Analytic Aspects of Periodic Instantons

by

Benoit Charbonneau

B.Sc., Université du Québec à Montréal, 1998
M.Sc., Université du Québec à Montréal, 1999

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2004

Author . . . .
Department of Mathematics
August 5, 2004

Certified by . . . .
Tomasz S. Mrowka
Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
Pavel Etingof
Chairman, Department Committee on Graduate Students

Analytic Aspects of Periodic Instantons
by
Benoit Charbonneau

Submitted to the Department of Mathematics
on August 5, 2004, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

The main result is a computation of the Nahm transform of a $\mathrm{SU}(2)$-instanton over $\mathbb{R} \times T^3$, called spatially-periodic instanton. It is a singular monopole over T^3, a solution to the Bogomolny equation, whose rank is computed and behavior at the singular points is understood under certain conditions.

A full description of the Riemannian ADHMN construction of instantons on $\mathbb{R}^4$ is given, preceding a description of the heuristic behind the theory of instantons on quotients of $\mathbb{R}^4$. The Fredholm theory of twisted Dirac operators on cylindrical manifolds is derived, the spectra of spin Dirac operators on spheres and on product manifolds are computed. A brief discussion on the decay of spatially-periodic and doubly-periodic instantons is included.

Thesis Supervisor: Tomasz S. Mrowka
Title: Professor of Mathematics

Acknowledgments

It all started on October 3, 1994, when I decided I would do math in life. But I already told that bit of the story in my masters thesis.

The readership of a thesis is usually quite small. This is in part because thesis are usually not well distributed. Soon indeed, this thesis will fall in the "microfilm zone," to be lost forever except to those of unfaltering determination. Thank you then, dear reader, for being here with me, and please accompany me in this section on a celebration of the many thanks I have to give.

Before continuing any further, it is imperative that I stop to warmly acknowledge my advisor and mentor, Tomasz Mrowka. By standing on the shoulders of a giant, it is much easier to see ahead. From him I learned many valuable lessons that I hope to retain and pass on to the next generation. One of these I included at the start of Chapter 8. His guidance — punctuated by his usual "So?" asked in the middle of a hallway — made this project a success. His patience, wisdom, and vision make him a very very good mentor to those that are willing to work hard without being told all the time what to do. The more I interact with Tom, the more I respect him, the more I understand how fortunate I am to be working with him.

The story of how he became my advisor is perhaps symptomatic of how easy the relationship can be with Tom. I remember that day very well: I was exiting the mail room as he walked by. We talked for a brief moment about an idea I had of working with one of his collaborator at Harvard. Tom told me that this collaborator had just had a baby and wasn't looking for more students at the moment, although I could still probably win him over should I push a little. And then he said, "Do you want a problem?" And as we walked to the subway, he gave me a problem. While I deviated a bit from this original problem, it is still on the back burner of my mind and permeates the research presented here.

Only a few months later I finally decided to choose Tom for advisor. We bumped into each other in front of the applied math common room, and I told him something along the lines of "I think it is about time for me to pick an advisor. I was thinking of you. What do I have to do?" He just replied, "Tell Linda." That was it! So easy!

I know that Tom's influence in my life doesn't stop with the completion of this thesis, it will persist, just as the influence of my earlier advisors, official or not, persisted. My style is greatly influenced by Tom and the four men I am about to describe.

Fernand Beaudet was my unofficial undergraduate advisor. He his the best ambassador for mathematics I have ever seen. He is the cause of my being a mathematician, and guided me through the early years. Back then, I never made a decision without consulting him.

François Bergeron is my first research advisor. I still think very highly of him as a research and teaching mentor. He taught me it is okay to say the same stupid mistake over and over again, because as time passes, we catch ourselves making that mistake quicker and quicker, and this is how we become experts. I saw many of my fellow PhD students being afraid of exhibiting their ignorance or incompetence to their advisor. That never scared me.

Pierre Bouchard, my masters thesis advisor, was always a great believer in me. His love of mathematics is vibrant, magnetic, and his generosity is exemplary.

François Lalonde, my masters thesis co-advisor believed in me back in the old days and sent me to MIT. I don't believe that we are admitted to this great institution based on what we have done, because frankly, the only thing most of us have done is get straight A's, and there is an enormous quantity of people with good grades on this planet. We are admitted because of what people believe we can do. I am extremely grateful for François's vision of what could be and what should be.

I wish to thank these four original mentors profusely. Thank you, thank you! I could feel you all with me all these years.

My gratitude also extends to Jacques Hurtubise and Chantal David whose letters of recommendation helped support François's letter.

My mathematical parents being acknowledged, I take this opportunity to recognize my real family, mom, dad, and my sisters. Although I have been away for a very long time, and only had a few chances to interact with my family, their unconditional love, their belief in who I am and who I can become, their trust in my ability to navigate through the hardship and to make good decisions most of the time mean a lot to me. I want to thank them for all they are.

Special thanks go to the special girls that shared my life, Anne-Marie when I came here and Caroline this past year. You've sure been a cause of a lot of non-productive time, as far as math goes, and thank you for that. Thank you for your great smiles and for believing in me. You've helped me become a better person.

Etienne Rassart has become through the years living with him a great friend and sort of a brother I never had. I thank him for his never ending support and his encouragements.

I am very fortunate to have very faithful friends up north. Thanks to all of those that came south regularly, Dominic Blais, Arianne Daoust, Sarah Dulude-Dufour, Jérémie Dupuis, Hélène Parent, Anne-Marie Roy-Boulard, Stéphane Tessier, Mathieu Villeneuve-Bélair, and my house-mates Geneviève Falardeau, Pierre Poissant-Marquis, and Yvon Cournoyer. Vos visites furent très précieuses.

And great friendships developed here too. In additions to those mathematicians I acknowledge further down, I salute Pramod Achar, Tilman Bauer, Roya Beheshti, Natasha Bushueva, Ana-Maria Castravet, Antoine Charbonneau, Nenad Dedić, James and Nicole Franklin, Morten Hønsen, Alyssa Ker, Kári Ragnarsson, Sarah Raynor, Joshua Weitz, Catherine Witt, and Wai Ling Yee. We will see each other somewhere across the world.

Je suis très heureux d'avoir de compter maintenant parmis mes amis les Tymon Barwicz, Cilanne Boulet, Patrick Charbonneau, Thomas Gervais, Na'ama Pat-el, Francis Poulin, Evelyne Robidoux, et Christine Sobczak.

The mathematics department at MIT is a fantastic place, very effervescent. With about 120 students and 40 or more professors and postdocs, it is dynamic and I cherish the time I spent here. Major thanks go to the staff for running this department so smoothly.

I want to thank the various persons with whom I had very useful and stimulating discussions about the math for this thesis. Among others these are Denis Auroux, Anda Degeratu, Larry Guth, David Jerison, Julien Keller, Kevin McGerty, Frédéric Rochon, Michael Singer, Jake Solomon, Dan Stroock, and Damiano Testa.

Thanks also go to Hugh Bray, Ilya Elson, Andreas Malmendier, Etienne Rassart, Carmen Young, and Aleksey Zinger for many other fun and useful math discussions.

Nataša Šešum encouraged me to attend the EDGE summer school and conference in Edinburgh a few summers ago. That conference was very inspiring to me. Thank you Nataša, for that and for your great friendship.

The jury evaluating my work was composed of Victor Guillemin and Peter Kronheimer. I want to thank them for taking their time to evaluate this work, and for their helpful comments.

The poet Darren Frey helped me verify the English of this thesis. For a foreigner whose first language is French, this help was really welcome. Thank you.

I would like to thank the Institute for Advanced Study for the time I spent there visiting Tom on his sabbatical this past year. Particularly, I would like to thank Momota Ganguli, the librarian, and her assistant Judy Wilson-Smith, for running what is probably the most efficient library in the universe. They certainly simplified my life numerous time. I wish there will be a librarian of that kind wherever I will end up later.

The financial benefactors deserve praise for encouraging fundamental research by saving us from starvation. I want to thank NSERC, the Rosenblith Fondation, and MIT for their direct support, and NSF for its support of my advisor.

I arrived in office 2-251 at a time when it was a party zone and only the imminent deadlines of homework were keeping me working. Left to myself once the classes were over, I had to learn to work even with a far away deadline. I had to grow. My friend Leo Jesudian told me once to raise the bar in all I do. Thanks go to him and his wife Tiffany, as well as to my great friend Maneesh Bhatnagar for teaching me focus, discipline, goal setting and work ethic. When I arrived at MIT, I was not a person who could do research. But I was able to change and become one.

Yanir Rubinstein and Frédéric Rochon, the unexpected readers of my master thesis, encouraged me to surpass myself for this PhD thesis. Fortunately, my writing skills have improved over the years, thanks to the chance I had to serve as a writing TA under the supervision and dedicated mentorship of Steven Kleiman. Thank you.

The completion of a PhD is a very slow and gruelling process. It is very natural to feel dejected, frustrated, inadequate, braindead, desperate, miserable, close to tears. This is also a time when people's brain works overtime about all the different things they could be doing instead. Its the brain's attempt to cope by escapism. Thank you, thank you, thank you so much to Martin Pinsonnault and Suparna Sanyal for helping me stay sane.

While I never met Marcos Jardim, his papers had a great impact on my vision of this project and of further projects I want to tackle. I want to thank him and all the others whose papers made me explore over the last few years a gorgeous piece of mathematical landscape.

To all of those who paved the way: thank you! Perched at the top of the wall of knowledge, sitting on the shoulders of so many giants, you lead us in the quest for truth and beauty. Never stop pushing back the frontiers of ignorance. You are doing a great work, don't you dare come down.

Contents

Introduction

This introduction is composed of three parts. There is first a description of the main result of this thesis, then a historical account of the ideas leading to this thesis, and finally a road map composed of a brief description of each of the chapters in this thesis.

A Yang-Mills instanton on a Riemannian four-manifold is a vector bundle E along with the gauge equivalence class of a connection A whose curvature F_A is anti-self dual and of finite L^2 norm.

The Nahm transform of an instanton (E, A) on $\mathbb{R} \times T^3$ consists of a bundle V over an open subset of T^3, a connection B on V, and an element Φ of $\mathrm{End}V$. These objects are constructed as follows. Each point $z \in T^3$ correspond to a flat line bundle L_z over $\mathbb{R} \times T^3$, and we consider the twisted spin Dirac operator

$$\not{D}^*_{A_z} : \Gamma(\mathbb{R} \times T^3, S^- \otimes E \otimes L_z) \to \Gamma(\mathbb{R} \times T^3, S^+ \otimes E \otimes L_z).$$

The bundle V is defined by the equation

$$V_z = L^2 \cap \ker(\not{D}^*_{A_z}).$$

Let t be the $\mathbb{R}$-coordinate in $\mathbb{R} \times T^3$, and m_t denote multiplication by t. Let P denote the L^2 projection on V, and d^z the trivial connection for the trivial bundle with infinite dimensional fiber $L^2(\mathbb{R} \times T^3, S^- \otimes E)$. Then the connection B and the Higgs field Φ are defined by the equations

$$B = Pd^z,$$
$$\Phi = -2\pi i P m_t.$$

The main result of the present thesis is the following theorem.

Theorem (8.0-1 in the text on page 84). *Outside of a set W consisting of at most four points, the family of vector spaces V described above defines a vector bundle of rank*

$$\frac{1}{8\pi^2} \int_{\mathbb{R} \times T^3} |F_A|^2,$$

and the couple (B, Φ) satisfies the Bogomolny equation

$$\nabla_B \Phi = *F_B.$$

For $w \in W$ and z close enough to w, unless we are in the Scenario 2 of page 91, there are maps

The author can be reached at *benoit@alum.mit.edu*

$\Phi^{\perp}$ *and* $\Phi^{\lrcorner}$ *such that*

$$\Phi = \frac{-i}{2|z-w|}\Phi^{\perp} + \Phi^{\lrcorner},$$

and $\Phi^{\perp}$ *is the* L^2*-orthogonal projection on the orthogonal complement of a naturally defined sub-bundle* $\underline{V}$ *of* V*.*

We know heuristically, as shown in Chapter 2, that (B, Φ) must satisfies the Bogomolny equation, up to a limiting term coming out of some integration by parts on $\mathbb{R} \times T^3$. As it is shown in Chapter 7, harmonic spinors are exponentially decaying outside of W, and it then must be that the limiting term just mentioned is 0.

The rank of V is not really a surprise and follows for some relative index theorem. It is a sharp contrast with the $S^1 \times \mathbb{R}^3$ case where the computation, and the formula itself, is slightly more involved; see [NS00].

The last part of the theorem follows from a careful analysis of some geometric splitting of V coming from considering the kernel of the Dirac operator in some weighted L^2-space

$$L^2_{\delta} := e^{\delta t} L^2,$$

and variants. Taking $\epsilon > 0$ small enough, we define the various spaces

$$\begin{aligned} \underline{V}_z &:= e^{-\epsilon|t|} L^2 \cap \ker(\not{\mathfrak{D}}^*_{A_z}), \quad & \overline{V}_z &:= e^{\epsilon|t|} L^2 \cap \ker(\not{\mathfrak{D}}^*_{A_z}), \\ \overline{K}_z &:= e^{\epsilon|t|} L^2 \cap \ker(\not{\mathfrak{D}}_{A_z}), \quad & \mathcal{H}_z &:= e^{\epsilon|t|} L^2 \cap \ker(\not{\mathfrak{D}}^*_{A_z} \not{\mathfrak{D}}_{A_z}). \end{aligned}$$

Then obviously $\underline{V} \subset V \subset \overline{V}$. But also, as shown in Section 8.2,

$$\overline{V}_z = \underline{V}_z \oplus \not{\mathfrak{D}}_{A_z} \mathcal{H}_z.$$

A progression of ideas

A concrete understanding of instantons played an important role in particle physics since their discovery in mid-'70 by Belavin *et al* [BPST75]. More importantly for us, it played an important role in four-dimensional topology and geometry. For example, Donaldson has shown in [Don83] how to extract information about the intersection form on a given manifold from its moduli space of instantons; see [FU84] and [DK90] for more details.

Finding a complete description of all instantons on a given space is not an easy task and we have a description for a limited number of spaces. In particular, we do not completely understand the moduli spaces for quotients of $\mathbb{R}^4$ by lattices. In that picture, a non-linear analog of the Fourier transform, the "Nahm transform," appears.

This present thesis takes place in the quest for a unified understanding of moduli spaces of instantons on $\mathbb{R}^4$ invariant under the action of a group of translations via the Nahm transform heuristic.

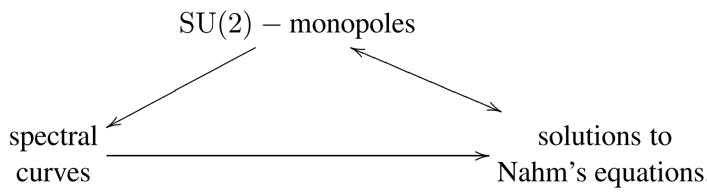
The problem of describing all instantons on $\mathbb{R}^4$ was addressed by Atiyah, Drinfeld, Hitchin and Manin in 1978 in [ADHM78]. Their description became known as the *ADHM construction*. Using twistor methods, they were able to equate the moduli space of instanton on $\mathbb{R}^4$ to a finite dimensional space of algebraic data, called the "ADHM data." Still using twistor methods, and using the relationship between monopoles on $\mathbb{R}^3$ (solutions to Bogomolny equation) and time-invariant instantons on $\mathbb{R}^4$, Hitchin [Hit82] proved in 1982 that every monopole can be constructed from some algebraic geometry data, the "spectral curve."

Nahm in 1981-1982 proposed a simplification which he thought would be better understood by physicists. As it turned out, his idea was very fruitful. The main idea is to construct the ADHM data by considering the kernel of the Dirac operator coupled to the instanton connection. By twisting the connection by a flat connection parameterized by t, Nahm also explained how monopoles can arise from solutions to a set of differential equations on $\mathbb{R}$, which we now call the "Nahm equations."

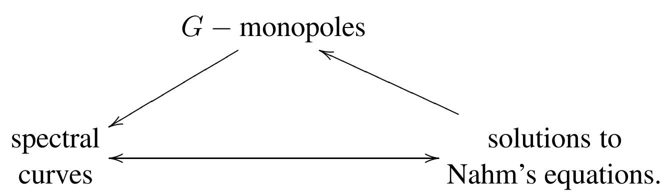
These ideas were rapidly exploited by Corrigan and Goddard in [CG84] who formalized the $\mathbb{R}^4$ story, a complete proof of which with some algebraico-geometric flavor can be found in [DK90, Chap. 3], and by Hitchin in [Hit83] who completed the $\mathrm{SU}(2)$-monopole story.

Around 1988, Braam noticed that Nahm's considerations can be used for instantons on flat tori. Exploiting Braam's observation, Schenk and Braam–van Baal in [Sch88] and [BvB89] proved independently a bijective correspondence between the moduli spaces of instantons over a flat torus and over its dual torus.

While the proofs of Corrigan–Goddard–Nahm and Schenk–Braam–van Baal are quite direct, it is not the case with Hitchin's construction, which sits in a triangle of equivalences:



In 1989, Hurtubise and Murray completed the monopole story for all classical groups, using in [HM89] a triangle of ideas similar to Hitchin's:



Note in both cases that not all arrows go both ways. While the "spectral curves" are interesting objects to study in themselves, it would be desirable to pass directly from monopoles to Nahm data, as we do for $\mathbb{R}^4$ and T^4. For $\mathrm{SU}(2)$-monopoles, this direct proof was accomplished in 1993 by Nakajima in [Nak93].

All those various correspondences fit in a more general framework. The Nahm transform takes an instanton over $\mathbb{R}^4$, invariant under the action of some group of translations Λ, and creates some Nahm data over $\mathbb{R}^{4*}$, invariant under the action of

$$\Lambda^* := \{t \in \mathbb{R}^{4*} \mid t(\Lambda) \subset \mathbb{Z}\},$$

or equivalently, over $\mathbb{R}^{4*}/\Lambda^*$.

More precisely, for each instanton A on a bundle E over $\mathbb{R}^4/\Lambda$, the Nahm transform creates a bundle $\hat{E}$ over $\mathbb{R}^{4*}/\Lambda^*$ less a few points and a connection $\hat{A}$. The self-dual part of the curvature $F_{\hat{A}}$ encodes the behavior of solutions to the Dirac equation in the non-compact directions. The bundle $\hat{E}$ is assembled from kernels of twisted Dirac operators for perturbations of A varying continuously over $\mathbb{R}^{4*}/\Lambda^*$, less those points where the associated Dirac operator is not Fredholm.

For an expanded version of the Nahm transform, as well as for examples of non-flat Nahm transforms and a survey of the literature, read the survey paper [Jar].

This idea has been exploited quite successfully by Marcos Jardim in his doctoral thesis [Jar99] and a series of papers [Jar01, Jar02a, Jar02b]. Some analytical details concerning asymptotics were tackled by Jardim and Biquard in [BJ01]. This work relates doubly-periodic instantons, or instantons on $T^2 \times \mathbb{R}^2$, with singular Higgs pairs on T^2. It is worth noting that Jardim's construction does use Hitchin's approach and goes through the spectral curves realm.

And so do Cherkis and Kapustin in [CK98, CK99, CK01] where they relate monopoles on $\mathbb{R}^2 \times S^1$ to solutions of Hitchin's equations on $S^1 \times \mathbb{R}$ using the Nahm transform and Hitchin's approach.

While Nye's doctoral thesis's work [Nye01] on the Nahm story for calorons, which are instantons on $S^1 \times \mathbb{R}^3$, does not directly use spectral curves, it relies on the construction of the Nahm data for monopoles of [HM89] which does use them. Nye's work, and the companion paper [NS00] with Singer, cover a lot of ground but bits and pieces are missing. As mentioned by Nye in his thesis, a direct proof of the $SU(n)$-monopole story through a careful analysis of the Dirac operator similar to Nakajima's proof for the $SU(2)$ case would help cover even more ground.

Of the four-dimensional quotients of $\mathbb{R}^4$, there remains only $\mathbb{R} \times T^3$. At this point in time, very little is known about instantons on $\mathbb{R} \times T^3$: some comments about the Nahm transform heuristic, and numerical approximations are found in [vB96]. This current thesis is a step forward.

Road Map

The heart of this present thesis is composed of Chapter 2, where the heuristic guiding our steps is presented, and Chapter 8, where the main result is described and proved. The experienced reader might want to pick and choose what he wants to read from the other chapters in order to get to the main result. To facilitate this approach, we now rapidly explore the whole thesis.

In Chapter 1, we explore the ADHM construction of instantons on $\mathbb{R}^4$, incorporating the idea of Nahm and using only Riemannian constructions and avoiding at all cost any use of the complex structure of $\mathbb{R}^4$. Acknowledging those facts, this chapter is called "The Riemannian ADHMN construction."

In Chapter 2, we explore in more details the Nahm Transform heuristic which guides the research in this field of study. The curvature computation presented in that chapter is the key ingredient in understanding why the pair (B, Φ) satisfies the Bogomolny equation on almost all of T^3.

In Chapter 3, we study the Dirac spectrum of product manifolds. Of particular interest is the Dirac Spectrum Formula given in Theorem 3.2-1; see page 41. This formula constructs the spectrum $\Sigma_{M\times N}$ of the Dirac operator on a spinor bundle of the product manifold $M^m \times N^n$ in terms of the spectra Σ_M on M^m and Σ_N on N. More precisely, we get

$$\Sigma_{M\times N} = \begin{cases} \pm\left|\Sigma_M \times \Sigma_N\right|, & \text{if } m \text{ and } n \text{ are odd;} \\ \pm\left|\Sigma_M^{>0} \times \Sigma_N\right| \cup (\Sigma_N)^{\#k_M^+} \cup (-\Sigma_N)^{\#k_M^-}, & \text{if } m \text{ is even.} \end{cases}$$

This formula might not be present in the literature. As a corollary, we derived in Theorem 3.4-1 a formula for the spectrum of the Dirac operator on the spinor bundle of T^3 twisted by a flat line bundle.

In Chapter 4, we derive formulas for eigenvalues and multiplicities of the Dirac operator on spheres. Section 4.1 computes the spectrum for S^3. The proof presented here is quite similar to a proof of Hitchin of which the author was not aware at the time of the writing. Knowledge of this spectrum is necessary to understand the asymptotic behavior of harmonic spinors on $\mathbb{R}^4$ proved in Chapter 7 and used in Chapter 1. Section 4.2 presents a construction of Trautman for the eigenvalues on all spheres and confirms to some extend the results of the other section.

In Chapter 5, we take note of certain results concerning the asymptotic decay of instantons on $\mathbb{R} \times T^3$. The proof exists elsewhere in the literature and is not included here. Should one be able to adapt the center manifold proof for instantons on cylindrical manifolds presented in [MMR94] to warped cylinders, one could use Theorem 5.2-2 on the decay of instantons on $T^2 \times \mathbb{R}^2$ living in the gauge group translates of the zero Fourier mode to prove a conjecture of Jardim on finite energy and quadratic decay.

In Chapter 6, we define weighted Sobolev spaces and study conditions on the weights for a Dirac operator twisted by an instanton to be Fredholm. An analysis of the time-independent case provides a formula for the difference of the indices for different weights. A short story of the concepts of weighted Sobolev spaces is presented to get the chapter off the ground.

In Chapter 7, we derive knowledge of the asymptotic behavior of harmonic spinor. To achieve that goal, the Fredholm theory of Chapter 6 is extended to weighted Sobolev spaces on half-cylinders. Once this task is accomplished, a diagram chase gives the desired result. This chapter closes with an analysis of the asymptotic behavior on $\mathbb{R}^4$. The knowledge of this behavior is necessary for part of the algebraic data in the ADHMN construction of Chapter 1.

In Chapter 8, we describe the Nahm transform of spatially periodic instantons. It is a singular monopole on T^3. The excision proof of Chapter 6 allows for a computation of the L^2-index of the Dirac operator, which is presented in Section 8.1. A geometric splitting of the bundle V given in Section 8.2 allows for an understanding of the behavior of the Higgs field at the singular points, which is given in Section 8.3. A derivation of a precise formula for the Green's operator on $S^{\pm} \otimes L_z$ presented in Section 8.4 constitutes some preliminary work on the behavior of the connection B at the singular points.

Four appendices complete this thesis. In Appendix A, we derive the various dimensional reductions of the anti-self-dual equation. In Appendix B, we study an excision principle for the index of Fredholm operators. In Appendix C, we state and prove an algebraic lemma useful for simplifying the exposition in Chapter 8. In Appendix D, we study how the Dirac operator changes under a conformal change of the metric. In Appendix E, we visit the treatment of Bartnik of weighted Sobolev spaces on $\mathbb{R}^n$ and Fredholm properties for operators asymptotic to the ordinary Laplacian, merely cleaning up a part of his paper [Bar86] by adding proofs where needed. The results presented in this appendix are used in Chapter 1 and parallel to a certain extend our treatment of Dirac operator on cylindrical manifolds of Chapters 6 and 7.

Chapter 1

The Riemannian ADHMN construction

On a four dimensional riemannian manifold X, a *G-instanton* is a G-bundle E equipped with the gauge equivalence class of a connection A which is such that its curvature F_A is anti-self-dual (written ASD for short)

$$*F_A = -F_A$$

and has finite energy

$$\|F_A\|_{L^2} < \infty.$$

In the case where X is compact, we can associate to the $\mathrm{SU}(n)$-instanton (E, A) its instanton number $c_2(E)$. In fact, the equalities

$$\begin{aligned} c_2(E) &= \frac{1}{8\pi^2} \int Tr(F_A)^2 \\ &= \frac{1}{8\pi^2} \int -|F_A^+|^2 + |F_A^-|^2 d\mu \end{aligned}$$

and

$$\|F_A\|_{L^2}^2 = \int |F_A^+|^2 + |F_A^-|^2 d\mu$$

indicate that

$$\|F_A\|_{L^2}^2 = 8\pi^2 c_2(E) \text{ if and only if } A \text{ is ASD.}$$

Hence not every bundle admit a ASD connection: an obstruction to the existence of a ASD connection on E is $c_2(E) \geq 0$.

In this chapter, we explore the ADHM construction of instantons on $\mathbb{R}^4$ from a strictly riemannian viewpoint. Most treatments found in the literature exploit the holomorphic possibilities stemming from the ASD condition. Nahm's [Nah84] and Corrigan–Goddard's [CG84] papers are unlike those, but provide more of a backbone than a complete construction.

1.1 The setting

In this chapter, we consider only instantons for the group $\mathrm{SU}(n)$ on the space $\mathbb{R}^4$.

Let $S = S^+ \oplus S^-$ be the spinor bundle of $\mathbb{R}^4$. Recall that S^+ and S^- are trivial bundles with quarternionic fiber $\mathbb{H}$. Let's denote the Clifford multiplication by ρ.

Let E be a complex vector bundle with structure group $\mathrm{SU}(n)$. Let E be equipped with a connection

A. We denote by D_A the Dirac operator $\Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$ and D_A^* its adjoint. The Laplacian $\nabla_A^* \nabla_A$ we denote Δ_A. Thus

$$\Delta_A f = -\sum_{i=1}^{4} (\partial_i)^2 - 2 \sum_{i=1}^{4} A_i \partial_i f - \sum_{i=1}^{4} ((\partial_i A_i) + A_i^2) f.$$

The main object we are studying are instantons on $\mathbb{R}^4$. An *instanton connection* is

$$\begin{gathered} \text{a } \mathrm{SU}(n) \text{ bundle } E, \text{ and} \\ \text{a connection } A \text{ on } E \text{ such that} \\ F_A^+ = 0 \text{ (ASD condition), and} \\ \|F_A\|_{L^2} < \infty. \end{gathered}$$

An *instanton* is the gauge equivalence class of an instanton connection. It must be in fact that $k := \|F_A\|_{L^2} / 8\pi^2$ is an integer that we call the *charge*. Let $\mathcal{M}_{k,n}^{\mathrm{ASD}}$ denote the moduli space of instantons of charge k and rank n.

Equally important are the *ADHM data*. They are

$$\begin{gathered} \text{a hermitian vector space } V \text{ of rank } k, \\ \text{a hermitian vector space } W \text{ of rank } n, \\ \text{a 1-form } a \text{ with values in hermitian endomorphisms of } V, \text{ and} \\ \text{a map } \Psi \colon V \to S^+ \otimes W. \end{gathered}$$

In the more general framework of Chapter 2, ADHM data are called *Nahm data*.

There is a natural notion of isomorphism of ADHM data. Of course, any two hermitian vector spaces of same rank are isomorphic, so a can be thought as a 1-form with values in hermitian $k \times k$ matrices, and Ψ as a $2n \times k$ matrix. The ADHM data (V, W, a, Ψ) and (V', W', a', Ψ') are to be considered equivalent if there exist $u \in \mathrm{SU}(n)$ and $v \in \mathrm{U}(k)$ for which

$$u a' u^{-1} = a, \text{ and } (1 \otimes u^{-1}) \Psi' v = \Psi. \tag{1.1}$$

The aim of the ADHM construction is to place in correspondence the space of instantons and the space of equivalence classes of ADHM data satisfying Conditions (1.3) and (1.4) described below.

We identify S^+ to its dual using a complex skewform ω on S^+:

$$\begin{aligned} S^+ &\to (S^+)^* \\ s &\mapsto \omega(\cdot, s). \end{aligned}$$

Hence we can associate to the map $\Psi \colon V \to S^+ \otimes W$ the map

$$\Phi = (\omega \otimes 1) \circ (1 \otimes \Psi) \colon S^+ \otimes V \to W.$$

We use a and Φ to define the map

$$\begin{aligned} Q_x &\colon S^+ \otimes V \to S^- \otimes V \oplus W \\ Q_x &= \begin{bmatrix} \sum_{i=1}^4 \rho(\partial_i) \otimes (a_i + x_i) \\ \Phi \end{bmatrix}. \end{aligned} \tag{1.2}$$

The conditions referred to above are the

$$-\rho([a,a]) + 2\Phi^*\Phi = 1 \otimes \Psi^*\Psi \quad \text{(ADHM equation)}, \tag{1.3}$$
$$Q_x \text{ is everywhere injective.} \quad \text{(non-degeneracy condition)} \tag{1.4}$$

Let $\mathcal{M}_{n,k}^{\mathrm{ADHM}}$ denote the space of ADHM data satisfying the ADHM and non-degeneracy conditions, modulo the equivalence relation of Equation (1.1).

The goal of this chapter is to prove the following theorem.

Theorem 1.1-1 (ADHM construction). *The map*

$$\mathfrak{N}\colon \mathcal{M}_{k,n}^{\mathrm{ASD}} \to \mathcal{M}_{n,k}^{\mathrm{ADHM}},$$

constructed in Section 1.2, and the map

$$\mathfrak{F}\colon \mathcal{M}_{n,k}^{\mathrm{ADHM}} \to \mathcal{M}_{k,n}^{\mathrm{ASD}},$$

constructed in Section 1.3 are inverses of each other.

1.2 From instanton to ADHM data

We build up the ADHM data bit by bit.

The Weitzenbock formula

$$D_A^* D_A = \Delta_A + \rho(F_A^+) + \frac{1}{4}\text{scalar curvature}$$

tells us that the ASD (anti-self-dual) condition for the connection A is equivalent to the condition that

$$D_A^* D_A \text{ commutes with quaternion multiplication.}$$

It also tells us that for an A connection whose curvature is ASD, $\ker(D_A) \cap L^2 = \{0\}$. Indeed, because of the Weitzenbock formula, when $D_A\phi = 0$ it must be that ϕ is parallel. But to be L^2 on $\mathbb{R}^4$, a parallel section must then be 0.

Set

$$V_E := L^2 \cap \ker(D_A^*),$$

and

$$W_E := \text{ bounded harmonic sections of } E.$$

Elements of W_E are in natural bijection with sections parallel at infinity. Set the scalar product on W_E to be

$$\langle w_1, w_2\rangle = 4\pi^2(w_1^\infty, w_2^\infty). \tag{1.5}$$

Let $\{\psi_1, \ldots, \psi_k\}$ be a L^2-orthonormal basis of V_E. We use the L^2 scalar product, which we also denote $\langle\ ,\ \rangle$, and define the projection Π by the formula

$$\Pi := \sum_{j=1}^{k} \langle \psi_j,\ \cdot\ \rangle \psi_j.$$

Let m_μ denote multiplication by x_μ. Consider the linear map

$$\begin{aligned} a_\mu \colon V_E &\to V_E \\ \psi &\mapsto -\Pi m_\mu \psi. \end{aligned} \tag{1.6}$$

The endomorphism a_μ has matrix

$$\Big[-\langle \psi_i, x_\mu \psi_j \rangle_{L^2}\Big]_{1\le i,j\le k}.$$

This matrix is clearly hermitian, that is $a_\mu^* = a_\mu$.

The L^2 condition imposes a particular asymptotic behavior to elements of V_E. We study in Chapter 7 how harmonic spinors decay on cylindrical manifolds. Since $\mathbb{R}^4 \setminus \{0\} = \mathbb{R} \times S^3$ conformally, we reprove in Section 7.3 the classical result that any element of V_E has an asymptotic expansion of the type

$$|x|^{-4}\rho(x)\hat{\phi} + O(|x|^{-4}) \tag{1.7}$$

for a parallel section $\hat{\phi}$ of $S^+ \otimes E$.

We define the map

$$\begin{aligned} \Psi \colon V_E &\to S^+ \otimes W_E \\ \phi &\mapsto \hat{\phi}/2. \end{aligned}$$

We package the obtained ADHM data as

$$\mathfrak{N}(E, A) = (V_E, W_E, a, \Psi). \tag{1.8}$$

The rest of this section is devoted to the analysis justifying the given description of $\mathfrak{N}(E, A)$ and preparing the way for the proof that $\mathfrak{N}(E, A)$ satisfies the conditions (1.3) and (1.4).

The Green's function G and Projections

As multiplication by x_i could potentially kick an element of L^2 out of it, we have to prove that on $\ker(\not{\mathfrak{D}}_A^*)$ it doesn't. To do so, we observe that the L^2-condition on V_E is actually too weak. This observation is best described in the realm of weighted Sobolev spaces see Appendix E or [Bar86] for conventions and results.

Again using the conformal identification of $\mathbb{R}^4 \setminus \{0\}$ with $\mathbb{R} \times S^3$, and using the fact that the interval $(-3/2, 3/2)$ contains no eigenvalue of the Dirac operator on S^3 (see Section 4.1), we can use the technology of Chapter 6 or of Appendix E to prove that

$$V_E = W_\delta^{1,2} \cap \ker(D_A^*)$$

for $\delta \in (-3, 0)$. In that range, the kernel is constant.

For $-2 < \delta < 0$ and $1 < p < \infty$, the operator

$$\Delta_A \colon W_\delta^{k+2,p}(S \otimes E) \to W_{\delta-2}^{k,p}(S \otimes E)$$

is invertible; see for example [KN90, lemma 5.1, p. 279]. Let G_A denote its inverse, the so called *Green's operator*. Observe that as Δ_A is defined independently of δ, so is G_A for $\delta < 0$.

Set

$$P_A := Id - D_A G_A D_A^*.$$

In a finite dimensional setting, it is obvious that $\Pi = P_A$. The next lemma tells us for which weighted Sobolev spaces these projections are indeed the same.

Lemma 1.2-1. *When $\delta \in (-3,-1)$, the projection $P_A\colon W_\delta^{1,2} \to V_E$ is a well-defined continuous map. When $\delta < -1$, the projection $\Pi\colon L_\delta^2 \to V_E$ is a well-defined continuous map. On the spaces $W_\delta^{1,2}$ for $\delta \in (-3,-1)$, we have $\Pi = P_A$.*

Proof: All the maps in the sequence

$$W_\delta^{1,2} \xrightarrow{D_A^*} L_{\delta-1}^2 \xrightarrow{G_A} W_{\delta+1}^{2,2} \xrightarrow{D_A} W_\delta^{1,2}$$

are continuous when $\delta + 1 \in [-2, 0]$, thus when $\delta \in [-3, -1]$. Since $D_A^* P_A = 0$ in the interior of that range, P_A maps into V_E for $\delta \in (-3,-1)$.
We have

$$|\langle \psi_j, \phi\rangle| \leq \|\psi_j\|_{2,\delta_1} \|\phi\|_{2,\delta}$$

with $\delta + \delta_1 = -4$. Since $\psi_j \in V_E \subset W_{\delta_1}^{1,2}$ for $\delta_1 > -3$, we have that the scalar product is finite when $\phi \in L_\delta^2$ for $\delta < -1$. We can clearly see that Π is continuous and maps into V_E in that range.
Now, suppose $\phi \in V_E^\perp \cap W_\delta^{1,2}$. Then

$$\langle P_A\phi, \psi_j\rangle = \langle \phi, \psi_j\rangle - \langle D_A G_A D_A^* \phi, \psi_j\rangle.$$

The first term of the right hand side is clearly 0 since $\psi_j \in V_E$. For $\phi \in W_\delta^{1,2}$ with $\delta \in (-3,-1)$, we have the equality

$$\langle D_A G_A D_A^* \phi, \psi_j\rangle = \langle G_A D_A^*\phi, D_A^*\psi_j\rangle = 0.$$

Hence $\langle P_A\phi, \psi_j\rangle = 0$ for all j and $P_A\phi \in V_E^\perp$. Since we already know that $P_A\phi \in V_E$, we must have $P_A\phi = 0$ and $P_A = \Pi$. □

Asymptotic for $G\phi$

We also need to know the asymptotic behavior of $G_A\phi$ for $\phi \in V_E$.

Lemma 1.2-2. *For $\phi \in V_E$, we have*

$$G_A\phi = r^2\frac{\phi}{4} + O(r^{-2}). \tag{1.9}$$

Proof: Notice first that

$$\nabla_A^*\nabla_A r^2\phi = -8\phi - 4\nabla_A^x\phi + r^2\nabla_A^*\nabla_A\phi. \tag{1.10}$$

Since $\phi \in V_E$, we have from Equation (1.7) that

$$\phi = |x|^{-4}\rho(x)\hat{\phi} + O(r^{-4}).$$

In fact, by doing the decomposition in some higher order Sobolev spaces, we see that

$$\nabla_A^x O(r^{-4}) = O(r^{-4}), \text{ and}$$
$$\nabla_A^* \nabla_A O(r^{-4}) = O(r^{-6}).$$

Note now that

$$\begin{aligned}
\nabla_A^x \rho(\nu) r^{-3}\hat{\phi} &= r\rho(\nabla_A^\nu \nu) r^{-3}\hat{\phi} + r\rho(\nu)(\nu \cdot r^{-3})\hat{\phi} + \rho(\nu) r^{-3} r \nabla_A^\nu \hat{\phi} \\
&= 0 + r\rho(\nu)(-3r^{-4})\hat{\phi} + 0 \\
&= -3\rho(\nu) r^{-3}\hat{\phi}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\nabla_A^x \phi &= -3\phi + O(r^{-4}) + \nabla_A^x O(r^{-4}) \\
&= -3\phi + O(r^{-4}). \qquad (1.11)
\end{aligned}$$

Similarly, since

$$\begin{aligned}
\sum_{i=1}^4 \nabla_A^i \nabla_A^i \big(r^{-4} \sum_{j=1}^4 x_j \rho(\partial_j)\hat{\phi}\big) &= \sum_{i=1}^4 \nabla_A^i \big(r^{-4} \sum_{j=1}^4 \nabla_A^i x_j \rho(\partial_j)\hat{\phi} + \partial_i(r^{-4}) \sum_{j=1}^4 x_j \rho(\partial_j)\hat{\phi}\big) \\
&= \sum_{i=1}^4 \nabla_A^i \big(r^{-4}\rho(\partial_i)\hat{\phi}\big) + \sum_{i=1}^4 \partial_i^2 (r^{-4})\rho(x)\hat{\phi} \\
&\qquad + \sum_{i=1}^4 \partial_i(r^{-4}) \nabla_A^i \big(\rho(x)\hat{\phi}\big) \\
&= 2\sum_{i=1}^4 \partial_i(r^{-4})\rho(\partial_i)\hat{\phi} + 8r^{-6}\rho(x)\hat{\phi} \\
&= 2\sum_{i=1}^4 (-4) r^{-6} x_i \rho(\partial_i)\hat{\phi} + 8r^{-6}\rho(x)\hat{\phi} \\
&= 0
\end{aligned}$$

we have

$$\begin{aligned}
\nabla_A^* \nabla_A \phi &= -\sum_{i=1}^4 \nabla_A^i \nabla_A^i \big(r^{-4}\rho(x)\hat{\phi}\big) + \nabla_A^* \nabla_A O(r^{-4}) \\
&= O(r^{-6}) \qquad (1.12)
\end{aligned}$$

Summing up what we know with Equations (1.10), (1.11), and (1.12), we find that

$$\nabla_A^* \nabla_A r^2 \phi = -8\phi - 4\big(-3\phi + O(r^{-4})\big) + r^2 O(r^{-6}).$$

Applying G_A on both sides, we get

$$G_A \phi = r^2 \frac{\phi}{4} + G_A O(r^{-4}).$$

Lemma 3.3.35 from [DK90, p. 105] tells us that

$$G_A O(r^{-4}) = O(r^{-4}) + O(r^{-2}) + O(r^{4-(2+4)}) = O(r^{-4}).$$

Substituting in the previous equation, we complete the proof. □

The Curvature of a

In view of the heuristic we explore in Chapter 2, we choose to temporarily view V_E as the fiber of a trivial bundle over $\mathbb{R}^4$. The endomorphisms a_μ then team up to produce the constant connection

$$a = a_1 dx^1 + \cdots + a_4 dx^4.$$

Lemma 1.2-3. *The curvature $F_a = \frac{1}{2}[a, a]$ of the connection a on the trivial bundle with fiber V_E over $\mathbb{R}^4$, seen as an element of $\operatorname{End}(V_E) \otimes \bigwedge^2 \mathbb{R}^4$ is given by*

$$F_a = \Pi G_A \sum_{i=1}^{3} \rho(\bar{\epsilon}_i) \otimes \bar{\epsilon}_i - \frac{1}{8} \sum_{i=1}^{3} \Psi^* \rho(\epsilon_i) \Psi \otimes \epsilon_i.$$

Proof: Component-wise, the curvature is

$$(F_a)_{ij} = \frac{1}{2}[a_i, a_j].$$

To compute this curvature, we need to to see that

$$[m_i, D_A] = -\rho(\partial_i), \tag{1.13}$$

and that

$$[\rho(\partial_i), G_A] = 0. \tag{1.14}$$

These two results are independent of the ASD condition on A. The second has to do with the fact that the ∂_i are parallel.

Let $\phi \in V_E$. We have

$$[a_i, a_j](\phi) = \Pi(m_i \Pi m_j \phi) - \Pi(m_j \Pi m_i \phi). \tag{1.15}$$

Since at this point we have two formulas for Π, let's use both and use P_A to denote the usage of the $1 - D_A G_A D_A^*$ formula, and Π for the scalar product type formula. We then compute

$$\Pi(m_i P_A m_j \phi) = \sum_{l=1}^{k} \lim_{r\to\infty} \left(\int_{B^4(r)} (m_i(1 - D_A G_A D_A^*) m_j \phi, \psi_l) \right) \psi_l, \tag{1.16}$$

but

$$(m_i(1 - D_A G_A D_A^*) m_j \phi, \psi_l) = (m_i m_j \phi, \psi_l) - (m_i D_A G_A D_A^* m_j \phi, \psi_l). \tag{1.17}$$

The first term gets killed when we antisymmetrize with respect to i and j. We thus compute only the second term. Equation (1.13) tells us that

$$\begin{aligned}(m_i D_A G_A D_A^* m_j \phi, \psi_l) &= (m_i D_A G_A m_j D_A^* \phi, \psi_l) + (m_i D_A G_A \rho(\partial_j)\phi, \psi_l) \\ &= (D_A m_i G_A \rho(\partial_j)\phi, \psi_l) - (\rho(\partial_i) G_A \rho(\partial_j)\phi, \psi_l).\end{aligned} \tag{1.18}$$

The second term gives the $\Pi G_A \rho(\partial_i)\rho(\partial_j)$ part of the curvature once we integrate, take the limit, substitute, antisymmetrize as asked by Equation (1.15) and divide by 2 to get its part in F_a.

As proved in [Roe98, p. 46], we have

$$(D_A s, \psi_l) - (s, D_A^* \psi_l) = \sum_{h=1}^{4} \partial_h(\rho(\partial_h)s, \psi_l).$$

Since $D_A^*\psi_l = 0$, the first term of Equation (1.18) transforms:

$$(D_A m_i G_A \rho(\partial_j)\phi, \psi_l) = \sum_{h=1}^{4} \partial_h(\rho(\partial_h) G_A \rho(\partial_j)\phi, \psi_l).$$

We now integrate by parts over the ball of radius r to obtain

$$\int_{B^4(r)} (D_A m_i G_A \rho(\partial_j)\phi, \psi_l) = \int_{S^3(r)} (\rho(\nu) m_i G_A \rho(\partial_j)\phi, \psi_l).$$

We now use the asymptotic given by Equations (1.7) and (1.9) to get

$$\psi_l = r^{-3}\rho(\nu)\hat{\psi}_l + O(r^{-4}), \text{ and}$$
$$G_A\phi = \rho(\nu)\hat{\phi}/4r + O(r^{-2}).$$

Then

$$\begin{aligned}
&\int_{S^3(r)} (\rho(\nu) m_i G_A \rho(\partial_j)\phi, \psi_l) \\
&\qquad = \sum_{h=1}^{4} \int_{S^3(r)} r^{-1} x_i x_h (\rho(\nu)\rho(\partial_j)\rho(\partial_h)\hat{\phi}/4r + O(r^{-2}), r^{-3}\rho(\nu)\hat{\psi}_l + O(r^{-4})) \\
&\qquad = \sum_{h=1}^{4} \int_{S^3(r)} (r^{-5}/4) x_i x_h (\rho(\partial_j)\rho(\partial_h)\hat{\phi}, \hat{\psi}_l) + \int_{S^3(r)} (O(r^{-5}) + O(r^{-4}) + O(r^{-4})) \\
&\qquad = (r^{-5}/4) \sum_{h=1}^{4} (\rho(\partial_j)\rho(\partial_h)\hat{\phi}, \hat{\psi}_l) \int_{S^3(r)} x_i x_h + O(r^{-4}) Vol(S^3(r)).
\end{aligned}$$

As $r \to \infty$, the volume of $S^3(r)$ is $O(r^3)$ thus the last term vanish in the limit. The integral $\int_{S^3(r)} x_i x_h$ vanishes when $i \neq l$ and is otherwise

$$r^2 Vol(S^3(r))/4 = r^5 Vol(S^3(1))/4 = r^5\pi^2/2.$$

Thus we have from Equations (1.17), (1.16) and (1.15) that

$$[a_i, a_j](\phi) = 2\Pi G_A \rho(\partial_i)\rho(\partial_j)\phi - \frac{\pi^2}{4} \sum_{l=1}^{k} (\rho(\partial_i)\rho(\partial_j)\hat{\phi}, \hat{\psi}_l)\psi_l.$$

Note that

$$\sum_{1 \leq i,j \leq 4} \rho(\partial_i)\rho(\partial_j) dx^i \wedge dx^j = \sum_{i=1}^{3} \rho(\epsilon_i)\epsilon_i + \rho(\bar{\epsilon}_i)\bar{\epsilon}_i.$$

Remember that $\bigwedge^+$ acts trivially on $V_E \subset \Gamma(S^- \otimes E)$ and $\bigwedge^-$ acts trivially on $\Gamma(S^+ \otimes E)$. Hence, the curvature is

$$\begin{aligned} F_a(\phi) &= \Pi G_A \sum_{i=1}^{3} \rho(\bar{\epsilon}_i)\phi\bar{\epsilon}_i - \frac{\pi^2}{8} \sum_{i=1}^{3} \sum_{l=1}^{k} (\rho(\epsilon_i)\hat{\phi}, \hat{\psi}_l)\psi_l \epsilon_i \\ &= \Pi G_A \sum_{i=1}^{3} \rho(\bar{\epsilon}_i)\phi\bar{\epsilon}_i - \frac{\pi^2}{2} \sum_{i=1}^{3} \sum_{l=1}^{k} (\rho(\epsilon_i)\Psi(\phi), \Psi(\psi_l))\psi_l \epsilon_i. \end{aligned}$$

Using the scalar product given by Equation (1.5), we complete the proof. □

The ADHM data satisfies the conditions

Before going further, let's walk through the association between Φ and Ψ in more details. Using the identification $\mathbb{R}^4 = \mathbb{H} = S^+ = S^-$, the Clifford multiplication $\rho(x)\colon S^+ \to S^-$ is multiplication by $-\bar{x}$ and $\rho(x)\colon S^- \to S^+$ is multiplication by x.

Let ϵ_i and $\bar{\epsilon}_i$ denote the usual basis of $\bigwedge^+$ and $\bigwedge^-$ respectively. The action of self-dual forms on S^+ is

$$\rho(\epsilon_1) = 2i, \quad \rho(\epsilon_2) = 2j, \quad \rho(\epsilon_3) = 2k.$$

We use the complex basis $s_1 = 1, s_2 = j$ of S^+, with the identification

$$\begin{aligned} \mathbb{C} \oplus \mathbb{C} &= S^+ \\ (z_1, z_2) &\mapsto z_1 + j z_2. \end{aligned} \tag{1.19}$$

Then,

$$\rho(\epsilon_1) = 2\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \rho(\epsilon_2) = 2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \rho(\epsilon_3) = 2\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}. \tag{1.20}$$

We split the map Ψ as

$$\begin{aligned} \Psi \colon V &\to S^+ \otimes W \\ \Psi &= s_1 \otimes \Psi_1 + s_2 \otimes \Psi_2. \end{aligned}$$

We identify S^+ to its dual using the skewform $\omega = s^1 \wedge s^2$:

$$\begin{aligned} S^+ &\to (S^+)^* \\ s &\mapsto \omega(\cdot, s). \end{aligned}$$

Thus

$$\begin{aligned} s_1 &\mapsto -s^2 \\ s_2 &\mapsto s^1. \end{aligned}$$

As mentioned before, in doing so we identify Ψ with the map

$$\begin{aligned} \Phi = (\omega \otimes 1) \circ (1 \otimes \Psi) &\colon S^+ \otimes V \to W \\ \Phi &= -s^2 \Psi_1 + s^1 \Psi_2. \end{aligned}$$

The adjoints are

$$\Psi^*\colon S^+ \otimes W \to V$$
$$\Psi^* = s^1\Psi_1^* + s^2\Psi_2^*,$$

and

$$\Phi^*\colon W \to S^+ \otimes V$$
$$\Phi^* = -s_2 \otimes \Psi_1^* + s_1 \otimes \Psi_2^*.$$

Thus

$$\Psi^* = -(\omega \otimes 1) \circ (1 \otimes \Phi^*).$$

We have

$$\Psi^*\Psi\colon V \to V,$$
$$\Psi^*\Psi = \Psi_1^*\Psi_1 + \Psi_2^*\Psi_2, \tag{1.21}$$

and

$$\Phi^*\Phi\colon S^+ \otimes W \to S^+ \otimes W$$

is given by $s_2 \otimes s^2 \otimes \Psi_1^*\Psi_1 + s_1 \otimes s^1 \otimes \Psi_2^*\Psi_2 - s_1 \otimes s^2 \otimes \Psi_2^*\Psi_1 - s_2 \otimes s^1 \otimes \Psi_1^*\Psi_2$. In matrix form, this expression becomes

$$\Phi^*\Phi = \begin{bmatrix} \Psi_2^*\Psi_2 & -\Psi_2^*\Psi_1 \\ -\Psi_1^*\Psi_2 & \Psi_1^*\Psi_1 \end{bmatrix}. \tag{1.22}$$

Theorem 1.2-4. *The ADHM data (V_E, W_E, a, Ψ) obtained from the $SU(n)$-instanton connection (E, A) satisfies the ADHM and nondegeneracy conditions (1.3) and (1.4).*

Proof: Let's first consider the action of $[a, a] = 2F_a$ on $S^+ \otimes V$. On that space, only the self-dual part matters. Recall from Lemma 1.2-3 that for $\phi \in V_E$, we have

$$[a, a]^+(\phi) = 2F_a^+ = \frac{1}{4}\sum_{i=1}^{3} \Psi^*\rho(\epsilon_i)\Psi(\phi)\epsilon_i.$$

Let's break it down using the identification of Equation (1.19) and the matrices of Equation (1.20). We have

$$\Psi^*\rho(\epsilon_1)\Psi = \begin{bmatrix} \Psi_1^* & \Psi_2^* \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = 2i(\Psi_1^*\Psi_1 - \Psi_2^*\Psi_2),$$

and similarly

$$\Psi^*\rho(\epsilon_1)\Psi = 2(\Psi_2^*\Psi_1 - \Psi_1^*\Psi_2),$$
$$\Psi^*\rho(\epsilon_1)\Psi = -2i(\Psi_2^*\Psi_1 + \Psi_1^*\Psi_2).$$

For the map $\rho([a, a])\colon S^+ \otimes V \to S^- \otimes V$, only the self-dual part matters and in term of the identification of Equation (1.19), it is given by

$$\begin{aligned}\rho([a,a]) &= \frac{1}{4}\left(2\begin{bmatrix} i & 0 \\ 0 & -i\end{bmatrix}\Psi^*\rho(\epsilon_1)\Psi + 2\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}\Psi^*\rho(\epsilon_2)\Psi + 2\begin{bmatrix} 0 & -i \\ -i & 0\end{bmatrix}\Psi^*\rho(\epsilon_3)\Psi\right) \\ &= \begin{bmatrix} \Psi_2^*\Psi_2 - \Psi_1^*\Psi_1 & -2\Psi_2^*\Psi_1 \\ -2\Psi_1^*\Psi_2 & \Psi_1^*\Psi_1 - \Psi_2^*\Psi_2 \end{bmatrix}.\end{aligned}$$

Using Equations (1.21) and (1.22), we see that

$$-\rho([a,a]) + 2\Phi^*\Phi = 1 \otimes \Psi^*\Psi \quad \text{(ADHM equation)}.$$

Hence the ADHM equation (1.3) is satisfied. □

1.3 From ADHM data to instanton

We now start with some ADHM data (V, W, a, Ψ) satisfying the ADHM and nondegeneracy conditions (1.3) and (1.4), and want to construct an instanton connection

$$\mathfrak{F}(V, W, a, \Psi) = (E, A).$$

Recall from Equation (1.2) that we define $Q_x \colon S^+ \otimes V \to S^- \otimes V \oplus W$ as

$$Q_x = \begin{bmatrix} \sum_{i=1}^4 \rho(\partial_i) \otimes (a_i + x_i) \\ \Phi \end{bmatrix}.$$

Since Q_x is injective for every x, the map $Q_x^* Q_x \colon S^+ \otimes V \to S^+ \otimes V$ is an isomorphism at for every x. Let

$$G_{Q_x} = (Q_x^* Q_x)^{-1} \tag{1.23}$$

be its inverse.

Let E be the bundle with fiber $\ker(Q_x^*)$ at x. The bundle E sits in the trivial bundle with fiber $S^- \otimes V \oplus W$. To simplify the notation, we drop the subscript x. The map

$$R := 1 - Q G_Q Q^* \tag{1.24}$$

is the orthogonal projection on E. We equip E with the induced connection

$$A := Rd.$$

Theorem 1.3-1. *The pair (E, A) is an instanton connection on $\mathbb{R}^4$.*

Proof: To compute the curvature, we first need a better grip on G_Q. We have

$$Q^* = \begin{bmatrix} -\sum_{i=1}^4 \rho(\partial_i) \otimes (a_i + x_i) & \Phi^* \end{bmatrix},$$

thus

$$
\begin{aligned}
Q^*Q &= -\sum_{1\le i,j\le 4} \rho(\partial_i)\rho(\partial_j)\otimes (a_i+x_i)(a_j+x_j) + \Phi^*\Phi \\
&= 1\otimes \sum_{i=1}^{4}(a_i+x_i)^2 - \frac{1}{2}\rho([a,a]) + \Phi^*\Phi \\
&= 1\otimes \Big(\sum_{i=1}^{4}(a_i+x_i)^2 + \frac{1}{2}\Psi^*\Psi\Big). \qquad (1.25)
\end{aligned}
$$

Thus, Q^*Q commutes with the quaternions and so does its inverse G_Q. We can then write

$$G_Q = 1\otimes g_Q$$

with g_Q the inverse of the map $\mathcal{q}\colon V\to V$ given by

$$\mathcal{q} = \sum_{i=1}^{4}(a_i+x_i)^2 - (1/2)\Psi^*\Psi.$$

Notice that

$$[\partial_\mu, Q] = \begin{bmatrix}\rho(\partial_\mu)\\ 0\end{bmatrix}.$$

The curvature acts on $\phi\in E$ as

$$RdRd\phi = \sum_{1\le i,j\le 4} R\partial_i R\partial_j\phi dx^i\wedge dx^j,$$

and

$$
\begin{aligned}
R\partial_i R\partial_j\phi &= R\partial_i\partial_j\phi - R\partial_i Q G_Q Q^*\partial_j\phi \\
&= R\partial_i\partial_j\phi - R\partial_i Q G_Q \partial_j Q^*\phi + R\partial_i Q F\begin{bmatrix}\rho(\partial_j) & 0\end{bmatrix}\phi \\
&= R\partial_i\partial_j\phi + 0 + RQ\partial_i G_Q\begin{bmatrix}\rho(\partial_j) & 0\end{bmatrix}\phi - R\begin{bmatrix}\rho(\partial_i)\\ 0\end{bmatrix} G_Q\begin{bmatrix}\rho(\partial_j) & 0\end{bmatrix}\phi \\
&= R\partial_i\partial_j\phi - R\begin{bmatrix}G_Q\rho(\partial_i)\rho(\partial_j) & 0\\ 0 & 0\end{bmatrix}
\end{aligned}
$$

Thus, we find

$$F_{\hat{a}} = -R\sum_{i=1}^{3}\begin{bmatrix}G_Q\rho(\bar\epsilon_i) & 0\\ 0 & 0\end{bmatrix}\bar\epsilon_i,$$

and the connection has ASD curvature.
Since $\mathcal{q} = r^2 + O(r)$ as r tends to ∞, we have that

$$g_Q = r^{-2} + O(r^{-3}) \text{ as } r\to\infty.$$

Then

$$
\begin{aligned}
R &= 1 - QG_QQ^* \\
&= 1 - \begin{bmatrix} \rho(x) + O(1) \\ O(1) \end{bmatrix} (r^{-2} + O(r^{-3})) \begin{bmatrix} -\rho(x) + O(1) & O(1) \end{bmatrix} \\
&= 1 - \begin{bmatrix} -\rho(x)^2 r^{-2} + O(r^{-1}) & O(r^{-1}) \\ O(r^{-1}) & O(r^{-2}) \end{bmatrix} \\
&= \begin{bmatrix} O(r^{-1}) & O(r^{-1}) \\ O(r^{-1}) & 1 + O(r^{-2}) \end{bmatrix},
\end{aligned}
$$

thus the curvature of the connection Rd on E satisfies

$$R \begin{bmatrix} G_Q & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} O(r^{-3}) & 0 \\ 0 & 0 \end{bmatrix} \text{ as } r \to \infty,$$

and is consequently in L^2.
The proof is now complete. □

In fact, we can prove even a better asymptotic formula for g_Q.

Lemma 1.3-2. *In fact, we even have*

$$g_Q = r^{-2} - 2\sum x_j a_j r^{-4} - \Big(\sum a_j^2 - \frac{1}{2}\Psi^*\Psi\Big) r^{-4} + 4\sum_{j,k} a_j a_k x_j x_k r^{-6} + O(r^{-5}). \tag{1.26}$$

Proof: The proof is mechanical. We build up the asymptotics of g_Q from the asymptotics of q_Q, term by term. □

1.4 Uniqueness

We now wish to prove that starting from some ADHM data, creating the associated instanton and looking at the ADHM data this instanton produce, we come back to where we started. In other words, we prove in this section that the composition

$$\begin{matrix} \text{ADHM data} & \xrightarrow{\text{Sect.1.3}} & \text{instanton} & \xrightarrow{\text{Sect.1.2}} & \text{ADHM data} \\ (V, W, a, \Psi) & \mathfrak{F} & (E, A) & \mathfrak{N} & (V', W', a', \Psi') \end{matrix}$$

gives $(V', W', a', \Psi') = (V, W, a, \Psi)$.
We are thus searching for a proof that V is isomorphic, in a somewhat canonical way, to $\ker(D^*)$ in sections of $S^- \otimes E$. As E is a subbundle of $S^- \otimes V \oplus W$, we would be happy to find a map

$$\psi \colon V \to S^- \otimes \big(S^- \otimes V \oplus W\big),$$

or equivalently

$$\tilde{\psi} \colon S^+ \otimes V \to S^- \otimes V \oplus W,$$

such that

$$\begin{gathered}\operatorname{Im}(\tilde{\psi}) \subset E,\\ D^*\psi = 0,\\ \psi \text{ is injective.}\end{gathered}$$

Using the identification $S^+ \equiv S^-$, define the map

$$\begin{gathered}b\colon S^+ \otimes V \to S^- \otimes V \oplus W\\ x \mapsto (x, 0).\end{gathered} \tag{1.27}$$

Our candidate is

$$\tilde{\psi} = RbG_Q. \tag{1.28}$$

Obviously, $Im(\tilde{\psi}) \subset E$ as R projects on E. To prove that $D^*\psi = 0$, we observe that

$$\begin{aligned}\widetilde{D_A^*\psi} &= \langle D_A^*\psi, \cdot\rangle\\ &= \sum_{j=1}^4 \langle \rho(\partial_j)\nabla_A^j\psi, \cdot\rangle\\ &= -\sum_{j=1}^4 \langle \nabla_A^j\psi, \rho(\partial_j)\cdot\rangle\\ &= -\sum_{j=1}^4 (\nabla_A^j\tilde{\psi}) \circ \rho(\partial_j).\end{aligned}$$

Hence

$$\begin{aligned}\widetilde{D^*\psi} &= -\sum_{j=1}^4 R(\partial_j\tilde{\psi})\rho(\partial_j)\\ &= -\sum_{j=1}^4 R\partial_j(RbG_Q)\rho(\partial_j)\\ &= -\sum_{j=1}^4 R(\partial_j R)bG_Q\rho(\partial_j) - \sum_{j=1}^4 Rb(\partial_j G_Q)\rho(\partial_j)\\ &= \sum_{j=1}^4 R\partial_j(QG_QQ^*)bG_Q\rho(\partial_j) - \sum_{j=1}^4 Rb(\partial_j G_Q)\rho(\partial_j)\\ &= \sum_{j=1}^4 R(\partial_j Q)G_QQ^*bG_Q\rho(\partial_j) + \sum_{j=1}^4 RQ\partial_j(G_QQ^*)bG_Q\rho(\partial_j) - \sum_{j=1}^4 Rb(\partial_j G_Q)\rho(\partial_j).\end{aligned}$$

On that last line, the second sum is obviously null as $RQ = 0$. As for the first sum, observe that $\partial_j Q = b\rho(\partial_j)$, while

$$G_QQ^*bG_Q = \frac{1}{2}\sum_{i=1}^4 \rho(\partial_i)\partial_i G_Q. \tag{1.29}$$

Indeed, from Equation (1.25), we derive that $\partial_i G_Q = -2G_Q(a_i + x_i)G_Q$. Equation (1.29) follows immediately once we recognize that $Q^*b = -\sum_{i=1}^4 \rho(\partial_i)(a_i + x_i)$.

Going back to where we left, we have

$$\begin{aligned}\widetilde{D_A^*\psi} &= \frac{1}{2}\sum_{1\le i,j\le 4} Rb\rho(\partial_j)\rho(\partial_i)(\partial_i G_Q)\rho(\partial_j) - \sum_{j=1}^4 Rb(\partial_j G_Q)\rho(\partial_j) \\ &= Rb\sum_{i=1}^4\Big(\frac{1}{2}\sum_{j=1}^4 \rho(\partial_j)\rho(\partial_i)\rho(\partial_j) - \rho(\partial_i)\Big)\partial_i G_Q \\ &= 0.\end{aligned}$$

Now that we know that ψ maps V to sections of $S^- \otimes E$ satisfying the Dirac equation, we would like to see that ψ is actually an isomorphism $V \to V'$. To prove this result, we use the following analytic lemma. But first let ∂^2 denote the Laplacian $\sum_i \partial_i^2$.

Lemma 1.4-1. *For ψ defined by Equation* (1.28)*, we have*

$$\psi^*\psi = -\frac{1}{4}\partial^2 g_Q. \tag{1.30}$$

Proof: Let tr_2 denote the trace along the spinor factor. Notice that

$$\psi^*\psi = tr_2(\tilde{\psi}^*\tilde{\psi}).$$

Indeed, as we have

$$\begin{aligned}\psi &= s_1 \otimes \psi_1 + s_2 \otimes \psi_2, \qquad & \psi^* &= s^1 \otimes \psi_1^* + s^2 \otimes \psi_2^*, \\ \tilde{\psi} &= s^1 \otimes \psi_1 + s^2 \otimes \psi_2, \qquad & \tilde{\psi}^* &= s_1 \otimes \psi_1^* + s_2 \otimes \psi_2^*,\end{aligned}$$

then

$$\begin{aligned}tr_2(\tilde{\psi}^*\tilde{\psi}) &= s_1 \otimes s^1\psi_1^*\psi_1 + s_1 \otimes s^2\psi_1^*\psi_2 + s_2 \otimes s^1\psi_2^*\psi_1 + s_2 \otimes s^2\psi_2^*\psi_2 \\ &= \psi_1^*\psi_1 + \psi_2^*\psi_2 = \psi^*\psi.\end{aligned}$$

On one hand, multiplying $\tilde{\psi}^*\tilde{\psi}$ by ${G_Q}^{-1}$ yields

$$\begin{aligned}\tilde{\psi}^*\tilde{\psi} &= G_Q b^* Rb G_Q \\ &= G_Q(b^*bG_Q - b^*QG_QQ^*bG_Q) \\ &= {G_Q}^2 - \frac{1}{2}\sum_{i=1}^4 G_Q b^* Q\rho(\partial_i)\partial_i G_Q.\end{aligned}$$

On the other hand, for each k, we have

$$\begin{aligned}\partial_i^2({G_Q}^{-1}G_Q) &= \partial_i(2(a_i + x_i)G_Q) + \partial_i({G_Q}^{-1}\partial_i G_Q) \\ &= 2G_Q + 4(a_i + x_i)\partial_i G_Q + {G_Q}^{-1}\partial_i^2 G_Q.\end{aligned}$$

As $\partial^2 = \sum_{i=1}^4 \partial_i^2$, those equalities sum up to

$$\partial^2 G_Q = -8{G_Q}^2 - 4\sum_{i=1}^4 G_Q(a_i + x_i)\partial_i G_Q. \tag{1.31}$$

We hence have

$$\begin{aligned}
\psi^*\psi &= tr_2(\tilde{\psi}^*\tilde{\psi}) \\
&= tr_2({G_Q}^2 - \frac{1}{2}\sum_{i=1}^4 G_Q b^* Q\rho(\partial_i)\partial_i G_Q) \\
&= 2{g_Q}^2 - \frac{tr_2}{2}(\sum_{1\le i,j\le 4} G_Q(a_j + x_j)\rho(\partial_j)\rho(\partial_i)\partial_i G_Q) \\
&= 2{g_Q}^2 + \sum_{i=1}^4 g_Q(a_i + x_i)\partial_i g_Q - \frac{tr_2}{2}(G_Q\sum_{i\ne j}(a_j + x_j)\rho(\partial_j)\rho(\partial_i)\partial_i G_Q).
\end{aligned}$$

The tr_2 part of this last line cancels as for $j \ne k$ we have $\rho(\partial_j)\rho(\partial_i) = -\rho(\partial_i)\rho(\partial_j)$ while $tr_2(\rho(\partial_j)\rho(\partial_i)) = tr_2(\rho(\partial_i)\rho(\partial_j))$. Equation (1.30) then follows from this computation and Equation (1.31). The proof of the lemma is now complete. □

Using that lemma, we show that ψ is an isomorphism. Recall from Lemma 1.3-2 that we have the asymptotic behavior $g_Q = r^{-2} + O(r^{-3})$ as $r \to \infty$, and more to the point, it is so that $\partial_r g_Q = -2r^{-3} + O(r^{-4})$ as $r \to \infty$. We then have

$$\begin{aligned}
\int_{\mathbb{R}^4} \psi^*\psi &= -\frac{1}{4}\int_{\mathbb{R}^4} \partial^2 g_Q \\
&= -\lim_{r\to\infty} \frac{1}{4}\int_{S^3(r)} \partial_r g_Q \\
&= \frac{1}{2} Vol(S^3) id_V \\
&= \pi^2 id_V.
\end{aligned}$$

Thus an orthonormal basis $v_1, \dots, v_k$ of V gives an orthonormal basis $\pi^{-1}\psi(v_1), \dots, \pi^{-1}\psi(v_k)$ of V'. Indeed, as $\psi = \sum_j \psi(v_j)v^j$, we have $\psi^* = \sum_j v_j \otimes \langle\psi(v_j), \cdot\rangle$ hence pointwise we have $\psi^*\psi = \sum_{i,j}\langle\psi(v_j), \psi(v_i)\rangle v_j \otimes v^i$, or once we integrate,

$$\int_{\mathbb{R}^4} \psi^*\psi = \sum_{1\le i,j\le 4} \langle\psi(v_j), \psi(v_i)\rangle_{L^2}\, v_j \otimes v^i.$$

Remembering that the $\psi(v_j)$ do not have norm 1, we compute the endomorphism a'_μ of V':

$$a'_\mu = \frac{1}{\pi^2}\sum_{1\le i,j\le 4} \langle\psi(v_j), -x_\mu\psi(v_i)\rangle_{L^2}\, v_j \otimes v^i = -\frac{1}{\pi^2}\int_{\mathbb{R}^4} x_\mu \psi^*\psi.$$

Using from Lemma 1.3-2 the asymptotic knowledge that

$$\partial_r g_Q = -2r^{-3} + 6\sum_{j=1}^4 x_j a_j r^{-5} + O(r^{-5}),$$

we find that

$$\begin{aligned}
\int_{\mathbb{R}^4} x_\mu \psi^* \psi &= -\frac{1}{4}\int_{\mathbb{R}^4} x_\mu \partial^2 g_Q \\
&= \frac{1}{4}\lim_{r\to\infty}\int_{S^3(r)} \partial_r(x_\mu) g_Q - x_\mu \partial_r g_Q \\
&= \frac{1}{4}\lim_{r\to\infty}\int_{S^3(r)} \frac{x_\mu}{r}\big(r^{-2} - 2\sum_{j=1}^4 x_j a_j r^{-4} + O(r^{-4})\big) \\
&\qquad\qquad + 2x_\mu r^{-3} - 6\sum_{j=1}^4 x_\mu x_j a_j r^{-5} + O(r^{-5}) \big) \\
&= \frac{1}{4}\lim_{r\to\infty}\Big(3r\int_{S^3(1)} x_j - 8\sum_{j=1}^4 a_j \int_{S^3(1)} x_\mu x_j + O(r^{-1})\Big) \\
&= -2a_\mu \int_{S^3} x_j^2 \\
&= -2a_\mu Vol(S^3)/4 \\
&= -\pi^2 a_\mu .
\end{aligned}$$

Hence we obtain back the same maps, $a'_\mu = a_\mu$.

1.5 Completeness

We now close this chapter by proving that every instanton arise from some ADHM data. In other words, we prove in this section that the composition

$$\begin{matrix} \text{instanton} \\ (E, A) \end{matrix} \xrightarrow[\mathfrak{N}]{\text{Sect.1.2}} \begin{matrix} \text{ADHM data} \\ (V', W', a', \Psi') \end{matrix} \xrightarrow[\mathfrak{F}]{\text{Sect.1.3}} \begin{matrix} \text{instanton} \\ (E', A') \end{matrix}$$

gives an instanton (E', A') gauge equivalent to (E, A).

This last fact establishes the validity of Theorem 1.1-1.

Since $(E', A') = \mathfrak{F}(V, W, a, \Psi)$, then E'_x sits in $S^- \otimes V \oplus W$ as $\ker(Q^*_x)$. Elements of $S^- \otimes V$, once contracted using the skewform ω, give sections of E. Hence the map

$$\begin{aligned}
\alpha_x \colon S^- \otimes V \oplus W &\to E_x \\
\begin{bmatrix} \psi \\ \phi \end{bmatrix} &\mapsto \omega G_A \psi(x) + \frac{1}{2}\phi(x)
\end{aligned}$$

is well defined. Its adjoint α^*_x gives the map we want between E and E'. To prove that fact, we need to show that $Q^*_x \alpha^*_x = 0$, or equivalently $\alpha_x Q_x = 0$. For any $s \otimes \psi \in S^+ \otimes V$, we have

$$\begin{aligned}
\alpha_x Q_x(s \otimes \psi) &= \sum_{j=1}^4 \omega(\rho(\partial_j) s \otimes G_A(a_j + x_j)\psi)(x) + \frac{1}{2}\Phi(s \otimes \psi)(x) \\
&= \omega\big(s, \sum_{j=1}^4 \rho(\partial_j) G_A(a_j + x_j)\psi)(x) + \frac{1}{2}\Psi(\psi)(x)\big),
\end{aligned}$$

hence it suffices to prove that for every $\psi \in V$,

$$\sum_{j=1}^{4} \rho(\partial_j) G_A(a_j + x_j)\psi)(x) + \frac{1}{2}\Psi(\psi)(x) = 0. \tag{1.32}$$

Since V sits in $L^2_{-3+\epsilon}$ for all small ϵ, it must be that $\rho(\partial_j)G_A a_j \psi \in L^2_{-1+\epsilon}$. Equation (1.9) guarantees that

$$\begin{aligned} G_A\psi &= \frac{r^2\psi}{4} + O(r^{-2}) \\ &= \frac{\rho(x)}{2r^2}\Psi(\psi) + O(r^{-2}) \end{aligned}$$

hence $\rho(x)G_A\psi + (1/2)\Psi(\psi) \in L^2_{-1+\epsilon}$ as well. Hence the left-hand-side of Equation (1.32) is all in $L^2_{-1+\epsilon}$, and thus must be 0 if harmonic.

Applying Δ kills the $\Psi(\psi)$ term, and for the rest we obtain

$$\begin{aligned} \Delta\Big(\sum_{j=1}^{4} \rho(\partial_j) G_A(a_j + x_j)\psi)(x)\Big) &= \sum_{j=1}^{4} \rho(\partial_j) a_j \psi + \rho(\partial_j) m_j \psi - 2\rho(\partial_j)\nabla_A^j G_A\psi \\ &= \Big(\sum_{j=1}^{4} \rho(\partial_j) D_A G_A D_A^* m_j \psi\Big) - 2D_A^* G_A\psi \\ &= \Big(\sum_{j=1}^{4} \rho(\partial_j)\Big(\sum_{k=1}^{4} \rho(\partial_i)\nabla_A^i\Big)\rho(\partial_j) G_A\psi\Big) - 2D_A^* G_A\psi, \end{aligned}$$

which is 0 since

$$\sum_{j=1}^{4} \rho(\partial_j)\rho(\partial_i)\rho(\partial_j) = 2\rho(\partial_i).$$

The validity of Equation (1.32) is now established, and so is the fact that α^* maps E to E'. Proving that $(\alpha^*)^* A'$ is gauge equivalent to A is an exercise in rewriting [KN90, Section 6a]. It is left to the author and the reader for further study.

Chapter 2

The Nahm transform heuristic

Heuristically, the Nahm transform places instantons on $\mathbb{R}^4/\Lambda$ in reciprocity with certain data on the dual $\mathbb{R}^{4^*}/\Lambda^*$. This short chapter describes in more details this heuristic.

Let Λ be a closed subgroup of $\mathbb{R}^4$. We associate two $\mathbb{R}$-vector spaces to Λ:

$$\begin{aligned} \Lambda_{\mathbb{R}} &:= \text{ maximal } \mathbb{R}\text{-linear subspace of } \Lambda, \\ \Lambda_{\mathbb{Z}} &:= \text{ orthogonal complement of } \Lambda_{\mathbb{R}} \text{ in } \Lambda, \\ \mathfrak{L}(\Lambda) &:= \mathbb{R}\text{-vector subspace of } \mathbb{R}^4 \text{ generated by } \Lambda. \end{aligned}$$

Obviously, Λ is isomorphic to some $\mathbb{R}^r \times \mathbb{Z}^s$ with $r + s \leq 4$, and then $\dim \Lambda_{\mathbb{R}} := r$, $\dim \Lambda_{\mathbb{Z}} := s$, and $\dim \mathfrak{L}(\Lambda) := r + s$.

The dual Λ^* is defined to be

$$\Lambda^* := \{z \in \mathbb{R}^{4^*} \mid z(\Lambda) \subset \mathbb{Z}\}.$$

Obviously, $\dim \Lambda^*_{\mathbb{R}} = 4 - r - s$, $\dim \Lambda^*_{\mathbb{Z}} = s$, and $\dim \mathfrak{L}(\Lambda^*) = 4 - r$.

We start with a $\mathrm{SU}(n)$-bundle E over $\mathbb{R}^4$ and a $\mathrm{SU}(n)$-connection A on E, both invariant under the action of Λ. We require the curvature of A, denoted F_A, to be anti-self-dual (ASD) and to have finite L^2-norm on the quotient $\mathbb{R}^4/\Lambda$. Equivalently, we start with a vector bundle E on $\mathbb{R}^4/\Lambda$, and a connection A and endomorphisms $a_1, \dots, a_r$ of E, such that $(A, a_1, \dots, a_r)$ satisfies the $(4-r)$-dimensional reduction of the ASD equation, as given in Appendix A, and such that its L^2-energy, to be defined appropriately, is finite.

It might happen that the finite L^2-norm condition is too strong to get any interesting solutions, in which case we need to search for a better condition. This need arises for example on $\mathbb{R}^2 \times S^1$, and Cherkis–Kapustin give in [CK98] an appropriate logarithmic decay condition for the endomorphism a_1. Anyhow, we are exploring a heuristic for studying instantons or their various dimensional reductions, not a precise recipe, and adjustments need to be made in many cases.

Suppose now z is an element of $\mathbb{R}^{4^*}$, the space of $\mathbb{R}$-valued linear functions on $\mathbb{R}^4$. We define the bundle L_z over $\mathbb{R}^4$ to be a trivial $\mathbb{C}$-bundle with connection

$$\omega_z := 2\pi i z = 2\pi i \sum_{j=1}^{4} z_j dx^j.$$

Notice that L_z is invariant under the action of Λ, and that it is flat. Furthermore, whenever $z' \in \Lambda^*$, the bundles with connections L_z and $L_{z+z'}$ are isomorphic over $\mathbb{R}^4/\Lambda$, or equivalently we can parameterize flat connections over $\mathbb{R}^4/\Lambda$ by elements of $\mathbb{R}^{4*}/\Lambda^*$.

Indeed, for $z \in \mathbb{R}^{4*}$, define the function $g_z\colon \mathbb{R}^4 \to \mathrm{U}(1)$ by

$$g_z(x) = e^{-2\pi i z(x)},$$

and notice that g_z is invariant under the action of Λ for each $z \in \Lambda^*$. But more importantly, we have

$$g_z \cdot \omega_{z'} = \omega_{z'-z}.$$

We write A_z for the connection $A \otimes 1 + 1 \otimes \omega_z$ on $E \otimes L_z = E$. For $z \in \mathbb{R}^{4*}$, consider the operator

$$D^*_{A_z}\colon \Gamma(\mathbb{R}^4, S^- \otimes E \otimes L_z) \to \Gamma(\mathbb{R}^4, S^+ \otimes E \otimes L_z).$$

A section of the bundle $S^- \otimes E \otimes L_z$ is said to be in L^2_Λ if it is invariant under the action of Λ and if its L^2-norm over $\mathbb{R}^4/\Lambda$ is finite.

We set

$$V_z := L^2_\Lambda \cap \ker(D^*_{A_z}).$$

By putting some restrictions on A, for example that (E, A) has no flat factor, we ensure via the use of the Weitzenbock formula that $L^2_\Lambda \cap \ker(D_{A_z}) = \{0\}$. At this point, we need to prove that $D^*_{A_z}$ is Fredholm to prove that V is a bundle, and compute the index of $D^*_{A_z}$ to find its rank.

It turns out in many cases that $D^*_{A_z}$ is not Fredholm for every z, which is a good thing. Without going into details, suppose for example that $D^*_{A_z}$ was Fredholm everywhere when $\Lambda = \mathbb{Z}^3$. As we explore in this present thesis, the object created by the Nahm transform would be a monopole over T^3. But as one can show (see [Pau98, Prop. 1]), monopoles over compact 3-manifolds are not very interesting.

Notice that for any section ϕ of $S^- \otimes E$, we have

$$\begin{aligned} D^*_{A_z}(g_z\phi) &= g_z(D^*_A\phi + 2\pi i cl(z)\phi) + cl(\mathit{grad}\; g_z)\phi \\ &= g_z D^*_A \phi. \end{aligned}$$

Then for all $z' \in \Lambda^*$, we have an isomorphism

$$g_{z'}\colon V_z \to V_{z+z'}, \tag{2.1}$$

hence V is a bundle over $\mathbb{R}^{4*}/\Lambda^*$.

It is important to keep two points of view in parallel, the full $\mathbb{R}^{4*}$ view and the quotient $\mathbb{R}^{4*}/\Lambda^*$ view. In the first view, we perform a curvature computation and observe how far the curvature of the connection B we introduce on V is from being anti-self-dual. In the second view, we can sometime reduce dimension, as in the $\mathbb{R}^4$-ADHM case.

Let's stick to the $\mathbb{R}^{4*}$ point of view for now. We define a connection B on V. Each fiber V_z is in fact contained in $L^2_\Lambda(S^- \otimes E)$. We can then consider the trivial connection d^z in the trivial bundle of fiber $L^2_\Lambda(S^- \otimes E)$, and its projection Pd^z to V.

The operator $D^*_{A_z} D_{A_z}$ should be invertible, and we use the Green's operator $G_{A_z} = (D^*_{A_z} D_{A_z})^{-1}$ to define the projection P by the formula

$$P = 1 - D_{A_z} G_{A_z} D^*_{A_z}.$$

To help ease the notation, set $\Omega := 2\pi i \sum_{j=1}^{4} cl(dx^j)dz^j$.
Let's now compute the curvature of B. Notice that

$$[d^z, D_{A_z}] = [d^z, D_A + 2\pi i \sum_{j=1}^{4} z_j cl(dx^j)] = \Omega,$$

and similarly for $D^*_{A_z}$.
The Leibnitz's rule tells us that

$$d^z \langle P d^z \phi, \psi \rangle = \langle d^z P d^z \phi, \psi \rangle - \langle P d^z \phi, d^z \psi \rangle,$$

but as $\langle P d^z \phi, \psi \rangle = \langle d^z \phi, \psi \rangle$, we also have

$$\begin{aligned} d^z \langle P d^z \phi, \psi \rangle &= d^z \langle d^z \phi, \psi \rangle \\ &= \langle d^{z2} \phi, \psi \rangle - \langle d^z \phi, d^z \psi \rangle \\ &= -\langle d^z \phi, d^z \psi \rangle, \end{aligned}$$

hence the curvature F_B can be computed as follows:

$$\begin{aligned} \langle (P d^z)^2 \phi, \psi \rangle &= \langle d^z P d^z \phi, \psi \rangle \\ &= \langle P d^z \phi, d^z \psi \rangle - \langle d^z \phi, d^z \psi \rangle \\ &= -\langle D_{A_z} G_{A_z} D_{A_z} d^z \phi, d^z \psi \rangle \\ &= \langle D_{A_z} G_{A_z} \Omega \phi, d^z \psi \rangle. \end{aligned}$$

Let ν be the normal vector field to $S^{r-1}(R) \times T^s$. The integration by parts necessary to bring D on the right-hand-side of the scalar product introduces a boundary term

$$\lim_\partial := \lim_{R \to \infty} \int_{S^{r-1}(R) \times T^s} \langle cl(\nu) G \Omega \phi, d^z \psi \rangle. \tag{2.2}$$

Performing the said integration by parts, we obtain

$$\begin{aligned} \langle F_B \phi, \psi \rangle &= \langle G_{A_z} \Omega \phi, D_{A_z} d^z \psi \rangle + \lim_\partial \\ &= -\langle G_{A_z} \Omega \phi, \Omega \psi \rangle + \lim_\partial \\ &= \langle G_{A_z} \phi, \Omega \wedge \Omega \psi \rangle + \lim_\partial. \end{aligned}$$

The first term is ASD since

$$\begin{aligned} \Omega \wedge \Omega &= -4\pi^2 \sum_{1 \le i,j \le 4} cl(dx^i \wedge dx^j) dz^i \wedge dz^j \\ &= -4\pi^2 \sum_{j=1}^{3} \big(cl(\epsilon_j)\epsilon_j + cl(\bar{\epsilon}_j)\bar{\epsilon}_j \big). \end{aligned}$$

and $\bigwedge^-$ acts on $S^- \otimes E$. (Remember from page 25 that the ϵ_j and $\bar{\epsilon}_j$ are the usual basis of $\bigwedge^+$ and $\bigwedge^-$ respectively.)
The key to the ADHMN construction of instantons on $\mathbb{R}^4$ as portrayed in Chapter 1 is to get a good grip of the boundary term $\lim_\partial$, which is precisely the role of Φ and the ADHM condition (1.3).

Let's now see how the heuristic described so far can lead to the ADHMN construction for $\mathbb{R}^4$, and explore what happen in the second point of view, where we look at everything on the quotient $\mathbb{R}^{4*}/\Lambda^*$.

Of course, our first task is now to interpret the connection B in that new setting. It passes really well to the quotient by $\Lambda^*_{\mathbb{Z}}$, the difficulty lies in quotienting by the remaining $\Lambda^*_{\mathbb{R}}$.

Suppose we have coordinates $(x_1, \ldots, x_4)$ on $\mathbb{R}^4$ and associated coordinates $(z_1, \ldots, z_4)$ on $\mathbb{R}^{4*}$ such that

$$\Lambda^*_{\mathbb{R}} = \{z_1 = \cdots = z_{r+s} = 0\}.$$

Let's introduce new coordinates $(u_1, \ldots, u_{r+s}, v_1, \ldots, v_{4-r-s}) = (z_1, \ldots, z_4)$. Using the gauge transformation of Equation (2.1), we regard the space V_u as $V_{(u,0)}$. We go back to the other point of view using the isomorphism

$$g_v \colon V_u \to V_{(u,v)}.$$

Suppose ϕ_u is a section of V on $\mathbb{R}^{4*}/\Lambda^*$. Then

$$\begin{aligned} g_v^{-1} B(g_v \phi_u) &= g_v^{-1} P\big((d^u + d^v) g_v \phi_u\big) \\ &= P d^u \phi_u - 2\pi i \sum_{j=r+s+1}^{4} P m_{x_j}(\phi_u). \end{aligned}$$

Hence the connection matrices B_j for $r + s + 1 \leq j \leq 4$ get replaced by $-2\pi i P m_{x_j}$. Of course, m_{x_j} is multiplication by x_j, as in Equation (1.6).

The heuristic presented in this chapter therefore allows us to say that, to any connection A on a bundle E over $\mathbb{R}^4/\Lambda$ satisfying the appropriate dimensional reduction (see Appendix A) of the ASD equation, we can associate the following objects:

1. a family V of vector spaces over $\mathbb{R}^{4*}/\Lambda^*$ defined by the kernel $V_z = \ker(D^*_{A_z})$ of the Dirac operator lifted to $\mathbb{R}^4$, family which forms a bundle over the open set on which $D^*_{A_z}$ is Fredholm;

2. a connection B on V defined by the projection Pd^z of the trivial connection d^z on $\mathbb{R}^{4*}/\Lambda^*$;

3. a family of endomorphisms a_α, as many as $\dim(\mathfrak{L}(\Lambda)^\perp) = \dim \Lambda^*_{\mathbb{R}}$, of V, defined by the projection $-2\pi i P m_{x_\alpha}$ of the multiplication m_{x_α} by the coordinate x_α on $\mathbb{R}^4$.

When the boundary term of Equation (2.2) is 0, the $(B, \{a_\alpha\})$ satisfies the appropriate dimensional reduction of the ASD equation on $\mathbb{R}^{4*}/\Lambda^*$. When it is not, further ad hoc analysis is required.

The various dimensional reductions of the ASD equations are presented in Appendix A.

Chapter 3

Dirac Spectrum of Product Manifolds

In this chapter, we compute the spectrum of the Dirac operator of manifolds which are products, using the spectrum of the Dirac operator on each factor.

The spectrum of the Dirac operator has been computed in many cases: spheres in [Tra93], three-dimensional Berger spheres in [Hit74], odd-dimensional Berger spheres in [Bär96], flat manifolds in [Pfä00], tori in [Fri84], simply connected Lie groups in [Feg87], fibrations over S^1 in [Kra01], etc. To learn more about the current state of affairs related to spectra and eigenvalue estimates for Dirac operator, see the survey paper [Bär00].

The explicit formula of Theorem 3.2-1 for the spectrum of the Dirac operator on a general product manifold seems to be missing in the literature.

As a special case, we compute in theorem 3.3-1 the spectrum of the Dirac operator on the n-dimensional torus. Our computation confirms the result in [Fri84].

In Section 3.1, we construct the spinor bundles on $M \times N$ from the spinor bundles on the factor. This section is somewhat inspired by [Kli]. In Section 3.2, we compute the Dirac spectrum of product manifolds $M \times N$. In Section 3.3, we use the acquired knowledge to compute the Dirac spectrum for a torus. In Section 3.4, we consider the special case where $N = S^1$ and we twist the spinor bundle by a flat line bundle on S^1. We extend the result to the torus.

For the reader interested in a shortcut to the main result of this thesis, most of this chapter can be skipped, only Theorem 3.4-1 and the remarks that follow it are necessary.

3.1 Complex Spinor Bundles of $M \times N$

Let M and N be two manifolds equipped with spinor bundles S_M and S_N respectively. Let p_1, p_2 be the projections on the first and second factor of $M \times N$. In this section, we construct a spinor bundle S for $M \times N$ using S_M and S_N.

For a vector v tangent to M, let ρ_v and c_v denote the Clifford multiplication on S_M and S respectively. For a vector w tangent to N, we similarly use ρ_w and c_w.

Suppose **at least one** of the manifolds, say M, is **even-dimensional**. Thus S_M splits as $S_M^+ \oplus S_M^-$. In that case, we set

$$S := p_1^* S_M \otimes p_2^* S_N,$$

and set $S_\pm := p_1^* S_M^\pm \otimes p_2^* S_N$.

Suppose on the contrary that **both** manifolds are **odd-dimensional**. In that case, we set

$$S := (p_1^* S_M \otimes p_2^* S_N) \oplus (p_1^* S_M \otimes p_2^* S_N).$$

Let S_+ denote the first factor, and S_- the second.
In both cases, with respect to the decomposition $S = S_+ \oplus S_-$, set

$$c_v := \begin{bmatrix} & \rho_v \otimes 1 \\ \rho_v \otimes 1 & \end{bmatrix}, \text{ and } c_w := \begin{bmatrix} 1 \otimes \rho_w & \\ & -1 \otimes \rho_w \end{bmatrix}.$$

Proposition 3.1-1. *The map c is a Clifford multiplication. The bundle S is a spinor bundle.*

Proof: First, for the Clifford multiplication. We have

$$c_w c_v = \begin{bmatrix} & -\rho_v \otimes \rho_w \\ \rho_v \otimes \rho_w & \end{bmatrix} = -c_v c_w,$$

hence $c_v c_w + c_w c_v = 0$, as wanted.
Let d_m be the dimension of a complex irreducible representation of $\mathbb{C}l_m$. From [LM89, Thm 5.8, p. 33], we have

$$d_{2k+1} = d_{2k} = 2^k.$$

Since S has the required dimension and is a Clifford bundle, it must be a spin bundle. □

Positive and Negative Spinors

At this point, a warning is necessary: the splitting $S = S_+ \oplus S_-$ on $M^m \times N^n$ when both m and n are odd is *not* the same as the splitting $S = S^+ \oplus S^-$ in terms of positive and negative spinors. The second splitting appears through the isomorphism

$$\begin{aligned} S^+ \oplus S^- &\to S_+ \oplus S_- \\ (a, b) &\mapsto (ia, a) + (b, ib), \end{aligned} \tag{3.1}$$

and its inverse

$$\begin{aligned} S_+ \oplus S_- &\to S^+ \oplus S^- \\ (a, b) &\mapsto \frac{1}{2}(b - ai, a - bi). \end{aligned} \tag{3.2}$$

Let's verify the accuracy of this last statement. The orientation class in $\mathbb{C}l_m$ is

$$\omega = \begin{cases} i^n vol & \text{for } m = 2n \\ i^{n+1} vol & \text{for } m = 2n + 1. \end{cases}$$

It satisfies $\omega^2 = 1$. We define $S^\pm = (1 \pm c(\omega))S$; see [LM89, Prop 5.15, p. 36], where his $\omega_{\mathbb{C}}$ is our ω. Note that $c(\omega)$ acts as ± 1 on $S^\pm$.
Since m is odd, we know from [LM89, Prop 5.9, p. 34] that $\rho(\omega_M)$ can be either ± 1 and that the corresponding representations are inequivalent. However, the definition of the complex spin representation is independent of which irreducible representation of $\mathbb{C}l_m$ is used; see [LM89, Prop 5.15]. For simplicity, let's fix a sign and always choose the spinor bundle for which the orientation class acts with that sign.

The orientation class is $\omega = -i\omega_M\omega_N$, and

$$c(\omega) = \begin{bmatrix} & i\rho(\omega_M)\otimes\rho(\omega_N) \\ -i\rho(\omega_M)\otimes\rho(\omega_N) & \end{bmatrix} = \begin{bmatrix} & i \\ -i & \end{bmatrix}.$$

The decomposition $S = S^+ \oplus S^-$ given by $S^+ = \{(ia, a)\}$ and $S^- = \{(b, ib)\}$ is thus accurate.

When m and n are both even, the orientation class is $\omega = \omega_M\omega_N$, and

$$c(\omega) = \begin{bmatrix} \rho(\omega_M)\otimes\rho(\omega_N) & \\ & \rho(\omega_M)\otimes\rho(\omega_N) \end{bmatrix}.$$

The splitting $S_N = S_N^+ \oplus S_N^-$ induces splittings for S_+ and S_-. Using the very evident notation coming from those splittings, we have

$$S^+ = S_{++} \oplus S_{--}, \text{ and}$$
$$S^- = S_{+-} \oplus S_{-+}.$$

3.2 Dirac Spectrum Formula

To describe the spectrum of the Dirac operator on $M \times N$, we need to work with multisets, as most eigenvalues appear with high multiplicity. For the multiset A, let $A^{\#a}$ be the union of a copies of A. Of course, the kernel is always a very special set and we need some notation for the multiplicity of 0 in the spectrum when m is even. For that purpose, set

$$k_M^\pm := \dim\ker D_M^\pm.$$

Theorem 3.2-1. *The spectrum of the Dirac operator on $M^m \times N^n$ on the spinor bundle constructed in Section 3.1 from chosen spinor bundles on M and N is given as a multiset in terms of the respective spectrum Σ_M and Σ_N by the formula*

$$\Sigma_{M\times N} = \begin{cases} \pm|\Sigma_M \times \Sigma_N|, & \textit{if } m \textit{ and } n \textit{ are odd;} \\ \pm|\Sigma_M^{>0} \times \Sigma_N| \cup (\Sigma_N)^{\#k_M^+} \cup (-\Sigma_N)^{\#k_M^-}, & \textit{if } m \textit{ is even.} \end{cases}$$

Proof: For the decomposition $S = S_+ \oplus S_-$, the Dirac operator on the spinor bundle of $M \times N$ is

$$D = \begin{bmatrix} D_N & D_M \\ D_M & -D_N \end{bmatrix}.$$

Suppose first that m and n are both odd.

Let $\{\psi_\mu\}_{\mu\in\Sigma_M}$ be a basis of eigenvectors of D_M on $L^2(S_M)$, with $D_M\psi_\mu = \mu\psi_\mu$. There might be more than one function called ψ_μ, but this abuse of notation should not cause any problems. Similarly, let $\{\phi_\nu\}_{\nu\in\Sigma_N}$ be a basis of eigenvectors of D_N for $L^2(S_N)$.

We have $L^2(S) = \mathbb{C}^2 \otimes L^2(S_M) \otimes L^2(S_N)$, and on $\mathbb{C}^2 \otimes \psi_\mu \otimes \phi_\nu$, we have

$$D = \begin{bmatrix} \nu & \mu \\ \mu & -\nu \end{bmatrix}.$$

This matrix has two eigenvectors, of respective eigenvalue $\pm\sqrt{\mu^2+\nu^2}$. The corresponding eigenvectors are respectively $[\nu \pm \sqrt{\mu^2+\nu^2} \;\; \mu]^T$ when $\mu \neq 0$ while for $\mu = 0$, the vector $[1\ 0]^T$ and $[0\ 1]^T$ correspond respectively to the eigenvalues ν and $-\nu$.

We just proved the theorem for m and n both odd.

Suppose now that m is even. As before, let $\{\psi_\mu\}$ and $\{\phi_\nu\}$ be eigenbasis for D_M and D_N. Recall that the positive and negative eigenspaces of any Dirac operator D_M (not just the spin one) are isomorphic via

$$\psi_\mu = \psi_\mu^+ + \psi_\mu^- \mapsto \psi_{-\mu} = \psi_\mu^+ - \psi_\mu^-.$$

For $\mu = 0$, we have the positive spinors $\psi_0^+ \in L^2(S_M^+)$, and the negative spinors $\psi_0^- \in L^2(S_M^-)$. We thus have a unique decomposition

$$f = f_0^+\psi_0^+ + f_0^-\psi_0^- + \sum_{\mu>0}(f_\mu + f_{-\mu})\psi_\mu^+ + \sum_{\mu>0}(f_\mu - f_{-\mu})\psi_\mu^-.$$

Note that

$$\begin{aligned} D(\psi_0^+ \otimes \phi_\nu) &= \nu\psi_0^+ \otimes \phi_\nu, \text{ and} \\ D(\psi_0^- \otimes \phi_\nu) &= -\nu\psi_0^- \otimes \phi_\nu. \end{aligned}$$

Then $\ker D_M$ contributes

$$(\Sigma_N)^{\#k_M^+} \cup (-\Sigma_N)^{\#k_M^-}.$$

Suppose now that $\mu \neq 0$. Then

$$D(\psi_\mu \otimes \phi_\nu) = \mu\psi_\mu \otimes \phi_\nu + \nu\psi_{-\mu} \otimes \phi_\nu.$$

Thus, for $\mu > 0$, the Dirac operator acts on the span of $\psi_\mu \otimes \phi_\nu$ and $\psi_{-\mu} \otimes \phi_\nu$ as

$$\begin{bmatrix} \mu & \nu \\ \nu & -\mu \end{bmatrix}.$$

This matrix has two eigenvectors, of respective eigenvalue $\pm\sqrt{\mu^2+\nu^2}$.

We just proved the theorem for m even. The proof is now complete. □

Corollary 3.2-2. *When both m and n are odd, we have*

$$\ker(D^+) \textit{ is isomorphic to } \ker(D^-).$$

When both m and n are even, we have

$$\begin{aligned} k_{M\times N}^+ &= k_M^+ k_N^+ + k_M^- k_N^-, \textit{ and} \\ k_{M\times N}^- &= k_M^+ k_N^- + k_M^- k_N^+. \end{aligned}$$

Proof: When m and n are both odd, a basis of $\ker(D)$ is given by all the

$$\begin{bmatrix} i \\ 1 \end{bmatrix} \otimes \psi_0 \otimes \phi_0 \text{ (sections of } S^+\text{), and } \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \psi_0 \otimes \phi_0 \text{ (sections of } S^-\text{).}$$

When m and n are both even, a basis of $\ker(D)$ is given by all the

$$\psi_0^+ \otimes \phi_0^+, \psi_0^- \otimes \phi_0^- \text{ (sections of } S^+\text{)},$$
$$\psi_0^+ \otimes \phi_0^-, \psi_0^- \otimes \phi_0^+ \text{ (sections of } S^-\text{)}.$$

The proof is now complete. □

Corollary 3.2-3. *The index of D^+ on an even-dimensional product $M^m \times N^n$ is given by*

$$ind(M \times N) = \begin{cases} 0, & \text{if } m \text{ and } n \text{ are both odd;} \\ ind(M) \cdot ind(N), & \text{if } m \text{ and } n \text{ are both even.} \end{cases}$$

3.3 Dirac Spectrum of T^n

The work we have done so far allows us to compute the spectrum of the Dirac operator on the n-torus $T^n = \mathbb{R}^n/\Lambda$.

Recall first that there are two possibilities for the spinor bundle S_1, the trivial, denoted $\mathbb{S}_0$, and the nontrivial, denoted $\mathbb{S}_1$. In either case, we have $\rho_\theta = i$.

Let $\mathbb{S}_{\epsilon_1 \cdots \epsilon_n}$ denote the spin structure constructed inductively using the procedure given in Section 3.1. Let $b_1^*, \ldots, b_n^*$ be a basis for the lattice

$$\Lambda^* := \{\lambda^* \in \mathbb{R}^* \mid \lambda^*(\Lambda) \subset \mathbb{Z}\}$$

dual to the lattice Λ defining T^n.

Theorem 3.3-1. *The Dirac Spectrum for the spin structure $\mathbb{S}_{\epsilon_1 \cdots \epsilon_n}$ on the torus $T^n = \mathbb{R}^n/\Lambda$ is the multiset of all the*

$$\pm 2\pi |b^* + \sum \epsilon_j b_j^*/2|,$$

for $b^* \in \Lambda^*$ given with multiplicity $2^{\lfloor n/2 \rfloor - 1}$.

Proof: Note that we can rewrite this theorem as saying that the spectrum is

$$\pm |\Sigma_{S^1} \times \cdots \times \Sigma_{S^1}|^{\# 2^{\lfloor n/2 \rfloor - 1}}.$$

When $n = 1$ and S^1 has length ℓ, we have

$$\Sigma_{S^1} = (2\pi/\ell)(\epsilon/2 + \mathbb{Z}).$$

The factor $1/2$ counterbalances the $\pm$ as $-\Sigma_{S^1} = \Sigma_{S^1}$.

In fact, we will use the more general fact that $-\Sigma_{T^n} = \Sigma_{T^n}$, and that $k_{T^{2k}}^+ = k_{T^{2k}}^-$.

Suppose now that $n = 2k+1$. Then $T^n = T^{2k} \times S^1$. From Theorem 3.2-1, we know the spectrum is

$$\pm |\Sigma_{T^{2k}}^{>0} \times \Sigma_{S^1}| \cup (\Sigma_{S^1})^{\# k_{T^{2k}}^+} \cup (-\Sigma_{S^1})^{\# k_{T^{2k}}^-},$$

which is

$$\pm |\Sigma_{T^{2k}} \times \Sigma_{S^1}|^{\# \frac{1}{2}}.$$

By induction, this multiset is

$$\pm\big|\pm|\Sigma_{S^1}\times\cdots\times\Sigma_{S^1}|\times\Sigma_{S^1}\big|^{\#\frac{1}{2}2^{k-1}}=\pm\big|\Sigma_{S^1}\times\cdots\times\Sigma_{S^1}\times\Sigma_{S^1}\big|^{\#2^{k-1}},$$

as wanted.

Suppose now that $n=2k$. Then $T^n=T^{2k-1}\times S^1$, and using Theorem 3.2-1, we find that this spectrum is

$$\begin{aligned}\pm|\Sigma_{T^{2k-1}}\times\Sigma_{S^1}|&=\pm\big|\pm|\Sigma_{S^1}\times\cdots\times\Sigma_{S^1}|\times\Sigma_{S^1}\big|^{\#2^{k-2}}\\&=\pm\big|\Sigma_{S^1}\times\cdots\times\Sigma_{S^1}\times\Sigma_{S^1}\big|^{\#2^{k-1}},\end{aligned}$$

as wanted.

The proof is now complete. □

3.4 Tensoring by L_z

Suppose now we change the Clifford bundle, tensoring it by the flat bundle L_z on S^1, which is trivial with connection $2\pi iz\,d\theta$. Since it is constant in the M direction, it doesn't affect Σ_M.

Whether m is odd or even, the new Dirac operator is

$$D=\begin{bmatrix}D_{S^1}-2\pi z & D_M\\ D_M & -(D_{S^1}-2\pi z)\end{bmatrix},$$

hence we just need to shift the eigenvalues of D_{S^1} by $-2\pi z$.

For m odd and even, the new spectrum is respectively

$$\pm|\Sigma_M\times(\Sigma_{S^1}-2\pi z)|,\ \text{and}$$
$$\pm|\Sigma_M^{>0}\times(\Sigma_{S^1}-2\pi z)|\cup(\Sigma_{S^1}-2\pi z)^{\#k_M^+}\cup(-\Sigma_{S^1}+2\pi z)^{\#k_M^-}.$$

Suppose now we tensor the spinor bundle on T^n by the flat bundle L_z with connection

$$2\pi i\sum_{j=1}^{n}z_j\,dx^j,$$

with $z\in\Lambda^*$. A modification of the proof used in Section 3.3 works to prove by induction that the spectrum of the Dirac operator D_z on T^n for $n\geq 2$ is

$$\Big\{\pm2\pi|b^*-z+\sum_{i=1}^{n}\epsilon_i b_i^*/2|\mid b^*\in\Lambda^*\Big\}^{\#2^{\lfloor n/2\rfloor-1}}.$$

This result implies that 0 is in the spectrum if and only if $z\in\sum_{i=1}^{n}\epsilon_i b_i^*/2+\Lambda^*$.

As we need it later on, let's summarize the situation for the three-dimensional torus T^3.

Theorem 3.4-1. *Choose the trivial spin structure S on $T^3=\mathbb{R}^3/\Lambda$. Pick $z\in\mathbb{R}^{3*}$. The spectrum of the Dirac operator on $S\otimes L_z$ is given by the multiset*

$$\pm2\pi|\Lambda^*-z|.$$

The eigenspaces coming from $0 \in \Lambda^*$ are found as follows. Notice first that $S = \mathbb{C}^2$ and that S^+ and S^- on T^2 are found using Maps (3.1) and (3.2) by the projections

$$P_+ = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}, \text{ and } P_- = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} D &= D_{T^2,z} - 2\pi z_3 P_+ + 2\pi z_3 P_- \\ &= 2\pi \begin{bmatrix} -z_2 & -(z_1 + z_3 i) \\ -(z_1 - z_3 i) & z_2 \end{bmatrix}. \end{aligned}$$

When $(z_1, z_3) = (0, 0)$, the eigenspaces are the

$$\begin{aligned} &\mathbb{C}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{of eigenvalue } -2\pi z_2, \\ &\mathbb{C}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{of eigenvalue } 2\pi z_2, \end{aligned}$$

while when $(z_1, z_3) \neq (0, 0)$, they are the

$$\mathbb{C}\begin{bmatrix} -(z_1 + z_3 i) \\ z_2 \pm |z| \end{bmatrix} \quad \text{of eigenvalue } \pm 2\pi z_2.$$

We can consider the bundle $V_{2\pi\epsilon}$ created from the eigenspace of D_z of eigenvalue $2\pi\epsilon$ on the sphere $|z| = \epsilon$. A non-zero section is obviously given by $[-(z_1 + z_3 i) \;\; z_2 + |z|]^T$. This section vanishes only at $(0, \epsilon, 0)$ and its multiplicity is obviously 1, hence $c_1(V_{2\pi\epsilon}) = e(V_{2\pi\epsilon}) = \pm 1$, depending on the choice of orientation class.

Chapter 4

Dirac Spectrum of S^n

Because of the splitting relation

$$D^{\pm}_{\mathbb{R}\times S^3} = \pm\frac{\partial}{\partial t} + D_{S^3},$$

and because of the conformal equivalence $\mathbb{R} \times S^3 \equiv \mathbb{R}^4 \setminus \{0\}$, the eigenvalues of the Dirac operator D on S^3 and the kernel of the Dirac operator on $\mathbb{R}^4$ are intimately related. So we first proceed to study the eigenvalues of D on S^3. We then exploit these results in Section 7.3 to prove the asymptotic behavior of Equation (1.7).

In Section 4.1, we compute the Spectrum of D_{S^3} in a way quite similar to Hitchin's [Hit74]. In Section 4.2, we confirm the results of Section 4.1, using a construction of Trautman for the spectrum of the Dirac operators of spheres. The drawback of Trautman's method is that it does not give easily the multiplicities, which is why we need the computations.

4.1 S^3: Spherical harmonics and representations

Let's start by writing down an explicit formula for D_{S^3} and $D^2_{S^3}$. Consider the left-invariant orthonormal frame on S^3 given by

$$\begin{aligned} e_1(x) &:= x \cdot i, \\ e_2(x) &:= x \cdot j, \text{ and} \\ e_3(x) &:= x \cdot k. \end{aligned}$$

As derivations, the e_i satisfy the commuting relations obtained by cyclicly permuting $\{1, 2, 3\}$ in the expression

$$[e_1, e_2] = 2e_3.$$

The Levi-Civita connection matrix in that orthonormal frame is

$$[\omega^a_b] = \begin{bmatrix} 0 & -e^3 & e^2 \\ e^3 & 0 & -e^1 \\ -e^2 & e^1 & 0 \end{bmatrix}.$$

The spinor bundle $S(S^3)$ of S^3 is a trivial $\mathbb{H}$-bundle. The vectors e_1, e_2 and e_3 act by Clifford multiplication on $S(S^3)$ simply by left-multiplication by i, j and $-k$ respectively, so that the volume element acts as $+1$. Thus the spin connection endomorphism is

$$\Omega = \frac{1}{2} \sum_{1 \le a < b \le 3} \omega^a_b \otimes cl(e_b) cl(e_a)$$
$$= \frac{1}{2}(e^1 \otimes i + e^2 \otimes j + e^3 \otimes k),$$

and the spin connection is $d + \Omega$. The Dirac operator hence, obeys the rule

$$D = ie_1 + je_2 - ke_3 + \frac{3}{2}.$$

Then, we have the formula

$$D^2 = -e_1e_1 - e_2e_2 - e_3e_3 + ie_1 + je_2 - ke_3 + \frac{9}{4}.$$

In this formula, the part which looks second order is actually the standard Laplacian on S^3:

$$\Delta := -e_1e_1 - e_2e_2 - e_3e_3.$$

The eigenvalues of D are distributed symmetrically with respect to 0, as we now establish.

Theorem 4.1-1. *Let $n \equiv 3 \mod 4$ and M be a riemannian manifold of dimension n. Let ϕ be an orientation-reversing isometry. Then the spectrum Σ of the Dirac operator D on the spinor bundle of M is symmetric: $\Sigma = -\Sigma$.*

Proof: In this case, the square of the Clifford volume element ω is $\omega^2 = 1$. Thus Cl_n splits as

$$Cl_n = V_+ \oplus V_-.$$

It turns out that the $V_\pm$ are invariant for the action of Cl_n.

The algebra map $\alpha\colon Cl_n \to Cl_n$, generated by $\alpha(v) = -v$ for $v \in \mathbb{R}^n$, exchange V_+ and V_-. It is an isomorphism of $Spin(n)$-representations since $Spin(n) \subset Cl^0_n$.

Note now that there is a canonical isomorphism $P_{SO}(M) \equiv \phi^* P_{SO}(M)$: on the fibers over a given point, it is given by $(e_1, \dots, e_n) \mapsto (d\phi(e_n), \dots, d\phi(e_1))$.

This isomorphism induces an isomorphism $P_{Spin}(M) \equiv \phi^* P_{Spin}(M)$.

Let $\ell\colon Spin(n) \to V_\pm$ be left-multiplication and set

$$S_\pm := P_{Spin}(M) \times_\ell V_\pm.$$

Both are "the" spinor bundle on M. They are isomorphic via α. Let's choose S_+ to work with.

Set $Cl_{Spin}(M) := P_{Spin}(M) \times_\ell Cl_n$. Then, as Clifford-modules, we have the isomorphism $\phi^* Cl_{Spin}(M) \equiv Cl_{Spin}(M)$. This isomorphism exchange $S_\pm$ and $S_\mp$.

Suppose now that $s \in \Gamma(S_+)$ and consider $\alpha(\phi^* s) \in \Gamma(S_+)$. At the point x, we have that $\alpha(\phi^* s)(x) = \alpha(s(\phi(x))) \in (S_-)_{\phi(x)} \equiv (S_+)_x$.

The connection on $Cl_{Spin}(M)$, being $1/4 \cdot \sum_{1 \le a,b \le n} \omega_{ba} e_a e_b$, is preserved by the canonical isomorphism. Thus

$$D(\alpha(\phi^* s)) = -\alpha(D(\phi^* s)) = -\alpha(\phi^*(Ds)).$$

Hence, if $Ds = \lambda s$, then $D(\alpha(\phi^* s)) = -\lambda \cdot \alpha(\phi^* s)$. The proof is now complete. □

Recall that $L^2(S^3)$ has a decomposition in eigenspaces of D and Δ; in fact, they are linked since the Laplacian commutes with our canonical basis as it is parallel:

$$[\Delta, e_a] = 0 \text{ for } a = 1, 2, 3.$$

Indeed, we have, for example, that

$$\begin{aligned}
[\Delta, e_1] &= -e_1e_1e_1 - e_2e_2e_1 - e_3e_3e_1 + e_1e_1e_1 + e_1e_2e_2 + e_1e_3e_3 \\
&= -e_2[e_2, e_1] - e_2e_1e_2 - e_3[e_3, e_1] - e_3e_1e_3 \\
&\qquad + [e_1, e_2]e_2 + e_2e_1e_2 + [e_1, e_3]e_3 + e_3e_1e_3 \\
&= 2e_2e_3 - 2e_3e_2 + 2e_3e_2 - 2e_2e_3 \\
&= 0.
\end{aligned}$$

Thus the eigenspaces are invariant under the action of $sp(1)$.

Let's review now some classical theory. Let f be a function on S^3 and F an extension of f to $\mathbb{R}^4$. Then

$$\Delta(f) = \Delta_{\mathbb{R}^4}(F) + 3\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial r^2}. \tag{4.1}$$

This decomposition is fantastically simple and allows for a complete description of the eigenvalues of Δ. Let $H_m(\mathbb{R}^4)$ denote the set of harmonic homogeneous polynomials of degree m and denote by $H_m(S^3)$ its restriction to S^3. It follows from Equation (4.1) that $H_m(S^3)$ consists of eigenvectors for the Laplacian Δ on function on S^3. The corresponding eigenvalues are $m(m+2)$. In fact, these are all the eigenvalues.

In fact, the eigenvectors of the Laplacian on S^n are always the corresponding $H_m(S^n)$ and the eigenvalues are correspondingly the $m(m+n-1)$; see [GHL90, Cor 4.49].

Since they correspond to different eigenvalues, the spaces $H_m(S^3)$ are invariant under the action of $sp(1)$. So we reduce our study of eigenvalues of D to the study of its eigenvalues on eigenspaces of Δ. For those we have the beautiful decomposition theorem that follows.

Theorem 4.1-2. *We have the following isomorphism of complex representation of the Lie algebra* $sp(1)$*:*

$$H_m(S^3; \mathbb{C}) \cong (m+1)Sym^m\mathbb{C}^2.$$

Proof: The left-invariant vector fields e_1, e_2, e_3 satisfy the same commuting relation as i, j, and k. Thus, we can view them in $su(2)$ as

$$\begin{aligned}
e_1 &\equiv \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \\
e_2 &\equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \text{ and} \\
e_3 &\equiv \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.
\end{aligned}$$

To study the representation theory of $su(2)$, it is convenient to use the standard basis H, X, and Y of sl_2 since $su(2)$ and sl_2 have the same irreducible representations. In terms of the e_a, we have

$$
\begin{aligned}
H &= -ie_1, \\
X &= \frac{1}{2}(-e_2 + ie_3), \text{ and} \\
Y &= \frac{1}{2}(e_2 + ie_3).
\end{aligned}
$$

We set $z_1 = x_1 + ix_2$, and $z_2 = x_3 + ix_4$. In these coordinates, we have

$$
\begin{aligned}
H &= -\frac{\partial}{\partial r} + 2\Big(z_1\frac{\partial}{\partial z_1} + \bar{z}_2\frac{\partial}{\partial \bar{z}_2}\Big) \\
X &= \bar{z}_2\frac{\partial}{\partial \bar{z}_1} - z_1\frac{\partial}{\partial z_2}, \\
Y &= -z_2\frac{\partial}{\partial z_1} + \bar{z}_1\frac{\partial}{\partial \bar{z}_2}, \text{ and} \\
\Delta_{\mathbb{R}^n} &= 4\Big(\frac{\partial}{\partial z_1}\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial z_2}\frac{\partial}{\partial \bar{z}_2}\Big)
\end{aligned}
$$

Consider now the $m+1$ homogeneous polynomials

$$
p_a := z_1^a \bar{z}_2^{m-a}, \quad a = 0, \dots, m.
$$

They are obviously killed by $\Delta_{\mathbb{R}^n}$ and thus are in $H_m(S^3, \mathbb{C})$.

Equally obvious is the fact that they are killed by X. Hence, each p_a generates an irreducible submodule of $H_m(S^3)$. Since

$$
\begin{aligned}
H(p_a) &= -mp_a + 2\big(ap_a + (m-a)p_a\big) \\
&= mp_a,
\end{aligned}
$$

this module is isomorphic to $Sym^m\mathbb{C}^2$ as a representation of $su(2)$.

We have so far establish the presence of $(m+1)Sym^m\mathbb{C}^2$ inside $H_m(S^3;\mathbb{C})$. Since both spaces have dimension $(m+1)^2$ (see [ABR01, p. 78, Prop. 5.8]) they must be equal. □

Now consider the isomorphism between $\mathbb{H}$ and $\mathbb{C}^2$ given by the natural decomposition $z_1 + jz_2$. This isomorphism induces an isomorphism between $Sp(1)$ and $SU(2)$ as follows:

$$
\begin{aligned}
&Sp(1) \equiv SU(2) \\
&\quad i \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \\
&\quad j \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \\
&\quad k \mapsto \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.
\end{aligned}
$$

Basically, the operator D restricts to a set of operators, one for every m:

$$\begin{aligned} H_m(S^3;\mathbb{C})^2 &\to H_m(S^3;\mathbb{C})^2 \\ \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} &\mapsto \begin{bmatrix} ie_1 & -e_2+ie_3 \\ e_2+ie_3 & -ie_1 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \end{aligned}$$

Following Hitchin, we let Q denote the operator

$$Q := \begin{bmatrix} -H & 2X \\ 2Y & H \end{bmatrix}.$$

The operator D is thus the sum $Q + 3/2$ on the invariant subspace $(Sym^m\mathbb{C}^2)^2$. Let's find the eigenvalues of Q on this subspace.

Let x, y be the standard basis of $\mathbb{C}^2$. Then $Sym^m\mathbb{C}^2$ is the set of homogeneous polynomials of degree m in x and y. As such, there is an obvious basis $h_a := x^a y^{m-a}$ for $Sym^m\mathbb{C}^2$ and we have

$$\begin{aligned} H(h_a) &= (2a-m)h_a, \\ X(h_a) &= (m-a)h_{a+1}, \text{ and} \\ Y(h_a) &= ah_{a-1}. \end{aligned}$$

The vectors

$$\begin{bmatrix} h_0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ h_m \end{bmatrix}$$

are eigenvectors of Q, of eigenvalues m. Each of them appear with multiplicity $m+1$ in the space $H_m(S^3;\mathbb{C})^2$. Consider now the vectors

$$\begin{bmatrix} (1+m-a)h_a \\ ah_{a-1} \end{bmatrix} \text{ and } \begin{bmatrix} h_a \\ -h_{a-1} \end{bmatrix}.$$

These $2m$ vectors, along with the two others, span $(Sym^m\mathbb{C}^2)^2$. Furthermore, they are eigenvectors of Q, of respective eigenvalues m and $-m-2$. The $m+1$ diagonal $(Sym^m\mathbb{C}^2)^2$ factors in $H_m(S^3;\mathbb{C})^2$ span the whole space. Hence these eigenvalues appear with multiplicities $m(m+1)$.

We just proved the following theorem.

Theorem 4.1-3. *The eigenvalues of the Dirac operator on the spinor bundle of S^3 are the*

$$\pm(k+3/2), \textit{ for } k \in \mathbb{N},$$

each $\pm(k+3/2)$ appearing with multiplicity $(k+1)(k+2)$.

This result is confirmed by a similar computation in [Hit74, Prop 3.2] and by a different method of Andrzej Trautman in [Tra93] which apply to all spheres, and which we describe in Section 4.2.

4.2 Trautman's construction

We now confirm the results of the previous two sections by a method of Trautman which apply to all spheres. This method appeared as a first paper [Tra93] in a projected series of paper of Trautman with E. Winkowska on the spectrum of the Dirac operator on hypersurface. The promised sequel *Spinors and the Dirac operator on hypersurfaces. II. The spheres as an example* was apparently never completed and does not appear in the literature.

Let S be the spinor bundle on $\mathbb{R}^{n+1}$. Let $i\colon S^n(r) \to \mathbb{R}^{n+1}$ be the inclusion. Then $i^*(S)$ is a Clifford bundle on $S^n(r)$. On this bundle, we have a spin connection, which gives us a Dirac operator D_r. Let D be the Dirac operator on S. We now look at the relationship between D and D_r.

Let $e_1, \ldots, e_n$ be an orthonormal frame on a patch of $S^n(1)$. We can extend this frame by radial parallel transport to a cone of $\mathbb{R}^{n+1} \setminus \{0\}$. Let $e_0 = \nu$ be the radial vector field. Let $I\!I_r$ be the second fundamental form of $S(r)$. Then

$$\begin{aligned} D &= \rho(e_0)\nabla_{e_0} + \sum_{0<i\leq n} \rho(e_i)\nabla_{e_i} \\ &= \rho(\nu)\frac{\partial}{\partial r} + \sum_{0<i\leq n} \rho(e_i)e_i + \frac{1}{2}\rho(e_i) \sum_{0\leq j<k\leq n} \omega_k^j(e_i)\rho(e_k e_j) \\ &= \rho(\nu)\frac{\partial}{\partial r} + \frac{1}{2}\sum_{0<i\leq n}\sum_{0<k\leq n} \omega_k^0(e_i)\rho(e_i e_k)\rho(\nu) + D_r \\ &= \rho(\nu)\frac{\partial}{\partial r} - \frac{1}{2} tr(I\!I_r)\rho(\nu) + D_r. \end{aligned}$$

A simple computation shows that $tr(I\!I_r) = -n/r$. Thus

$$D = D_r + \rho(\nu)\frac{\partial}{\partial r} + \frac{n}{2r}\rho(\nu). \tag{4.2}$$

Let $p\colon \mathbb{R}^{n+1} \to S$ be a spinor-valued homogeneous harmonic polynomial of degree $l+1$. The polynomial Dp has degree l and is killed by D. Consider then

$$s_\pm = \frac{(1 \mp \rho(\nu))}{2} Dp.$$

We have $Ds_\pm = 0$. Since Dp is homogeneous of degree l, we have $\partial s_\pm/\partial r = (l/r)s_\pm$. Thus

$$\begin{aligned} D_r s_\pm &= -\rho(\nu)\frac{\partial s_\pm}{\partial r} - \frac{n}{2r}\rho(\nu)s_\pm \\ &= \frac{(l+n/2)}{r}\rho(-\nu)s_\pm \\ &= \pm\frac{(l+n/2)}{r}s_\pm. \end{aligned}$$

For $n = 3$, we see the spectrum described by Theorem 4.1-3.

Now, $i^*(S)$ splits as a direct sum of irreducible spinor bundles. Because of dimensional reason and because n-spheres ($n > 1$) have only one spin structure, for $n = 3$, the bundle $i^*(S)$ splits as two copies of the spinor bundle of S^3, and for $n = 2$, the bundle $i^*(S)$ is the spinor bundle of S^2.

As $i^*(S)$ splits, the eigenspaces of D_r split too. So we are building genuine eigenspaces for the Dirac operator on the spinor bundle of S^n.

Chapter 5

Decay of instantons

On a cylindrical manifold $\mathbb{R}\times Y$ with any warped product metric, the ASD equation for a connection in temporal gauge is

$$\partial_t A = - *_3 F_A^{(3)}, \tag{5.1}$$

On $\mathbb{R} \times T^3$ with coordinates (t, θ), we can expand any connection according to its Fourier modes:

$$A(\theta, t) = \sum_{\nu \in \mathbb{Z}^3} A^\nu(t) e^{i\theta\cdot\nu}.$$

While the ASD equation mixes terms from different Fourier modes, the zero-mode behaves particularly nicely. Set

$$H := \{A \mid A^\nu = 0 \text{ for } \nu \neq 0\}.$$

For the product metric, Equation (5.1) is autonomous, and it turns out that $\mathcal{G}H$ is then a center manifold for the flow of that equation. Hence, every flow line with finite energy approaches exponentially a flow line in $\mathcal{G}H$.

Since finite energy correspond here to

$$\int_{[1,\infty)\times T^3} |F_A|^2 < \infty,$$

it remains to study the flow lines in the finite dimensional space H and in order to understand the decay of instantons.

This material is well known to [MMR94], where it is proved that every instanton converges to a flat connection, and the decay to that instanton is exponential if the flat limit is irreducible.

For the warped metric on $[1, \infty) \times T^3$ coming from $T^2 \times \mathbb{R}^2$ by polar coordinate on the $\mathbb{R}^2$ factor, the Flow Equation (5.1) is not autonomous and consequently we cannot use the traditional center manifold theorem. It is worthwhile however to study the behavior of flow lines on $\mathcal{G}H$ as well.

Any element of H can be expressed as

$$A = a_1 d\theta^1 + a_2 d\theta^2 + a_3 d\theta^3,$$

with $a_j \in \mathfrak{su}(n)$.

The Flow Equation (5.1), once written with the a_j, gives rise to the equations

$$\left.\begin{aligned} a_1' &= [a_2, a_3] \\ a_2' &= [a_3, a_1] \\ a_3' &= [a_1, a_2] \end{aligned}\right\} \text{ for the product metric } \mathbb{R} \times T^3, \tag{5.2}$$

and

$$\left.\begin{aligned} a_1' &= t^{-1}[a_2, a_3] \\ a_2' &= t^{-1}[a_3, a_1] \\ a_3' &= t \quad [a_1, a_2] \end{aligned}\right\} \text{ for the warped metric } T^2 \times \mathbb{R}^2. \tag{5.3}$$

These equations are quite symmetrical, and we can reduce the study of those systems to the study of their invariants.

5.1 On $\mathbb{R} \times T^3$

Let's restrict our attention to $\mathfrak{su}(2)$, and let's first deal with System (5.2). Define the following real valued functions:

$$\begin{aligned} f &:= |a_1'|^2 + |a_2'|^2 + |a_3'|^2, \\ g &:= |a_1|^2 + |a_2|^2 + |a_3|^2, \\ d &:= \langle a_1, [a_2, a_3] \rangle. \end{aligned} \tag{5.4}$$

Lemma 5.1-1. *For a_1, a_2, a_3 flowing according to System (5.2), and for f, g, d defined by Equations (5.4), we have*

$$\begin{aligned} f' &= 16dg, \\ g' &= 6d, \\ d' &= f. \end{aligned}$$

Proof: We equip $\mathrm{SU}(2)$ with its bi-invariant metric that gives it the riemannian structure of S^3. For elements X, Y, Z, W of the Lie algebra, the covariant derivative and Riemann tensor have the simple expressions

$$\begin{aligned} \nabla_X Y &= \frac{1}{2}[X, Y], \text{ and} \\ R(X, Y, Z, W) &= \frac{1}{4}\langle [X, Y], [Z, W] \rangle; \end{aligned}$$

see for example [GHL90, 3.17].

Since the sectional curvature is 1 everywhere, we have

$$R(X, Y, Z, W) = \langle X, Z \rangle \langle Y, W \rangle - \langle Y, Z \rangle \langle X, W \rangle.$$

Notice, then, that for any permutation ijk of 123, we have

$$\langle a_i, a_k' \rangle = \pm \langle a_i, [a_i, a_j] \rangle = \pm 2 \langle a_i, \nabla_{a_i} a_j \rangle = \mp 2 \langle \nabla_{a_i} a_i, a_j \rangle = \mp \langle [a_i, a_i], a_j \rangle = 0,$$

while $d = \langle a_1, a_1' \rangle = \langle a_2, a_2' \rangle = \langle a_3, a_3' \rangle$. It is then quite obvious that $g' = 6d$.

We have

$$\begin{aligned}(|a_1'|^2)' &= 2\langle [a_2,a_3],[a_2',a_3]\rangle + 2\langle [a_2,a_3],[a_2,a_3']\rangle \\ &= 8R(a_2,a_3,a_2',a_3) + 8R(a_2,a_3,a_2,a_3') \\ &= 8\langle a_2,a_2'\rangle |a_3|^2 + 8|a_2|^2\langle a_3,a_3'\rangle \\ &= 8d(|a_3|^2+|a_2|^2).\end{aligned}$$

This equation together with similar equations for $(|a_2'|^2)'$ and $(|a_3'|^2)'$ yield $f' = 16dg$.
As for d, we have

$$\begin{aligned}d' &= \langle a_1',[a_2,a_3]\rangle + \langle a_1,[a_2',a_3]\rangle + \langle a_1,[a_2,a_3']\rangle \\ &= |a_1'|^2 - \langle [a_1,a_3],a_2'\rangle - \langle [a_2,a_1],a_3'\rangle \\ &= |a_1'|^2 + |a_2'|^2 + |a_3'|^2 = f.\end{aligned}$$

The proof is now complete. □

We are now ready to prove decay properties of instantons.

Theorem 5.1-2 (Decay of instantons). *Let A be an instanton on $\mathbb{R}\times T^3$. Then*

$$|F_A| = o(t^{-1})$$

as $t\to\infty$.

Proof: As we mentioned in the introduction to this chapter, we only have to study the flow lines in H as any other is exponentially decaying to a flow line in $\mathcal{G}H$.
Notice first the $||F_A||^2 = \int f = \int d'$ hence $\lim d$ exist as $t\to\infty$. Suppose $\lim d = 2l > 0$. For some T and $t > T$, we have $d > l$. Hence $g' > l$, or once we integrate, $g(t) > lt + C$. Thus $f' = 16dg > 16l^2t + C$ and $\lim f' = \infty$. But then surely $\lim f = \infty$ and f cannot be integrable, which clearly contradicts the finite energy condition. Hence we proved

$$\lim d \le 0.$$

In fact, as $d' = f \ge 0$, we have

$$d \le 0 \text{ always.}$$

Consequently, $f' = 16dg \le 0$. Thus f must have a finite limit since $f\ge 0$. Since $\int f$ converges, we have

$$\lim f = 0.$$

As $g \ge 0$ and $g' = 6d \le 0$, we have

$$\lim g \text{ exists.}$$

Then we judiciously apply l'Hospital's rule, denoted HR below. Since

$$(\lim g) = \lim \frac{g/t}{1/t} \overset{\mathrm{HR}}{=} \lim \frac{-g/t^2 + 6d/t}{-1/t^2} = (\lim g) - 6\lim td$$

we have

$$d = o(t^{-1}).$$

But then,

$$0 = \lim td = \lim \frac{d}{1/t} \overset{\text{HR}}{=} \lim \frac{f}{-1/t^2} = -\lim t^2 f.$$

Since $f = |F_A|^2$, the conclusion follows. □

We can actually pull out more decay from those equations, even exponential decay in [MMR94]. Here we prove polynomial decay to infinite order for non-zero limits.

Theorem 5.1-3 (Extra decay for non-zero limits). *Let A be an instanton on $\mathbb{R} \times T^3$. Suppose that the flat connection to which A converges at infinity is not gauge equivalent to $0 \in H$. Then for all k,*

$$|F_A| = o(t^{-k})$$

as $t \to \infty$.

Proof: We work in H. We already proved in Theorem 5.1-2 and its proof that $f = o(t^{-2})$ and $d = o(t^{-1})$.
Once we suppose $d = o(t^{-k})$, we find $0 = \lim d/t^{-k} = -k \lim f/t^{-k-1}$ using l'Hospital's rule. Hence

$$d = o(t^{-k}) \text{ implies } f = o(t^{-k-1}), \tag{5.5}$$

Suppose now $\lim g \neq 0$, and $f = o(t^{-k})$. Then

$$0 = \lim \frac{f}{t^{-k}} = -\frac{1}{k} \lim \frac{dg}{t^{-k-1}} \overset{\text{HR}}{=} -\frac{1}{k} (\lim g) \lim \frac{d}{t^{-k-1}}.$$

Hence

$$f = o(t^{-k}) \text{ implies } d = o(t^{-k-1}), \tag{5.6}$$

under the condition $\lim g \neq 0$.
The conclusion follows by pumping up Equations (5.5) and (5.6). □

5.2 On $T^2 \times \mathbb{R}^2$

We keep our attention on $\mathfrak{su}(2)$, and deal now with System (5.3). Define the following real valued functions:

$$\begin{aligned} f &:= |a_1'|^2 + |a_2'|^2 + \frac{1}{t^2}|a_3'|^2, \\ u &:= \frac{1}{t^2}|a_3'|^2 \\ g_1 &:= |a_1|^2 + |a_2|^2 \\ g_2 &:= |a_3|^2, \\ d &:= \langle a_1, [a_2, a_3] \rangle. \end{aligned} \tag{5.7}$$

Lemma 5.2-1. *For a_1, a_2, a_3 flowing according to System (5.3), and for f, u, g_1, g_2, d defined by*

Equations (5.7), we have

$$g_1' = \frac{4}{t}d, \qquad u' = \frac{8}{t}g_1 d,$$
$$g_2' = 2td, \qquad d' = tf$$

Proof: We proceed as in the proof of Lemma 5.1-1.
Using the Leibnitz rule, we find

$$\begin{aligned} d' &= \langle a_1', [a_2, a_3]\rangle + \langle a_1, [a_2', a_3]\rangle + \langle a_1, [a_2, a_3']\rangle \\ &= t|a_1'|^2 - \langle [a_1, a_3], a_2'\rangle - \langle [a_2, a_1], a_3'\rangle \\ &= t|a_1'|^2 + t|a_2'|^2 + \frac{1}{t}|a_3'|^2, \end{aligned}$$

hence proving $d' = tf$.
While $g_2' = 2\langle a_3, a_3'\rangle = 2td$, we have

$$\begin{aligned} g_1' &= 2\langle a_1, a_1'\rangle + 2\langle a_2, a_2'\rangle \\ &= \frac{2}{t}(\langle a_1, [a_2, a_3]\rangle + \langle a_2, [a_3, a_1]\rangle) \\ &= \frac{4}{t}d, \end{aligned}$$

thus proving the differential equations for g_1 and g_2.
We have

$$\begin{aligned} u' &= (|[a_1, a_2]|^2)' \\ &= 2\langle [a_1, a_2], [a_1', a_2]\rangle + 2\langle [a_1, a_2], [a_1, a_2']\rangle \\ &= \frac{8}{t}(R(a_1, a_2, [a_2, a_3], a_2) + R(a_1, a_2, a_1, [a_3, a_1])) \\ &= \frac{8}{t}(\langle a_1, [a_2, a_3]\rangle |a_2|^2 - \langle a_2, [a_2, a_3]\rangle\langle a_1, a_2\rangle + |a_1|^2\langle a_2, [a_3, a_1]\rangle - \langle a_2, a_1\rangle\langle a_1.[a_3, a_1]\rangle) \\ &= \frac{8}{t}g_1 d. \end{aligned}$$

The proof is now complete. □

Theorem 5.2-2. *In the gauge group translates of the zero Fourier mode on $T^2 \times \mathbb{R}$, finite energy instantons have quadratically decaying curvature.*

Proof: We of course aim to prove that $f = o(t^{-4})$.
Notice first the $||F_A||^2 = \int tf = \int d'$ hence $\lim d$ exist as $t \to \infty$.
Suppose $\lim d = 2l > 0$. For some T and $t > T$, we have $d > l$. Hence $g_1' = 4t^{-1}d > 4t^{-1}l$, or once we integrate, $g_1(t) - g_1(T) > 4l\log(t/T)$, or $g_1(t) > 4l\log(\gamma t)$ for some $\gamma > 0$.

Then $u' = 8t^{-1}g_1 d > 32l^2 t^{-1}\log(\gamma t)$. Since $\left(\log^2(\gamma t)\right)' = t^{-1}\log(\gamma t)$, we have

$$\begin{aligned} u(t) - u(T) &= \int_T^t u' \\ &> 32l^2 \int_T^t \left(\log^2(\gamma t)\right)' \\ &= 32l^2 \left(\log^2(\gamma t) - \log^2(\gamma T)\right), \end{aligned}$$

or for some constant ϵ, we get $u(t) > 32l^2\log^2(\gamma t) + \epsilon$. Hence $\lim u = \infty$, and then surely $\lim f = \infty$ and f cannot be integrable, which clearly contradicts the finite energy condition. Hence we proved

$$\lim d \leq 0.$$

In fact, as $d' = tf \geq 0$, we have

$$d \leq 0 \text{ always}.$$

Since $g_a' = 2td \leq 0$ and $g_2 \geq 0$, the quantity $\lim g_2$ exist and is finite.
Then we judiciously apply l'Hospital's rule. Since

$$(\lim g_2) = \lim \frac{g_2/t}{1/t} \overset{\text{HR}}{=} \lim \frac{-g_2/t^2 + g_2'/t'}{-1/t^2} = (\lim g) - 2\lim t^2 d$$

we have

$$d = o(t^{-2}).$$

But then,

$$0 = \lim t^2 d = \lim \frac{d}{1/t^2} \overset{\text{HR}}{=} \lim \frac{tf}{-2/t^3} = -\frac{1}{2}\lim t^4 f.$$

Since $f = |F_A|^2$, the conclusion follows. □

Theorem 5.2-2 is perhaps a first step in a dynamical system approach to proving a conjecture of Jardim that every doubly-periodic instanton connection of finite action $\|F_A\|_{L^2} < \infty$ has quadratic curvature decay. This conjecture appeared in [Jar02a, p. 433] and is supported in part by Nahm transform considerations in [BJ01].

5.3 Notes on different quotients

We proved in Section 5.1 that given an instanton A on $\mathbb{R} \times T^3$, its curvature F_A decays like $o(r^{-1})$. It was first observed by Mrowka that there are instantons who decay like r^{-1}. For example, let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the usual basis of $\mathfrak{su}(2)$, and consider

$$A = \frac{\mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz}{2r}.$$

The curvature of that connection is

$$F_A = \frac{-\mathbf{i}dr \wedge dx - \mathbf{j}dr \wedge dy - \mathbf{k}dr \wedge dz + \mathbf{k}dx \wedge dy + \mathbf{j}dz \wedge dx + \mathbf{i}dy \wedge dz}{2r^2},$$

which is quite stronger than $o(r^{-1})$.

On $\mathbb{R}^2 \times T^2$, Jardim gave the example of the connection

$$A = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \frac{d\theta}{\log r} + \frac{1}{r \log r} \begin{bmatrix} 0 & e^{-i\theta}(dx - idy) \\ -e^{i\theta}(dx + idy) & 0 \end{bmatrix}$$

with

$$F_A = O\Big(\frac{1}{r^2 \log r}\Big),$$

which again is a bit stronger than the conjectured $O(r^{-2})$ of Section 5.2.

For the classical $\mathbb{R}^4$ case, it was proved originally by Uhlenbeck in [Uhl82, Cor. 4.2] that the condition $\|F_A\|_{L^2} < \infty$ implies that $|F_A| = O(r^{-4})$; see also the appendix of the seminal work of Donaldson [Don83]. This decay is achieved by the connection

$$A = \frac{1}{1 + r^2}(\theta_1 \mathbf{i} + \theta_2 \mathbf{j} + \theta_3 \mathbf{k}),$$

with

$$\begin{aligned} \theta_1 &= x_1 dx^2 - x_2 dx^1 - x_3 dx^4 + x_4 dx^3 \\ \theta_2 &= x_1 dx^3 - x_3 dx^1 - x_4 dx^2 + x_2 dx^4 \\ \theta_3 &= x_1 dx^4 - x_4 dx^1 - x_2 dx^3 + x_3 dx^2. \end{aligned}$$

While this connection in this particular gauge is $O(r^{-1})$, its curvature

$$F_A = \frac{1}{1 + r^4}(d\theta_1 \mathbf{i} + d\theta_2 \mathbf{j} + d\theta_3 \mathbf{k})$$

is exactly of order $1/r^4$.

As for $S^1 \times \mathbb{R}^3$, by taking a monopole (A, Φ) on $\mathbb{R}^3$, we get an example of an instanton $\Phi dr + A$ whose curvature $(\nabla_A \Phi) \wedge dr + F_A$ is exactly of order $1/r^2$.

Chapter 6

Fredholm theory on $\mathbb{R} \times Y$

Let Y be a three-dimensional compact orientable manifold. Let (E, A) be a $SU(2)$-instanton over $\mathbb{R} \times Y$. We suppose for this section that A is in temporal gauge, that is it has no dt term. This assumption allows us to consider the restriction $A(t)$ to a cross-section $\{t\} \times Y$. The Dirac operator on $\{t\} \times Y$ is denoted D_A. We consider the Dirac operator

$$\not{\mathfrak{D}}_A = \partial_t + D_A$$

on sections of $S^+ \otimes E$.

Our aim in this chapter is to find spaces on which $\not{\mathfrak{D}}_A$ is a Fredholm operator, and on those spaces compute its index. It is quite natural for such problems to consider Sobolev spaces, as in the compact case. While it is quite natural, it is perhaps too restrictive and what is happening on cylindrical manifolds in terms of Fredholmness is better understood in the realm of weighted Sobolev spaces.

It was observed long ago that for the usual Laplacian on $\mathbb{R}^n$, the classical Sobolev spaces are the wrong spaces for domains: the Laplacian is not Fredholm for those domains. The same is true for other elliptic partial differential operators on $\mathbb{R}^n$.

As an attempt to remedy the situation, Homer Walker in [Wal71, Wal72, Wal73] proves that for certain domains, first order elliptic partial differential operators obtained from constant coefficients operators by adding a perturbation on a compact set are practically Fredholm, in the sense that the dimension of the kernel is finite and that the range can be described by a finite number of orthogonality condition.

In [NW73], the results are extended to a broader class of elliptic operators, perturbed in a less restrictive way, and L^p-type domains replace the L^2-type domains presents in the papers just described. We are presented with a sort of Gårding inequality decorated with weights, the treatment of which is not fully systematized at this point, and finite dimensionality of the kernel is proved.

The use of weights to describe which type of behavior is allowed at infinity was systematized in [Can75] with the introduction of weighted Sobolev spaces. Around the same time, Atiyah, Patodi and Singer in [APS75] made the crucial observation that the condition for an operator to be Fredholm in L^2 on a cylindrical manifold is that the restriction to the slice at infinity must have an empty kernel.

In [Loc81], and independently in [McO79], we are given specific APS-like conditions on the weights for an elliptic partial differential operator of any order with a certain type of asymptotic behavior to be Fredholm on the given weighted Sobolev spaces. The result and proof of that paper are extensions of [NW73], and partial results along these lines can be found in [Can75], where isomorphism properties of the Laplacian were derived.

Choquet-Bruhat and Christodoulou in [CBC81] remove restriction on p from another work of Cantor [Can79] and prove semi-Fredholmness, finite dimensionality of kernel and isomorphism theorems for operators on non-compact manifolds while giving improvements on imbedding and multiplication results for weighted Sobolev spaces. Those two papers constitute partial results along the lines of the more advanced and complete joint work [LM83, LM84] of Lockhart and McOwen. Their work extends the results of [Loc81] and [McO79] for systems of partial differential operators which are elliptic in the sense of Douglis–Nirenberg, and similar conditions on the weights are described to ensure Fredholmness.

Very similarly to what we do in this chapter, [LM85] study a much larger class of elliptic operator $C^\infty(E) \to C^\infty(f)$ for bundles E and F over a manifold with cylindrical ends. Conditions on weights to obtain Fredholmness and wall crossing formulas are derived. The paper also treats boundary valued problems with Lopatinski–Shapiro boundary conditions; see [APS75] along those lines.

As a prelude to proving that the mass of an asymptotically flat manifold is a geometric invariant, [Bar86] reviews the theory of weighted Sobolev spaces with an emphasis on two basic ideas which underlie the subject: the use of scaling arguments to pass from local estimates to global estimates and the derivation of sharp estimate from explicit formulas for Greens functions. Bartnik's paper add a number of technical improvements and some new observations to the theory. A simple example is that the indexing chosen for the weights is different from the one used by his predecessors, but it clearly reflects the growth at infinity allowed. An expanded version of his presentation, with complete proofs, can be found in Appendix E.

These results can be put in a geometric form following Melrose; see [Mel93]. Melrose's technique involves adding a boundary at infinity to the underlying non-compact complete riemannian manifold. Later work of Mazzeo–Melrose [MM98] was used by Singer–Nye in [NS00] for computing the index of the Dirac operator twisted by a caloron on $S^1 \times \mathbb{R}^3$.

6.1 Fredholmness

Suppose first that A is independent of t.

The space $W^{k,p}(X, F)$ is the space of L^p sections of the vector bundle F over X, whose derivatives up to order k for a given reference connection are also in L^p on X. Because $\not{\mathfrak{D}}_A$ is obviously defined on sections of $S^+ \otimes E$ and $\not{\mathfrak{D}}_A^*$ is obviously defined on sections of $S^- \otimes E$, we lighten up the notation by omiting the F, and most of time we omit the X as well, in which case it is assumed to be $\mathbb{R} \times Y$, or $\mathbb{R} \times T^3$ when appropriate.

Lemma 6.1-1. *Suppose the connection A does not depend of t. Then*

$$\not{\mathfrak{D}}_A \colon W^{1,2} \to L^2$$

is Fredholm if and only if $0 \notin \mathrm{Spec}(D_A)$.

We actually prove a stronger version of the lemma, as this more powerful version is useful later on.

Lemma 6.1-2. *Let T be a formally self-adjoint elliptic 1st order operator on a compact manifold Y. The operator $\hat{T} := \partial_t + T$ seen as*

$$\hat{T} \colon W^{1,2} \to L^2$$

is Fredholm if and only if $0 \notin \mathit{Spec}(T)$. *Furthermore, it is an isomorphism when Fredholm.*

Proof: Suppose $0 \notin \mathit{Spec}(T)$. Then we have the estimate $\|T\phi\|_{L^2(Y)} \geq C\|\phi\|_{W^{1,2}(Y)}$. Suppose ϕ is compactly supported. Then

$$\begin{aligned} \|\hat{T}\phi\|^2 &= \|\partial_t\phi\|^2 + \|T\phi\|^2 + 2\int_{\mathbb{R}} \partial_t \langle \phi, T\phi\rangle_{L^2(Y)} \\ &\geq (1+C^2)\Big(\|\partial_t\phi\|^2 + \big\|\|\phi\|_{W^{1,2}(Y)}\big\|^2_{L^2(\mathbb{R})}\Big) \\ &\geq (1+C^2)\|\phi\|^2_{W^{1,2}}, \end{aligned} \tag{6.1}$$

thus $\hat{T}$ has closed range and no kernel.

If $\phi \in L^2$ is orthogonal to the image of $\hat{T}$, then it is a weak solution to $\hat{T}\phi = 0$. Elliptic theory, for example in [LM89, Thm III.5.2(i), p. 193], implies that ϕ is C^∞. But the only C^∞ solution to be L^2 is 0. Thus $\hat{T}$ is an isomorphism.

Suppose now $0 \in \mathit{Spec}(T)$, and let ϕ_0 be in $\ker(T)$. We show that $\hat{T}$ does not have closed range. It suffices to do so to prove that $\partial_t\colon W^{1,2}(\mathbb{R}) \to L^2(\mathbb{R})$ does not have closed image. Set

$$f(x) := \begin{cases} 1/x, & \text{for } |x| \geq 1; \\ x, & \text{for } |x| \leq 1; \end{cases}$$

$$F(x) := \begin{cases} 1/2 - \log|x|, & \text{for } |x| \geq 1; \\ x^2/2, & \text{for } |x| \leq 1. \end{cases}$$

The function f clearly belongs to L^2. In fact, $\|f\|_{L^2} = 2\sqrt{2/3}$. We have $\partial_t F = f$, but $c + F \notin L^2$ for all $c \in \mathbb{R}$. Hence f is not in the image of ∂_t.

Choose $\chi_R\colon \mathbb{R} \to [0,\infty)$ with $\chi_R(x) = \chi_R(-x)$, and

$$\chi_R(x) = \begin{cases} 0, & \text{when } |x| \geq 2R; \\ 1, & \text{when } |x| \leq R. \end{cases}$$

Set $f_R := \chi_R f$. It is obvious that $f_R \to f$ in L^2, and $f_R(x) = -f_R(-x)$. This last property ensures that the function

$$F_R := \begin{cases} \int_{-3R}^{x} f_R, & \text{for } x \leq 0; \\ -\int_{x}^{3R} f_R, & \text{for } x \geq 0, \end{cases}$$

is well-defined at $x = 0$. The function F_R satisfies $\partial_t F_R = f_R$, and thus, since F_R is compactly supported, f_R is in the image of ∂_t. The image is therefore not closed. The proof is now complete. □

We now add a number of weighted Sobolev spaces to our arsenal. The weight function we use here, denoted σ_δ, depends only on t, and its definition depends on whether $\delta \in \mathbb{R}$ or $\delta \in \mathbb{R}^2$. For $\delta = (\delta_-, \delta_+) \in \mathbb{R}^2$, we want $\sigma_\delta > 0$ with

$$\sigma_\delta = \begin{cases} e^{-\delta_- t}, & \text{for } t < -1; \\ e^{-\delta_+ t}, & \text{for } t > 1. \end{cases}$$

To achieve such a weight function, choose c smooth and positive with

$$c(t) = \begin{cases} 1, & \text{for } t \leq -1; \\ 0, & \text{for } t \geq 1. \end{cases}$$

Then set

$$\sigma_\delta := e^{-\left(c\delta_- + (1-c)\delta_+\right)t}.$$

For $\delta \in \mathbb{R}$, set

$$\sigma_\delta := e^{-\delta t}.$$

The weighted Sobolev spaces are defined by the equation

$$W^{k,p}_\delta := \{ f \mid \|\sigma_\delta f\|_{W^{k,p}} < \infty \} = \sigma_{-\delta} W^{k,p}.$$

As usual, $L^p_\delta = W^{0,p}_\delta$. For $\delta \in \mathbb{R}$, notice that $\sigma_\delta = \sigma_{(\delta,\delta)}$ hence $W^{k,p}_\delta = W^{k,p}_{(\delta,\delta)}$.

Theorem 6.1-3. *Suppose A is translation invariant (it does not depend on t). Then*

$$\not{\mathfrak{D}}_A \colon W^{1,2}_\delta \to L^2_\delta$$

is Fredholm if and only if $\delta \notin \mathit{Spec}(D_A)$. Moreover, it is an isomorphism if Fredholm.

Proof: The following diagram is commutative:

$$\begin{array}{ccc} W^{1,2}_\delta & \xrightarrow{\not{\mathfrak{D}}} & L^2_\delta \\ \cong\downarrow & & \downarrow\cong \\ W^{1,2} & \xrightarrow[\sigma\not{\mathfrak{D}}\sigma^{-1}]{} & L^2 \end{array}$$

Because the columns are isomorphisms, the top row is Fredholm if and only if the bottom row is. But

$$\sigma_\delta \not{\mathfrak{D}}_A \sigma_\delta^{-1} = \partial_t + (\not{\mathfrak{D}}_A + \delta).$$

Using Lemma 6.1-2, we see it is Fredholm if and only if $0 \notin \mathit{Spec}(\not{\mathfrak{D}}_A + \delta)$, or equivalently when $-\delta \notin \mathit{Spec}(\not{\mathfrak{D}}_A)$. Since $\mathit{Spec}(\not{\mathfrak{D}}_A) = -\mathit{Spec}(\not{\mathfrak{D}}_A)$, the conclusion follows. □

Our ultimate goal is to find Fredholmness conditions for $\not{\mathfrak{D}}_A$, with the only hypothesis that A is an instanton. As we know, being an instanton forces A to have flat limits at $\pm\infty$, say Γ_- and Γ_+. As a notational convenience, we define the grid

$$\mathfrak{G}_A := \left(\mathit{Spec}(\Gamma_-) \times \mathbb{R}\right) \cup \left(\mathbb{R} \times \mathit{Spec}(\Gamma_+)\right). \tag{6.2}$$

Theorem 6.1-4. *Let Γ_+ and Γ_- be two flat connections on Y. Suppose A is a connection on $\mathbb{R} \times Y$ such that*

$$A = \begin{cases} \Gamma_-, & \text{on } (-\infty, -R) \times Y; \\ \Gamma_+, & \text{on } (R, \infty) \times Y. \end{cases}$$

Then for a weight $\delta \in \mathbb{R}^2$,

$$\not{\mathfrak{D}}_A \colon W^{1,2}_\delta \to L^2_\delta$$

is Fredholm if and only if $\delta \notin \mathfrak{G}_A$.

Proof: Consider the three following manifolds:

$$X_1 = \mathbb{R} \times Y,$$
$$X_2 = \big([-R-2, R+2]/_{(R+2)\sim(-R-2)}\big) \times Y,$$
$$X_3 = \mathbb{R} \times Y.$$

Using a path from Γ_- to Γ_+, we can find $\tilde{A}$ on X_2 such that $\tilde{A} = A$ on $[-R-1, R+1] \times Y$, and we can also find a function $\tilde{\sigma}$ defined on X_2 which restrict to σ_δ on that same subspace.
Choose a square root of a partition of unity

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 1$$

subordinate to the covering

$$\big((-\infty, -R) \times Y, (-R-1, R+1) \times Y, (R, \infty) \times Y\big).$$

Consider the operators

$$D_1 := \sigma_{\delta_-} \not{\mathfrak{D}}_{\Gamma_-} \sigma_{\delta_-}^{-1},$$
$$D_2 := \tilde{\sigma} \not{\mathfrak{D}}_{\tilde{A}} \tilde{\sigma}^{-1}, \text{ and}$$
$$D_3 := \sigma_{\delta_+} \not{\mathfrak{D}}_{\Gamma_+} \sigma_{\delta_+}^{-1}$$

defined on the spaces X_1, X_2, and X_3 respectively.
When $\delta \notin \mathfrak{G}_A$, all the D_i are Fredholm. In fact, D_1 and D_3 are even isomorphisms. Hence there exist

$$P_i \colon L^2(X_i) \to W^{1,2}(X_i), \; i = 1, 2, 3, \text{ and}$$
$$K_2 \colon L^2(X_2) \to L^2(X_2)$$

with K_2 compact such that

$$D_1 P_1 = 1, \; D_3 P_3 = 1,$$
$$D_2 P_2 = 1 + K_2.$$

Set

$$P := \phi_1 P_1 \phi_1 + \phi_2 P_2 \phi_2 + \phi_3 P_3 \phi_3.$$

Notice that P is a well defined operator $L^2(\mathbb{R} \times Y) \to W^{1,2}(\mathbb{R} \times Y)$. Then

$$\begin{aligned}
\sigma_\delta \not{\mathfrak{D}}_A \sigma_\delta^{-1}(Pf) &= \sum_i D_i \phi_i P_i \phi_i f \\
&= \big(\sum_i \phi_i D_i P_i \phi_i f\big) + \big(\sum_i [D_i, \phi_i] P_i \phi_i f\big) \\
&= \big(\sum_i \phi_i^2 f\big) + \big(\phi_2 K_2 \phi_2 f + \sum_i [D_i, \phi_i] P_i \phi_i f\big) \\
&= f + Kf
\end{aligned}$$

with K compact.

Similarly, we can find left-parametrices for the D_i and construct a left parametrix for $\sigma_\delta \mathfrak{D}_A \sigma_\delta^{-1}$ using them. Hence $\delta \notin \mathfrak{G}_A$ implies $\mathfrak{D}_A \colon W_\delta^{1,2} \to L_\delta^2$ is Fredholm.
The converse is a corollary of Theorem 6.4-1. It should be noted that we do not use this part of the result to establish Theorem 6.4-1. □

This last theorem now allows us to prove at last what we are really after.

Theorem 6.1-5. *Let (E, A) be a $SU(2)$-instanton on $\mathbb{R} \times Y$. Suppose that A is in temporal gauge and that it converges to flat connections Γ_+ at $+\infty$ and Γ_- at $-\infty$. Then the operator*

$$\mathfrak{D}_A \colon W_\delta^{1,2} \to L_\delta^2$$

is Fredholm if and only if $\delta \notin \mathfrak{G}_A$.

Proof: Let

$$(\chi_R^+, \chi_R^-, \chi_R^0)$$

be a partition of unity subordinate to the covering

$$\Big((R, \infty) \times Y, (-\infty, -R) \times Y, (-R-1, R+1) \times Y\Big).$$

Suppose $\Gamma_\pm = d + \gamma_\pm$ and $A = d + a$. Then a tends to γ_+ and γ_- when t tends to $+\infty$ and $-\infty$ respectively. Set

$$a_R = \chi_R^+ \gamma_+ + \chi_R^- \gamma_- + \chi_R^0 a. \tag{6.3}$$

For simplicity, we bring the discussion back to the classical Sobolev spaces $W^{1,2}$ and L^2 as we did in the proof of Theorem 6.1-3. Set

$$\begin{aligned} E_n &:= \sigma_\delta \mathfrak{D}_{a_n} \sigma_\delta^{-1}, \text{ and} \\ E &:= \sigma_\delta \mathfrak{D}_A \sigma_\delta^{-1}. \end{aligned}$$

All the E and E_n are operators from $W^{1,2}$ to L^2. Our aim is to show that E is Fredholm if and only if $\delta \notin \mathfrak{G}_A$. By virtue of Theorem 6.1-4, it is precisely out of that grid that E_1 is Fredholm. We now prove that $E - E_1$ is compact, whence the result.
Define the operator $K_n := \mathfrak{D}_{a_n} - \mathfrak{D}_{a_1}$. Then

$$\begin{aligned} K_n &= cl(a_n - a_1) \\ &= (\chi_n^+ - \chi_1^+) cl(\gamma_+) + (\chi_n^- - \chi_1^-) cl(\gamma_-) + (\chi_n^0 - \chi_1^0) cl(a). \end{aligned}$$

As it is a zeroth order operator, K_n is continuous $W^{1,2} \to W^{1,2}$. Observe that the coefficients in K_n have compact support:

$$\begin{aligned} supp(\chi_n^- - \chi_1^-) &= [-n-1, -1] \times Y \\ supp(\chi_n^+ - \chi_1^+) &= [1, n+1] \times Y \\ supp(\chi_n^0 - \chi_1^0) &= supp(\chi_n^- - \chi_1^-) \cup supp(\chi_n^+ - \chi_1^+). \end{aligned}$$

Hence K_n factorizes through the compact inclusion

$$W^{1,2}\big([-n-1, n+1] \times Y\big) \subset L^2\big([-n-1, n+1] \times Y\big).$$

Thus K_n is compact.

Suppose without loss of generality that $n < m$. Since

$$K_n - K_m = (\chi_n^+ - \chi_m^+)cl(\gamma_+) + (\chi_n^- - \chi_m^-)cl(\gamma_-) + (\chi_n^0 - \chi_m^0)cl(a),$$

we have

$$\|(K_n - K_m)\phi\|^2 = (\int_n^{m+1} \quad) + (\int_{-m-1}^{-n} \quad).$$

The first integral involves only γ_+ and a. On that domain, $\chi_j^0 + \chi_j^+ = 1$ for any j. Hence on $[n, m+1]$,

$$(K_n - K_m)\phi = (\chi_n^0 - \chi_m^0)cl(a - \gamma_+)\phi.$$

Since $a = \gamma_+ + O(1/t)$, we have

$$\|(K_n - K_m)\phi\| \leq C\frac{1}{n}\|\phi\|$$

for any Sobolev norm. Hence the sequence of compact operator K_n is Cauchy and its limit K is compact. Now obviously $\mathfrak{D}_A - \mathfrak{D}_{a_1} = K$ hence $\mathfrak{D}_A$ is Fredholm if and only if $\mathfrak{D}_{a_1}$ is Fredholm. The proof is now complete. □

6.2 Elliptic estimates

As for the compact case, we do have elliptic estimates but those are not sufficient to prove the Fredholmness of $\mathfrak{D}$, which is why we need the more involved proofs of the previous section. We do however need those inequalities for finding the asymptotic behavior of harmonic spinors in the Chapter 7. Let's derive them.

Theorem 6.2-1 (Gårding Inequality). *Let A be an instanton on $\mathbb{R} \times Y$. If $s \in L^2$ and $\mathfrak{D}_A s \in L^2$, then $s \in W^{1,2}$ and*

$$\|s\|_{W^{1,2}} \leq C(\|\mathfrak{D}_A s\|_{L^2} + \|s\|_{L^2}). \tag{6.4}$$

Proof: Let s_c denote the scalar curvature of Y. We start with the Weitzenbock formula:

$$\mathfrak{D}_A^* \mathfrak{D}_A s = \nabla_A^* \nabla_A s + (cl(F_A) + \frac{s_c}{4})s.$$

Suppose s has compact support. Then $\|\mathfrak{D}_A s\|_{L^2}^2 = \|\nabla_A s\|_{L^2}^2 + \langle (cl(F_A) + s_c/4)s, s\rangle_{L^2}$, thus

$$\begin{aligned}\|\nabla_A s\|_{L^2}^2 &\leq \|\mathfrak{D}_A s\|_{L^2}^2 + \sup(|F_A + s_c/4|)\|s\|_{L^2}^2 \\ &\leq \max(\sup(|F_A + s_c/4|), 1)(\|\mathfrak{D}_A s\|_{L^2} + \|s\|_{L^2})^2.\end{aligned}$$

While this inequality is good, we must not forget that the $W^{1,2}$-norm is defined using the trivial connection ∇. Fortunately, for C being, say, $2 + \sup(\sqrt{|F_A + s_c/4|}) + \sup(|A|)$, we have

$$\begin{aligned}\|\nabla s\|_{L^2} &\leq \|\nabla_A s\|_{L^2} + \|As\|_{L^2} \\ &\leq \|\nabla_A s\|_{L^2} + \sup(|A|)\|s\|_{L^2} \\ &\leq C(\|\mathfrak{D}_A s\|_{L^2} + \|s\|_{L^2}).\end{aligned}$$

Since A is in radial gauge, the ASD and L^2 conditions on the curvature imply that C is finite. Thus Equation (6.4) is proved for s with compact support.

Suppose now that s does not have compact support. We use now a trick used also in [LM89, p. 117] to show that $\ker(\not{\mathfrak{D}}) = \ker(\not{\mathfrak{D}}^2)$ on a complete manifold.

Choose $\chi \in C^\infty(\mathbb{R})$ such that

$$0 \leq \chi \leq 1,$$
$$\chi(t) = 1 \text{ for } |t| \leq 1,$$
$$\chi(t) = 0 \text{ for } |t| \geq 2, \text{ and } |\chi'| \leq 2.$$

We set $\chi_n(x,t) := \chi(t/n)$ on $\mathbb{R} \times Y$, and set $s_n := \chi_n s$. The sequence s_n has compact support and

$$s_n \to s \text{ in } L^2. \tag{6.5}$$

We know that

$$\not{\mathfrak{D}}_A s_n = cl(\mathrm{grad}\, \chi_n) s + \chi_n \not{\mathfrak{D}}_A s.$$

Obviously $\chi_n \not{\mathfrak{D}}_A s$ converges to $\not{\mathfrak{D}}_A s$ in L^2, and $\|cl(\mathrm{grad}\, \chi_n)s\|_{L^2} \leq (2/n)\|s\|_{L^2}$, hence

$$\not{\mathfrak{D}}_A s_n \to \not{\mathfrak{D}}_A s \text{ in } L^2. \tag{6.6}$$

Consequently, because of (6.5) and (6.6), and because Equation (6.4) is true for the s_n, we see that s_n is a Cauchy sequence in $W^{1,2}$. Hence s_n converges to, say, $\tilde{s}$ in $W^{1,2}$, whence it converges to $\tilde{s}$ in L^2. Thus $\tilde{s} = s$, and $s \in W^{1,2}$ as wanted, with norm bounded as in Equation (6.4). The proof is now complete. □

We push things up the scale a tiny bit with the next result.

Corollary 6.2-2 (Elliptic Estimate). *Let A be an instanton on $\mathbb{R} \times Y$ If $s \in W^{k,2}$ and $\not{\mathfrak{D}}_A s \in W^{k,2}$, then $s \in W^{k+1,2}$ and*

$$\|s\|_{W^{k+1,2}} \leq C(\|\not{\mathfrak{D}}_A s\|_{W^{k,2}} + \|s\|_{W^{k,2}}). \tag{6.7}$$

Proof: We prove it by induction, the first step being the result of Theorem 6.2-1. Suppose the result is true for $k-1$, and suppose $s \in W^{k,2}$ and $\not{\mathfrak{D}}_A s \in W^{k,2}$. Then $\nabla s \in W^{k-1,2}$ and

$$\not{\mathfrak{D}}_A \nabla s = \nabla \not{\mathfrak{D}}_A s + [\not{\mathfrak{D}}_a, \nabla] s.$$

The first term of the right hand side is in $W^{k-1,2}$, while the second,

$$[\not{\mathfrak{D}}_a s, \nabla] s = -\sum_j cl(\partial_j) \nabla (A(\partial_j)) s,$$

is in $W^{k-1,2}$ if $A \in W^{k-1,\infty}$, which is the case for a good choice of gauge as A is an instanton.

We hence have, by induction, that $\nabla s \in W^{k-1,2}$, which means that $s \in W^{k,2}$ and

$$\begin{aligned}
\|s\|_{W^{k,2}} &= \|s\|_{L^2} + \|\nabla s\|_{W^{k-1,2}} \\
&\leq \|s\|_{L^2} + C(\|\not{\mathfrak{D}}_A \nabla s\|_{W^{k-1,2}} + \|\nabla s\|_{W^{k-1,2}}) \\
&\leq C(\|s\|_{L^2} + \|\nabla \not{\mathfrak{D}}_A s\|_{W^{k-1,2}} + \|[\not{\mathfrak{D}}_A, \nabla] s\|_{W^{k-1,2}} + \|\nabla s\|_{W^{k-1,2}}) \\
&\leq C'(\|\not{\mathfrak{D}}_A s\|_{W^{k,2}} + \|s\|_{W^{k,2}}).
\end{aligned}$$

The proof is now complete. □

Remark 6.2-3. Note that in Estimates (6.4) and (6.7), we can choose a uniform C for any family of connections A_z parameterized by z in a compact set, independently of whether or not $\mathfrak{D}_{A_z}$ is Fredholm everywhere.

We can even prove a better estimate, which also hints to Fredholmness properties for $\delta \notin \mathfrak{G}_A$.

Theorem 6.2-4 (Gårding plus). *Suppose $\delta \notin \mathfrak{G}_A$. Then there exist a compact subcylinder K large enough so that for any $s \in W^{1,2}$, we have*

$$\|s\|_{W^{1,2}_\delta} \leq C\big(\|\mathfrak{D}_A s\|_{L^2_\delta} + \|s\|_{L^2_\delta(K)}\big), \tag{6.8}$$

with K and C depending only on A and δ.

Proof: First set $K_R := (-\infty, R] \times T^3$, and suppose $supp(s) \cap K_R = \varnothing$. Then

$$\|(\mathfrak{D}_A - \mathfrak{D}_\Gamma)s\|_{L^2} \leq \Big(\sup_{t>R} |A - \Gamma|\Big)\|s\|_{L^2}$$

hence as $R \to \infty$, the operator norm of the restriction of $\mathfrak{D}_A - \mathfrak{D}_\Gamma$ on elements with support out of K_R, denoted $\|\mathfrak{D}_A - \mathfrak{D}_\Gamma\|_{op,R}$, decreases to 0.

Let χ_R be a cut off function, with $\chi_R(t) = 0$ for $t > R + 1$, and $\chi_R(t) = 1$ for $t \leq R$. Write $s = s_0 + s_\infty$, with $s_0 = \chi_R s$ and $s_\infty = (1 - \chi_R)s$. Since $\mathfrak{D}_\Gamma$ is an isomorphism, we have

$$\begin{aligned}
\|s_\infty\|_{W^{1,2}} &\leq C\|\mathfrak{D}_\Gamma s_\infty\|_{L^2} \\
&\leq C\big(\|\mathfrak{D}_A s_\infty\|_{L^2} + \|(\mathfrak{D}_A - \mathfrak{D}_\Gamma)s_\infty\|_{L^2}\big) \\
&\leq C\big(\|\mathfrak{D}_A s_\infty\|_{L^2} + \|\mathfrak{D}_A - \mathfrak{D}_\Gamma\|_{op,R}\|s_\infty\|_{L^2}\big) \\
&\leq C\big(\|\mathfrak{D}_A s_\infty\|_{L^2} + \|\mathfrak{D}_A - \mathfrak{D}_\Gamma\|_{op,R}\|s_\infty\|_{W^{1,2}}\big).
\end{aligned}$$

But now,

$$\begin{aligned}
\|\mathfrak{D}_A s_\infty\|_{L^2} &\leq \|\mathfrak{D}_A s\|_{L^2} + \|\mathfrak{D}_A s_0\|_{L^2} \\
&\leq \|\mathfrak{D}_A s\|_{L^2} + \|\chi_R \mathfrak{D}_A s\|_{L^2} + \|cl(\mathrm{grad}\, \chi_R)s\|_{L^2} \\
&\leq C\big(\|\mathfrak{D}_A s\|_{L^2} + \|s\|_{L^2(K_{R+1})}\big).
\end{aligned}$$

Hence

$$\|s_\infty\|_{W^{1,2}} \leq C\big(\|\mathfrak{D}_A s\|_{L^2} + \|s\|_{L^2(K_{R+1})} + \|\mathfrak{D}_A - \mathfrak{D}_\Gamma\|_{op,R}\|s_\infty\|_{W^{1,2}}\big).$$

For R big enough, we can rearrange to obtain

$$\|s_\infty\|_{W^{1,2}} \leq C\big(\|\mathfrak{D}_A s\|_{L^2} + \|s\|_{L^2(K_{R+1})}\big).$$

We can now play the same game at $-\infty$, splitting s_0 as $s_{-\infty} + \tilde{s}_0$ and we obtain a similar estimate. Once we patch all those estimates together, we find that there is a R big enough so that for the compact subcylinder $K := [-R, R] \times T^3$, we have the desired Inequality (6.8). □

Note that the assumption $s \in W^{1,2}$ is important. This Theorem does not prove that $\mathfrak{D}_A \phi \in L^2$ and $\phi|_K \in L^2$ implies $\phi \in W^{1,2}$. If that implication were true, then in the language of Chapter 8, it would rule out the possibility that $\mathfrak{D}\mathcal{H} \cap V \neq \{0\}$, hence would imply that V and $V_\square$ are always equal.

6.3 Invariance of the kernels

We define the spaces

$$\begin{aligned} \ker(\delta) &:= \ker(\mathfrak{D}_A \colon W^{1,2}_\delta \to L^2_\delta), \\ \ker^*(\delta) &:= \ker(\mathfrak{D}^*_A \colon W^{1,2}_\delta \to L^2_\delta), \end{aligned} \tag{6.9}$$

and the integers

$$\begin{aligned} \mathrm{ind}(\delta) &:= \mathrm{ind}(\mathfrak{D}_A \colon W^{1,2}_\delta \to L^2_\delta) \\ N(\delta) &:= \dim \ker(\delta), \text{ and} \\ N^*(\delta) &:= \dim \ker^*(\delta). \end{aligned} \tag{6.10}$$

Since $(L^2_\delta)^* = L^2_{-\delta}$, Theorem 6.2-1 tells us that $\dim \mathrm{coker}(\mathfrak{D}_A) = N^*(-\delta)$, hence

$$\mathrm{ind}(\delta) = N(\delta) - N^*(-\delta).$$

That the formal adjoint $\mathfrak{D}^*_A$ on $W^{1,2}_{-\delta}$ is really the adjoint of $\mathfrak{D}_A$ on $W^{1,2}_\delta$ is guaranteed by the following lemma.

Lemma 6.3-1. *The subspace* $\ker^*(-\delta)$ *of* $L^2_{-\delta} = (L^2_\delta)^*$ *kills* $\mathrm{Im}(\delta)$ *in the* L^2 *natural pairing.*

Proof: Suppose ϕ is a smooth function with compact support. Then for all $\psi \in \ker^*(-\delta)$, we have $\langle \psi, \mathfrak{D}\phi \rangle = \langle \mathfrak{D}^*\psi, \phi \rangle = 0$. Since C^∞_c is dense in $W^{1,2}_\delta$, the lemma holds. □

Let's say our instanton A has limit $\Gamma_\pm$ as t tends to $\pm\infty$. Recall from Equation (6.2) the definition of the grid

$$\mathfrak{G}_A = \big(\mathit{Spec}(\Gamma_-) \times \mathbb{R}\big) \cup \big(\mathbb{R} \times \mathit{Spec}(\Gamma_+)\big)$$

in $\mathbb{R}^2$. As we have shown in Theorem 6.1-5, the operator $\mathfrak{D}_A \colon W^{1,2}_\delta \to L^2_\delta$ is Fredholm if and only if $\delta \notin \mathfrak{G}_A$. In fact, we have more, as in shown by the next theorem.

Theorem 6.3-2. *In each open square of* $\mathbb{R}^2$ *delimited by the grid* $\mathfrak{G}_A$*, the quantities*

$$\mathrm{ind}(\delta), N(\delta), \text{ and } N^*(\delta)$$

are constant. In fact, for δ, η *in a same square,*

$$\ker(\delta) = \ker(\eta), \text{ and } \ker^*(\delta) = \ker^*(\eta).$$

Proof: We use the family $D_\delta \colon W^{1,2} \to L^2$ of operators defined as

$$D_\delta := \sigma_\delta \mathfrak{D}_A \sigma_\delta^{-1}.$$

The family is linear in δ as

$$\begin{aligned} D_\delta &= \mathfrak{D}_A + \sigma_\delta cl(\mathit{grad}\, \sigma_\delta^{-1}) \\ &= \mathfrak{D}_A + \big(c\delta_- + (1-c)\delta_+ + tc'(\delta_- - \delta_+)\big). \end{aligned}$$

Hence the family depends continuously in the operator topology on the parameter $\delta \in \mathbb{R}^2$. Since

$$\operatorname{ind}(\delta) = \operatorname{ind}(D_\delta),$$

we see that indeed $\operatorname{ind}(\delta)$ is constant on each open square.
Let's define a partial ordering on $\mathbb{R}^2$ as follows

$$\delta \leq \eta \iff \delta_- \geq \eta_-, \text{ and } \delta_+ \leq \eta_+.$$

This ordering is designed so that

$$\delta \leq \eta \implies W^{k,p}_\delta \subset W^{k,p}_\eta.$$

Suppose for the moment that δ, η in the same open square are such that $\delta \leq \eta$. We then have

$$\begin{aligned} \ker(\delta) \subset \ker(\eta), \text{ hence} \\ N(\delta) \leq N(\eta). \end{aligned} \tag{6.11}$$

Similarly, as $-\delta \geq -\eta$, we have

$$\begin{aligned} \ker^*(-\delta) \supset \ker^*(-\eta), \text{ hence} \\ N^*(-\delta) \geq N^*(-\eta). \end{aligned} \tag{6.12}$$

But then, $\operatorname{ind}(\delta) = \operatorname{ind}(\eta)$ implies

$$N(\delta) - N(\eta) = N^*(-\delta) - N^*(-\eta).$$

Inequality (6.11) shows that the left-hand-side is nonpositive, while Inequality (6.12) shows that the right-hand-side is nonnegative. Hence both sides must be zero, and moreover

$$\begin{aligned} \ker(\delta) = \ker(\eta), \\ \ker^*(-\delta) = \ker(-\eta). \end{aligned}$$

The proof is not complete yet, as δ and η could be incomparable. In that case, we can find γ in the same open square smaller than both. We then have

$$\ker(\delta) = \ker(\gamma) = \ker(\eta),$$

and similarly for $\ker^*, N$ and N^*. The proof is now complete. □

6.4 Wall crossing

The following theorem tells us how the index changes as we cross a wall to change square. This theorem is quite useful for our main purpose on $\mathbb{R} \times T^3$ especially once we know the index of $\not{\mathfrak{D}}_A$ on weighted Sobolev spaces for weights contained in the open square around 0, which we compute in Section 8.1,

Theorem 6.4-1 (Wall Crossing). *For an ASD connection A on $\mathbb{R} \times Y$ converging to the flat con-*

nections $\Gamma_\pm$ on Y at $\pm\infty$, the index of $\not{\mathfrak{D}}_A$ and $\not{\mathfrak{D}}^*_A$ changes as follows:

$$\begin{aligned} \operatorname{ind}(\delta) &= \operatorname{ind}(\eta) + \dim\{D_{\Gamma_+}\phi = -\lambda\phi\}, \text{ and} \\ \operatorname{ind}^*(\delta) &= \operatorname{ind}^*(\eta) + \dim\{D_{\Gamma_+}\phi = \lambda\phi\} \end{aligned}$$

when $\delta_+ < \eta_+$, and δ and η are in adjacent open squares separated by the wall $\mathbb{R} \times \{\lambda\} \subset \mathfrak{G}_A$;

$$\begin{aligned} \operatorname{ind}(\delta) &= \operatorname{ind}(\eta) + \dim\{D_{\Gamma_-}\phi = -\lambda\phi\}, \text{ and} \\ \operatorname{ind}^*(\delta) &= \operatorname{ind}^*(\eta) + \dim\{D_{\Gamma_-}\phi = \lambda\phi\} \end{aligned}$$

when $\delta_- > \eta_-$, and δ and η are in adjacent open squares separated by the wall $\{\lambda\} \times \mathbb{R} \subset \mathfrak{G}_A$.

Proof: We start by considering that A is constant in t; say $A = \Gamma$. For simplicity, set

$$\begin{aligned} W_\lambda &= \{D_\Gamma\phi = \lambda\phi\}, \text{ and} \\ d_\lambda &= \dim W_\lambda. \end{aligned}$$

We have

$$\begin{aligned} \ker(\not{\mathfrak{D}}_\Gamma) &= \bigoplus_\lambda e^{-\lambda t} W_\lambda, \\ \ker(\not{\mathfrak{D}}^*_\Gamma) &= \bigoplus_\lambda e^{\lambda t} W_\lambda. \end{aligned}$$

Hence

$$\begin{aligned} \ker(\not{\mathfrak{D}}_\Gamma) \cap W^{1,2}_\delta &= \bigoplus_{\delta_- < -\lambda < \delta_+} e^{-\lambda t} W_\lambda, \\ \ker(\not{\mathfrak{D}}^*_\Gamma) \cap W^{1,2}_\delta &= \bigoplus_{\delta_- < \lambda < \delta_+} e^{\lambda t} W_\lambda. \end{aligned}$$

Now for $\delta \notin \mathfrak{G}_\Gamma$, we know $\not{\mathfrak{D}}_\Gamma$ is Fredholm hence

$$\begin{aligned} \operatorname{ind}(\delta) &= N(\delta) - N^*(-\delta) \\ &= \sum_{\delta_- < -\lambda < \delta_+} d_\lambda - \sum_{-\delta_- < \lambda < -\delta_+} d_\lambda \\ &= \sum_{\delta_- < -\lambda < \delta_+} d_\lambda - \sum_{\delta_+ < -\lambda < \delta_-} d_{-\lambda}. \end{aligned}$$

Suppose δ and η are in adjacent open squares delimited by $\mathbb{R}^2 \setminus \mathfrak{G}_\Gamma$, say δ is in the square to the left of the square containing η, and both squares are separated by $\{a\} \times \mathbb{R} \subset \mathfrak{G}_\Gamma$.
Since the index is constant in each open square, we can pick δ and η such that

$$\begin{aligned} \delta &= (a - \epsilon, b) \\ \eta &= (a + \epsilon, b) \end{aligned}$$

with $a + \epsilon < b$ or $b < a - \epsilon$.

Suppose $a+\epsilon<b$. Then $N^*(-\delta)=N^*(-\eta)=0$ and

$$N(\delta)=\sum_{a-\epsilon<-\lambda<b} d_\lambda = d_{-a}+\sum_{a+\epsilon<-\lambda<b} d_\lambda = d_{-a}+N(\eta).$$

Suppose on the contrary that $b<a-\epsilon$. Then $N(\delta)=N(\eta)=0$ and

$$N^*(-\delta)=\sum_{b<-\lambda<a-\epsilon} d_{-\lambda}=-d_a+\sum_{b<-\lambda<a+\epsilon} d_\lambda = -d_a+N^*(-\eta).$$

Hence in both cases, we find

$$\operatorname{ind}(\delta)=d_{-a}+\operatorname{ind}(\eta).$$

This formula also holds when δ is in the square above the one containing η.

Now suppose A has limiting connections Γ_+ and Γ_- at $+\infty$ and $-\infty$. We bring all the different operators we want to deal with on $W^{1,2}$ and L^2, and set

$$\begin{aligned} D_1 &:= \sigma_\delta \not{\mathfrak{D}}_A \sigma_\delta^{-1}, \\ D_2 &:= \sigma_\eta \not{\mathfrak{D}}_A \sigma_\eta^{-1}, \\ D_3 &:= \sigma_\delta \not{\mathfrak{D}}_{\Gamma_+} \sigma_\delta^{-1}, \\ D_4 &:= \sigma_\eta \not{\mathfrak{D}}_{\Gamma_+} \sigma_\eta^{-1}. \end{aligned}$$

Recall that $D_1=\not{\mathfrak{D}}_A+\big(c\delta_-+(1-c)\delta_++tc'(\delta_--\delta_+)\big)$, and similarly for the others.

Suppose that $\delta_-=\eta_-$. Notice that $D_1-D_2=D_3-D_4$. We can make up a compact operator K so that $D_1-D_2=K$ for $t\le 1$. Notice also that $D_1=D_3$ for $t>1$, and $D_2=D_4$ for $t>1$. Set

$$\begin{aligned} \tilde{D}_2 &:= D_2+K, \\ \tilde{D}_4 &:= D_4+K. \end{aligned}$$

Then we have

$$D_1=\begin{cases} \tilde{D}_2, & \text{for } t\le 1 \\ D_3, & \text{for } t>1; \end{cases}$$

$$\tilde{D}_4=\begin{cases} D_3, & \text{for } t\le 1; \\ \tilde{D}_2, & \text{for } t>1. \end{cases}$$

The excision principles for indices (see Theorem B-1) tells us that

$$\operatorname{ind}(D_1)-\operatorname{ind}(\tilde{D}_2)=\operatorname{ind}(D_3)-\operatorname{ind}(\tilde{D}_4).$$

Since $\operatorname{ind}(\tilde{D}_2)=\operatorname{ind}(D_2)$ and $\operatorname{ind}(\tilde{D}_4)=\operatorname{ind}(D_4)$, we see that the index changes the same way for ASD connections and time-independent connections. The proof is now complete. □

Remark 6.4-2. Notice that we proved something better for time-independent Γ. Indeed the analysis of $\ker(\delta)$ is such that we know its dimension $N(\delta)$ is lower semicontinuous: for

$$\delta_t := (\delta_-, \lambda-\epsilon+t) \quad \text{or} \quad \delta_t := (\delta_-+\epsilon-t, \lambda),$$

and for small ϵ and t, and some a and b, we have that

$$N(\delta_t) = a \text{ for } t \leq 0,$$
$$N(\delta_t) = b \text{ for } t > 0.$$

On $\mathbb{R} \times T^3$, the same is actually true for connections A when A decays exponentially to its limits. Suppose

$$\lambda \in \mathit{Spec}(D_{\Gamma_-}) \times \mathit{Spec}(D_{\Gamma_+}),$$
$$\delta \text{ is in the upper left open square adjacent to } \lambda,$$
$$\eta \text{ is in the lower right open square adjacent to } \lambda.$$

We have $\ker(\lambda) = \ker(\eta)$.

Indeed, suppose now $\phi \in \ker(\lambda)$. Then $\phi \in \ker(\delta)$ hence by Theorem 7.2-1, we expand ϕ for $t > 0$ as $\phi = e^{-\lambda_+ t}\psi_{\lambda_+} + \bar{\phi}$, with $\bar{\phi} \in W^{1,2}_{\eta_+}([0,\infty) \times T^3)$. Since ϕ and $\bar{\phi}$ are both in $W^{1,2}_{\lambda_+}$, so is the term $e^{-\lambda_+ t}\psi_{\lambda_+}$. This fact implies that $\psi_{\lambda_+} = 0$. Using a similar proof at $-\infty$, we find $\phi \in W^{1,2}_{\eta}$.

Chapter 7

Asymptotic behavior of harmonic spinors

In this chapter, we study how L^2-harmonic spinors on various spaces decay with time. For a 3-manifold Y of scalar curvature s_c, the Weitzenbock formula (see [Roe98, Prop 3.18, p. 48]) says

$$\not{\mathfrak{D}}_A^* \not{\mathfrak{D}}_A = \nabla_A^* \nabla_A + cl(F_A^+) + \frac{s_c}{4}.$$

For an ASD connection A, we hence see that should $s_c \geq 0$, every positive harmonic spinor is parallel for the connection A, hence has constant norm. This conclusion certainly prevents it from being L^2 on the manifold $\mathbb{R} \times Y$ of infinite volume.

In view of theorem 6.3-2, any negative L^2 harmonic spinor can be seen in $W_\delta^{1,2}$ for any δ in the open square delimited by the grid $\mathfrak{G}_A$ (see Equation (6.2) and containing $(0,0)$. The elliptic bootstrapping of Corollary 6.2-2 and the associated Sobolev embedding of $W^{3,2}$ in bounded C^0 functions (see [Heb99, Thm 3.4, p. 68]) tells us that if

$$\varphi \in W_\delta^{1,2}(\mathbb{R} \times Y), \text{ and } \not{\mathfrak{D}}_A^* \varphi = 0,$$

then

$$\varphi \leq C_\alpha e^{\alpha t}$$

for all α shy of the first negative eigenvalue of D_{Γ_+} on Y when $t \to \infty$ and shy of the first positive eigenvalue of D_{Γ_-} on Y when $t \to -\infty$.

While this result is nice, we can get a better knowledge of the asymptotic behavior. We first build up the theory on half-cylinders, which we need, and then apply it to T^3 and S^3 in place of Y.

7.1 Translation invariant operators on half-cylinders

For a more compact notation, we introduce the following shorthands:

$$Y_a := \{a\} \times Y$$
$$Y_{a+} := [a, \infty) \times Y$$
$$Y^+ := [0, \infty) \times Y$$

We hope to construct the asymptotic expansion of harmonic spinors by comparing the $W^{1,2}_{\delta}$-kernel of $\not{\mathfrak{D}}^*$ for varying δ. To compare then, we need finite dimensionality, or better, finite index. Before studying what gives us those properties, let's first eliminate options that won't.

Given a Dirac operator D on Y, with no zero eigenvalue, we have

$$\|D\phi\|_{L^2(Y)} \geq C\|\phi\|_{W^{1,2}(Y)}.$$

On the full cylinder $Y \times \mathbb{R}$, this estimate was enough to ensure that

$$\not{\mathfrak{D}}\colon W^{1,2}(\mathbb{R} \times Y) \to L^2(\mathbb{R} \times Y)$$

is an isomorphism. Working now on the half cylinder $[a, \infty) \times Y$, this estimate is not sufficient, as we now check. Define the ψ_λ by the eigenvalue equation

$$D\psi_\lambda = \lambda\psi_\lambda.$$

Then all the $e^{\lambda t}\psi_\lambda$ with $\lambda < 0$ are in $W^{1,2}([a, \infty) \times Y) \cap \ker(\not{\mathfrak{D}})$. So much for Fredholmness.

Another option would be to consider the operator

$$\not{\mathfrak{D}}\colon W^{1,2}_{\mathrm{tr}=0}([a, \infty) \times Y) \to L^2([a, \infty) \times Y)$$

on the space of sections whose restriction to Y_a is 0 The elliptic estimate (7.2) that we prove below still holds, but as $\phi(a) = 0$, we have

$$\|\phi\|_{W^{1,2}} \leq \|\not{\mathfrak{D}}\phi\|_{L^2}.$$

Hence $\not{\mathfrak{D}}$ is injective and has close range. It is therefore semi-Fredholm but it isn't Fredholm: its adjoint is the usual $\not{\mathfrak{D}}^*$ with no boundary condition and it has infinite dimensional kernel on a half-space.

The space $L^2(Y)$ splits according to the finite dimensional eigenspaces W_λ for D. Let

$$\Pi^+_\delta : L^2(Y) \to \bigoplus_{\lambda > \delta} W_\lambda$$

$$\Pi^-_\delta : L^2(Y) \to \bigoplus_{\lambda < \delta} W_\lambda$$

$$\Pi_\delta\colon L^2(Y) \to W_\delta$$

be the projections. To simplify notation we omit δ when it is 0 and set $\phi^\pm := \Pi^\pm(\phi)$.

For every $\phi \in L^2(Y)$, let ϕ_λ be its W_λ component. Thus

$$\phi = \sum \phi_\lambda.$$

Using this decomposition, we can define the space $W^{\frac{1}{2},2}(Y)$ using the norm

$$\|\phi\|^2_{W^{\frac{1}{2},2}} = \sum (1 + |\lambda|)\|\phi_\lambda\|^2_{L^2}. \tag{7.1}$$

Because Y is compact, the space $W^{\frac{1}{2},2}(Y)$ defined by two different Dirac operators are equal, with commensurate norms. The $+$ and $-$ part of L^2, however, depend highly on D.

Theorem 7.1-1. *The operator*

$$\not{\mathfrak{D}}\colon W^{1,2}(Y_{a+}) \to L^2(Y_{a+}) \oplus \Pi^+ W^{\frac{1}{2},2}(Y_a)$$
$$\phi \mapsto (\mathfrak{D}\phi, \Pi^+\phi(a))$$

is an isomorphism.

Keep in mind that D has no kernel.

Proof: It all starts as in the full cylinder case:

$$\begin{aligned}\|\mathfrak{D}\phi\|_{L^2}^2 &= \|\partial_t\phi\|_{L^2}^2 + \|D\phi\|_{L^2}^2 + \int_a^\infty \partial_t\langle\phi, D\phi\rangle_{L^2(Y)}\\ &\geq C\|\phi\|_{W^{1,2}}^2 - \langle\phi(a), D\phi(a)\rangle_{L^2(Y)}.\end{aligned}$$

Contrary to the full cylinder case, the boundary term here cannot be made to vanish and henceforth helps control the $W^{1,2}$-norm of ϕ. For the decomposition $\phi = \sum \phi_\lambda$, we have

$$\begin{aligned}\|\phi\|_{W^{1,2}}^2 &\leq C\big(\|\mathfrak{D}\phi\|_{L^2}^2 + \langle\phi(a), D\phi(a)\rangle_{L^2(Y)}\big)\\ &\leq C\big(\|\mathfrak{D}\phi\|_{L^2}^2 + \sum \lambda\|\phi_\lambda(a)\|_{L^2(Y)}^2\big)\\ &\leq C\big(\|\mathfrak{D}\phi\|_{L^2}^2 + \sum_{\lambda>0} |\lambda|\|\phi_\lambda(a)\|_{L^2(Y)}^2\big)\\ &\leq C\big(\|\mathfrak{D}\phi\|_{L^2}^2 + \|\phi^+(a)\|_{W^{\frac{1}{2},2}(Y)}^2\big).\end{aligned} \tag{7.2}$$

We just proved that $\|\phi\|_{W^{1,2}} \leq C\|\not{\mathfrak{D}}\phi\|$, hence $\not{\mathfrak{D}}$ is semi-Fredholm and injective.
Suppose that (ψ, η) is perpendicular to $\mathrm{Im}(\not{\mathfrak{D}})$. For all $\phi \in W^{1,2}([a,\infty)\times Y)$, we have

$$\begin{aligned}0 &= \langle\mathfrak{D}\phi, \psi\rangle + \langle\eta, \phi^+(a)\rangle\\ &= \langle\phi, \mathfrak{D}^*\psi\rangle - \langle\phi(a), \psi(a)\rangle + \langle\eta, \phi^+(a)\rangle\\ &= \langle\phi, \mathfrak{D}^*\psi\rangle - \langle\phi^-(a), \psi^-(a)\rangle + \langle\eta - \psi^+(a), \phi^+(a)\rangle.\end{aligned}$$

Going through all the ϕ with $\phi(a) = 0$ in a first time, $\phi^+(a) = 0$ then, and finally $\phi^-(a) = 0$, we prove

$$\begin{aligned}\mathfrak{D}^*\psi &= 0,\\ \eta &= \psi^+(a),\\ \psi^-(a) &= 0.\end{aligned}$$

Thus we have $-\partial_t\psi + D\psi = 0$, which means that ψ is a linear combination of the $e^{\lambda t}\psi_\lambda$. The condition $\psi^-(a) = 0$ forces out all the negative λ, while the positive ones are forced out by the L^2 condition. Hence $\psi = 0$ and $\not{\mathfrak{D}}$ is surjective. The proof is now complete. □

While $\mathfrak{D}\colon W^{1,2}([a,\infty)\times Y) \to L^2([a,\infty)\times Y)$ is not Fredholm, an easy corollary of Theorem 7.1-1 is that it is surjective. Hence

$$\mathfrak{D}\colon W^{1,2}([a,\infty)\times Y) \to L^2([a,\infty)\times Y)$$

is semi-Fredholm with

$$\operatorname{ind}(\slashed{\mathfrak{D}}) = \infty.$$

Weighted version

As in the full cylinder case, we can look at weighted version of $\slashed{\mathfrak{D}}$ and $\slashed{\mathbb{D}}$. For computing the asymptotic expansion of harmonic spinors, we actually need to consider the dual $\slashed{\mathfrak{D}}^*$ and its counterpart

$$\begin{aligned} \slashed{\mathbb{D}}^* \colon W^{1,2}(Y_{a+}) &\to L^2(Y_{a+}) \oplus \Pi^- W^{\frac{1}{2},2}(Y) \\ \phi &\mapsto (\slashed{\mathfrak{D}}^*\phi, \Pi^-\phi(a)) \end{aligned}$$

Staring at the diagrams

$$\begin{array}{ccc} W^{1,2}_\delta & \xrightarrow{\slashed{\mathfrak{D}}^*} & L^2_\delta \\ \cong\downarrow & & \downarrow\cong \\ W^{1,2} & \longrightarrow & L^2 \end{array} \qquad \begin{array}{ccc} W^{1,2}_\delta & \xrightarrow{\slashed{\mathbb{D}}^*} & L^2_\delta \oplus \Pi^-_\delta W^{\frac{1}{2},2}(Y) \\ \cong\downarrow & & \downarrow\cong \\ W^{1,2} & \xrightarrow[(\slashed{\mathfrak{D}}^*-\delta,\Pi)]{} & L^2 \oplus \Pi^-_\delta W^{\frac{1}{2},2}(Y) \end{array}$$

shows that the top row $\slashed{\mathfrak{D}}^*$ and $\slashed{\mathbb{D}}^*$ are respectively semi-Fredholm and Fredholm if and only if $\delta \notin \mathit{Spec}(D)$. Moreover, when $\delta \notin \mathit{Spec}(D)$ they are surjective and an isomorphism respectively.

Independence of the norm

For any operator $T\colon W^{1,2}(Y) \to L^2(Y)$, the operator

$$\hat{T} := \partial_t + T$$

has a norm independent of the half-cylinder on which that norm is taken. In other words,

$$\begin{aligned} \hat{T}\colon W^{1,2}(Y_{a+}) \to L^2(Y_{a+}) \\ \hat{T}\colon W^{1,2}(Y_{b+}) \to L^2(Y_{b+}) \end{aligned}$$

have the same operator norm. This fact is a manifestation of the translation invariance of $\hat{T}$. To prove this claim, consider the following characterization of the norm:

$$\|\hat{T}\| = \sup\{\|\hat{T}f\| : \|f\| = 1\}.$$

Shifting a function in t by $b-a$ doesn't change its L^2 or $W^{1,2}$ norm and shifts its value under $\hat{T}$. So if $f_{b-a}(y,t) := f(y, t+b-a)$, then

$$\begin{aligned} \|\hat{T}\|_{op,a} &= \sup\{\|\hat{T}f\|_{L^2(Y_{a+})} : \|f\|_{W^{1,2}(Y_{a+})} = 1\} \\ &= \sup\{\|(\hat{T}f)_{b-a}\|_{L^2(Y_{b+})} : \|f\|_{W^{1,2}(Y_{a+})} = 1\} \\ &= \sup\{\|\hat{T}(f_{b-a})\|_{L^2(Y_{b+})} : \|f\|_{W^{1,2}(Y_{a+})} = 1\} \\ &= \|\hat{T}\|_{op,b}. \end{aligned}$$

7.2 The commutative diagram

In this section, we derive the asymptotic behavior of harmonic spinors in the case where the connection A decays exponentially to its limit Γ, with decay rate β,

$$|A - \Gamma| \leq Ce^{-\beta t}.$$

This feat is be achieved by some diagram chase. We first introduce maps to compose our diagram. Suppose $\eta < \delta$ and $\mathcal{S}pec(D) \cap [\eta, \delta] = \{\lambda\}$. Then the map

$$\begin{aligned} I \colon \Pi_\eta^- W^{\frac{1}{2},2}(Y_a) \oplus W_\lambda &\to \Pi_\delta^- W^{\frac{1}{2},2}(Y_a) \\ (\phi, \psi) &\mapsto \phi + e^{a\lambda}\psi \end{aligned}$$

is obviously an isomorphism.

Similarly, the map

$$\begin{aligned} J \colon W_\eta^{1,2}(Y_{a+}) \oplus W_\lambda &\to W_\delta^{1,2}(Y_{a+}) \\ (\phi, \psi) &\mapsto \phi + e^{\lambda t}\psi \end{aligned}$$

is obviously an injection.

Consider now the map

$$\begin{aligned} K \colon W_\eta^{1,2}(Y_{a+}) \oplus W_\lambda &\to L_\eta^2(Y_{a+}) \oplus \Pi_\eta^- W^{\frac{1}{2},2}(Y_a) \oplus W_\lambda \\ (\phi, \psi) &\mapsto \big(\not{\mathfrak{D}}_A(\phi + e^{\lambda t}\psi), \Pi_\eta^- \phi, \psi + e^{-a\lambda}\Pi_\lambda \phi(a)\big). \end{aligned}$$

Let's verify that this map is well-defined. As $\not{\mathfrak{D}}_A^*(e^{\lambda t}\psi) = \not{\mathfrak{D}}_\Gamma^*(e^{\lambda t}\psi) + cl(A - \Gamma)e^{\lambda t}\psi$, we have $\big|\not{\mathfrak{D}}_A^*(e^{\lambda t}\psi)\big| \leq Ce^{(\lambda-\beta)t}|\psi|$. Hence, if

$$\lambda - \beta < \eta, \tag{7.3}$$

then $\not{\mathfrak{D}}_A^*(e^{\lambda t}\psi) \in L_\eta^2(Y_{a+})$, and K is well-defined.

We put all these maps in a diagram

$$\begin{array}{ccc} W_\delta^{1,2}(Y_{a+}) & \xrightarrow{\not{\mathbb{D}}^*} & L_\delta^2(Y_{a+}) \oplus \Pi_\delta^- W^{\frac{1}{2},2}(Y_a) \\ {\scriptstyle J}\uparrow & & \uparrow{\scriptstyle \iota \oplus I} \\ W_\eta^{1,2}(Y_{a+}) \oplus W_\lambda & \xrightarrow[K]{} & L_\eta^2(Y_{a+}) \oplus \Pi_\eta^- W^{\frac{1}{2},2}(Y_a) \oplus W_\lambda \end{array} \tag{7.4}$$

which is commutative as

$$\begin{aligned} \not{\mathbb{D}}^* J(\phi, \psi) &= \big(\not{\mathfrak{D}}_A^*(\phi + e^{\lambda t}\psi), \Pi_\delta^- \phi(a) + e^{a\lambda}\psi\big) \\ &= \big(\not{\mathfrak{D}}_A^*(\phi + e^{\lambda t}\psi), \Pi_\eta^- \phi(a) + \Pi_\lambda \phi(a) + e^{a\lambda}\psi\big) \\ &= \Big(\not{\mathfrak{D}}_A^*(\phi + e^{\lambda t}\psi), \Pi_\eta^- \phi(a) + e^{a\lambda}\big(\psi + e^{-a\lambda}\Pi_\lambda \phi(a)\big)\Big) \\ &= (\iota \oplus I)K(\phi, \psi). \end{aligned}$$

Now that we know that the diagram is commutative, we want to exploit the fact that its rows are isomorphisms. While Theorem 7.1-1 assures us that $\not{\mathbb{D}}^*$ is an isomorphism, we still have to prove that K is one as well. Using the identification

$$\not{\mathfrak{D}}^* \colon W^{1,2}_\eta(Y_{a+}) \equiv L^2_\eta(Y_{a+}) \oplus \Pi^-_\eta W^{\frac{1}{2},2}(Y_a),$$

we see that K has the form

$$\begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix}$$

for the splitting $W^{1,2}_\eta(Y_{a+}) \oplus W_\lambda$ of the domain and codomain. Hence $K-1$ is a compact operator, and K is thus Fredholm of index 0. If $K(x) = K(y)$, then $\not{\mathfrak{D}}^* J(x) = \not{\mathfrak{D}}^* J(y)$ as the diagram is commutative, hence $x = y$ and K is injective. Being of index 0, it henceforth must be an isomorphism.

Let's now exploit this fantastic diagram. Suppose

$$\phi \in \ker(\not{\mathfrak{D}}^*_A) \cap W^{1,2}_\delta(\mathbb{R} \times Y).$$

Then for a big enough, the diagram (7.4) has rows which are isomorphism for δ close to the first negative eigenvalue of D_{Γ_+} and η past it, and satisfying condition (7.3). Theorem 6.3-2 guarantees us that $\phi \in W^{1,2}_\delta(Y_{a+})$ for that particular δ.

We now chase around the diagram. Since I is an isomorphism, we know there exist $(\chi, \nu) \in \Pi^-_\eta W^{\frac{1}{2},2}(Y_a) \oplus W_\lambda$ such that

$$\iota \oplus I(0, \chi, \nu) = \not{\mathfrak{D}}^*(\phi).$$

But as K is an isomorphism, there is $(\bar\phi, \bar\psi) \in W^{1,2}_\eta(Y_{a+}) \oplus W_\lambda$ such that

$$K(\bar\phi, \bar\psi) = (0, \chi, \nu).$$

By commutativity of the diagram, we have

$$\not{\mathfrak{D}}^* J(\bar\phi, \bar\psi) = \not{\mathfrak{D}}^*(\phi)$$

but $\not{\mathfrak{D}}^*$ is an isomorphism hence $\phi = e^{\lambda t}\bar\psi + \bar\phi$ for $t > a$.

Of course, at this point the choice of a is artificial and we can choose $a = 0$. We hence proved the following result.

Theorem 7.2-1. *Suppose $\phi \in \ker(\not{\mathfrak{D}}^*_A) \cap W^{1,2}_\delta(\mathbb{R} \times Y)$. Suppose $\lambda - \beta < \eta < \delta$ and that λ is the only eigenvalue of D between η and δ: $\mathrm{Spec}(D) \cap [\eta, \delta] = \{\lambda\}$. Then there exist $\bar\psi \in W_\lambda$ and $\bar\phi \in W^{1,2}_\eta(Y^+)$ such that*

$$\phi = e^{\lambda t}\bar\psi + \bar\phi \text{ for } t > 0. \tag{7.5}$$

Furthermore, $\bar\phi = O(e^{\eta t})$ as $t \to \infty$.

7.3 Asymptotic on $\mathbb{R}^4$

In the spirit of this chapter, we want to study $\mathbb{R}^4$ as a cylindrical manifold. Let's then use the conformal equivalence

$$\begin{aligned} \mathbb{R} \times S^3 &\to \mathbb{R}^4 \setminus \{0\} \\ (t, x) &\mapsto e^t x. \end{aligned}$$

The respective metrics of those spaces are related by the formula

$$g_{\mathbb{R}^4\setminus\{0\}} = (e^{2t})g_{\mathbb{R}\times S^3} \quad \text{or} \quad g_{\mathbb{R}^4\setminus\{0\}} = |x|^2 g_{\mathbb{R}\times S^3}.$$

It then follows that $dvol_{\mathbb{R}^4\setminus\{0\}} = |x|^4 dvol_{\mathbb{R}\times S^3}$, hence

$$\|\phi\|_{L^2(\mathbb{R}^4\setminus\{0\})} = \||x|^2\phi\|_{L^2(\mathbb{R}\times S^3)}.$$

The spinor bundles of $\mathbb{R}\times S^3$ and $\mathbb{R}^4\setminus\{0\}$ are isomorphic. Once we fix a spinor bundle to work with, we can compare the Dirac operators given for the two metrics. The correct relation, as seen in Appendix D, is

$$D_{\mathbb{R}^4\setminus\{0\}} = |x|^{-5/2} D_{\mathbb{R}\times S^3} |x|^{3/2}.$$

Thus

$$D_{\mathbb{R}^4\setminus\{0\}}\phi = 0 \quad \text{iff} \quad D_{\mathbb{R}\times S^3}(|x|^{3/2}\phi) = 0. \tag{7.6}$$

Let p be the projection $\mathbb{R}\times S^3 \to S^3$. Let $S(S^3)$ be the spinor bundle of S^3. Set S^+ and S^- to be $p^*(S(S^3))$. The spinor bundle on $\mathbb{R}\times S^3$ is $S^+\oplus S^-$.

The Clifford multiplication exchanges S^+ and S^-. For vectors tangent to S^3, the Clifford multiplication is already defined. The vector $\partial/\partial t$ acts as $id\colon S^+\to S^-$ and $-id\colon S^-\to S^+$.

In this decomposition, the Dirac operator splits nicely:

$$D^{\pm}_{\mathbb{R}\times S^3} = \pm\frac{\partial}{\partial t} + D_{S^3}. \tag{7.7}$$

We use now the knowledge of the eigenvalues of the D_{S^3} on S^3 obtained in Theorem 4.1-3 to understand the asymptotic expansion of solutions ϕ to the equation

$$D^-_{\mathbb{R}^4\setminus\{0\}}\phi = 0$$

under the constraint of being L^2.

We have here a basis of the kernel of $D^-_{\mathbb{R}^4\setminus\{0\}}$. Indeed, if

$$D_{S^3}\psi_\lambda = \lambda\psi_\lambda$$

then, as suggested by Equation (7.7), we have

$$D^-_{\mathbb{R}\times S^3}(e^{\lambda t}\psi_\lambda) = 0$$

and thus, because of the conformal relation 7.6, we have

$$D^-_{\mathbb{R}^4\setminus\{0\}}(|x|^{\lambda-3/2}\psi_\lambda) = 0.$$

Let's now use the notation of Chapter 1. Hence A is an instanton connection on a bundle E over $\mathbb{R}^4$ and $V_E = L^2(\mathbb{R}^4, S^-\otimes E)\cap\ker(D^-_A)$. Theorem 7.2-1 then tells us any $\phi\in V_E$ has an asymptotic behavior

$$\phi = |x|^{-3}\psi_{-3/2} + O(|x|^{-4}).$$

Theorem 4.1-3 tells us the space of possible $\psi_{-3/2}$ has dimension $2\dim(E)$. We build this space using parallel sections of $S^+\otimes E$ for the trivial connection on E. Let $a\in\Gamma(S^+\otimes E)$ be parallel.

Consider the section $\phi_a := \rho(\nu)a/r^3$ of $S^- \otimes E$. In coordinates, we have

$$\phi_a = \sum_{i=1}^{4} \frac{x_i}{r^4}\rho(\partial_i)a$$

and we compute

$$\begin{aligned} D_{\mathbb{R}^4\setminus\{0\}}\phi_a &= \sum_{1\le i,j\le 4} \rho(\partial_j)\partial_j(\frac{x_i}{r^4})\rho(\partial_i)a \\ &= \sum_{1\le i,j\le 4} (\frac{\delta_{ij}r^4 - 4x_ix_jr^2}{r^8})\rho(\partial_j)\rho(\partial_i)a \\ &= -(4r^{-4} - 4r^{-4})a \\ &= 0. \end{aligned}$$

But then, formula 4.2 implies that

$$\begin{aligned} D_r(\phi_a) &= -\rho(\nu)\frac{\partial\phi_a}{\partial r} - \frac{3}{2r}\rho(\nu)\phi_a \\ &= -\frac{3}{r^4}a + \frac{3}{2r^4}a \\ &= -\frac{3}{2}a/r^4 \\ &= \rho(-\nu)\Big(-\frac{3}{2}\phi/r\Big). \end{aligned}$$

Recall now that S^+ and S^- are actually pullbacks of the spinor bundle of S^3. In this setting, $\rho(\nu)\colon S^- \to S^+$ is $-id$, as explained in the beginning of this section. So restricting to $r = 1$, we really find

$$D_{S^3}\phi = -\frac{3}{2}\phi.$$

So far we proved that for any $\phi \in V_E$, we have

$$\phi = |x|^4\rho(x)a + O(|x|^{-4})$$

for a parallel section a of $S^+ \otimes E$, parallel that is for the trivial connection on E. This result is not exactly Equation (1.7), but leads to it. Indeed, the same analysis we did works for the Laplacian. Hence parallel sections of $S^+ \otimes E$ for A or for the trivial connection are the same to leading order, hence we have

$$\phi = |x|^4\rho(x)\hat{\phi} + O(|x|^{-4})$$

for some $\hat{\phi} \in W_E$, and we proved Equation (1.7).

Chapter 8

Nahm Transform: Instantons to singular monopoles

"It doesn't matter what you write
as long as you write the truth.
Then we can figure out what it means."
TOMASZ S. MROWKA

Following the heuristic of Chapter 2, we show in this chapter that the Nahm transform

$$\mathfrak{N}(E, A) = (V, B, \Phi)$$

of a $SU(2)$-instanton (E, A) on $\mathbb{R} \times T^3$ is a singular monopole (V, B, Φ) over over T^3.

As we found out in Chapter 5, once in a temporal gauge, the connection A has limiting flat connections over the cross-section T^3 at $+\infty$ and $-\infty$, say

$$\lim_{t \to \pm\infty} A = \Gamma_{\pm}.$$

The flat connection $\Gamma_{\pm}$ gives a splitting $L_{w_{\pm}} \oplus L_{-w_{\pm}}$ of the restriction of E to T^3 at the infinities, for some $w_{\pm} \in T^3$. Let W denote the set

$$W := \{w_+, -w_+, w_-, -w_-\}.$$

As before, we denote A_z the connection on $E \otimes L_z$. We consider the Dirac operator

$$\mathfrak{D}^*_{A_z} : L^2 \to L^2.$$

Outside of W, Theorem 6.1-5 guarantees that $\mathfrak{D}^*_{A_z}$ is Fredholm. Since $\ker(\mathfrak{D}_{A_z}) = 0$ as F_{A_z} is ASD, we have a bundle V over $T^3 \setminus W$ whose fiber at z is

$$V_z := \ker(\mathfrak{D}^*_{A_z}) \cap L^2.$$

By a gauge transformation, we can make the connection Pd^z independent of the $\mathbb{R}$ factor. We can thus see it as

$$\text{a connection } B \text{ on } T^3 \setminus W,$$
$$\text{a Higgs field } \Phi \in \Gamma(T^3 \setminus W, \mathrm{End}V).$$

The main result of this present thesis is the following theorem.

Theorem 8.0-1. *Outside of a set W consisting of at most four points, the family of vector spaces V described above defines a vector bundle of rank*

$$\frac{1}{8\pi^2}\int |F_A|^2,$$

and the couple (B, Φ) satisfies the Bogomolny equation

$$\nabla_B \Phi = *F_B.$$

For $w \in W$ and z close enough to w, unless we are in the Scenario 2 of page 91, there are maps $\Phi^\perp$ and $\Phi^\lrcorner$ such that

$$\Phi = \frac{-i}{2|z-w|}\Phi^\perp + \Phi^\lrcorner,$$

and $\Phi^\perp$ is the L^2-orthogonal projection on the orthogonal complement of a naturally defined sub-bundle $V_\lrcorner$ of V.

The last part of the theorem is made clearer by the introduction of some notation in Section 8.2. The assumption that we are not in the Scenario 2 of page 91 can most probably be dropped.

Proof: The rank of V is computed in Section 8.1.
The limit term $\lim_\partial$ of Equation (2.2) is

$$\lim\nolimits_\partial = \langle \nu\Omega G\phi, d^z\psi\rangle_{T^3}\Big|_{-\infty}^{\infty}.$$

For $z \notin W$, both $G\phi$ and $d^z\psi$ decay exponentially by Equation (7.5) hence

$$\lim\nolimits_\partial = 0,$$

and the connection Pd^z on $\mathbb{R} \times (T^3 \setminus W)$ is ASD. Thus, as explained in Chapter 2, the pair (B, Φ) satisfies outside of W the appropriate dimensional reduction of the ASD equation, which is in this case the Bogomolny Equation (A.3):

$$\nabla_B \Phi = *F_B.$$

The last part of the theorem is the content of Section 8.3 and rest on the splitting of Section 8.2. □

8.1 An L^2-index theorem for $\mathbb{R} \times T^3$

The following theorem is reminiscent of the similar result for $\mathbb{R}^4$.

Theorem 8.1-1. *For a $SU(2)$-instanton (E, A) on $\mathbb{R} \times T^3$, the index of the Dirac operator*

$$\not{D}_A \colon W^{1,2}(\mathbb{R} \times T^3) \to L^2(\mathbb{R} \times T^3)$$

when A has nonzero limits at $\pm\infty$ is given by the formula

$$\mathrm{ind}(\not{\mathfrak{D}}_A) = -\frac{1}{8\pi^2}\int |F_A|^2.$$

Proof: As seen earlier, the fact that A has nonzero limits guarantees that the operator $\not{\mathfrak{D}}_A$ is Fredholm on $W^{1,2}$. Moreover, A decays exponentially to its limits.

Recall now that $\mathrm{ind}(\not{\mathfrak{D}}_A) = \mathrm{ind}(\not{\mathfrak{D}}_{a_R})$ for all $R > 0$. We now compute $\mathrm{ind}(\not{\mathfrak{D}}_{a_R})$ using the relative index theorem. It could be that $\Gamma_- \neq \Gamma_+$, but this case is easily converted to a situation where $\Gamma_- = \Gamma_+$, as we now see.

Choose a path Γ_s in the space of flat connections on T^3 starting at Γ_+ and ending at Γ_-, and avoiding the trivial connection. Hence $0 \notin Spec(D_{\Gamma_s})$ for all s. To define the family of connections a_R^s, replace Γ_+ by Γ_s in the definition of a_R given by Equation (6.3).

The family $\not{\mathfrak{D}}_{a_R^s}$ of Fredholm operator depends continuously on s. Hence

$$\mathrm{ind}(\not{\mathfrak{D}}_A) = \mathrm{ind}(\not{\mathfrak{D}}_{a_R}) = \mathrm{ind}(\not{\mathfrak{D}}_{a_R^0}) = \mathrm{ind}(\not{\mathfrak{D}}_{a_R^1}).$$

Note now that the connection a_R^1 equals Γ_- outside $[-R-1, R+1]\times T^3$. Hence the relative index theorem tells us

$$\mathrm{ind}(\not{\mathfrak{D}}_{a_R^1}) - \mathrm{ind}(\not{\mathfrak{D}}_{\Gamma_-}) = \mathrm{ind}(\tilde{\not{\mathfrak{D}}}_{a_R^1}) - \mathrm{ind}(\tilde{\not{\mathfrak{D}}}_{\Gamma_-}), \tag{8.1}$$

where the tilded operators are extensions to some compact manifold of the restriction of the operators $\not{\mathfrak{D}}_{a_R^1}$ and $\not{\mathfrak{D}}_{\Gamma_-}$ to $[-R-1, R+1]\times T^3$.

Lemma 6.1-1 and Theorem 3.4-1 tell us that $\mathrm{ind}(\not{\mathfrak{D}}_{\Gamma_-}) = 0$. Hence the left-hand-side of Equation (8.1) is equal to $\mathrm{ind}(\not{\mathfrak{D}}_A)$.

To compute the right-hand-side, we embed $[-R-1, R+1]\times T^3$ in some flat T^4. The spinor bundles S^+ and S^- on $[-R-1, R+1]\times T^3$ agree very nicely with those of T^4. We extend both a_R^1 and Γ_- by the trivial bundle with connection Γ_-.

The Atiyah-Singer index theorem (see [Roe98, Thm 12.27, p.164] or [LM89, Thm III.12.10, p. 256]) tells us that

$$\begin{aligned}
\mathrm{ind}(\tilde{\not{\mathfrak{D}}}_{\Gamma_-}) &= \{ch(\Gamma_-)\cdot \hat{\mathbf{A}}(T^4)\}[T^4] \\
\mathrm{ind}(\tilde{\not{\mathfrak{D}}}_{a_R^1}) &= \{ch(a_R^1)\cdot \hat{\mathbf{A}}(T^4)\}[T^4] \\
&= (\frac{c_1^2}{2} - c_2)[T^4].
\end{aligned}$$

Since a_R^1 is in $SU(2)$, we have $c_1 = 0$, while

$$c_2[T^4] = \frac{1}{8\pi^2}\int_{T^4} |F_{a_R^1}|^2.$$

Note that on the complement of $[-R-1, R+1]\times T^3$ in T^4, the connection a_R^1 equals Γ_- hence is flat there. Furthermore, on $[-R, R]\times T^3$, we have $a_R^1 = A$. On $[R, R+1]\times T^3$ and $[-R-1, -R]\times T^3$, the curvature $F_{a_R^1}$ involves cut off functions, their derivatives and $(A - \Gamma_-)$ terms. Since A tends to Γ_- exponentially fast, we therefore have constant C and β such that

$$\left|\mathrm{ind}(\not{\mathfrak{D}}_A) + \frac{1}{8\pi^2}\int_{[-R,R]\times T^3} |F_A|^2\right| \le Ce^{-\beta R}.$$

As $R \to \infty$, we have the wanted result. $\square$

Now suppose (E, A) is a $SU(2)$-instanton on $\mathbb{R} \times T^3$. As mentioned before,

$$\lim_{t \to \pm\infty} = \Gamma_\pm.$$

The flat connection $\Gamma_\pm$ gives a splitting $L_{w_\pm} \oplus L_{-w_\pm}$, for some $w_\pm \in \Lambda^*$, of the bundle E at $\pm\infty$ respectively.

We twist the connection A by the flat connection parameterized by $z \in T^3$. Hence

$$\begin{aligned} \mathit{Spec}(\Gamma_+ \otimes L_z) &= \pm 2\pi|\Lambda^* - z + w_+| \cup \pm 2\pi|\Lambda^* - z - w_+|, \\ \mathit{Spec}(\Gamma_- \otimes L_z) &= \pm 2\pi|\Lambda^* - z + w_-| \cup \pm 2\pi|\Lambda^* - z - w_-|. \end{aligned}$$

Thus $\not{\mathfrak{D}}_{A_z}$ is Fredholm as long as $z \pm w_+ \notin \Lambda^*$ and $z \pm w_- \notin \Lambda^*$. Moreover, when it is Fredholm, the elements of its L^2-kernel decay exponentially.

8.2 A Geometric Splitting and Exact Sequences

In this section, we analyse a splitting of V in a neighborhood of a point $w \in W$ where the solution (B, ϕ) to Bogomolny equation is singular. This point w is associated to the limit $\Gamma = \Gamma_+$ of A at, say, $+\infty$, in the sense that Γ splits E as $L_w \oplus L_{-w}$ on T^3.

Suppose the connection A decays at most with rate β, as in $|A - \Gamma_+| \leq Ce^{-\beta t}$ for $t > 0$ and $|A - \Gamma_-| \leq Ce^{\beta t}$ for $t < 0$. Set

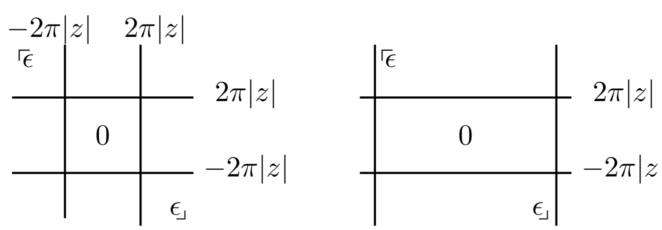
$$\epsilon := \frac{1}{4} \min\Big(\beta, \mathrm{dist}\big(w, \Lambda^* + W \setminus \{w\}\big)\Big),$$

and define the six weights

$$\begin{array}{lll} {}^{\ulcorner}\epsilon := (-\epsilon, \epsilon) & \bar{\epsilon} := (0, \epsilon) & \epsilon^{\urcorner} := (\epsilon, \epsilon) \\ {}_{\llcorner}\epsilon := (-\epsilon, -\epsilon) & \underline{\epsilon} := (0, -\epsilon) & \epsilon_{\lrcorner} := (\epsilon, -\epsilon) \end{array}$$

displayed here in a way which is reminiscent of their position in $\mathbb{R}^2$.

Consider the ball $B^3(w)$ of radius 2ϵ around w. As z varies in $B^3(w)$, and depending on whether $\Gamma_+ = \Gamma_-$ or not, there are two or one walls to cross to pass from 0 to ${}^{\ulcorner}\epsilon$ and from $\epsilon_{\lrcorner}$ to 0. In a picture, we have



$\Gamma_+ = \Gamma_-$ $\qquad\qquad$ $\Gamma_+ \neq \Gamma_-$

As z varies in $B^3(w)$, those walls move around without ever touching $\epsilon_{\lrcorner}$ and ${}^{\ulcorner}\epsilon$. Hence for $L^2_{\epsilon_{\lrcorner}}$ and $L^2_{{}^{\ulcorner}\epsilon}$, the operators $\not{\mathfrak{D}}_{A_z}$, $\not{\mathfrak{D}}^*_{A_z}$ and $\not{\mathfrak{D}}^*_{A_z}\not{\mathfrak{D}}_{A_z}$ are Fredholm for all $z \in B^3(w)$.

Hence for $z \in B^3(w)$, the six vector spaces

$$\begin{aligned} \overline{V}_z &:= \ker(\not{\mathfrak{D}}^*_{A_z}) \cap L^2_{\overline{\epsilon}}, & \overline{K}_z &:= \ker(\not{\mathfrak{D}}_{A_z}) \cap L^2_{\overline{\epsilon}}, & \mathcal{H}_z &:= \ker(\nabla^*_{A_z} \nabla_{A_z}) \cap L^2_{\overline{\epsilon}}, \\ \underline{V}_z &:= \ker(\not{\mathfrak{D}}^*_{A_z}) \cap L^2_{\underline{\epsilon}}, & \underline{K}_z &:= \ker(\not{\mathfrak{D}}_{A_z}) \cap L^2_{\underline{\epsilon}}, & K_z &:= \ker(\not{\mathfrak{D}}_{A_z}) \cap L^2 \end{aligned}$$

are kernels of Fredholm operators. By contrast, the space V_z, already defined as $\ker(\not{\mathfrak{D}}^*_{A_z}) \cap L^2$, is not the kernel of a Fredholm operator at w.

Notice that none of those vector space form a priori a bundle over $B^3(w)$ as the dimensions could jump at random. However, for $L^2_{\overline{\epsilon}}$ and $L^2_{\underline{\epsilon}}$, the operators $\not{\mathfrak{D}}_{A_z}$, $\not{\mathfrak{D}}^*_{A_z}$, and $\nabla^*_{A_z} \nabla_{A_z}$ are Fredholm operators for all $z \in B^3(w)$. The various indices are therefore constant and we have that, for example,

$$\dim \underline{V}_z - \dim \overline{K}_z \text{ is constant on } B^3(w).$$

We have the following obvious results:

$$\begin{aligned} \underline{V} \subset V &\subset \overline{V}, & \underline{K} \subset K &\subset \overline{K}, \\ \not{\mathfrak{D}}\mathcal{H} &\subset \overline{V}, & \overline{K} &\subset \mathcal{H}, \\ & & \underline{K} = K &= \{0\}. \end{aligned}$$

It was remarked on page 73 that $\underline{V}_w = V_w$. The following few lemmas describe in more detail the relationship between the various spaces.

We saw in Section 3.4 that the smallest eigenvalues of D_{Γ_z} are $\pm 2\pi|z-w|$. For simplicity, we set

$$\lambda := 2\pi|z-w|,$$

and define

$$W_\lambda := \lambda \text{ eigenspace of } D_{\Gamma_z} \text{ on } T^3.$$

The family W_λ defines a bundle over the sphere $|z-w| = \lambda/2\pi$ around w. Its rank is given by

$$\mathrm{rk} W_\lambda = \begin{cases} 1, & \text{if } \lambda \neq 0 \text{ and } 2w \notin \Lambda^*; \\ 2, & \text{if } \lambda \neq 0 \text{ and } 2w \in \Lambda^*, \text{ or } \lambda = 0 \text{ and } 2w \notin \Lambda^*; \\ 4, & \text{if } \lambda = 0 \text{ and } 2w \in \Lambda^*. \end{cases} \tag{8.2}$$

As suggested by Theorem 7.2-1, this W_λ plays an important role in understanding the relations between the various spaces just introduced.

For any instanton connection A' on $\mathbb{R} \times T^3$, set

$$\begin{aligned} V(\delta) &:= \ker(\not{\mathfrak{D}}^*_{A'}) \cap L^2_\delta, \\ K(\delta) &:= \ker(\not{\mathfrak{D}}_{A'}) \cap L^2_\delta, \end{aligned}$$

and let $[\delta]$ denote the open square in $\mathbb{R}^2 \setminus \mathfrak{G}_{A'}$ containing δ.

Lemma 8.2-1 (one wall). *Suppose $\delta, \eta \in \mathbb{R}^2 \setminus \mathfrak{G}_{A'}$ are weights for which $[\delta]$ and $[\eta]$ are adjacent and separated by the wall $\{\mu\} \times \mathbb{R}$ or $\mathbb{R} \times \{\mu\}$. Then the sequence*

$$0 \longrightarrow V(\delta) \longrightarrow V(\eta) \xrightarrow{\lim(e^{-\mu t}\cdot)} W_\mu \xrightarrow{\left(\lim(e^{\mu t}\cdot)\right)^*} K(-\delta)^* \longrightarrow K(-\eta)^* \longrightarrow 0, \tag{8.3}$$

where the limits are both evaluated at $+\infty$ when $[\eta]$ is above $[\delta]$ and at $-\infty$ when $[\eta]$ is to the left of $[\delta]$, is exact.

Proof: Theorem 7.2-1 ensures that the limits give functions α and β^* which are well defined, and that

$$0 \longrightarrow V(\delta) \longrightarrow V(\eta) \longrightarrow W_\mu \quad \text{and} \quad 0 \longrightarrow K(-\eta) \longrightarrow K(-\delta) \longrightarrow W_\mu$$

are exact.

It only remains to prove that Sequence (8.3) is exact at W_μ. Suppose $\phi \in V(\eta)$ and $\psi \in K(-\eta)$. Then

$$\begin{aligned} 0 &= \langle \not{\mathfrak{D}}_{A'}^* \phi, \psi \rangle - \langle \phi, \not{\mathfrak{D}}_{A'} \psi \rangle \\ &= \lim_{t \to \infty} \langle \phi, \nu\psi \rangle - \lim_{t \to -\infty} \langle \phi, \nu\psi \rangle \\ &= \lim_{t \to \infty} \langle e^{-\mu t}\phi, \nu e^{\mu t}\psi \rangle - \lim_{t \to -\infty} \langle e^{-\mu t}\phi, \nu e^{\mu t}\psi \rangle. \end{aligned}$$

One of those limits is $\beta^*\alpha(\phi)(\psi)$ while the other one vanishes as we now see. Suppose $[\eta]$ is above $[\delta]$, and suppose $\{\mu'\} \times \mathbb{R}$ is the wall to their right. Then $\phi = O(e^{\mu' t})$ as $t \to -\infty$ by Theorem 7.2-1. But for some $\mu'' < \mu'$, the wall $\{-\mu''\} \times \mathbb{R}$ is exactly to the right of $[-\eta]$ hence $\psi = O(e^{-\mu'' t})$ as $t \to -\infty$. But then

$$\beta^*\alpha(\phi)(\psi) = \lim_{t \to -\infty} O(e^{(\mu' - \mu'')t}) = 0,$$

hence $\mathrm{Im}(\alpha) \ker(\beta^*)$. A similar argument establish the same fact when $[\eta]$ is to the left of $[\delta]$.

The sequence is then exact if $\dim \mathrm{Im}(\alpha) = \dim \ker(\beta^*)$. We have two short exact sequences:

$$\begin{gathered} 0 \longrightarrow V(\delta) \longrightarrow V(\eta) \longrightarrow \mathrm{Im}(\alpha) \longrightarrow 0, \quad \text{and} \\ 0 \longrightarrow W_\mu / \ker(\beta^*) \longrightarrow K(-\delta)^* \longrightarrow K(-\eta)^* \longrightarrow 0. \end{gathered}$$

Using those short exact sequences and notation from Equations (6.10), we have

$$\begin{aligned} \dim \mathrm{Im}(\alpha) - \dim \ker(\beta^*) &= N^*(\eta) - N^*(\delta) - \dim W_\mu + N(-\delta) - N(-\eta) \\ &= \mathrm{ind}^*(\eta) - \mathrm{ind}^*(\delta) - \dim W_\mu. \end{aligned}$$

The Wall Crossing Theorem 6.4-1 forces the last line to be 0. The proof is thus complete. □

Lemma 8.2-2. *Suppose $\Gamma_+ \neq \Gamma_-$. Then the sequences*

$$0 \longrightarrow V_z \longrightarrow {}^{\ulcorner}V_z \longrightarrow W_\lambda \longrightarrow 0, \qquad \textit{for } \lambda \neq 0, \tag{8.4}$$

$$0 \longrightarrow V_{\lrcorner z} \longrightarrow V_z \longrightarrow W_{-\lambda} \longrightarrow {}^{\ulcorner}K_z \longrightarrow 0, \qquad \textit{for } \lambda \neq 0, \tag{8.5}$$

$$0 \longrightarrow V_w \longrightarrow {}^{\ulcorner}V_w \longrightarrow W_0 \longrightarrow {}^{\ulcorner}K_w \longrightarrow 0, \tag{8.6}$$

are exact.

Proof: Apply Lemma 8.2-1 to the choice of weights $\{{}^{\ulcorner}\epsilon, 0\}$ and $\{0, \epsilon_{\lrcorner}\}$ for the connection $A' = A_z$, and remember that $K_{\lrcorner} = K = \{0\}$. □

Building up on that knowledge, we work out in Appendix C a technology used to deal with the two walls involved in passing from V to ${}^{\ulcorner}V$ in the case $\Gamma_+ = \Gamma_-$. The result of the beautiful abstract non-sense taking place there is summarized in the following lemma, which should be compared to Lemma 8.2-2.

Lemma 8.2-3. *Suppose* $\Gamma_+ = \Gamma_-$. *Then the sequences*

$$0 \longrightarrow V_z \longrightarrow \ulcorner\!V_z \longrightarrow W_\lambda \oplus W_{-\lambda} \longrightarrow 0, \qquad \textit{for } \lambda \neq 0, \tag{8.7}$$
$$0 \longrightarrow V_{\lrcorner z} \longrightarrow V_z \longrightarrow W_\lambda \oplus W_{-\lambda} \longrightarrow \ulcorner\!K_z \longrightarrow 0, \qquad \textit{for } \lambda \neq 0, \tag{8.8}$$
$$0 \longrightarrow V_w \longrightarrow \ulcorner\!V_w \longrightarrow W_0 \oplus W_0 \longrightarrow \ulcorner\!K_w \longrightarrow 0, \tag{8.9}$$

are exact.

Proof: See Appendix C. □

An analysis for $\nabla^*_{A_z}\nabla_{A_z}$ parallel to the one of Chapter 6 for $\not{D}_{A_z}$ brings a very similar wall crossing formula

$$\operatorname{ind}(\nabla^*_{A_z}\nabla_{A_z}, \ulcorner\epsilon) - \operatorname{ind}(\nabla^*_{A_z}\nabla_{A_z}, \epsilon_\lrcorner) = \begin{cases} 2\dim W_0, & \text{for } \Gamma_+ \neq \Gamma_-; \\ 4\dim W_0, & \text{for } \Gamma_+ = \Gamma_-. \end{cases}$$

However, since $\nabla^*_{A_z}\nabla_{A_z}$ is self-adjoint, $\operatorname{ind}(\nabla^*_{A_z}\nabla_{A_z}, \ulcorner\epsilon) = -\operatorname{ind}(\nabla^*_{A_z}\nabla_{A_z}, \epsilon_\lrcorner)$, whence

$$\operatorname{rk}\mathcal{H} = \begin{cases} \dim W_0, & \text{for } \Gamma_+ \neq \Gamma_-; \\ 2\dim W_0, & \text{for } \Gamma_+ = \Gamma_-. \end{cases}$$

Using Equation (8.2), we can even say

$$\operatorname{rk}\mathcal{H} = \begin{cases} 2, & \text{for } \Gamma_+ \neq \Gamma_- \text{ and } 2w \notin \Lambda^*; \\ 4, & \text{for } \Gamma_+ \neq \Gamma_- \text{ and } 2w \in \Lambda^*, \text{ or } \Gamma_+ = \Gamma_- \text{ and } 2w \notin \Lambda^*; \\ 8, & \text{for } \Gamma_+ = \Gamma_- \text{ and } 2w \in \Lambda^*. \end{cases}$$

For $z \neq w$, an analysis parallel to the one of Chapter 7 gives injective maps

$$0 \longrightarrow \mathcal{H}_z \longrightarrow W_\lambda \oplus W_{-\lambda} \longrightarrow 0, \qquad \text{for } z \neq w \text{ and when } \Gamma_+ \neq \Gamma_-, \tag{8.10}$$
$$0 \longrightarrow \mathcal{H}_w \longrightarrow W_0 \longrightarrow 0, \qquad \text{when } \Gamma_+ \neq \Gamma_-, \tag{8.11}$$
$$0 \longrightarrow \mathcal{H}_z \longrightarrow \left(W_\lambda \oplus W_{-\lambda}\right)^2 \longrightarrow 0, \qquad \text{for } z \neq w \text{ and when } \Gamma_+ = \Gamma_-, \tag{8.12}$$
$$0 \longrightarrow \mathcal{H}_w \longrightarrow W_0 \oplus W_0 \longrightarrow 0, \qquad \text{when } \Gamma_+ = \Gamma_-, \tag{8.13}$$

which are surjective for dimensional reasons.
Bringing all of those sequences together allows us to conclude the following.

Theorem 8.2-4. *On* $B^3(w)$, *we have*

$$\ulcorner\!V = V_\lrcorner \oplus \not{D}\mathcal{H}.$$

Proof: Denote W'_λ the space

$$W'_\lambda := \begin{cases} W_\lambda \oplus W_{-\lambda}, & \text{if } \Gamma_+ = \Gamma_-; \\ W_\lambda, & \text{if } \Gamma_+ \neq \Gamma_-. \end{cases}$$

Let $p\colon W'_\lambda \oplus W'_{-\lambda} \to W'_\lambda$ denote the map $p(a,b) = 2\lambda b$.

For $\lambda \neq 0$, we use the Snake Lemma on the diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \mathcal{H} & \longrightarrow & {W'}_{\lambda} \oplus {W'}_{-\lambda} & \longrightarrow & 0 \\ & & \big\downarrow & & \not{\mathfrak{D}}\big\downarrow & & \big\downarrow p & & \\ 0 & \longrightarrow & V & \longrightarrow & \bar{V} & \longrightarrow & {W'}_{\lambda} & \longrightarrow & 0 \end{array}$$

coming from Sequences (8.4), (8.7), (8.10), and (8.12), to produce an exact sequence

$$\begin{array}{ccccccccccc} \ker(0) & \longrightarrow & \ker(\not{\mathfrak{D}}) & \longrightarrow & \ker(p) & \longrightarrow & \operatorname{coker}(0) & \longrightarrow & \operatorname{coker}(\not{\mathfrak{D}}) & \longrightarrow & \operatorname{coker}(p) \\ 0 & \longrightarrow & \bar{K}_z & \longrightarrow & {W'}_{-\lambda} & \longrightarrow & V_z & \longrightarrow & \operatorname{coker}(\not{\mathfrak{D}}) & \longrightarrow & 0 \end{array} \tag{8.14}$$

Note that the map $V \to \operatorname{coker}(\not{\mathfrak{D}})$ being surjective forces $\bar{V}$ to be spanned by V and $\not{\mathfrak{D}}\mathcal{H}$.
Sequences (8.5) and (8.8) imply

$$\dim V_z = \dim \underline{V}_z + \dim {W'}_{\lambda} - \dim \bar{K}_z$$

while sequences (8.4) and (8.7) imply

$$\dim \bar{V}_z = \dim V_z + \dim {W'}_{\lambda}.$$

Thus

$$\dim \bar{V}_z = \dim \underline{V}_z + 2 \dim {W'}_{\lambda} - \dim \bar{K}_z = \dim \underline{V}_z + \dim \not{\mathfrak{D}}\mathcal{H}.$$

Since Lemma 6.3-1 guarantees that $\langle \not{\mathfrak{D}}\mathcal{H}, \underline{V} \rangle = \{0\}$, we have $V \cap \not{\mathfrak{D}}\mathcal{H}$ perpendicular to $\underline{V}$ for the L^2 inner product. Hence $\not{\mathfrak{D}}\mathcal{H} \cap \underline{V} = \{0\}$, and $\bar{V}_z = \underline{V}_z \oplus \not{\mathfrak{D}}\mathcal{H}$.
It remains to prove the theorem for $z = w$. We already know $\underline{V}_w = V_w$ and $\not{\mathfrak{D}}\mathcal{H}_w \subset \bar{V}_w$. We also know from Sequences (8.6) (8.9) that

$$\begin{aligned} \dim \bar{V}_w &= \dim V_w + \dim {W'}_0 - \dim \bar{K}_w \\ &= \dim \underline{V}_w + \dim \not{\mathfrak{D}}\mathcal{H}_w. \end{aligned}$$

We therefore only have to prove that the intersection $V_w \cap \not{\mathfrak{D}}_{A_w}\mathcal{H}_w$ is $\{0\}$ to complete the proof.
The asymptotic behavior of $\phi \in \mathcal{H}_w$ is

$$\phi = \begin{cases} t\phi_0^+ + \phi_1^+ + o(1), & \text{as } t \to \infty; \\ t\phi_0^- + \phi_1^- + o(1), & \text{as } t \to -\infty; \end{cases}$$

for some $\phi_0^\pm, \phi_1^\pm \in W_0$. If $\Gamma_+ \neq \Gamma_-$, we must have $\phi_0^- = \phi_1^- = 0$, as w is associated to Γ_+.
The asymptotic behavior of $\not{\mathfrak{D}}_{A_w}\phi$ is

$$\not{\mathfrak{D}}_{A_w}\phi = \begin{cases} \phi_0^+ + o(1), & \text{as } t \to \infty; \\ \phi_0^- + o(1), & \text{as } t \to -\infty. \end{cases}$$

Suppose $\not{\mathfrak{D}}_{A_w}\phi \in L^2$. Then

$$\begin{aligned} \|\not{\mathfrak{D}}_{A_w}\phi\|_{L^2}^2 &= \langle \not{\mathfrak{D}}_{A_w}^* \not{\mathfrak{D}}_{A_w}\phi, \phi \rangle + \lim_{t\to\infty} \langle \not{\mathfrak{D}}_{A_w}\phi, \nu\phi \rangle + \lim_{t\to-\infty} \langle \not{\mathfrak{D}}_{A_w}\phi, \nu\phi \rangle \\ &= \langle \phi_0^+, \phi_1^+ \rangle + \lim_{t\to\infty} t|\phi_0^+|^2 - \langle \phi_0^-, \phi_1^- \rangle - \lim_{t\to-\infty} t|\phi_0^-|^2. \end{aligned}$$

For $\|\not{\mathfrak{D}}_{A_w}\phi\|_{L^2}$ to be finite, we must get rid of the limits, thus forcing $\phi_0^{\pm} = 0$ and consequently we have $\not{\mathfrak{D}}_{A_w}\phi = 0$. The proof is now complete. □

For a continuous family of Fredholm operators, like $\not{\mathfrak{D}}_{A_z}$ on $L^2_{\mathfrak{E}}$ parameterized on $B^3(w)$, the dimension of the kernel can only drop in a small neighborhood of a given point, it cannot increase. However, not any random behavior is acceptable.

Lemma 8.2-5 (also found in [Kat95, p. 241]). *Let $T\colon X \to Y$ be Fredholm and $S\colon X \to Y$ a bounded operator. Then the operator $T + tS$ is Fredholm and $\dim\ker(T + tS)$ is constant for small $|t| > 0$.*

Before spelling out the proof of this lemma, which we obviously use with $T = \not{\mathfrak{D}}_{A_w}, X = W^{1,2}_{\mathfrak{E}}, Y = L^2_{\mathfrak{E}}$, and $S = cl(e)$ for some direction $e \in \mathbb{R}^3$, let's note that three scenarios are possible.

1. $\dim \mathbb{K}_z$ is constant on a neighborhood around w, say $B^3(w)$;
2. $\dim \mathbb{K}_z$ is constant for $z \in B^3(w) \setminus \{w\}$, but is smaller than $\dim \mathbb{K}_w$;
3. $\dim \mathbb{K}_{w+\lambda e} \neq \dim \mathbb{K}_{w+\lambda' e'}$ for small $\lambda, \lambda' > 0$ and some $e \neq e'$.

We close this section with the proof of Lemma 8.2-5.

Proof: The proof is a simplified proof of the one provided by Kato in [Kat95, p. 241] for more general T and S.

Define the sequences $M_n \subset X$ and $R_n \subset Y$ by

$$\begin{aligned} M_0 &:= X, & R_0 &:= Y, \\ M_n &:= S^{-1}R_n, & R_{n+1} &:= TM_n. \end{aligned}$$

All the M_n and R_n are imbricated as

$$M_0 \supset M_1 \supset M_2 \supset \cdots \quad \text{and} \quad R_0 \supset R_1 \supset R_2 \supset \cdots .$$

That the M_n are closed is a trivial fact once it is established that the R_n are closed. But define $\tilde{X} := X/\ker(T)$ and $\tilde{M}_n$ to be the set of corresponding $\ker(T)$-cosets. Then for the map $\tilde{T}$ defined as $\tilde{T}(x + \ker(T)) = T(x)$, we have $TM_n = \tilde{T}\tilde{M}_n$. Since $\tilde{T}$ is injective and Fredholm, and since $\tilde{M}_n$ is closed in $\tilde{X}$, then $R_{n+1} = \tilde{T}\tilde{M}_n$ is closed as well.

Define now

$$X' := \bigcap_n M_n \quad \text{and} \quad Y' := \bigcap_n R_n,$$

and let T', S' be the restriction to X'.

If $x \in X'$, then $x \in M_n$ for all n hence $T'x = Tx \in TM_n = R_{n+1}$ for all n, and by definition $Sx \in R_n$. So both T' and S' are bounded operators $X' \to Y'$.

We now prove $\mathrm{Im}(T') = Y'$. Suppose $y \in Y'$, then $y \in R_n = TM_{n-1}$ for all n, hence $T^{-1}y \cap M_{n-1} \neq \varnothing$. Since T is Fredholm, $T^{-1}y$ is closed and finite dimensional. We hence have a descending sequence $T^{-1}y \cap M_n \supset T^{-1}y \cap M_{n+1} \supset \cdots$ of finite dimensional nonempty affine spaces, which must then be stationary after a finite number of steps. The limit, which is then nonempty, must be $T^{-1}y \cap X'$, hence $y \in \mathrm{Im}(T')$, and T' is surjective.

Notice that trivially, $\ker(T' + tS') \subset \ker(T + tS)$. But more interestingly, those kernels are equal for $t \neq 0$. Indeed, take $x \in \ker(T + tS)$. We know $x \in X = M_0$, and prove by induction that $x \in M_n$ for all n, hence proving that $x \in \ker(T' + tS')$. The induction step is proved by staring at the definitions: being in the kernel forces $S(-tx) = Tx \in R_{n+1}$ if $x \in M_n$; but then $-tx \in M_{n+1}$ and for $t \neq 0$, we then have $x \in M_{n+1}$. We thus established that

$$\ker(T' + tS') = \ker(T + tS) \text{ for } t \neq 0.$$

Obviously, for t small enough, $T' + tS'$ is Fredholm and surjective, hence for small enough $|t| > 0$,

$$\dim \ker(T + tS) = \dim \ker(T' + tS') = \operatorname{ind}(T' + tS')$$

is constant. $\square$

8.3 Asymptotic of the Higgs field

We know study the behavior of the Higgs field Φ as z approaches of a point in $w \in W$.

We know w is associated to the limit Γ of A at ∞ or $-\infty$, in the sense that Γ splits E as $L_w \oplus L_{-w}$. Without loss of generality, we suppose

$$\Gamma_+ = \Gamma.$$

We can break up the analysis depending on which scenario happens; see page 91.

When $\Gamma_+ \neq \Gamma_-$, and for $2\pi|z - w| < \epsilon$, notice that

$$\begin{aligned} {}^{\ulcorner}V_z &= L^2_{\ulcorner\epsilon} \cap \ker(\not{\mathfrak{D}}^*_{A_z}) = L^2_{\overline{\epsilon}} \cap \ker(\not{\mathfrak{D}}^*_{A_z}) = L^2_{\epsilon\urcorner} \cap \ker(\not{\mathfrak{D}}^*_{A_z}), \text{ and} \\ V_{\lrcorner z} &= L^2_{\llcorner\epsilon} \cap \ker(\not{\mathfrak{D}}^*_{A_z}) = L^2_{\underline{\epsilon}} \cap \ker(\not{\mathfrak{D}}_{A_z^*}) = L^2_{\epsilon\lrcorner} \cap \ker(\not{\mathfrak{D}}^*_{A_z}). \end{aligned}$$

When $\Gamma_+ = \Gamma_-$, those spaces are a priori all different.

Theorem 8.3-1. *Suppose $\dim {}^{\ulcorner}K_z$ is constant in a neighborhood of w. On a closed ball $B^3(w)$ around w, there exists families of operators $\Phi^\perp$ and $\Phi^\lrcorner$, bounded independently of z , such that*

$$\Phi = -\frac{-i}{2|z - w|}\Phi^\perp + \Phi^\lrcorner. \tag{8.15}$$

Furthermore, $\Phi^\perp$ is the L^2-orthogonal projection on $\not{\mathfrak{D}}_{A_z}\mathcal{H}_z \cap V_z$.

Proof: When $\dim {}^{\ulcorner}K_z$ is constant on $B^3(w)$, so is $\dim V_{\lrcorner z}$. Hence $V_\lrcorner$ is a bundle on $B^3(w)$, say of rank l. Obviously, $V_\lrcorner$ supports many different norms, and amongst those are the L^2 and $L^2_{\epsilon\lrcorner}$ norms, which must be equivalent since $V_\lrcorner$ is finite rank on a compact space.

Hence, for $\phi \in V_{\lrcorner z}$, observe that

$$\|t\phi\|_{L^2} \leq C_\epsilon \|\phi\|_{\epsilon\lrcorner} \leq C\|\phi\|_{L^2}.$$

Denote $P^\lrcorner$ the L^2-orthogonal projection of V on $V_\lrcorner$. We just proved that

$$\Phi \circ P^\lrcorner \text{ is bounded independently of } z \in B^3(w).$$

It is part of the map $\Phi^\lrcorner$ announced in the statement of the theorem.

As suggested above, let $\Phi^{\perp}$ denote the L^2-orthogonal projection on $\not{\mathfrak{D}}_{A_z}\mathcal{H}_z \cap V_z$. Then

$$\begin{aligned}\Phi = -2\pi i P m_t &= \Phi P^{\lrcorner} - 2\pi i(P^{\lrcorner} + \Phi^{\perp}) m_t \Phi^{\perp} \\ &= \Phi P^{\lrcorner} + 2\pi i P^{\lrcorner} m_t \Phi^{\perp} - 2\pi i \Phi^{\perp} m_t \Phi^{\perp}.\end{aligned}$$

As it turns out, $P^{\lrcorner} m_t \Phi^{\perp}$ is also bounded independently of $z \in B^3(w)$. Indeed, suppose we have an L^2-orthonormal frame $\phi_1, \dots, \phi_l$ of $V_{\llcorner}$ in some open subset of $B^3(w)$, then

$$\begin{aligned}\|P^{\lrcorner} m_t \Phi^{\perp}(\phi)\|_{L^2} &= \Big\|\sum_{j=1}^{l} \langle \phi_j, t\Phi^{\perp}(\phi)\rangle \phi_j\Big\|_{L^2} \\ &= \Big\|\sum_{j=1}^{l} \langle t\phi_j, \Phi^{\perp}(\phi)\rangle \phi_j\Big\|_{L^2} \\ &\leq \sum_{j=1}^{l} C\|\phi_j\|_{L^2}^2 \|\Phi^{\perp}(\phi)\|_{L^2} \\ &\leq C\|\phi\|_{L^2}.\end{aligned}$$

It remains only to analyze $\Phi^{\perp} m_t \Phi^{\perp}$.

Pick a vector $e \in \mathbb{R}^3$ of length 1. Let

$$Ray = \{w + \frac{\lambda}{2\pi} e\} \subset B^3(w)$$

be a ray inside $B^3(w)$ emerging from w. As the notation suggests, we parameterize this ray by $\lambda = 2\pi|z - w|$. Pick a family $\phi_z \in \not{\mathfrak{D}}_{A_z}\mathcal{H}_z$ for $z \in Ray$, with

$$\begin{aligned}\phi_z &\in V_z \text{ for } \lambda > 0, \\ \|\phi_z\|_{L^2_{\ulcorner\epsilon}} &= 1. \end{aligned} \tag{8.16}$$

But then,

$$\|\phi_z\|_{L^2} \to \infty \text{ as } \lambda \to 0.$$

To prove this claim, suppose it is not true. Then there is a subsequence $\phi_{z_j} \rightharpoonup \tilde{\phi}_w$ weakly in L^2. Hence $\langle \phi_{z_j}, f\rangle \to \langle \tilde{\phi}_w, f\rangle$ for all $f \in L^2$, in particular for all $f \in L^2_{\epsilon\lrcorner} = (L^2_{\ulcorner\epsilon})^*$, whence $\phi_{z_j} \rightharpoonup \tilde{\phi}_w$ weakly in $L^2_{\ulcorner\epsilon}$. Since $\phi_z \to \phi_w$ in $L^2_{\ulcorner\epsilon}$, we have $\tilde{\phi}_w = \phi_w$, which is impossible as $\tilde{\phi}_w$ is in L^2 while ϕ_w is not.

Theorem 6.1-3 guarantees that the operator $\not{\mathfrak{D}}^*_{\Gamma_w}$ is an isomorphism $W^{1,2}_{\llcorner\epsilon} \to L^2_{\llcorner\epsilon}$, and $W^{1,2}_{\epsilon\urcorner} \to L^2_{\epsilon\urcorner}$, hence there exist a constant C such that

$$\|u\|_{W^{1,2}_{\llcorner\epsilon}} \leq C\|\not{\mathfrak{D}}^*_{\Gamma_w} u\|_{L^2_{\llcorner\epsilon}}, \quad \text{for } u \in W^{1,2}_{\llcorner\epsilon}, \tag{8.17}$$

$$\|u\|_{W^{1,2}_{\epsilon\urcorner}} \leq C\|\not{\mathfrak{D}}^*_{\Gamma_w} u\|_{L^2_{\epsilon\urcorner}}, \quad \text{for } u \in W^{1,2}_{\epsilon\urcorner}. \tag{8.18}$$

Because $\phi_z \in V_z$ for $\lambda > 0$, Theorem 7.2-1 tells us that for $t > 0$, we can write $\phi_z = e^{-\lambda t}\psi_{-\lambda} + g_z$ for some eigenvector $\psi_{-\lambda}$ of eigenvalue $-\lambda$ of D_{Γ_z} and some $g_z \in W^{1,2}_{-\epsilon}([0,\infty) \times T^3)$.

When $\Gamma_- = \Gamma_+$, we also have interest in understanding the asymptotic behavior at $-\infty$. Theorem

7.2-1 tells us that for $t < 0$, we can write $\phi_z = e^{\lambda t}\psi_\lambda + j_z$ for some eigenvector ψ_λ of eigenvalue λ of D_{Γ_z} and some $j_z \in W^{1,2}_\epsilon((-\infty, 0] \times T^3)$.

While g_z and j_z appear to be defined only for $t > 0$ and $t < 0$ respectively, let's define them globally on $\mathbb{R} \times T^3$ by $g_z = \phi_z - e^{-\lambda t}\psi_{-\lambda}$ and $j_z = \phi_z - e^{\lambda t}\psi_\lambda$.

Notice that

$$\not{\mathfrak{D}}^*_{\Gamma_z} g_z = \not{\mathfrak{D}}^*_{\Gamma_z}\phi_z = (\not{\mathfrak{D}}^*_{\Gamma_z} - \not{\mathfrak{D}}^*_{A_z})\phi_z = cl(\Gamma - A)\phi_z, \tag{8.19}$$

and similarly

$$\not{\mathfrak{D}}^*_{\Gamma_z} j_z = \not{\mathfrak{D}}^*_{\Gamma_z}\phi_z = (\not{\mathfrak{D}}^*_{\Gamma_z} - \not{\mathfrak{D}}^*_{A_z})\phi_z = cl(\Gamma - A)\phi_z, \tag{8.20}$$

Remember that we decided that w is associated to $\Gamma = \Gamma_+$. Hence for $t > 0$, we know that $|A - \Gamma| \leq Ce^{-\beta t}$. For $t < 0$, we have

$$\begin{aligned} |A - \Gamma| &\leq |A - \Gamma_-| + |\Gamma_- - \Gamma| \\ &\leq Ce^{\beta t} + C'. \end{aligned}$$

Hence overall, there is a constant such that $|cl(A - \Gamma)| \leq C\sigma_{(0,\beta)}$, and this estimate can be improved to $|cl(A - \Gamma)| \leq C\sigma_{(-\beta,\beta)}$ when $\Gamma_- = \Gamma_+$. Hence $cl(A - \Gamma)$ gives a bounded map $L^2_{\ulcorner\epsilon} \to L^2_{\llcorner\epsilon}$ in all cases and $L^2_{\ulcorner\epsilon} \to L^2_{\epsilon\urcorner}$ when $\Gamma_- = \Gamma_+$. Thus Equation (8.19) yields

$$\|\not{\mathfrak{D}}^*_{\Gamma_z} g_z\|_{L^2_{\llcorner\epsilon}} \leq C\|\phi_z\|_{L^2_{\ulcorner\epsilon}}, \tag{8.21}$$

and for the special case $\Gamma_- = \Gamma_+$, Equation (8.20) yields

$$\|\not{\mathfrak{D}}^*_{\Gamma_z} j_z\|_{L^2_{\epsilon\urcorner}} \leq C\|\phi_z\|_{L^2_{\ulcorner\epsilon}}. \tag{8.22}$$

From Equations (8.17), and (8.21), we derive

$$\begin{aligned} \|g_z\|_{W^{1,2}_{\llcorner\epsilon}} &\leq C\|\not{\mathfrak{D}}^*_{\Gamma_w} g_z\|_{L^2_{\llcorner\epsilon}} \\ &= C\|\not{\mathfrak{D}}^*_{\Gamma_z} g_z + \lambda cl(e) g_z\|_{L^2_{\llcorner\epsilon}} \\ &\leq C\|\phi_z\|_{L^2_{\ulcorner\epsilon}} + C\lambda\|g_z\|_{L^2_{\llcorner\epsilon}}, \end{aligned}$$

hence for λ small enough, we can rearrange and obtain

$$\|g_z\|_{W^{1,2}_{\llcorner\epsilon}} \text{ is bounded independently of small } z, \tag{8.23}$$

while from Equations (8.18) and (8.22), we similarly obtain

$$\|j_z\|_{W^{1,2}_{\epsilon\urcorner}} \text{ is bounded independently of small } z. \tag{8.24}$$

This last fact is also true for $\Gamma_- \neq \Gamma_+$, for in that case $j_z = \phi_z$ and its $L^2_{\ulcorner\epsilon}$-norm is equivalent to the $L^2_{\epsilon\urcorner}$-norm, as both as defined on $\mathbb{V}$ over $B^3(w)$.

While it is agreeable to work with a smooth splitting, nothing prevents us from considering the function

$$h_\lambda = \begin{cases} e^{\lambda t}\psi_\lambda, & \text{for } t < 0, \\ e^{-\lambda t}\psi_{-\lambda}, & \text{for } t > 0, \end{cases}$$

and the associate splitting

$$\phi_z = h_\lambda + r_z,$$

for which, obviously,

$$r_z = \begin{cases} j_z, & \text{for } t < 0, \\ g_z, & \text{for } t > 0. \end{cases} \tag{8.25}$$

From the bounds of Equations (8.23) and (8.24), we have that

$$\|r_z\|_{L^2_{\epsilon_j}} \text{ is bounded independently of small } z. \tag{8.26}$$

Consider the families

$$\bar{\phi}_z := \phi_z / \|\phi_z\|_{L^2},$$
$$\bar{h}_\lambda := h_\lambda / \|\phi_z\|_{L^2},$$
$$\bar{r}_z := r_z / \|\phi_z\|_{L^2}.$$

The bound (8.26), and the fact that $\|\phi_z\|_{L^2} \to \infty$ imply that

$$\|\bar{r}_z\|_{L^2_{\epsilon_j}} \to 0 \text{ as } \lambda \to 0.$$

A fortiori, $\|\bar{r}_z\|_{L^2} \to 0$.
The triangle inequality guarantees

$$\left| \|\bar{h}_\lambda\|_{L^2} - \|\bar{r}_z\|_{L^2} \right| \leq \|\bar{\phi}_z\|_{L^2} \leq \|\bar{h}_\lambda\|_{L^2} + \|\bar{r}_z\|_{L^2}.$$

Since $\|\bar{\phi}_z\|_{L^2} = 1$, and $\|\bar{r}_z\|_{L^2} \to 0$, we must have

$$\|\bar{h}_\lambda\|_{L^2} \to 1 \text{ as } \lambda \to 0.$$

Let's now come back to our main worry. We study

$$\langle t\bar{\phi}_z, \bar{\phi}_z \rangle = \langle t\bar{h}_\lambda, \bar{h}_\lambda \rangle + 2\langle \bar{h}_\lambda, t\bar{r}_z \rangle + \langle t\bar{r}_z, \bar{r}_z \rangle.$$

The last two terms are bounded by a multiple of $\|t\bar{r}_z\|_{L^2}$. But

$$\|t\bar{r}_z\|_{L^2} \leq C\|\bar{r}_z\|_{L^2_{\epsilon_j}} = C\|\tilde{r}_z\|_{L^2_{\epsilon_j}} / \|\phi_\lambda\|_{L^2},$$

hence it is going to 0.
As for the first term, we have

$$\begin{aligned} \langle t\bar{h}_\lambda, \bar{h}_\lambda \rangle &= \frac{1}{\|\phi_\lambda\|^2_{L^2}} \Big(\int_0^\infty t e^{-2\lambda t} |\psi_{-\lambda}|^2 + \int_{-\infty}^0 t e^{2\lambda t} |\psi_\lambda|^2 \Big) \\ &= \frac{1}{2\lambda} \frac{1}{\|\phi_\lambda\|^2_{L^2}} \Big(\int_0^\infty e^{-2\lambda t} |\psi_{-\lambda}|^2 + \int_{-\infty}^0 e^{2\lambda t} |\psi_\lambda|^2 \Big) \\ &= \frac{1}{2\lambda} \|\bar{h}_\lambda\|^2_{L^2}, \end{aligned}$$

hence

$$\langle t\bar{\phi}_\lambda, \bar{\phi}_\lambda \rangle = \frac{1}{2\lambda} + o(1) \text{ as } \lambda \to 0.$$

Suppose now $\bar{\phi}_z^1$ and $\bar{\phi}_z^2$ are two such families, but so that

$$\langle \bar{\phi}_z^1, \bar{\phi}_z^2 \rangle_{L^2} = 0.$$

Then

$$\begin{aligned}\langle t\bar{\phi}_z^1, \bar{\phi}_z^2 \rangle &= \langle t\bar{h}_\lambda^1, \bar{h}_\lambda^2 \rangle + \langle \bar{h}_\lambda^1, t\bar{r}_z^2 \rangle + \langle t\bar{r}_z^1, \bar{h}_\lambda^2 \rangle + \langle t\bar{r}_z^1, \bar{r}_z^2 \rangle \\ &= \frac{1}{2\lambda}\langle \bar{h}_\lambda^1, \bar{h}_\lambda^2 \rangle + o(1),\end{aligned}$$

and of course $\langle \bar{h}_\lambda^1, \bar{h}_\lambda^2 \rangle \to 0$, hence the result. □

One of the crucial feature of this proof is our ability to find a uniform bound for m_t on $\underline{V}$. Such a bound exist in the case where $\dim \overline{K}_z$ is constant precisely because this constant rank condition implies that $\underline{V}$ is a bundle over $B^3(w)$, allowing us to say that the L^2-norm and $L^2_{\epsilon_j}$-norm are equivalent.

We can take the trace of (B, Φ) to obtain an abelian monopole (a, φ) on $B^3(w) \setminus \{w\}$. The Bogomolny equation reduces to

$$d\,\varphi = *d\,a,$$

and thus $\Delta\varphi = 0$. Since φ is harmonic, not every possible behavior as $z \to w$ is acceptable. For one thing, there is a unique set of homogeneous harmonic polynomials p_m and q_m of degree m which give a decomposition of φ on $B^3(w) \setminus \{w\}$ as a Laurent series

$$\varphi = \sum_{m=0}^{\infty} p_m(z-w) + \sum_{m=0}^{\infty} \frac{q_m(z-w)}{|z-w|^{2m+1}};$$

see [ABR01, Thm 10.1, p. 209].

Whether or not the rank is constant, we can find for any sequence of points approaching w a subsequence of points $z_j \to w$ for which the decomposition of Equation 8.15 is valid. We then have

$$\lim_{j\to\infty} 2|z_j - w|\varphi_{z_j} = i \dim \not{\mathfrak{D}}_{A_{z_j}} \mathcal{H}_{z_j} = i(\mathrm{rk}\mathcal{H} - \dim \overline{K}_{z_j}).$$

By the Laurent series decomposition given above, this number must be the same in any way we approach w, hence $\dim \overline{K}_z$ must be constant on $B^3(w) \setminus \{w\}$, thus eliminating Scenario 3 of page 91.

Scenario 2 remains to be dealt with.

8.4 Preliminary work: Green's operator on $S^\pm \otimes L_z$

This section consists of preliminary work on the study of the behavior of the connection B at the singular points. Because of the formula $B = Pd^z = (1 - \not{\mathfrak{D}}_{A_z} G_{A_z} \not{\mathfrak{D}}^*_{A_z})d^z$, getting an explicit formula for G_{A_z} would greatly help in understanding the asymptotic behavior of B at singular points. As a first step into achieving that goal, we compute the Green's operator for the Laplacian on $S^\pm \otimes L_z$ on $\mathbb{R} \times T^3$.

Define the operator

$$\begin{aligned} T_\lambda \colon L^2(\mathbb{R}) &\to L^2(\mathbb{R}) \\ g &\mapsto \frac{1}{\lambda}\int_{-\infty}^{\infty} e^{-\lambda|t-s|}g(s)ds \end{aligned}$$

Let ϕ_ν be a basis of the eigenspaces of the Laplacian on the spinor bundle of T^3. Then every section ϕ decomposes as

$$\phi = \sum_{\nu\in\mathbb{Z}^3} g_\nu(t)\phi_\nu. \tag{8.27}$$

Here multiplicities are hidden but keep them in mind.

Lemma 8.4-1. *For all $z \in \hat{T}^3$, we have*

$$G_{L_z}(\phi) = \frac{1}{2}\sum_{\nu\in\mathbb{Z}^3} T_{2\pi|\nu-z|}(g_\nu)\phi_\nu.$$

Proof: First notice that

$$T_\lambda g = \frac{1}{\lambda}\Big(\int_{-\infty}^{t} e^{-\lambda t}e^{\lambda s}g(s)ds + \int_{t}^{\infty} e^{\lambda t}e^{-\lambda s}g(s)ds\Big),$$

hence $\partial_t(T_\lambda g) = g(T)/\lambda - (\int_{-\infty}^{t}) + (\int_t^\infty) - g(t)/\lambda$, and

$$\begin{aligned} \partial_t^2(T_\lambda g) &= -g(t) + \lambda(\int_{-\infty}^{t}) + \lambda(\int_t^\infty) - g(t)/\lambda \\ &= -2g + \lambda^2 T_\lambda g. \end{aligned}$$

Remember now that on $S^\pm \otimes L_z$ on $\mathbb{R}\times T^3$, the Laplacian splits as

$$\Delta_{L_z} = -\partial_t^2 + \Delta_{T^3,L_z}.$$

Recall also that for $\nu \in \mathbb{Z}^3$, we have $\Delta_{T^3}\phi_\nu = (2\pi|\nu|)^2\phi_\nu$ and $\Delta_{T^3,L_z}\phi_\nu = (2\pi|\nu-z|)^2\phi_\nu$. Hence for the proposed G, we have

$$\begin{aligned} \Delta_{L_z}G_{L_z}\phi &= \frac{1}{2}\sum_{\nu\in\mathbb{Z}^3}\Big(-\partial_t^2 T_{2\pi|\nu-z|}(g_\nu)\phi_\nu + (2\pi|\nu-z|)^2 g_\nu\phi_\nu\Big) \\ &= \frac{1}{2}\sum_{\nu\in\mathbb{Z}^3}(2g_\nu - (2\pi|\nu-z|)^2 g_\nu + (2\pi|\nu-z|)^2 g_\nu)\phi_\nu \\ &= \phi. \end{aligned}$$

The proof is now complete. □

It appears very important then to understand T_λ carefully. Let $m_\lambda \colon \mathbb{R}\to\mathbb{R}$ be the multiplication by λ. We have the following identities.

Lemma 8.4-2. *For different values of $\lambda, \eta > 0$, we have*

$$T_\lambda = \frac{\eta^2}{\lambda^2} m^*_{\lambda/\eta} T_\eta m^*_{\eta/\lambda}, \tag{8.28}$$

and in particular

$$T_\lambda = \frac{1}{\lambda^2} m^*_\lambda T_1 m^*_{\lambda^{-1}}.$$

Proof: We just compute

$$\begin{aligned}
T_\lambda(g) &= \frac{1}{\lambda}\int_{-\infty}^{\infty} e^{-\lambda|t-s|} g(s)ds \\
&= \frac{1}{\lambda}\int_{-\infty}^{\infty} e^{-\eta|\frac{\lambda}{\eta}t-\frac{\lambda}{\eta}s|} g\Big(\frac{\lambda s/\eta}{\lambda/\eta}\Big)\frac{d(\lambda s/\eta)}{\lambda/\eta} \\
&= \frac{\eta^2}{\lambda^2} T_\eta(g\circ m_{\eta/\lambda})\circ m_{\lambda/\eta},
\end{aligned}$$

whence the conclusion. The proof is now complete. □

Lemma 8.4-3. *Viewed as operators $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, the operators m^*_λ and T_λ have norm*

$$\|m^*_\lambda f\| = \frac{1}{\sqrt{\lambda}}\|f\|,\ \forall f,$$
$$\|T_\lambda\| = \frac{1}{\lambda^2}\|T_1\|.$$

Furthermore for z close to 0, the Green's operator has norm

$$\|G_{L_z}\| = \frac{\|T_1\|}{2|z|^2}$$

as an operator $L^2 \to L^2$.

Proof: First notice

$$||m^*_\lambda f||^2 = \int |f(\lambda t)^2|dt = \int |f(s)|^2 ds/\lambda = \frac{1}{\lambda}||f||^2.$$

Then we compare. On one hand

$$\begin{aligned}
||T_\lambda g|| &= \frac{1}{\lambda^2}||m^*_\lambda T_1(m^*_{\lambda^{-1}}g)|| \\
&= \frac{1}{\lambda^2}\frac{1}{\sqrt{\lambda}}||T_1(m^*_{\lambda^{-1}}g)|| \\
&\le \frac{1}{\lambda^2}\frac{1}{\sqrt{\lambda}}||T_1||\ ||m^*_{\lambda^{-1}}g|| \\
&= \frac{1}{\lambda^2}||T_1||\ ||g||,
\end{aligned}$$

hence

$$||T_\lambda|| \le \frac{1}{\lambda^2}||T_1||. \tag{8.29}$$

On the other hand, we find in a similar fashion that

$$||T_1|| \le \lambda^2||T_\lambda||. \tag{8.30}$$

From Inequalities (8.29) and (8.30), we obtain the desired result for T_λ.

To find the norm of G_{L_z}, we use the decomposition given by Equation (8.27). We have

$$\begin{aligned}||G_{L_z}(\phi)||^2 &= \frac{1}{4}\sum ||T_{2\pi|\nu-z|}(g_\nu)||^2 \\ &\leq \frac{1}{4}\sum \frac{1}{(2\pi)^4|\nu-z|^4}||T_1||^2\ ||g_\nu||^2 \\ &\leq \frac{||T_1||^2}{4}\sup\Big(\frac{1}{(2\pi)^4|\nu-z|^4}\Big)||\phi||^2,\end{aligned}$$

hence for z close to 0,

$$||G_{L_z}|| \leq \frac{||T_1||}{8\pi^2|z|^2}.$$

To prove equality, note that for $\phi = g_0\phi_0$, with $g_0 \in L^2$, and $||g_0|| = 1$, we have

$$G_{L_z}(\phi) = \frac{1}{2}T_{2\pi|z|}(g_0)\phi_0.$$

But then

$$\begin{aligned}||G_z|| &= \sup_{||\phi||=1} ||G_z(\phi)|| \\ &\geq \sup_{||g_0||=1} ||G_z(g_0\phi_0)|| \\ &= \frac{1}{2}\sup_{||g_0||=1} ||T_{2\pi|z|}(g_0)|| \\ &= \frac{1}{8\pi^2|z|^2}||T_1||.\end{aligned}$$

The proof is now complete. □

Lemma 8.4-4. *We have*

$$G_{L_z} = \frac{\mathfrak{L}_z}{|z|^2} + \mathfrak{M}_z,$$

with the $L^2 \to L^2$ operator norms of $\mathfrak{L}_z$ and $\mathfrak{M}_z$ bounded independently of z for $\mathrm{dist}(z, \mathbb{Z}^3) < 1/2$*.*

Proof: Let p_0 be the projection $\phi \mapsto g_0\phi_0$ and $p_1 = 1 - p_0$. Set

$$\begin{aligned}\mathfrak{L}_z(\phi) &:= |z|^2 G_z p_0(\phi), \\ \mathfrak{M}_z(\phi) &= G_z p_1(\phi).\end{aligned}$$

Obviously, $G_{L_z} = \mathfrak{L}_z/|z|^2 + \mathfrak{M}_z$. It remains to show that $\mathfrak{L}_z$ and $\mathfrak{M}_z$ are uniformly bounded for z close to Λ^*. For $\mathfrak{M}_z$ and $\mathrm{dist}(z, \mathbb{Z}^3) < 1/2$, we have

$$\begin{aligned}\|G_{L_z}p_1(\phi)\|^2 &= \frac{1}{4}\sum_{\nu\neq 0} \|T_{2\pi|\nu-z|}(g_\nu)\|^2 \\ &\leq \frac{\|T_1\|^2}{4}\sup_{\nu\neq 0}\frac{1}{(2\pi|\nu-z|)^4}\|\phi\|^2 \\ &\leq \frac{4}{\pi^2}\|T_1\|^2\|\phi\|^2,\end{aligned}$$

proving the claim for $\mathfrak{M}_z$.

Obviously,

$$G_{L_z} p_0(\phi) = \frac{1}{8\pi^2|z|^2} m^*_{2\pi|z|} T_1(m^*_{(2\pi|z|)^{-1}} g_0)\phi_0,$$

hence the claims for $\mathfrak{L}_z$ follows from Lemma 8.4-3. The proof is now complete. □

While $\|\mathfrak{L}_z\|_{op}$ is constant and not 0, the family of operators $\mathfrak{L}_z$ in a very weak sense converges to 0.

Lemma 8.4-5. *Let $g \in C_c^\infty$. we have that*

$$\mathfrak{L}_z(g) \to 0 \text{ in } L^2 \text{ norm as } \lambda \to 0.$$

Proof: Suppose the support of g is $[m, M]$. Then

$$\begin{aligned}
\|\mathfrak{L}_z(g)\|^2 &= (2\pi|z|)^2 \int_{-\infty}^{\infty} \Big| \int_{-\infty}^{\infty} e^{-2\pi|z|\cdot|t-s|} g(s) ds \Big|^2 dt \\
&\le (2\pi|z|)^2 \int_{-\infty}^{\infty} (M-m)^2 \max(g)^2 e^{-4\pi|z| dist(t, supp(g))} \\
&= 2\pi|z| \big((M-m)\max(g)\big)^2.
\end{aligned}$$

The result follows. □

Appendix A

Reduction of ASD equation to lower dimension

The curvature of the connection $A = A_1 dx^1 + \cdots A_4 dx^4$ is given by

$$\begin{aligned} F &= dA + A \wedge A \\ &= \sum_{i,j} \partial_j A_i dx^j \wedge dx^i + \sum_{i,j} A_i A_j dx^i \wedge dx^j \\ &= \sum_{i<j} F_{ij} dx^i \wedge dx^j \end{aligned}$$

with $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$.

To convert to the standard self-dual $\epsilon_1, \epsilon_2, \epsilon_3$, and anti-self-dual $\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3$ basis of $\bigwedge^2$, we collect terms. For example

$$\begin{aligned} F_{12} dx^{12} + F_{34} dx^{34} &= (\frac{F_{12} + F_{34}}{2}) dx^{12} + (\frac{F_{12} - F_{34}}{2}) dx^{12} \\ &\quad + (\frac{F_{12} + F_{34}}{2}) dx^{34} + (\frac{F_{34} - F_{12}}{2}) dx^{34} \\ &= (\frac{F_{12} + F_{34}}{2}) \epsilon_1 + (\frac{F_{12} - F_{34}}{2}) \bar{\epsilon}_1. \end{aligned}$$

We keep collecting terms, and get

$$\begin{aligned} F = &(\frac{F_{12} + F_{34}}{2}) \epsilon_1 + (\frac{F_{12} - F_{34}}{2}) \bar{\epsilon}_1 + (\frac{F_{13} - F_{24}}{2}) \epsilon_2 \\ &+ (\frac{F_{13} + F_{24}}{2}) \bar{\epsilon}_2 + (\frac{F_{14} + F_{23}}{2}) \epsilon_3 + (\frac{F_{14} - F_{23}}{2}) \bar{\epsilon}_3. \end{aligned}$$

So the ASD equations are

$$\begin{aligned} F_{12} + F_{34} &= 0, \\ F_{13} - F_{24} &= 0, \\ F_{14} + F_{23} &= 0. \end{aligned} \tag{A.1}$$

We now peel off dimensions one at a time.

Dimension 3: the Bogomolny equation

Let the A_i be independent of x_1, and set $\Phi := A_1$. Then the Equations (A.1) reduce to

$$\begin{aligned} -\partial_2\Phi + [\Phi, A_2] + F_{34} &= 0, \\ -\partial_3\Phi + [\Phi, A_3] - F_{24} &= 0, \\ -\partial_4\Phi + [\Phi, A_4] + F_{23} &= 0. \end{aligned} \tag{A.2}$$

Set $B := A_2 dx^2 + A_3 dx^3 + A_4 dx^4$. It is a connection on the (x_2, x_3, x_4)-space. On that space, the Hodge star works as follows:

$$\begin{aligned} *dx^2 &= dx^3 \wedge dx^4, \\ *dx^3 &= -dx^2 \wedge dx^4, \\ *dx^4 &= dx^2 \wedge dx^3, \text{ and} \\ *^2 &= 1 \text{ on } \textstyle\bigwedge^1. \end{aligned}$$

Furthermore, the connection B extends to endomorphisms by the formula

$$\nabla_B \Phi = d\Phi + [B, \Phi].$$

Hence the Equations (A.2) can be written as a single equation as

$$\nabla_B \Phi = *F_B, \tag{A.3}$$

the *Bogomolny equation*.

Dimension 2: the Hitchin equations

Let the A_i be independent of x_3 and x_4, and set $\phi_1 := A_3$, $\phi_2 := A_4$. The the Equations (A.1) reduce to

$$\begin{aligned} F_{12} + [\phi_1, \phi_2] = 0, \\ \partial_1\phi_1 + [A_1, \phi_1] - \partial_2\phi_2 - [A_2, \phi_2] = 0, \\ \partial_1\phi_2 + [A_1, \phi_2] + \partial_2\phi_1 + [A_2, \phi_1] = 0. \end{aligned}$$

In other words, set $B := A_1 dx^1 + A_2 dx^2$, and we have

$$\begin{aligned} F_{12} &= -[\phi_1, \phi_2], \\ \nabla_B^1 \phi_1 &= \nabla_B^2 \phi_2, \\ \nabla_B^2 \phi_1 &= -\nabla_B^1 \phi_2. \end{aligned} \tag{A.4}$$

Since all orientable 2-manifolds are complex, let $dz = dx_1 + i dx_2$, and

$$\Phi := \frac{1}{2}(\phi_1 + i\phi_2) dz.$$

Should the connection A be on a bundle E, then Φ is a section of $\bigwedge^{1,0} End(E)$ and is called a *Higgs* *field*. On 1-forms, we consider the graded commutator

$$\begin{aligned}[\Phi, \Phi^*] &= \Phi\Phi^* + \Phi^*\Phi \\ &= \frac{1}{2}(\phi_1 + i\phi_2)(\phi_1 - i\phi_2) dz \wedge d\bar{z} + \frac{1}{2}(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) d\bar{z} \wedge dz \\ &= -\frac{i}{2}[\phi_1, \phi_2] dz \wedge d\bar{z} \\ &= -[\phi_1, \phi_2] dx^1 \wedge dx^2.\end{aligned}$$

Hence

$$F_B = [\Phi, \Phi^*].$$

Consider the operator $\bar{\partial}_B = \frac{1}{2}(\nabla_B^1 + i\nabla_B^2)d\bar{z}$, and we have

$$\begin{aligned}\bar{\partial}_B \Phi &= \big(\bar{\partial}_B \phi_1 + i(\bar{\partial}_B \phi_2)\big) d\bar{z} \wedge dz \\ &= \frac{1}{2}(\nabla_B^1 \phi_1 + i\nabla_B^2 \phi_1 + i\nabla_B^1 \phi_2 - \nabla_B^2 \phi_2) d\bar{z} \wedge dz \\ &= 0.\end{aligned}$$

Hence the Equations (A.4) can be written as two equations

$$\begin{gathered} F_B = \tfrac{1}{4}[\Phi, \Phi^*], \\ \bar{\partial}_B \Phi = 0, \end{gathered} \tag{A.5}$$

which we call the *Hichin equations*.

Dimension 1: The Nahm Equations

Let the A_i be independent of x_2, x_3, x_4, and set

$$\begin{gathered} t := x_1, \\ B := A_1 dt, \\ T_1 := A_2,\ T_2 := A_3,\ T_3 := A_4. \end{gathered}$$

Then the Equations (A.1) reduce to

$$\begin{gathered} \nabla_B^t T_{\sigma(1)} + [T_{\sigma(2)}, T_{\sigma(3)}] = 0, \\ \text{for all even permutation } \sigma. \end{gathered} \tag{A.6}$$

We call those equations the *Nahm equations*. These equations first appeared in [Nah83].

Appendix B

Excision principle for the index of Fredholm operators

Let

$$\begin{aligned} D_1 \colon L^2(X_1) &\to L^2(X_1) \\ D_2 \colon L^2(X_2) &\to L^2(X_2) \end{aligned}$$

be unbounded Fredholm operators, defined locally.
Let $X_1 = A_1 \cup B_1$, and $X_2 = A_2 \cup B_2$, with compact intersections

$$A_1 \cap B_1 = A_2 \cap B_2,$$

and suppose $D_1 = D_2$ on that intersection.
We construct $\tilde{X}_1 = A_1 \cup B_2$ and $\tilde{X}_2 = A_2 \cup B_1$. Let

$$\begin{aligned} \tilde{D}_1 \colon L^2(\tilde{X}_1) &\to L^2(\tilde{X}_1) \\ \tilde{D}_2 \colon L^2(\tilde{X}_2) &\to L^2(\tilde{X}_2) \end{aligned}$$

be unbounded Fredholm operators, defined locally, such that

$$\tilde{D}_1 = \begin{cases} D_1, & \text{on } A_1; \\ D_2, & \text{on } B_2; \end{cases}$$

$$\tilde{D}_2 = \begin{cases} D_2, & \text{on } A_2; \\ D_1, & \text{on } B_1. \end{cases}$$

Theorem B-1. *Under the hypothesis just described, we have*

$$\operatorname{ind}(D_1) + \operatorname{ind}(D_2) = \operatorname{ind}(\tilde{D}_1) + \operatorname{ind}(\tilde{D}_2).$$

Proof: Choose square roots of partitions of unity

$$\phi_1^2 + \psi_1^2 = 1 \quad \phi_2^2 + \psi_2^2 = 1$$

subordinate to (A_1, B_1) and (A_2, B_2). Choose them so that

$$\phi_1 = \phi_2 \text{ and } \psi_1 = \psi_2 \text{ on } A_1 \cap B_1 = A_2 \cap B_2. \tag{B.1}$$

We define maps

$$\begin{aligned} \Phi &\colon L^2(X_1) \oplus L^2(X_2) \to L^2(\tilde{X}_1) \oplus L^2(\tilde{X}_2) \\ \Psi &\colon L^2(\tilde{X}_1) \oplus L^2(\tilde{X}_2) \to L^2(X_1) \oplus L^2(X_2) \end{aligned}$$

which in matrix form are written as

$$\Phi = \begin{bmatrix} \phi_1 & \psi_2 \\ -\psi_1 & \phi_2 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} \phi_1 & -\psi_1 \\ \psi_2 & \phi_2 \end{bmatrix}.$$

Notice that outside of $A_i \cap B_i$, we clearly have $\psi_1\phi_1 = \psi_2\phi_2$. Equation (B.1) shows that this equality is also true in the intersection. Hence

$$\Phi\Psi = \begin{bmatrix} \phi_1^2 + \psi_2^2 & -\psi_1\phi_1 + \psi_2\phi_2 \\ -\psi_1\phi_1 + \psi_2\phi_2 & \psi_1^2 + \phi_2^2 \end{bmatrix} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix},$$

and Φ and Ψ are inverse of each other. They are in fact isometries. Indeed, we have

$$\begin{aligned} \|\Phi(f_1, f_2)\|^2 &= \int_{\tilde{X}_1} |\phi_1 f_1 + \psi_2 f_2|^2 + \int_{\tilde{X}_2} |\phi_2 f_2 - \psi_1 f_1|^2 \\ &= \int_{\tilde{X}_1} \phi_1^2 |f_1|^2 + \int_{\tilde{X}_1} \psi_2^2 |f_2|^2 + 2\int_{\tilde{X}_1} \phi_1\psi_2 \langle f_1, f_2\rangle \\ &\quad + \int_{\tilde{X}_2} \phi_2^2 |f_2|^2 + \int_{\tilde{X}_2} \psi_1^2 |f_1|^2 - 2\int_{\tilde{X}_2} \phi_2\psi_1 \langle f_2, f_1\rangle \\ &= \int_{X_1} |f_1|^2 + \int_{X_2} |f_2|^2 = \|(f_1, f_2)\|^2. \end{aligned}$$

Consider now $D = D_1 \oplus D_2$ and $\tilde{D} = \tilde{D}_1 \oplus \tilde{D}_2$. Then

$$\operatorname{ind}(D) = \operatorname{ind}(D_1) + \operatorname{ind}(D_2) \quad \text{and} \quad \operatorname{ind}(\tilde{D}) = \operatorname{ind}(\tilde{D}_1) + \operatorname{ind}(\tilde{D}_2).$$

We pull back $\tilde{D}$ to $L^2(X_1) \oplus L^2(X_2)$ and compare it to D. Should the difference $\Psi\tilde{D}\Phi - D$ be compact, the theorem would be proved. We proceed:

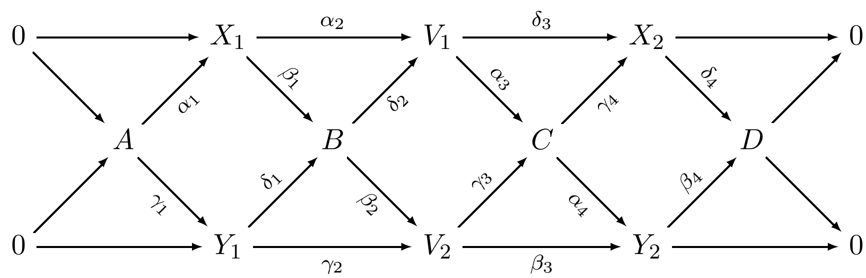
$$\begin{aligned} \Psi\tilde{D}\Phi &= \begin{bmatrix} \phi_1 & -\psi_1 \\ \psi_2 & \phi_2 \end{bmatrix} \begin{bmatrix} \tilde{D}_1 & \\ & \tilde{D}_2 \end{bmatrix} \begin{bmatrix} \phi_1 & \psi_2 \\ -\psi_1 & \phi_2 \end{bmatrix} \\ &= \begin{bmatrix} \phi_1\tilde{D}_1\phi_1 + \psi_1\tilde{D}_2\psi_1 & \phi_1\tilde{D}_1\psi_2 - \psi_1\tilde{D}_2\phi_2 \\ \psi_2\tilde{D}_1\phi_1 - \phi_2\tilde{D}_2\psi_1 & \psi_2\tilde{D}_1\psi_2 + \phi_2\tilde{D}_2\phi_2 \end{bmatrix} \\ &= \begin{bmatrix} \phi_1 D_1\phi_1 + \psi_1 D_2\psi_1 & \phi_1 D_1\psi_2 - \psi_1 D_2\phi_2 \\ \psi_2 D_1\phi_1 - \phi_2 D_2\psi_1 & \psi_2 D_1\psi_2 + \phi_2 D_2\phi_2 \end{bmatrix} \\ &= D + K. \end{aligned}$$

Since each entry of K is supported on the product $(A_1 \cap B_1) \times (A_2 \cap B_2)$, which is compact, the operator K is compact and the proof is now complete. □

Appendix C

An abstract non-sense lemma

Suppose we are given four exact sequences $\alpha, \beta, \gamma, \delta$ interlaced in the braided diagram



(C.1)

and such that all triangles and squares commute.

Lemma C-1. *The sequence*

$$0 \longrightarrow A \xrightarrow{\epsilon_1} B \xrightarrow{\epsilon_2} V_1 \oplus V_2 \xrightarrow{\epsilon_3} C \xrightarrow{\epsilon_4} D \longrightarrow 0$$

coming out of Diagram (C.1), with the obvious choice of maps

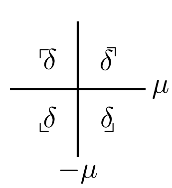
$$\epsilon_1 = \beta_1\alpha_1 = \delta_1\gamma_1 \qquad \epsilon_2 = \begin{bmatrix} \delta_2 \\ \beta_2 \end{bmatrix} \qquad \epsilon_3 = \begin{bmatrix} \alpha_3 & -\gamma_3 \end{bmatrix} \qquad \epsilon_4 = \beta_4\alpha_4 = \delta_4\gamma_4,$$

is exact.

Before proving this lemma, let's observe how it is used in the main text of this thesis on page 88. Lemma 8.2-2 tells us that for the weights δ, η situated in adjacent open squares of $\mathbb{R}^2 \setminus \mathfrak{G}_{A'}$, and separated by the wall $\{\mu\} \times \mathbb{R}$ or $\mathbb{R} \times \{\lambda\}$, we have an exact sequence

$$0 \longrightarrow V(\delta) \longrightarrow V(\eta) \longrightarrow W_\mu \longrightarrow K(-\delta)^* \longrightarrow K(-\eta)^* \longrightarrow 0. \qquad \text{(C.2)}$$

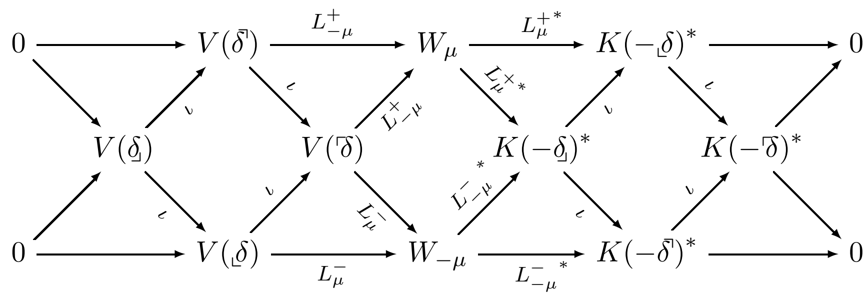
Suppose now we have the following choice of weights:



Denote ι any inclusion map, and $L_\mu^\pm$ the maps

$$L_\mu^+(\phi) = \lim_{t\to\infty} e^{\mu t}\phi, \quad \text{and} \quad L_\mu^- = \lim_{t\to-\infty} e^{\mu t}\phi.$$

Then sequences akin to Sequence (C.2) fit in a diagram



similar to Diagram (C.1).

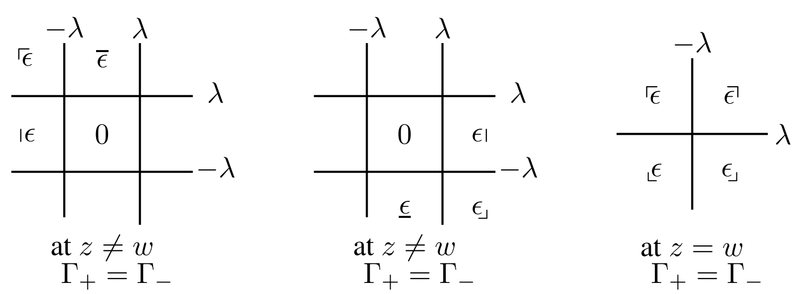
Suppose $\phi V({}^{\ulcorner}\delta)$, and $\psi \in K(-\delta_{\lrcorner})$. Then

$$\begin{aligned} 0 &= \langle \not{\mathfrak{D}}_{A'}\phi, \psi\rangle - \langle \phi, \not{\mathfrak{D}}_{A'}\psi\rangle \\ &= \langle \psi, \nu\psi\rangle|_{-\infty}^{\infty} \\ &= \lim_{t\to\infty}\langle e^{-\mu t}\phi, \nu e^{\mu t}\psi\rangle - \lim_{t\to-\infty}\langle e^{\mu t}\phi, \nu e^{-\mu t}\psi\rangle \\ &= \left(L_\mu^{+*} L_{-\mu}^+(\phi) - L_{-\mu}^{-\,*} L_\mu^-(\phi)\right)(\psi), \end{aligned}$$

hence the middle square commutes. It is quite obvious that all the other squares and triangles commute. The application of Lemma C-1 gives an exact sequence

$$0 \longrightarrow V(\delta_{\lrcorner}) \longrightarrow V({}^{\ulcorner}\delta) \longrightarrow W_\mu \oplus W_{-\mu} \longrightarrow K(-\delta_{\lrcorner})^* \longrightarrow K(-{}^{\ulcorner}\delta)^* \longrightarrow 0.$$

In particular, the sets of weights



yield for $A' = A_z$ the exact sequences

$$\begin{aligned} &0 \longrightarrow V_z \longrightarrow {}^{\ulcorner}V_z \longrightarrow W_\lambda \oplus W_{-\lambda} \longrightarrow {K_z}^* \longrightarrow {K_{\lrcorner z}}^* \longrightarrow 0, && \text{for } \lambda \neq 0, \\ &0 \longrightarrow V_{\lrcorner z} \longrightarrow V_z \longrightarrow W_\lambda \oplus W_{-\lambda} \longrightarrow {}^{\ulcorner}{K_z}^* \longrightarrow {K_z}^* \longrightarrow 0, && \text{for } \lambda \neq 0, \\ &0 \longrightarrow V_w \longrightarrow {}^{\ulcorner}V_w \longrightarrow W_0 \oplus W_0 \longrightarrow {}^{\ulcorner}{K_w}^* \longrightarrow {K_w}^* \longrightarrow 0. \end{aligned}$$

We can now proceed to the postponed proof of Lemma C-1.

Proof: There is nothing deep in this proof: it is only a diagram chase. It is included here for completeness.

The sequence is obviously exact at A and D since any composition of injective maps is surjective, and any composition of injective maps is surjective.

The compositions

$$\begin{aligned}
\epsilon_2\epsilon_1 &= \begin{bmatrix}\delta_2\\ \beta_2\end{bmatrix}\epsilon_1 = \begin{bmatrix}\delta_2\delta_1\gamma_1\\ \beta_2\beta_1\alpha_1\end{bmatrix} = 0,\\
\epsilon_3\epsilon_2 &= \begin{bmatrix}\alpha_3 & -\gamma_3\end{bmatrix}\begin{bmatrix}\delta_2\\ \beta_2\end{bmatrix} = \alpha_3\delta_2 - \gamma_3\beta_2 = 0, \text{ and}\\
\epsilon_4\epsilon_3 &= \begin{bmatrix}\epsilon_4\alpha_3 & -\epsilon_4\gamma_3\end{bmatrix} = \begin{bmatrix}\beta_4\alpha_4\alpha_3 & -\delta_4\gamma_4\gamma_3\end{bmatrix} = 0
\end{aligned}$$

ensure that $\mathrm{Im}(\epsilon_j)\ker(\epsilon_j)$. We now prove the other inclusions.

To simplify notation, every element denoted by a small letter belongs to the space denoted by the corresponding capital letter. For example, $b \in B, x_1 \in X_1, c_1 \in C$.

Suppose $b \in \ker(\epsilon_2)$. Then $\delta_2(b) = \beta_2(b) = 0$ hence $\beta_1(x_1) = b = \delta_1(y_1)$. But then $\alpha_2(x_1) = \delta_2\beta_1(x_1) = 0$ hence $x_1 = \alpha_1(a)$, and thus $b = \beta_1\alpha_1(a)$ and $\ker(\epsilon_2) \subset \mathrm{Im}(\epsilon_1)$ and the sequence is exact at B.

Suppose $(v_1, v_2)\ker(\epsilon_3)$, or equivalently $\alpha_3(v_1) - \gamma_3(v_2) = 0$. Then

$$\begin{aligned}
0 &= \alpha_4(\alpha_3(v_1) - \gamma_3(v_2)) = -\beta_3(v_2),\\
0 &= \gamma_4(\alpha_3(v_1) - \gamma_3(v_2)) = -\delta_3(v_1).
\end{aligned}$$

But then, because the β and δ sequences are exact, we have $v_2 = \beta_2(b_2)$ and $v_1 = \delta_2(b_1)$. As $\gamma_3\beta_2(b_1) = \alpha_3\delta_2(b_1) = \alpha_3(v_1) = 0$, we have $\beta_2(b_1) \in \ker(\gamma_3) = \mathrm{Im}(\gamma_2)$ hence $\beta_2(b_1) = \gamma_2(y_1)$. Similarly, $\delta_2(b_2) = \alpha_2(x_1)$. But then

$$\begin{aligned}
\epsilon_2(b_1 - \delta_1(y_1) + b_2 - \beta_1(x_1)) &= \begin{bmatrix}\delta_2(b_1) - \delta_2\delta_1(y_1) + \delta_2(b_2) - \delta_2\beta_1(x_1)\\ \beta_2(b_1) - \beta_2\delta_1(y_1) + \beta_2(b_2) - \beta_2\beta_1(x_1)\end{bmatrix}\\
&= \begin{bmatrix}v_1 - 0 + \alpha_2(x_1) - \delta_2\beta_1(x_1)\\ \gamma_2(y_1) - \beta_2\delta_1(y_1) + v_2 - 0\end{bmatrix}\\
&= \begin{bmatrix}v_1\\ v_2\end{bmatrix}.
\end{aligned}$$

Hence $\ker(\epsilon_3) \subset \mathrm{Im}(\epsilon_2)$ and the sequence is exact at $V_1 \oplus V_2$.

Suppose $\epsilon_4(c) = 0$. Then $\delta_4\gamma_4(c) = 0$ but $\ker(\delta_4) = \mathrm{Im}(\delta_3)$ hence $\gamma_4(c) = \delta_3(v_1) = \gamma_4\alpha_3(v_1)$ and thus $\alpha_3(v_1) - c \in \ker(\gamma_4) = \mathrm{Im}(\gamma_3)$. Hence $\alpha_3(v_1) - c = \gamma_3(v_2)$ and then $c = \epsilon_3(v_1, v_2)$. Hence $\ker(\epsilon_4) \subset \mathrm{Im}(\epsilon_3)$ and the sequence is exact at D.

The proof is now complete. □

Appendix D

Dirac operators and conformal change of metric

Consider a spin manifold M of dimension n. The Dirac operators D and D' associated to the conformally related metrics g and $g' = e^{2f}g$ on M are, once a spinor bundle is chosen, linked by the formula

$$D' = e^{-\frac{n+1}{2}f} \circ D \circ e^{\frac{n-1}{2}f}. \tag{D.1}$$

In 1986, Bourguignon in [Bou86, p. 339, Prop. 10] had the above formula and claimed that Hitchin in [Hit74] had it wrong in 1974. Then, Lawson and Michelsohn, in their wonderful book [LM89, p. 134], had a $n-1$ on the left hand side instead of a $n+1$. Finally, in 1990, in their inspiring book, Donaldson and Kronheimer, in the case $n = 4$, had a factor of $-1/2$ on the left hand side instead of a $-5/2$; see [DK90, p. 102]. The formula was however only used in [DK90] and [LM89] to see how the kernels of D and D' are related, so no harm was done.

A proof of the Formula

We now prove the formula D.1. Let's denote the spinor bundles for g and g' by S and S'. Let $\mu\colon Spin(n) \to Aut(\Delta)$ be the spin representation. Then

$$\begin{aligned} S &= P_{Spin}(M, g) \times_\mu \Delta, \text{ and} \\ S' &= P_{Spin}(M, g') \times_\mu \Delta. \end{aligned}$$

We assume that they are the "same" spin structure. Hence S and S' are isomorphic as vector bundles but not as Clifford bundles.

The bundles S and S' come equipped with extra structure. The Clifford multiplication for g is a map $\rho\colon TM \to Aut(S)$ satisfying $\rho(v)^2 = -g(v, v)$. Define a new Clifford multiplication $\rho'\colon TM \to Aut(S)$ for g' by the formula $\rho' = e^f \rho$. It is still skew-adjoint.

Let e_i be an orthonormal frame for g over an open set U and ω_{ij} be the Levi-Civita connection matrix for that frame. Over U, the spin connection (see [LM89, p. 110]) is $d + \Omega$ with

$$\Omega(V) = \frac{1}{4} \sum_{i,j} \omega_{ij}(V) \rho(e_j) \rho(e_i).$$

Set $e'_j = e^{-f} e_j$. The e'_j form an orthonormal frame for g'. Note that $\rho'(e'_j) = \rho(e_j)$.
As shown in [LM89, p. 133–134], we have

$$\nabla'_X Y = \nabla_X Y + (Xf)Y + (Yf)X - g(X,Y)\mathrm{grad}\,(f), \text{ and} \qquad \text{(D.2)}$$
$$\omega'_{ij}(V) = \omega_{ij}(V) + (e_j f)g(V,e_i) - (e_i f)g(V,e_j).$$

Duplicating the definition of Ω, we set

$$\Omega'(V) := \frac{1}{4}\sum_{i,j}(\omega_{ij}(V)\rho(e_j)\rho(e_i) + (e_j f)g(V,e_i)\rho(e_j)\rho(e_i) - (e_i f)g(V,e_j)\rho(e_j)\rho(e_i)).$$

This expression simplifies to

$$\begin{aligned}\Omega'(V) &= \Omega(V) + \frac{1}{4}\rho(\mathrm{grad}\,(f))\rho(V) - \frac{1}{4}\rho(V)\rho(\mathrm{grad}\,(f))\\ &= \Omega(V) - \frac{1}{2}\rho(V)\rho(\mathrm{grad}\,(f)) - \frac{1}{2}g(V,\mathrm{grad}\,(f)).\end{aligned} \qquad \text{(D.3)}$$

Let's now check that the connection ∇' induced by Ω' is compatible with the Levi-Civita connection of (M,g').
Notice first that

$$\begin{aligned}\nabla_X(\rho'(Y)s) &= \nabla_X(e^f\rho(Y)s)\\ &= (Xf)\rho'(Y)s + e^f\nabla_X(\rho(Y)s).\end{aligned}$$

Since S is a Clifford bundle for (M,g), we have

$$\nabla_X(\rho'(Y)s) = \rho'((Xf)Y)s + \rho'(\nabla_X Y)s + \rho'(Y)\nabla_X s. \qquad \text{(D.4)}$$

Notice now that

$$\begin{aligned}\rho(X)\rho(\mathrm{grad}\,(f))\rho'(Y)s &= -\rho(X)\rho'(Y)\rho(\mathrm{grad}\,(f)) - 2e^f\rho(X)(Yf)s\\ &= \big(\rho'(Y)\rho(X) + 2e^f g(X,Y)\big)\rho(\mathrm{grad}\,(f))s - 2\rho'\big((Yf)X\big)s,\end{aligned}$$

so that

$$-\frac{1}{2}\rho(X)\rho(\mathrm{grad}\,(f))\rho'(Y)s = -\frac{1}{2}\rho'(Y)\rho(X)\rho(\mathrm{grad}\,(f))s + \rho'\big((Yf)X - g(X,Y)\mathrm{grad}\,(f)\big)s. \qquad \text{(D.5)}$$

Putting all these computations together, we find

$$\begin{aligned}\nabla'_X(\rho'(Y)s) &\overset{D.3}{=} \nabla_X(\rho'(Y)s) - \frac{1}{2}\rho(X)\rho(\mathrm{grad}\,(f))s - \frac{1}{2}(Xf)\rho'(Y)s\\ &\overset{D.4}{=} \rho'(\nabla_X Y + (Xf)Y)s + \rho'(Y)\nabla_X s - \frac{1}{2}\rho(X)\rho(\mathrm{grad}\,(f))s - \frac{1}{2}(Xf)\rho'(Y)s\\ &\overset{D.2,D.5}{=} \rho'(\nabla'_X Y)s + \rho'(y)(\nabla_X s - \frac{1}{2}\rho(X)\rho(\mathrm{grad}\,(f))s - \frac{1}{2}(Xf)s)\\ &\overset{D.3}{=} \rho'(\nabla'_X Y)s + \rho'(Y)\nabla'_X s.\end{aligned}$$

We also need to check that the connection ∇' is compatible with the hermitian metric on S. Notice first that

$$\begin{aligned}&\langle\rho(V)\rho(\mathit{grad}\,(f))s_1,s_2\rangle+\langle s_1,\rho(V)\rho(\mathit{grad}\,(f))s_2\rangle\\&\qquad=\langle s_1,\rho(\mathit{grad}\,(f))\rho(V)s_2\rangle+\langle s_1,\rho(V)\rho(\mathit{grad}\,(f))s_2\rangle=-2(Vf)\langle s_1,s_2\rangle.\end{aligned}$$

Thus, we have

$$\begin{aligned}\langle\nabla'_V s_1,s_2\rangle+\langle s_1,\nabla'_V s_2\rangle&=\langle\nabla_V s_1,s_2\rangle+\langle s_1,\nabla_V s_2\rangle\\&\qquad-\frac{1}{2}\big(\langle\rho(V)\rho(\mathit{grad}\,(f))s_1,s_2\rangle+\langle s_1,\rho(V)\rho(\mathit{grad}\,(f))s_2\rangle\big)\\&\qquad-\frac{1}{2}\big(\langle(Vf)s_1,s_2\rangle+\langle s_1,(Vf)s_2\rangle\big)\\&=\langle\nabla_V s_1,s_2\rangle+\langle s_1,\nabla_V s_2\rangle+(Vf)\langle s_1,s_2\rangle-(Vf)\langle s_1,s_2\rangle\\&=V\langle s_1,s_2\rangle,\end{aligned}$$

since ∇ is compatible with $\langle\ ,\ \rangle$.

We are now finished proving that S with the connection ∇' and the Clifford multiplication ρ' is really a Clifford bundle for (M,g').

Let $V:=Hom_{Cl}(S',S)$. We have that as Clifford bundles for (M,g'), the bundle S and $S'\otimes V$ are isomorphic.

By Schur's lemma, the bundle V is a smooth complex line bundle. Since S and S' are isomorphic as smooth complex vector bundles, V is trivial. But more than that, the connection on V is trivial as we now show.

Let ρ' also denote the Clifford multiplication on S' and Ω' denote the spin connection on S'. Such an abuse of notation is harmless if we are careful. We have again

$$\Omega'(V)=\frac{1}{4}\sum_{i,j}\omega'_{ij}(V)\rho'(e'_j)\rho'(e'_i).$$

Let f be a section of V and s a section of S'. Then since f commutes with Clifford multiplication, we have

$$\begin{aligned}(\nabla f)(s)&=\nabla\big(f(s)\big)-f(\nabla s)\\&=d\big(f(s)\big)+\Omega'\cdot f(s)-f(ds)-f(\Omega'\cdot s)\\&=df(s),\end{aligned}$$

and thus our claim is proved.

So we can really work with the bundle S when studying the relationship between D and D'. Let's do that.

We have

$$D'\phi=e^{-f}\Big(D\phi+\frac{n-1}{2}\rho(\mathit{grad}\,(f))\phi\Big).$$

Indeed,

$$\begin{aligned}
D'\phi &= \sum \rho'(e_i')\nabla'_{e_i'}\phi \\
&= e^{-f}\sum \rho(e_i)\nabla'_{e_i}\phi \\
&= e^{-f}\Big(D\phi - \frac{1}{2}\sum \rho(e_i)\rho(e_i)\rho(\mathrm{grad}\,(f))\phi - \frac{1}{2}\sum \rho(e_i)g(e_i,\mathrm{grad}\,(f))\phi\Big) \\
&= e^{-f}\Big(D\phi + \frac{n}{2}\rho(\mathrm{grad}\,(f))\phi - \frac{1}{2}\rho(\mathrm{grad}\,(f))\phi\Big) \\
&= e^{-f}\Big(D\phi + \frac{n-1}{2}\rho(\mathrm{grad}\,(f))\phi\Big).
\end{aligned}$$

But then,

$$\begin{aligned}
D\big(e^{\frac{n-1}{2}f}\phi\big) &= e^{\frac{n-1}{2}f}\Big(D\phi + \frac{n-1}{2}\rho(\mathrm{grad}\,(f))\phi\Big) \\
&= e^{\frac{n-1}{2}f}e^{f}D'\phi,
\end{aligned}$$

whence $D'\phi = e^{-\frac{n+1}{2}f}D\big(e^{\frac{n-1}{2}f}\phi\big)$, as wanted.

A confirmation

The proof just presented should at least convince us that there exists such a formula. To confirm that the factors are correct, suppose that D and D' are linked by formula D.1 and that D is self-adjoint for the L^2 inner product on (M, g). We want to prove now that D' is self-adjoint on (M, g'). Recall that $dvol' = e^{nf}dvol$. Thus,

$$\begin{aligned}
\langle D'\phi,\psi\rangle' &= \int \langle D'\phi,\psi\rangle\ dvol' \\
&= \int \langle e^{-(n+1)f/2}De^{(n-1)f/2}\phi,\psi\rangle e^{nf}\ dvol \\
&= \langle De^{(n-1)f/2}\phi, e^{(n-1)f/2}\psi\rangle \\
&= \langle e^{(n-1)f/2}\phi, De^{(n-1)f/2}\psi\rangle \\
&= \langle \phi, D'\psi\rangle'.
\end{aligned}$$

That is it.

In fact, this computation and the triviality of V show that any formula which has one of the factors must also have the other up to a constant.

Appendix E

Weighted Sobolev spaces on $\mathbb{R}^n$, Bartnik's presentation

In this appendix, we visit and work through a part of Robert Bartnik's paper "The Mass of an Asymptotically Flat Manifold," [Bar86]. The part we are concerned with deals with weighted Sobolev spaces on $\mathbb{R}^n$ for $n \geq 3$, and Fredholmness of certain 2nd order elliptic partial differential operators. This appendix is an companion to Bartnik's writing, merely adding proofs that were lacking.

Weighted Sobolev Spaces

Set

$$\sigma(x) := (1 + |x|^2)^{1/2}.$$

The space L^p_δ is the space of measurable functions in L^p_{loc} which are finite in the $\| \ \|_{p,\delta}$-norm:

$$\|u\|_{p,\delta} = \begin{cases} \left(\int_{\mathbb{R}^n} \sigma^{-\delta p - n}(x) |u(x)|^p \, dx \right)^{1/p}, & 1 \geq p < \infty; \\ ess\ sup_{\mathbb{R}^n} \sigma^{-\delta} |u|, & p = \infty. \end{cases}$$

When $p < \infty$, the usual L^p space arise as $L^p_{-n/p}$. In fact, we have an even stronger proposition.

Proposition E-1. *The following map is an isometry:*

$$\begin{aligned} L^p &\to L^p_\delta \\ f &\mapsto \sigma^{\delta + n/p} f. \end{aligned}$$

The space $W^{k,p}_\delta$ is then defined as the space of functions in L^p_δ with weak derivatives in the appropriate weighted L^p space. The norm is

$$\|u\|_{k,p,\delta} = \sum_{l \leq k} \|D^l u\|_{p,\delta - l}.$$

These spaces are nicely set-up for the interesting theorems to be in some sense independent of n, as will become apparent later with the various inequalities and embeddings.

For one thing, we have that

$$\|\sigma^a\|_{\infty,\delta} = ess\ sup\, \sigma^{a-\delta} = \begin{cases} 1, & \text{if } a \leq \delta; \\ \infty, & \text{otherwise.} \end{cases}$$

Similarly, we have, for $1 \leq p < \infty$,

$$\begin{aligned} \|\sigma^a\|_{p,\delta}^p &= \int_{\mathbb{R}^n} \sigma^{(a-\delta)p-n} \\ &= \omega_n \int_0^\infty \sigma(r)^{(a-\delta)p-n} r^{n-1}\, dr. \end{aligned}$$

The integral on the last line is finite if and only if its $[1,\infty)$ part is finite. It is the case if and only if $a < \delta$. In fact, we have the following stronger proposition.

Proposition E-2. *We have*

$$\sigma^a \in W_\delta^{k,\infty} \quad \Longleftrightarrow \quad a \leq \delta,$$

and for $1 \leq p < \infty$*, we have*

$$\sigma^a \in W_\delta^{k,p} \quad \Longleftrightarrow \quad a < \delta.$$

Proof: We already proved the case $k = 0$. Notice now that

$$\frac{\partial}{\partial x_i}(\sigma^a) = a\sigma^{a-2}x_i$$

and similarly, these exist homogeneous polynomials p_α of degree $|\alpha|$ such that

$$\frac{\partial}{\partial x^\alpha}(\sigma^a) = \sigma^{a-2|\alpha|}p_\alpha.$$

Thus

$$\left\|\frac{\partial\sigma^a}{\partial x^\alpha}\right\|_{p,k,\delta-|\alpha|} \leq C\|\sigma^{a-|\alpha|}\|_{p,\delta-|\alpha|}.$$

This inequality imply that $\sigma^a \in L_\delta^p$ implies $\sigma^a \in W_\delta^{k,p}$. $\square$

Many different choices for weight function σ could have been considered. Apparently some work has been done with exponential weights instead of the "polynomial" weight that we use here.

We define similarly the norm $\| \ \|'_{k,p,\delta}$ and the spaces L'^p_δ and $W'^{k,p}_\delta$ of functions on $\mathbb{R}^n \setminus \{0\}$ by changing the weight function to $r(x) := |x|$.

In these modified Sobolev spaces, scaling becomes homogeneous. Indeed, set

$$u_R(x) := u(Rx);$$

then

$$\|u_R\|'_{k,p} = R^\delta\|u\|'_{k,p,\delta}.$$

Of course any norm without a weight refers to a usual Sobolev norm.

Set $A_R = B_{2R} \setminus B_R$ and $E_R = \mathbb{R}^n \setminus B_R$. We use an obvious notation for restriction over subset of $\mathbb{R}^n$. Then the norm $u \mapsto \|u\|_{p,\delta;A_R}$ is equivalent to the norm $u \mapsto R^{-\delta}\|u_R\|_{p;A_1}$, and Bartnik writes

$$\|u\|_{p,\delta;A_R} \approx R^{-\delta}\|u_R\|_{p;A_1},$$

with constants not depending on R but depending on δ. This equivalence allows us to rescale and apply local estimates to prove part of the following theorem.

Theorem E-3. *We have the following inequalities:*

$$\text{If } p \leq q \text{ and } \delta_2 < \delta_1 \text{ then } \|u\|_{p,\delta_1} \leq c\|u\|_{q,\delta_2}. \tag{E.1}$$

$$\text{If } 1/p = 1/q + 1/r \text{ and } \delta = \delta_1 + \delta_2 \text{ then } \|uv\|_{p,\delta} \leq \|u\|_{q,\delta_1} \|v\|_{r,\delta_2}. \tag{E.2}$$

$$\text{For any } \epsilon > 0, \text{ there exists } C(\epsilon) \text{ s.t. } \|u\|_{1,p,\delta} \leq \epsilon\|u\|_{2,p,\delta} + C(\epsilon)\|u\|_{0,p,\delta}. \tag{E.3}$$

Proof: To prove Inequality (E.2), we write, using Proposition E-1,

$$\|uv\|_{p,\delta} = \|\sigma^{-\delta-n/p}uv\|_p = \|\sigma^{-\delta_1-n/q}u\sigma^{-\delta_2-n/r}v\|_p.$$

Using the usual Hölder inequality, we obtain the result.

In the conditions under which we wish Inequality (E.1) to be true, there exists r such that $1/p = 1/q + 1/r$. Thus, we can again use Proposition E-1 and the usual Hölder inequality to get

$$\begin{aligned}
\|u\|_{p,\delta_1} &= \|\sigma^{-\delta_1-n/p}u\|_p \\
&= \|(\sigma^{-n/r}\sigma^{\delta_2-\delta_1})(\sigma^{-\delta_2-n/q}u)\|_p \\
&\leq \|\sigma^{\delta_2-\delta_1}\|_{r,0}\|u\|_{q,\delta_2}
\end{aligned}$$

Since $\delta_2 < \delta_1$, we know by Proposition E-2 that $\sigma^{\delta_2-\delta_1} \in L^r_0$ hence Inequality (E.1) is true.

We now rescale and apply local estimates to prove Inequality (E.3). We know that there exists constants C_1, C_2 not depending on R such that

$$\|u\|_{1,p,\delta;A_R} \leq C_1 R^{-\delta}\|u_R\|_{1,p;A_1}, \text{ and}$$

$$R^{-\delta}\|u_R\|_{2,p;A_1} \leq C_2\|u\|_{2,p,\delta;A_R}.$$

Set $\epsilon' = \epsilon/C_1C_2$. We know from the local interpolation inequality that for some $C(\epsilon')$,

$$\|u_R\|_{1,p;A_1} \leq \epsilon'\|u_R\|_{2,p;A_1} + C\|u_R\|_{0,p;A_1}.$$

Thus

$$\begin{aligned}
\|u\|_{1,p,\delta;A_R} &\leq C_1 R^{-\delta}\|u_R\|_{1,p;A_1} \\
&\leq C_1 R^{-\delta}\epsilon'\|u_R\|_{2,p;A_1} + R^{-\delta}C\|u_R\|_{0,p;A_1} \\
&\leq \epsilon\|u\|_{2,p,\delta;A_R} + C\|u\|_{0,p,\delta;A_R}.
\end{aligned}$$

Now, we need to patch all these interpolations together. Recall first that for all $p > 0$, we have

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

Set $D_0 := B_1$ and, for $i > 0$, set $D_i = A_{2^{i-1}}$. The D_i are disjoint and cover $\mathbb{R}^n$. Set $u_i = u|_{D_i}$.

The fact that
$$\|u\|_{1,p,\delta} \le \epsilon \|u\|_{2,p,\delta} + C(\epsilon)\|u\|_{0,p,\delta}$$
follows from the computation
$$\begin{aligned}\|u\|^p_{1,p,\delta} &= \sum \|u_i\|^p_{1,p,\delta;D_i} \\ &\le \sum \Big(\epsilon \|u_i\|_{2,p,\delta;D_i} + C\|u_i\|_{0,p,\delta;D_i}\Big)^p \\ &\le 2^{p-1}\epsilon^p \|u\|^p_{2,p,\delta} + C\|u\|^p_{0,p,\delta}.\end{aligned}$$
The proof for $p = \infty$ goes along the same lines. □

The inequalities are great and useful, and, apart from the first one, are generalizations of what we have for usual Sobolev spaces.

Before studying the generalization of other classical powerful estimates, the Sobolev embedding theorems, we must generalize yet another type of space, the Hölder spaces. Define first, for $x \in \mathbb{R}^n$, the punctured ball $B(x)$ to be the set of all y such that $0 < 4|x-y| < \sigma(x)$. For $0 < \alpha \le 1$, the weighted Hölder norm is defined by the equation
$$\|u\|_{C^{0,\alpha}_\delta} = \sup_{x\in\mathbb{R}^n}\Big(\sigma^{-\delta}(x)|u(x)|\Big) + \sup_{x\in\mathbb{R}^n}\Big(\sigma^{-\delta+\alpha}(x)\sup_{y\in B(x)}\frac{|u(x)-u(y)|}{|x-y|^\alpha}\Big).$$
Note that if $\|u\|_{C^{0,\alpha}_\delta}$ is finite, then u is continuous. Indeed, close to x_0, we have
$$|u(x_0) - u(y)| \le C\sigma(x_0)^{-\alpha+\delta}\,|x_0 - y|^\alpha.$$
We may now proceed.

Theorem E-4. *Suppose $u \in W^{k,p}_\delta$. We have the following inequalities:*

$$\text{If } n-kp>0 \text{ and } p\le q\le np/(n-kp) \text{ then } \|u\|_{np/(n-kp),\delta} \le C\|u\|_{k,q,\delta}. \tag{E.4}$$
$$\text{If } n-kp<0 \text{ then } \|u\|_{\infty,\delta} \le C\|u\|_{k,p,\delta} \tag{E.5}$$
$$\text{and } |u(x)| = o(r^\delta) \text{ as } r\to\infty. \tag{E.6}$$
$$\text{If } n-kp<0 \text{ and } 0<\alpha\le\min(1,k-n/p) \text{ then } \|u\|_{C^{0,\alpha}_\delta} \le C\|u\|_{k,p,\delta} \tag{E.7}$$
$$\text{and } \|u\|_{C^{0,\alpha}_\delta(A_R)} = o(1) \text{ as } R\to\infty. \tag{E.8}$$

Proof: Suppose first that $n - kp > 0$. Then set $p^* := np/(n-kp) < \infty$. We have
$$\begin{aligned}\|u\|_{p^*,\delta;A_R} &\le CR^{-\delta}\|u_R\|_{p^*;A_1} \\ &\le CR^{-\delta}\|u_R\|_{k,q;A_1} \text{ (by the usual Sobolev inequality)} \\ &\le C\|u\|_{k,q,\delta;A_R}.\end{aligned}$$

Writing $u = \sum u_i$ as in the proof of Theorem E-3, we obtain

$$\begin{aligned} \|u\|_{p^*,\delta} &= \big(\sum \|u_i\|_{p^*,\delta}^{p^*}\big)^{1/p^*} \\ &\leq C\big(\sum \|u_i\|_{k,q,\delta}^{p^*}\big)^{1/p^*} \\ &\leq C\big(\sum \|u_i\|_{k,q,\delta}^{q}\big)^{1/q} \\ &= C\|u\|_{k,q,\delta}. \end{aligned}$$

Inequality (E.4) is now proved.

Maybe it is worthwhile noting down the proof of the last inequality of this proof. In fact, it is sufficient to prove that $(1+x^{p^*})^{1/p^*} \leq (1+x^q)^{1/q}$ for $x \geq 0$. The function

$$f(x) = \frac{(1+x^{p^*})^q}{(1+x^q)^{p^*}}$$

has derivative

$$\frac{x^{p^*-1}p^*q(1+x^{p^*})^{q-1}(1+x^q)^{p^*} - p^*qx^{q-1}(1+x^{p^*})^q(1+x^q)^{p^*-1}}{(1+x^q)^{2p^*}}.$$

Once we remove the common factors, which are anyway strictly positive, we have

$$x^{p^*-1}(1+x^q) - x^{q-1}(1+x^{p^*}) = x^{p^*-1} - x^{q-1} \geq 0$$

since $q \leq p^*$. Thus f always increases. But obviously, it takes the value 1 at infinity.

The same scaling argument and application of the usual Sobolev inequality apply to prove Inequality (E.5). Of course, $u \in W_\delta^{k,p}$ imply then that

$$|r(x)^{-\delta}u(x)| \leq C\|u\|_{\infty,\delta;A_{|x|}} \leq C\|u\|_{k,p,\delta;A_{|x|}}$$

which converges to 0. Thus the asymptotic behavior of Equation (E.6) is now proved.

To prove Inequality (E.7), we would like use the decomposition $u = \sum u_i$. That cannot work however. Yet something of the sort works.

As before, we have that

$$\|u\|_{C_\delta^{0,\alpha}(A_R)} \approx R^{-\delta}\|u_R\|_{C_\delta^{0,\alpha}(A_1)}$$

with constants not depending on R. But the usual theorem can be used and we have

$$\begin{aligned} \|u\|_{C_\delta^{0,\alpha}(A_R)} &\leq CR^{-\delta}\|u_R\|_{C_\delta^{0,\alpha}(A_1)} \\ &\leq CR^{-\delta}\|u_R\|_{k,p;A_1} \\ &\leq C\|u\|_{k,p,\delta;A_R}. \end{aligned}$$

Since $\| \, \|_{k,p;B_1}$ is equivalent to $\| \, \|_{k,p,\delta;B_1}$, we have

$$\|u\|_{C_\delta^{0,\alpha}(D_i)} \leq C\|u\|_{k,p,\delta;D_i},$$

with the constant C not depending on i.

Now the condition $y \in B(x)$ in the inner supremum becomes really essential. Indeed, we can see that if $x \in D_i$ then $B(x)$ is contained in $D_{i-1} \cup D_i \cup D_{i+1}$. Because of that, we can actually bound the $C^{0,\alpha}_\delta$-norm of a function by norms of restrictions.

Let's write $a(x,y)$ for $\sigma(x)^{-\delta+\alpha}|u(x)-u(y)|/|x-y|^\alpha$. Thus,

$$\|u\|_{C^{0,\alpha}_\delta} = \|u\|_{\infty,\delta} + \sup_x \sup_{y\in B(x)} a(x,y).$$

Let $x \in D_i$. We split the "ball" $B(x)$ in three parts.

Suppose first that $y \in D_{i-1} \cap B(x)$. There is a point $z \in [x,y] \cap D_{i-1} \cap D_i$. As for any point in D_i, we have the relationship $4^{-1}\sigma(z) \le \sigma(x) \le 4\sigma(z)$. Thus

$$\begin{aligned} a(x,y) &\le a(x,z) + 4^{|-\delta+\alpha|} a(z,y) \\ &\le \sup_{z\in B(x)\cap D_i} a(x,z) + 4^{|-\delta+\alpha|} \sup_{y\in B(x)\cap D_{i-1}} a(z,y). \end{aligned}$$

But $B(x) \cap D_{i-1} \subset B(z) \cap D_{i-1}$. Thus

$$a(x,y) \le \|u\|_{C^{0,\alpha}_\delta(D_i)} + 4^{|-\delta+\alpha|} \|u\|_{C^{0,\alpha}_\delta(D_{i-1})}.$$

Similarly, we have for $y \in D_{i+1} \cap B(x)$ that

$$a(x,y) \le \|u\|_{C^{0,\alpha}_\delta(D_i)} + 4^{|-\delta+\alpha|} \|u\|_{C^{0,\alpha}_\delta(D_{i+1})},$$

and for $y \in D_i \cap B(x)$ that $a(x,y) \le \|u\|_{C^{0,\alpha}_\delta(D_i)}$.

Hence, for $x \in D_i$,

$$\begin{aligned} \sup_{y\in B(x)} a(x,y) &\le \max(1, 4^{|-\delta+\alpha|})\big(\|u\|_{C^{0,\alpha}_\delta(D_{i-1})} + \|u\|_{C^{0,\alpha}_\delta(D_i)} + \|u\|_{C^{0,\alpha}_\delta(D_{i+1})}\big) \\ &\le C \sum \|u\|_{C^{0,\alpha}_\delta(D_i)}. \end{aligned}$$

whence $\|u\|_{C^{0,\alpha}_\delta} \le C \sum \|u\|_{C^{0,\alpha}_\delta(D_i)}$.

But then,

$$\begin{aligned} \|u\|_{C^{0,\alpha}_\delta} &\le C \sum \|u\|_{C^{0,\alpha}_\delta(D_i)} \\ &\le C \sum \|u\|_{k,p,\delta;D_i} \\ &\le C \|u\|_{k,p,\delta}, \end{aligned}$$

proving Equation (E.7).

Estimate (E.8) is a consequence of Inequality (E.7) for the domain A_R, and of the finiteness of p, which implies that $\|u\|_{k,p,\delta;A_R}$ tends to 0. □

Fredholm theory for second order operators asymptotic to Δ

Of course, we introduce this big machinery in order to do some Fredholm theory for certain partial differential operators on $\mathbb{R}^n$. We consider in Bartnik's paper second order operator which are "asymptotic" to the Laplacian in the following sense.

Definition: The operator $u \to Pu$ defined by

$$Pu = \sum_{i,j} a^{ij}(x)\partial^2_{i,j}u + \sum_i b^i(x)\partial_i u + c(x)u$$

is *asymptotic* to Δ at rate $\tau \geq 0$ if there exist $n < q < \infty$ and constants C_1, λ such that

$$\lambda|\xi|^2 \leq \sum_{i,j} a^{ij}\xi_i\xi_j \leq \lambda^{-1}|\xi|^2, \text{ and}$$
$$\|a^{ij} - \delta_{ij}\|_{1,q,-\tau} + \|b^i\|_{0,q,-1-\tau} + \|c\|_{0,q,-2-\tau} \leq C_1.$$

Let's note that Bartnik uses the norm $L^{q/2}_{-2-\tau}$-norm for c instead of the $L^q_{-2-\tau}$-norm like we do.
We should first check what are natural domain and codomain for such operators.

Theorem E-5. *If P is asymptotic to Δ, then*

$$P\colon W^{2,p}_\delta \to W^{0,p}_{\delta-2}$$

is bounded for $1 \leq p \leq q$ and every $\delta \in \mathbb{R}$.

Proof: We show that each piece is bounded. In the first place, we have

$$\begin{aligned}
\|a^{ij}\partial^2_{ij}u\|_{0,p,\delta-2} &\leq \|(a^{ij} - \delta_{ij})\partial^2_{ij}u\|_{0,p,\delta-2} + \|\delta_{ij}\partial^2_{ij}u\|_{0,p,\delta-2} \\
&\leq (\|a^{ij} - \delta_{ij}\|_{\infty,0} + 1)\|\partial^2_{ij}u\|_{0,p,\delta-2} \\
&\leq C(\|a^{ij} - \delta_{ij}\|_{\infty,-\tau} + 1)\|u\|_{2,p,\delta} \\
&\leq C(\|a^{ij} - \delta_{ij}\|_{1,q,-\tau} + 1)\|u\|_{2,p,\delta} \\
&\leq C\|u\|_{2,p,\delta}.
\end{aligned}$$

When $p = q$, we have

$$\begin{aligned}
\|b^i\partial_i u\|_{0,p,\delta-2} &\leq \|b^i\|_{p,-1}\|\partial_i u\|_{\infty,\delta-1} \\
&\leq C\|b^i\|_{q,-1-\tau}\|\partial_i u\|_{1,p,\delta-1} \\
&\leq C\|u\|_{2,p,\delta}.
\end{aligned}$$

Suppose now that $p < q$ and let $1/p = 1/r + 1/q$. Note that $r - p = rp/q$ thus the inequality $n < q = rp/(r-p)$ implies $nr/(n+r) < p$. Of course, we have $p < r$. Thus the Sobolev embedding theorem says

$$\|\partial_i u\|_{r,\delta-1+\tau} \leq C\|\partial_i u\|_{r,\delta-1} \leq \|\partial_i u\|_{1,p,\delta-1} \leq C\|u\|_{2,p,\delta}, \text{ and}$$
$$\|u\|_{r,\delta+\tau} \leq C\|u\|_{r,\delta} \leq C\|u\|_{1,p,\delta} \leq C\|u\|_{2,p,\delta}.$$

Note that these inequalities are true even for $\tau = 0$. We can now complete the proof by seeing that

$$\begin{aligned}\|b^i \partial_i u\|_{0,p,\delta-2} &\leq \|b^i\|_{q,-1-\tau} \|\partial_i u\|_{r,\delta-1+\tau} \\ &\leq C\|u\|_{2,p,\delta},\end{aligned}$$

and $\|cu\|_{0,p,\delta-2} \leq \|c\|_{0,q,-2-\tau} \|u\|_{0,r,\delta+\tau} \leq C\|u\|_{2,p,\delta}$. □

Since P is asymptotic to Δ and is continuous, it is natural to hope and daringly expect that even though we are not on a compact set, some elliptic estimate is anyhow available for P.

Proposition E-6 ([Bar86, Prop 1.6]). *Suppose P is asymptotic to Δ, and $1 < p \leq q$, and $\delta \in \mathbb{R}$. There is a constant $C = C(n, p, q, \delta, C_1, \lambda)$ such that if $u \in L^p_\delta$ and $Pu \in L^p_{\delta-2}$ then $u \in W^{2,p}_\delta$ and*

$$\|u\|_{2,p,\delta} \leq C(\|Pu\|_{0,p,\delta-2} + \|u\|_{0,p,\delta}) \tag{E.9}$$

Proof: Let's define

$$P_R = a^{ij}_R \partial_{ij} + R b^i_R \partial_i + R^2 c_R,$$

and note that $P_R u_R = R^2 (Pu)_R$.

Here we need to use the usual L^p estimates for A_1. We have, see [GT83, p. 235, Thm 9.11], that for a fattened domain $\tilde{A}_1$, we have

$$\|u_R\|_{2,p;A_1} \leq C(\|u_R\|_{p;\tilde{A}_1} + \|P_R u_R\|_{p;\tilde{A}_1}).$$

Going through the proof of [GT83, p. 235, Thm 9.11], we see that the constant in this inequality depends in particular on

$$\|a^{ij}_R - \delta_{ij}\|_{\infty;\tilde{A}_1} \leq C\|a^{ij} - \delta_{ij}\|_{\infty,0} \leq C\|a^{ij} - \delta_{ij}\|_{C^{0,\alpha}_{-\tau}} \leq C(C_1).$$

It turns out that this dependance is independent of R.

It also depends on

$$\|R b^i_R\|_{\infty;\tilde{A}_1} \leq C\|b^i\|_{\infty,-1} \leq C\|b^i\|_{0,q,-1-\tau},$$

which is bounded independently of R as well.

The constant also depends on c, but this term is harder to bound. Reading the proof, we reach a point where we want to reduce the number of derivative on u_R to use some interpolation estimate. We have

$$\|D^2 u_R\|_p \leq C\|a^{ij}_R \partial_{ij}\|_p \leq C(\|P_R u_R\|_p + \|R b^i_R \partial_u u_R\|_p + \|R^2 c_R u\|_p),$$

and the last term must be bounded somehow. We already did the needed work while proving the continuity of P: our proof that multiplication by c is bounded $W^{2,p} \to L^p$ actually works for $W^{1,p} \to L^p$. Thus the constant in the L^p estimate depends on

$$\|R^2 c_R\|_{q;\tilde{A}_1} \leq C\|c\|_{q,-2} \leq C\|c\|_{q,-2-\tau}$$

which is bounded independently of R.

But then, we have

$$\begin{aligned}\|u\|_{2,p,\delta;A_R} &\le CR^{-\delta}\|u_R\|_{2,p;\tilde{A}_1}\\ &\le CR^{-\delta}(\|u_R\|_{p;\tilde{A}_1}+\|P_Ru_R\|_{p;\tilde{A}_1})\\ &= CR^{-\delta}(\|u_R\|_{p;\tilde{A}_1}+R^2\|(Pu)_R\|_{p;\tilde{A}_1})\\ &\le C(\|u\|_{p,\delta;\tilde{A}_R}+\|Pu\|_{p,\delta-2;\tilde{A}_R}).\end{aligned}$$

The trick we have done so often now with the domains D_i completes the proof. □

We are interested in the "Fredholmness" of P. But the estimate given to us in the previous proposition is not sufficient: we need some compactness of the right-hand-side term.

To understand the Fredholmness of P, we first deal with the Laplacian. The orders of growth of harmonic functions in $\mathbb{R}^n \setminus B_1$ are $\mathbb{Z} \setminus \{-1, \dots, 3-n\}$ and are called *exceptional values*.

For nonexceptional weighing parameter δ, we have a very strong Fredholmness result given by the next theorem. Before reaching it, we extract a lemma from a paper of Nirenberg and Walker.

Lemma E-7 ([NW73, lemma 2.1]). *Fix $p \in (1,\infty)$, and set $p' = p/(p-1)$. Let $a, b \in \mathbb{R}$ be such that $a+b>0$. Set*

$$K'(x,y) = |x|^{-a}|x-y|^{a+b-n}|y|^{-b} \text{ for } x \neq y$$

and for $u \in L^p$ define

$$K'u(x) = \int K'(x,y)u(y)dy.$$

Then there is a constant $c = c(n,p,a,b)$ such that

$$\|K'u\|_p \le c\|u\|_p$$

if and only if $a < n/p$ and $b < n/p'$.

Proof: See [NW73, p. 273] for the proof that the conditions on a and b are necessary.

We can assume that a and b are nonnegative. Indeed, at least one of then is, say b. Suppose $a<0$. Then the inequality

$$\frac{|x|}{|x-y|} \le 1 + \frac{|y|}{|x-y|}$$

implies that

$$\begin{aligned}K'(x,y) &\le |x-y|^{b-n}|y|^{-b}\Big(1+\frac{|y|}{|x-y|}\Big)^{-a}\\ &\le C|x-y|^{b-n}|y|^{-b} + C|x-y|^{a+b-n}|y|^{-a-b}.\end{aligned}$$

For nonnegative a and b satisfying $a < n/p$ and $b < n/p'$, the inequality $|x|^n \ge \prod |x_i|$ yields

$$K'(x,y) \le \prod_{i=1}^{n} |x_i|^{-a/n}|x_i-y_i|^{(a+b)/n-1}|y_i|^{-b/n}.$$

The problem is thus reduced to one dimensional.

Now for the one-dimensional result, [NW73] cite a lemma whose origin is really unclear.

Suppose that $K(x,y)$ is nonnegative and homogeneous of degree -1 for $x \ge 0$ and $y \ge 0$, and that the (necessarily identical) quantities

$$\int_0^\infty K(x,1)x^{-1/p'}dx \quad \text{and} \quad \int_0^\infty K(1,y)x^{-1/p}dy$$

are equal to some number $C < \infty$. Then the integral operator

$$Ku(x) = \int_0^\infty K(x,y)u(y)dy$$

is bounded on $L_p((0,\infty))$ with norm not greater than C.

Before proving this result, let's see that K' satisfies the hypotheses. It is obviously positive and of the correct homogeneity. Now let $I_s(\alpha,\beta) := \int_0^s r^\alpha(1-r)^\beta dr$. Set $\alpha = -1/p' - a$ and $\beta = a+b-1$. We then have

$$\begin{aligned}
\int_0^\infty K'(x,1)x^{-1/p'}dx &= \int_0^\infty r^\alpha |r-1|^\beta dr \\
&= \int_0^1 r^\alpha (1-r)^\beta dr + \int_1^\infty r^\alpha (r-1)^\beta dr \\
&= I_1(\alpha,\beta) + I_1(\beta,\alpha) \\
&= I_{1/2}(\alpha,\beta) + I_{1/2}(\beta,\alpha) + I_{1/2}(-\alpha-\beta,\beta) + I_{1/2}(\beta,-\alpha-\beta),
\end{aligned}$$

as $\int_{1/2}^1 r^\alpha(1-r)^\beta dr = \int_0^{1/2}(1-s)^\alpha s^\beta ds$.

But on $[0,1/2]$, we have $1/2 < 1-r < 1$ hence $I_{1/2}(\alpha,\beta)$ is comparable to $\int_0^{1/2} r^\alpha dr$ which converges if and only if $\alpha > -1$. Thus $\int_0^\infty K'(x,1)x^{-1/p'}dx$ converges iff $\alpha,\beta > -1$ and $\alpha+\beta < 1$. Certainly, $b < 1/p'$ and $a < 1/p$ imply all these requirements.

To prove the general result for K, we use the homogeneity of K to turn the problem into a convolution problem. Let $\mathcal{M}$ be the multiplicative group $\mathbb{R}_{>0}$. Let $L^P_{\mathcal{M}}$ denote the L^p-space for the Haar measure dx/x on $\mathcal{M}$. Note that multiplication by $x^{1/p}$ is an isometry $L^p \to L^p_{\mathcal{M}}$. Furthermore, since the $L^1_{\mathcal{M}}$-norm of $K(x,1)x^{1/p}$ is finite, convolution with that function is continuous $L^p_{\mathcal{M}} \to L^p_{\mathcal{M}}$. Thus,

$$\begin{aligned}
\|Ku\|^p_{L^p} &= \int \Big(\int K(x,y)u(y)dy \Big)^p dx \\
&= \int \Big(\int K(x/y,1)u(y)\frac{dy}{y} \Big)^p dx \\
&= \int \Big(x^{-1/p} \int K(x/y,1)(x/y)^{1/p}\; y^{1/p}u(y)\frac{dy}{y} \Big)^p dx \\
&= \|x^{-1/p}(K(\cdot,1)(\cdot)^{1/p}) * (y^{1/p}u))\|^p_{L^p} \\
&= \|K(\cdot,1)(\cdot)^{1/p}) * (y^{1/p}u)\|^p_{L^p_{\mathcal{M}}} \\
&\leq C^p \|y^{1/p}u\|^p_{L^p_{\mathcal{M}}} \\
&= C^p \|u\|^p_{L^p},
\end{aligned}$$

and the claim is proved. □

We define

$$k^-(\delta) = \max\{k \text{ exceptional}, k < \delta\}$$

and then move on to the theorem.

Theorem E-8 ([Bar86, Thm 1.7]). *Suppose* δ *is nonexceptional,* $1 < p < \infty$ *and* $s \in \mathbb{N}$. *Then*

$$\Delta\colon W'^{s+2,p}_{\delta} \to W'^{s,p}_{\delta-2}$$

is an isomorphism and there is a constant $C = C(n, p, \delta, s)$ *such that*

$$\|u\|'_{s+2,p,\delta} \leq C\|\Delta u\|'_{s,p,\delta-2}.$$

Proof: Set $k = k^-(\delta)$. Let $\mu = (x \cdot y)/|x||y|$ and P_j^λ denote the ultraspherical function arising in the Taylor expansion of $|x-y|^{2-n}$ with respect to $|y|/|x|$ when $|y| < |x|$:

$$|x-y|^{-2\lambda} = |x|^{-2\lambda} \sum_{0}^{\infty} P_j^\lambda(\mu)(|y|/|x|)^j. \tag{E.10}$$

Set $\lambda = (n-2)/2$.

We first show that the inverse of $\Delta\colon W'^{2,p}_{\delta} \to W'^{0,p}_{\delta-2}$ has kernel $K(x,y)$:

$$c_n K(x,y) = \begin{cases} |x-y|^{2-n}, & \text{if } 2-n < \delta < 0; \\ |x-y|^{2-n} - |y|^{2-n} \sum_0^k P_j^\lambda(\mu)(|x|/|y|)^j, & \text{if } k \geq 0; \\ |x-y|^{2-n} - |x|^{2-n} \sum_0^{2-n-k} P_j^\lambda(\mu)(|y|/|x|)^j, & \text{if } k < 2-n. \end{cases}$$

We will refer to these three cases and the three corresponding definitions of $K(x,y)$ as K1, K2 and K3. We now go through a series of step that lead to the proof that $K(x,y)$ defines a bounded operator from $W'^{0,p}_{\delta-2}$ to $W'^{0,p}_{\delta}$.

Note first that in the cases K2 and K3, we have $k < \delta < k+1$, and that in the case K1, we have $k = 2-n$.

We have the estimates

$$|K(x,y)| \leq c(n,k)|x-y|^{2-n} \begin{cases} (|x|/|y|)^{k+1}, & \text{if } |x| < |y|/2; \\ (|x|/|y|)^{n+k-2}, & \text{if } |x| \geq |y|/2. \end{cases} \tag{E.11}$$

We need here $n \geq 3$. Then the estimates for K1 are trivial. Indeed, $k+1 \leq 0$ thus $|x| < |y|/2$ imply $1 < 2^{3-n}(|x|/|y|)^{k+1}$ and $n+k-2 = 0$ thus $|x| \geq |y|/2$ imply $1 \leq 2^{n+k-2}(|x|/|y|)^{n+k-2}$.

Let's first prove Estimate (E.11) for K2 in the case $|x|/|y| < 1/2$. We have in that case that $|x-y| \leq 3|y|/2$, hence $|x-y|^2 - n \geq c(n)|y|^{2-n}$. Since we are exactly in the case where the expansion of Equation (E.10) converges (swapping y and x), we have

$$\begin{aligned} |K(x,y)| &= \left| |y|^{-2\lambda} \sum_{k+1}^{\infty} P_j^\lambda(\mu)(|x|/|y|)^j \right| \\ &\leq |y|^{2-n}(|x|/|y|)^{k+1} \sum_{j \geq 0} \max_{\mu \in B_1}(|P_{j+k+1}^\lambda(\mu)|)2^{-j} \\ &\leq c(n,k)|x-y|^{2-n}(|x|/|y|)^{k+1}, \end{aligned}$$

as wanted.

We prove the case $|x| > |y|/2$ for $K2$ term by term. As $n+k-2 \geq 1$, it must be that

$(|x|/|y|)^{n+k-2} \geq 2^{2-k-n}$ thus $|x-y|^{2-n} \leq c(n,k)|x-y|^{2-n}(|x|/|y|)^{n+k+2}$. For $j \leq k$, we have

$$\left||y|^{2-n} P_j^\lambda(\mu)(|x|/|y|)^j\right| \leq \Big(\max_{i\leq k}\max_{\mu\in B_1}|P_i^\lambda(\mu)|\Big)\frac{|x|^j}{|y|^{j+n-2}}, \text{ and}$$
$$\frac{|x|^j}{|y|^{j+n-2}} \leq 2\frac{|x|^{j+1}}{|y|^{j+1+n-2}} \leq 2^{k-j}\frac{|x|^k}{|y|^{n+k-2}}.$$

These inequalities, along with the fact that $|x-y| \leq 3|x|/2$ implies

$$1 \leq c(n)|x-y|^{2-n}|x|^{n-2},$$

can be used to prove the second estimate for K2.

Now let K_1 be the operator kernel $|x-y|^{2-n}(|x|/|y|)^\alpha$. Set $a = \delta + n/p - \alpha$ and $b = -b - n/p + 2 + \alpha$. Then $a + b = 2$ and Lemma E-7 shows that

$$K_1'(x,y) = |x|^{-\delta-n/p} K_1(x,y)|y|^{\delta-2+n/p}$$

defines a bounded operator $L^p \to L^p$ when $a < n/p$ and $b < n/p'$, that is when $\delta < \alpha$ and $\delta > 2 - n + \alpha$.

Since the composition

$$L'^p_{\delta-2} \xrightarrow{|y|^{-\delta+2-n/p}} L^p \xrightarrow{K_1'} L^p \xrightarrow{|x|^{\delta+n/p}} L'^p_\delta$$

is precisely K_1, then K_1 is continuous when $\alpha + 2 - n < \delta < \alpha$.

We use simultaneously $\alpha = k+1$ and $\alpha = n+k-2$ along with Estimate (E.11) to see that K is bounded when $k < \delta < k+1$, which correspond to the cases K2 and K3. The case K1 is dealt with in a slightly different fashion, without the use of Estimate (E.11): we just use $\alpha = 0$.

Now that we know that K is bounded $L'^p_{\delta-2} \to L'^p_\delta$, we use K to show the surjectivity of Δ. First recall that

$$\Delta_x|x-y|^{2-n} = \Delta_y|x-y|^{2-n} = \delta(x-y).$$

Furthermore, the right-hand-side terms in K2 and K3 are harmonic in $\mathbb{R}^n \setminus \{0\}$. Thus

$$\Delta_x K = \Delta_y K = \delta(x-y) \text{ in } D'(\mathbb{R}^n \setminus \{0\}).$$

Hence $K(\Delta u) = u$ for all $u \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$. Since $C_c^\infty(\mathbb{R}^n \setminus \{0\})$ is dense in $W'^{k,p}_\delta$, the continuity of K implies

$$\|u\|'_{p,\delta} \leq C\|\Delta u\|'_{p,\delta-2}.$$

Using Estimate (E.9), we find

$$\begin{aligned}\|u\|'_{2,p,\delta} &\leq C(\|\Delta u\|'_{p,\delta-2} + \|u\|'_{p,\delta}) \\ &\leq C\|\Delta u\|'_{\delta-2},\end{aligned}$$

as wanted for the case $s = 0$. This Δ is injective and has closed range. Since it is surjective on $C_c^\infty(\mathbb{R}^n \setminus \{0\})$, it must be surjective by density. □

With this result in our pocket, we are close to seeing the Fredholmness of P. The more powerful elliptic estimate presented in the next theorem is the tool we need.

Theorem E-9 ([Bar86, Thm 1.10]). *Suppose P is asymptotic to Δ and $\delta \in \mathbb{R}$ is nonexceptional. For $1 < p \leq q$, the map $P\colon W^{2,p}_\delta \to W^{0,p}_{\delta-2}$ has finite-dimensional kernel and closed range and, for $u \in W^{2,p}_\delta$, we have constants C and R depending only on P, δ, n, p, q such that*

$$\|u\|_{2,p,\delta} \leq C(\|Pu\|_{0,p,\delta-2} + \|u\|_{L^p(B_R)}).$$

Proof: Let $\| \ \|_{op}$ denote the operator norm for bounded linear functions $W^{2,p}_\delta \to W^{0,p}_{\delta-2}$ and $\| \ \|_{op,R}$ denote the same norm but restricted to functions with support in $E_R = \mathbb{R}^n \setminus B_R$.
Suppose $supp(u) \subset E_R$, then

$$\|(P-\Delta)u\|_{0,p,\delta-2} \leq \Big(\sup_{|x|>R}\{|a^{ij}(x) - \delta_{ij}|\} + C\|b\|_{0,q,-1;E_R} + C\|c\|_{0,q,-2;E_R}\Big)\|u\|_{2,p,\delta}.$$

Since P is asymptotic to Δ, we thus have

$$\|P-\Delta\|_{op,R} = o(1) \text{ as } R \to \infty.$$

Let $\chi \in C^\infty_c(B_2)$ be such that $o \leq \chi \leq 1$ with $\chi = 1$ in B_1. Set $\chi_R(x) = \chi(x/R)$. Given u, write $u_0 = \chi_R u$ and $u_\infty = (1-\chi_R)u$. Thus $u = u_0 + u_\infty$.
We have

$$\begin{aligned}\|u_\infty\|_{2,p,\delta} &\leq C\|\Delta u_\infty\|_{0,p,\delta-2} \\ &\leq C\Big(\|Pu_\infty\|_{0,p,\delta-2} + \|P-\Delta\|_{op,R}\|u_\infty\|_{2,p,\delta}\Big)\end{aligned}$$

and we estimate

$$\begin{aligned}\|Pu_\infty\|_{0,p,\delta-2} &\leq \|Pu\|_{0,p,\delta-2} + \|Pu_0\|_{0,p,\delta-2} \\ &\leq \|Pu\|_{0,p,\delta-2} + \|\chi_R Pu\|_{0,p,\delta-2} + \|[P,\chi_R]u\|_{0,p,\delta-2} \\ &\leq C\|Pu\|_{0,p,\delta-2} + C\|u\|_{1,p,\delta-1;A_R}.\end{aligned}$$

By throwing in a factor of R in C, this last norm can be considered with weight δ. Since $\|P - \Delta\|_{op,R} = o(1)$, for R sufficiently large we have

$$\|u_\infty\|_{2,p,\delta} \leq C(\|Pu\|_{0,p,\delta-2} + \|u\|_{1,p,\delta;A_R}).$$

We have the exact same estimate for u_0.
But then

$$\begin{aligned}\|u\|_{2,p,\delta} &\leq \|u_0\|_{2,p,\delta} + \|u_\infty\|_{2,p,\delta} \\ &\leq C(\|Pu\|_{0,p,\delta-2} + \|u\|_{1,p,\delta;A_R}).\end{aligned}$$

Using the Interpolation Inequality (E.3), we get the wanted estimate.
Now suppose $\{u_k\} \in \ker P$ satisfy $\|u_k\|_{2,p,\delta} = 1$. By Rellich we may assume that $\{u_k\}$ converges

in $L^p(B_R)$. Thus

$$\|u_j - u_k\|_{2,p,\delta} \to 0 \text{ as } \min(j,k) \to \infty$$

and $\{u_k\}$ is Cauchy hence convergent in $W^{2,p}_\delta$. Hence $\ker P$ is finite dimensional.
Since $\dim\ker P < \infty$, there is a closed subspace Z such that $W^{2,p}_\delta = Z + \ker P$ and

$$\|u\|_{2,p,\delta} \le C\|Pu\|_{0,p,\delta-2} \text{ for all } u \in Z.$$

Indeed, should there be no such bound, we could find a sequence $\{u_i\} \in Z$ with $\|u_i\|_{2,p,\delta} = 1$ but $Pu_i \to 0$. But then using the estimate proved earlier and the Rellich lemma on B_R, there would be a subsequence of the u_i which is Cauchy. By closedness, the limit $u = \lim u_i$ is in Z. But then $Pu = 0$ and $\|u\|_{2,p,\delta} = 1$: contradiction!
The fact that P has closed range follows directly. □

We are interested in the dimension of the kernel of P.

Theorem E-10. *The number* $\dim\ker(P\colon W^{2,p}_\delta \to W^{0,p}_{\delta-2})$ *is independent of* p *for* $1 < p \le q$.

Proof: We split the range $1 < p \le q$ into three zones:

zone 1: $n < p \le q$,
zone 2: $n/2 < p \le n$, and
zone 3: $1 < p \le n/2$.

Suppose that $Pu = 0$ and $u \in W^{2,p}_\delta$.
Suppose first that p is in zone 1. As $n/p < 1$, we have $0 < 1 - n/p \le 1$. Take any α with $0 < \alpha \le 1 - n/p$. Then $u \in W^{2,p}_\delta$ implies $\|u\|_{C^{0,\alpha}_\delta} \le C\|u\|_{1,p,\delta}$ hence u is continuous.

Also, since $n - p < 0$, we have $|u(x)| = o(r^\delta)$ as $r \to \infty$. In conjunction with the continuity of u, this asymptotic behavior indicates that $u \in L^s_\delta$ for every s. Hence $u \in W^{2,s}_\delta$ for every s by Proposition E-6.

Suppose now that p is in zone 2. Then $n - 2p < 0$ and $2 - n/p \le 1$. Thus, again,

$$\|u\|_{C^{0,\alpha}_\delta} \le C\|u\|_{2,p,\delta}$$

and u is continuous, and

$$|u(x)| = o(r^\delta) \text{ as } r \to \infty.$$

Again, we have $u \in W^{2,s}_\delta$ for every s.
Suppose now that p is in zone 3. Then $n - p \ge n/2$. Thus $p < 2p \le np/(n-p)$ and

$$\|u\|_{np/(n-p),\delta} \le C\|u\|_{1,p,\delta}.$$

Since $Pu = 0$, we have by Proposition E-6 that $u \in W^{2,np/(n-p)}_\delta$. Iterating this reasoning a finite number of time, we push p out of zone 3 and once in zone 1 or 2, we know that $u \in W^{2,s}_\delta$ for every s. □

Note that the role of q is absolutely artificial here. The only reason we need it is to be able to use Proposition E-6.

Because of this last theorem, it is natural to define

$$N(P,\delta) := \dim\ker(P\colon W^{2,p}_{\delta} \to W^{0,p}_{\delta-2}).$$

While there is more in Bartnik's paper that could be done, we end by studying a Theorem quite similar to Theorem 7.2-1 of this thesis.

Theorem E-11 ([Bar86, Thm 1.17]). *Suppose $P \sim \Delta$ at rate $\tau > 0$. Suppose δ is nonexceptional and that $u \in W^{2,q}_{\delta}$ satisfies $Pu = 0$ in E_R. Then there is an exceptional value $k \le k^-(\delta)$ and $h_k \in C^\infty(\mathbb{R}^n)$ such that h_k is harmonic and homogeneous of degree k in E_R and*

$$u - h_k = o(r^{k-\tau}) \text{ as } r \to \infty.$$

Proof: Set $F := \Delta u$. Since $Pu = 0$ in E_R, we have that $F := (\delta_{ij} - a^{ij})\partial^2_{ij}u - b^i\partial_i u - cu$ in $|x| > R$; thus $F \in W^{0,q}_{\delta-2\tau}$.
We can take $\epsilon < \tau/2$ small enough so that $\delta - \tau + \epsilon$ and $\delta - \tau + 2\epsilon$ are nonexceptional. Then $F \in W^{0,q}_{\delta-2-\tau}$ implies that $F \in W^{0,q}_{\delta-2-\tau+\epsilon}$. But $\Delta\colon W^{2,q}_{\delta-\tau+\epsilon} \to W^{0,q}_{\delta-2-\tau+\epsilon}$ is Fredholm. So let $\beta_1, \dots, \beta_n$ be a basis of $\ker(\Delta^*) \subset (W^{0,q}_{\delta-2-\tau+\epsilon})^*$. An element f is in $Im(\Delta)$ if and only if $\beta_1(f) = \cdots = \beta_n(f) = 0$.
Notice that $(W^{0,q}_{\delta-2-\tau+\epsilon})^* = (L^q_{\delta-2-\tau+\epsilon})^* = L^{q'}_{-\delta+2+\tau-\epsilon-n}$ by integration against each other. So the β_i are functions.
We want to modify F in B_R so that it becomes an element of $Im(\Delta)$. We are thus looking for f with $f = 0$ in E_R such that

$$\beta_i(f) = \beta_i(F), \text{ for } i = 1, \dots, n.$$

Restrict β_i to B_R. Since $L^{q'}_{-\delta+2+\tau-\epsilon-n}(B_R) = L^q_{\delta-2-\tau+\epsilon}(B_R)^*$, there are $f_i \in L^q_{\delta-2-\tau+\epsilon}(B_R)$ with

$$\beta_i(f_j) = \delta_{ij}.$$

Extend f_i to $\mathbb{R}^n$ by 0 on E_R. Then $f_i \in L^q_{\delta-2-\tau+\epsilon}$. The function

$$F - \beta_1(F)f_1 - \cdots - \beta_n(F)f_n$$

is killed by all the β_i thus it is in the image of Δ.
Thus, there exists a v in $W^{2,q}_{\delta-\tau+\epsilon}$ such that

$$\Delta(u - v) = 0, \text{ for } |x| > R.$$

The classical expansion for harmonic functions now shows that

$$u - v = h_k + O(r^{k-1})$$

for some $k \le k^-(\delta)$ and h_k harmonic and of degree k in E_R. The decay estimate for v is improved by iteration: $u - h_k \in W^{2,q}_{\delta-\tau+\epsilon}$ implies $F \in W^{0,q}_{\delta-2-2\tau+\epsilon}$ □

Bibliography

[ABR01] Sheldon Axler, Paul Bourdon, and Wade Ramey. *Harmonic function theory*, volume 137 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001.

[ADHM78] M. F. Atiyah, V. G. Drinfel′d, N. J. Hitchin, and Yu. I. Manin. Construction of instantons. *Phys. Lett. A*, 65(3):185–187, 1978.

[APS75] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.

[Bar86] Robert Bartnik. The mass of an asymptotically flat manifold. *Comm. Pure Appl. Math.*, 39(5):661–693, 1986.

[Bär96] C. Bär. Metrics with harmonic spinors. *Geom. Funct. Anal.*, 6(6):899–942, 1996.

[Bär00] Christian Bär. Dependence of the Dirac spectrum on the Spin structure. In *Global analysis and harmonic analysis (Marseille-Luminy, 1999)*, volume 4 of *Sémin. Congr.*, pages 17–33. Soc. Math. France, Paris, 2000.

[BJ01] Olivier Biquard and Marcos Jardim. Asymptotic behaviour and the moduli space of doubly-periodic instantons. *J. Eur. Math. Soc. (JEMS)*, 3(4):335–375, 2001.

[Bou86] Jean-Pierre Bourguignon. L'opérateur de Dirac et la géométrie riemannienne. *Rend. Sem. Mat. Univ. Politec. Torino*, 44(3):317–359 (1987), 1986.

[BPST75] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin. Pseudoparticle solutions of the Yang-Mills equations. *Phys. Lett. B*, 59(1):85–87, 1975.

[BvB89] Peter J. Braam and Pierre van Baal. Nahm's transformation for instantons. *Comm. Math. Phys.*, 122(2):267–280, 1989.

[Can79] M. Cantor. Some problems of global analysis on asymptotically simple manifolds. *Compositio Math.*, 38(1):3–35, 1979.

[Can75] M. Cantor. Spaces of functions with asymptotic conditions on R^n. *Indiana Univ. Math. J.*, 24:897–902, 1974/75.

[CBC81] Y. Choquet-Bruhat and D. Christodoulou. Elliptic systems in $H_{s,\delta}$ spaces on manifolds which are Euclidean at infinity. *Acta Math.*, 146(1-2):129–150, 1981.

[CG84] E. Corrigan and P. Goddard. Construction of instanton and monopole solutions and reciprocity. *Ann. Physics*, 154(1):253–279, 1984.

[CK98] Sergey A. Cherkis and Anton Kapustin. Singular monopoles and supersymmetric gauge theories in three dimensions. *Nuclear Phys. B*, 525(1-2):215–234, 1998.

[CK99] S.A. Cherkis and A. Kapustin. Singular Monopoles and Gravitational Instantons. *Commun. Math. Phys.*, 203:713–728, 1999.

[CK01] Sergey Cherkis and Anton Kapustin. Nahm transform for periodic monopoles and $\mathcal{N} = 2$ super Yang-Mills theory. *Comm. Math. Phys.*, 218(2):333–371, 2001.

[DK90] Simon K. Donaldson and Peter B. Kronheimer. *The geometry of four-manifolds*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1990.

[Don83] S. K. Donaldson. An application of gauge theory to four-dimensional topology. *J. Differential Geom.*, 18(2):279–315, 1983.

[Feg87] Howard Fegan. The spectrum of the Dirac operator on a simply connected compact Lie group. *Simon Stevin*, 61(2):97–108, 1987.

[Fri84] Th. Friedrich. Zur Abhängigkeit des Dirac-operators von der Spin-Struktur. *Colloq. Math.*, 48(1):57–62, 1984.

[FU84] Daniel S. Freed and Karen K. Uhlenbeck. *Instantons and four-manifolds*, volume 1 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1984.

[GHL90] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, 1990.

[GT83] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1983. QA 377.G49.

[Heb99] Emmanuel Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, volume 5 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 1999.

[Hit74] Nigel Hitchin. Harmonic spinors. *Advances in Math.*, 14:1–55, 1974.

[Hit82] N. J. Hitchin. Monopoles and geodesics. *Comm. Math. Phys.*, 83(4):579–602, 1982.

[Hit83] N. J. Hitchin. On the construction of monopoles. *Comm. Math. Phys.*, 89(2):145–190, 1983.

[HM89] Jacques Hurtubise and Michael K. Murray. On the construction of monopoles for the classical groups. *Comm. Math. Phys.*, 122(1):35–89, 1989.

[Jar] Marcos Jardim. A survey on Nahm transform. *Journal of Geometry and Physics, In Press*.

[Jar99] Marcos B. Jardim. *Nahm transform of doubly-periodic instantons*. PhD thesis, Oxford, 1999.

[Jar01] Marcos Jardim. Construction of doubly-periodic instantons. *Comm. Math. Phys.*, 216(1):1–15, 2001.

[Jar02a] Marcos Jardim. Classification and existence of doubly-periodic instantons. *Q. J. Math.*, 53(4):431–442, 2002.

[Jar02b] Marcos Jardim. Nahm transform and spectral curves for doubly-periodic instantons. *Comm. Math. Phys.*, 225(3):639–668, 2002.

[Kat95] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[Kli] F. Klinker. The spinor bundle of Riemannian products. arXiv: math.DG/0212058.

[KN90] Peter B. Kronheimer and Hiraku Nakajima. Yang-Mills instantons on ALE gravitational instantons. *Math. Ann.*, 288(2):263–307, 1990.

[Kra01] Margarita Kraus. Eigenvalues of the Dirac operator on fibrations over S^1. *Ann. Global Anal. Geom.*, 19(3):235–257, 2001.

[LM83] Robert B. Lockhart and Robert C. McOwen. On elliptic systems in $\mathbf{R}^n$. *Acta Math.*, 150(1-2):125–135, 1983.

[LM84] Robert B. Lockhart and Robert C. McOwen. Correction to: "On elliptic systems in $\mathbf{R}^n$" [Acta Math. **150** (1983), no. 1-2, 125–135]. *Acta Math.*, 153(3-4):303–304, 1984.

[LM85] Robert B. Lockhart and Robert C. McOwen. Elliptic differential operators on noncompact manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):409–447, 1985.

[LM89] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[Loc81] Robert B. Lockhart. Fredholm properties of a class of elliptic operators on noncompact manifolds. *Duke Math. J.*, 48(1):289–312, 1981.

[McO79] Robert C. McOwen. The behavior of the Laplacian on weighted Sobolev spaces. *Comm. Pure Appl. Math.*, 32(6):783–795, 1979.

[Mel93] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters Ltd., Wellesley, MA, 1993.

[MM98] Rafe Mazzeo and Richard B. Melrose. Pseudodifferential operators on manifolds with fibred boundaries. *Asian J. Math.*, 2(4):833–866, 1998.

[MMR94] John W. Morgan, Tomasz Mrowka, and Daniel Ruberman. *The L^2-moduli space and a vanishing theorem for Donaldson polynomial invariants*. Monographs in Geometry and Topology, II. International Press, Cambridge, MA, 1994.

[Nah83] Wener Nahm. All self-dual multimonopoles for arbitrary gauge groups. In *Structural elements in particle physics and statistical mechanics (Freiburg, 1981)*, volume 82 of *NATO Adv. Study Inst. Ser. B: Physics*, pages 301–310. Plenum, New York, 1983.

[Nah84] Werner Nahm. Self-dual monopoles and calorons. In *Group theoretical methods in physics (Trieste, 1983)*, volume 201 of *Lecture Notes in Phys.*, pages 189–200. Springer, Berlin, 1984.

[Nak93] Hiraku Nakajima. Monopoles and Nahm's equations. In *Einstein metrics and Yang-Mills connections (Sanda, 1990)*, volume 145 of *Lecture Notes in Pure and Appl. Math.*, pages 193–211. Dekker, New York, 1993.

[NS00] Tom M. W. Nye and Michael A. Singer. An L^2-index theorem for Dirac operators on $S^1 \times \mathbf{R}^3$. *J. Funct. Anal.*, 177(1):203–218, 2000.

[NW73] Louis Nirenberg and Homer F. Walker. The null spaces of elliptic partial differential operators in $\mathbf{R}^n$. *J. Math. Anal. Appl.*, 42:271–301, 1973.

[Nye01] Tom M. W. Nye. *The geometry of calorons*. PhD thesis, University of Edinburgh, 2001.

[Pau98] Marc Pauly. Monopole moduli spaces for compact 3-manifolds. *Math. Ann.*, 311(1):125–146, 1998.

[Pfä00] Frank Pfäffle. The Dirac spectrum of Bieberbach manifolds. *J. Geom. Phys.*, 35(4):367–385, 2000.

[Roe98] John Roe. *Elliptic operators, topology and asymptotic methods*, volume 395 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, 1998.

[Sch88] H. Schenk. On a generalised Fourier transform of instantons over flat tori. *Comm. Math. Phys.*, 116(2):177–183, 1988.

[Tra93] Andrzej Trautman. Spin structures on hypersurfaces and the spectrum of the Dirac operator on spheres. In *Spinors, twistors, Clifford algebras and quantum deformations (Sobótka Castle, 1992)*, volume 52 of *Fund. Theories Phys.*, pages 25–29. Kluwer Acad. Publ., Dordrecht, 1993.

[Uhl82] Karen K. Uhlenbeck. Removable singularities in Yang-Mills fields. *Comm. Math. Phys.*, 83(1):11–29, 1982.

[vB96] Pierre van Baal. Instanton moduli for $T^3 \times \mathbb{R}$. *Nuclear Phys. B Proc. Suppl.*, 49:238–249, 1996. arXiv:hep-th/9512223.

[Wal71] Homer F. Walker. On the null-spaces of first-order elliptic partial differential operators in $R^{\mathbf{n}}$. *Proc. Amer. Math. Soc.*, 30:278–286, 1971.

[Wal72] Homer F. Walker. A Fredholm theory for a class of first-order elliptic partial differential operators in $\mathbf{R}^n$. *Trans. Amer. Math. Soc.*, 165:75–86, 1972.

[Wal73] Homer F. Walker. A Fredholm theory for elliptic partial differential operators in R^n. In *Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971)*, pages 225–230. Amer. Math. Soc., Providence, R.I., 1973.

Index