APPLICATION OF LINEAR ROUTING SYSTEMS TO

REGIONAL GROUNDWATER PROBLEMS

by

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ABSTRACT

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Work in groundwater analysis goes back to the last century. Only in the last decade, however, has there been an increase of interest in applying a linear systems approach to the problem of routing groundwater flow. This thesis applies linear systems to the routing of groundwater within a regional basin.

The research reported here has been devoted to the following:

1. Developing a fast convolution technique through the use of the Fast Fourier Transforms.

2. Developing a method for determining the system response parameters through linearizing the governing equation for groundwater flow by applying Laplace transforms and using the Method of Moments.

3. Developing a groundwater routing model using the above techniques applied to a regional groundwater basin.

The results from the Harmonic Analysis have been compared with those generated by the complete solution for open channel flow. The hydrograph generated with the use of the parameters determined from the parameter estimation technique are compared to those resulting from a finite difference scheme.

The techniques developed in the use of Harmonic Analysis and parameter estimation are incorporated into a model for analyzing a regional groundwater problem and the results discussed.

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Additional appreciation goes to Mr. Darío Valencia and Mr. Guillermo Vicens, Research Assistants for their helpful ideas generated through enlightening discussions. Also to Miss Elba Rosso for enduring the dreadful script in the typing of this thesis, and to Mr. David Njus, graduate student, Harvard, who as penman and scholar, with careful thought, drafted the figures herein.

Finally but not least, thanks goes to my wife, Rosemary, and our daughter, for the loneliness endured during the period of this thesis.

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Chapter I

INTRODUCTION

I-1 Problem Statement

The theory to groundwater flow representation goes back to 1856 when Henry Darcy first developed an empirical relationship for steady-state saturated flow. Jules Dupuit and many others have since expanded Darcy's relationship into one representing unsteady conditions. Through expanding knowledge in subsurface hydro-geology and soil mechanics, the complexities of the subterranean region have become enormous. The 'real world' conditions which are non-homogeneous, non-isotopic and contain cracks, fissures, etc., make it impossible to represent, in detail, the behaviour of a subsurface environment.

The complexities of the subsurface terrain also implies that the response of such a system is non-linear. However, if the necessity of a non-linear solution is accepted, a unique solution for each soil condition, recharge pattern and the many other facits of the system is required. Thus, with the linear systems approach of modeling the groundwater system, we desire to find a simple but functional procedure for determining the general behaviour pattern of such a system. This theme will be further discussed in Section I-3.

I-2 Background to Groundwater Flow Modeling

Over the past decade, a tremendous effort has been devoted to the understanding of groundwater flow. The techniques vary widely but may be categorized into theoretical, analytical, experimental and numerical. Initial attempts were theoretical, going back to the early 1900s, when the dispersive effects of the groundwater systems were noticed. Since that time researchers have delved more deeply into the relationship of the various subsurface parameters to the dispersive effects caused by the soil characteristics. Breitenbach [1971], in a paper presented on groundwater simulation, pointed out the various analytical techniques used to-day. These analysts use Fourier Series, Laplace Transforms, conformal mapping or graphical approximations. The data are obtained, generally, by methods of well withdrawals or parallel drains of a variety of configurations.

Simulation techniques used, range from physical models using sand or other porous media, viscous fluids, electric means (relating Ohm's Law to Darcy's Relationship) and membranes, to numerical methods. It is interesting to note that many modern methods or theories have developed from other fields of study, for instance, the well known heat flow (Carslaw and Jaeger (1959)) relation to dispersion, as well as Ohm's Law in electrical theory to mass flux (Darcy's Law). The background and theory involved in these areas are discussed by Reddell and Sunada [1971].

A Frenchman by the name of D'Andrimont introduced concepts which lead to simulation of small groundwater basins by Toth [1962] and Freeze and Witherspoon [1966]. With the advent of the digital computer came a rapid increase of basin studies, for example, those done by Bittinger, et al. [1967] and Tyson and Weber [1964] as well as many others.

As the vastness of groundwater storage reservoirs unveils, researchers are beginning to widen the scope of subterranean flows to a regional basin. Nelson and Cearlock [1967] discuss the various methods applied to large heterogeneous systems. Schneider [1966] and Megnien [1964] also have done work in analysing regional flow patterns of groundwater.

I-3 Introduction to Linear Systems

As many researchers turned to numerical methods in an attempt to by-pass the complexities of the analytical solutions for computing the reactivity of a groundwater system, so have many turned to the linear systems approach. Originally, linear systems were developed for overland flow and were accepted in groundwater because, to quote Kraijenhoff Van De Leur [1966], "... the unit hydrograph methods are in complete accord with the nature of the simplifying assumptions that have been accepted in order to find analytical solutions for the equations describing the flow of groundwater."

The linear systems approach to routing groundwater in the subterranean region is a subset to work done by Sherman [1932], who advanced the unit hydrograph theory which later was used in routing of surface flows. It was an attempt by hydrologists to estimate the overall effect of an 'ideal' system and compare the result to an actual system. The hope was that a close approximation to that system would be obtained. The basic assumption underlying linear system theory is that the series of simple inputs may be used in conjunction with a characterizing function of the system to simulate the effects of a complex inflow pattern. Obviously, then, the characterizing function must implicitly contain all the variable process characteristics necessary for such a representation - an ideological condition to be sure. Should such a simplifying technique be used at all? A good justification for using linear systems is provided by Rodriguez [1972] when he says that a linear system "... may provide less information where information is not wanted and better information where it is wanted, all at less cost in time and effort."

The work that has evolved from linear systems in groundwater flow can be found in Chapter II.

I-4 Scope of Work

The work carried out in this thesis will be:

a) to develop the use of Harmonic Analysis within the linear systems approach for a fast computational scheme of convolution,

b) To use this method to develop a general model that can be used under regional consideration,

c) to apply the model to a regional area.

I-5 Brief Summary of Results

A convolution technique is discussed in Chapter III which utilizes a Fast Fourier Transform program developed at M.I.T.. This procedure was found to be highly efficient in terms of time and accuracy. In Chapter IV, a groundwater routing model is presented which is capable of utilizing any configuration of system response which might be encountered in a groundwater zone. This model utilizes the convolution technique in an effective procedure for analyzing such a groundwater system. In application of this model it was found to be better practice to isolate the different flow processes discussed in Chapter IV since the substantial damping effect of the groundwater aquifer produced time steps incompatible for aggregating those processes into one outflow hydrograph. Use of the fast Fourier transform technique for predicting the response to an input provides a highly efficient procedure for analysing both the transient and the periodic situations. This is found especially useful in studying the behavior of slowly responding aquifer systems to periodic inputs.

Chapter II

DEVELOPMENTS TO LINEAR SYSTEMS ANALYSIS

II-1 Hydrograph Theory

In 1929 Folse presented the ideas of base-flow separation, reduction of rainfall due to the variance of infiltration rates and the derivation of physical constants for representing hydrologic systems. Sherman, in 1932, used these ideas to develop the well known hydrograph theory. The basic assumptions for use with the unit hydrograph which is the result of surface runoff or effective rainfall, are:

a) Effective rainfall is uniformly distributed within its duration.

b) The effective rainfall is distributed uniformly over the entire drainage basin.

c) The time duration is constant for a direct runoff hydrograph due to an effective rainfall of unit duration.

d) Those direct runoff hydrographs that have the same time duration have ordinates which are directly proportional to the total amount of direct runoff represented by each hydrograph. Note that this implies use of the principles of linearity, superposition and proportionability.

e) The runoff hydrograph from a given rainfall period reflects all the physical characteristics of the given drainage basin.

The two significant features in linear systems application that are invoked by the above assumptions are those of time invariance and of superposition. Time invariance, i.e., stationarity with time, implies that the basin response will not vary with time - in other words, the resulting hydrographs of an effective runoff of the same duration will be the same. Superposition refers to the property that a hydrograph resulting from a given pattern of rainfall excess can equivalently be generated by superimposing the hydrographs from separate amounts of rainfall excess that occur during each period of the same duration. Thus, in order to use the principle of superposition only those systems that consist of linear elements may be considered. The most effective way of characterizing the behaviour of such systems is to allow the effective input to become a delta input (or unit impulse). The resulting output is known as the instantaneous unit hydrograph, designated by h(0,t) or I.U.H. The properties are:

$$h(0,t) = 0 \qquad t < 0$$

$$h(0,t) \rightarrow 0 \qquad t \rightarrow \infty \qquad \text{II-1}$$

$$\int_{0}^{\infty} h(0,t)dt = 1.0 = \text{volume of runoff.}$$

II-2 Linear Systems

A system, as defined by Eagleson, in 1967, is any set of inter-related components, material or conceptual, that are identified by their state variables. When the components are isolated from the 'real' system and provide the state variables, the result is an 'idealized' system since it excludes some of the parameters or characteristics found in the environment. If this were not done, the task would either be impossible or so complex that it would be economically infeasible. A schematic of a hydrologic system might be as shown in Figure II-1.

The Instantaneous Unit Hydrograph, I.U.H., is the basis for the linear systems theory since it represents the response of a system to a unit impulse (delta function), and completely characterizes the system. The output resulting from the application of a known input can be uniquely determined by convolution of the input with the I.U.H.

The characteristic function of a linear system, e.g. the I.U.H., can be of two types, one being time invariant and the other being time variant. If the system is time invariant, then the system may be represented by a differential equation with constant coefficients as

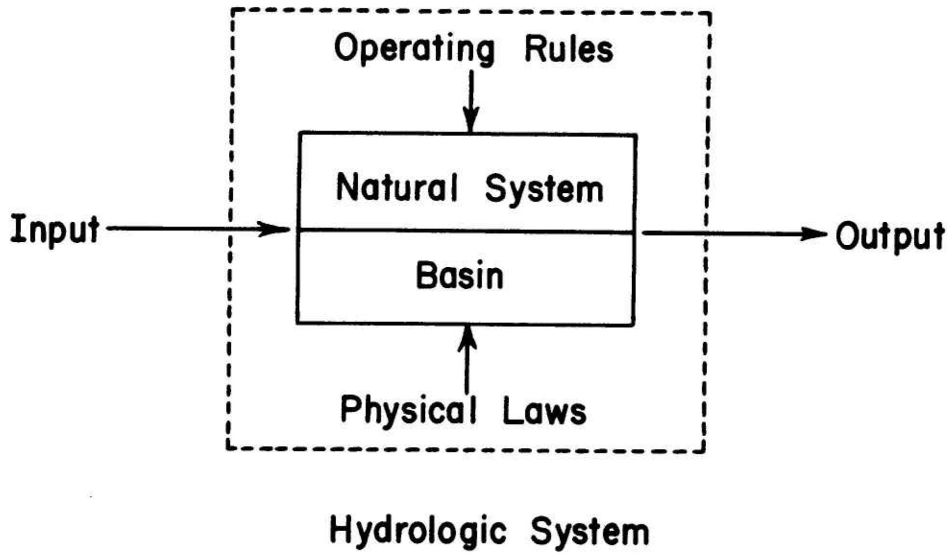


Figure II-1

Schematic of a Hydrologic System

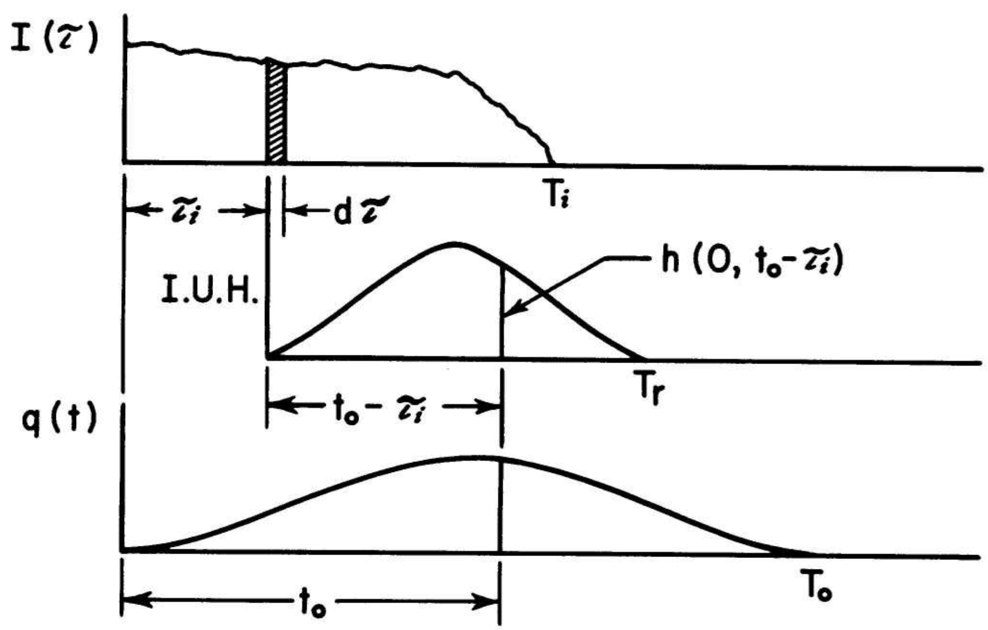


Figure II-2

Linear System Approach in Deriving an outflow hydrograph in Equation II-2.

$$I(t) = A_n \frac{d^n q(t)}{dt^n} + A_{n-1} \frac{d^{n-1} q(t)}{dt^{n-1}} + - - - - + A_o q(t) \qquad \text{II-2}$$

where $I(t)$ = Time varying input

$q(t)$ = Time varying output

This equation implies that the response to a sum of inputs is the same as if the inputs were individually computed and the responses summed. The difference between a time invariant system and a time variant system is that the coefficients for a time variant system are time dependent. The behaviour of a typical L.T.I. (Linear Time Invariant) System is shown in Figure II-2. This figure also represents the use of the convolution integral (or Duhamel's Integral) for a causal system, viz

$$q(t) = \int_o^t I(\tau)\, h(t-\tau)\, d\tau \qquad \text{II-3}$$

where $I(\tau)$ = Inflow Rate

$h(t)$ = System's impulse response function

II-3 Some Typical Linear Systems

Previous sections have discussed the use of a characteristic function which when convoluted with simple inputs will produce an output hydrograph representative of the system. This characteristic function may consist of one, two or three parameter models that are used to represent the system responses to an input function. The following sections describe the basic models that are presently used in representing a linear system response.

II-3.1 Linear Reservoir Model

In a groundwater system, one would normally expect heterogeneous soil conditions, as well as extremely small (in relation to those found in surface hydrology) transmissivities or diffusivities. Therefore, one might assume that the translational effects of subsurface flow might be neglected and treat the system as a storage reservoir. The reservoir is what is known as a one parameter model where the one parameter, K, is used to represent the total hydraulic characteristics of open channels for surface flow routing or the soil characteristics for groundwater flow. The conceptual storage reservoir is shown in Figure II-3. The linear storage is related to the outflow by:

$$S = K\ q(t)^x \qquad \text{II-4}$$

where $x = 1$ for linear systems

< 1 for sublinear systems

> 1 for supralinear systems

The continuity equation for the storage reservoir is given by Equation II-5, where $x = 1$.

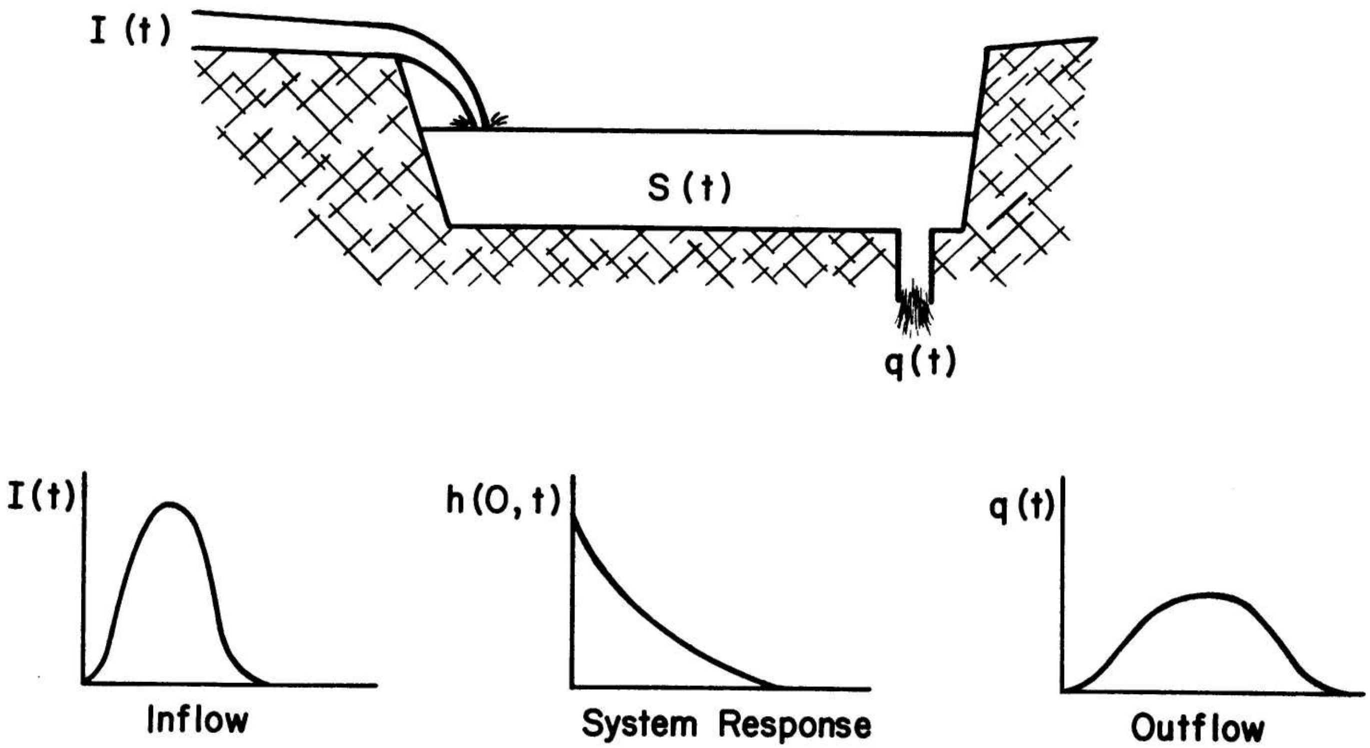


Figure II-3

Conceptual Storage Reservoir

$$I(t) = q(t) + \frac{dS}{dt}$$

or $$I(t) = q(t) + K\,\frac{d\,q(t)}{dt} \qquad \text{II-5}$$

where $S(t)$ = represents the reservoir storage

K = time constant or lag between the input centroid and the output centroid

$q(t)$ = rate of discharge

Equation II-5 can be rewritten as:

$$\frac{d\,q(t)}{dt} + \frac{q(t)}{K} = \frac{I(t)}{K} \qquad \text{II-6}$$

This is a first order linear equation and the total solution may be determined from the homogeneous and particular solutions.

The complete solution to equation II-6 is given by

$$q = e^{-t/K} \left[\int I\, e^{t/K}\, dt + C \right] \qquad \text{II-7}$$

assuming a constant input, the complete solution becomes:

$$q = C_1\, e^{-t/K} + I \qquad \text{II-8}$$

Introducing the boundary conditions which are:

$$q = 0,\ t = 0 \qquad \text{II-9}$$

results in

$$C_1\, e^{o} + I = 0 \qquad \text{II-10}$$

thus $$C_1 = -I \qquad \text{II-11}$$

The complete solution to a constant inflow to a storage reservoir then becomes

$$Q = I\,(1-e^{-t/K}) \qquad \text{II-12}$$

II-3.1.1 Application to Time Varying Inflow

If we apply a constant input of rate I to such a linear reservoir, the resulting outflow rate is as shown in Figure II-4, or as given by

$$\begin{aligned} q &= I \int_o^t \frac{1}{K} e^{-\frac{(t-\tau)}{K}} \, d\tau \\ &= I\,(1-e^{-t/K}) \end{aligned} \qquad \text{II-13}$$

where $t \leq T$, the input time period of I.

If equal time periods are assumed with block inputs, I_n, then the outflow rates, q_n, at the end of periods 1, 2,.....n, would be:

$$\begin{aligned} q_1 &= I_1\,(1-e^{-1/K}) \\ q_2 &= I_2\,(1-e^{-1/K}) + I_1\,e^{-1/K} \\ q_n &= I_n\,(1-e^{-1/K}) + I_{n-1}\,e^{-1/K} + \\ &\quad I_{n-2}\,e^{-2/K} \ldots\ldots I_1\,e^{-(\frac{n-1}{K})} \end{aligned} \qquad \text{II-14}$$

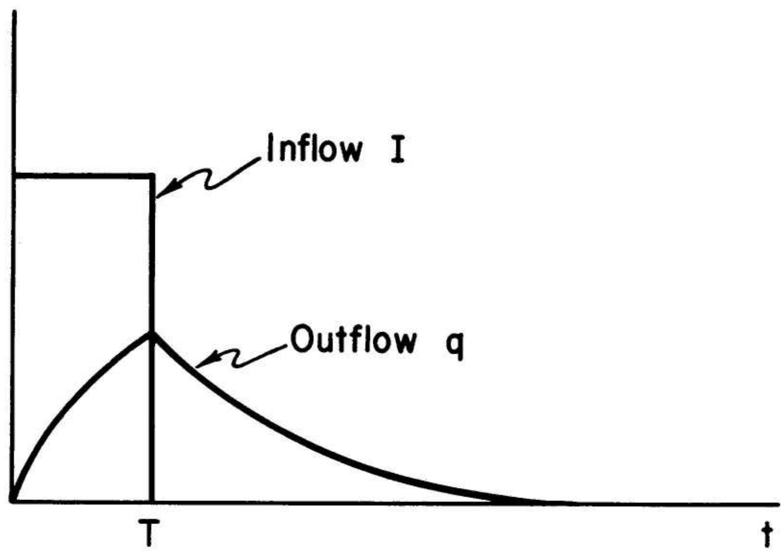


Figure II-4

Outflow Hydrograph Resulting from a Constant Input to a Linear Reservoir

II-3.2 Linear Channel Model

Another single parameter model, known as the lag model or the linear channel, is used for the purpose of translation of a flood wave. The linear channel requires a constant velocity at any point in the channel for all discharges such that the relationship between the inflow and outflow at that point is merely:

$$Q\ (t) = I\ (t-\tau) \qquad \text{II-15}$$

where τ represents the translation in time of the flood wave with no attenuation of the wave.

Dooge [1959] first presented the linear channel concept and pointed out that it can also be considered to be as a cascade of an infinite number of infinitesimal storages. As shown in the above section, the lag to a single reservoir (single storage) is represented by K. Then, if we have n reservoirs in series, the lag would be nK. Thus if n goes to infinity as K goes to 0, while nK remains constant, the variance about the mean, nK^2, goes to zero, implying that an instantaneous input of unit input will cause an instantaneous output of the same volume after the mean travel time of nK.

II-3.3 Two Parameter Models

When considering a groundwater system, one must be realistic in choosing a model for representing that system. Common sense tells us that a pure translation or the linear reservoir (exponential distribution) which lacks the property of having adequate 'memory' (Hillier [1967]), will fail to represent the groundwater system. Thus the tendency has been to incorporate these elementary, single parameter models into a variety of configurations. This led to the two parameter models such as the Lag and Route Model or the Nash Model. The former is represented by the block diagram in Figure II-5. It has the following impulse response:

$$q\ (t) = \frac{1}{K} e^{-\left(\frac{t-\tau}{K}\right)} \qquad \text{II-16}$$

where K = delay time of the linear reservoir.

τ = translation time of the linear channel.

Noting the obvious increase in flexibility by applying such models in series, Nash [1958] developed what has become known as the Nash Model - (Also developed by Kalinin - Milynkov [1958]). Nash conceptually applied a series of n reservoirs each of delay time K, represented by the block diagram in Figure II-6, in order to represent the systems I.U.H.. Thus the total lag to the system can be shown to be nK, since in a series configuration the outflow of one reservoir is the inflow to the succeeding reservoir. The impulse response for this model is:

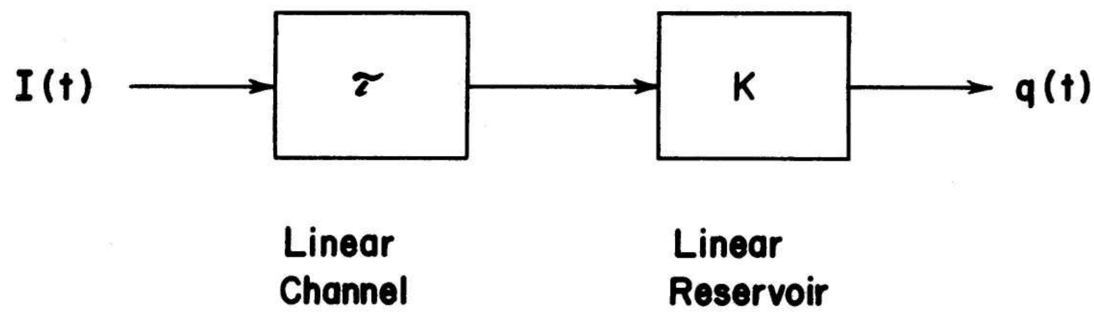


Figure II-5

Block Diagram of a Lag and Route Model

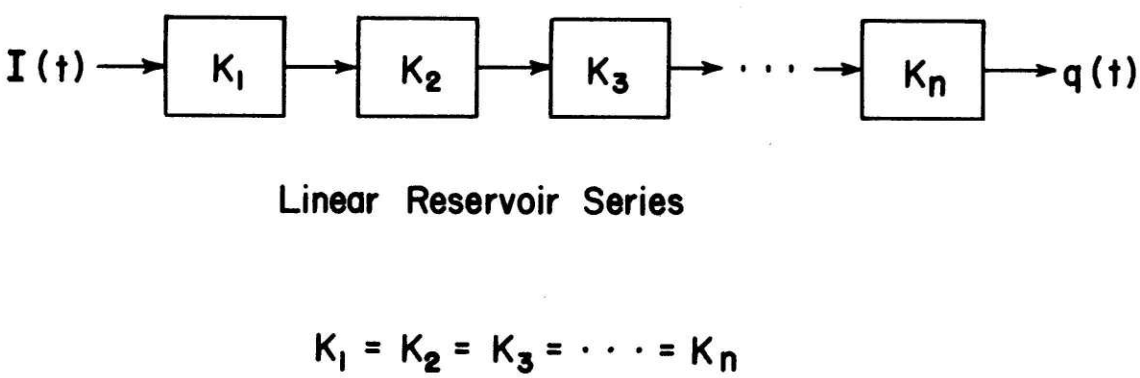


Figure II-6

Block Diagram of a Nash Model

$$h(0,t) = \frac{1}{K} \left(\frac{t}{K}\right)^{n-1} \frac{1}{\Gamma(n)} e^{-t/K} \qquad \text{II-17}$$

Notice that the Nash Model is also a modified gamma distribution. Nash simply used the tools developed earlier by Zoch [1934] in linear reservoirs, Clark [1945] in linear storage routing and Edson [1951] in two parameter model development.

II-3.4 Three Parameter Models

Many three parameter models merely add the linear channel, a translational effect, to the two parameter models. In this paper this is accomplished with the Nash Model as discussed above. However, Harley [1967] also uses the translation with a Muskingum Model, and the Diffusion Analogy.

The advantage of the three parameter models is that they are more capable of simulating a complex system as in the natural highly damped groundwater system.

II-4 Model Formulation Using Linear Systems

With the basic tools now available to linear systems researchers, an infinite number of configurations become available to represent the complex systems of the real world. Many of the following models were presented by Kraijenhoff [1966].

In 1955 Lyshede related a series of exponential functions to the effect of runoff from rainfall and the basin characteristics. This pointed out the possible use of linear reservoirs in series which form the cascade effect of the Gamma distribution.

Singh, in 1964, used the time - area hydrograph and routed it through two linear reservoirs in series to represent the effect of overland and channel flows. Singh's System is shown in Figure II-7.

Diskin, also in 1964, proposed a model using two Nash Models in parallel, each branch consisting of a different number of equal reservoirs and both branches having different lag characteristics in the reservoir series such as shown by Figure II-8. Thus by splitting the input hydrograph, Diskin was able to develop a system that would lag the output by:

$$\alpha\, n_1\, K_1 + (1-\alpha)\, n_2\, K_2 \qquad \text{II-18}$$

The I.U.H., of this system then would by represented by

$$h\,(0,t) = \frac{\alpha}{K_1\,(n_1 - 1)} \left(\frac{t}{K_1}\right)^{n_1 - 1} e^{-t/K_1} + \qquad \text{II-19}$$

$$\frac{(1-\alpha)}{K_2\,(n_2 - 1)} \; \frac{t}{K_2}\, n_2^{-1} \; e^{-t/K_2}$$

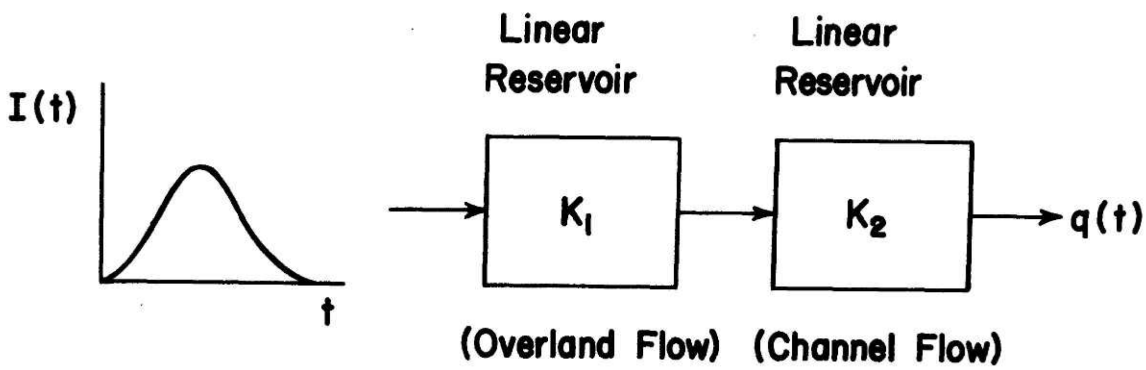


Figure II-7

Singh's Model in Simulating Overland and Channel Flows

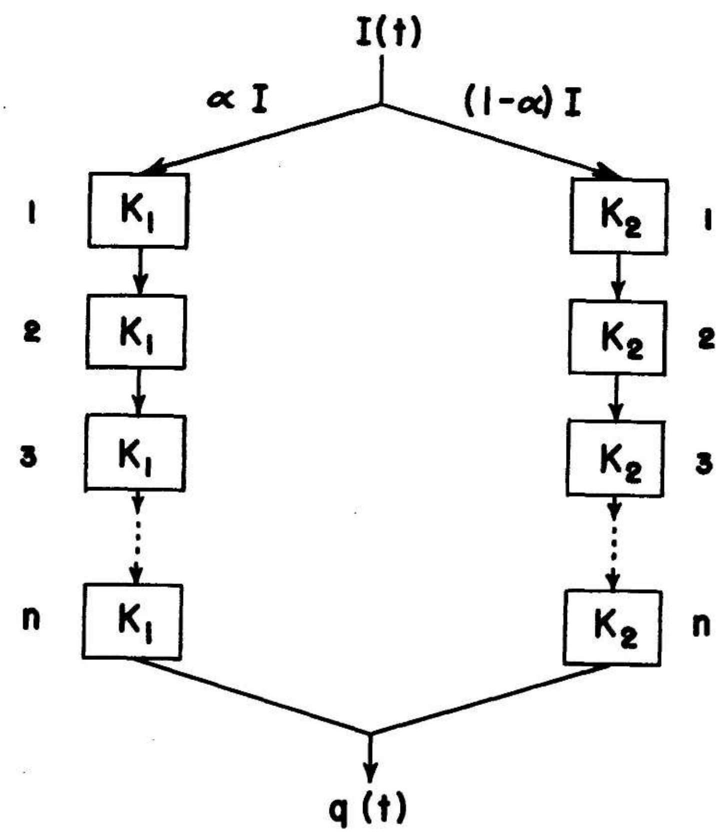


Figure II-8

Diskin's Parallel Nash Model Configuration

II-5 Groundwater Systems

The Netherlands has done work in groundwater systems for many years using the Dupuit - Forchheimer approximations.

As mentioned in Chapter I, there was skepticism in using a linear system approach until it was realized that the basic assumptions for analytical solutions to groundwater flow were equivalent to those used in the unit hydrograph theory. Therefore, the development stages of groundwater flow in linear systems are described briefly in the following paragraphs.

In 1947, Edelman developed equations for two dimensional groundwater flow into a unit length of channel and applied the convolution integral to determine the effect on the groundwater flow-rate of a constant infiltration rate into the phreatic zone as shown in Figure II-9, resulting in the equation

$$Q\,(t) = - \int_o^t \frac{P}{\sqrt{\pi}} \; \frac{KD}{\mu} \, (t-\tau)^{-\frac{1}{2}} \; d\,(-\tau)$$

$$= P \; \frac{2}{\sqrt{\pi}} \; \sqrt{\frac{KD}{\mu}} \; t^{\frac{1}{2}} \qquad \text{II-20}$$

where P = constant percolation rate

KD = transmissivity

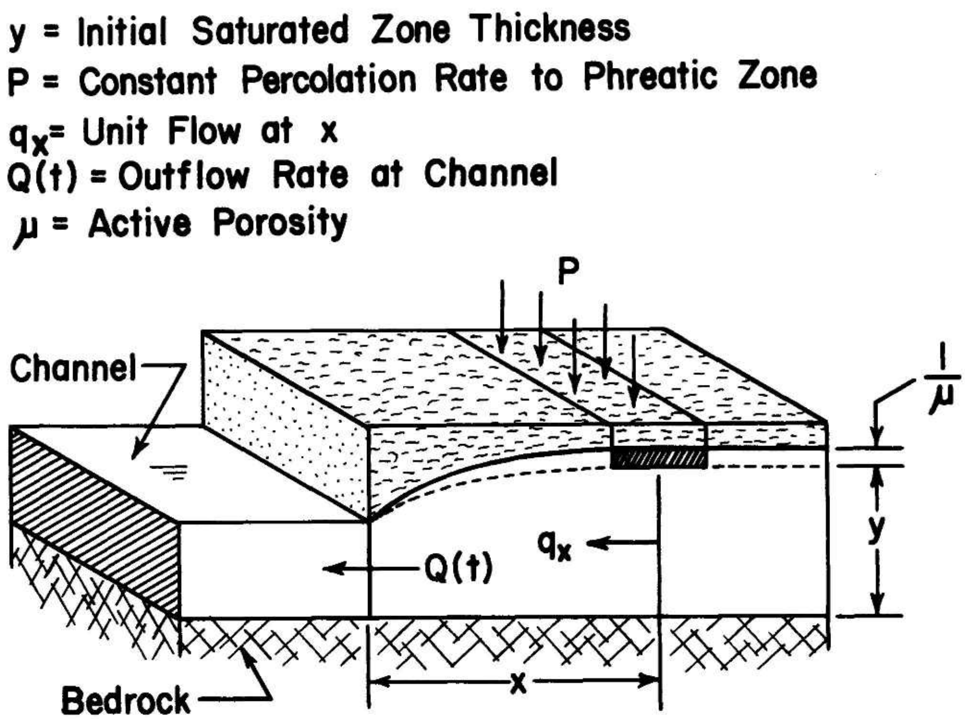


Figure II-9

Edelman's One-Sided Groundwater Flow to a Unit Width Channel

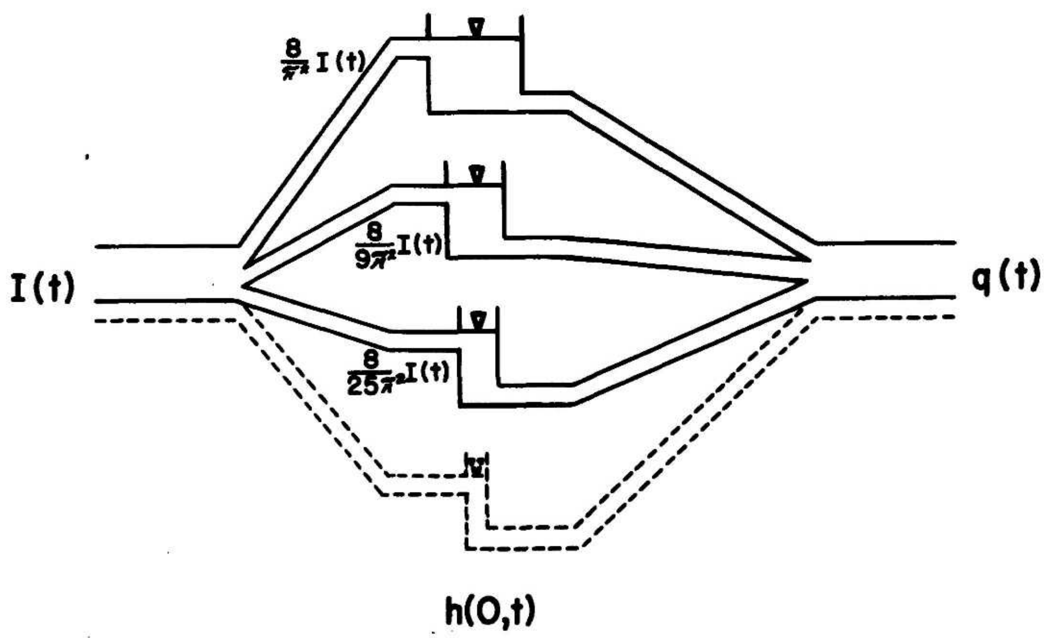


Figure II-10

Reservoir Representation of Kraijenhoff's Flow to Drainage Canals

μ = active porosity

In studying the effects of the phreatic zone in irrigation areas, Glover [1954] developed an equation which relates the spacial and time change of the free water surface to an instantaneous irrigation inflow, s, in equation II-21

$$y(x,t) = \frac{s}{\mu} \frac{4}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} e^{-n^2 t/j} \sin \frac{n\pi x}{L} \qquad \text{II-21}$$

where $j = \frac{1}{\pi^2} \frac{\mu L^2}{KD}$

From this relationship, Kraijenhoff [1958] developed the I.U.H., for flow into parallel drainage channels.

$$h(0,t) = \frac{8}{\pi^2} \frac{1}{j} \sum_{n=1,3,5}^{\infty} e^{-n^2/j} \qquad \text{II-22}$$

Expanding and setting the lags, K, equal to functions of j, results in

$$h(0,t) = \frac{8}{\pi^2} \frac{1}{K_1} e^{-t/K_1} + \frac{1}{9} \frac{8}{\pi^2} \frac{1}{K_2} e^{-t/K_2} + \frac{1}{25} \frac{8}{\pi^2} \frac{1}{K_3} e^{-t/K_3} + \ldots \qquad \text{II-23}$$

We may see that this equation represents the behaviour of a system of linear reservoirs in parallel as shown in Figure II-10. The lag for such a system is given by equation II-24.

$$LAG = \frac{8}{\pi^2} K_1 + \frac{1}{9}\frac{8}{\pi^2} K_2 + \ldots$$

$$= \frac{8}{\pi^2} j \left(1 + \frac{1}{3^4} + \frac{1}{5^4} \ldots\right) \qquad \text{II-24}$$

$$= \frac{8}{\pi^2}\frac{\pi^4}{96} j = \frac{\pi^2}{12} j$$

where $K_1 = j$; $K_2 = j/9$; $K_3 = j/25$ etc.

De Jager [1965], Wesseling [1969] and Wemelsfelden [1963] used other modified configurations to represent flows to parallel drains or river channels.

In an attempt to develop the use of linear systems in groundwater application, Dooge [1960] used the concepts introduced so far to derive coefficients in a simplifying technique. By accepting the work done by Thornthwaite and Penman in estimating infiltration, evaporation and other soil characteristics that determine the flow of groundwater in the unsaturated zone, Dooge developed coefficients for use under a number of conditions, these being:

a) Water table close to the surface where there is a direct effect on recharge by rainfall and evapotranspiration.

b) Water table well below the ground surface where the recharge to the groundwater system is accomplished only after the upper soil region reaches field capacity.

c) Composite type where the groundwater table reacts as a shallow table until the groundwater 'storage' decreases producing an effect more in line with a deep water table.

Dooge's procedure is based on a constant recharge over a given time period. Using constant time periods and the storage concept generated by the linear reservoir discussion in Section II-3, he derived three routing coefficients and three coefficients required under the conditions of negative recharge. These basic equations are

$$Q_n = C_o R_n + C_1 R_{n-1} + C_2 Q_{n-1} \qquad \text{II-25}$$

where Q_n = Outflow due to contributions by the past n recharges

R_n = Recharge in period n

R_{n-1} = Recharge in period n-1

Q_{n-1} = Outflow due to contributions from the past n-1 number of recharges.

The coefficients are given by:

$$C_o = 1 - \frac{K}{T} (1 - e^{-T/K})$$

$$C_1 = \frac{K}{T} (1-e^{-T/K}) - e^{-T/K} \qquad \text{II-26}$$

$$C_2 = e^{-T/K}$$

where T = time period of the recharge.

K = the linear reservoir lag coefficient.

The negative recharge computation is based on a storage calculation at the end of period n, viz

$$S_n = R_n \frac{K}{T} (1-e^{-T/K}) + (Q_n - C_o R_n) \frac{1}{e^{T/K}-1} \qquad \text{II-27}$$

If for any period this goes to zero, he calculates two additional coefficients:

$$S_n = C_3 R_n + C_4 Q_n$$

where $C_3 = \frac{K}{T} - \frac{1}{e^{T/K}-1}$ II-28

$$C_4 = \frac{1}{e^{T/K}-1}$$

Thus if one knows the parameters required by linear reservoir theory, simplified coefficients may be calculated for routing through a groundwater system.

Though the computation scheme would become more complex, one could conceptually consider using a time varying period that could be accepted as being more realistic, i.e., to maintain a constant input, which is acceptable under certain restrictions, and vary the time over which the inflow is constant.

II-6 Parameter Estimation

Definitions, derivations and configurations have been offered in the previous sections but the most important and perhaps the most significant aspect to linear systems theory is that of parameter estimation. It should be obvious that a simulation procedure requires a highly selective method of correlating the I.U.H. parameters as dependent variables with the basin characteristics as the independent variables. Methods available for parameter estimation include:

a) Fourier Coefficients.
b) Laguerre Coefficients.
c) Method of least squares.
d) Method of maximum likelihood.
e) Method of moments.
f) Wiener - Hopf equations

O'Donnell [1960] presented an approach to develop the I.U.H., by means of Harmonic Analysis, which produced Fourier Coefficients.

He used the fact that the Fourier expansion can be used if the input/output hydrographs are assumed periodic. These expansions may be represented by

Inflow Expansion:

$$I(\tau) = \sum_{n=o}^{\infty} a_n \cos\left(n \frac{2\pi\tau}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(n \frac{2\pi\tau}{T}\right) \qquad \text{II-29}$$

I.U.H. Expansion:

$$h(t-\tau) = \sum_{m=o}^{\infty} \alpha_m \cos\left(m \frac{2\pi (t-\tau)}{T}\right) + \sum_{m=1}^{\infty} \beta_m \sin\left(m \frac{2\pi (t-\tau)}{T}\right) \qquad \text{II-30}$$

Outflow Expansion:

$$Q(t) = \sum_{r=o}^{\infty} A_r \cos\left(r \frac{2\pi\tau}{T}\right) + \sum_{r=1}^{\infty} B_r \sin\left(r \frac{2\pi\tau}{T}\right) \qquad \text{II-31}$$

By applying the convolution integral and considering the n^{th} harmonic, he was able to derive the kernel coefficients with respect to the input/output coefficients, thus giving:

$$\alpha_o = \frac{1}{T} \frac{A_o}{a_o}$$

$$\alpha_n = \frac{2}{T} \frac{a_n A_n + b_n B_n}{a_n^2 + b_n^2} \quad \text{for } n \geq 1 \qquad \text{II-32}$$

$$\beta_n = \frac{2}{T} \frac{a_n B_n - b_n A_n}{a_n^2 + b_n^2}$$

Thus, if a long enough record is available and accurate, then a simple means for determining the instantaneous unit hydrograph is available.

Dooge [1965] proposed another scheme for the analysis of heavily damped linear systems using Laguerre functions, similar to the method proposed by O'Donnell [1960] in using coefficients derived by means of harmonic analysis. The equations derived by means of the Laguerre functions are:

Input function

$$I(t) = \sum_{n=o}^{\infty} a_n f_n(t) \qquad \text{II-33}$$

Response function

$$h(t) = \sum_{n=o}^{\infty} \alpha_n f_n(t) \qquad \text{II-34}$$

Output function

$$Q(t) = \sum_{n=o}^{\infty} A_n f_n(t) \qquad \text{II-35}$$

The linkage function coefficient are given by

$$A_p = \sum_{k=o}^{p} \alpha_k \, a_{p-k} - \sum_{k=o}^{p-1} \alpha_k \, a_{p-1-k} \qquad \text{II-36}$$

Eagleson [1965] presents the Methods of Least Squares and the Weiner - Hopf Equation as procedures for determining the instantaneous unit hydrographs, while Hillier and Lieberman [1967] present the method of Maximum Likelihood and others in determining parameter estimators.

The Method of Moments for estimating parameters was first applied to hydrologic systems by Nash [1959]. The accuracy of this procedure is dependant on the number of samples taken of the system and that these samples are truly representative of the basin characteristics. Maddaus [1969] offers what he considered to be disadvantages to the use of this method, and are as follows.

a) Non-linearity is filtered out of the lower moments but the non-linearity tends to concentrate in the higher moments.

b) Inconsistency may occur between the parameters and the assumed model resulting in negative parameters.

c) They tend to be biased at the extremities, thereby causing the greatest error at the peak.

Since the Method of Moments is an effective parameter technique in hydrologic systems, where our concern is with the lower moments, then any non-linearities of the natural system must reside in the upper moments. When applying this procedure to unit hydrograph theory where only positive causal systems are considered, any inconsistency producing a negative parameter would, indeed, reduce the effectiveness of this procedure. The effect of c) will be shown in Chapter IV where the peak is shown to have the greatest error when utilizing the Method of Moments. Since the lower moments do provide the more significant results in modeling hydrologic systems, it has become an accepted fact that the first three or four moments only, be used in parameter estimation. Nash [1959] recommends the use of dimensionless parameters in allowing an independence between the parameters and the I.U.H. This is accomplished by dividing all moments exclusive of the first by the first moment. To present an example of the Method of Moments, the Nash Model will be considered. The I.U.H. of a Nash Model may be represented by:

$$h\ (0,t) = \frac{1}{K}\ (t/K)^{n-1}\ \frac{1}{\Gamma(n)}\ e^{-t/K} \qquad \text{II-37}$$

where n = number of equal linear reservoirs in series

K = the time constant or lag of a single reservoir.

Then the first moment about the origin, or lag of the system is given by:

$$M_1' = \int_o^{\infty} h\ (0,t)\ t\ dt$$

$$= n\ K \int_o^{\infty} (t/K)^n \frac{1}{\Gamma(n)}\ e^{-t/K}\ d\ (t/K) \qquad \text{II-38}$$

$$= n\ K$$

where M_1' = first moment about the origin.

The second moment about the mean (or centroid) known as the variance, is determined by the equation:

$$M_2 = M_2' - M_1'^2$$

where $$M_2' = \int_o^{\infty} h\ (0,t)\ t^2\ dt$$

$$= K^2\ n\ (n+1) \qquad \text{II-39}$$

then $$M_2 = K^2\ n^2 + K^2\ n - K^2\ n^2$$

$$= nK^2$$

where M_1 = first moment about the mean

M_2 = second moment about the mean

M_2' = second moment about the origin

By the same procedure, the third moment (Skewness) and the fourth moment (Kurtosis) may be derived to be:

$$M_3 = 2\ n\ K^3$$

$$M_4 = 6\ n\ K^4 \qquad \text{II-40}$$

A similar procedure may be used for the moments of the Linear Reservoir and Lag and Route Models. These are represented by Equation II-41 and II-42.

Linear Reservoir Moments

$$M_1 = K$$

$$M_2 = K^2$$

$$M_3 = 2\ K^3 \qquad \text{II-41}$$

$$M_4 = 6\ K^4$$

Lag and Route Moments

$$M_1 = \tau + K$$

$$M_2 = K^2$$

II-42

$M_3 = 2 K^3$

$M_4 = 6 K^4$

Appendix B.3 develops moments in greater detail.

Another parameter that sometimes proves important is the time to peak, which is found by taking the first derivative of the I.U.H., and equating to zero, since in hydrology the work is with causal systems. Thus:

The time to peak is that at which

$$\frac{d}{dt} h(0,t) = 0 \qquad \text{II-43}$$

Then for the various models discussed in Section II-3:

$$\begin{aligned} &\text{Nash Model} && T_p = (n-1) K \\ &\text{Single Reservoir} && T_p = 0 \qquad \text{II-44} \\ &\text{Lag and Route Model} && T_p = \tau \end{aligned}$$

II-7 Theoretical Development to the Regional Groundwater Routing Model

The basic equations for unsteady one dimensional flow will be considered assuming the following conditions apply:

a) Unconfined flow

b) Incompressable fluid flow

c) Darcy's Law applies

d) Sloping bedrock

The continuity equation is given by:

$$\frac{\partial q}{\partial x} + Sc \frac{\partial h}{\partial t} = - \frac{q_i}{\Delta x} \qquad \text{II-45}$$

where q_i = inflow

Δx = area of inflow

Sc = storage coefficient

h = piezometric head

However, the response to a Dirac delta function of inflow is desired such that the continuity equation will be:

$$\frac{\partial q}{\partial x} + Sc \frac{\partial h}{\partial t} = 0 \qquad t > 0 \qquad \text{II-46}$$

The groundwater flow will be represented by a modified form of the Darcy equation, incorporating the advective velocity, as in Equation II-47.

$$q = a\,h - K_p\,h_m \frac{\partial h}{\partial x} \qquad \text{II-47}$$

where q = groundwater flow (L^3/T)

a = advective velocity due to the sloping bedrock (L/T)

h = piezometric head (L)

K_p = permeability (L/T)

h_m = mean depth of the saturated water zone

The advective velocity may be more adequately shown to be

$$a = -K_p \frac{dh}{dL} \text{ from Darcy's Law} \qquad \text{II-48}$$

$$= -K_p \cdot \text{slope}$$

where the minus sign indicates the direction of flow. Then equation II-47 can be rewritten as

$$\frac{\partial q}{\partial x} = \frac{a\,\partial h}{\partial x} - K_p\, h_m \frac{\partial^2 h}{\partial x^2} \qquad \text{II-49}$$

and the continuity equation II-46 will then become

$$a \frac{\partial h}{\partial x} - K_p\, h_m \frac{\partial^2 h}{\partial x^2} + Sc \frac{\partial h}{\partial t} = 0 \qquad \text{II-50}$$

or

$$\frac{K_p\, h_m}{Sc} \frac{\partial^2 h}{\partial x^2} = \frac{a}{Sc} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} \qquad \text{II-51}$$

By using Laplace Transforms, Harley [1967] shows that the resulting system response to the Dirac delta function input (of a similar relationship) will be:

$$h\ (x,t) = \frac{1}{2\sqrt{\pi K_p}} \cdot \frac{x}{t^{3/2}} \cdot \exp\left(-\frac{(ct-x)^2}{4 K_p t}\right) \qquad \text{II-52}$$

thus equation II-51 results in:

$$h\ (x,t) = \frac{\sqrt{Sc}}{2\sqrt{\pi K_p h_m}} \cdot \frac{x}{t^{3/2}} \cdot \exp\left(-\frac{(at/Sc-x)^2}{4 K_p t}\right) \qquad \text{II-53}$$

Appendix B.2 proves a similar result for a horizontal bedrock condition.

The cumulants for the above response function are shown in Appendix B.2 to be derived simply from the Laplace Transform. Harley [1967] shows the first four cumulants to be given by Equations II-54

$$C_1 = x/c$$

$$C_2 = \frac{2 K_p x}{c^3} \qquad \text{II-54}$$

$$C_3 = \frac{12\ K_p^{\ 2}\ x}{c^5}$$

$$C_4 = \frac{120\ K_p^{\ 3}\ x}{c^7}$$

Equations II-55 are the four cumulants derived from the governing equation of groundwater flow.

$$Z_1 = x\ Sc/a$$

$$Z_2 = 2\ K_p\ h_m\ Sc^2\ x/a^3 \qquad \text{II-55}$$

$$Z_3 = 12\ K_p^{\ 2}\ h_m^{\ 2} Sc^3\ x/a^5$$

$$Z_4 = 120\ K_p^{\ 3}\ h_m^{\ 3}\ Sc^4\ x/a^7$$

In Chapter IV, the first three cumulants will be used to determine the lag of a single reservoir, the number of single reservoirs as well as the lag, τ, required to simulate the system response based on the input parameters used to compute the cumulants shown above.

Chapter III

FOURIER TRANSFORMS IN LINEAR HYDROLOGIC SYSTEMS

III-1 Introduction to Fourier Analysis

A linear system is characterized by the following equation:

$$f_o(t) = \int_{-\infty}^{\infty} f_i(\tau)\, h\,(t-\tau)\, d\,\tau \qquad \text{III-1}$$

where $f_i(t)$ = an input function

$h\,(t)$ = the system response function, characterized as the response to a unit impulse

$f_o(t)$ = response of the system to the input $f_i(t)$

Applying a Fourier transformation to equation III-1 yields

$$F_o(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_o(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega t}\, dt \cdot \int_{-\infty}^{\infty} f_i(\tau)\, h\,(t-\tau)\, d\,\tau \qquad \text{III-2}$$

Letting $s=t-\tau$ and changing the order of integration, the above equation reduces to :

$$F_o(\omega) = \int_{-\infty}^{\infty} h(s)\, e^{-j\omega}\, d\, s \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} f_i(\tau)\, e^{-j\omega\tau}\, d\tau \qquad \text{III-3}$$

which can be further stated as

$$F_o(\omega) = H(\omega) \cdot F_i(\omega) \qquad \text{III-4}$$

where $F_i(\omega)$ = the Fourier transform of input function

$H(\omega)$ = the Fourier transform of the system response, (multiplied by 2π)

$F_o(\omega)$ = the Fourier transform of the output.

Therefore a convolution integral can be reduced to a simple multiplication of Fourier transforms.

Traditionally in hydrology the whole series of linear models, such as the linear Reservoir, Muskingum, Nash and linear solutions to the momentum and continuity equations, have been utilized by obtaining expressions for the system response function and performing the lengthy and time consuming numerical (or sometimes analytical) convolution in a computer.

The purpose of this chapter is to combine the knowledge of the analytical Fourier transforms of these linear systems with the availability of numerical computer techniques to obtain Fourier transforms in order to utilize equation II-4 to find the outflow from a system resulting from a known inflow.

By reducing the complicated convolution procedures to a simple multiplication and by utilizing an efficient numerical transform scheme the time required to obtain the output function should be reduced significantly.

In this chapter two available computer programs to carry out Fourier transformations are investigated. One is based on traditional finite numerical integration of the Fourier transform equations; the other is based on the theory of Fast Fourier Transforms. The accuracy, speed and ease of use of each of these programs are evaluated and compared.

III-2 Fourier Transform Technique

Two numerical, computational techniques are considered in this paper. One is based on the Cooley - Tukey Fast Fourier Transform Theory which was available at M.I.T. as Subroutine FOURT, (Appendix A-2). The other is based on finite numerical integration of the Fourier Transform equation and will be noted by Subroutine FOURTRAN (see reference Eagleson and Goodspeed (1970)).

The discrete form of the finite transform used in Subroutine FOURTRAN is:

$$F\ (\omega) = \frac{1}{\sqrt{2\pi}} \sum_{L=1}^{NF} f(t)\ \exp\ (-j\omega\ L\Delta t)\ \Delta t \qquad \text{III-5}$$

where NF = number of points in time at which f(t) is given

f(t) = input function at interval Δt

Δt = time interval

ω = angular frequency

This equation is evaluated at different equally spaced $\Delta\omega$'s up to some cutoff frequency, ω_o, which must be less or equal to $\pi/\Delta t$. If the given ω_o value exceeds $\pi/\Delta t$ it is automatically adjusted to this value. In order to return to the time domain (by taking the inverse transform) the procedure is as follows:

a) change sign of the exponential

b) integrate over the angular frequencies ($\Delta\omega$)

c) evaluate at different times (t).

III-2.1 Characteristics of Subroutine FOURTRAN --- Numerical Integration Technique

The following are characteristics of Subroutine FOURTRAN:

a) Two options are available to the user for the output, a complex spectrum or the normalized amplitude and phase of the spectrum.

b) The program does not require the input function to be periodic.

c) The input function is assumed to start at a value of zero, to be sampled at equal intervals, and to be zero after the sampled period.

d) For simplicity, the forward and inverse transform equation are made similar by multiplying by $1/\sqrt{2\pi}$, thus allowing a simple transition from the forward transform to the inverse transform.

e) Since the complex spectrum of a time series is symmetrical about the origin and if FOURTRAN is used to find the inverse transform, only one portion of the symmetrical transform is input and thus the resulting time domain series must be doubled in order to keep the proper scale.

III-2.1.1 Test and Results for Subroutine FOURTRAN

The tests performed on Subroutine FOURTRAN consisted of entering and exiting the program with one function in order to determine the effectiveness of the program in returning the identical function. The function chosen was that of the linear reservoir as presented in Chapter II.

The forward and inverse Fourier transforms from Subroutine FOURTRAN are dependent on the following parameters:

Δt = sampling time interval

ω_o = maximum angular frequency, Nyquist frequency

$\Delta\omega$ = angular frequency interval.

Table III-1 is a tabulation of the significant parameters in the forward transform (frequency domain). In considering Table III-1 and comparing the analytical and computational transforms, these indicated:

a) Accuracy increases as the integration steps decreased, i.e., Δt in the forward transform and $\Delta\omega$ in the inverse transform.

b) Figure III-1 shows the aliasing effect in the forward transform when compared to the analytical transform, i.e., as the transforms approach the higher frequencies, the complex spectrum obtained, using the program, diverges from the theoretical transform.

c) As the integrating variable, Δt, decreases the time of execution increases significantly. An

Table III-1

FOURIER TRANSFORM (FORWARD) RESULTS -- SUBROUTINE FOURTRAN

[Input Function: $1/K\ e^{-(t/K)}$]

Sample Period	ΔT Time Step	No. of Points	K (Time)	ω_o (Rad/Time)	Freq. (Cy/Time)	$\Delta\omega$ (Rad/Time)	Δ Freq. (Cy/Time)	Execution Time (Sec)
36.0	.5485	65	5.0	5.727	.9115	.0873	.0139	-
36.0	0.2	65	5.0	5.727	.9115	.0873	.0139	6.84
36.0	0.1	65	5.0	5.727	.9115	.0873	.0139	12.84
36.0	0.1	4	5.0	5.727	.9115	1.2	.1910	1.81
36.0	0.1	74	5.0	6.5	1.0345	.0873	.0139	14.49
36.0	0.05	65	5.0	5.727	.9115	.0873	.0139	24.53
36.0	0.1	11	5.0	5.727	.9115	0.50	.0796	2.7
36.0	0.1	22	5.0	5.727	.9115	0.25	.0398	4.73
42.0	0.1	65	5.0	5.727	.9115	.0873	.0139	14.58
42.0	0.1	74	5.0	5.727	1.0345	.0873	.0139	16.72

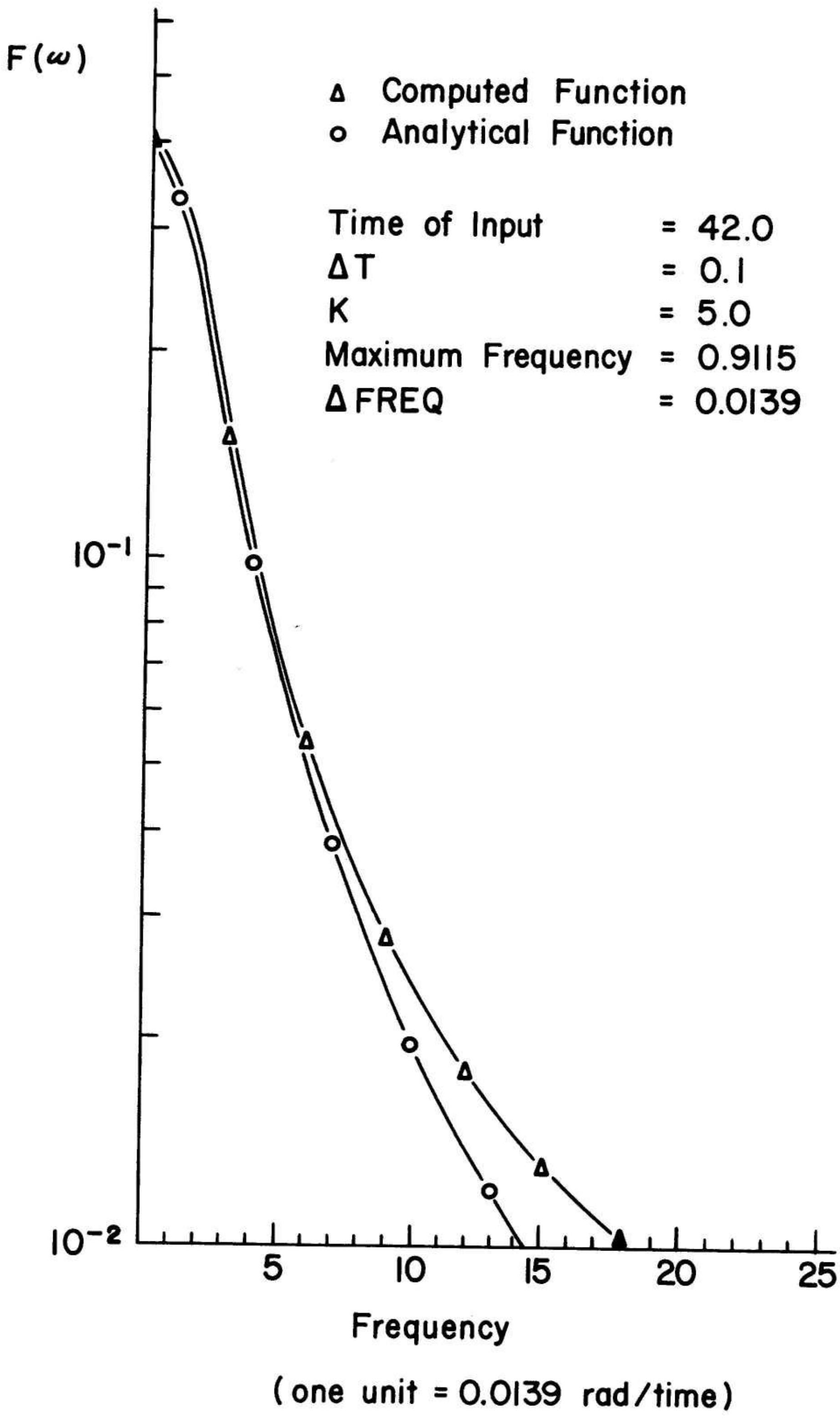
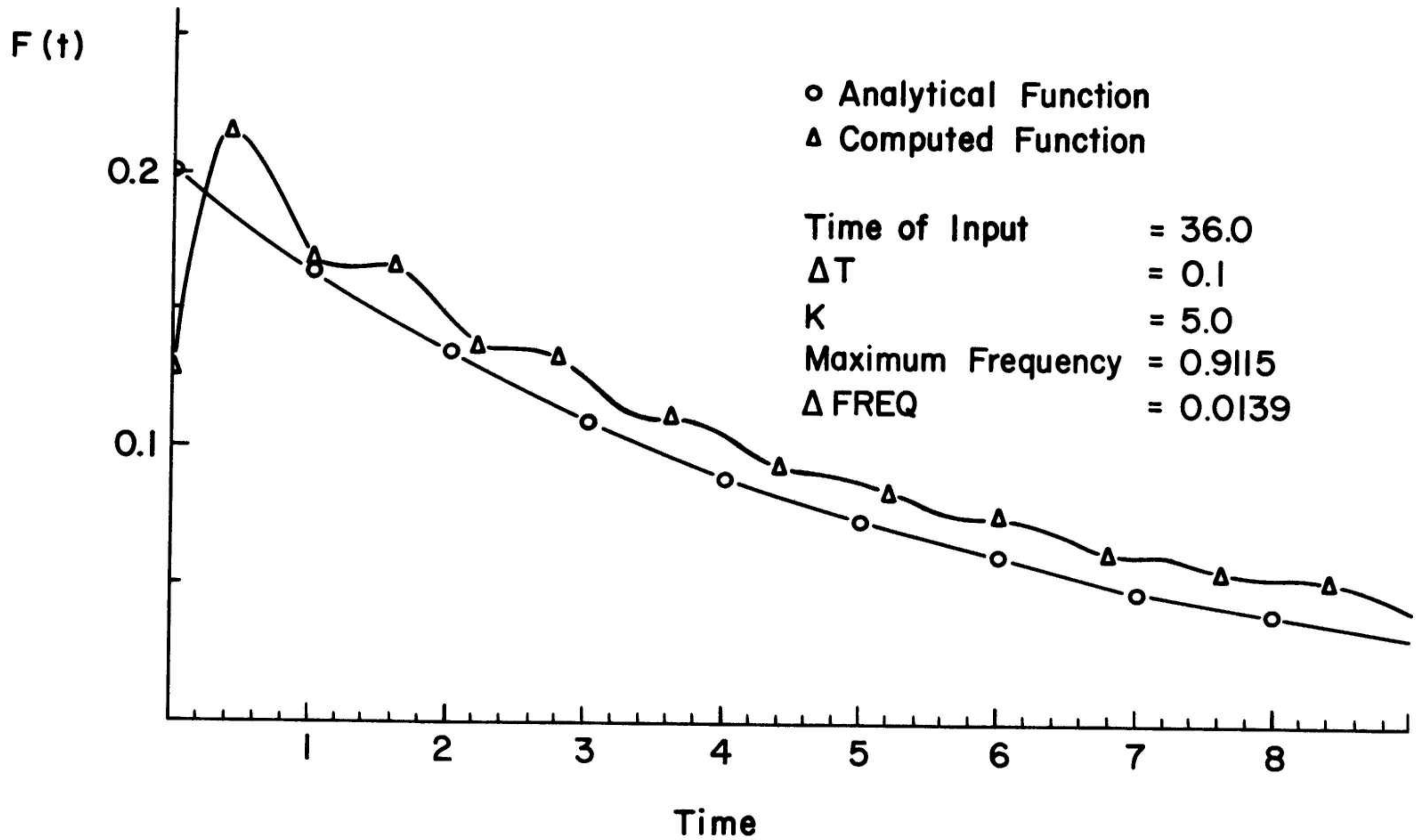


Figure III-1

Aliasing Effect in the Forward Transform--Subroutine FOURTRAN attempt was made to decrease the aliasing effect by increasing the Nyquist frequency, ω_o; as this decreased the integrating step, Δt, however the execution time again increased. Increasing $\Delta\omega$, on the other hand, provides an inverse relationship with the execution time. The problem is that in order to provide good results in the inverse transform an adequate number of points in the frequency domain must be provided. This implies use of a smaller $\Delta\omega$ and therefore higher execution times in the inverse and forward transform calculations.

The results obtained when finding the inverse of a complex spectrum using FOURTRAN are shown in Figure III-2. As the figure shows, the numerical integration method of finding the Fourier transform fails to reproduce its original input, i.e., if a forward transform is performed on an input and then the corresponding inverse on that transform, FOURTRAN fails to reproduce the original function. This is due to the finite integration technique. The problem that we are faced with is when a system response is to be represented by a series system of models as discussed in Chapter II. In this case, the output of one model response is the input into the next, thus requiring a multiple use of a convolution technique. Thus, if Subroutine FOURTRAN was recalled a number of times, the integration error would compound itself.



<u>Figure III-2</u>

Inverse Transform (Time Domain)--Subroutine FOURTRAN

III-2.2 Characteristics of Subroutine FOURT --- Fast Fourier Transform Technique

III-2.2.1 Data Requirements

This subprogram assumes periodicity, i.e., the input values represent one cycle of a periodic function. The input values must be at even time (frequency) intervals for a forward (inverse) transform and may be real or complex. However, when returning from the frequency domain the data must always be complex. If the number of data points is a power of two this subprogram will run at its maximum efficiency. The only data the program requires are the input values, the number of input values and information indicating if the inverse or forward transforms is desired.

III-2.2.2 Subroutine FOURT --- Forward Transform

The Nyquist frequency, ω_o, is determined analytically as $\pi/\Delta t$, thus defining the frequency interval, $\Delta\omega$, at which the transform will be evaluated. A property of the FOURT returned forward transform is the symmetry of the transform about the Nyquist frequency, with that frequency as the midpoint (plus one if the function has an even number of points) of the transform. Since FOURT evaluates over a frequency range of $2\pi(N-1)/N\Delta t$, at frequency intervals of $2\pi/N\Delta t$, the Nyquist frequency will be located at point N/2 due to FOURT's symmetric representation in the frequency domain. The number of output points in

Subroutine FOURT is identical to those input.

III-2.2.3 Subroutine FOURT --- Inverse Transform

The output of the inverse transform is a regular time series and has the same time intervals as the original input since it is based on the same number of input points. The resulting time domain function must be divided by the number of points used in the calculation in order to obtain the correct results - a property of FOURT. In finding the inverse transform the user must ensure that the complex spectrum is input in the symmetrical conjugate form described above.

III-2.2.4 Implications of the Input-output Requirements

As mentioned above, the number of points returned after transforming with the subprogram FOURT is the same as input initially. Since the program assumes a periodic function this implies that in using the program to convolute, by multiplication of input and response transforms, the same number of points must be in each of the transforms.

It is important to note that FOURT does not consider the $1/2\pi$ factor usually found in Fourier transforms so caution must be used in interpreting FOURT's results.

III-2.2.5 Tests and Results

In finding the inverse Fourier transform of a FOURT obtained complex spectrum, the program was able to reproduce the original time domain function identically, thus indicating a good computational scheme. When the theoretical Fourier transform was input in the frequency domain and the inverse was taken the results were very close to the true function, Figure III-3. However, the aliasing effect still exists as shown in Figure III-4.

III-3 Selection of an Efficient Fourier Transformation Technique

Subroutine FOURT, the Fast Fourier Transformation technique is chosen over Subroutine FOURTRAN for the following reasons:

a) it is considerably faster

b) the accuracy is maintained in the inverse transform when using the theoretical forward transform.

c) the exact reproduction of the function in the time domain is obtained when the forward and inverse transforms are computed in succession.

III-4 Convoluting with Fourier Transforms

Implementing the algorithm to convolute by multiplying Fourier transforms presents some immediate problems.

First, the selection of the Fast Fourier transform program, FOURT requires that the functions being transformed be defined as

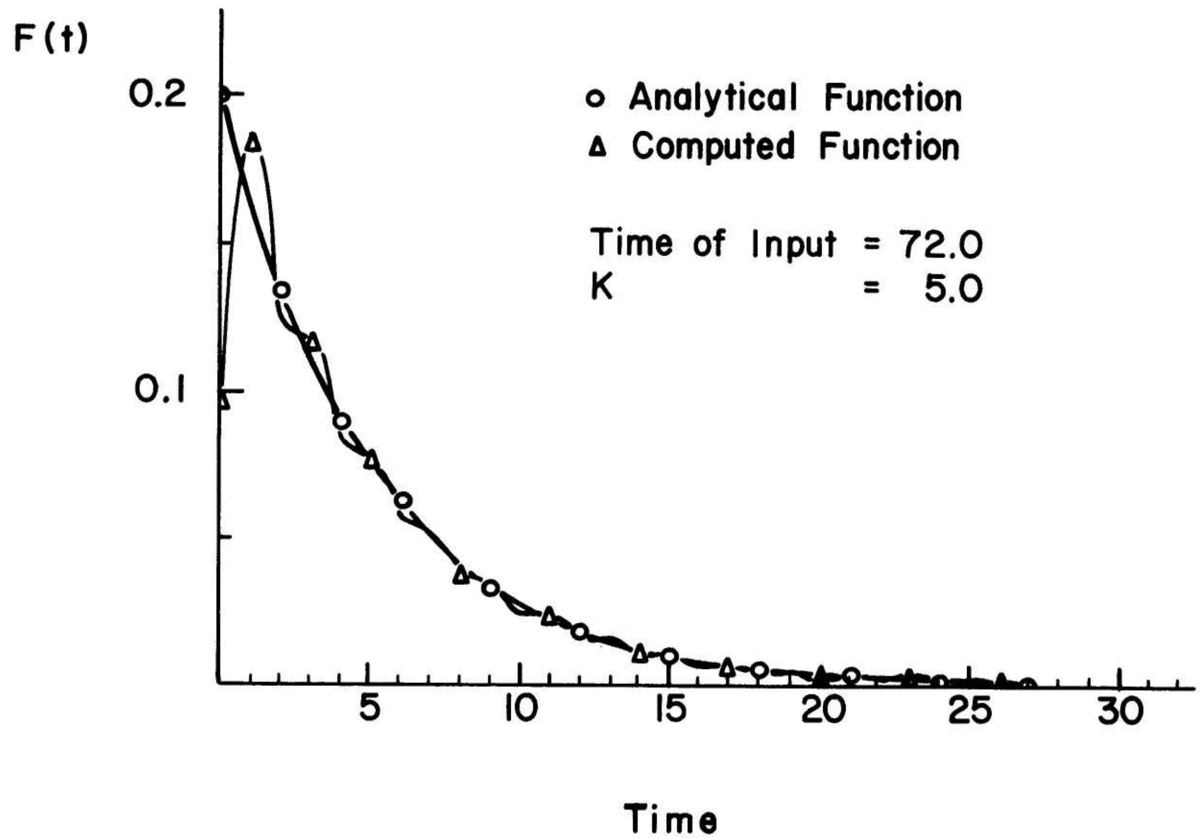


Figure III-3

Inverse Transform (Time Domain)--Subroutine FOURT

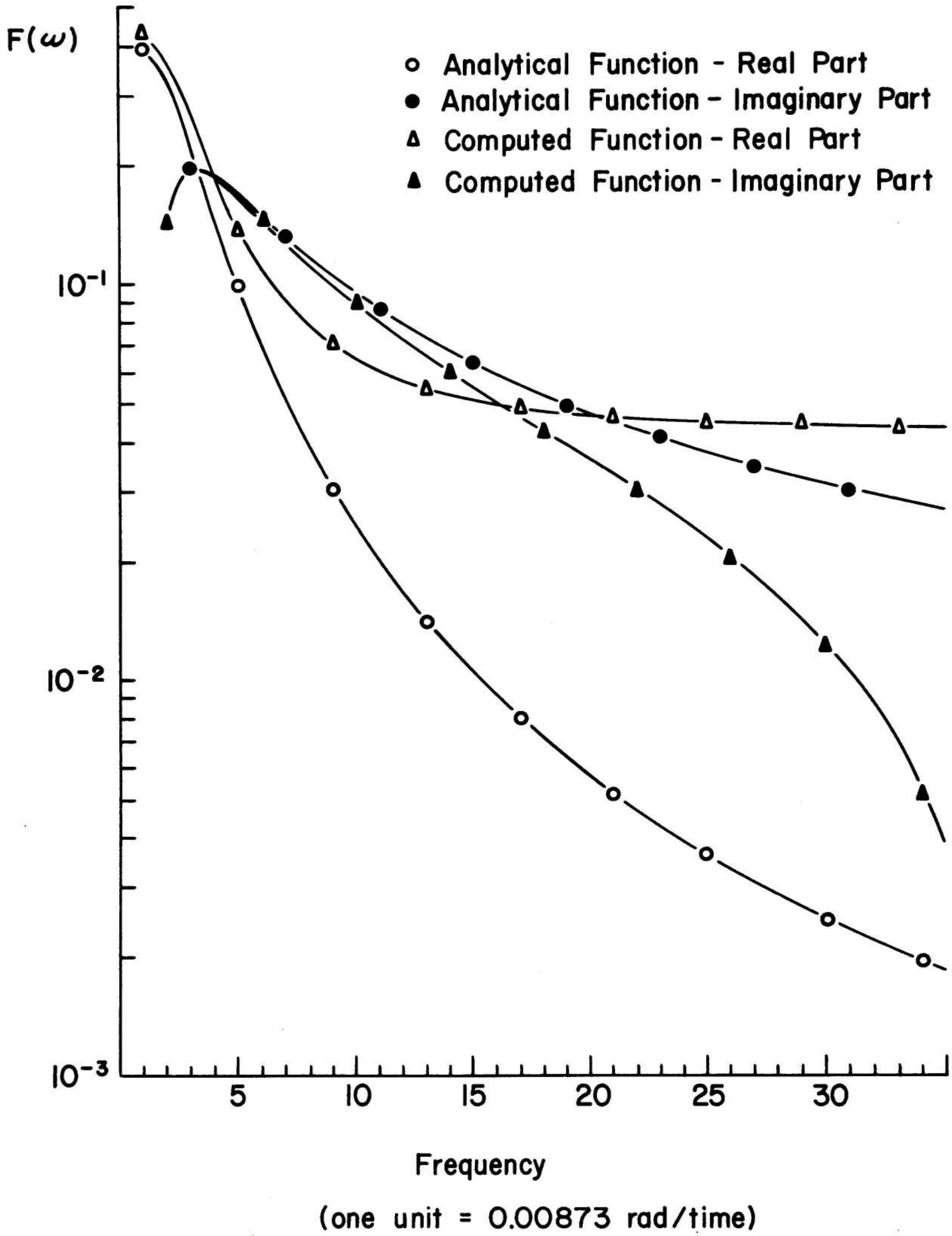


Figure III-4

Forward Transform Indicating Aliasing Effect--Subroutine FOURT periodic functions. Selecting this period so that the resulting output would not be affected by the assumed repeating portions of the input and response functions, was one important task.

Secondly, the response functions of all linear models approach zero at infinity. Since a finite function was required in order to be able to define a time period for the input, response and output functions all of which must be input as the same time period, it was necessary to develop a criteria for adequate definition of a cut off time for the response function.

Finally, the Fourier transform of a function may include infinitely many terms requiring that the frequency range of the transforms also go to infinity. It is necessary, then, to develop a method to find the Nyquist frequency, ω_o, which will limit the frequency bandwidth to the frequencies of interest. By fixing the Nyquist frequency, the time increment, Δt, that the input function must be sampled at in order to detect frequencies up to ω_o, is defined.

III-4.1 Defining the Nyquist Frequency, ω_o

The requirement for defining the Nyquist frequency, ω_o, is to ascertain the 'energy' required if the response function is sufficient to produce accurate results. To determine the 'energy' retained by the response function, the 'power' spectrum of the response function and NOT the input function is used for this purpose.

The logic here is that the frequencies of the obtained output are limited by the dominating frequencies of the system response function. Hydrologic systems, generally, pass a significant amount of the input energy within the lower frequencies. As this is also the case of the linear models used to represent the system response, most of the energy within the system may be retained without the consideration of very high frequencies.

The procedure of fixing ω_o is illustrated using the well known linear reservoir model whose response function is given by

$$h(t) = \frac{1}{K} e^{-t/K} \qquad \text{III-6}$$

where t = time

K = the delay time constant

The normalized amplitude spectrum of this function is given by:

$$\left| H_o(\omega K) \right| = (1 + (\omega K)^2)^{-1/2} \qquad \text{III-7}$$

The power density spectrum is defined as the square of the amplitude, or:

$$\left| H_o(\omega K) \right|^2 = \frac{1}{1+(\omega K)^2} \qquad \text{III-8}$$

Since a unit impulse input function is being considered, its amplitude density spectrum is given by:

$$\left| P_\delta \ (\omega K) \right| = \frac{1}{2\pi} \qquad \text{III-9}$$

Then the energy density spectrum of the output is the resultant multiplication of the input and response power spectrums, or:

$$\phi_o \ (\omega K) = \left| H_o \ (\omega K) \right|^2 \cdot \left| P_\delta \ (\omega K) \right|^2$$

$$= \frac{1}{2\pi} \left| H_o \ (\omega K) \right|^2 \qquad \text{III-10}$$

$$= \frac{1}{2\pi} \ \left(\frac{1}{1+(\omega K)^2} \right)$$

The energy of the output that must be preserved is chosen to be 98% of the total energy. Then, the Nyquist frequency, ω_o, must be found which will assure that an energy loss greater than 2% does not occur. The total energy of this system is, given by:

$$E = 2 \int_0^\infty \Phi_o \ (\omega K) \ d \ (\omega K)$$

$$= \frac{1}{\pi} \int_0^\infty \frac{d(\omega K)}{1+(\omega K)^2} \qquad \text{III-11}$$

$$= 0.5$$

Thus, in order to keep 98% of the energy, we need to integrate over the area of interest, as in Equation III-12 ,such that

$$E = \frac{1}{\pi} \int_{o}^{\omega K} \frac{d(\omega K)}{1+(\omega K)^2} = 0.49$$

$$= \frac{1}{\pi} [\tan^{-1} (\omega K) - n \pi] = 0.49 \qquad \text{III-12}$$

But for n=1, Equation III-12 becomes

$$\frac{\tan^{-1}(\omega K)}{\pi} = 1.49 \qquad \text{III-13}$$

thus

$$\begin{aligned} \omega K &= \tan (4.68) \\ &= \tan (268^\circ) \\ &= 28.636 \end{aligned} \qquad \text{III-14}$$

Thus, 98% of the output energy of a linear reservoir model will be passed if the Nyquist frequency, ω_o, is determined by the expression III-15, which is in terms of the model parameters K.

$$\omega_o = 28.636/K \qquad \text{III-15}$$

III-4.2 Effects of Complex Responses on the Nyquist Frequency

Theoretically this procedure should be applied to each model in order to obtain their respective expressions for ω_o. Unfortunately the power spectrums of other model response functions get fairly complicated, especially as the number of parameters increase. Due to this difficulty, it was decided to use the Nyquist frequency determined for the linear reservoir as a basis for all linear systems used. Making such an assumption should assure that the selection of ω_o is on the conservative side. Of all the linear models the linear reservoir can pass the highest frequency components.

Figure III-5 demonstrates how the linear reservoir normalized amplitude spectrum has higher frequency components than Nash Models of order greater than 1 (which is the linear reservoir).

III-4.3 Selection of Response Function Duration

The time period for the system response was chosen arbitrarily to be that time which would allow 99% of the response to have occurred.

Again this was done by setting up an integral equation. As an example the linear reservoir formulation was used. The total area under the linear reservoir response curve is unity, so to find the time which should be used, the integrated system response for a linear reservoir was equated to .99, thus representing an area of 99%.

$$\int_o^t \frac{1}{K} e^{-t/K} = .99 \qquad \text{III-16}$$

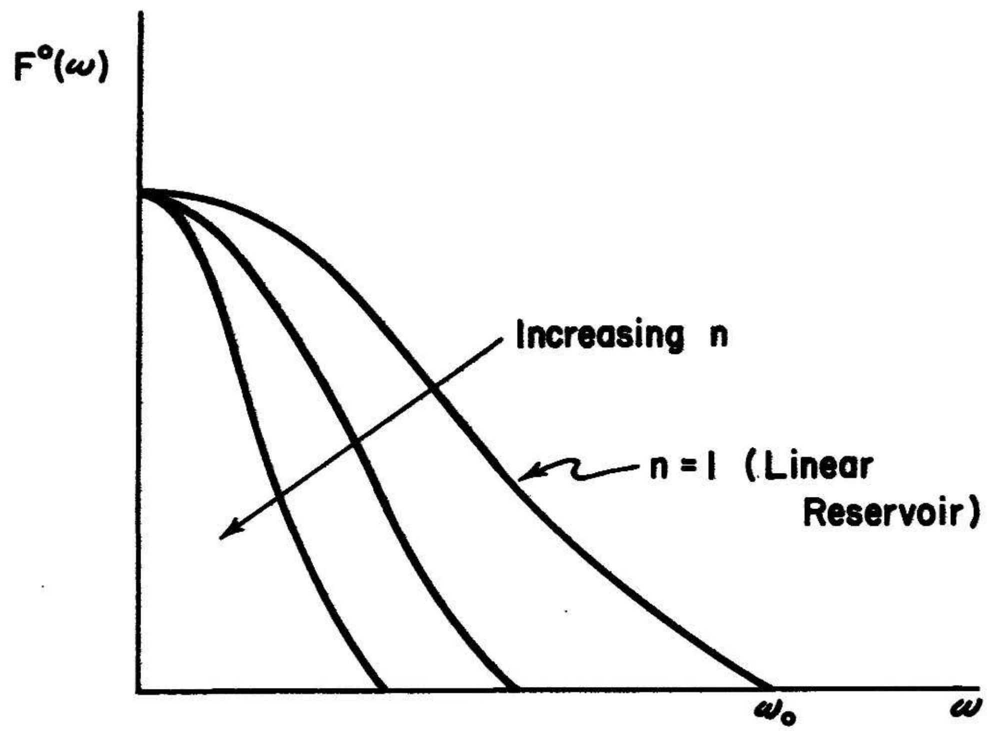


Figure III-5

Normalized Amplitude Spectrum Relationships for Nash Model which is

$$1 - e^{-t/K} = .99 \qquad \text{III-17}$$

so

$$T_r = -K \cdot \ln (.01)$$
$$\qquad\qquad\qquad \text{III-18}$$
$$= 4.605\ K$$

Again, due to the difficulty of integrating some of the more complicated response functions, it was decided to use the linear reservoir criteria with other linear models.

If this procedure were followed for the more complex system response models, it would be found that the integration increases in complexity. To alleviate this problem, the 'lag' of the complex system responses was used in place of the lag for the linear reservoir response in Equation III-18. Although a large part of the area under the linear reservoir response curve is concentrated at the origin, this procedure provides a conservative but efficient solution for the response time period. The results of this procedure when applying a Nash Model response is found in Appendix B-1.

III-4.4 Selection of the Output Period

As mentioned in Section III-2.2 the selection of FOURT for calculating the Fourier transformations required choosing a time period representative of the output hydrograph. The response and input functions are then obliged to have the same period, so zeros must be added in order to extend these functions to the required time period. The algorithm utilized to do this is the following:

The duration of the output, T_o, according to convolution theory, is the sum of the input duration, T_i, and response duration T_r

so $$T_o = T_i + T_r \qquad \text{III-19}$$

where T_r = the duration of the response to the system as given in Section III-4.2

T_i = the time of input duration.

Let N_i be the number of points in the input given at intervals Δt_i.

$$N_i = {}^{T}i/\Delta t_i + 1 \qquad \text{III-20}$$

Let N be the total number of points in the output which must be at the same interval, Δt_i, as the input, then

$$N = \frac{T_o}{\Delta t_i} + 1$$

$$= {}^{T}i/\Delta t_i + {}^{T}r/\Delta t_i + 1 \qquad \text{III-21}$$

$$= {}^{N}i + {}^{T}r/\Delta t_i$$

Therefore the number of zeros to be added to the input function is given by:

$$\text{Number of zeros} = T_r/\Delta t_i \qquad \text{III-22}$$

The time period required for the input, response and output functions must be defined by the period $T_r + T_i$. Since the system responses will not be utilized in the time domain but only in the forward transform, this procedure will transpose the time period into the number of input values as required by FOURT. For a further discussion, refer to Section III-2.2.

III-4.5 An Example - The Theoretical Solution

Having solved the implementation problems, an example was tried. In order to have a basis for comparison, the theoretical solution of the example is obtained initially.

The example utilized was the response of a linear reservoir to a square wave input of amplitude I_o and duration T_i.

The output function for this example can be found by convolution. The convolution integral for this example is:

$$f_o(t) = 1/K \int_o^t I_o \, e^{-\left(\frac{t-\tau}{K}\right)} \, d\tau \qquad \text{III-23}$$

This can be divided into two regions:

$$f_o(t) = \frac{1}{K} \int_0^t I_o \, e^{-\left(\frac{t-\tau}{K}\right)} d\tau \quad \text{For } t \leq T_i$$

III-24

$$= [I_o \, (1 - e^{-t/K})]$$

and

$$f_o(t) = \frac{1}{K} \int_0^{T_i} I_o \, e^{-\left(\frac{t-\tau}{K}\right)} d\tau \quad \text{For } t > T_i$$

III-25

$$= I_o \, e^{-t/K} \, [e^{T_i/K} - 1]$$

The same result can be reached by finding the Fourier transform of the input and the response functions and multiplying them together. The Fourier transform of the input is given by:

$$F_i(\omega) = \frac{1}{2\pi} \int_0^{T_i} I_o \, e^{-j\omega t} \, dt$$

$$= I_o \frac{1}{2\pi} \left[\frac{e^{-j\omega t}}{-j\omega}\right] \Bigg|_0^{t_i} \qquad \text{III-26}$$

$$= \frac{I_o}{2\pi(j\omega)} \quad (1 - e^{-j\omega T_i})$$

The Fourier transform of the response function is defined as:

$$H(\omega) = 2\pi \left(\frac{1}{2\pi}\right) \int_o^{\infty} \frac{1}{K} e^{-t/K} e^{-j\omega t} dt$$

III-27

$$= \frac{1}{1+j\omega K}$$

The Fourier transform of the output function is obtained by multiplying these two results, i.e.

$$F_o(\omega) = F_i(\omega) \cdot H(\omega)$$

III-28

$$= \frac{I_o}{2\pi} \left[\frac{1}{j\omega - \omega^2 K}\right] (1 - e^{-j\omega T_i})$$

III-4.6 Obtained Results

A computer program was written to test the example discussed in Section III-4.5. The program used as input, the desired forcing function and the parameter, K, describing the linear reservoir model used. The program obtained the desired Nyquist frequency, ω_o, the corresponding time intervals at which the input must be given, $\Delta t=\pi/\omega_o$, and interpolated the input to the desired time interval if not given at that Δt. The program also evaluated the theoretical Fourier transform of the response function, found the transform of the input by using FOURT, multiplied them together, found the inverse of the resulting transform using FOURT to obtain the output function and finally plotted the resulting data together with the theoretical result. A copy of the program is included in Appendix A-1.

The program was tested with a square wave input having a maximum value of 3.0 and duration of 2 time units. The linear reservoir model used a parameter K equal to 1.5. The plot of the resulting output function and its theoretical value can be seen in Figure III-6. As shown, the results are extremely accurate. The program with all the plotting, interpolating etc., took 2.67 seconds to execute on the I.B.M. 360-67 computer system.

It is interesting to note that even though FOURT forward transforms showed marked aliasing effect (Figure III-4) and also failed to reproduce, exactly, the correct function when finding the inverse of a transform that was not its own (Figure III-3), that when the inverse of the forward transform is taken, after convolution, resulted in such an accurate solution. This is due to the fact that the

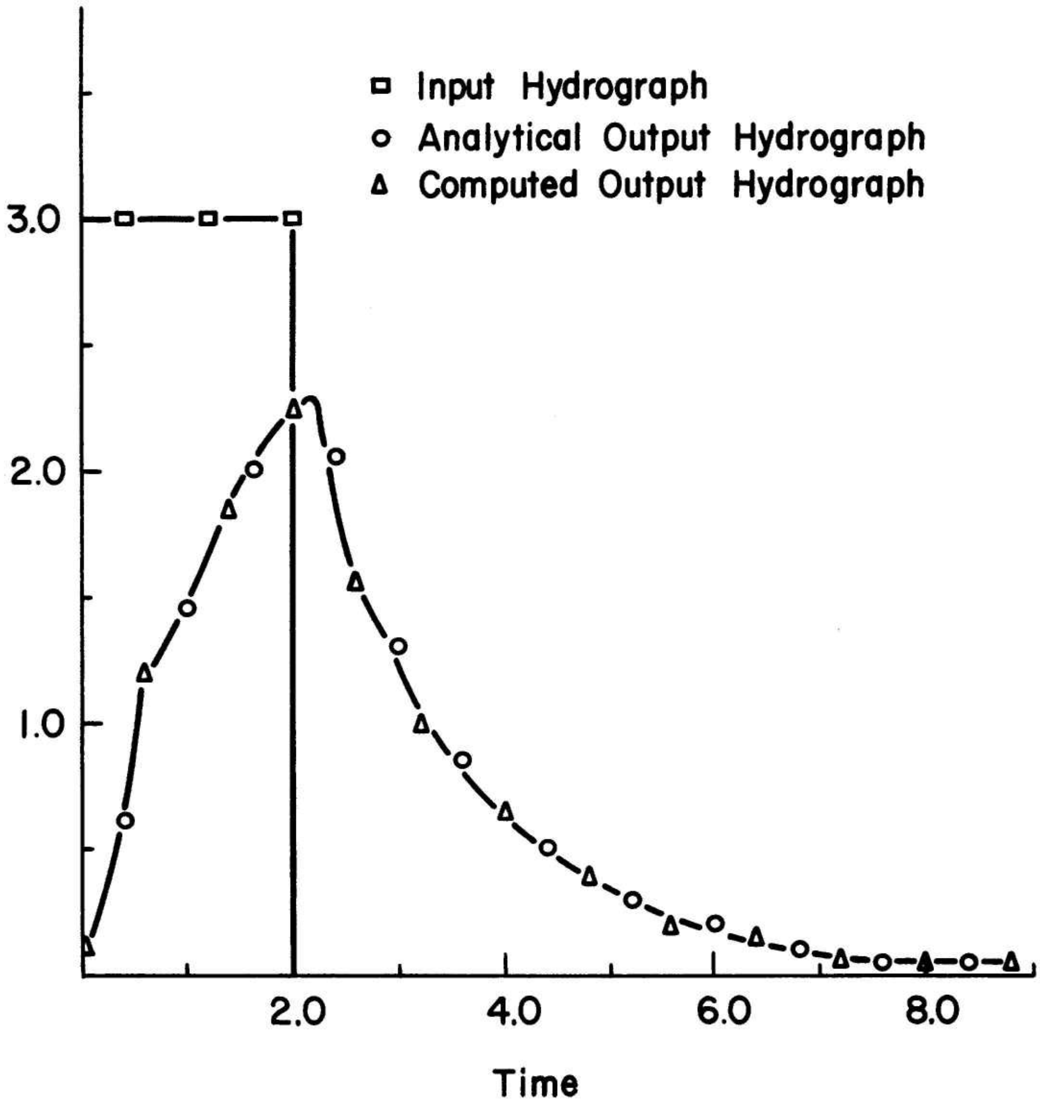


Figure III-6

Outflow Hydrograph of a Constant Inflow Convoluted with a Nash Model Response

response is most sensitive to the low frequencies. In the areas of low frequencies the aliasing effect and the error in finding an inverse of a non FOURT transform are minimal.

III-5 Application to Surface Routing

Harley [1967], in part, utilized two parameter simulation models to represent flood routing in open channels. This he accomplished through simplifying assumptions of the various complexities in the system. These complexities he listed as being in the field of physics, geometry and inflows. The complex physics was satisfied by taking the complete equations for open channel hydraulics; the complex geometry was handled by assuming a uniformly wide rectangular chezy channel; and lastly, linearization provided the means of simplification of complex inflows. This means that the response of the channel may be characterized by the response to a delta function.

In order to relate the complete linear equation to the parameters of the simplified two and three parameter models, Harley compared the cumulants or moments, these being the Lag, M_1, the variance, M_2, the skewness, M_3 and the Kurtosis, M_4. The parameters estimated by this method are expressed in terms of the hydraulic parameters of the original channel while in its reference steady - state condition. The lag, M_1, was equated to the first moment about the origin and the variance, M_2, was equated to the second moment about the center, these being derived from field data.

In the derivation of these parameters, Harley used the cumulants in the dimensionless form known as the shape factors. These are:

$$
\begin{aligned}
S_1 &= C_1 \\
S_2 &= C_2/C_1^{\,2} \\
S_3 &= C_3/C_1^{\,3} \\
S_4 &= C_4/C_1^{\,4}, \text{ etc.}
\end{aligned}
\qquad \text{III-29}
$$

This in effect, removes the time scale effect from the second and following cumulants, making them dimensionless.

The two linear models that will be used to represent the system response of Harley's complete linear channel equation will be the two parameter models, Lag and Route Model and Nash Model.

The properties of the moments with respect to the distribution are:

M_o = Area

M_1' = Lag (or mean), with respect to the origin.

M_2 = Variance, i.e. measure of dispersion of the distribution about the mean.

M_3 = Skewness, i.e. measure of the shift of the peak from the midpoint of the time axis of the distribution.

M_4 = Kurtosis, i.e. measure of the peakness of the distribution.

Harley notes that the cumulants except the first, are invariant under a change of origin and are expressed in the relation:

$$\text{Exp} \quad (C_1 t + C_2 \frac{t^2}{2} + \ldots . C_n \frac{t^n}{n} + \ldots .)$$

$$= 1 + M_1' t + M_2' \frac{t^2}{2} + \ldots . M_n' \frac{t^n}{n} + \ldots . \qquad \text{III-30}$$

$$= \int_{-\infty}^{\infty} e^{tx} \, dF$$

and may be related to the moment by:

$$C_1 = M_1'$$
$$C_2 = M_2$$
$$C_3 = M_3 \qquad \text{III-31}$$
$$C_4 = M_4 - 3M_2^2$$

He goes on to show that the cumulants for the complete solution of the linear channel equation are:

$$C_1 = \frac{2}{3} \frac{x}{V_o} = \frac{x}{1.5 \, V_o}$$

$$C_2 = \frac{2}{3} (1 - \frac{F^2}{4}) (\frac{y_o}{S_o x}) (\frac{x}{1.5 V_o})^2$$

III-32

$$C_3 = \frac{4}{3} (1 - \frac{F^2}{4}) (1 + \frac{F^2}{2}) (\frac{y_o}{S_o x})^2 (\frac{x}{1.5 V_o})^3$$

$$C_4 = \frac{40}{9} (1 - \frac{F^2}{4}) (1 + \frac{11}{20} F^2 + \frac{1}{4} F^4) (\frac{y_o}{S_o x})^3 (\frac{x}{1.5 V_o})^4$$

where V_o = mean velocity in the steady state

F = Froude number

y_o = depth of water in the steady state

S_o = slope of the channel bottom.

These then can be related to the cumulants of the systems that are selected to simulate the channel routing of the flood wave. The cumulants for the two parameter models, the Lag and Route and the Nash, are presented in Section II-6.

The parameters and conditions that Harley used in the Channel Routing are as follows

Flow conditions:	Reference Discharge	=	150.0	cfs.
	Reference Velocity	=	4.14126	ft/sec.
	Reference Celerity	=	6.21189	ft/sec.
	Reference Depth	=	36.22086	ft.
	$D (S_o x/y)$	=	5.52167	

Channel Parameters:	Length	=	200 miles.
	Slope	=	1.0 ft/mile.
	Friction coefficient	=	50.0 (Chezy).
	Froude No.	=	0.12129.

This configuration yields the following cumulation and shape factors

S_1 = 47.22126 hrs

S_2 = 0.120292

S_3 = 0.0438913

S_4 = 0.0265171

Inflow Parameters:

Type = Thomas Wave

$$[q\ (0,t) = \frac{q\ max}{2}\ (1 - \cos(2\pi\ t/f)]$$

where f = wave frequency of recurrence

q max = Maximum flow over t

with

Time to peak = 48 hrs.

Peak discharge = 200 cfs.

Base discharge = 50 cfs.

By relating the shape factors to the cumulants, the parameters of the linear simulation models can be related to the hydraulic characteristics. For instance, in the case of the Nash Model we have:

$K = 5.68034$

Thus $n = 8.31309$

The same procedure can be used to obtain the parameters for the Lag and Route Model yielding

$K = 16.37782$

$\tau = 30.84344.$

Obviously, one must be careful of placing physical significance to these parameters. For instance, in the case of the Nash parameter, n, which represents the number of linear reservoirs, this requires a fractional reservoir thus pointing out the discrepancy between the physical significance and the parameter.

The program used to calculate the output hydrograph by means of the harmonic analysis may be found in Appendix A-1. Designed for use on the IBM 370-155 computer system the program uses less than 120K of core. If the program was compiled (machine Language), and required a plotted output, the results would be obtained in 2.64 secs. for the Lag and Route Model and 3.9 secs. for the Nash Model.

The results are plotted in Figure III-7 and III-8. Comparing the results obtained by Harley of the linear solution together with those obtained using the FFT approach, Figure III-7, indicates the results of the Lag and Route Model to be as good as or better than Harley's solution, resulting in an RMS error of 0.00638 or better.

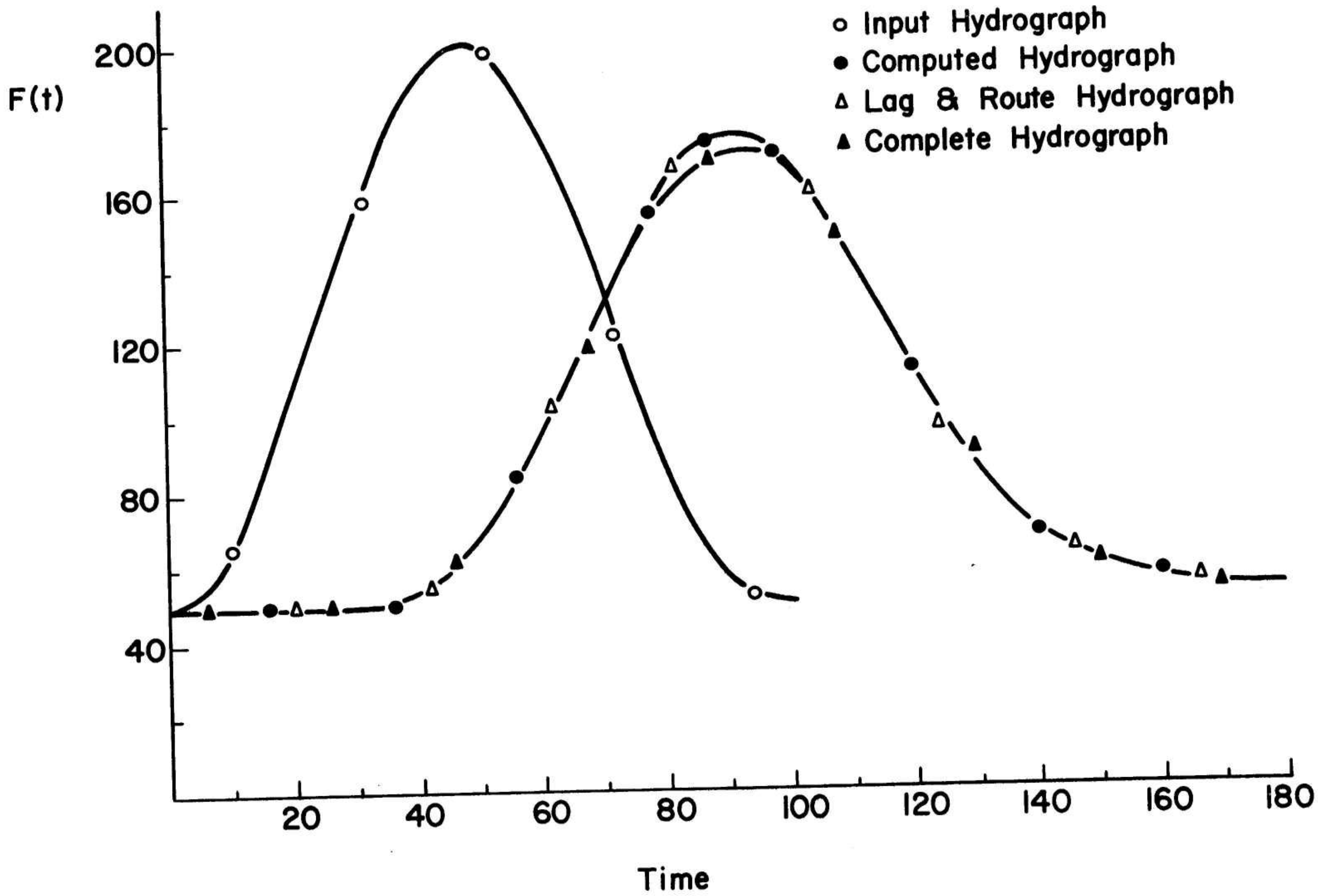


Figure III-7

Outflow Hydrograph of a Lag and Route System Response-FFT Technique

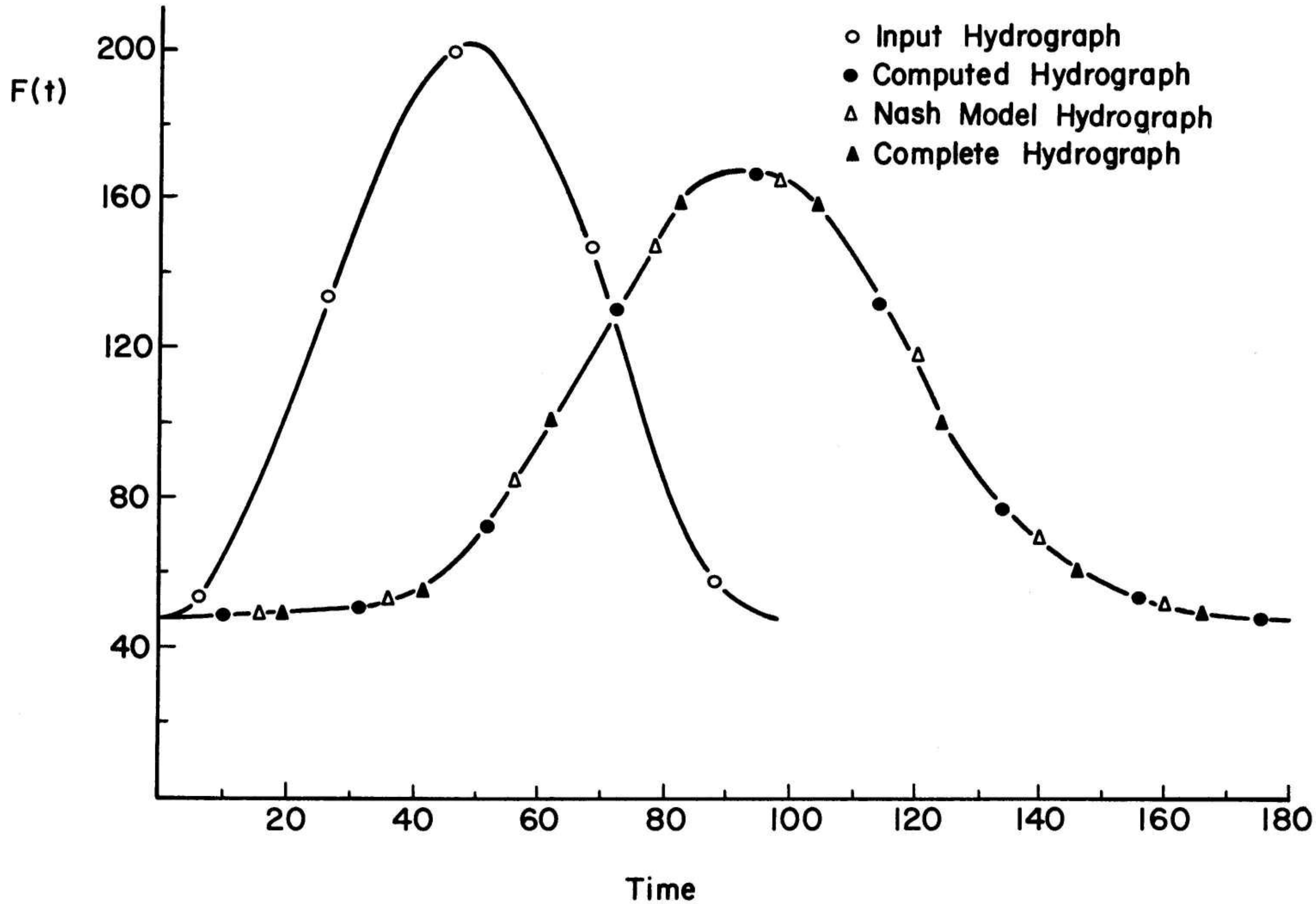


Figure III-8

Outflow Hydrograph of a Nash System Response-FFT Technique

Figure III-8, indicates the results of the Nash Model also resulting in RMS error as good as or better than 0.00089 for Harley's solution.

Chapter IV

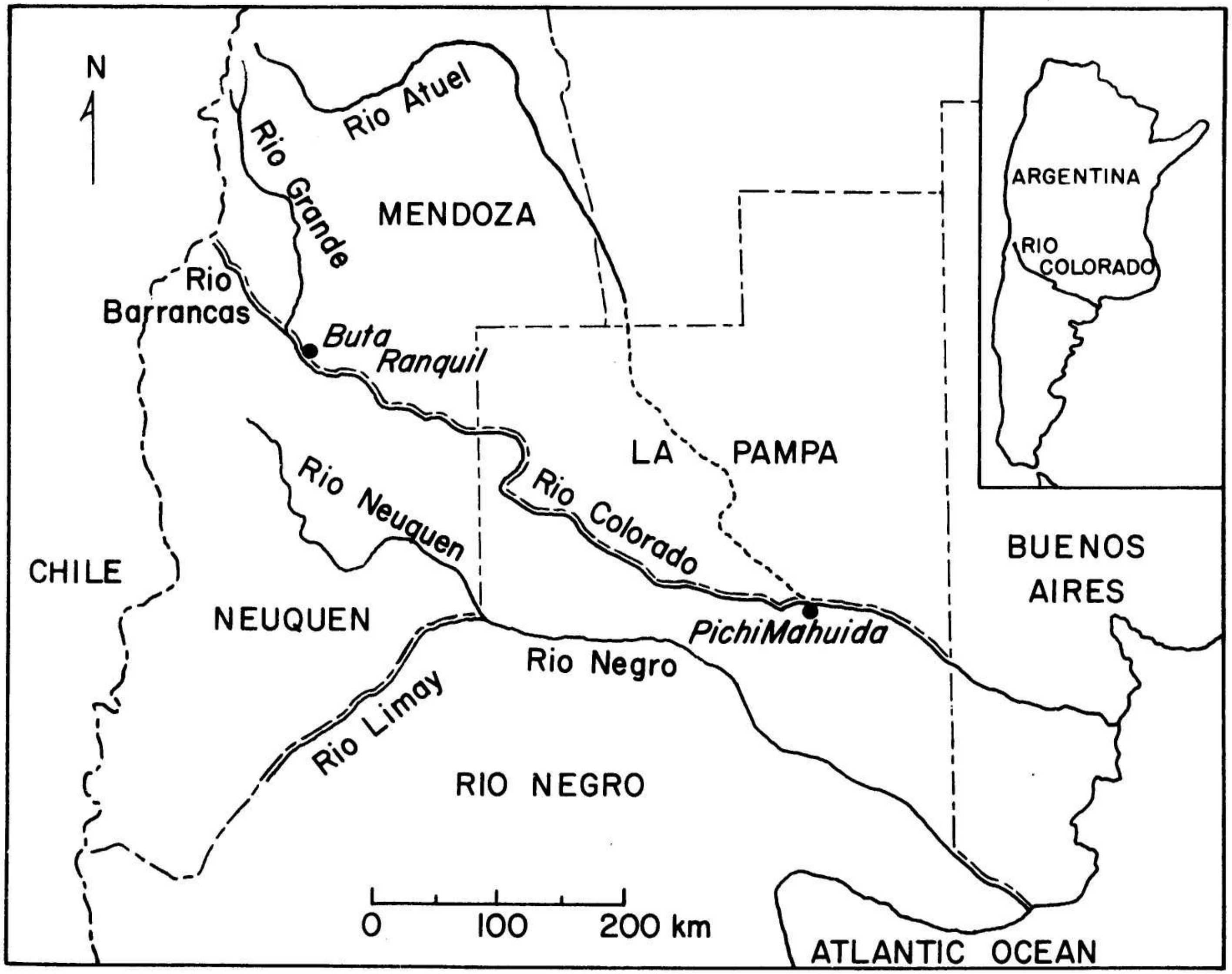
APPLICATION TO A REGIONAL RIVER BASIN

IV-1 Discussion of the Selected Regional River Basin

The river basin selected for study in this chapter is the Rio Colorado River Basin in Argentina where water is supplied almost entirely by the melting snows of the Andes Mountain Range. The tributaries that drain the catchment area within the Andes are: the Rio Grande, the Barrancas, the Arroyo Butaco and the Arroyito Chachaico - Buta Ranquil. The Rio Colorado then, carries these waters from the Andes (about 72° west longitude) to the Atlantic Ocean (about 62° west longitude) flowing through the great Patagonian Plain which consists primarily of sedimentary material, Figure IV-1. Very little precipitation occurs in the central region located in La Pampa province. The mountainous region receives the greatest precipitation, mostly as snow. The third region is the Eastern Coastal Region in the province of Buenos Aires which has considerable vegetation due to the moderate precipitation and temperate climate.

IV-1.1 Geology and Soil Description within the River Basin

The geology in the head regions of the Rio Colorado is a



<u>Figure IV-1</u> Rio Colorado River Basin, Argentina

conglomerate of three primary formations, these being tertiary, cretaceous and jurassic. The tertiary consists of upper continental deposits, basalts, undifferentiated eruptive rocks, acid and mesosilicic intrusive facies, lower continental deposits and marine deposits. The cretaceous era consists of marine deposits, continental deposits and marine & continental deposits, while the jurassic era consists of marine & continental deposits and marine deposits. Below the headwaters of the Rio Colorado the basin dates almost entirely to the quaternary era consisting of glacial and fluvioglacial continental deposits, marine deposits, basalts and other undifferentiated volcanic rocks. There are outcrops of the tertiary, cretaceous and precambrian eras, also.

The three regions, mountainous, central and eastern may be used in segregating the soil types. The mountainous area is predominately fine sandy soil and due to the lack of organic material, provides a rapid infiltrating system. The more complex central region may be divided into three major categories the first being a mantle of sand as found in the mountainous region, then narrow beds of non-consolidated river pebbles or gravels and lastly, two horizons with the upper one consisting of a sandy silt material low in organic material and therefore being adequately drained, and the lower comprised of loam and clay or a silty loam which tends to retain the salts lost from the upper horizon through leaching. The lower horizon sometimes appears on the surface due to the erosion of the upper. A third material which is found in the eastern region of the La Pampa district is a hard pan material called 'Tosca' which restricts both infiltration and root passage. The eastern region which has soils that are suitable for cultivation, has two soil horizons as well, the upper being a sandy or sandy loam, while the lower is sandy with fine silts held together by a calcareous cement. With the high rate of irrigation and moderate infiltration rate in this area, the water table has risen, thereby increasing the salinity in the upper horizons. Hard pan material has also been found in this region.

IV-1.2 Hydrologic and Agricultural Discussion

The Rio Colorado begins at an elevation of 4,800 m at the source of the Rio Grande, and flows to the Atlantic Ocean over a distance of 750 Kilometers. In the upper reaches, the river averages a slope of 2 - 0.4 m/km but decreases to an average of 0.4 m/km in the lower reaches. Due to the sediment transport capacity of the river and the small slope in the eastern region, a delta was formed. The average flow at Buta Ranquil is 143 m^3/sec. but has a recorded minimum flow of 44.0 m^3/sec. and a maximum flow of 678 m^3/sec..

The subsurface conditions have not been thoroughly studied within this river basin. The trend in the Buenos Aires region tends to show that basalt fractures allow the water to enter into a deeper zone. In the western portion of the La Pampa province a shallow bedrock situation produces a high water table that is found to have an elevation of about 2 meters below the surface. This bedrock strata is lower in the eastern sector of the province, thus lowering the water table.

Agriculturally, the region is sparse but there are plans to develop irrigation sites along the river. Generally, the production consists of 70% alfalfa, 19% vegetables and the rest fruit or other minor crops. The general procedure for irrigation of an agricultural cultivated area is to provide enough water so as to meet the consumptive use requirment for the crops as well as the leaching requirment for the soil. In the Rio Colorado Basin the growing season spans an eight month period therefore there is a four month period when there are no water requirements. This procedure presents a cyclical water demand very similar to 1/2 of a sine wave. For the purpose of this work it is assumed that the leaching demand is met with the water reaching the phreatic zone in the same distribution pattern.

IV-2 Conceptual Discussion of a Regional Groundwater Routing Model

With the background having been discussed in the above sections, the logic behind a regional groundwater model may be discussed. There are three areas of interest in the groundwater area:

a) A quick responding system which exists in the upper root zone where a condition of interflow may occur.
b) A rapidly responding system representative of the shallow zone in the soil structure, resulting from a drainage system in an irrigation site or as a natural phenomena.
c) A slowly responding system that exists in the deep reaches of the soil structure (deep water zone), which may represent the flow of groundwater in an aquifer laterally to a river or parallel to the river interacting with the river at some distance further down stream. The conceptual logic here may be seen in Figure IV-2, which is a profile of the eastern portion of the basin. Here a loss to the river resulting from a zone of higher permeability, over which the river flows, will, possibly, provide additional groundwater flow in a direction parallel to or diverging from the river due to the geological structure of the soil. Or a groundwater flow may result in an old river bed after that river was diverted for other purposes or possibly as a result of a change in the bedrock strata. Figure IV-1, indicates a tributary that no longer provides flow to the Rio Colorado due to a diversion in an upstream province.

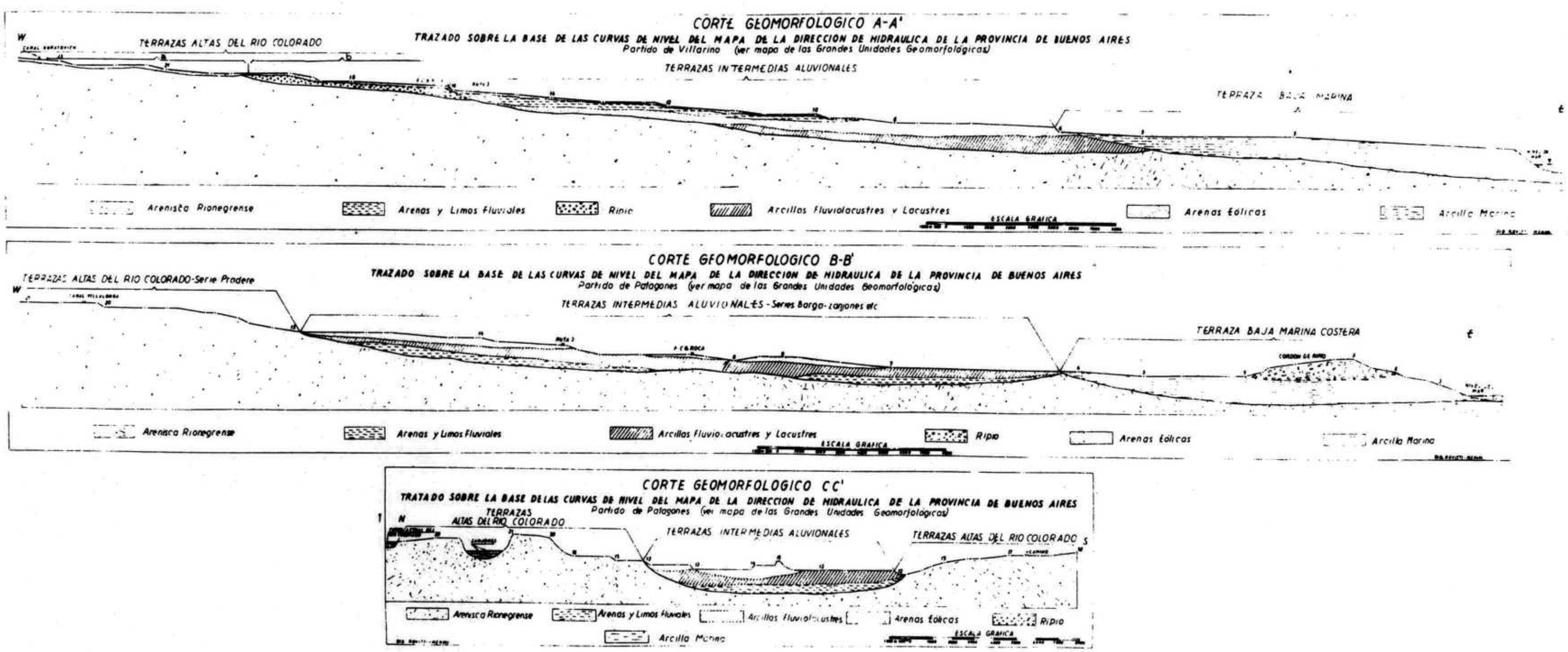


Figure IV-2

Profile of the Eastern Sector of the Rio Colorado River

A conceptual schematic of a groundwater routing model for an irrigation site is shown in Figure IV-3, which also indicates some typical linear model for each process.

IV-3 Model Development

The model which will be developed in this section is similar to one suggested by Diskin [1964], where he distributed his input into two parallel Nash chain series thus providing a more flexible system response. The block diagram used in this model is shown in Figure IV-4.

IV-3.1 General Linear Groundwater Routing Model

It was found in Chapter III that the use of the Fast Fourier Transform in Harmonic Analysis for convoluting an input with the system response to a delta function was not only fast but also very accurate in representing an output hydrograph. The parameters used for the system response in Chapter III were derived through the use of moments and the general governing equations for open channel flow.

In a groundwater regime, there are many complex processes that can never be completely understood either by reason of mathematical theory or by the many unknowns in the subsurface zones such as cracks, fissures, non-homogeneous and non-isotopic conditions. Thus a method was needed that would be fast and represent, to the best available means, the response of such a subsystem. Considerable work

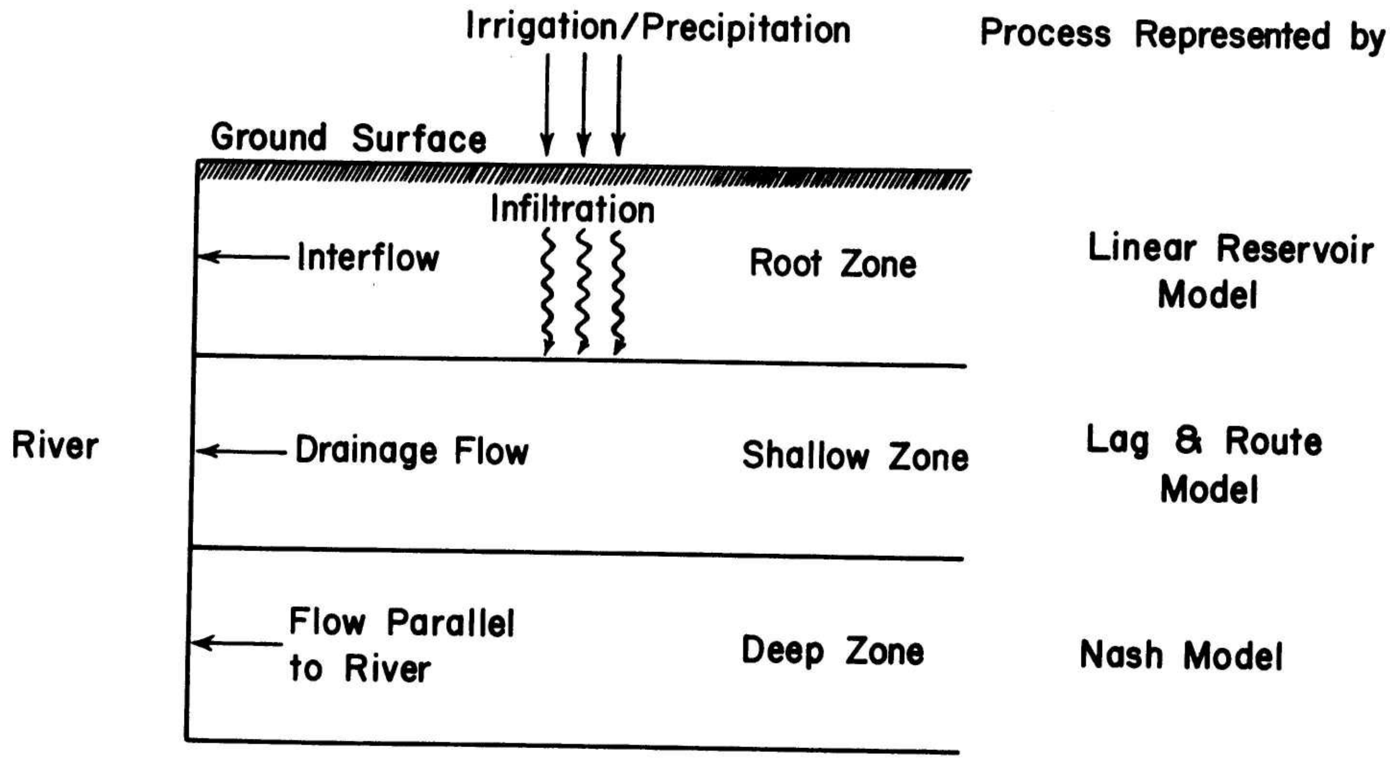
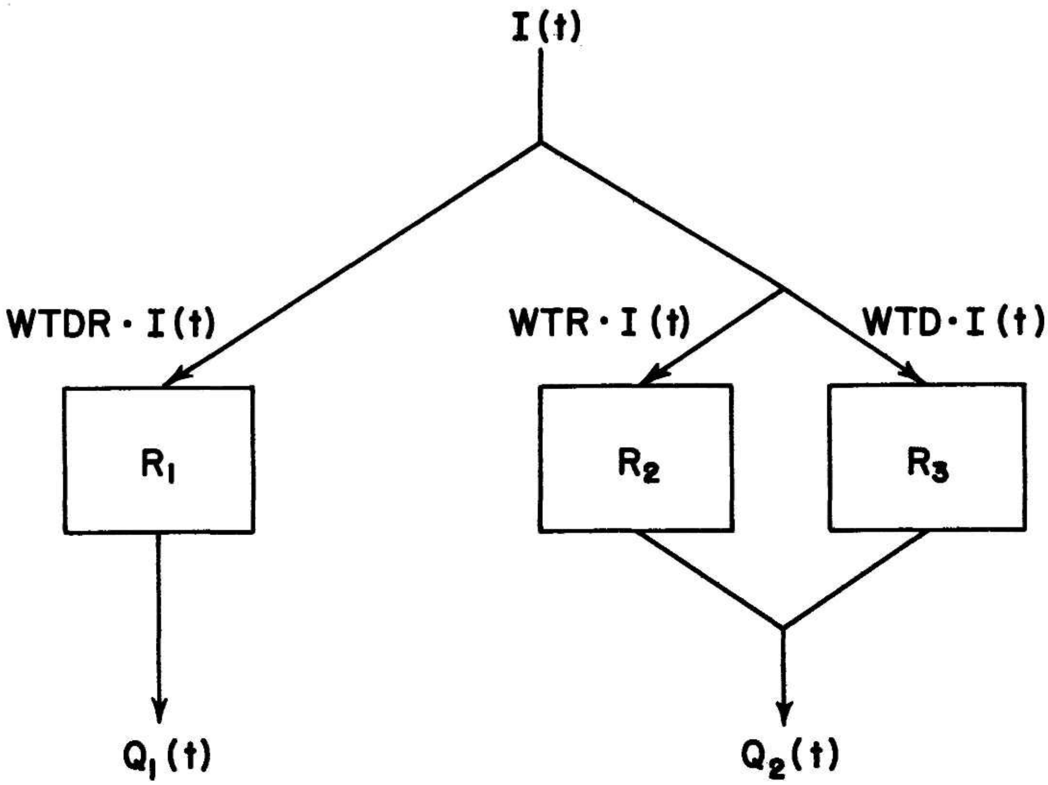


Figure IV-3

Conceptual Groundwater Routing Model



$I(t)$	Inflow to Groundwater System
R_1	Slow Response System for Flow Parallel to the River
R_2	Medium Response to River - Aquifer Relationship
R_3	Rapid Response to Drainage System
WTDR, WTR, WTD	Weight of Distribution, Where the Sum Totals 1.0
$Q_1(t)$	Outflow at Some Point Downstream
$Q_2(t)$	Outflow Resulting from Groundwater Flow and Drainage, Laterally at the River

Figure IV-4

Block Diagram of a Regional Groundwater Routing Model has been done in understanding flow through porous media either theoretically or using an empirical systems approach. The Dupuit approximation and Darcy's Law have both been used in finite difference schemes to determine what might be expected in groundwater movement. Dooge [1960], used a linear system approach to develop coefficients that could be used to represent flow into and out of a river system. This was discussed in Section II-6, Parameter Estimation. As this uses the requirement of constant recharge over the time period, it is felt that this is too restrictive. Therefore, in an attempt to avoid the constant coefficient concept as presented by Dooge and O'Donnell (discussed in Section II-6), a procedure similar to that used by Harley [1967] for open channel flow will be considered here for groundwater flow.

To represent the various processes that might take place, as in a shallow water table with flow to drainage ditches as well as a deeper system that would provide for flow to a river, lake or ocean, we must be able to use various system responses to the same delta function. Conceptually, this might require the three known system responses, the Nash, Lag and Route and Linear Reservoir Models, to act in series, in parallel or in a complex configuration of both. With an increasing complexity of the system response, in order to prevent a decrease in efficiency of the program mentioned in Chapter III, a procedure will be used to conserve the time of computation. As an example which will be considered later, a Nash Model is used in series with a Lag and Route Model. We would expect the time required for computation to be 4.6 seconds plus 3.9 seconds or a total of 8.5 seconds, thus reaching an infeasible point in cost control due to a large requirement for computer time.

Consider the prospect of manipulating the inputs and system responses in the frequency domain for the entire record of interest before using the Fast Fourier Transform, F.F.T., program to re-enter the time domain.

It was shown in Chapter III that the frequency response to a Dirac delta function for a linear reservoir is simply

$$H(\omega) = \frac{1}{1+j\omega K} \qquad \text{IV-1}$$

where K = linear reservoir time constant

ω = angular velocity

j = $\sqrt{-1}$

Since the Fourier Transform can be obtained by the relation:

$$F(\omega) = \frac{1}{2\pi} \int_0^\infty e^{-j\omega t} f(t)\, dt \qquad \text{IV-2}$$

then the frequency response for the Lag and Route Model is simply

$$H(\omega) = \exp(-j\omega\tau) / (1 + j\omega K) \qquad \text{IV-3}$$

By the same procedure the Fourier transform for the Nash Model is:

$$H(\omega) = (1 + \omega^2 K^2)^{-n/2} \exp(-jn \tan^{-1}(\omega K)) \qquad \text{IV-4}$$

Since the Nash Model is a series of n linear reservoirs, let us put this transform into a more suitable form. The frequency transform for a Nash Model may be represented as:

$$H(\omega) = \left(\frac{1}{1+j\omega K}\right)^n \qquad \text{IV-5}$$

or more illustratively by:

$$H(\omega) = \underset{1}{\left(\frac{1}{1+j\omega K}\right)} \underset{2}{\left(\frac{1}{1+j\omega K}\right)} \underset{3}{\left(\frac{1}{1+j\omega K}\right)} \cdots\cdots \underset{n}{\left(\frac{1}{1+j\omega K}\right)} \qquad \text{IV-6}$$

This is equally applicable to unequal linear reservoirs which would be represented by:

$$H(\omega) = \left(\frac{1}{1+j\omega K_1}\right) \left(\frac{1}{1+j\omega K_2}\right) \cdots\cdots \left(\frac{1}{1+j\omega K_n}\right) \qquad \text{IV-7}$$

The important point shown above is that if we have a series of individual components that represent the system all we need to do is simply multiply the Fourier Transforms together to get the transform of the whole system.

If we have a situation where the system response is to be represented by parallel responses instead of in series, as shown in Figure IV-5 we know from Chapter III, that the Fourier Transform for the outputs, Q, can be determined in the frequency domain by multiplying the input transform, $I(\omega)$, with the response transform, $H(\omega)$, or:

$$Q_1(\omega) = H_1(\omega) \cdot I_1(\omega)$$

$$Q_2(\omega) = H_2(\omega) \cdot I_2(\omega) \qquad \text{IV-8}$$

The inverse Fourier Transforms are given by:

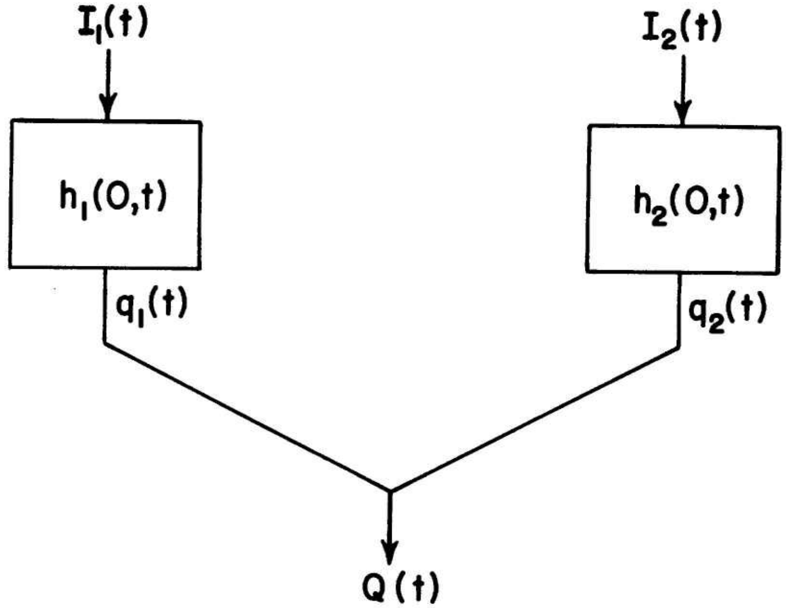
$$q_1(t) = \int_0^\infty Q_1(\omega)\, e^{j\omega t}\, d\omega$$

$$q_2(t) = \int_0^\infty Q_2(\omega)\, e^{j\omega t}\, d\omega \qquad \text{IV-9}$$

but as shown by Figure IV-5

$$Q(t) = q_1(t) + q_2(t)$$

$$= \int_0^\infty (Q_1 + Q_2)\, e^{j\omega t}\, d\omega \qquad \text{IV-10}$$



$I_i(t)$	Inflow Rate to System i
$h_i(O,t)$	Instantaneous Unit Hydrograph for System i
$q_i(t)$	Outflow Rate from System i
$Q(t)$	Total Outflow

Figure IV-5

Parallel Systems

therefore:

$$Q(\omega) = Q_1(\omega) + Q_2(\omega) \qquad \text{IV-11}$$

Thus by merely adding the Fourier Transforms of the Output, Q, the total output hydrograph in the frequency domain can be obtained. Therefore, with only one operation it is possible to return the total output hydrograph to the time domain.

Keeping these concepts in mind and returning to the program discussed in Chapter III, a relatively simple and efficient model may be generated to determine an output hydrograph to a groundwater routing model.

The input to such a model may be whatever one desires to represent the inflow to a groundwater system, using possibly a different input for each leg of a parallel system or one input segregated, by weights, for each leg of the system.

There is a restriction however to the use of the F.F.T. program (see Appendix A-2), that is, as discussed in Chapter III, the number of points that enter the transform program is the same number as the points that are returned from the program. Therefore, it is desirable to use the total time period initially. Then by simply adding zeros to the shorter 'legs', the time period for each parallel segment, can be brought up to the necessary time period. There are two important considerations to be looked at here, the time period to use and the Nyquist frequency, ω_o, that must be used. As indicated in Chapter III, these are both dependent on the system 'lag'.

The system lag must be considered by the three possible conditions, whether it be a series system, a parallel system or a combination system. The lag as shown in the last chapter is represented by the first moment about the origin, or for the three cases are:

$$\text{Linear Reservoir} = K_{LR}$$

$$\text{Lag and Route} = \tau + K_L \qquad \text{IV-12}$$

$$\text{Nash} = nK_N$$

Thus if all three models were to be in series, the system lag, K_s would be represented by:

$$K_s = nK_N + (\tau + K_L) + K_{LR} \qquad \text{IV-13}$$

However, if the system consists of parallel members then the greatest system lag K_s would be used to give the largest time period, or

$$K_T = \text{Max}(K_s) \qquad \text{IV-14}$$

In Chapter III, a method was developed that determined the minimum response time period for which 98% of the area under the distribution (or response) curve was assured. This method was based on the linear reservoir or the exponential distribution. Since we are now considering the Nash Model, the procedure for determining the time period must be altered. The significant point in such a determination was the integration of the response function. For the exponential distribution this was a simple task. However, the Nash Model being a form of the gamma distribution, proved to be more complex - not so much from the theoretical standpoint as the fact that computationally, the time would increase. A small program was generated to test the significance of such a computational scheme for use within the program. The test program and the results may be found in Appendix B-1. It was found that conservative results would be obtained for the time period of the system response if the lag (first moment) of the Nash Model, nK, was used in lieu of K as used in the computational technique for the Linear Reservoir case. Therefore it was unnecessary to change the procedure for determining the time period of the response of the system. Additionally, this required the determination of the greatest 'lag' in a series response system. By using the greatest 'lag' of a series system we were assured that the response time period would meet the requirements of the entire model and yet not prove to be extravangant on the time necessary to execute the model.

The second requirement is that the folding frequency, or Nyquist frequency, is great enough to retain most of the energy of the system in the frequency domain. As discussed in Chapter III, by using the Nyquist frequency for a linear reservoir, i.e. n = 1, most frequency requirements would be met for any of the three types of models. This was shown to be true by Figure III-5. Also by acknowledging the fact that as the number of reservoirs in series increase there is an increased damping mechanism which reduces the effect of the higher frequencies then a conservative estimate would be obtained if the Nyquist frequency were evaluated in the same manner as in the Linear Reservoir Model.

Three parameters that are used in the Groundwater Routing Model need to be discussed. These are the translational lag, τ, the effects of baseflow on a hydrograph, and the spacial parameter, WDTH. For the special configuration, as indicated by Figure IV-4, implemented into this model, the Lagged Nash Model was used thus requiring the use of a translational lag. As discussed in Section II-3, the translational lag will pass all frequencies. Therefore, no effect is produced on the outflow hydrograph except a shift in time. The model accounts for these lags by shifting the time array by the lag and adjusting the points to each system accordingly. However, it is the input hydrographs which are shifted by this time lag while in the time domain. If this were not done in the time domain, it would not be possible to sum two parallel legs that had different translational lags while in the frequency mode. As mentioned above no error is introduced by this procedure. The second parameter indicates a baseflow in the system. Normally this baseflow would be removed from the input hydrograph and added to the outflow. However, if the parameter estimation technique discussed in Section II-7 is used, this procedure requires the use of sloping bedrock and, in turn, has an advective velocity term incorporated. Therefore, only in the case where the parameters are input individually, may the baseflow parameter be utilized. The spacial parameter, WDTH, is used to transform the unit flow into total flow for the area considered. Therefore it represents the area over which the input exists.

The program listing for the Groundwater Routing Model may be found in Appendix A-3. A restriction in the model requires that all 'series' systems be computed by type of response model, i.e., all Nash response models will be computed before the next response model is considered in that series.

IV-3.1.1 Discussion of the Shallow Zone System Implemented into the Model

There is a theoretical problem encountered in the drainage process. The parameters used for the routing of groundwater laterally to the river as well as parallel to the river were determined by a procedure derived from a Dirac delta function. An irrigation area requires that water be applied over a large surface with respect to the travel distance to the drainage canals, thus extending the assumption beyond its practical limit. Therefore a finite difference scheme was selected to assist in deriving the parameters required by the Linear Routing Model.

IV-3.1.1.1 Drainage Spacing

The Bureau of Reclamation (R.D. Glover [1967]), developed an empirical relationship for relating the soil characteristics with the drainage canal spacing. Equation IV-15, represents this relationship, which was used in analyzing the irrigation sites in the Rio Colorado basin to determine the drainage canal spacing, L.

$$L = \left(\frac{2\pi \, K_p \, Y_{Max} \, DEPTH}{PERC_{Max}} \right)^{1/2} \qquad \text{IV-15}$$

where K_p = Permeability

Y_{Max} = Max lens height allowed above drainage ditch

$DEPTH$ = Depth to botton of drainage ditch from ground surface

$PERC_{Max}$ = Maximum percolated water over period of interest

By using this relationship and the available soil characteristics within the Rio Colorado basin, a mean drainage spacing of 50 meters was calculated and used with the finite difference scheme to generate an outflow hydrograph. This hydrograph was based on an input representative of the mean leaching requirements in the basin and a permeability of 47 meters/month.

IV-3.1.1.2 Finite Difference Scheme

The finite difference scheme uses Darcy's Law in conjunction with the continuity of flow equation. The resulting finite difference equation is given by Equation IV-16.

$$H_j^{t+\Delta t} - H_j^{t} = \sum_{i \varepsilon I_j} \frac{\Delta t \; a_{ij}}{Sc_j \; A_j} H_i^{t+\Delta t} + \frac{\Delta t \; Q_j^{t}}{Sc_j \; A_j} \qquad \text{IV-16}$$

where

a_{ij} ≡ Specific permeability (L^2/T)

≡ (Flow area x Permeability)/L

Sc_j ≡ Storage Coefficient, cell j (L/L)

Q_j ≡ Flow input into cell j (L/T)

H_i ≡ G.W. Elevation initially, cell i (L)

H_j ≡ G.W. Elevation initially, cell j (L)

HT ≡ Final G.W. Elevation (L)

I_j ≡ Index of cell adjacent to cell j

The flow across a boundary cell may similarly be determined to be:

$$QF(j) = \sum_{i\varepsilon j}^{N} a_{ij} (HT(i\varepsilon j,j) - HT(j)) \qquad \text{IV-17}$$

where QF(j) = Flow into boundary cell j

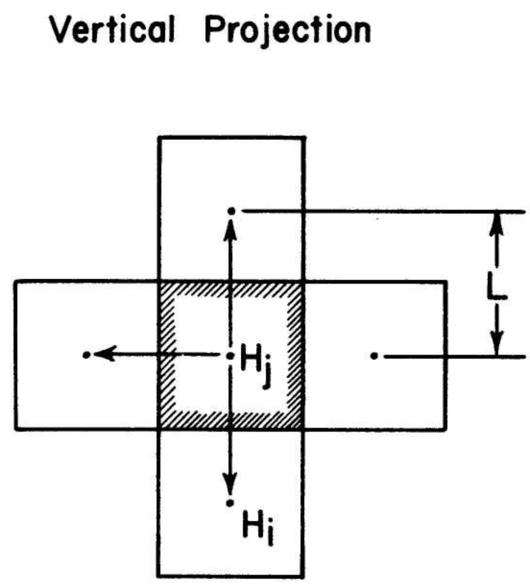
N = Number of cells adjacent to boundary

Figure IV-6, shows the physical interpretation of the parameters listed above.

A program was written to solve the simultaneous differential equations that are set up in matrix notation with the use of equation IV-16. The procedure used by the program in solving the simultaneous equations is similar to the Gauss-Jordan method of elimination. The time of execution is extremely fast for solving a matrix with a small numbers of nodes, however, this time increases exponentially with an increase in the number of nodes. Elinger [1972] uses this method to study the entire irrigation process, inclusive of a salinity analysis.

IV-3.1.1.3 Parameter Estimation with the Use of the Finite Difference Scheme

The configuration has two boundary cells providing the limits for three nodes used to represent the 50 meters drainage spacing. It was found that a steady state flow condition would be established



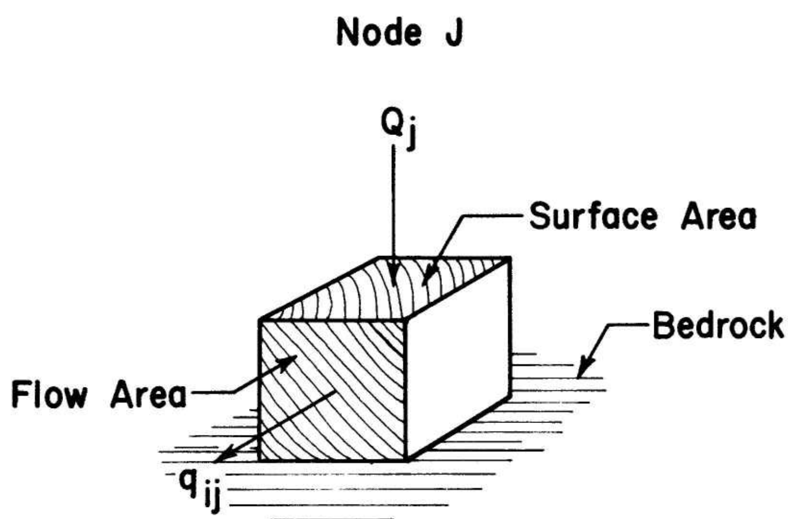


Figure IV-6

Physical Schematic of the Finite Difference Scheme after running the model for three years, or three irrigation cycles. A typical hydrograph was extracted from this steady state condition which was used, along with the Method of Moments, for estimating the parameters required by the Linear Groundwater Routing Model. Figures IV-7, 8, and 9, show the relationship between the parameters of the Nash and Lagged Nash models for the number of single reservoirs, n, the lag to a single reservoir, K, and the translational lag, τ, used in the Lagged Nash model, respectively. These parameters may be estimated from the figures and serve as input to the Linear Routing Model, if the conditions are such that the permeability is about 47 m/month, the drainage canals are about 3 meters deep and spaced about 50 meters. In this derivation it was assumed that the bedrock slope produced little or no effect to the results.

IV-4 Linear Groundwater Routing

Model Discussion

Since the parameter estimation procedure was developed with the assumption that a delta function serve as the forcing function, consideration must be given to the actual input to the system. If the input area is small with respect to the distance of flow to the river, then the errors of assumption in the derivation are acceptable. However, if the area of application is large, e.g., a large irrigation site or rainfall distributed over a wide area, with respect to the lateral distance from the river, then significant errors are intro-

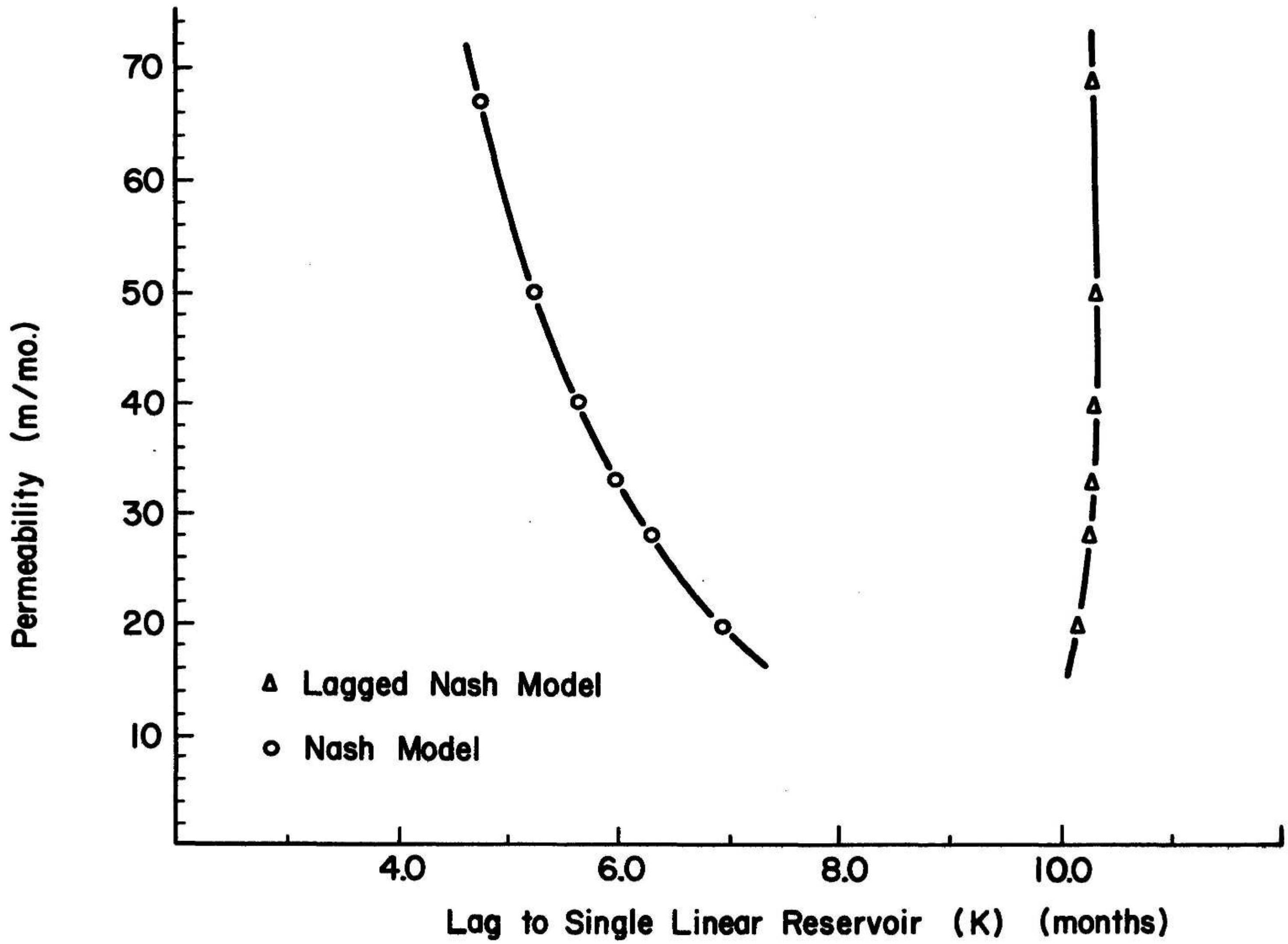


Figure IV-7

Nomograph for Parameter K-Lagged Nash and Nash Models

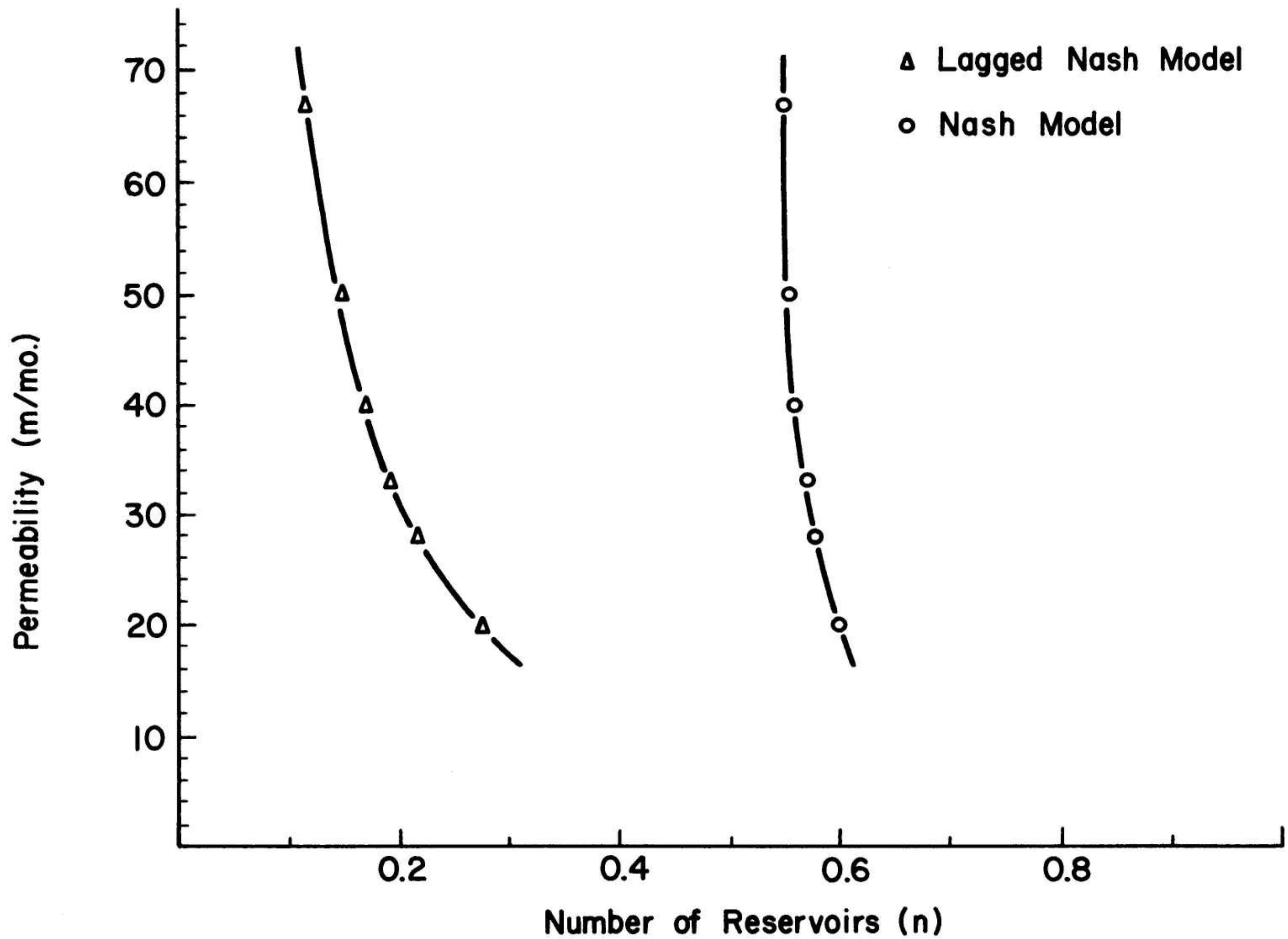
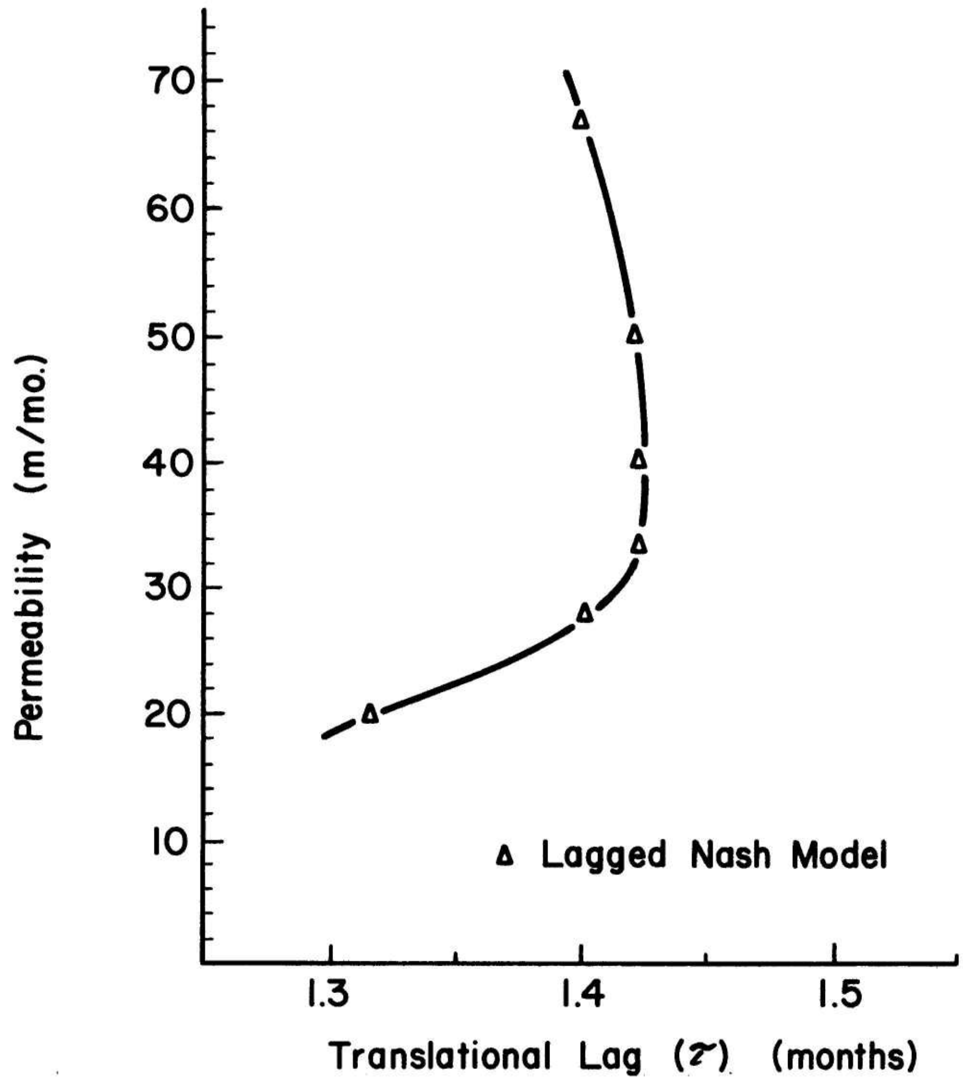


Figure IV-8

Nomograph for Parameter n-Lagged Nash and Nash Models



<u>Figure IV-9</u>

Nomograph for Parameter τ-Lagged Nash Model duced and a re-derivation under more applicable assumptions is desirable. For this report, it is assumed that the assumptions are indeed met.

So far the discussion has concerned only positive outflows from the aquifer to the river. Unless a condition exists that would never allow the reverse to take place, a rare situation, the more common circunstances would require a negative outflow due to a rising river stage or evapotranspiration. When an irrigation site is considered, normally drainage ditches are used to allow most of the leached water to return to the river via surface flows. The drainage ditches in turn restrict the height reached by the water table. Also, Philips [1957] shows that for bare dry soil (light clay) the evaporation loss to the water table, in terms of free water, will be about 1 centimeter per year when the water table is at a depth of 1.5 to 2.0 meters. Therefore the most important consideration to make, when analysing water losses to the water table would be the effects of evapotranspiration which might set up a negative recharge condition. This would depend on the type of crop with its depth of influence and consumptive use requirements. In the case of the Rio Colorado in Argentina the normal plant growth and thus the evapotranspiration is relatively small except in the irrigation sites. By considering the consumptive use of the crops within each site and the irrigation water applied, the leaching water may be determined which is, essentially, the water that will percolate to the groundwater system.

The other condition mentioned is a rise in the river stage that will increase the 'bank storage'. The negative outflow caused by such a situation could be determined by checking the river stage at each time step. If the stage is above the water table, then change the sign of the system response and the depth of the saturated flow, thus allowing flow to enter the groundwater zone. Obviously, engineering judgement plays a major factor in making a change such as this.

IV-4.1 Model Results

The model was used to analyse a situation when a single irrigation area lies in relatively close proximity to a stream reach. The tests are divided into two segments:

a) the prediction of the input to the drainage ditches resulting from applied irrigation water and

b) the prediction of the discharge to the river as a result of this same irrigation water.

In the simulation of the drainage to the ditches a standard drainage ditch spacing of 50 meters was assumed, while for the computation of the seepage from the total area to the river a number of situations were examined. These ranged from a 200 square meter site at distances of 100 to 500 meters from the river to a 1500 meters square area located 6750 meters from the river.

The standard input for most runs is what will be called the three year irrigation cycle. This cycle represents the leached irrigation water which is assumed to reach the groundwater table and therefore represent the forcing function to the groundwater system. This was discussed, to some degree, in Section IV-3.1.1. A one year input cycle represented by a depth of water applied to an area is shown in Table IV-1.

The drainage system, represented by the block diagram in Figure IV-10, was first tested with this three year irrigation cycle. The parameters were determined by the procedure discussed in Section IV-3.1.1, and are based on the Lagged Nash system response model. These parameters, as shown below:

$$\begin{aligned} n &= 0.7841 \\ K &= 3.7690 \text{ months} \\ \tau &= 0.5483 \text{ months} \end{aligned} \qquad \text{IV-18}$$

indicate that a fraction of one linear reservoir with a mean, or lag, of 3.769 months together with a translational lag of 0.5483 months would represent the drainage system under study. The input with the resulting outflow hydrograph for this system is shown in Figure IV-11.

For comparison, the outflow hydrograph from the Linear Groundwater Routing Model, LGRM, is related to the comparable hydrograph generated by the finite difference scheme. Figure IV-12 presents the typical cycle for the finite difference scheme and the LGRM.

Table IV-1

ONE YEAR OF THE IRRIGATION INPUT CYCLE

Month	Water Input (m) Over Area
1 (September)	0.1595
2	0.1806
3	0.2446
4	0.2711
5	0.2854
6	0.2569
7	0.2107
8	0.1804
9	0.0
10	0.0
11	0.0
12	0.0

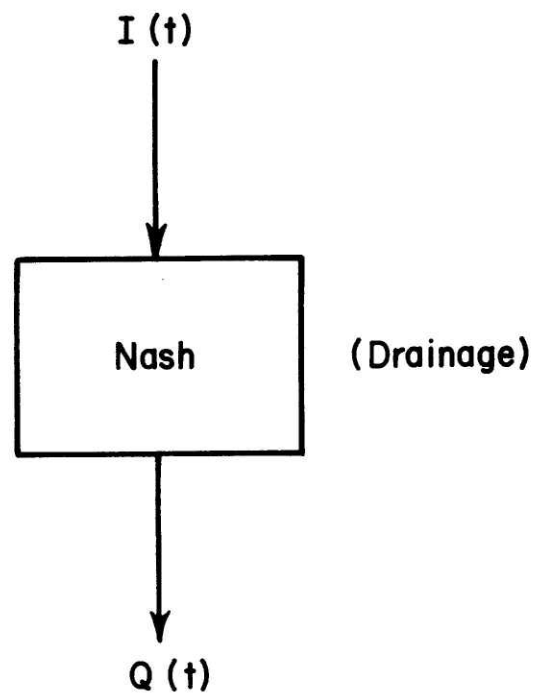


Figure IV-10

Block Diagram of the Drainage System

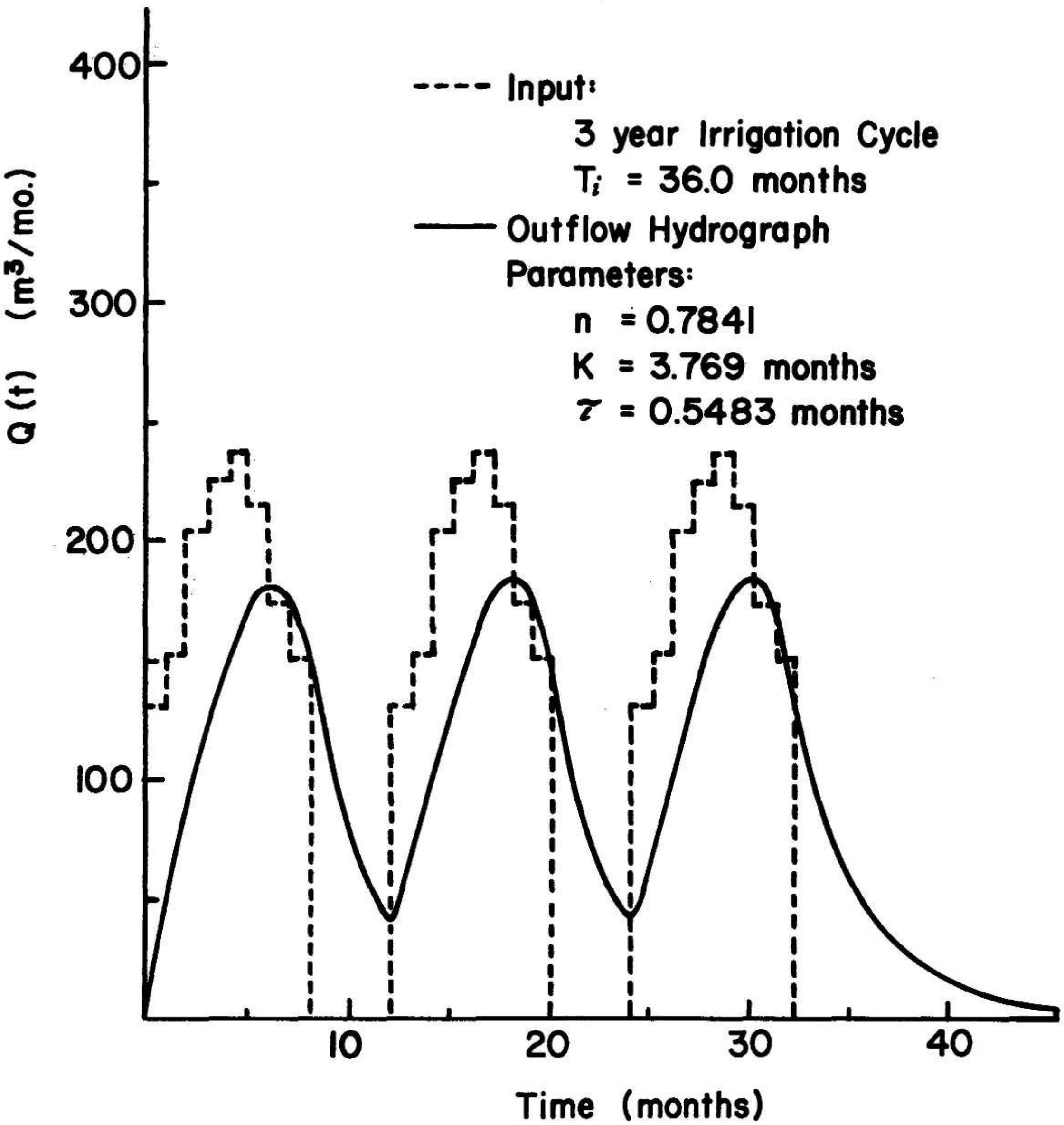


Figure IV-11

Outflow Hydrograph to Drainage Using Nash System Response

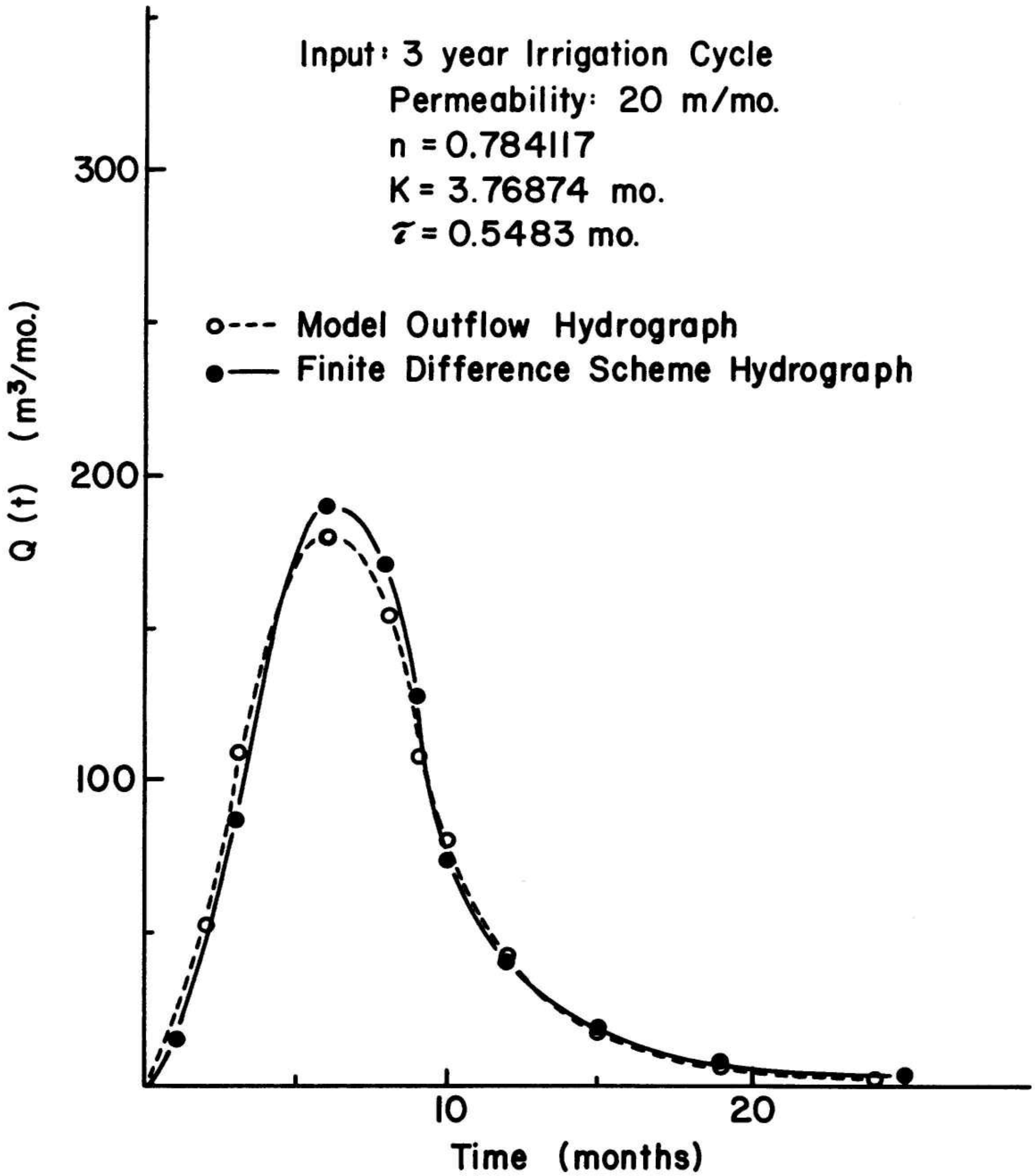


Figure IV-12

Typical Cycle of the Linear Routing Model and

Finite Difference Scheme

Notice the discrepancy at the peak confirming the discussion in Section II-6 which states that the greatest error results at the extremities when the parameters are generated by the Method of Moments. This could possibly be reduced if the fourth moment, the Kurtosis, was used in conjunction with a four parameter model. The typical cycles are derived from a series of cycles which are approximately at a steady state condition by removing the effects of the adjacent cycles. Figure IV-13 compares these same hydrograph but represents the third and final cycle of the output hydrograph. The effect of the adjacent cycle is apparent at the origin.

As discussed in Chapter III, the model is based on an aperiodic input. However, all the inputs used in the model are represented by a three cycle series. In addition, irrigation sites will receive water in a repeated pattern for perpetuity, unless changes are made to the irrigation policy. Because of these two reasons, it might prove beneficial to return to the concept of a periodic signal. The aperiodic signal was maintained by adding zeros (the length of the response signal) to the input signal. This procedure prevents the system from being effected by adjacent signals. Return to a periodic signal should simply require the removal of those zeros. Figure IV-14 indicates the result of doing so when the same input is applied to an area with the same parameters that are representative of the drainage system. Notice that each cycle is identical to the next. By comparing Figures IV-14 with IV-13, we can see that, over the length of

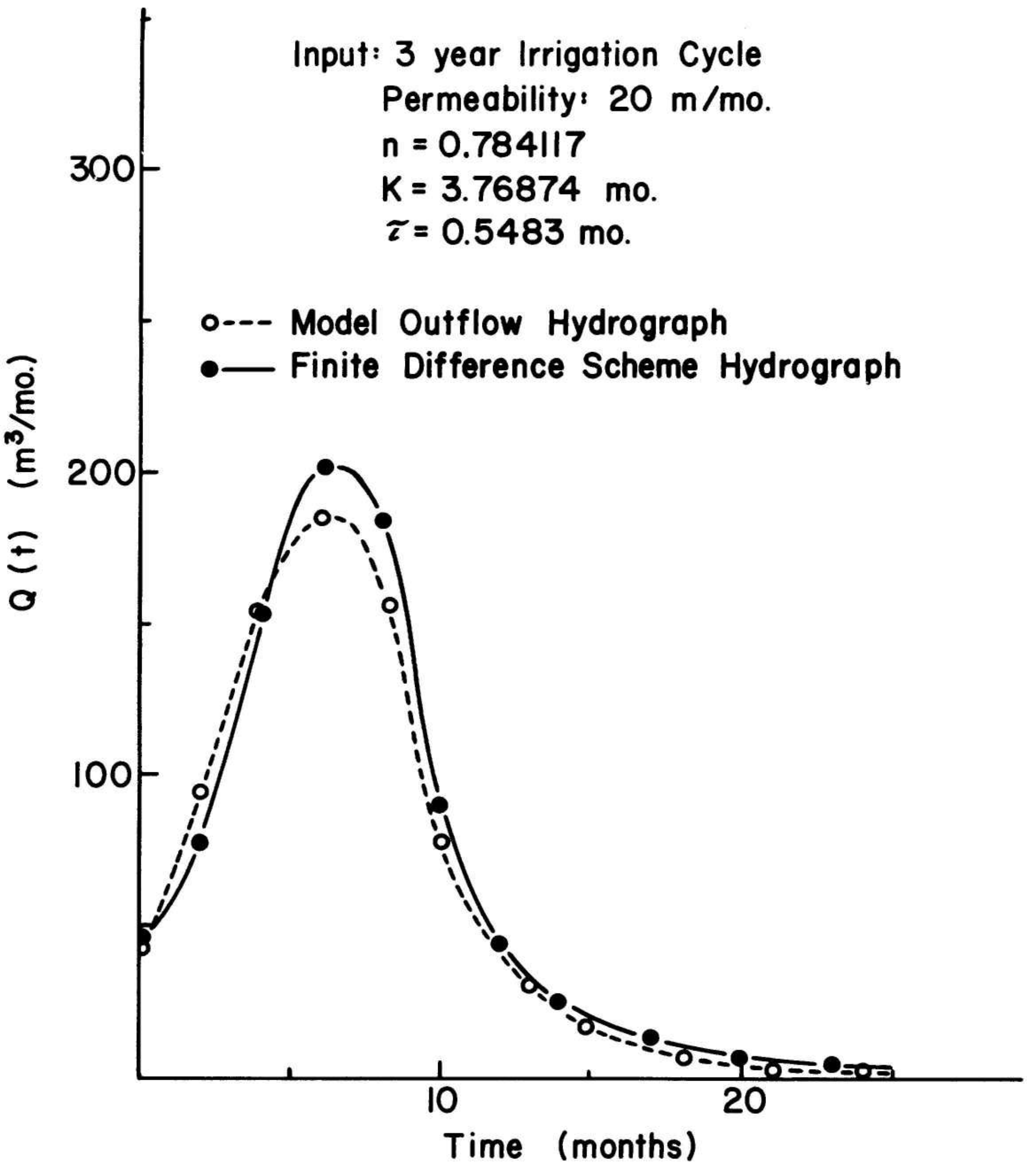


Figure IV-13

Final Cycle of a Three Year Input to the

Linear Routing Model and Finite Difference Scheme

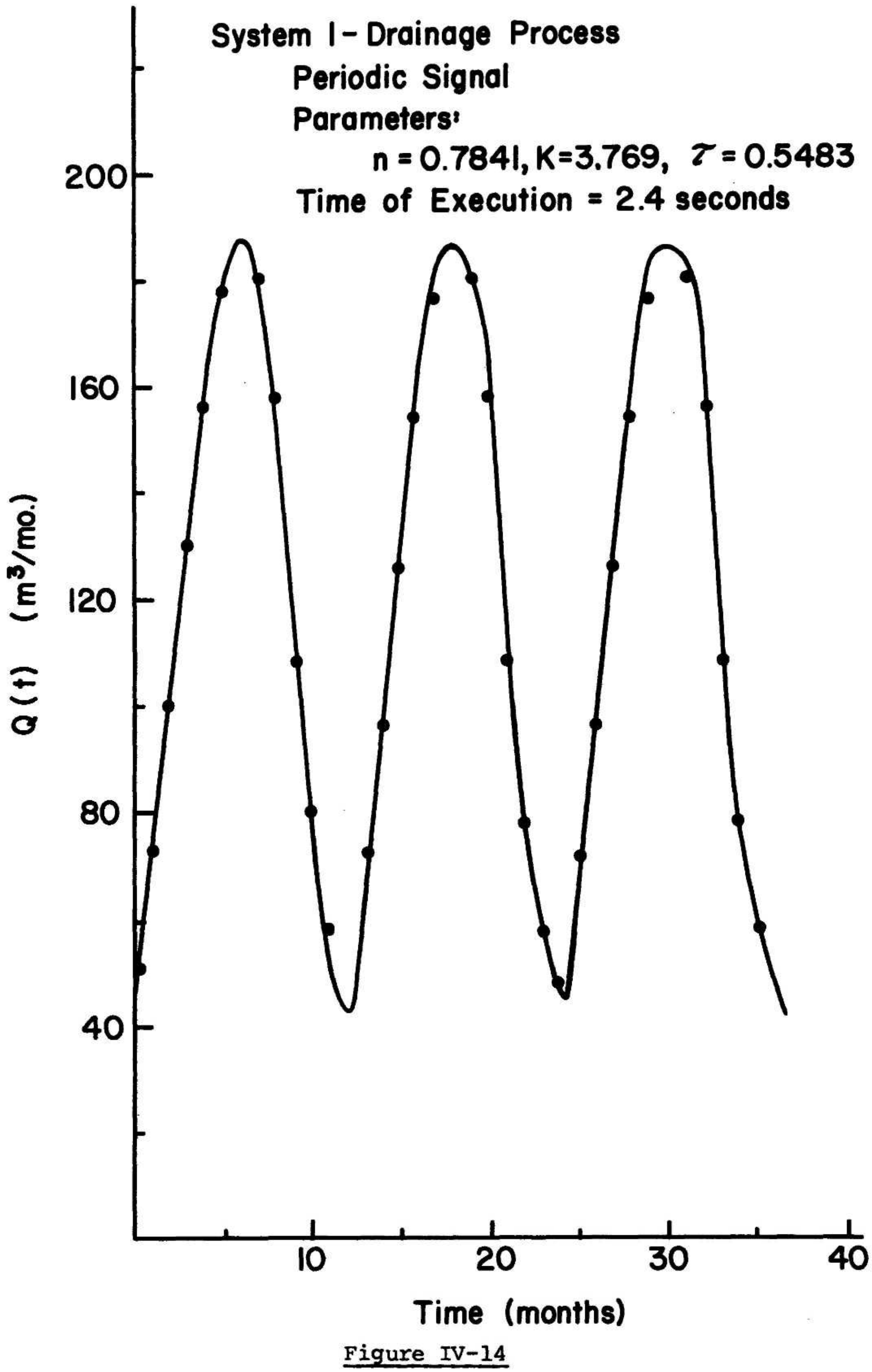


Figure IV-14

Response of the Drainage System to a Periodic Input the input period the hydrographs are identical; an interesting as well as useful result. The importance of this capability is readily appreciated especially when steady state or periodic responses to slowly responding systems are of interest.

The flexibility incorporated into the model for the system response is significant in representing a more realistic situation. In this case, as reflected in Figure IV-15, one chain of Nash elements would represent an interflow process while a parallel chain represents the same drainage process as presented in the above paragraphs. For this example 60% of the inflow is assumed to go to interflow while the remaining 40% percolates to a deeper zone. This was taken to be a reasonable assumption based on the soil and hydraulic conditions of the area. The outflow hydrograph resulting from the two inputs and system responses is shown in Figure IV-16. The system parameters shown below are selected to indicate the effect of an actual situation. The drainage parameters provide the same system lag as in previous paragraphs but with the translational lag effect reduced to zero. The interflow process parameters were chosen to yield a more rapidly responding system. These parameters are:

Drainage Process

$n = 0.8$

$K = 4.73929$

$\tau = 0.0$

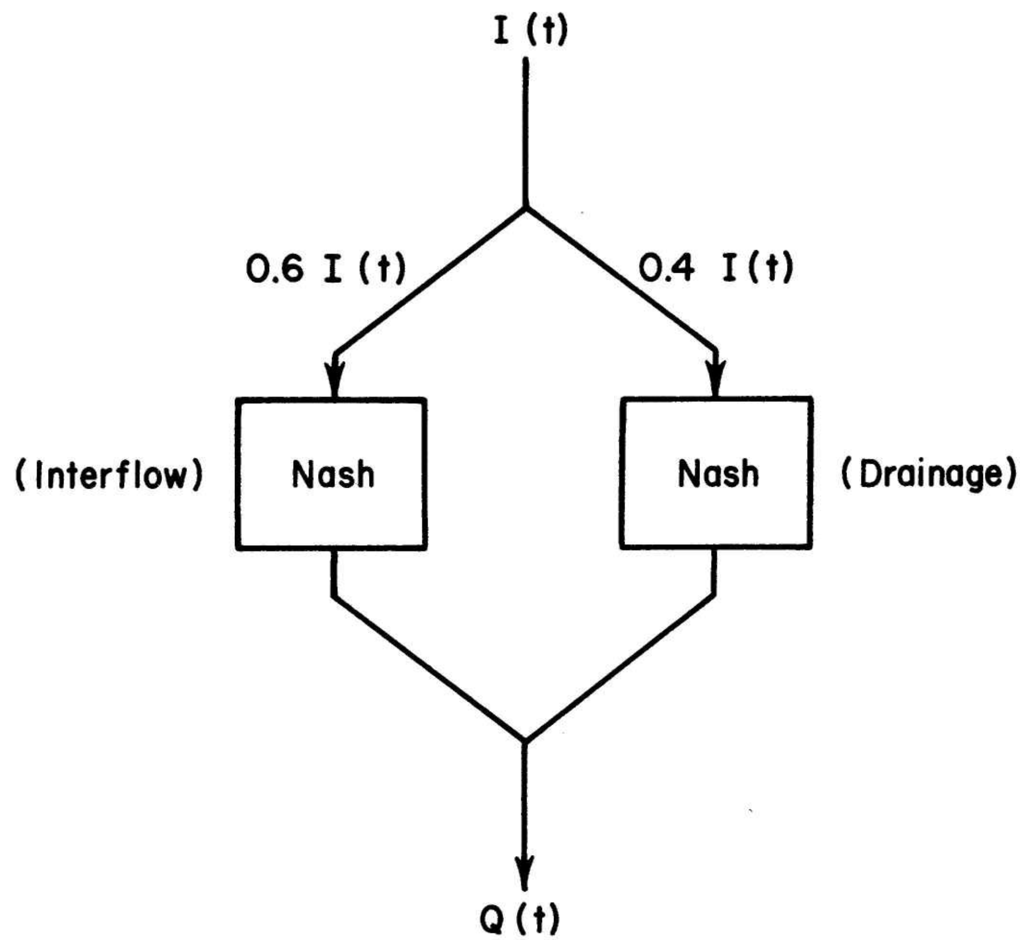


Figure IV-15

Block Diagram of a Parallel Nash System Response Representing and Interflow and Drainage Process

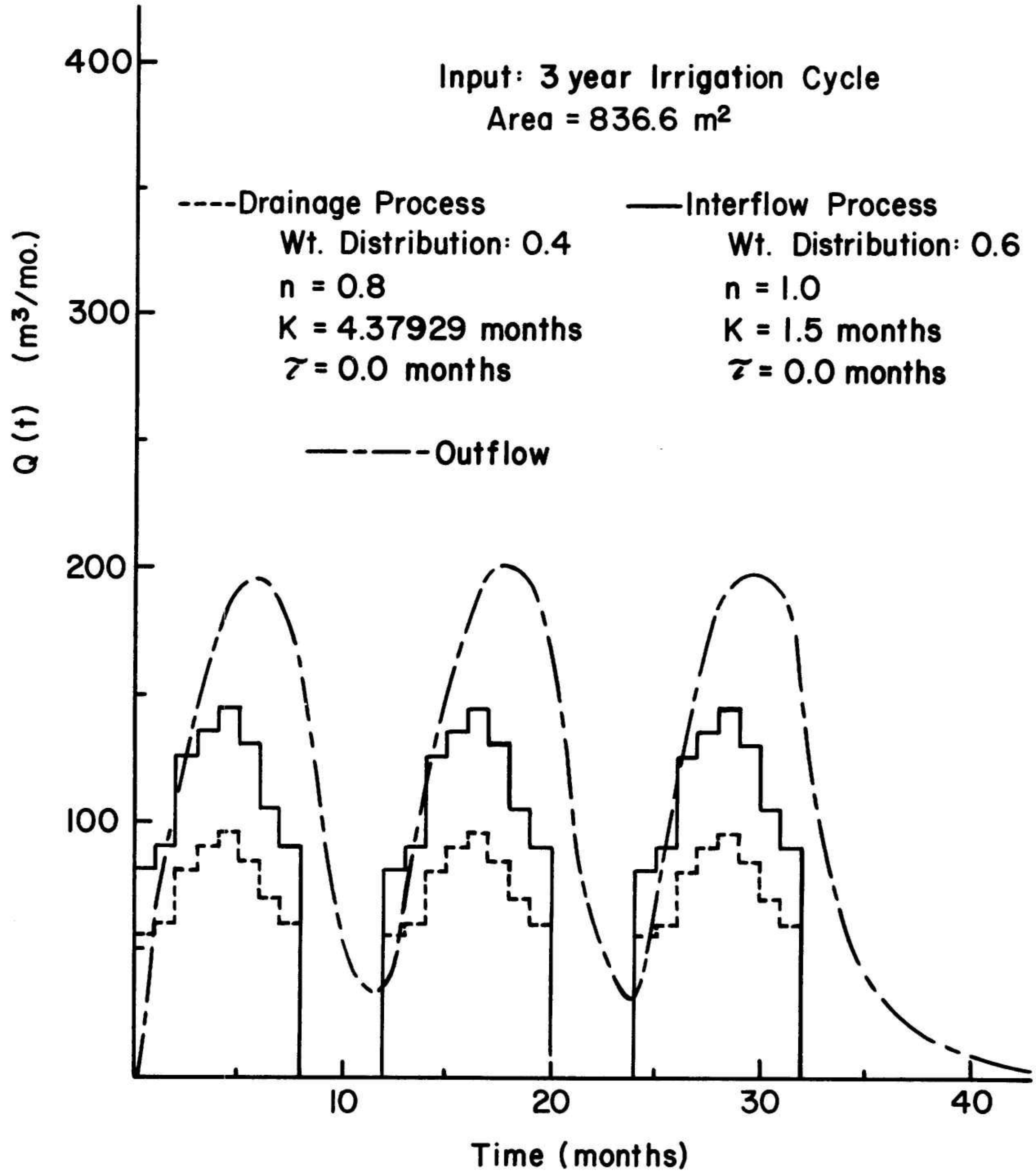


Figure IV-16

Outflow Hydrograph of the Interflow and Drainage Processes

Interflow Process

$n = 1.0$

$K = 1.5$

$\tau = 0.0$

The effect of this combination of flows (Figure IV-17) becomes apparent when this outflow hydrograph is plotted concurrently with the drainage system hydrograph shown in Figure IV-11. Since the interflow process dominates the combined system the result is as expected, i.e., the peak is increased and occurs earlier than that indicated in the drainage process (Figure IV-II). If the lag of the combined processes was calculated it would result in a system lag of 2.08 months. Since the lag of the drainage process above is 3.50 months, a difference of about 1.4 months is expected. A lag of this magnitude is clearly illustrated in Figure IV-17.

Figures IV-18 and 19 show the results of two other system configurations. Figure IV-18 represents the outflow hydrograph when a block input is convoluted with a system response of Nash and Lag and Route Models in series. As previously mentioned (Section IV-3.1) the execution time required to convolute an input with a Nash model in series with a Lag and Route model might be expected to be 8.5 seconds. The result of this model with that exact system response was executed in 3.42 seconds, inclusive of the output requirements and the plotting routine which is a substantial savings. Figure IV-19 represents a

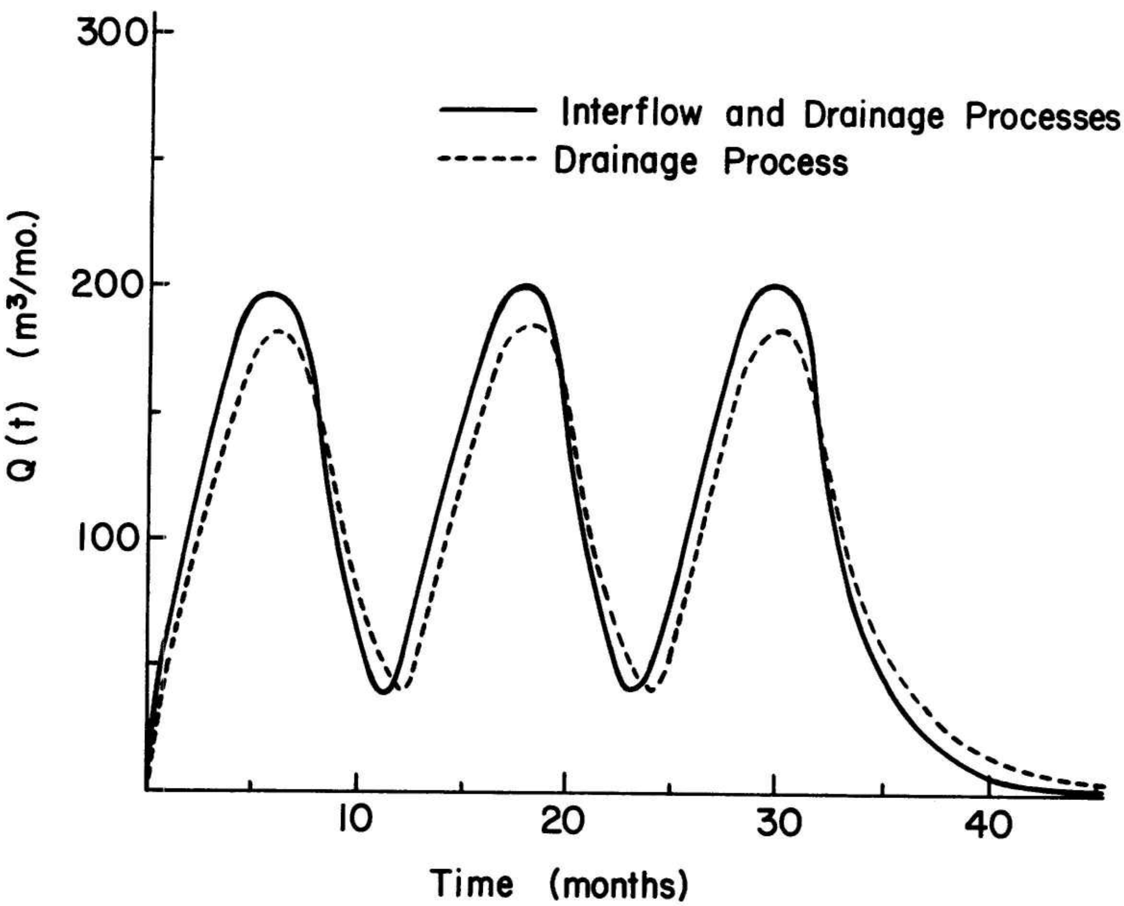


Figure IV-17

Comparison of the Drainage System Outflow Hydrograph

to the Interflow/Drainage Outflow Hydrograph

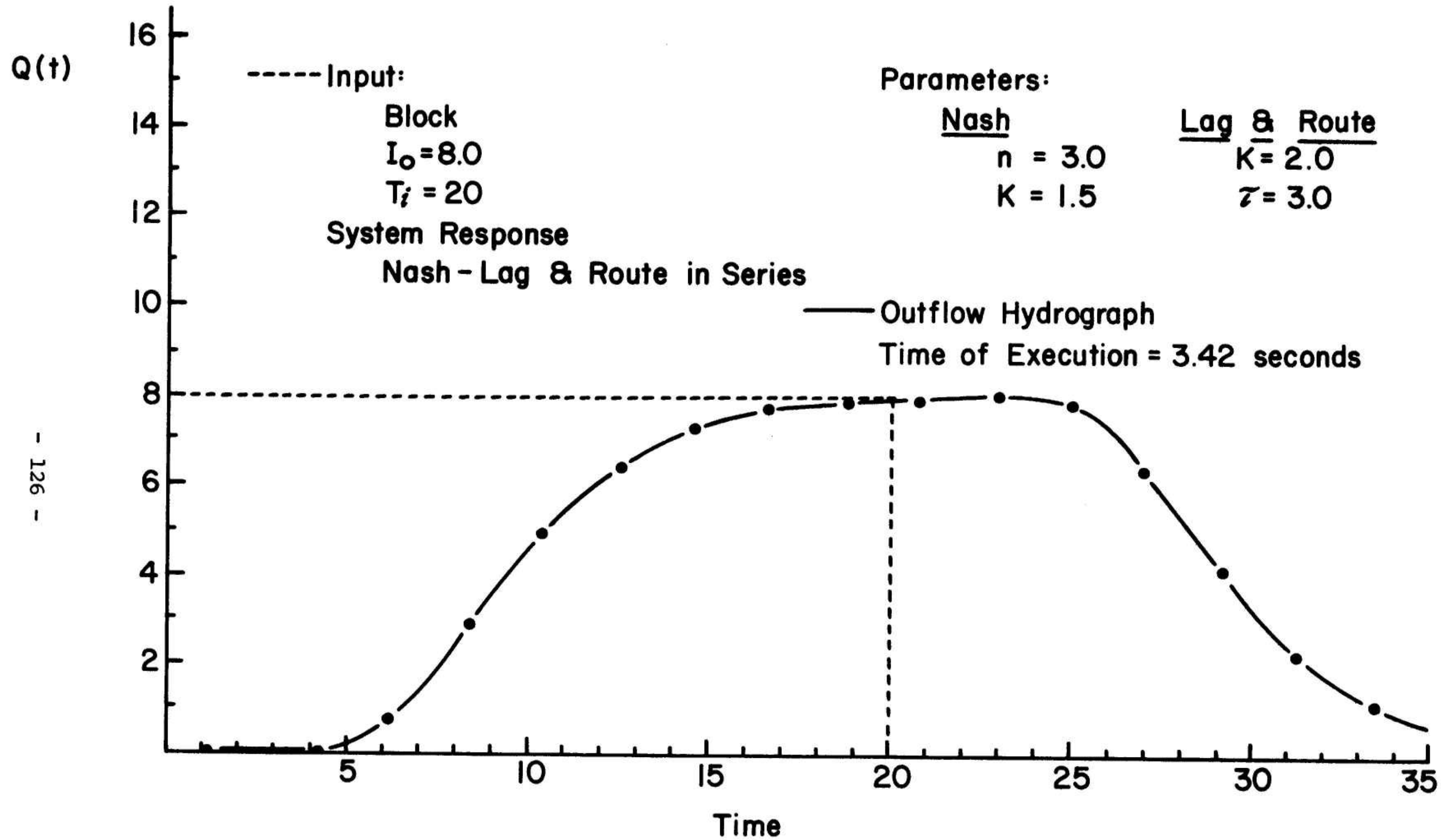


Figure IV-18

Results of a System Response Represented by Nash

and Lag and Route Models in Series

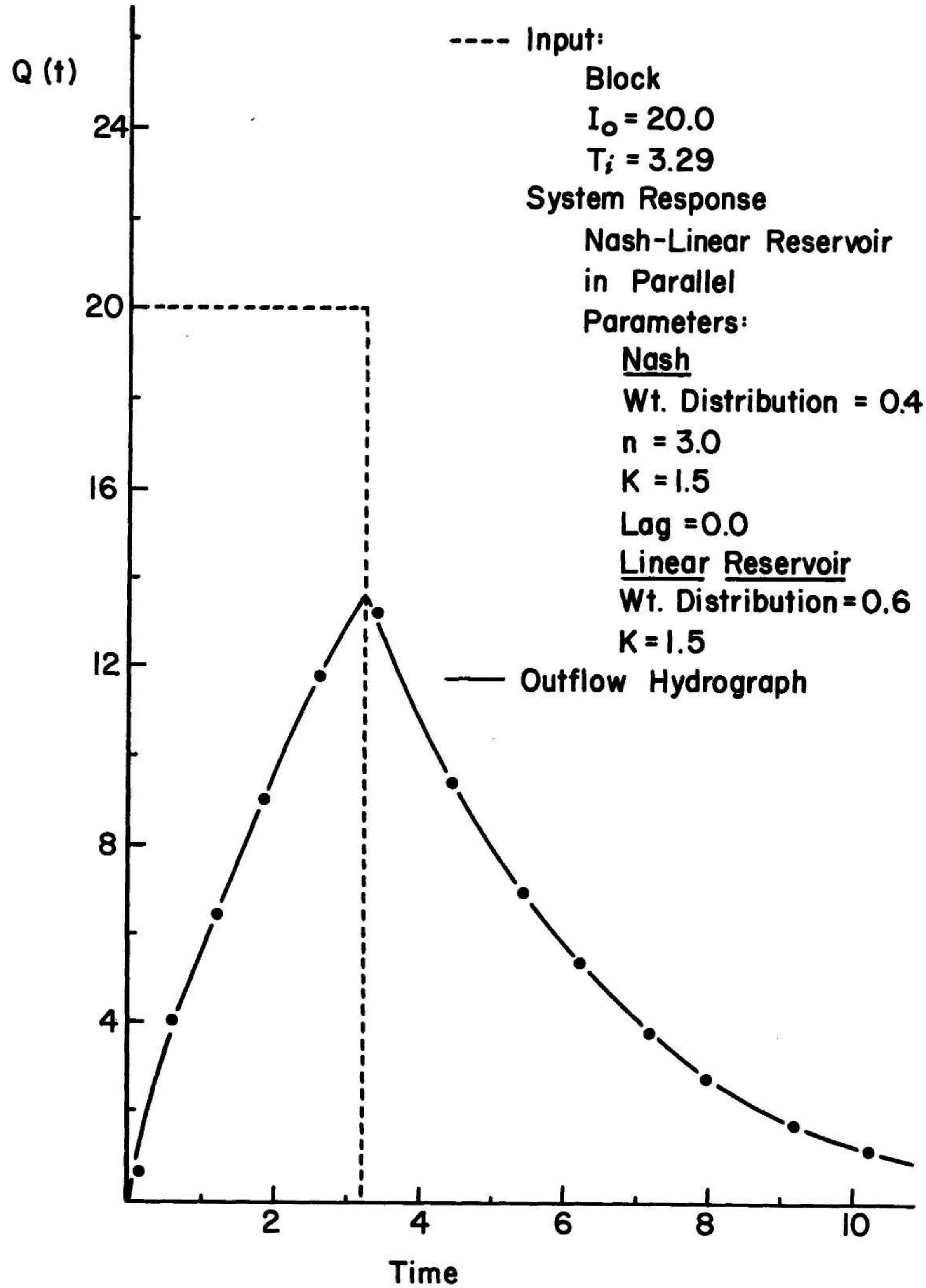


Figure IV-19

Results of a System Response Represented by Nash and Linear Reservoir Models in Parallel

parallel system using a Nash model and a Linear Reservoir model for the response characteristics with the parameters indicated.

The same three year irrigation cycle was further used as an input to this system, representing discharge from the site back to the river. This system is a very slowly responding one and its behavior contrasts sharply with the rapid response systems described above. Figures IV-20 and 21 will serve to illustrate the results. The input variables again represent a sampling of data obtained from the Rio Colorado river basin. The system represents the lateral flow to the river from a 1500 meters square irrigation area located about 6750 meters from the river. Figure IV-20 is the resulting hydrograph for a groundwater aquifer which has a bedrock slope of .01 m/m. This slope plays a significant part in determining the parameters (refer to SectionIV-3.1) as indicated by comparison with Figure IV-21 which is the outflow hydrograph to an aquifer with a bedrock slope of .001 m/m. One important fact must be noted here - that of the magnitude of the time step. In both cases the time step greatly exceeds that of the input time period which means that the result indicates merely a transient response of a pulsed input. Notice that the outflow hydrograph in the lesser sloped system closely resembles the linear reservoir response. This verifies the comments of Dooge [1960] who states that the translational effect is so small in a groundwater system that in effect, the system may be represented by a storage reservoir. With such a large time step it is not possible to determine the periodic

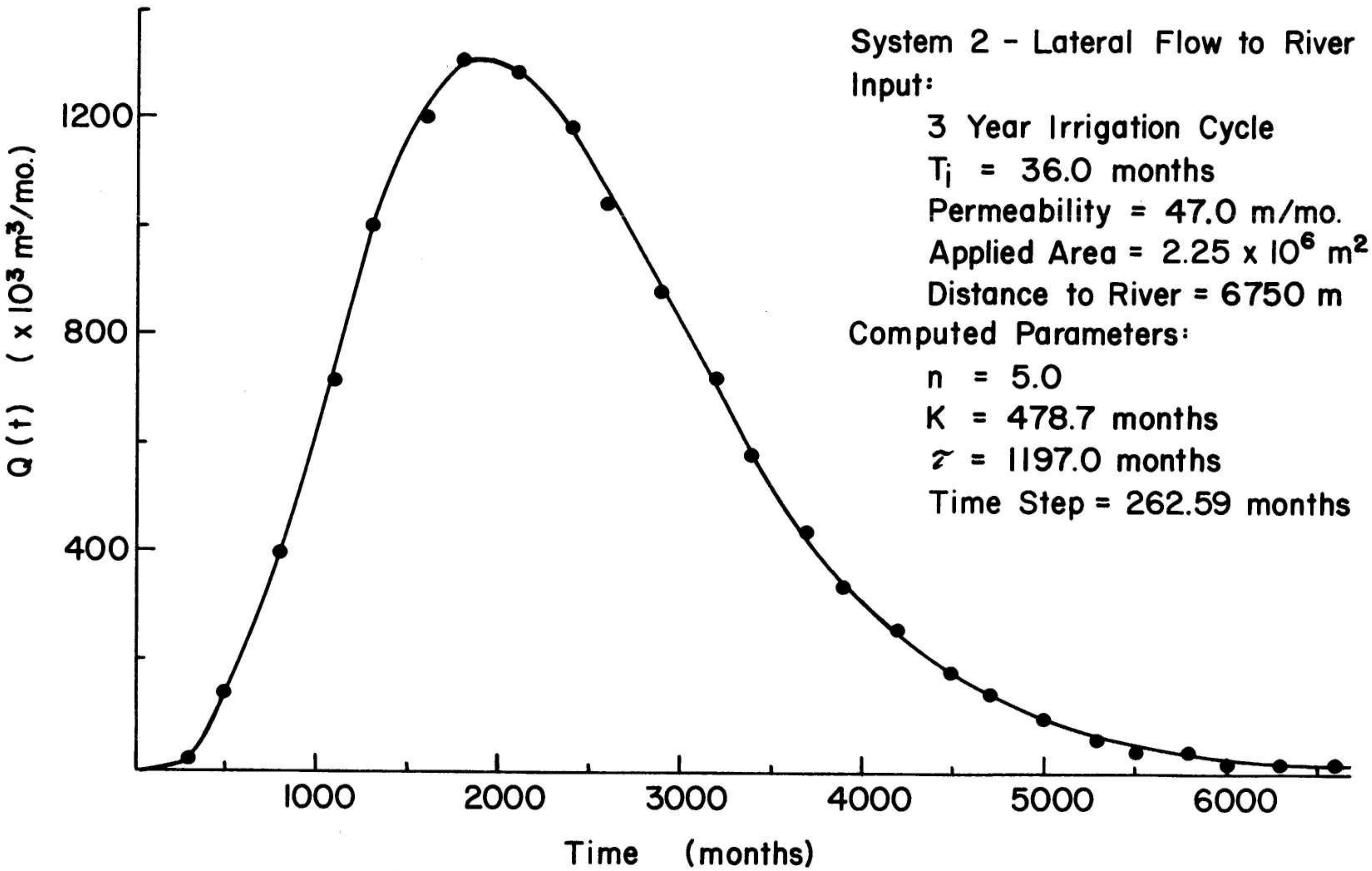


Figure IV-20

Response at the River of an Input to the Irrigation

Site-Bedrock Slope = .01 m/m

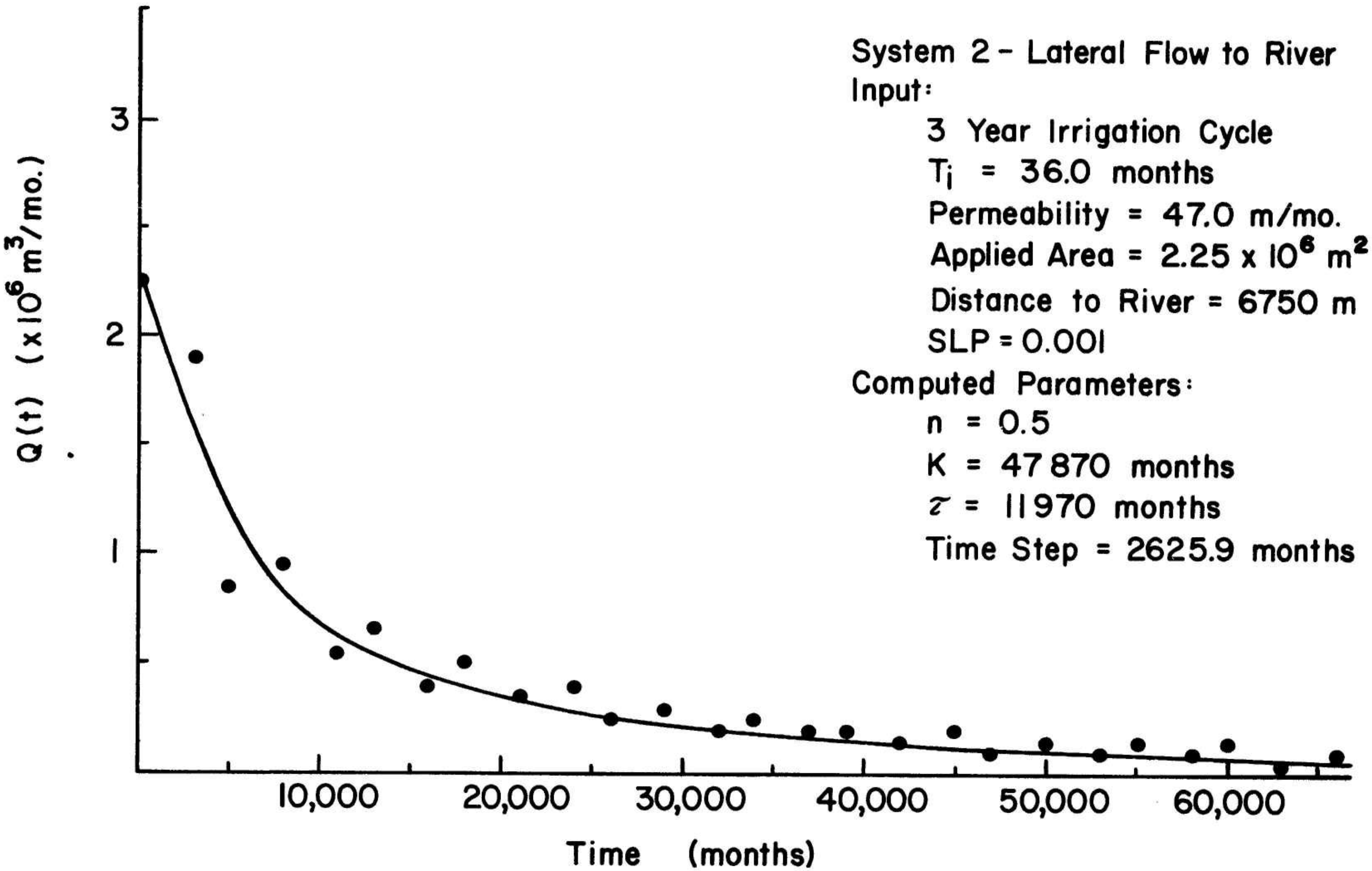


Figure IV-21

Response at the River of an Input to the Irrigation

Site-Bedrock Slope = .001 m/m

signal response.

The effectiveness of the 'medium' slowly responding system system cannot be demonstrated when such great travel distances are considered. Therefore, shorter distances will be used in order to present the significance of the theoretical techniques in use. As mentioned earlier, the slope is a significant parameter in representing the system response. Figure IV-21 shows the system outflow approaching a Linear Reservoir response when the slope was .001 m/m for a distance of 6750 meters. However, when this travel distance is reduced to 100 meters the outflow hydrograph responds very rapidly. The number of points is small which prevents an exact analysis of this hydrograph. This technique does provide a decreasingly accurate result as the slope approaches zero. A slope of .05 m/m was selected to test the system; the slope though relatively small, is sufficient to demonstrate the effects of travel distance. Figures IV-22, 23 and 24 exhibit the damping effect of travel distance in the system on a cyclical input. In the case of the two parameters, the number of reservoirs, n, increase linearly from 0.370 to 1.852, while the translational lag, τ, varies from 3.54 months to 17.73 months. The damping effect may be understood more clearly by considering the system lag, nK. Notice in Figure IV-22 that this lag amounts to 7.09 months, which is much less than one input cycle. As expected the response indicates three clear cycles which rapidly approach a steady state condition. Figure IV-25, shows the effects of a periodic signal indicating that the aperiodic

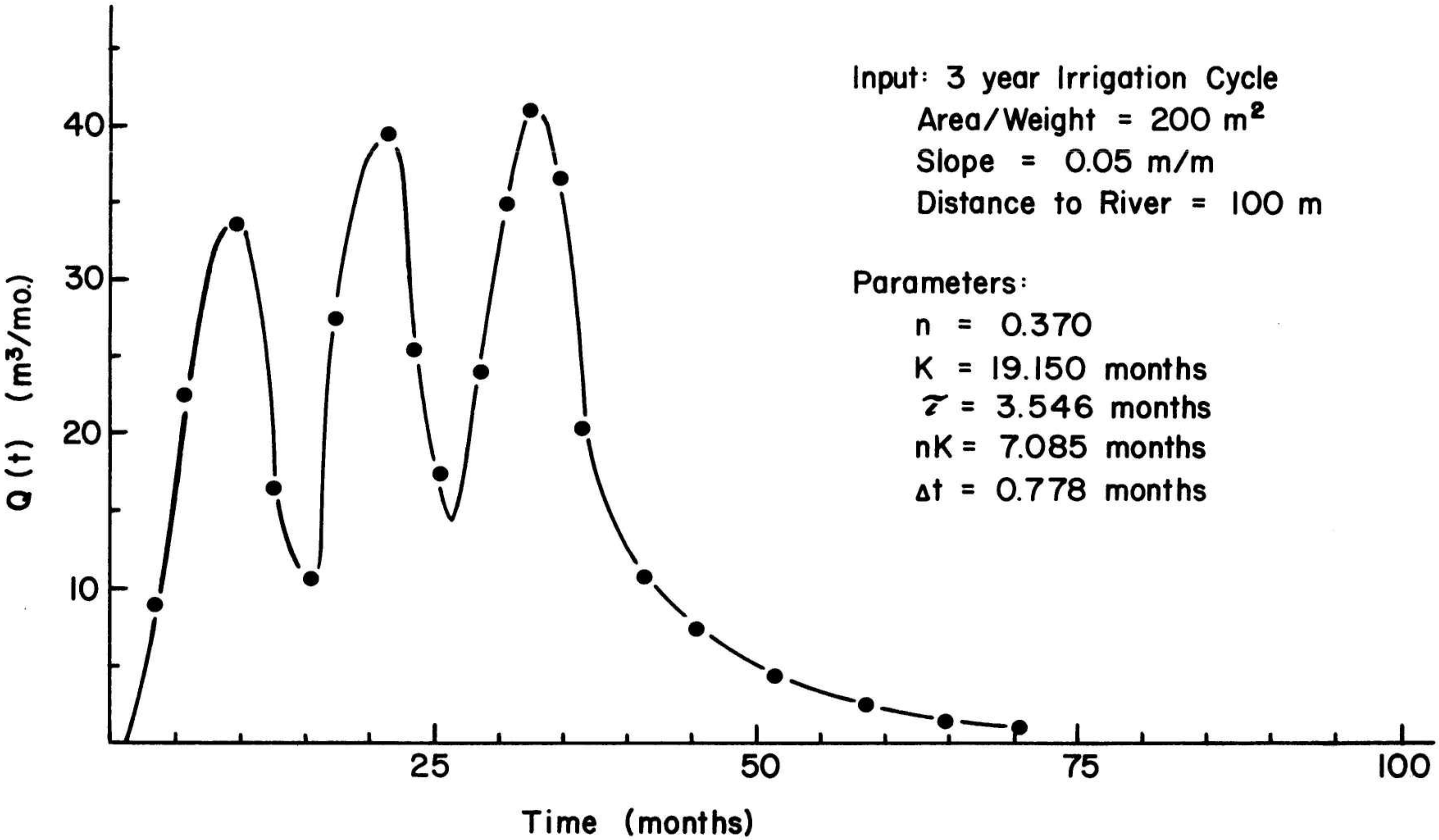


Figure IV-22

Outflow Hydrograph of a Three Cycle Irrigation Input at 100 meters

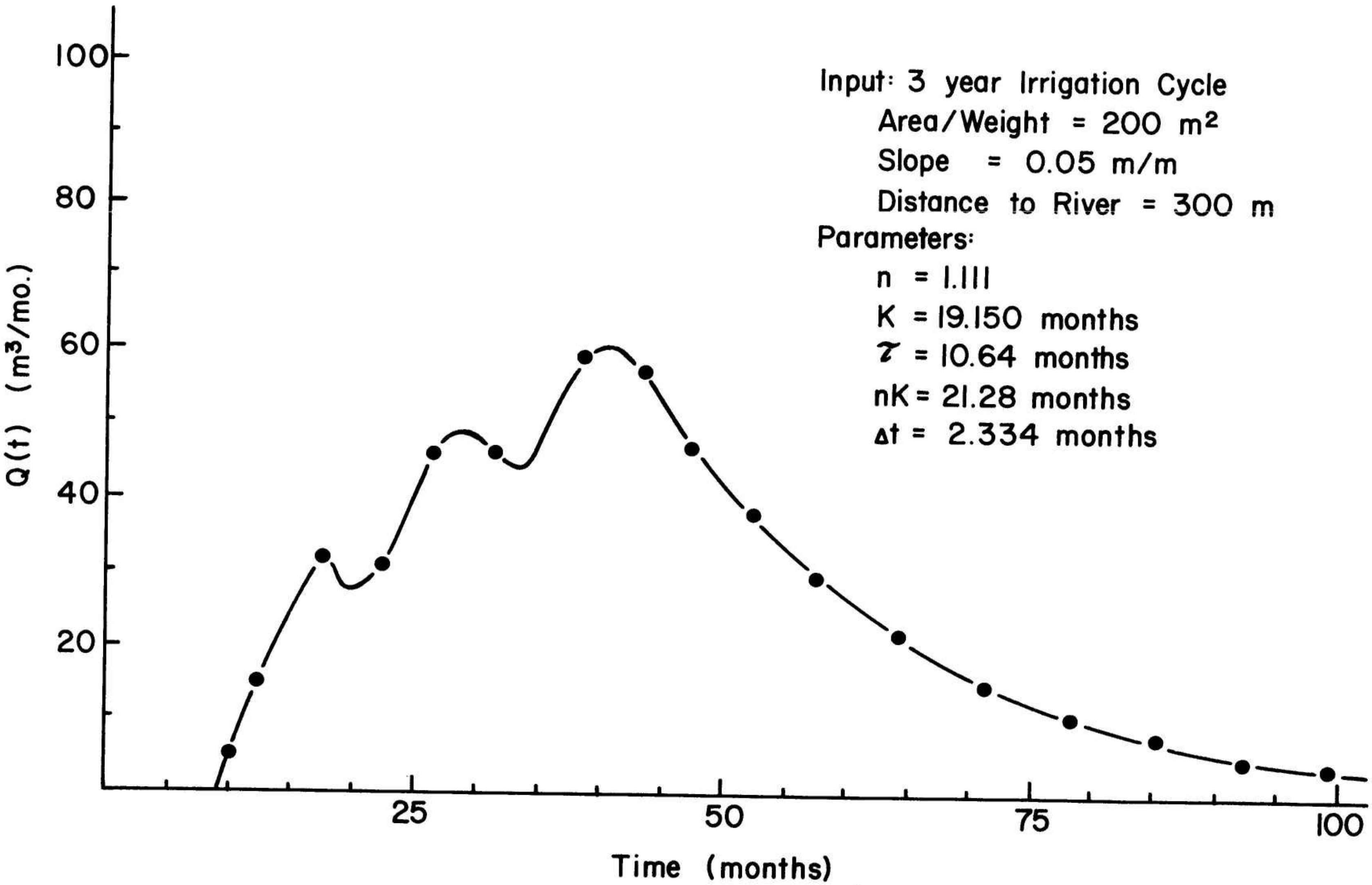


Figure IV-23

Outflow Hydrograph of a Three Cycle Irrigation Input at 300 meters

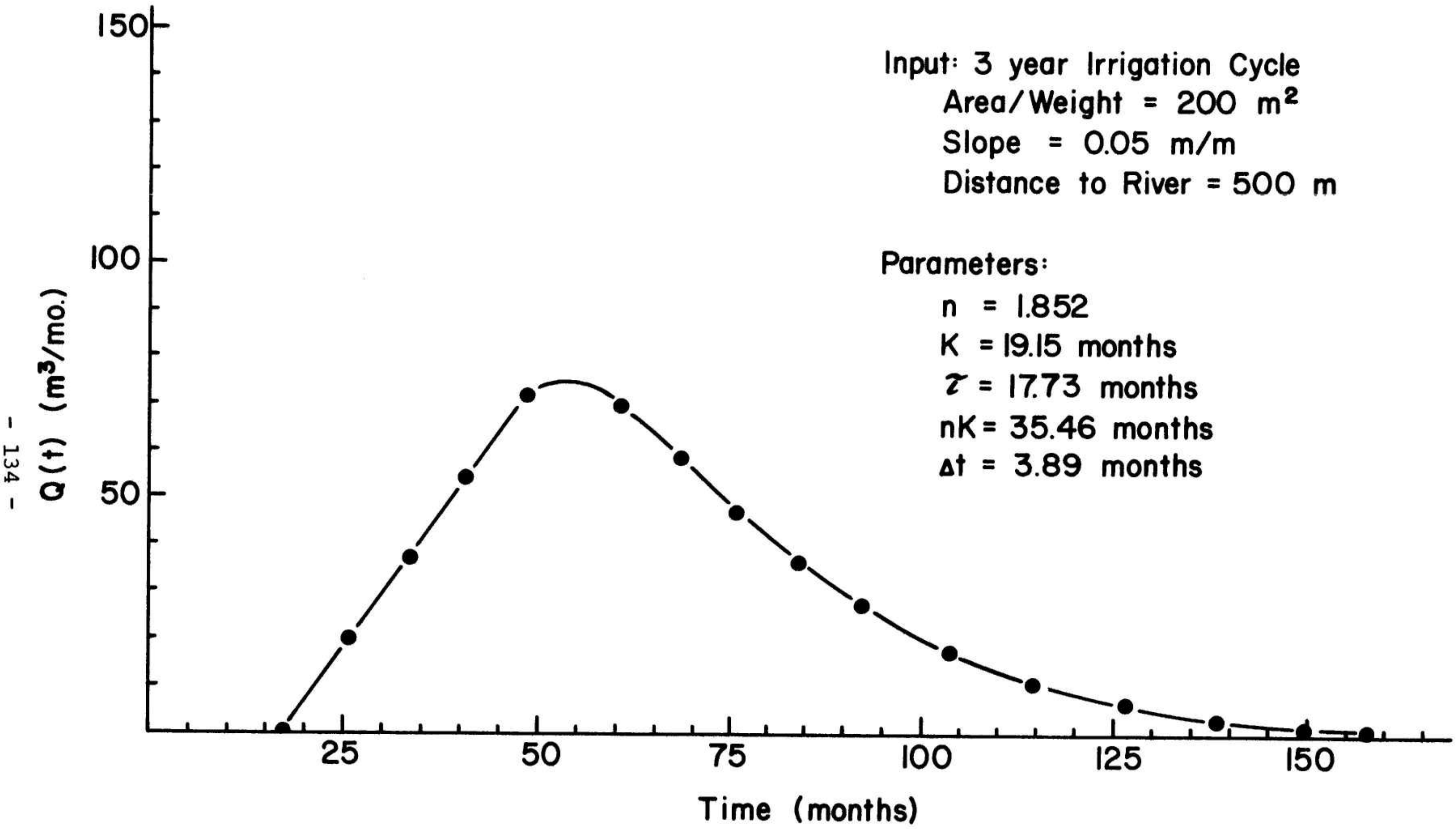


Figure IV-24

Outflow Hydrograph of a Three Cycle Irrigation Input at 500 meters

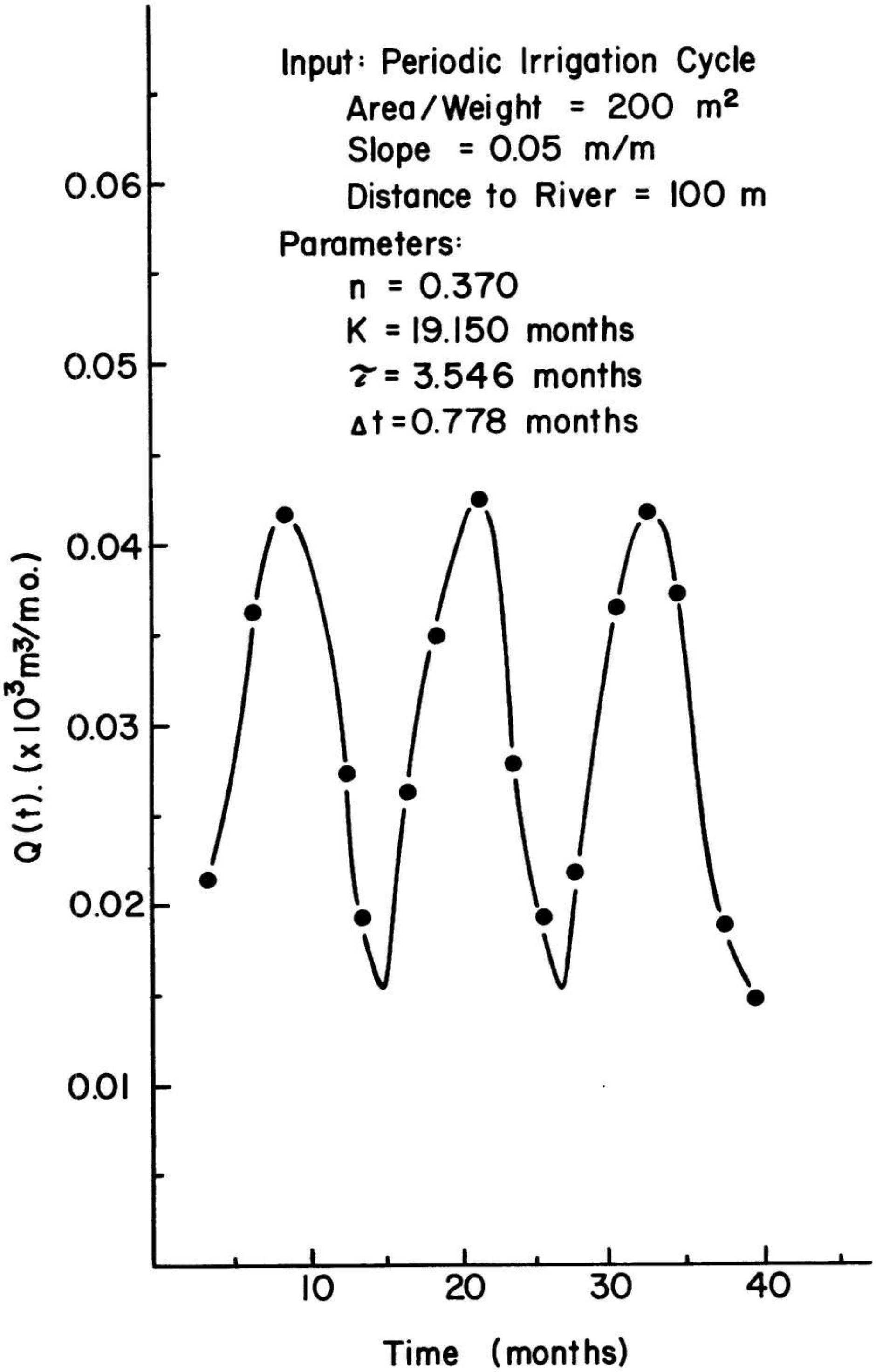


Figure IV-25

Response of a Periodic Signal at 100 meters signal of Figure IV-22 does indeed reach a steady state. In Figure IV-23 the system lag increases to 21.3 months which means that each input cycle will produce considerable effect on the following year. Thus, the figure indicates a much greater transient effect on the system for the same input period. Finally we see that a system lag of 35.5 months in Figure IV-24 almost exceeds the input time period. In turn the damping effect of the system on the input has all but reduced the cyclical input to that of a simple transient. Figure IV-26 indicates that if the same input were extended to perpetuity the system would hardly be effected by the input configuration - an interesting result. From Figure IV-20 we can see that as the system lag exceeds the input time period, the transient result is relatively uneffected by the shape of the input signal.

The third basic system of flow parallel to the river (See Section IV-3.1) is not presented here since its behavior is expected to be similar to that of the flow to the river but with longer distances and lesser bedrock slopes. Therefore it was not felt necessary to include it within this discussion.

The program was run on an I.B.M. 370/155 computer system. The core storage requirement for the program is about 120K.

We have seen that the Groundwater Routing Model is capable of providing results to a periodic or an aperiodic signal using highly efficient techniques. Such a flexible, efficient model as this could

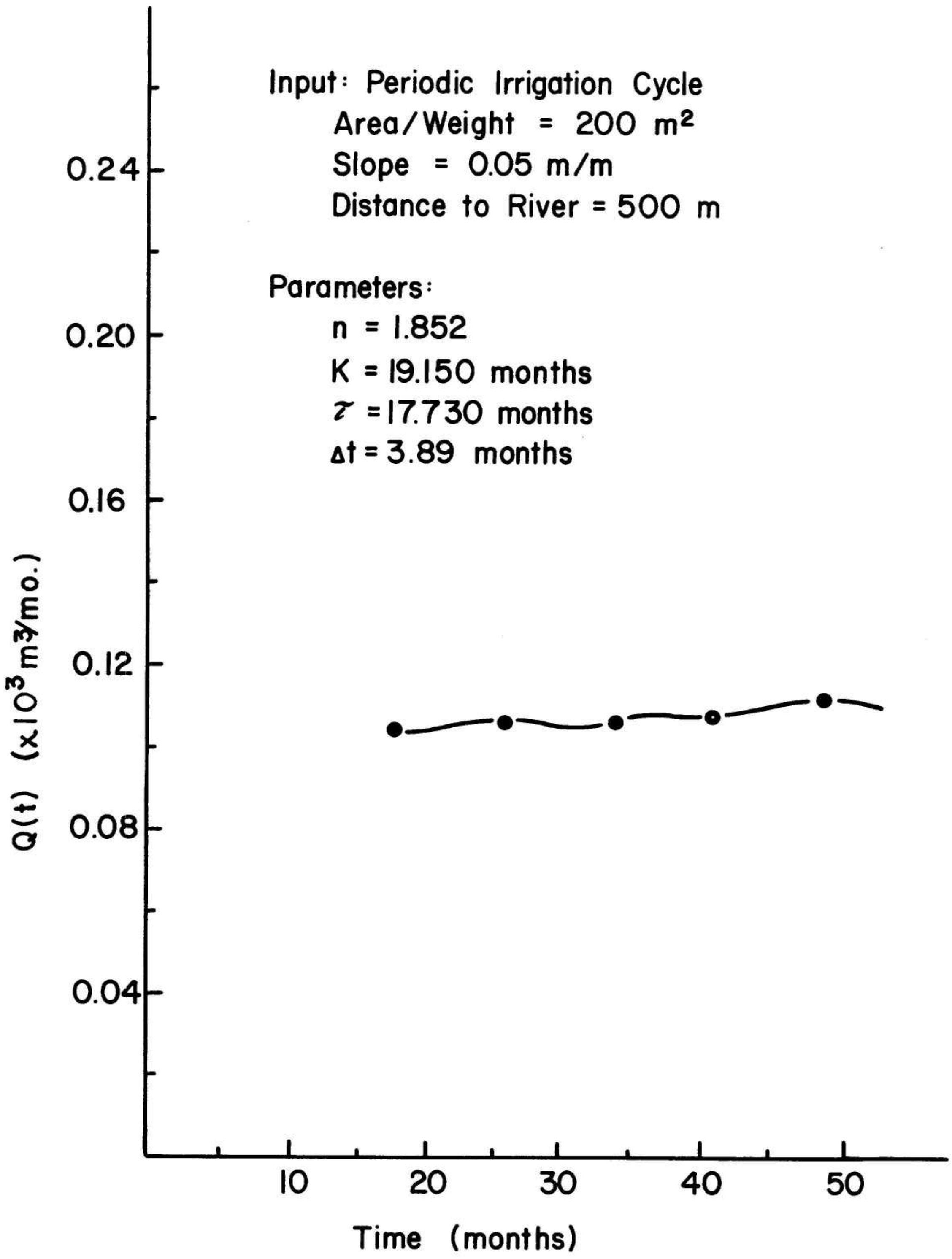


Figure IV-26

Response of a Periodic Signal at 500 meters provide the Engineer or Manager with a low cost model capable of representing a wide variation of groundwater problems.

Chapter V

CONCLUDING REMARKS

V-1 Summary

Chapter II summarized the developments within the field of linear systems as well as the work carried out in groundwater flow modeling.

Chapter III developed an efficient technique of convolution and compared the results to those obtained by a linearized solution of the complete equation for open channel flow.

Chapter IV discussed a river basin in Argentina which was used as an example of application of a groundwater routing model to an actual basin. A model was developed using a parameter estimation technique of the system response based on the governing equation for groundwater flow. The results of this model were discussed.

V-2 Conclusions

The groundwater routing model discussed in Chapter IV has a fast computational scheme which can be utilized to analyze a groundwater system. It is a highly flexible technique, capable of any system response desirable with little variation in computer time. As the model represents an approximation to a groundwater system it is, at best, to be used as a tool to understanding the sensitivity of the system being considered.

As in any technique being developed, the procedure for the routing of groundwater has a number of benefits as well as disadvantages to other methods being used to accomplish the same ends. The benefits for such a system are:

a) the convolution technique is highly efficient
b) the cost of implementation is small
c) any response configuration may be utilized to represent the desired groundwater aquifer system
d) the input may be of any design, whether it be a periodic or aperiodic signal
e) in considering the longer time periods necessary when analysing slowly responding systems, such as very long aquifers, the procedure automatically adjusts the time increment to the minimum level of interest to the system which also yields benefits in terms of inexpensive analysis of that groundwater system.

The problem areas include:

a) a technique is required to determine the weight distribution on the input to each process being considered

b) 'parallel' flows which represent two processes, such as a drainage system and a deep water zone, have a wide variance in their computed time steps. For this reason, if it is desired to combine these flows, a separate procedure is required to do so

c) the parameter estimation technique for the drainage system of the routing model requires the establishment of parameters based on (1) a nomograph given similar conditions to previously analyzed situations; (2) using known data and the Method of Moments; or (3) the use of a finite difference scheme, with the Method of Moments, to develop the desired parameters. The accuracy of such estimations has not been determined

d) the parameter estimation technique for a deep water zone process requires the input to be over a small area with respect to the travel distance. Additionally it is based on an advective velocity which is dependent of the slope of the bedrock resulting in problems to this procedure as the slope approaches zero.

V-3 Future Work

To be brief, the work required in developing linear groundwater models in the future might include the following:

a) Incorporate the lateral flow to the river with a linear stream routing scheme, as in the M.I.T. catchment model, in order to determine the total groundwater outflow hydrograph at some point downstream.

b) Develop a technique for distributing the flow to the various processes represented by the system responses. This could include a linear reservoir or a similar response model to represent the infiltration process.

c) Improve the parameter estimation process and determine the sensitivities of the parameters when data is available for doing so.

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LIST OF FIGURES

LIST OF SYMBOLS

Symbols	Description
a	advective velocity due to sloping bedrock (L/T)
a_i	constant coefficients of inflow cosine term or Laguerre functions in time period i
a_{ij}	specific permeability from node i to node j (L^2/T)
A	coefficient for differential equation [Appendix B]
A_i	constant coefficients of outflow cosine term or Laguerre functions in time period i
b_i	constant coefficients of inflow sine term in time period i
B	coefficient for differential equation [Appendix B]
B_i	constant coefficients for outflow sine term in time period i
c	coefficient for differential equation [Appendix B]
c	wave celerity in open channel flow (L/T)
C_i	constant coefficient of integration
C_j	j^{th} cumulant

D	differential operator
DEPTH	distance from ground surface to invert of drainage system (L)
E	total 'energy' of the system
$f_i(t)$	input function
$f_n(t)$	Laguerre function
$f_o(t)$	output function
F	Froude number
F_i	I^{th} moment of the input [Appendix B]
$F(\omega)$	discrete forward transform of input function
$F_i(\omega)$	Fourier transform of input function
$F_o(\omega)$	Fourier transform of output function
G_i	I^{th} moment of output [Appendix B]
h	piezometric head (L)
h_m	mean height of saturated water zone (L)
h(t)	impulse response

$h(0,t)$	instantaneous unit hydrograph
$h(t-\tau)$	I.U.H. lagged by time τ
H_i	I^{th} moment of response function
H_i	initial groundwater elevation in cell i (L)
H_j	initial groundwater elevation in cell j (L)
HT	Final groundwater elevation (L)
H( )	Laplace transform of piezometric head [Appendix B]
$H(\omega)$	response function
$H_o(\omega)$	normalized amplitude spectrum
I	constant inflow or input rate
I_i	inflow or recharge for period i
I_j	index to cell adjacent to cell j
I_o	input amplitude of constant input
$I(t)$	time varying input
$I(\tau)$	inflow rate at time τ
$I(t-\tau)$	inflow distribution lagged by τ

$I(\omega)$ input function (frequency mode)

j parameter used in empirical equations [Chapter II]

j $\sqrt{-1}$

K time constant of a linear reservoir (T)

K_i reservoir lag coefficient for system i (T)

K_L lag coefficient of a Lag and Route Model (T)

K_{LR} lag coefficient of a Linear Reservoir Model (T)

K_N lag coefficient of a Nash Model (T)

K_p permeability (L/T)

K total 'lag' to a series system (T)

K_T maximum system lag for use in a parallel system (T)

KD transmissivity (L^3/T)

L drainage spacing (L)

L travel distance of flow in aquifer (L)

M_i I^{th} moment about the centroid

M_i'	I^{th} moment about the origin
n	constant
n	time period
n	number of equal linear reservoirs in series
n_i	number of equal linear reservoirs in series i
N	number of cells adjacent to boundary node
N	number of points in output hydrograph from subroutine FOURT
NF	number of points within time period of discrete Fourier transform
N_i	number of input points to subroutine FOURT
P	constant percolation rate
$P_\delta(\omega)$	impulse amplitude function
$PERC_{Max}$	maximum percolated water over period of interest (L^3)
q	outflow
q	groundwater flow resulting from the diffusion analogy
q_i	inflow due to advective velocity

$q(t)$	time varying output
$q_i(t)$	outflow for period i
Q	outflow resulting from a constant inflow
Q_j	flow input into cell j (L/T)
Q_n	volume outflow during period n
QF(j)	flow into boundary cell j
$Q(s)$	Laplace transform of outflow function
$Q(t)$	outflow at time t
$Q_i(t)$	outflow for period i
$Q(\omega)$	output function (frequency mode)
R_i	volume recharge to aquifer during period i (L^3)
s	Laplace operator
s	time transformation variable (T)
s	instantaneous irrigation inflow
S	reservoir storage
S_n	reservoir storage for period n

$S(t)$	reservoir storage at time t
S_i	shape factors with index i
S_c	storage coefficient (L/L)
S_o	slope of channel bottom (L/L)
t	time (T)
T_i	time period for input function (T)
T_o	time period for output function (T)
T_p	time to peak (T)
T_r	time period for system response (T)
T'	new time period [Appendix B] (T)
V_o	mean longitudinal velocity in the steady state condition of open channel flow (L/T)
x	travel length of flow (L)
x	superscript to outflow rate
$\underline{\bar{X}}$	area of function outside time period [Appendix B]

$y(x,t)$ free surface height at x and time t (L)

y_o depth of water in the steady state of open channel flow (L)

Y_{Max} maximum lens height allowed above drainage ditch (L)

z_i I^{th} cumulant to Groundwater Routing Model

α weight distribution parameter

α_i constant coefficients for kernel cosine term or Laguerre function in time period i

β_i constant coefficients for kernel sine term in time period i

$\Gamma(n)$ gamma function of n

Δ increment operator

μ active porosity

τ translational lag of a linear channel (T)

Φ_o energy density spectrum of the output

ω angular frequency (rad/T)

ω_o Nyquist, or folding, frequency (rad/T)

LIST OF TABLES

APPENDIX A

Computer Programs

A-1 MODEL GENERATED FOR HARMONIC ANALYSIS

```
C*****************************************************************************
C
C     PROGRAM FOR CONVOLUTING AN INPUT FUNCTION
C     WITH FAST FOURIER TRANSFORM
C
C     APRIL 17, 1972 R. BRAS AND D. EVANS
C
C*****************************************************************************
      COMPLEX XINPU2(300),OUTPUT(300),FT(300),XINSTO(300),TINP(300),J1
      REAL*8 DATE, TIMEX
      REAL*8 TIT1E,TIT2E,TIT3E,TIT4E,TIT5E,TIT6E,TIT7E,TIT8E,TIT12,TIT22
     1,TIT32,TIT42,TIT52,TIT62,TIT13,TIT23,TIT33,TIT43,TIT53,TITLE(10,10
     1),TIT14,TIT24,TIT34,TIT44,TIT54,TIT64,TIT74,TIT84,TIT94
      DATA TITLE/100*'        '/
      REAL K,NRES,KRES,LAG
      DIMENSION TINPU2(300)
     1,WORK(300),JJ(10,300),ISH(10)
      DIMENSION TINPU3(300),XINPU3(300),XINPU4(300),XIN(300),YINSTO(300)
      COMMON /IOPUT/IREAD,IRITE,ITERM,IPLCT
      COMMON/IOS/DATE,TIMEX
      DATA TIT1E,TIT2E,TIT3E,TIT4E,TIT5E,TIT6E,TIT7E,TIT8E/'        ','
     1     L','INEAR   ','SOLUTICN','        C','OMPLETE ','        ','
     2     '/
      DATA TIT12,TIT22,TIT32,TIT42,TIT52,TIT62/'TIME DOM','AIN-INPU','T
     1HYDROG','RAPH    ','        ',' -OUTPU'/
      DATA TIT13,TIT23,TIT33,TIT43,TIT53/'ANALYTIC','AL SOLUT','ION-OUTP
     1','UT HYDRO','GRAPH   '/
      DATA TIT14,TIT24,TIT34,TIT44,TIT54,TIT64,TIT74,TIT84,TIT94/'INPUT
     1(F','REQ DOMA','IN) - AN','ALYTICAL',' SOLUTIO','N - REAL','
     2- ','COMPUTED','N - IMAG'/
      PI=3.141593
      IRITE=6
      IREAD=5
      ITERM=6
      IPLOT=6
      I=0
      DO 2 M=1,10
2     ISH(M)=0
C
C     READ PARAMETERS
C
      READ(IREAD,3)DATE,TIMEX
3     FORMAT(2A8)
      READ(IREAD,13)NO,NRES
13    FORMAT(I3,F10.0)
      READ(IREAD,14)K,KRES,LAG,BFL
14    FORMAT(F10.0)
C     FIND W NEEDED TO KEEP 98 O/O RESPONSE ENERGY
      WNOT=28.636/KRES
      TINPMA=96.
C     FIND NUMBER OF ZEROS TO ADD TO FUNCTIONS AND FIGURE
C     OUT TMAX FOR RESPONSE FUNCTION TO KEEP 99 O/O OF AREA
C     INTERPOLATE AT RIGHT TIMER INTERVALS IF NECESSARY
18    TRESMA=-KRES*ALOG(.01)
      DELT=PI/WNOT
      M=1
      WTIME=-DELT
C
C     CALC. INPUT FUNCTION
C
1050  WTIME=WTIME+DELT
      ARG=(PI*WTIME)/(TINPMA/2)
      XINPU2(M)=125.-75.*COS(ARG)
      XINPU2(M)=XINPU2(M)-BFL
      TINPU2(M)=WTIME
      M=M+1
      IF(WTIME.LT.TINPMA) GO TO 1050
      NINPU2=M-1
C     LENGTH OF OUTPUT=TIME INPUT+TIME RESP
C     SO NUMBER OF PTS IN OUTPUT IS (TINPU2(NINPU2)+TRESMA)/DELT)
C     OR N=NINPU2+TRESMA/DELT)
C      THE NUMBER OF ZEROS TO BE ADDED IS THEN TRESMA/DEL
      NZEROS=TRESMA/DELT
      N=NINPU2+NZEROS
      DO 27 M=1,NZEROS
      XINPU2(NINPU2+M)=0.
27    TINPU2(NINPU2+M)=TINPU2(NINPU2)+(M*DELT)
      WRITE(IRITE,503)
      DO 500 L=1,N
      IF(L.LE.NINPU2) XINSTO(L)=XINPU2(L)+BFL
      IF(L.GT.NINPU2) XINSTO(L)=XINPU2(L)
500   WRITE(IRITE,502)XINSTO(L),TINPU2(L)
502   FORMAT(' ',10X,2(E13.6,5X),10X,E13.6)
503   FORMAT(' INPUT FUNCTION TO BE CONVOLUTED',//,T10,'FUNC VALUE',T50,
     1'TIME')
C
C     OBTAIN FORWARD TRANSFORM OF INPUT USING FOURT
C
      CALL FOURT(XINPU2,N,1,-1,0,WORK)
C
C     EVALUATE ANALYTICAL TRANSFORM OF RESPONSE FUNCTION
C
      DW=(2*PI)/(N*DELT)
C     DO FIRST HALF OF TRANSFORM, AT LAST PT SET IMAG. PART=0
      N1=(N/2)+1
      OM=0.
      OM1=0.
      J1=CMPLX(0.,1.)
      DO 6 J=1,N1
C
C     ANALYTICAL RESPONSE TRANSFORMS FOR LINEAR RESERVOIR, LAG AND ROUTE
C     MODEL, AND NASH MODEL RESPECTIVELY
C
      GO TO (1100,1105,1110),NO
1100  FT(J)=(1./CMPLX(1.,OM))
      GO TO 505
```

```
1105  FT(J)=CEXP(-J1*CM1*LAG)/(1+(J1*OM))
      GO TO 505
1110  FT(J)=(1+OM**2)**(-NRES/2)*CEXP(-J1*NRES*ATAN(OM))
505   NCK=MOD(N,2)
      IF(NCK.GT.0 .AND.J.EQ.N1) GO TO 4
      IF(J.NE.N1) GO TO 4
      FT1=REAL(FT(J))
      FT(J)=CMPLX(FT1,0.)
4     IF(J.EQ.N1) GO TO 7
      OM1=OM1+OW
6     OM=OM1*K
C     SECOND HALF OF TRANSFORM=CONJUGATE IDENTITY OF 1ST HALF
7     IF (NCK.GT.0) GO TO 103
      GO TO 102
103   N2=N1
      L=N1+1
      GO TO 101
102   L=N1+1
      N2=N1-1
101   DO 8 J=L,N
      FT(J)=FT(N2)
      FT2=REAL(FT(J))
      FT3=AIMAG(FT(J))
      FT(J)=CMPLX(FT2,-FT3)
C
C     MULTIPLY TRANSFORM FUNCTIONS
C
8     N2=N2-1
      DO 35 L=1,N
      OUTPUT(L)=XINPU2(L)*FT(L)
35    CONTINUE
      WRITE(IRITE,120)
120   FORMAT('1',T20,'OUTPUT FUNCTION,INVERSE OF TRANSFORM MULTIPLICATIO
     1N'////)
C
C     INVERSE TRANSFORM OF OUTPUT HYDROGRAPH
C
      CALL FOURT(OUTPUT,N,1,1,1,WORK)
      DO 150 J=1,N
      OUTPUT(J)=OUTPUT(J)/N
      OUTPUT(J)=OUTPUT(J)+BFL
150   WRITE(IRITE,125)OUTPUT(J),TINPU2(J)
125   FORMAT(' ',T40,2(3X,E13.6),10X,E13.6)
C
C     READ IN TEST HYDROGRAPH AND TIME ARRAY TO COMPARE TO COMPUTED HYDROGRAPH
C
      DO 41 I=1,30
      READ(IREAD,42)TINPU3(I)
42    FORMAT(F10.0)
41    CONTINUE
      DO 43 I=1,30
      READ(IREAD,44)XINPU3(I)
44    FORMAT(F10.0)
43    CONTINUE
      DO 45 I=1,30
      READ(IREAD,46)XINPU4(I)
46    FORMAT(F10.0)
45    CONTINUE
      INP=30
      WRITE(IRITE,50)
50    FORMAT('1','INTERPOLATED RESULTS FROM COMPLETE SOLUTION(TIME,FUNC
     1)'//)
      DO 51 I=1,INP
      XIN(I)=XINPU3(I)
51    CONTINUE
C
C     INTERPOLATE TEST HYDROGRAPH TO BE COMPATABLE WITH COMPUTED HYDROGRAPH
C     TIME STEPS
C
      DO 52 I=1,N
      DEL=TINPU2(I)
      CALL INTRPL(INP,TINPU3,XIN,DEL,YINTP,2)
      WRITE(IRITE,53)DEL,YINTP
53    FORMAT(' ',10X,2(E13.6,5X))
      YINSTO(I)=YINTP
52    CONTINUE
      DO 80 J=1,N
80    XINPU3(J)=YINSTO(J)
      DO 54 I=1,INP
      XIN(I)=XINPU4(I)
54    CONTINUE
      WRITE(IRITE,55)
55    FORMAT('1','INTERPOLATED RESULTS FOR HARLEY SOLUTION(TIME,FUNC)'//
     1)
      DO 56 I=1,N
      DEL=TINPU2(I)
      CALL INTRPL(INP,TINPU3,XIN,DEL,YINTP,2)
      WRITE(IRITE,53)DEL,YINTP
      YINSTO(I)=YINTP
56    CONTINUE
      DO 81 J=1,N
81    XINPU4(J)=YINSTO(J)
C
C     PLOTTER
C
      TITLE(1,1)=TIT12
      TITLE(1,2)=TIT22
      TITLE(1,3)=TIT32
      TITLE(1,4)=TIT42
      TITLE(2,1)=TIT52
      TITLE(2,2)=TIT62
      TITLE(2,3)=TIT32
      TITLE(2,4)=TIT42
      TITLE(3,1)=TIT1E
      TITLE(3,2)=TIT2E
      TITLE(3,3)=TIT3E
```

```
      TITLE(3,4)=TIT4E
      TITLE(4,1)=TIT1E
      TITLE(4,2)=TIT5E
      TITLE(4,3)=TIT6E
      TITLE(4,4)=TIT4E
C     XINPU4= HARLEY SOLUTION...XINPU3= COMPLETE SOLUTION
      DO  915 J=1,N
      QO(4,J)=XINPU3(J)
      QO(3,J)=XINPU4(J)
      QO(2,J)=REAL(OUTPUT(J))
915   QO(1,J)=REAL(XINSTO(J))
      WRITE(IRITE,949)
949   FORMAT('1')
      CALL GPLOX(1J,TINPU2,QO,4,N,0.,TITLE,ISH)
950   FORMAT(' ',5(2X,I5))
      CALL EXIT
      END
```

A-2 FAST FOURIER TRANSFORM PROGRAM

```
C     THE COOLEY-TUKEY FAST FOURIER TRANSFORM IN USASI BASIC FORTRAN
C     TRANSFORM(J1,J2,...) = SUM(DATA(I1,I2,...)*W1**((I1-1)*(J1-1))
C                                 *W2**((I2-1)*(J2-1))*...),
C     SQRT(-1)/NN(1)), ETC.  THERE IS NO LIMIT ON THE DIMENSIONALITY
C     (NUMBER OF SUBSCRIPTS) OF THE DATA ARRAY.  IF AN INVERSE
C     WHERE I1 AND J1 RUN FROM 1 TO NN(1) AND W1=EXP(ISIGN*2*PI*
C     TRANSFORM (ISIGN=+1) IS PERFORMED UPON AN ARRAY OF TRANSFORMED
C     (ISIGN=-1) DATA, THE ORIGINAL DATA WILL REAPPEAR,
C     MULTIPLIED BY NN(1)*NN(2)*...  THE ARRAY OF INPUT DATA MAY BE
C     REAL OR COMPLEX, AT THE PROGRAMMERS OPTION, WITH A SAVING OF
C     UP TO FORTY PER CENT IN RUNNING TIME FOR REAL OVER COMPLEX.
C     (FOR FASTEST TRANSFORM OF REAL DATA, NN(1) SHOULD BE EVEN.)
C     THE TRANSFORM VALUES ARE ALWAYS COMPLEX, AND ARE RETURNED IN THE
C     ORIGINAL ARRAY OF DATA, REPLACING THE INPUT DATA.  THE LENGTH
C     OF EACH DIMENSION OF THE DATA ARRAY MAY BE ANY INTEGER.  THE
C     PROGRAM RUNS FASTER ON COMPOSITE INTEGERS THAN ON PRIMES, AND IS
C     PARTICULARLY FAST ON NUMBERS RICH IN FACTORS OF TWO.
C     TIMING IS IN FACT GIVEN BY THE FOLLOWING FORMULA.  LET NTOT BE THE
C     TOTAL NUMBER OF POINTS (REAL OR COMPLEX) IN THE DATA ARRAY, THAT
C     IS, NTOT=NN(1)*NN(2)*...  DECOMPOSE NTOT INTO ITS PRIME FACTORS,
C     SUCH AS 2**K2 * 3**K3 * 5**K5 * ...  LET SUM2 BE THE SUM OF ALL
C     THE FACTORS OF TWO IN NTOT, THAT IS, SUM2 = 2*K2.  LET SUMF BE
C     THE SUM OF ALL OTHER FACTORS OF NTOT, THAT IS, SUMF = 3*K3+5*K5+..
C     THE TIME TAKEN BY A MULTIDIMENSIONAL TRANSFORM ON THESE NTOT DATA
C     IS T = T0 + NTOT*(T1+T2*SUM2+T3*SUMF).  ON THE CDC 3300 (FLOATING
C     POINT ADD TIME = SIX MICROSECONDS), T = 3000 + NTOT*(600+40*SUM2+
C     175*SUMF) MICROSECONDS ON COMPLEX DATA.
C
C     IMPLEMENTATION OF THE DEFINITION BY SUMMATION WILL RUN IN A TIME
C     PROPORTIONAL TO NTOT*(NN(1)+NN(2)+...).  FOR HIGHLY COMPOSITE NTOT
C     THE SAVINGS OFFERED BY THIS PROGRAM CAN BE DRAMATIC.  A ONE-DIMEN-
C     SIONAL ARRAY 4000 IN LENGTH WILL BE TRANSFORMED IN 4000*(600+
C     40*(2+2+2+2+2)+175*(5+5+5)) = 14.5 SECONDS VERSUS ABOUT 4000*
C     4000*175 = 2800 SECONDS FOR THE STRAIGHTFORWARD TECHNIQUE.
C
C     THE CALLING SEQUENCE IS--
C     CALL FOURT(DATA,NN,NDIM,ISIGN,IFORM,WORK)
C
C     DATA IS THE ARRAY USED TO HOLD THE REAL AND IMAGINARY PARTS
C     OF THE DATA ON INPUT AND THE TRANSFORM VALUES ON OUTPUT.  IT
C     IS A MULTIDIMENSIONAL FLOATING POINT ARRAY, WITH THE REAL AND
C     IMAGINARY PARTS OF A DATUM STORED IMMEDIATELY ADJACENT IN STORAGE
C     (SUCH AS FORTRAN IV PLACES THEM).  THE EXTENT OF EACH DIMENSION
C     IS GIVEN IN THE INTEGER ARRAY NN, OF LENGTH NDIM.  ISIGN IS -1
C     TO INDICATE A FORWARD TRANSFORM (EXPONENTIAL SIGN IS -) AND +1
C     FOR AN INVERSE TRANSFORM (SIGN IS +).  IFORM IS +1 IF THE DATA AND
C     THE TRANSFORM VALUES ARE COMPLEX.  IT IS 0 IF THE DATA ARE REAL
C     BUT THE TRANSFORM VALUES ARE COMPLEX.  IF IT IS 0, THE IMAGINARY
C     PARTS OF THE DATA SHOULD BE SET TO ZERO.  AS EXPLAINED ABOVE, THE
C     TRANSFORM VALUES ARE ALWAYS COMPLEX AND ARE STORED IN ARRAY DATA.
C     WORK IS AN ARRAY USED FOR WORKING STORAGE.  IT IS NOT NECESSARY
C     IF ALL THE DIMENSIONS OF THE DATA ARE POWERS OF TWO.  IN THIS CASE
C     IT MAY BE REPLACED BY 0 IN THE CALLING SEQUENCE.  THUS, USE OF
C     POWERS OF TWO CAN FREE A GOOD DEAL OF STORAGE.  IF ANY DIMENSION
C     IS NOT A POWER OF TWO, THIS ARRAY MUST BE SUPPLIED.  IT IS
C     FLOATING POINT, ONE DIMENSIONAL OF LENGTH EQUAL TO TWICE THE
C     LARGEST ARRAY DIMENSION (I.E., NN(I) ) THAT IS NOT A POWER OF
C     TWO.  THEREFORE, IN ONE DIMENSION FOR A NON POWER OF TWO,
C     WORK OCCUPIES AS MANY STORAGE LOCATIONS AS DATA.  IF SUPPLIED,
C     WORK MUST NOT BE THE SAME ARRAY AS DATA.  ALL SUBSCRIPTS OF ALL
C     ARRAYS BEGIN AT ONE.
C
C     THE FAST FOURIER ALGORITHM PLACES TWO RESTRICTIONS UPON THE
C     NATURE OF THE DATA BEYOND THE USUAL RESTRICTION THAT
C     THE DATA FORM ONE CYCLE OF A PERIODIC FUNCTION.  THEY ARE--
C     1.  THE NUMBER OF INPUT DATA AND THE NUMBER OF TRANSFORM VALUES
C     MUST BE THE SAME.
C     2. CONSIDERING THE DATA TO BE IN THE TIME DOMAIN,
C     THEY MUST BE EQUI-SPACED AT INTERVALS OF DT.  FURTHER, THE TRANS-
C     FORM VALUES, CONSIDERED TO BE IN FREQUENCY SPACE, WILL BE EQUI-
C     SPACED FROM 0 TO 2*PI*(NN(I)-1)/(NN(I)*DT) AT INTERVALS OF
C     2*PI/(NN(I)*DT) FOR EACH DIMENSION OF LENGTH NN(I).  OF COURSE,
C     DT NEED NOT BE THE SAME FOR EVERY DIMENSION.
C
C     THERE ARE NO ERROR MESSAGES OR ERROR HALTS IN THIS PROGRAM.  THE
C     PROGRAM RETURNS IMMEDIATELY IF NDIM OR ANY NN(I) IS LESS THAN ONE.
C     EXAMPLE 1.  THREE-DIMENSIONAL FORWARD FOURIER TRANSFORM OF A
C     COMPLEX ARRAY DIMENSIONED 32 BY 25 BY 13 IN FORTRAN IV.
C     DIMENSION DATA(32,25,13),WORK(50),NN(3)
C     COMPLEX DATA
C     DATA NN/32,25,13/
C     DO 1 I=1,32
C     DO 1 J=1,25
C     DO 1 K=1,13
C   1 DATA(I,J,K)=COMPLEX VALUE
C     CALL FOURT(DATA,NN,3,-1,1,WORK)
C
C     EXAMPLE 2.  ONE-DIMENSIONAL FORWARD TRANSFORM OF A REAL ARRAY OF
C     LENGTH 64 IN FORTRAN II.
C     DIMENSION DATA(2,64)
C     DO 2 I=1,64
C     DATA(1,I)=REAL PART
C   2 DATA(2,I)=0.
C     CALL FOURT(DATA,64,1,-1,0,0)
C
C     PROGRAM BY NORMAN BRENNER FROM THE BASIC PROGRAM BY CHARLES
C     RADER (BOTH OF MIT LINCOLN LABORATORY).  MAY 1967.  THE IDEA
C     FOR THE DIGIT REVERSAL WAS SUGGESTED BY RALPH ALTER (ALSO MIT LL).
C     THIS IS THE FASTEST AND MOST VERSATILE VERSION OF THE FFT KNOWN
C     TO THE AUTHOR.  A PROGRAM CALLED FOUR2 IS AVAILABLE THAT ALSO
C     PERFORMS THE FAST FOURIER TRANSFORM AND IS WRITTEN IN USASI BASIC
C     FORTRAN.  IT IS ABOUT ONE THIRD AS LONG AND RESTRICTS THE
C     DIMENSIONS OF THE INPUT ARRAY (WHICH MUST BE COMPLEX) TO BE POWERS
C     OF TWO.  ANOTHER PROGRAM, CALLED FOUR1, IS ONE TENTH AS LONG AND
C     RUNS TWO THIRDS AS FAST ON A ONE-DIMENSIONAL COMPLEX ARRAY WHOSE
C     LENGTH IS A POWER OF TWO.
```

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C
C     REFERENCE--
C     FAST FOURIER TRANSFORMS FOR FUN AND PROFIT, W. GENTLEMAN AND
C     G. SANDE, 1966 FALL JOINT COMPUTER CONFERENCE.
C
C     THE WORK REPORTED IN THIS DOCUMENT WAS PERFORMED AT LINCOLN LAB-
C     ORATORY, A CENTER FOR RESEARCH OPERATED BY MASSACHUSETTS INSTITUTE
C     OF TECHNOLOGY, WITH THE SUPPORT OF THE U.S. AIR FORCE UNDER
C     CONTRACT AF 19(628)-5167.
C
C     THE FAST FOURIER TRANSFORM IN USASI BASIC FORTRAN
C
      SUBROUTINE FOURT(DATA,NN,NDIM,ISIGN,IFORM,WORK)
      DIMENSION DATA(1),NN(1),IFACT(32),WORK(1)
      TWOPI=6.2831853307
      RTHLF=.70710 67812
      IF(NDIM-1)920,1,1
1     NTOT=2
      DO 2 IDIM=1,NDIM
      IF(NN(IDIM))920,920,2
2     NTOT=NTOT*NN(IDIM)
C
C     MAIN LOOP FOR EACH DIMENSION
C
      NP1=2
      DO 910 IDIM=1,NDIM
      N=NN(IDIM)
      NP2=NP1*N
      IF(N-1)920,900,5
C
C     IS N A POWER OF TWO AND IF NOT, WHAT ARE ITS FACTORS
C
5     M=N
      NTWO=NP1
      IF=1
      IDIV=2
10    IQUOT=M/IDIV
      IREM=M-IDIV*IQUOT
      IF(IQUOT-IDIV)50,11,11
11    IF(IREM)20,12,20
12    NTWO=NTWO+NTWO
      IFACT(IF)=IDIV
      IF=IF+1
      M=IQUOT
      GO TO 10
20    IDIV=3
      INON2=IF
30    IQUOT=M/IDIV
      IREM=M-IDIV*IQUOT
      IF(IQUOT-IDIV)60,31,31
31    IF(IREM)40,32,40
32    IFACT(IF)=IDIV
      IF=IF+1
      M=IQUOT
      GO TO 30
40    IDIV=IDIV+2
      GO TO 30
50    INON2=IF
      IF(IREM)60,51,60
51    NTWO=NTWO+NTWO
      GO TO 70
60    IFACT(IF)=M
70    NON2P=NP2/NTWO
C
C     SEPARATE FOUR CASES--
C        1. COMPLEX TRANSFORM
C        2. REAL TRANSFORM FOR THE 2ND, 3RD, ETC. DIMENSION.  METHOD--
C           TRANSFORM HALF THE DATA, SUPPLYING THE OTHER HALF BY CON-
C           JUGATE SYMMETRY.
C        3. REAL TRANSFORM FOR THE 1ST DIMENSION, N ODD.  METHOD--
C           SET THE IMAGINARY PARTS TO ZERO.
C        4. REAL TRANSFORM FOR THE 1ST DIMENSION, N EVEN.  METHOD--
C           TRANSFORM A COMPLEX ARRAY OF LENGTH N/2 WHOSE REAL PARTS
C           ARE THE EVEN NUMBERED REAL VALUES AND WHOSE IMAGINARY PARTS
C           ARE THE ODD NUMBERED REAL VALUES.  SEPARATE AND SUPPLY
C           THE SECOND HALF BY CONJUGATE SYMMETRY.
C
      IFMIN = 1
      IIRNG = NP1
      IF (IFORM .LE. 0 .AND. IDIM .LT. 4) GO TO 71
      ICASE = 1
      GO TO 100
C
71    IF (IDIM .LE. 1) GO TO 72
      ICASE = 2
      IIRNG = NP0 * (1 + NPREV / 2)
      GO TO 100
C
72    IF (NTWO .GT. NP1) GO TO 73
      ICASE = 3
      GO TO 100
C
73    ICASE = 4
      IFMIN = 2
      NTWO = NTWO / 2
      N = N / 2
      NP2 = NP2 / 2
      NTOT = NTOT / 2
C
      I = - 1
      DO 80 J = 1, NTOT
         I = I + 2
         DATA(J) = DATA(I)
80    CONTINUE
C
C     SHUFFLE DATA BY BIT REVERSAL, SINCE N=2**K.  AS THE SHUFFLING
```

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C     CAN BE DONE BY SIMPLE INTERCHANGE, NO WORKING ARRAY IS NEEDED
C
100   IF(NON2P-1)101,101,200
101   NP2HF=NP2/2
      J=1
      DO 150 I2=1,NP2,NP1
      IF(J-I2)121,130,130
121   I1MAX=I2+NP1-2
      DO 125 I1=I2,I1MAX,2
      DO 125 I3=I1,NTOT,NP2
      J3=J+I3-I2
      TEMPR=DATA(I3)
      TEMPI=DATA(I3+1)
      DATA(I3)=DATA(J3)
      DATA(I3+1)=DATA(J3+1)
      DATA(J3)=TEMPR
125   DATA(J3+1)=TEMPI
130   M=NP2HF
140   IF(J-M)150,150,141
141   J=J-M
      M=M/2
      IF(M-NP1)150,140,140
150   J=J+M
      GO TO 300
C
C     SHUFFLE DATA BY DIGIT REVERSAL FOR GENERAL N
C
200   NWORK=2*N
      DO 270 I1=1,NP1,2
      DO 270 I3=I1,NTOT,NP2
      J=I3
      DO 260 I=1,NWORK,2
      IF(ICASE-3)210,220,210
210   WORK(I)=DATA(J)
      WORK(I+1)=DATA(J+1)
      GO TO 240
220   WORK(I)=DATA(J)
      WORK(I+1)=0.
240   IFP2=NP2
      IF=IFMIN
250   IFP1=IFP2/IFACT(IF)
      J=J+IFP1
      IF(J-I3-IFP2)260,255,255
255   J=J-IFP2
      IFP2=IFP1
      IF=IF+1
      IF(IFP2-NP1)260,260,250
260   CONTINUE
      I2MAX=I3+NP2-NP1
      I=1
      DO 270 I2=I3,I2MAX,NP1
      DATA(I2)=WORK(I)
      DATA(I2+1)=WORK(I+1)
270   I=I+2
C
C     MAIN LOOP FOR FACTORS OF TWO.
C     W=EXP(ISIGN*2*PI*SQRT(-1)*M/(4*MMAX)).  CHECK FOR W=ISIGN*SQRT(-1)
C     AND REPEAT FOR W=W*(1+ISIGN*SQRT(-1))/SQRT(2).
C
300   IF(NTWO-NP1)600,600,305
305   NP1TW=NP1+NP1
      IPAR=NTWO/NP1
310   IF(IPAR-2)350,330,320
320   IPAR=IPAR/4
      GO TO 310
330   DO 340 I1=1,I1RNG,2
      DO 340 K1=I1,NTOT,NP1TW
      K2=K1+NP1
      TEMPR=DATA(K2)
      TEMPI=DATA(K2+1)
      DATA(K2)=DATA(K1)-TEMPR
      DATA(K2+1)=DATA(K1+1)-TEMPI
      DATA(K1)=DATA(K1)+TEMPR
340   DATA(K1+1)=DATA(K1+1)+TEMPI
350   MMAX=NP1
360   IF(MMAX-NTWO/2)370,600,600
370   LMAX=MAX0(NP1TW,MMAX/2)
      DO 570 L=NP1,LMAX,NP1TW
      M=L
      IF(MMAX-NP1)420,420,380
380   THETA=-TWOPI*FLOAT(L)/FLOAT(4*MMAX)
      IF(ISIGN)400,390,390
390   THETA=-THETA
400   WR=COS(THETA)
      WI=SIN(THETA)
410   W2R=WR*WR-WI*WI
      W2I=2.*WR*WI
      W3R=W2R*WR-W2I*WI
      W3I=W2R*WI+W2I*WR
420   DO 530 I1=1,I1RNG,2
      KMIN=I1+IPAR*M
      IF(MMAX-NP1)430,430,440
430   KMIN=I1
440   KDIF=IPAR*MMAX
450   KSTEP=4*KDIF
      IF(KSTEP-NTWO)460,460,530
460   DO 520 K1=KMIN,NTOT,KSTEP
      K2=K1+KDIF
      K3=K2+KDIF
      K4=K3+KDIF
      IF(MMAX-NP1)470,470,480
470   U1R=DATA(K1)+DATA(K2)
      U1I=DATA(K1+1)+DATA(K2+1)
      U2R=DATA(K3)+DATA(K4)
      U2I=DATA(K3+1)+DATA(K4+1)
      U3R=DATA(K1)-DATA(K2)
```

```
      U3I=DATA(K1+1)-DATA(K2+1)
      IF(ISIGN)471,472,472
471   U4R=DATA(K3+1)-DATA(K4+1)
      U4I=DATA(K4)-DATA(K3)
      GO TO 510
472   U4R=DATA(K4+1)-DATA(K3+1)
      U4I=DATA(K3)-DATA(K4)
      GO TO 510
480   T2R=W2R*DATA(K2)-W2I*DATA(K2+1)
      T2I=W2R*DATA(K2+1)+W2I*DATA(K2)
      T3R=WR*DATA(K3)-WI*DATA(K3+1)
      T3I=WR*DATA(K3+1)+WI*DATA(K3)
      T4R=W3R*DATA(K4)-W3I*DATA(K4+1)
      T4I=W3R*DATA(K4+1)+W3I*DATA(K4)
      U1R=DATA(K1)+T2R
      U1I=DATA(K1+1)+T2I
      U2R=T3R+T4R
      U2I=T3I+T4I
      U3R=DATA(K1)-T2R
      U3I=DATA(K1+1)-T2I
      IF(ISIGN)490,500,500
490   U4R=T3I-T4I
      U4I=T4R-T3R
      GO TO 510
500   U4R=T4I-T3I
      U4I=T3R-T4R
510   DATA(K1)=U1R+U2R
      DATA(K1+1)=U1I+U2I
      DATA(K2)=U3R+U4R
      DATA(K2+1)=U3I+U4I
      DATA(K3)=U1R-U2R
      DATA(K3+1)=U1I-U2I
      DATA(K4)=U3R-U4R
520   DATA(K4+1)=U3I-U4I
      KDIF=KSTEP
      KMIN=4*(KMIN-I1)+I1
      GO TO 450
530   CONTINUE
      M=M+LMAX
      IF(M-MMAX)540,540,570
540   IF(ISIGN)550,560,560
550   TEMPR=WR
      WR=(WR+WI)*RTHLF
      WI=(WI-TEMPR)*RTHLF
      GO TO 410
560   TEMPR=WR
      WR=(WR-WI)*RTHLF
      WI=(TEMPR+WI)*RTHLF
      GO TO 410
570   CONTINUE
      IPAR=3-IPAR
      MMAX=MMAX+MMAX
      GO TO 360
```

```
C
C     MAIN LOOP FOR FACTORS NOT EQUAL TO TWO.
C     W=EXP(ISIGN*2*PI*SQRT(-1)*(J1+J2-I3-1)/IFP2)
C
600   IF(NON2P-1)700,700,601
601   IFP1=NTWO
      IF=INON2
610   IFP2=IFACT(IF)*IFP1
      THETA=-TWOPI/FLOAT(IFACT(IF))
      IF(ISIGN)612,611,611
611   THETA=-THETA
612   WSTPR=COS(THETA)
      WSTPI=SIN(THETA)
      DO 650 J1=1,IFP1,NP1
      THETM=-TWOPI*FLOAT(J1-1)/FLOAT(IFP2)
      IF(ISIGN)614,613,613
613   THETM=-THETM
614   WMINR=COS(THETM)
      WMINI=SIN(THETM)
      I1MAX=J1+I1RNG-2
      DO 650 I1=J1,I1MAX,2
      DO 650 I3=I1,NTOT,NP2
      I=1
      WR=WMINR
      WI=WMINI
      J2MAX=I3+IFP2-IFP1
      DO 640 J2=I3,J2MAX,IFP1
      TWOWR=WR+WR
      J3MAX=J2+NP2-IFP2
      DO 630 J3=J2,J3MAX,IFP2
      JMIN=J3-J2+I3
      J=JMIN+IFP2-IFP1
      SR=DATA(J)
      SI=DATA(J+1)
      OLDSR=0.
      OLDSI=0.
      J=J-IFP1
620   STMPR=SR
      STMPI=SI
      SR=TWOWR*SR-OLDSR+DATA(J)
      SI=TWOWR*SI-OLDSI+DATA(J+1)
      OLDSR=STMPR
      OLDSI=STMPI
      J=J-IFP1
      IF(J-JMIN)621,621,620
621   WORK(I)=WR*SR-WI*SI-OLDSR+DATA(J)
      WORK(I+1)=WI*SR+WR*SI-OLDSI+DATA(J+1)
630   I=I+2
      WTEMP=WR*WSTPI
      WR=WR*WSTPR-WI*WSTPI
640   WI=WI*WSTPR+WTEMP
      I=1
      DO 650 J2=I3,J2MAX,IFP1
```

```
      J3MAX=J2+NP2-IFP2
      DO 650 J3=J2,J3MAX,IFP2
      DATA(J3)=WORK(I)
      DATA(J3+1)=WORK(I+1)
  650 I=I+2
      IF=IF+1
      IFP1=IFP2
      IF(IFP1-NP2)610,700,700
C
C     COMPLETE A REAL TRANSFORM IN THE 1ST DIMENSION, N EVEN, BY CON-
C     JUGATE SYMMETRIES.
C
  700 GO TO (900,800,900,701),ICASE
  701 NHALF=N
      N=N+N
      THETA=-TWOPI/FLOAT(N)
      IF(ISIGN)703,702,702
  702 THETA=-THETA
  703 WSTPR=COS(THETA)
      WSTPI=SIN(THETA)
      WR=WSTPR
      WI=WSTPI
      IMIN=3
      JMIN=2*NHALF-1
      GO TO 725
  710 J=JMIN
      DO 720 I=IMIN,NTOT,NP2
      SUMR=(DATA(I)+DATA(J))/2.
      SUMI=(DATA(I+1)+DATA(J+1))/2.
      DIFR=(DATA(I)-DATA(J))/2.
      DIFI=(DATA(I+1)-DATA(J+1))/2.
      TEMPR=WR*SUMI+WI*DIFR
      TEMPI=WI*SUMI-WR*DIFR
      DATA(I)=SUMR+TEMPR
      DATA(I+1)=DIFI+TEMPI
      DATA(J)=SUMR-TEMPR
      DATA(J+1)=-DIFI+TEMPI
  720 J=J+NP2
      IMIN=IMIN+2
      JMIN=JMIN-2
      WTEMP=WR*WSTPI
      WR=WR*WSTPR-WI*WSTPI
      WI=WI*WSTPR+WTEMP
  725 IF(IMIN-JMIN)710,730,740
  730 IF(ISIGN)731,740,740
  731 DO 735 I=IMIN,NTOT,NP2
  735 DATA(I+1)=-DATA(I+1)
  740 NP2=NP2+NP2
      NTOT=NTOT+NTOT
      J=NTOT+1
      IMAX=NTOT/2+1
  745 IMIN=IMAX-2*NHALF
      I=IMIN
      GO TO 755
  750 DATA(J)=DATA(I)
      DATA(J+1)=-DATA(I+1)
  755 I=I+2
      J=J-2
      IF(I-IMAX)750,760,760
  760 DATA(J)=DATA(IMIN)-DATA(IMIN+1)
      DATA(J+1)=0.
      IF(I-J)770,780,780
  765 DATA(J)=DATA(I)
      DATA(J+1)=DATA(I+1)
  770 I=I-2
      J=J-2
      IF(I-IMIN)775,775,765
  775 DATA(J)=DATA(IMIN)+DATA(IMIN+1)
      DATA(J+1)=0.
      IMAX=IMIN
      GO TO 745
  780 DATA(1)=DATA(1)+DATA(2)
      DATA(2)=0.
      GO TO 900
C
C     COMPLETE A REAL TRANSFORM FOR THE 2ND, 3RD,
C     CONJUGATE SYMMETRIES.
C
  800 IF(I1RNG-NP1)805,900,900
  805 DO 860 I3=1,NTOT,NP2
      I2MAX=I3+NP2-NP1
      DO 860 I2=I3,I2MAX,NP1
      IMAX=I2+NP1-2
      IMIN=I2+I1RNG
      JMAX=2*I3+NP1-IMIN
      IF(I2-I3)820,820,810
  810 JMAX=JMAX+NP2
  820 IF(IDIM-2)850,850,830
  830 J=JMAX+NP0
      DO 840 I=IMIN,IMAX,2
      DATA(I)=DATA(J)
      DATA(I+1)=-DATA(J+1)
  840 J=J-2
  850 J=JMAX
      DO 860 I=IMIN,IMAX,NP0
      DATA(I)=DATA(J)
      DATA(I+1)=-DATA(J+1)
  860 J=J-NP0
C
C     END OF LOOP ON EACH DIMENSION
C
  900 NP0=NP1
      NP1=NP2
  910 NPREV=N
  920 RETURN
      END
```

A-3 GROUNDWATER ROUTING MODEL

```
C*******************************************************************************
C
C      PROGRAM FOR ROUTING OF GROUND WATER USING A
C      FAST FOURIER TRANSFORM TECHNIQUE
C
C      JUNE 1972, D.H.EVANS
C
C*******************************************************************************
C
C      THIS PROGRAM IS DESIGNED FOR USE WITH SUBROUTINES RESPF AND SET
C
C*******************************************************************************
      COMPLEX XINPU2(500),FT(500),FCOM(500),J1,FSTO(3,500)
      REAL*8 DATE, TIMEX
      REAL*8 TL1,TL2,TL3,TL4,TL8,TL5,TL6,TL7,TITLE(10,10)
      REAL*8 TL9,TL10,TL11,TL12,TL13,TL14,TL15
      REAL K(10,3,5),NRES(10,3,5),LAG(10,3,5)
      INTEGER SYS(5,3),U,Z
      DIMENSION TINPU2(500),AXIN(50),TIN(50)
     1,WORK(500),QO(10,500),ISH(10)
      DIMENSION XINSTO(3,500),ICON(10),XIN(500)
      DIMENSION BFL(10),WDTH(3)
      COMMON/COMP/XINPU2,XINSTO,TINPU2
      COMMON/INPT/LAG,NRES,K,SYS,ICON,BFL,AXIN,TIN,NINPU2,NP1,NZEROS,
     1WTD,WTR,WTDR,WDTH,SFT1,SFT2,SFT3,SFT,NSFT
      COMMON/HARC/FT,N,NO,DELT,IX,Z,PI
      COMMON /IOPUT/IREAD,IRITE,ITERM,IPLOT
      COMMON/IDS/DATE,TIMEX
      DATA TL1,TL2,TL3,TL4,TL5,TL6,TL7,TL8/'OUTPUT H','YDROGRAP','H FROM
     1 ','FREQUENC','Y STORAG','E        ','TIME DOM','AIN      '/
      DATA TL9/'INPUT  H'/
      DATA TL10,TL11,TL12,TL13/'SYSTEM 1','SYSTEM 2','PARALLEL',' FLOW
     1H'/
      IRITE=6
      IREAD=5
      ITERM=6
      IPLOT=6
      PI=3.141593
      DO 1 I=1,10
1     ISH(I)=0
      Z=1
C
C      READ VARIABLES AND INITIALIZE PARAMETERS
C
      CALL SET
      DO 2 I=1,N
2     FCOM(I)=0.0
C*****...NO,TYPE MODEL.....IX,WHICH MODEL TYPE IS BEING CALC IN THAT SERIES
      DO 50 Z=1,NP1
C
C      LOOP PARALLEL SYSTEMS
C
C***** SET UP INPUT BASED ON THAT LEG OF THE PARALLEL CONFIGURATION
C
      GO TO (4,5,6),Z
C*****CHANGE THIS TO SUIT THE WEIGHTED INPUT DISTRIBUTION
4     XMUL=WTD*WDTH(1)
      NFT=SFT1/DELT+0.5
      GO TO 7
5     XMUL=WTR*WDTH(2)
      NFT=SFT2/DELT+0.5
      GO TO 7
6     XMUL=WTD*WDTH(3)
      NFT=SFT3/DELT+0.5
7     DO 10 I=1,NINPU2
      IF(Z.EQ.1)XIN(I)=XINSTO(1,I)
C
C      BASEFLOW SHOULD BE REMOVED HERE IF NECESSARY AND SHIFT THE TIME
C      SCALE BY THE TRANSLATIONAL LAG, NFT
C
      XINPU2(I)=XIN(I)*XMUL
      XINPU2(I+NFT)=XINPU2(I)
      IF(NFT.GT.0.AND.I.LE.NFT)XINPU2(I)=0.0
      XINSTO(Z,I)=XIN(I)*XMUL
      IF(ICON(1).EQ.0) GO TO 10
      WRITE(IRITE,8)I,XINPU2(I),TINPU2(I)
8     FORMAT(' ',I3,5X,3(E12.6,2X))
10    CONTINUE
C****************THOSE SECTIONS SURROUNDED BY :: ARE FOR APERIODIC FUNCTIONS
C:::::::::::::::::::::::
      GO TO 500
501   NZRO=NZEROS+NSFT
      DO 12 M=1,NZRO
      IN=NINPU2+M+NFT
      XINPU2(IN)=0.0
      XINSTO(Z,IN)=REAL(XINPU2(IN))
      TINPU2(IN)=TINPU2(NINPU2)+(M*DELT)
12    CONTINUE
500   CONTINUE
C:::::::::::::::::::::::
C
C      OBTAIN FORWARD TRANSFORM OF INPUT USING FOURT
C
      WRITE(IRITE,13)XMUL,NFT
13    FORMAT(' XMUL,NFT',2X,E13.6,5X,I5)
      CALL FOURT(XINPU2,N,1,-1,0,WORK)
      NL1=SYS(Z,1)
      NL2=SYS(Z,2)
      NL3=SYS(Z,3)
C     NXT, MAINTAINS COUNTER OF SERIES FORWITHDRAWING FLOW WITHIN EACH LEG
      NXT=0
C
C***** BEGIN THE SERIES LOOP
C
14    IF(NL1.EQ.0) GO TO 15
      LOOP=NL1
```

```
      NO=1
      NL1=0
      GO TO 18
15    IF(NL2.EQ.0) GO TO 16
      LOOP=NL2
      NO=2
      NL2=0
      GO TO 18
16    IF(NL3.EQ.0) GO TO 17
      LOOP=NL3
      NO=3
      NL3=0
      GO TO 18
17    GO TO 25
18    NXT=NXT+1
      DO 22 U=1,LOOP
      IX=U
      IF(ICON(4).EQ.0) GO TO 20
      WRITE(IRITE,19)Z,NO,IX,NL1,NL2,NL3
19    FORMAT(' Z,NO,IX,NL1,NL2,NL3=',6(2X,I4))
C
C     COMPUTE RESPONSE FUNCTIONS
C
20    CONTINUE
      CALL RESPF
C***** NOTE THAT FSTO() IS REINITIALIZED THROUGHOUT THE SERIES
C     THOUGH NOT MULTIPLYING RESPONSE BY RESPONSE TO OBTAIN THE SYSTEM
C     RESPONSE, BASICLY THE SAME BUT MULT. BY THE INPUT TRANSFORM,AS WELL,
C     AS THE SERIES DEVELOPS
C
C     MULTIPLY FUNCTIONS TO OBTAIN OUTPUT TRANSFORM
C
      DO 21 L=1,N
      FSTO(Z,L)=XINPU2(L)*FT(L)
      XINPU2(L)=FSTO(Z,L)
C*****FSTO MAY BE USED TO REMOVE THE SUM OF VOL OF EACH PARALLEL LEG
21    CONTINUE
C     SERIES LOOP-INNER TRANSFORM PART
22    CONTINUE
      IF(ICON(4).EQ.0) GO TO 24
      WRITE(IRITE,23)NL1,NL2,NL3
23    FORMAT(' LOOPING THRU TYPES:NL1,NL2,NL3',3I5)
24    CONTINUE
      GO TO 14
C     GONE THRU ALL CONFIGURATIONS OF SERIES READY FOR NEXT PARALLEL LEG
25    CONTINUE
      IF(ICON(4).EQ.0) GO TO 27
      WRITE(IRITE,26)
26    FORMAT(' COMPLETED LEG OF PARALLEL LOOP')
27    CONTINUE
C     PARALLEL  LOOP
50    CONTINUE
      IF(ICON(5).EQ.1) NP1=2
```

```
C
C     TOTALING THE FLOW FOR CONFIGURATION, INSURE NP1 REPRESENTS TOTAL DESIRED
C*****FROM HERE ON INSURE THAT EACH SYSTEM IS CORRECTLY TRANSFORMED  AND OUTPUT
C
28    DO 30 I=1,NP1
      DO 29 M=1,N
C     FCO1,2,3 MAY BE USED TO MAINTAIN AN INDIVIDUAL ACCOUNT OF EACH LEG
C     IF USED
C
C     COMBINE PARALLEL MEMBERS
C
29    FCOM(M)=FCOM(M)+FSTO(I,M)
30    CONTINUE
C
C     RETURN TO TIME DOMAIN
C
      CALL FOURT(FCOM,N,1,1,1,WORK)
      IF(ICON(2).EQ.0) GO TO 32
      WRITE(IRITE,31)
31    FORMAT(' PT',10X,'NASH OUTPUT TOTAL',10X,'TIME')
32    CONTINUE
      DO 35 I=1,N
C
C     ADD BASEFLOW HERE IF NECESSARY AND DIVIDE BY NUMBER OF POINTS
C     INPUT OT FOURT
C
      FCOM(I)=FCOM(I)/N+BFL(2)
      IF(ICON(2).EQ.0) GO TO 34
C
C     PRINT RESULTS
C
      WRITE(IRITE,33)I,FCOM(I),TINPU2(I)
33    FORMAT(' ',I3,3(3X,E13.6))
34    CONTINUE
35    CONTINUE
      IF(ICON(3).EQ.0) GO TO 100
C
C     PLOTTER
C
      TITLE(1,1)=TL1
      TITLE(1,2)=TL2
      TITLE(1,3)=TL3
      TITLE(1,4)=TL4
      TITLE(1,5)=TL5
      TITLE(1,6)=TL6
      TITLE(2,1)=TL9
      TITLE(2,2)=TL2
      TITLE(2,3)=TL3
      TITLE(2,4)=TL7
      TITLE(2,5)=TL8
      TITLE(2,6)=TL10
      TITLE(3,1)=TL9
      TITLE(3,2)=TL2
```

```
      TITLE(3,3)=TL3
      TITLE(3,4)=TL7
      TITLE(3,5)=TL8
      TITLE(3,6)=TL11
      DO 36 J=1,N
      QD(3,J)=XINSTO(2,J)
      QD(2,J)=XINSTO(1,J)
36    QD(1,J)=REAL(FCCM(J))
      IF(DELT.GT.1.) SCALE=1.0
      IF(DELT.LT.1.) SCALE=5.0
      IF(DELT.GT.10.) SCALE=0.0
      DELT=1.0
      WRITE(IRITE,37)
37    FORMAT('1')
      CALL GPLOX(10,TINPU2,QD,3,N,SCALE,TITLE,ISH)
100   CALL EXIT
      END
```

```
      SUBROUTINE SET
C
C     SUBROUTINE USED TO READ ALL PARAMETERS FOR 'MAIN' AND TO COMPUTE
C     THE CUMULANTS OF THE RESPONSE FUNCTION AND TO SET UP THE CONTROL PARAM.
C
      COMPLEX FT(500),XINPU2(500)
      REAL*8 DATE,TIMEX
      REAL K(10,3,5),NRES(10,3,5),LAG(10,3,5)
      REAL K1MIN,K2MIN,K3MIN,K1MAX,K2MAX,K3MAX,K1RES,KMAX,KMIN
      REAL KT(3)
      INTEGER SYS(5,3),Z
      DIMENSION SLP(2),BFL(10),ICON(10),AXIN(50),TIN(50),WDTH(3)
      DIMENSION XYN(50),TINPU2(500)
      DIMENSION XINSTO(3,500)
      COMMON/COMP/XINPU2,XINSTO,TINPU2
      COMMON/HARC/FT,N,NO,DELT,IX,Z,PI
      COMMON/INPT/LAG,NRES,K,SYS,ICON,BFL,AXIN,TIN,NINPU2,NP1,NZEROS,
     1WTD,WTR,WTDR,WDTH,SFT1,SFT2,SFT3,SFT,NSFT
      COMMON/IOPUT/IREAD,IRITE,ITERM,IPLOT
      COMMON/IDS/DATE,TIMEX
C
C
C**********CAUTION****USE PARALLEL SYSTEMS WHICH ARE COMPATABLE,I.E.
C**********************ONLY  THOSE WITH TIME STEPS ABOUT THE SAME, FOR SPACE
C
C      ICON(1),DATA;(2),RESULTS;(3),PLOTTER;(4),TEST PRINTS
C      (5) SPECIAL CONFIGURATION
C      RETURN WTS.: WTD=TO DRAINAGE, WTR=TO RIVER, WTDR= TO DOWN STREAM
C       -TINPMA- MAX TIME PERIOD OF INPUT (YEARS IF NEED BE)
C****  INSURE THAT 'NP1,NO AND IX' CORRESPOND TO DESIRED SYSTEM
C*****NP1,NO PARALLEL LEGS;NO AND IX DEFINED BELOW
C
C      READ VARIABLES
C
       READ(IREAD,50)DATE,TIMEX,ICON(1),ICON(2),ICON(3),ICON(4),ICON(5)
     1,ICODE
       READ(IREAD,51)TINPMA,NP1
       READ(IREAD,53) WTD,WTR,WTDR,(WDTH(I),I=1,3)
       READ(IREAD,53) PERM,(SLP(I),I=1,2),DR,DS,SCOEF,SMH
       NX=TINPMA+1.5
       DO 20 I=1,NX
 20    READ(IREAD,54)AXIN(I),TIN(I)
       DO 1  I=1,NP1
 1     READ(IREAD,55)SYS(I,1),SYS(I,2),SYS(I,3),BFL(I)
       KMIN=99999.
       KMAX=1.0
       K1MIN=9999.
       K2MIN=9999.
       K3MIN=9999.
       K1MAX=1.00
       K2MAX=1.00
       K3MAX=1.00
       SFT=0.0
       SFT1=0.0
       SFT2=0.0
       SFT3=0.0
C
C      IF DESIRE TO COMPUTE PARAMETERS BASED ON THEORETICAL SOLUTION
C      SET ICON(5)=1
C
       IF(ICON(5).EQ.1) GO TO 11
       DO 10 I=1,NP1
       S1=0.0
       S2=0.0
       S3=0.0
       KT(I)=0.0
C      CONTROL VARIABLE WHERE 2ND DIMENSION:1,NO. OF NASH MODELS,2, NO. OF
C      LAG AND ROUTE MODELS AND 3, RESERVOIR MODELS
       NL1=SYS(I,1)
       NL2=SYS(I,2)
       NL3=SYS(I,3)
C      INPUT PARAMETERS: 1ST DIM.= CONFIG., 2ND DIM.= MODEL TYPE,3RD DIM= NO.
C      IN THAT SERIES
       IF(NL1.EQ.0) GO TO 4
       DO 3 J=1,NL1
C
C      INPUT PARAMETERS BASED ON MODEL TYPE AND CONFIGURATION
C
       READ(IREAD,54)NRES(I,1,J),K(I,1,J),LAG(I,1,J)
       IF(ICON(1).EQ.0) GO TO 2
       WRITE(IRITE,70)NRES(I,1,J),K(I,1,J),LAG(I,1,J),I,J
 2     CONTINUE
       K1RES=NRES(I,1,J)*K(I,1,J)
       KT(I)=KT(I)+K1RES
       S1=S1+LAG(I,1,J)
       IF(K1MAX.LT.K1RES)K1MAX=K1RES
 3     IF(K1MIN.GT.K1RES) K1MIN=K1RES
 4     IF(NL2.EQ.0) GO TO 7
       DO 6 J=1,NL2
       NRES(I,2,J)=0.0
       READ(IREAD,54)K(I,2,J),LAG(I,2,J)
       IF(ICON(1).EQ.0) GO TO 5
       WRITE(IRITE,71)K(I,2,J),LAG(I,2,J),I,J
 5     CONTINUE
       X2RES=K(I,2,J)
       KT(I)=KT(I)+X2RES
       S2=S2+LAG(I,2,J)
       IF(K2MAX.LT.X2RES)K2MAX=X2RES
 6     IF(K2MIN.GT.X2RES)K2MIN=X2RES
 7     IF(NL3.EQ.0) GO TO 21
       DO 9 J=1,NL3
       NRES(I,3,J)=0.0
       LAG(I,3,J)=0.0
       READ(IREAD,54)K(I,3,J)
       IF(ICON(1).EQ.0) GO TO 8
       WRITE(IRITE,72)K(I,3,J),I,J
```

```
9       CONTINUE
        KT(I)=KT(I)+K(I,3,J)
        SFT3=0.0
        IF(K3MAX.LT.K(I,3,J))K3MAX=K(I,3,J)
9       IF(K3MIN.GT.K(I,3,J))K3MIN=K(I,3,J)
C       CHECKING FOR LONGEST SERIES STRING
21      GO TO (22,23,24),I
22      SFT1=S1+S2+S3
        IF(SFT.LT.SFT1)SFT=SFT1
        IF(KMAX.LT.K1MAX)KMAX=K1MAX
        IF(KMIN.GT.K1MIN)KMIN=K1MIN
        GO TO 25
23      SFT2=S1+S2+S3
        IF(SFT.LT.SFT2)SFT=SFT2
        IF(KMAX.LT.K2MAX)KMAX=K2MAX
        IF(KMIN.GT.K3MIN)KMIN=K3MIN
        GO TO 25
24      SFT3=S1+S2+S3
        IF(SFT.LT.SFT3)SFT=SFT3
        IF(KMAX.LT.K3MAX)KMAX=K3MAX
        IF(KMIN.GT.K2MIN)KMIN=K2MIN
25      CONTINUE
        IF(KMIN.GT.KT(I)) KMIN=KT(I)
        IF(KMAX.LT.KT(I)) KMAX=KT(I)
10      CONTINUE
        GO TO 15
C
C       DETERMINING THE PARAMETER OF RESPONSE FUNC. BASED ON GOVERNING EQN
C       AND THE USE OF LAPLACE TRANSFORMS
C       K=CUM3/(2*CUM2);N=CUM2/K**2;TAU=CUM1-NK
C
11      IF(DR.EQ.0.0) GO TO 26
        VS=PERM*SLP(1)
        K(2,1,1)=3*PERM*SMH*SCOEF/VS**2
        NRES(2,1,1)=2*DR*VS/(S*PERM*SMH)
        LAG(2,1,1)=DR*SCOEF/(3*VS)
        KMAX=K(2,1,1)*NRES(2,1,1)
        KMIN=KMAX
26      IF(DS.EQ.0.0) GO TO 27
        VS=PERM*SLP(2)
        K(3,1,1)=3*PERM*SMH*SCOEF/VS**2
        NRES(3,1,1)=2*DS*VS/(S*PERM*SMH)
        LAG(3,1,1)=DS*SCOEF/(3*VS)
        K1MIN=K(3,1,1)*NRES(3,1,1)
        IF (KMIN.GT.K1MIN)KMIN=K1MIN
        IF (KMAX.LT.K1MIN)KMAX=K1MIN
C*************PUT IN PARAMETERS FOR DRAINAGE SYSTEM
C        IF DRAINAGE SYSTEM IS NOT REQUIRED HERE PUT IN BLANK CARD
27      READ(IREAD,54)NRES(1,1,1),K(1,1,1),LAG(1,1,1)
        IF(NRES(1,1,1).EQ.0.0) GO TO 28
        K1MIN=NRES(1,1,1)*K(1,1,1)
        IF (KMIN.GT.K1MIN)KMIN=K1MIN
        IF (KMAX.LT.K1MIN)KMAX=K1MIN
28      SFT=0.0
        SFT1=LAG(1,1,1)
        SFT2=LAG(2,1,1)
        SFT3=LAG(3,1,1)
        IF(SFT.LT.SFT1)SFT=SFT1
        IF(SFT.LT.SFT2)SFT=SFT2
        IF(SFT.LT.SFT3)SFT=SFT3
C       FIND W NEEDED TO KEEP 98 O/O RESPONSE ENERGY
15      WNOT=28.636/KMIN
C       FIND NUMBER OF ZEROS TO ADD TO FUNCTIONS AND FIGURE
C       CUT TMAX FOR RESPONSE FUNCTION TO KEEP 99 O/O OF AREA
        TRESMA=-KMAX*ALOG(.01)
        DELT=PI/WNOT
C       LENGTH OF OUTPUT=TIME INPUT+TIME RESP
C       SO NUMBER OF PTS IN OUTPUT IS (TINPMA)+TRESMA)/DELT
C       OR N=(TINPMA+TRESMA)/DELT.  THEREFORE THE NUMBER
C       OF ZEROS TO ADD=TRESMA/DELT
        NINPU2=TINPMA/DELT+1.5
C:::::::::::::::::::::::
        NZEROS=TRESMA/DELT+0.5
        NSFT=SFT/DELT+0.5
        NZEROS=0
        NSFT=0
        WRITE(IRITE,999)WNOT,TRESMA,DELT,NINPU2,NZEROS,NSFT
999     FORMAT(' WNOT,TRESMA,DELT,NINPU2,NZEROS,NSFT',3(E11.4,1X),3(I5,1X)
       1)
        N=NINPU2+NZEROS+NSFT
C:::::::::::::::::::::::
C
C       SET UP INPUT AND TIME ARRAYS
C
        IF(NINPU2.GE.NX) GO TO 65
        NN=0
        I=1
61      TX=DELT*I
        XYN(I)=0.0
62      NN=NN+1
        IF(NN.EQ.NX)TIN(NN+1)=999.0
        XYN(I)=XYN(I)+AXIN(NN)
        NA=NN+1
        IF(TIN(NA).GT.TX) GO TO 63
        IF(NA.LE.NX) GO TO 62
63      XINSTO(Z,I)=XYN(I)
        XINPU2(I)=XYN(I)
        TINPU2(I)=DELT*(I-1)
        I=I+1
        NA=NN+1
        IF(NA.LE.NX) GO TO 61
        NOX=NX-(I-1)
        DO 64 L=1,NOX
        XINPU2(I-1+L)=0.0
64      TINPU2(I-1+L)=(I-1+L)*DELT
        GO TO 43
```

```
65    DO 42 I=1,NINPU2
      IF(NINPU2.EQ.NX) GO TO 41
      TI=DELT*(I-1)
      TINPU2(I)=TI
      IF(ICODE.EQ.0) CALL INTRPL(NX,TIN,AXIN,TI,YINTP,1)
      IF(ICODE.EQ.1) CALL BLKINT(NX,TIN,AXIN,TI,YINTP,1.0)
      IF(YINTP.LT.0.)YINTP=0.0
      XINSTO(Z,I)=YINTP
41    IF(NINPU2.EQ.NX)XINSTO(Z,I)=AXIN(I)
42    XINPU2(I)=XINSTO(Z,I)
43    CONTINUE
      IF(ICON(1).EQ.0) GO TO 18
C
C     WRITE VARIABLES IF SO DESIRED
C
      WRITE(IRITE,73)TINPMA,NP1
      WRITE(IRITE,74)PERM,(SLP(I),I=1,2),DP,DS,SCOEF,SMH
      DO 16 I=1,NX
16    WRITE(IRITE,75)I,NX,AXIN(I),TIN(I)
      DO 17 I=1,NP1
17    WRITE(IRITE,76)SYS(I,1),SYS(I,2),SYS(I,3)
      WRITE(IRITE,77)TRESMA,DELT,KMAX,KMIN,WNOT,NINPU2
      WRITE(IRITE,78) N,NZEROS,NSFT
      DO 19 I=1,NP1
19    WRITE(IRITE,79)K(I,1,1),NRES(I,1,1),LAG(I,1,1),I
      WRITE(IRITE,80)WTD,WTF,WTDR,WDTH(1),WDTH(2),WDTH(3)
18    CONTINUE
      RETURN
50    FORMAT(2A8,6I1)
51    FORMAT(1F10.0,I2)
52    FORMAT(7F10.0)
54    FORMAT(3F10.0)
55    FORMAT(3I2,F10.0)
70    FORMAT(' NRES,K,LAG,I,J',3(3X,E13.6),2(2X,I4))
71    FORMAT(' K,LAG,I,J',2(3X,E13.6),2(2X,I4))
72    FORMAT(' K,I,J',1(2X,E13.6),2(2X,I4))
73    FORMAT(' TINPMA,NP1',1(3X,E13.6),2X,I4)
74    FORMAT(' PERM,SLP(1),SLP(2),DR,DS,SCOEF,SMH=',7(2X,E11.4))
75    FORMAT(' ',' I=',I6,'NX=',I6,'INPUT',2X,E13.6,'TIME',2X,E13.6)
76    FORMAT(' SYS(1),(2),(3)',3(4X,I5))
77    FORMAT(' TRESMA,DELT,KMAX,KMIN,WNOT,NINPU2',5(2X,E13.6),2X,I5)
78    FORMAT(' N,NZEROS,NSFT',2X,I4,2(5X,I4))
79    FORMAT( ' K,NRES,LAG,I',3(E11.4,2X),I2)
80    FORMAT(' WTD,WTR,WTDR,WDTH-1-2-3-',6(E11.4,2X))
81    FORMAT(' J,NU,XNU,XYN(J)',2I6,2F10.0)
      END
```

```
      SUBROUTINE RESPF
C
C     SUBROUTINE TO COMPUTE RESPONSE FUNCTION
C
      COMPLEX J1,FT(500)
      REAL K(10,3,5),NRES(10,3,5),LAG(10,3,5)
      INTEGER Z
      COMMON/INPT/LAG,NRES,K
      COMMON/HARC/FT,N,NO,DELT,IX,Z,PI
C     EVALUATE ANALYTICAL TRANSFORM OF RESPONSE FUNCTION
      DW=(2*PI)/(N*DELT)
C     DO FIRST HALF OF TRANSFORM, AT LAST PT SET IMAG. PART=0IF ODD
      N1=(N/2)+1
      OM=0.
      OM1=0.
      J1=CMPLX(0.,1.)
      DO 20 J=1,N1
C
C     ANALYTICAL RESPONSE TRANSFORMS FOR NASH MODEL, LAG AND ROUTE MODEL,
C     AND LINEAR RESERVOIR RESPECTIVELY
C
      GO TO (3,2,1),NO
1     FT(J)=(1./CMPLX(1.,OM))
      GO TO 10
2     XLG=LAG(Z,NO,IX)
      FT(J)=CEXP(-J1*OM1*XLG)/(1+(J1*OM))
      GO TO 10
3     XNS=NRES(Z,NO,IX)
      FT(J)=(1+OM**2)**(-XNS/2.)*CEXP(-J1*XNS*ATAN(OM))
10    NCK=MOD(N,2)
      IF(NCK.GT.0 .AND.J.EQ.N1) GO TO 15
      IF(J.NE.N1) GO TO 15
      FT1=REAL(FT(J))
      FT(J)=CMPLX(FT1,0.)
15    IF(J.EQ.N1) GO TO 21
      OM1=OM1+DW
20    OM=OM1*K(Z,NO,IX)
C     SECOND HALF OF TRANSFORM=CONJUGATE IDENTITY OF 1ST HALF
21    IF (NCK.GT.0) GO TO 22
      GO TO 23
22    N2=N1
      L=N1+1
      GO TO 24
23    L=N1+1
      N2=N1-1
24    DO 25 J=L,N
      FT(J)=FT(N2)
      FT2=REAL(FT(J))
      FT3=AIMAG(FT(J))
      FT(J)=CMPLX(FT2,-FT3)
25    N2=N2-1
      RETURN
      END
```

APPENDIX B-1

Procedure for Determining the Time Period for the Nash Model

$$\text{Nash Model} = \frac{1}{K}\left(\frac{t}{K}\right)^{n-1} \frac{e^{-t/K}}{\Gamma(n)} \qquad \text{B-1.1}$$

The integral of the Nash Model, giving the area under the distribution, is given by:

$$\text{Area} = \int_{o}^{\infty} \frac{1}{K}\left(\frac{t}{K}\right)^{n-1} \frac{e^{-t/K}}{\Gamma(n)}$$

$$= \frac{1}{K\Gamma(n)} e^{-T/K} \sum_{r=o}^{n-1} (-1)^{r} \frac{(n-1)!\,(T/K)^{n-1-r}}{(n-1-r)!\,(-1)^{r+1}}$$

B-1.2

The steps to optimize the time of convergence for obtaining a percentage of the area follows:

a) Start with the T for a single reservoir as an approximation

i.e. $T = - K \cdot \text{ALOG}\ (.01)$ B-1.3

b) Using T from step a, determine the area under the curve for Time Period T using equation B-1.2 above.

c) If $\underline{\overline{X}}$ = (1 - Results of b)) is less than .01, or 1% of the total area, then use the present Time Period T. Otherwise determine the ordinate at time T by using equation B-1.1. Then assuming a rectangular area as shown in Figure B-1. Solve for Δt (the time increment to T).

$$\Delta t = \frac{\bar{X}}{f(T)} \qquad \text{B-1.4}$$

d) Increment T by Δt and return to b)

$$T' = T + \Delta t$$

Results

		TIME PERIOD	
N	K	Nash Integration	Linear Reservoir Theory
3	5	44.07	69.08
5	5	61.42	115.13
5	2	24.57	46.05
6	5	213.97	230.03
8	5	69.21	138.16
10	5	83.60	184.20

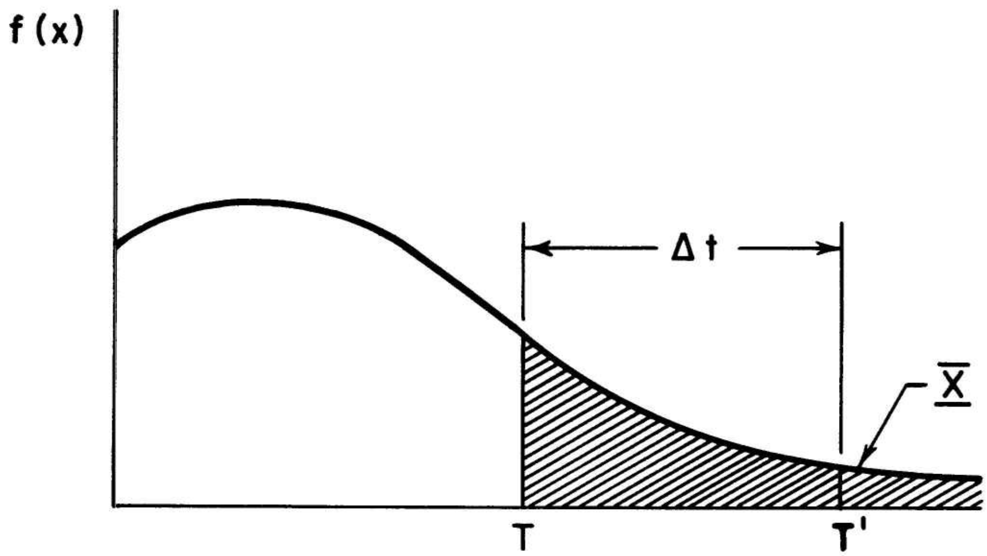

Figure B-1

Time Period Approximation

APPENDIX B-2

Derivation of the System Response to a Delta Function

Assumptions:

a) Two dimensional flow

b) No advective velocity (i.e. horizontal bedrock)

c) One side of response is considered, $|X|$.

Governing Equation:

$$h_m \frac{\partial^2 h}{\partial x^2} - \frac{Sc\partial h}{K_p dt} = \frac{q_i}{K_p \Delta x} \qquad \text{B-2.1}$$

where h_m = mean water saturated zone height

h = water saturation elevation

Sc = storage coefficient

K_p = permeability

q_i = flow due to advective velocity

Dynamic Equation: (Darcy's Equation)

$$q = - K_p h_m \frac{\partial h}{\partial x} \qquad \text{B-2.2}$$

Continuity:

$$\frac{\partial q}{\partial x} + Sc \frac{\partial h}{\partial t} = 0 \qquad \text{B-2.3}$$

A diagram of the system considered is shown in Figure B-2. Taking the Laplace transform of Equation B-2.1, we obtain:

$$A \frac{\partial^2 H}{\partial x^2} (x,s) - Bs\, H (x,s) + B\, h\, (x,0) = C\, Q(x,s) \qquad \text{B-2.4}$$

Assuming $q\ (0,t) = \delta\ (t)$

then $Q\ (0,s) = 1$ when $t = 0$, for all S

$Q\ (x,s) = 0 \qquad x > 0$

the coefficients represent:

$A = h_m$

$B = Sc/K_p$

$C = 1/K_p \Delta x$

assuming

$h\ (x,0) = 0$

then the characteristic equation for the system response to the delta function in the frequency domain is

$Ar^2 - Bs = 0$

thus $r^2 = Bs/A$

or $r = \pm \sqrt{Bs/A}$

The homogeneous solution to the differential equation B-2.2 is

$$H\ (x,s) = C_1\, e^{\sqrt{Bs/A}\,|X|} + C_2\, e^{-\sqrt{Bs/A}\,|X|} \qquad \text{B-2.5}$$

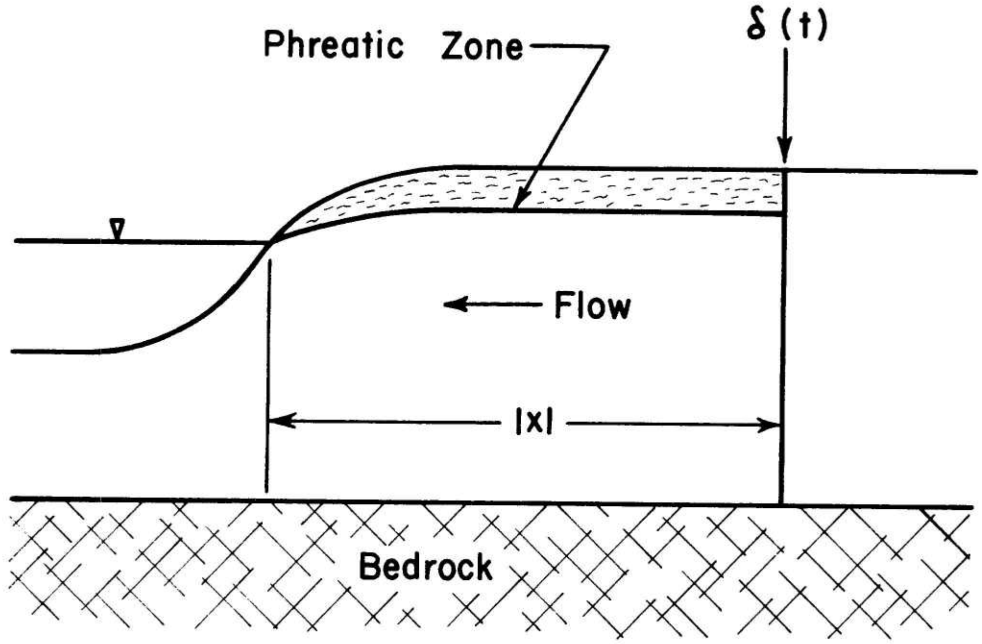

Figure B-2

System Response to Delta Function

But Hildebrand [1962], for finite intervals, shows that

$$\lim_{t \to \infty} h(x,t) = \lim_{s \to \infty} s\, H(x,s) \qquad \text{B-2.6}$$

which implies that $C_1 = 0$

Thus $$H(x,s) = C_2\, e^{-\sqrt{Bs/A}\,|X|} \qquad \text{B-2.7}$$

Applying the boundry condition @ X = 0 to equation B-2.5

$$H(0,s) = C_2 = C\, Q(0,s) = C \qquad \text{B-2.8}$$

To check this result:

$$A \frac{\partial^2 H}{\partial x^2}(x,s) = A \left(\frac{Bs}{A}\right) C_2 \exp\left(-\sqrt{\frac{Bs}{A}}\,|X|\right)$$

$$-Bs\, H(x,s) = -Bs\, C_2 \exp\left(-\sqrt{\frac{Bs}{A}}\,|X|\right) \qquad \text{B-2.9}$$

therefore

$$A \frac{\partial^2 H(x,s)}{\partial x^2} - Bs\, H(x,s) = 0$$

Thus $$H(x,s) = C \exp\left(-\sqrt{Bs/A}\,|X|\right) \qquad \text{B-2.10}$$

From Selby [1970] (Page 497, Eq. 82), the response to the delta function is

$$h(x,t) = \frac{C\sqrt{B/A}\ |X|}{2\sqrt{\pi}\ t^{3/2}} \exp\left(-\frac{B/A\ |X|X}{4t}\right) \qquad \text{B-2.11}$$

but

$$A = h_m$$

$$B = Sc/K_p$$

$$C = 1/K_p \Delta x$$

so

$$h(x,t) = \frac{\frac{1}{K_p \Delta x}\sqrt{\frac{Sc}{K_p h_m}}\ |X|}{2\sqrt{\pi}\ t^{3/2}} \exp\left(-\frac{(Sc/K_p h_m)|X|X}{4t}\right) \qquad \text{B-2.12}$$

See Appendix B-3.2 for the moment derivation of this system response.

APPENDIX B-3

Method of Moments-Parametric Analysis

B-3.1 Moments Analysis

The method of moments is a curve fitting procedure used in linear systems. Moments are normally referenced about the mean or about the origin, but may be referenced about any point.

The n^{th} order moments about the origin are defined by:

$$M'_n = \int_{-\infty}^{\infty} t^n F(t) \, dt \qquad \text{B-3.1}$$

where $F(t)$ = function or distribution

Thus, the first moment about the origin would be:

$$M'_1 = \int_{-\infty}^{\infty} t \, F(t) \, dt \qquad \text{B-3.2}$$

If the function, F(t) were a probability distribution then the Zeroth moment (area) would be unity, i.e.

$$M'_o = \int_{-\infty}^{\infty} t^o F(t) \, dt \qquad \text{B-3.3}$$

$$= 1$$

The n^{th} order moments about the centroid are defined as

$$M_n = \int_{-\infty}^{\infty} (t-M_1')^n \, F(t) \, dt$$

The interrelationship between the moments at the two referenced points may be given as the binomial theorem. A complete analysis of moments and cumulants is given by Harley [1967], i.e.

$$M_n = \sum_{i=o}^{j} \binom{j}{i} M_{j-i}' (- M_1')^i$$

$$M_n' = \sum_{i=o}^{j} \binom{j}{i} M_{j-i} (M_1')^i \qquad \text{B-3.4}$$

For example:

$$M_2' = M_2 + M_1'^2$$

$$M_3' = M_3 + 3M_1' M_2 + M_1'^3 \qquad \text{B-3.5}$$

or

$$M_2 = M_2' - M_1'^2$$

$$M_3 = M_3' - 3M_1' M_2 - M_1'^3 \qquad \text{B-3.6}$$

The effectiveness of the Methods of Moments in linear systems stems from the fact that a simple relationship exists between the input, response and output functions in the lower order moments. If we represent F as the input moments, H as the response moments and G as the output moments, we would have:

First Moment $G_1 = F_1 + H_1$

This simple property is true for the first three moments, beyond that the interactions become more complex.

B-3.2 Derivation of Moments Using Laplace Transforms

The Laplace transform of the flow function is given by

$$Q(x,s) = \int_o^\infty e^{-st} q(x,t) \, dt \qquad \text{B-3.7}$$

then
$$\frac{dQ(x,s)}{ds} = \int_o^\infty - t \, e^{-st} q(x,t) \, dt \qquad \text{B-3.8}$$

also
$$\frac{d^n Q(x,s)}{ds^n} = \int_o^\infty (-t)^n e^{-st} q(x,t) \, dt$$

$$= (-1)^n \int_o^\infty t^n e^{-st} q(x,t) \, dt \qquad \text{B-3.9}$$

thus
$$\left. \frac{d^n Q(x,s)}{ds^n} \right|_{s=o} = (-1)^n \int_o^\infty t^n q(x,t) \, dt \qquad \text{B-3.10}$$

Equation B-3.10 can also be stated as

$$\left. \frac{d^n Q(x,s)}{ds^n} \right|_{s=o} = (-1)^n \; M_n \qquad \text{B-3.11}$$

where $M_n = n^{th}$ Moment about the centroid

Thus, the derivative of the Laplace transform evaluated at s = 0 are the moments.

The cumulants may be determined similarly by taking the derivative of the logarithm of the Laplace transforms and evaluating at s = 0.

Applying this technique to Equation B-2.10

$$M_1 = -\frac{d}{ds} \left. (\log H\,(x,s)) \right|_{s=o}$$

$$= -\log C + \left(-\frac{\sqrt{Bs}}{A} |X|\right) \qquad \text{B-3.12}$$

$$= \log C$$

Note that any moments/ cumulants, excluding the first, will result in the moments being infinite. This result led us to model the differential equation as discussed in Section II-7.

APPENDIX C-1

Computer Implementation of the Convolution Technique by Means of Harmonic Analysis

The program developed to implement the convolution technique using the harmonic analysis concept requires only one subroutine for operation. However, the program presently includes a plotter routine and an interpolation routine to compare the computed hydrograph with a known test hydrograph. A listing of this program may be found in Appendix A-1. The plotter and interpolation routines are not discussed here but the user may implement his own programs to satisfy this requirement if deemed necessary. The subroutine essential for this program is subroutine FOURT, the FFT program developed at M.I.T.. Subroutine FOURT is fully explained by comment statements in Appendix A-2 but additional information may be found in Chapter III.

C-1.1 Program Input/Output Procedures

As Figure C-1 indicates, the program simply generates an input function, computes the analytical response functions which are placed in the format required by FOURT, manipulates the transforms of the input and response functions and returns the resulting output function to the time domain. In this case the parameters to the system response models are assumed known and are input to the program.

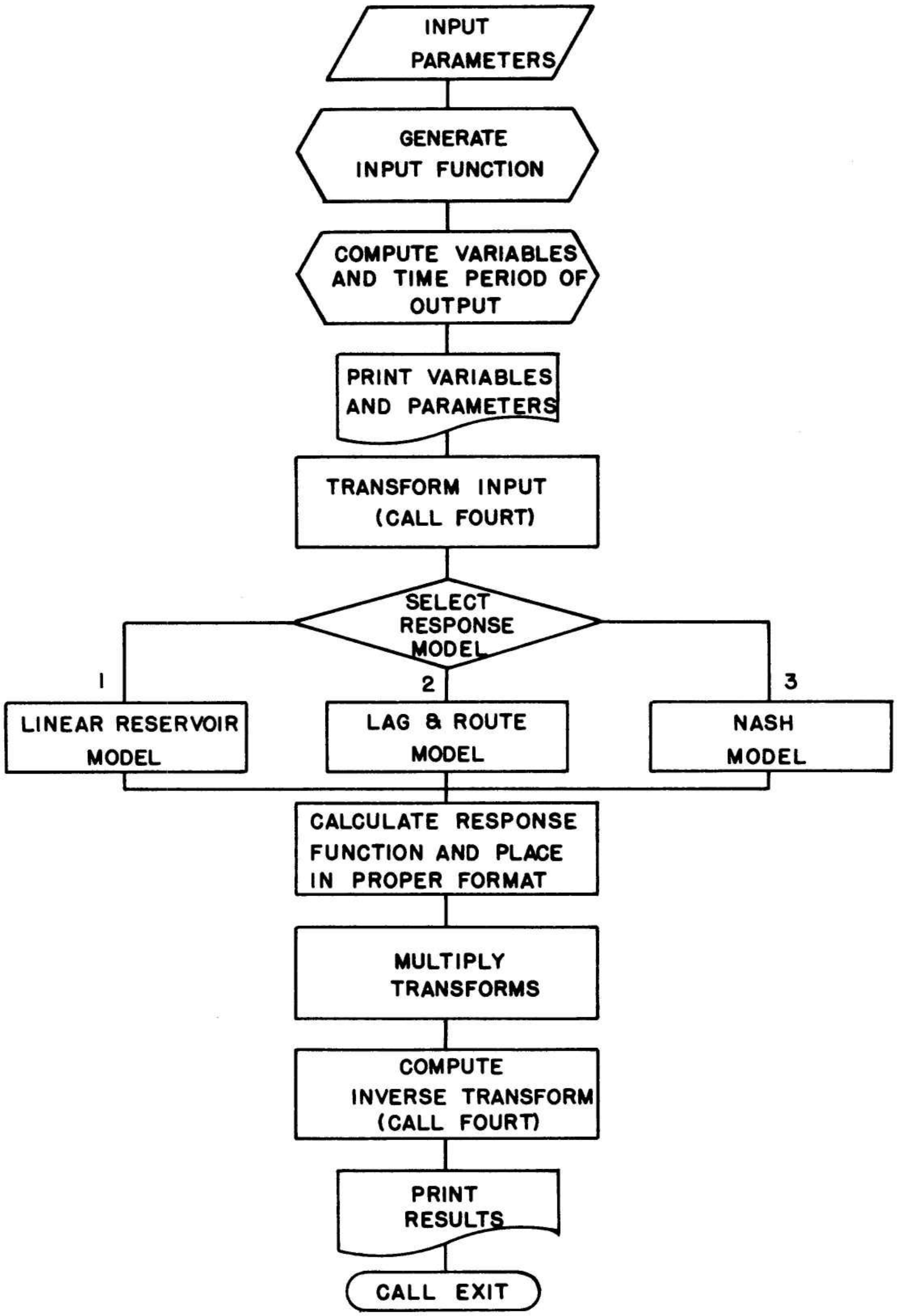

Figure C-1

Flow Chart of Convolution Program

C-1.1.1 Input Requirements

Two cards are used to input the necessary parameters for the input and the system response. These are:

Card 1

Variables	Description	Format
NO,NRES	NO (Col. 1-3) is the model desired to represent the system response and may be the integer values 1, 2, or 3. 1 represents the Linear Reservoir model, 2 the Lag and Route model, and 3 the Nash model as discussed in Chapter III. NRES (Col. 4-13) indicates the number of equal linear reservoirs used in series by the Nash model. If this system response is not required then this real variable may be ignored.	I3,F10.0

Card 2

K,KRES,LAG,BFL	K (Col. 1-10) a real variable indicating the first moment or 'lag' to the Linear Reservoir model. KRES (Col. 11-20) a real variable indicating the system lag required to determine the time period of the response function. For the Linear Reservoir and the Lag and Route models this is the same value as K above. For the Nash model, however, this would represent the value resulting from NRES · K(nK) or the lag of the Nash model. LAG (Col. 21-30) the real variable representing the translational lag of the	4F10.0

Lag and Route model. This variable may be set to zero if this model is not required.

BFL (Col. 31-40) is the real variable indicating a baseflow of the input hydrograph. This variable, too, may be set to zero or left out if the input condition warrants it.

C-1.1.1.1 Input Function

The function indicated in the program listing is that of a Thomas wave, being:

$$f(t) = \frac{q_{Max}}{2} \left[(1 - \text{Cos} \left(\frac{2\pi t}{f}\right)) \right] \qquad \text{C-1}$$

where q_{Max} = maximum amplitude of the input

f = time period of the input

a function more suitable to ones needs is easily substituted at this point in the program.

C-1.1.1.2 Subroutine FOURT

The listing for this program may be found in Appendix A-2. The calling sequence is discussed by the comment statements in that listing. The user of the convolution program need not understand the variables required by the calling statement of FOURT since all the requirements are satisfied by the main program, thus FOURT is not discussed except as found in Chapter III.

C-1.1.2 Output Presentation

The input function that will be transformed into the frequency domain is printed along with the corresponding time for each point to be used by the FFT program.

The resulting output function is printed with the same correct time array after the output is transformed into a time series from the frequency mode.