fibre F. Similarly, the complex conjugate of c^{-1} determines an operator $\bar{\Lambda}$. By means of the Plancherel formula mentioned one proves (cf. [11, §6]).

THEOREM 2.1. *There exist constants $c, c' > 0$ such that*

$$\int_X |f(x)|^2 dx = c' \int_\Xi |\Lambda \hat{f}(\xi)|^2 d\xi, \tag{2}$$

$$f = c(\Lambda\bar{\Lambda}\hat{f})^\vee \tag{3}$$

for all $f \in C_c^\infty(X)$.

If all Cartan subgroups of G are conjugate, the operators j and Λ are differential operators (c^{-1} is a polynomial). Considering $j\bar{j}$ is an element in $I(A)$ we put $\square = \Gamma^{-1}(j\bar{j}) \in \boldsymbol{D}(X)$. Then (3) can be written in the form

$$f = c\,\square\,((\check{f})^\vee), \qquad f \in C_c^\infty(X),$$

which is more convenient for applications [10, §7]. For the case when G is complex a formula closely related to (3) was given by Gelfand-Graev [2, §5.5].

Let x_0 and ξ_0 denote the origins in X and Ξ, respectively. The space $B = K/M$ can be viewed as the set of Weyl chambers emanating from x_0 in X. If $\xi = ka\cdot\xi_0$ $(k \in K,\ a \in A)$ we say that the Weyl chamber kM is *normal* to ξ and that a is the *complex distance* from x_0 to ξ. If $x \in X$, $b \in B$ let $\xi(x, b)$ be the horocycle with normal b passing through x, and let $A(x, b)$ denote the complex distance from x_0 to $\xi(x, b)$.

THEOREM 2.2. *For $f \in C_c^\infty(X)$ define the Fourier transform $\tilde{f}$ by*

$$\tilde{f}(\lambda, b) = \int_X f(x) \exp[(-i\lambda + \rho)(A(x, b))]dx \qquad (\lambda \in \mathfrak{A}^*,\ b \in B).$$

Then

$$f(x) = \int_{\mathfrak{A}^* \times B} \tilde{f}(\lambda, b) \exp(i\lambda + \rho)(A(x, b))] \, | \, c(\lambda) \, |^{-2} d\lambda db \tag{4}$$

$$\int_X | f(x) |^2 dx = \int_{\mathfrak{A}^* \times B} | \tilde{f}(\lambda, b) |^2 \, | \, c(\lambda) \, |^{-2} d\lambda db,$$

where db is a suitably normalized K-invariant measure on B.

REMARKS. (i) In view of the analogy between horocycles in X and hyperplanes in R^n formula (4) corresponds exactly to the Fourier inversion formula in R^n when written in polar coordinate form.

(ii) If f is a K-invariant function on X, Theorem 2.2 reduces to Harish-Chandra's Plancherel formula [6, p. 612]. Nevertheless, Theorem 2.2 can be derived from Harish-Chandra's formula.

(iii) A "plane wave" on X is by definition a function on X which is constant on each member of a family of parallel horocycles. Writing (4) in the form

$$f(x) = \int_B f_b(x) \, db \tag{4'}$$

we get a continuous decomposition of f into plane waves. On the other hand, if we write (4) in the form

$$f(x) = \int_{\mathfrak{A}^*} f_\lambda(x) \, | \, c(\lambda) \, |^{-2} \, d\lambda \tag{4''}$$

we obtain a decomposition of f into simultaneous eigenfunctions of all $D \in D(X)$.

We now define for Ξ the analogs of the spherical functions on X.

DEFINITION. A distribution (resp. C^∞ function) on $\Xi = G/MN$ is called *conical* if it is (1) MN-invariant; (2) eigendistribution (resp. eigenfunction) of each $D \in D(\Xi)$.

Let $\xi_0 = MN$, $\xi^* = m^*MN$, where m^* is any element in M' such that the automorphism $a \to m^* a m^{*-1}$ of A maps ρ into $-\rho$. By the Bruhat lemma, Ξ will consist of finitely many MNA-orbits; exactly one, namely $\Xi^* = MNA \cdot \xi^*$, has maximum dimension and given $\xi \in \Xi^*$ there exists a unique element $a(\xi) \in A$ such that $\xi \in MNa(\xi) \cdot \xi^*$ [10, p. 673]. Using Theorem 1.2 we find:

THEOREM 2.3. *Let T be a conical distribution on Ξ. Then there exists a $\psi \in C^\infty(\Xi^*)$ such that $T = \psi$ on Ξ^* and a linear function μ: $\mathfrak{A} \to C$ such that*

$$\psi(\xi) = \psi(\xi^*) \exp[\mu(\log a(\xi))] \qquad (\xi \in \Xi^*). \tag{5}$$

In general ψ is singular on the lower-dimensional MNA-orbits. However, we have:

THEOREM 2.4. *Let* $\mu: \mathfrak{A} \rightarrow \mathbf{C}$ *be a linear function and let* $\psi \in C^\infty(\Xi^*)$ *be defined by* (5). *Then* ψ *is locally integrable on* Ξ *if and only if*

$$\text{(6)} \qquad \operatorname{Re}(\langle \alpha, \mu + \rho \rangle) > 0 \qquad (\operatorname{Re} = \textit{real part})$$

for each restricted root $\alpha > 0$; *here* $\langle \ , \ \rangle$ *denotes the inner product on* $\mathfrak{A}^*$ *induced by the Killing form of* $\mathfrak{G}$. *If* (6) *is satisfied then* ψ, *as a distribution on* Ξ, *is a conical distribution.*

THEOREM 2.5. *The conical functions on* Ξ *are precisely the functions* ψ *given by* (5) *where for each restricted root* $\alpha > 0$,

$$\text{(7)} \qquad \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \textit{ is an integer } \geqq 0.$$

DEFINITION. A representation π of G on a vector space E will be called (1) *spherical* if there exists a nonzero vector in E fixed by $\pi(K)$; (2) *conical* if there exists a nonzero vector in E fixed by $\pi(MN)$.

The correspondence between spherical functions on X and spherical representations is well known. In order to describe the analogous situation for Ξ, for an arbitrary function ϕ on Ξ, let E_ϕ denote the vector space spanned by the G-translates of ϕ and let π_ϕ denote the natural representation of G on E_ϕ.

THEOREM 2.6. *The mapping* $\psi \rightarrow \pi_\psi$ *maps the set of conical functions on* Ξ *onto the set of finite-dimensional, irreducible conical representations of* G. *The mapping is one-to-one if we identify proportional conical functions and identify equivalent representations. Also*

$$\psi(g \cdot \xi_0) = (\pi_\psi(g^{-1})\mathsf{e}, \mathsf{e}'),$$

where e *and* e', *respectively, are contained in the highest weight spaces of* π_ψ *and of its contragredient representation. Finally,* μ *in* (5) *is the highest weight of* π_ψ.

COROLLARY 2.7. *Let* π *be a finite-dimensional irreducible representation of* G. *Then* π *is spherical if and only if it is conical.*

The highest weights of these representations are therefore characterized by (7). Compare Sugiura [**13**], where the highest weights of the spherical representations are determined.

3. **The case of a complex** G. If G is complex, M is a torus and some of the results of §2 can be improved. Let $\mathfrak{H}$ be a Cartan subalgebra

of $\mathfrak{G}$ containing $\mathfrak{A}$ and H the corresponding analytic subgroup of G. Now we assume G simply connected.

Let $\mathbf{D}(G/N)$ denote the algebra of all G-invariant differential operators on G/N. Let ν_0, $\nu^* \in G/N$ be constructed similarly as ξ_0 and ξ^* in §2. Then §1 applies to the submanifold $W = H\cdot\nu^*$ of $V = NH\cdot\nu^*$ and for each differential operator D on G/N, $\Delta(D)$ is defined and can be viewed as a differential operator on H.

THEOREM 3.1. *The mapping $D \rightarrow \Delta(D)$ is an isomorphism of $\mathbf{D}(G/N)$ onto the (real) symmetric algebra $S(\mathfrak{H})$. In particular, $\mathbf{D}(G/N)$ is commutative.*

As a consequence one finds that the N-invariant eigenfunctions $f \in C^\infty(G/N)$ of all $D \in \mathbf{D}(G/N)$ have a representation analogous to (5) in terms of the characters of H. Let E_f denote the vector space spanned by the G-translates of f and let π_f be the natural representation of G on E_f.

THEOREM 3.2. *The mapping $f \rightarrow \pi_f$ is a one-to-one mapping of the set of N-invariant holomorphic eigenfunctions of all $D \in \mathbf{D}(G/N)$ (proportional f identified) onto the set of all finite-dimensional[2] irreducible holomorphic representations of G (equivalent representations identified). Moreover*

$$f(g\cdot\nu_0) = (\pi_f(g^{-1})\mathrm{e}, \mathrm{e}'),$$

where e and e', respectively, are contained in the highest weight spaces of π_f and of its contragredient representation.

REFERENCES

1. F. A. Berezin, *Laplace operators on semisimple Lie groups*, Trudy Moskov. Mat. Obšč. **6** (1957), 371–463; Amer. Math. Soc. Transl. **21** (1962), 239–299.

2. I. M. Gelfand and M. I. Graev, *The geometry of homogeneous spaces, group representations in homogeneous spaces and questions in integral geometry related to them.* I, Trudy Moskov. Mat. Obšč. **8** (1959), 321–390; Amer. Math. Soc. Transl. **37** (1964), 351–429.

3. R. Godement, *A theory of spherical functions.* I, Trans. Amer. Math. Soc. **73** (1952), 496–556.

4. Harish-Chandra, *The characters of semisimple Lie groups*, Trans. Amer. Math. Soc. **83** (1956), 98–163.

5. ———, *Differential operators on a semisimple Lie algebra*, Amer. J. Math. **79** (1957), 87–120.

6. ———, *Spherical functions on a semisimple Lie group.* I, II, Amer. J. Math. **80** (1958), 241–310, 553–613.

[2] Compare the problem indicated in [**3**, p. 553].

7. ———, *Invariant distributions on Lie algebras*, Amer. J. Math. **86** (1964), 271–309.

8. S. Helgason, *Differential operators on homogeneous spaces*, Acta Math. **102** (1959), 239–299.

9. ———, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

10. ———, *Duality and Radon transform for symmetric spaces*, Amer. J. Math. **85** (1963), 667–692.

11. ———, *A duality in integral geometry; some generalizations of the Radon transform*, Bull. Amer. Math. Soc. **70** (1964), 435–446.

12. P-D. Methée, *Sur les distributions invariantes dans le groupe des rotations de Lorentz*, Comment. Math. Helv. **28** (1954), 225–269.

13. M. Sugiura, *Representations of compact groups realized by spherical functions on symmetric spaces*, Proc. Japan Acad. **38** (3) (1962), 111–113.

Institute for Advanced Study