Integration in Finite Terms with Elementary Functions and Dilogarithms

by

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Abstract

In this thesis, we report on a new theorem that generalizes Liouville's theorem on integration in finite terms. The new theorem allows dilogarithms to occur in the integral in addition to elementary functions. The proof is based on two identities for the dilogarithm, that characterize all the possible algebraic relations among dilogarithms of functions that are built up from the rational functions by taking transcendental exponentials, dilogarithms, and logarithms.

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Chapter 1

Introduction

In 1967, M.Rosenlicht [11] published an algebraic proof of Liouville's theorem on the problem of integration in finite terms with elementary functions, based on the notions of differential algebra. In 1972, J.Moses [7] started discussing the problem of extending Liouville's result to include non-elementary functions in the integral. He asked whether a given expression has an integral within a class of expressions of the form $F(V_i)$, where F is a given special function and (V_i) is a finite set of functions lying in the ground field. Singer, Saunders and Caviness [14] proved an extension of Liouville's theorem allowing logarithmic integrals and error functions to occur in the integral in addition to elementary functions. However the techniques used in their proofs don't apply directly to special functions such as the dilogarithm since the later has "non-elementary" identities. Also the dilogarithm is more complex than logarithmic integrals and error functions, in the sense that if an integrand has an integral which can be expressed using dilogarithms, these can have derivatives which contain logarithms transcendental over the field of integrand.

R.Coleman[3] produced an analytic characterization of the identities of the dilogarithm for rational functions. We show that two identities of the dilogarithm, in addition to the identities among primitives and the identities among exponentials, are required to generate all algebraic relations among dilogarithms and logarithms of functions built up from the rational functions by taking transcendental exponentials[1], logarithms and dilogarithms. Our proof uses Ostrowski's theorem[8] in several places. Given these two identities, we generalize Liouville's theorem to include dilogarithms in the integral, in addition to elementary functions. The basic conclusion is that an associated function to the dilogarithm, if dilogarithms appear in the integral, appears linearly, with logarithms appearing in a possible non-linear way.

[1]Transcendental exponentials are defined recursively as exponentials which are transcendental over the previous field which could contain logarithms, dilogarithms, and other transcendental exponentials.

Chapter 2

Dilogarithmic Elementary Extensions

Definition : A differential ring is a commutative ring R together with a derivation of R into itself,that is,a map $R \to R$ which if $x \mapsto x'$ satisfies the two rules

$$(x+y)' = x' + y'$$

$$(xy)' = x'y + xy'$$

In a differential ring we have $(x^n)' = nx^{n-1}x'$ for $n = 1, 2, 3, \ldots$. In particular setting $x = 1, n = 2$ we have $1' = 0$.

Definition : A differential field is a differential ring that is a field. If u, v are elements of a differential field and $v \neq 0$ we have the relation $(u/v)' = (u'v - uv')/v^2$.

Elements of derivative zero are called constants and in a differential field the totality of constants is itself a field, the subfield of constants.

If u, v are elements of a differential field such that $v \neq 0$ and $u' = v'/v$, in analogy with the classical situation we say that u is a logarithm of v or that v is an exponential of u.

Definition : If k is a differential field of characteristic zero, we call K a differential extension of k if K is a field extension of k and is itself a differential field such that the derivation on K, when restricted to k, is identical to the derivation on k.

Let k be a differential field of characteristic zero. The subfield of constants of k will be denoted by C. Let K be a differential extension such that $K = k(t)$ for some $t \in K$. An element $t \in K$ is called elementary over k if the field of constants of k is the same field of constants of K and t satisfies one of the following:

(1") $t' = a'/a$ for some $a \in k^*$. In this case, we write $t = log\ a$ and call t logarithmic over k.

(2") $t' = a't$ for some $a \in k$. In this case, we write $t = exp\ a$ and call t exponential over k.

(3") t is algebraic over k.

Definition : A differential extension field of a differential field is said to be elementary if this extension has the same subfield of constants as the base field and if there exists a finite tower of intermediate fields starting with the given base field and ending with the given extension field, such that each field in the tower after the first is obtained from its predecessor by the adjunction of a single element that is elementary over the preceding field.

That is a differential field F of k is said to be elementary over k if F and k have the same field of constants and if F can be resolved into a tower :

$$F = F_n \supseteq F_{n-1} \supseteq \cdots \supseteq F_1 \supseteq F_0 = k$$

such that $F_i = F_{i-1}(\theta_i)$, where, for each i, $1 \leq i \leq n$ one of the following holds :

(i) $\theta_i' = \phi'/\phi$ for some nonzero ϕ in F_{i-1}, which we write $\theta_i = log\phi$. We say that θ_i is logarithmic over F_{i-1}.

(ii) $\theta_i' = \phi'\theta_i$ for some ϕ in F_{i-1}, which we write $\theta_i = exp\phi$. We call θ_i exponential over F_{i-1}.

(iii) θ_i is algebraic over F_{i-1}.

Proposition: see [11] Let F be a differential field of characteristic zero and K an extension field of F. Then there exists a differential field structure on K that is compatible with that of F and with the field structure of K. This differential field structure on K is unique if K is algebraic over F and in any case induces a differential field structure on any subfield of K that contains and is algebraic over F.

Proof: Let D be a derivation on F. We want to show that D extends to a derivation on K. Assume first that $K = F(X)$, with X transcendental over F and consider the map :

$$D_0 : F[X] \to F[X]$$

defined by :

$$D_0(\sum_{i=0}^{n} a_i X^i) = \sum_{i=0}^{n} D(a_i) X^i$$

if $a_0, a_1, \ldots, a_n \in F$, is a derivation of $F[X]$ extending D.

We extend D_0 to the field $K = F(X)$ by setting, for $u, v \in F[X]$, $v \neq 0$,

$$D_0(u/v) = ((D_0 u)v - (D_0 v)u)/v^2$$

Suppose next that $K = F(x)$ with x algebraic over F. Let X be an indeterminate over F and let $f(X) \in F[X]$ be the minimal polynomial of x over F. The map

$$\frac{\partial}{\partial X} : F[X] \to F[X]$$

defined by :

$$\frac{\partial}{\partial X} \sum_{i=0}^{n} a_i X^i = \sum_{i=0}^{n} i a_i X^{i-1}$$

if $a_0, a_1, \ldots, a_n \in F$, is a derivation of $F[X]$ that annuls each element of F. So for any $g(X) \in F[X]$ the additive map $D_0 + g(X)\frac{\partial}{\partial X}$ is a derivation of $F[X]$ that extends D. Setting $f'(X) = (\frac{\partial}{\partial X})f$, we have $f'(x) \neq 0$ and since $F(x) = F[x]$ we can find a particular $g(X) \in F[X]$ such that

$$(D_0 f)(x) + g(x)f'(x) = 0$$

So $D_0 + g(X)\frac{\partial}{\partial X}$ maps $f(X)$ into a multiple of itself, hence maps the ideal $F[X]f(X)$ into itself, hence induces a derivation on the factor ring $F[X]/F[X]f(X)$ which is isomorphic to $F(x)$. This gives us the desired extension of D to $K = F(x)$. Thus D can be extended to a derivation of any simple extension field of F. If K is an arbitrary extension field of F then using the above and Zorn's lemma D can be extended to K. To complete the proof it suffices to show that if D_1 and D_2 are two derivations of the field K that agree on the subfield F and $x \in K$ is algebraic over F then $D_1 x = D_2 x$. Considering the derivation $D_1 - D_2$ of K, we have to show that any derivation of K which annuls all of F also annuls each $x \in K$ that is algebraic over F. For this we note that if $f(X) \in F[X]$ is the minimal polynomial of x over F then we have $0 = (f(x))' = f'(x).x'$, so that $x' = 0$. ■

Let k be a differential field of characteristic zero. A differential field extension F of k is said to be dilogarithmic-elementary over k if F and k have the same subfield of constants and if F can be resolved into a tower :

$$F = F_n \supseteq F_{n-1} \supseteq \cdots \supseteq F_1 \supseteq F_0 = k$$

such that $F_i = F_{i-1}(\theta_i', \theta_i)$, where for each i, $1 \leq i \leq n$ one of the following holds :

(i) $\theta_i' = \phi'/\phi$ for some nonzero ϕ in F_{i-1}, which we write $\theta_i = log\phi$. We say that θ_i is logarithmic over F_{i-1}.

(ii) $\theta_i' = \phi'\theta_i$ for some ϕ in F_{i-1}, which we write $\theta_i = exp\phi$. We call θ_i exponential over F_{i-1}.

(iii) $\theta_i' = -(\phi'/\phi)u$, where $\phi \in F_{i-1} - \{0, 1\}$, and u is such that $u' = (1-\phi)'/(1-\phi)$. In this case, we write $\theta_i = l_2(\phi)$ and call θ_i dilogarithmic over F_{i-1}. We note, in this case, that θ_i is defined up to the addition of a constant multiple of a logarithm over F_{i-1} since u is defined up to a constant.We don't assume , however, that u lies in F_{i-1}.

(iv) θ_i is algebraic over F_{i-1}.

Roughly speaking condition (iii) means that θ_i is the composition of the function ϕ with the dilogarithmic function $l_2(x)$ defined as :

$$l_2(x) = -\int_0^x \frac{log(1-t)}{t} dt$$

If K is a differential extension of k such that $K = k(t)$ for some $t \in K$ and $t' = a \in k$, we call t primitive over k and write $t = \int a$.

And finally **Definition** : If k is a differential field of characteristic zero, K a differential field extension of k such that $K = k(t, u, v)$, we say that $t = D(\phi)$, D is the Bloch-Wigner-Spence function, if ϕ is an element of $k - \{0, 1\}$ and:

$$t' = -\frac{1}{2}\frac{\phi'}{\phi}u + \frac{1}{2}\frac{(1-\phi)'}{(1-\phi)}v$$

where $u' = (1-\phi)'/(1-\phi)$ and $v' = \phi'/\phi$. From this definition, since u and v are defined up to additive constants, it follows that t is defined up to the addition of a linear combination of $log\phi$ and $log(1-\phi)$ with constant coefficients. Informally, t is equal to:

$$l_2(\phi) + \frac{1}{2} log\phi log(1-\phi)$$

Definition : For two differential fields k and K we say that K is a Liouvillian extension of k if there exist $t_1, \ldots, t_n \in K$ such that $K = k(t_1, \ldots, t_n)$ and each t_i is either algebraic, elementary, or primitive over $k(t_1, \ldots, t_{i-1})$.

Here are two results that are used repeatedly in what follows. First, we recall a version of Ostrowski's theorem [8] .

Theorem : Let k be a differential field of characteristic zero and let $K = k(log\ v_1, \ldots, log\ v_n)$ where $v_i \in k$,$1 \leq i \leq n$ and K, k have the same field of constants. Assume that $logv_1, \ldots, logv_r$ $(0 \leq r \leq n)$ are algebraically independent over k and that K and $k(logv_1, \ldots, logv_r)$ have the same transcendence degree r over k.

Then, there exist constants c_{ij} $(1 \leq i \leq r,\ r < j \leq n)$, $s_j \in k$ $(r < j \leq n)$ such that :

$$logv_j = \sum_{i=1}^{r} c_{ij} logv_i + s_j \ , \quad for \ \ j \in \{r+1, \ldots, n\}$$

and if $r = 0$, $logv_j \in k$ for all $j \in \{1, \ldots, n\}$.

The second result is a useful lemma due to Rosenlicht and Singer [12]:

Lemma : Let $k \subset K$ be differential fields of characteristic zero with the same field of constants C supposed to be algebraically closed. Assume that k is a Liouvillian extension of C and that K is algebraic over k.

Suppose that $c_1, \ldots, c_n \in C$ are linearly independent over Q, that $u_1, \ldots, u_n \in K^*$, $v \in K$, and that we have :

$$\sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v' \ \in k$$

Then, $v \in k$ and there is a non zero integer N such that $u_i^N \in k$, $i = 1, \ldots, n$.

The rest of this chapter is devoted to the statement and proof of one of the main results of this dissertation.

Definition : Let k be a differential field of characteristic zero. We call an expression S a simple elementary-dilogarithmic expression over k if :

$$S = g + \sum_{i \in I} c_i logw_i + \sum_{j \in J} [s_j log(1 - h_j) + t_j logh_j + d_j D(h_j)]$$

where I and J are finite sets , g, w_i, s_j, t_j, $h_j \in k$ and c_i, d_j are constants.

Lemma 1 : Let k be a differential field of characteristic zero , which is a Liouville extension of its subfield of constants C assumed algebraically closed. Suppose that we have an expression of the form :

$$\int f = g + \sum_{i \in I} c_i logw_i + \sum_{j \in J} [s_j log(1 - h_j) + t_j logh_j + d_j D(h_j)] \qquad (2.1)$$

where I and J are finite sets , $f \in k$, s_j, t_j, g, and w_i are algebraic over k, $h_j \in k$, and c_i, d_j are constants.

Then, we can write $\int f = S$, where S is a simple elementary-dilogarithmic expression over k. (So , we get g, w_i,s_j,t_j in k instead of being algebraics.)

Proof:

Let K be a finite normal algebraic extension field of k that contains g, $w_i(i \in I)$,s_j, t_j $(j \in J)$ (the smallest normal extension containing $k(g, w_1, \ldots, w_i, \ldots, s_1, \ldots, s_j, \ldots, t_j \ldots))$. Consider the vector space E over k spanned by the vectors

$$1, logh_1, \ldots, logh_j, \ldots\ log(1-h_1), \ldots, log(1-h_j), \ldots$$

Then, we choose among these vectors a k-basis $(1, e_1, \ldots, e_N)$ for E. By Ostrowski's theorem, we can write :

$$\left.\begin{array}{lcll} logh_j & = & \sum_{m=1}^{N} a_{jm} e_m + p_j & , a_{jm} \in C, p_j \in k \\ log(1-h_j) & = & \sum_{m=1}^{N} b_{jm} e_m + q_j & , b_{jm} \in C, q_j \in k \end{array}\right] \qquad (*)$$

We claim that $1, e_1, \ldots, e_N$ are still linearly independent over K.

Otherwise, and by Ostrowski's theorem, there exist constants α_m $(2 \leq m \leq N)$ and $Q_0 \in K$ such that :

$$e_1 = \sum_{m=2}^{N} \alpha_m e_m + Q_0 \quad \Rightarrow \quad e_1' = \sum_{m=2}^{N} \alpha_m e_m' + Q_0' \qquad (2.2)$$

By assumption, $e_m = logH_m$ $(1 \leq m \leq N)$, where $H_m \in \{(1-h_1), \ldots, h_1, \ldots\}$.

Let $\gamma_0 = 1$, γ_1, ... , γ_r be a vector space basis for the Q-span of $1, \alpha_1$, ... ,α_n, and write :

$$\alpha_m = \sum_{i=0}^{r} n_{mi} \gamma_i$$

with each $n_{mi} \in Q$. Replacing each γ_i by $\gamma_i / LCD(n_{mi})$ if necessary, we can assume $n_{mi} \in Z$ (where LCD means Least Common Denominator).

So we can write (2.2) as :

$$\frac{(H_1)'}{H_1}\gamma_0 = \sum_{i=0}^{r} \gamma_i \frac{(H_2^{n_{2i}} H_3^{n_{3i}} \dots H_N^{n_{Ni}})'}{H_2^{n_{2i}} H_3^{n_{3i}} \dots H_N^{n_{Ni}}} + Q_0'$$

which can also be written as :

$$\gamma_0 \frac{(H_1^{-1} H_2^{n_{20}} \dots H_N^{n_{N0}})'}{H_1^{-1} H_2^{n_{20}} \dots H_N^{n_{N0}}} + \sum_{i=1}^{r} \gamma_i \frac{(H_2^{n_{2i}} H_3^{n_{3i}} \dots H_N^{n_{Ni}})'}{H_2^{n_{2i}} H_3^{n_{3i}} \dots H_N^{n_{Ni}}} + Q_0' = 0 \qquad (2.3)$$

Using Singer's lemma, we deduce that $Q_0 \in k$. Investigating (2.2) again, we get :

$$e_1 = \sum_{m=2}^{N} \alpha_m e_m + Q_0$$

with $Q_0 \in k$, $\alpha_m \in k \cap C = C$. This is a contradiction , since the e_m $(1 \leq m \leq N)$ and 1 were assumed to be linearly independent over k.

So, $1, e_1, \dots, e_N$ are linearly dependent over K.

Now, we write (2.1) in terms of the relations (*) :

$$\int f = g_0 + \sum_{m=1}^{N} r_m e_m + \sum_{j \in J} d_j D(h_j) + \sum_{i \in I} c_i log w_i \qquad (2.4)$$

(where $g_0 \in K, r_m \in K$).

Taking the derivative of the previous relation, we obtain :

$$f = g_0' + \sum_{i \in I} c_i \frac{w_i'}{w_i} + \sum_{m=1}^{N} r_m e_m' + \sum_{m=1}^{N} r_m' e_m - \frac{1}{2} \sum_{j \in J} d_j \frac{h_j'}{h_j} log(1-h_j) + \frac{1}{2} \sum_{j \in J} d_j \frac{(1-h_j)'}{(1-h_j)} log h_j \qquad (2.5)$$

Using again the relation (*) for $log(1-h_j)$ and $log h_j$, and assembling coefficients of (2.5) according to the K-basis $(1, e_1, \dots, e_N)$, we obtain :

$$f = g_0' + \sum_{i \in I} c_i \frac{w_i'}{w_i} + \sum_{m=1}^{N} r_m e_m' - \frac{1}{2} \sum_{j \in J} d_j \frac{h_j'}{h_j} q_j + \frac{1}{2} \sum_{j \in J} d_j \frac{(1-h_j)'}{(1-h_j)} p_j \qquad (2.6)$$

(the above is the coefficient of the vector 1) , and :

$$r'_m - \sum_{j\in J}\frac{1}{2}d_j b_{jm}\frac{h'_j}{h_j} + \sum_{j\in J}\frac{1}{2}d_j a_{jm}\frac{(1-h_j)'}{(1-h_j)} = 0, \quad 1\le m\le N \tag{2.7}$$

(the above is the coefficient of the vector e_m).

From (2.7), we deduce that $r_m \in k$ (using Singer's lemma and exactly the same argument used in the above proof).

Assume that $M = [K:k]$. For any $\sigma \in Aut(K/k)$, we have (using 1.6):

$$f = \sigma(f) = \sigma(g'_0) + \sum_{i\in I} c_i \frac{\sigma(w_i)'}{\sigma(w_i)} + \sum_{m=1}^{N} r_m e'_m - \frac{1}{2}\sum_{j\in J} d_j \frac{h'_j}{h_j} q_j + \frac{1}{2}\sum_{j\in J} d_j \frac{(1-h_j)'}{(1-h_j)} p_j$$

Taking the sum over all the σ's in $Aut(K/k)$, we obtain :

$$Mf = \sum_{\sigma}\sigma(g'_0) + \sum_{i\in I} c_i \sum_{\sigma}\frac{\sigma(w_i)'}{\sigma(w_i)} + M[\sum_{m=1}^{N} r_m e'_m - \frac{1}{2}\sum_{j\in J} d_j q_j \frac{h'_j}{h_j} + \frac{1}{2}\sum_{j\in J} d_j p_j \frac{(1-h_j)'}{(1-h_j)}]$$

which implies that :

$$-f + (\frac{Tr(g_0)'}{M}) + \sum_{i\in I}\frac{c_i}{M}\frac{N(w_i)'}{N(w_i)} + [\sum_{m=1}^{N} r_m e'_m - \frac{1}{2}\sum_{j\in J} d_j q_j \frac{h'_j}{h_j} + \frac{1}{2}\sum_{j\in J} d_j p_j \frac{(1-h_j)'}{(1-h_j)}] = 0 \tag{2.8}$$

Where $Tr(\ )$ and $N(\ )$ are the trace and norm maps, respectively, from K to k.

Now, multiplying (2.8) by 1 and each (2.7) by e_m, adding them using again relations (*), and integrating, we get :

$$\int f = \frac{Tr(g_0)}{M} + \sum_{i\in I}\frac{c_i}{M} log N(w_i) + \sum_{m=1}^{N} r_m e_m + \sum_{j\in J} d_j D(h_j) \tag{2.9}$$

Note that $Tr(g_0)/M \in k$ and $N(w_i) \in k$, and also $e_m = log H_m$, where $H_m \in \{h_1, h_2, \ldots, 1-h_1, 1-h_2, \ldots\}$. So, the right-hand side of (2.9) is a simple elementary-dilogarithmic expression over k, which is what we wanted to prove. ■

Definition : Let k be a differential field of characteristic zero. K is a finite algebraic extension of k , and $log h_1, \ldots, log h_m$ are logarithmics over k (that is, $h_1, \ldots, h_m \in k$). Assume that the fields k and $K(log h_1, \ldots, log h_m)$ have the same field of constants C. We call L a linear logarithmic expression over $K(logh_1, \ldots, logh_m)$ if :

$$L = \sum_{i=1}^{m} c_i logh_i + r$$

where the c_i are constants and $r \in K$. L is said to be dependent on $logh_j$ $(1 \leq j \leq m)$ if $c_j \neq 0$.

Proposition 1: (See [2]) Let k be a differential field of characteristic zero which is a Liouville extension of its field of constants C assumed algebraically closed. Suppose that $f \in k$; $h_1, \ldots, h_n \in k$;K a finite algebraic extension of k; $a_1, \ldots, a_m \in C$; $d_1, \ldots, d_n \in C$; and $L_1, \ldots, L_m$ are linear logarithmic expressions over

$$K(log(1-h_1), \ldots, log(1-h_n)).$$

Then, if :

$$\int f - \sum_{j=1}^{n} d_j l_2(h_j) - \sum_{i=1}^{m} a_i logL_i \ \in K(log(1-h_1), \ldots, log(1-h_n)) \qquad (2.10)$$

$\int f$ is a simple elementary-dilogarithmic expression over k.

Proof:

Let r = trans-degree $K(log(1-h_1), \ldots, log(1-h_n))$ over k. If $r = 0$, then, by Ostrowski's theorem, $log(1-h_j) \in k(1 \leq j \leq n) \Rightarrow K(log(1-h_1), \ldots, log(1-h_n)) = K$, and $L_i \in K(1 \leq j \leq m)$. So, (2.10) implies that :

$$\int f = \sum_{j=1}^{n} d_j l_2(h_j) + \sum_{i=1}^{m} a_i logL_i + g, \quad g \in K, L_i \in K$$

$$\Rightarrow \int f = \sum_{j=1}^{n} d_j D(h_j) + g + \sum_{i=1}^{m} a_i logL_i - \frac{1}{2}\sum_{j=1}^{n} d_j log(1-h_j)logh_j$$

So, if :

$$s_j = -\frac{1}{2} d_j log(1-h_j) \in k, \ we \ get \ :$$

$$\int f = \sum_{j=1}^{n} d_j D(h_j) + g + \sum_{i=1}^{m} a_i log L_i + \sum_{j=1}^{n} s_j log h_j, \quad s_j \in k, g \in K, L_i \in k$$

So, by lemma 1, $\int f$ is a simple elementary-dilogarithmic expression over k and the proposition is proved for $r = 0$. Let r be greater than 0 and assume without loss of generality that $log(1 - h_1), \ldots, log(1 - h_r)$ are algebraically independent over K so that by Ostrowski's theorem again we find constants c_{jp} such that :

$$log(1 - h_j) = \sum_{p=1}^{r} c_{jp} log(1 - h_p) + R_j \qquad (**)$$

where : $R_j \in k$, $r < j \leq n$

So, $K(log(1 - h_1), \ldots, log(1 - h_n)) = K(log(1 - h_1), \ldots, log(1 - h_r))$.

Let $K_{i_0} = K(\, log(1 - h_1), \ldots, log(1 - h_{i_0 - 1}), log(1 - h_{i_0 + 1}), \ldots, log(1 - h_r))$ $(1 \leq i_0 \leq r)$. Clearly, $t_{i_0} = log(1 - h_{i_0})$ is transcendental over K_{i_0} since we have assumed that $log(1 - h_j)(1 \leq j \leq r)$ are algebraically independent over K.

For each $i_0 \in \{1, 2, \ldots, r\}$, let I_0 be the subset of $\{1, 2, \ldots, m\}$ such that, for all $i \in I_0$, L_i is dependent on $t_{i_0} = log(1 - h_{i_0})$. Then, (2.10) implies that :

$$\int f - \sum_{j=1}^{n} d_j l_2(h_j) - \sum_{i=1}^{m} a_i log L_i \in K_{i_0}(t_{i_0}) \qquad (2.11)$$

We want to prove that :

$$[\sum_{i \in I_{i_0}} a_i log L_i]' = 0$$

and that :

$$\int f - \sum_{j=1}^{n} d_j l_2(h_j) - \sum_{i \notin I_{i_0}} a_i log L_i \in K_{i_0}[t_{i_0}] \qquad (2.12)$$

Once (2.12) is proved for each index $i_0 \in \{1, 2, \ldots, r\}$, we deduce that :

$$\int f - \sum_{j=1}^{n} d_j l_2(h_j) - \sum_{i \in I_{00}} a_i log L_i \in \bigcap_{i_0 \in \{1,2,\ldots,r\}} K_{i_0}[t_{i_0}] = K[t_1, t_2, \ldots, t_r]$$

where I_{00} is such that, for all $i \in I_{00}$, L_i is not dependent on any $t_j = log(1 - h_j)$, for all $j \in \{1, 2, \ldots, r\}$. So, $L_i \in K$ for all $i \in I_{00}$, and :

$$\int f - \sum_{j=1}^{n} d_j l_2(h_j) - \sum_{i \in I_{00}} a_i log L_i = P(t_1, \ldots, t_r)$$

where P is a polynomial.

So, let $K_0 = K_{i_0} = K(t_1, \ldots, t_{i_0-1}, t_{i_0+1}, \ldots, t_r)$ and $t = t_{i_0}$. Then, if L_i depends on t, $L_i = b_i t + r_i$, where $r_i \in K_0$, and $b_i$is a constant, $b_i \neq 0$. By assumption, we had :

$$\int f - \sum_{j=1}^{n} d_j l_2(h_j) - \sum_{i=1}^{m} a_i log L_i = g(t) \in K_0(t) \tag{2.13}$$

If K^0 is a finite algebraic extension of K_0 where $g(t)$ splits into linear factors, we write :

$$g(t) = g_0(t) + \sum_{\alpha,\beta} \frac{r_{\alpha,\beta}}{(t - T_\alpha)^\beta}, \quad r_{\alpha,\beta} \in K^0, T_\alpha \in K^0, \beta \in N^*$$

α and β range over a finite set of positive integers, and $g_0(t) \in K^0[t]$. (2.13) yields :

$$f + \sum_{j=1}^{n} d_j \frac{h_j'}{h_j} log(1 - h_j) - \sum_{i=1}^{m} a_i \frac{L_i'}{L_i} - g_0'(t) - \sum_{\alpha,\beta} \frac{r'_{\alpha,\beta}}{(t - T_\alpha)^\beta} + \sum_{\alpha,\beta} \frac{\beta r_{\alpha,\beta}(t' - T'_\alpha)}{(t - T_\alpha)^{\beta+1}} = 0 \tag{2.14}$$

The key idea in the on-going proof is that,when we use the relations $(**)$ the expression:

$$f + \sum_{j=1}^{n} d_j \frac{h_j'}{h_j} log(1 - h_j)$$

is a linear polynomial in t over K_0 . Also, $g_0'(t)$ is a polynomial in t since $t' = (1 - h_{i_0})'/(1 - h_{i_0}) \in k$. So :

$$\sum_{L_i \text{ depends on } t} -a_i \frac{L_i'}{L_i} - \sum_{\alpha,\beta} \frac{r'_{\alpha,\beta}}{(t - T_\alpha)^\beta} + \sum_{\alpha,\beta} \frac{\beta r_{\alpha,\beta}(t' - T'_\alpha)}{(t - T_\alpha)^{\beta+1}}$$

must cancel.

Let $I_t = \{i \text{ such that } L_i = b_i t + r_i,\ b_i \neq 0\}$ and $I_t^0 = \{1, \ldots, m\} - I_t$. (2.14) then becomes :

$$
\begin{aligned}
f+\sum_{j=1}^{n} d_j \frac{h'_j}{h_j} log(1-h_j) \quad - & \quad \sum_{i\in I_t^0} \frac{r'_i}{r_i} - \sum_{i\in I_t} a_i \frac{(b_i t' + r'_i)}{(b_i t + r_i)} - g'_0(t) \\
- & \quad \sum_{\alpha,\beta} \frac{r'_{\alpha,\beta}}{(t-T_\alpha)^\beta} + \sum_{\alpha,\beta} \frac{\beta r_{\alpha,\beta}(t'-T'_\alpha)}{(t-T_\alpha)^{\beta+1}} = 0 \qquad (2.15)
\end{aligned}
$$

where $r_i \in K_0$.

First, $t' - T'_\alpha \neq 0$, otherwise we would have $t' = T'_\alpha$; and for each $\sigma \in Aut(K^0/K_0)$ we have $t' = \sigma(T_\alpha)' \Rightarrow [K^0 : K_0]t' = Tr(T_\alpha)' \Rightarrow t = 1/[K^0 : K_0]Tr(T_\alpha) + c$, where c is a constant and Tr is the trace map from K^0 to K_0. But this gives a contradiction since t was supposed to be transcendental over K_0.

So, if we look at the partial fraction decomposition we have in (2.15), we deduce that $r_{\alpha,\beta} = 0$ for all α, β, and we get :

$$
f+\sum_{j=1}^{n} d_j \frac{h'_j}{h_j} log(1-h_j) - \sum_{i\in I_t^0} a_i \frac{r'_i}{r_i} - \sum_{i\in I_t} a_i \frac{(b_i t' + r'_i)}{(b_i t + ri)} - g'_0(t) = 0
$$

which also implies that :

$$
\sum_{i\in I_t} a_i \frac{(b_i t' + r'_i)}{(b_i t + ri)} = 0
$$

(by looking at partial fraction decomposition). Also, $g_0 \in K^0[t] \cap K_0(t) \Rightarrow g_0 \in K_0[t]$, and :

$$
\sum_{i\in I_t} a_i log L_i
$$

is a constant.

By induction on $t = t_{i_0} \in \{log(1-h_1), \ldots, log(1-h_r)\}$, we deduce that :

$$
g(t_1, \ldots, t_r) = \int f - \sum_{j=1}^{n} d_j l_2(h_j) - \sum_{i\in I_{00}} a_i log L_i \ \in K[t_1, \ldots, t_r] \ and \ L_i \in K \qquad (2.16)
$$

We claim that g is a polynomial of degree 2 with constant coefficients, for all terms in $t_1, \ldots, t_r$ of degree 2. In fact, let $A_{\alpha_1\alpha_2\ldots\alpha_r} t_1^{\alpha_1} \ldots t_r^{\alpha_r}$ be one monomial in the leading homogenous term of g, with $A_{\alpha_1\alpha_2\ldots\alpha_r} \neq 0$. Then :

$$(A_{\alpha_1\alpha_2\ldots\alpha_r} t_1^{\alpha_1} \ldots t_r^{\alpha_r})' = A'_{\alpha_1\alpha_2\ldots\alpha_r} t_1^{\alpha_1} \ldots t_r^{\alpha_r} + \sum_{j=1}^{r} A_{\alpha_1\alpha_2\ldots\alpha_r} \alpha_j t'_j t_1^{\alpha_1} \ldots t_j^{\alpha_j - 1} \ldots t_r^{\alpha_r}$$

Assuming :

$$\sum_{j=1}^{r} \alpha_j \geq 2$$

and noticing that the derivative of the right-hand side of (2.16) is of degree 1 in $t_1, \ldots, t_r$, we deduce that :

$$A'_{\alpha_1\alpha_2\ldots\alpha_r} = 0 \quad \Rightarrow \quad A_{\alpha_1\alpha_2\ldots\alpha_r} \textit{ is a constant}$$

If :

$$\sum_{j=1}^{r} \alpha_j > 2$$

then there exists i_0 such that $\alpha_{i_0} \neq 0$. The coefficient of $t_1^{\alpha_1} \ldots t_{i_0}^{\alpha_{i_0}-1} \ldots t_r^{\alpha_r}$ must be zero in the derivative of $g(t_1, \ldots, t_r)$. So :

$$A_{\alpha_1\alpha_2\ldots\alpha_r} \alpha_{i_0} t'_{i_0} + A'_{\alpha_1\alpha_2\ldots\alpha_{i_0-1}\ldots\alpha_r} = 0$$

$$\Rightarrow \quad t_{i_0} = -\frac{1}{A_{\alpha_1\alpha_2\ldots\alpha_r}} \frac{1}{\alpha_0} A_{\alpha_1\alpha_2\ldots\alpha_{i_0-1}\ldots\alpha_r} + c$$

where c is a constant. But this is a contradiction since t is transcendental over K.

So, we deduce that g is a polynomial of degree 2 with constant coefficients, for all the terms in $t_1, \ldots, t_r$ of degree 2. That is :

$$g(t_1, \ldots, t_r) = A_0 + \sum_{p=1}^{r} A_p t_p + \sum_{\alpha,\beta \;\; \beta \geq \alpha} A_{\alpha,\beta} t_\alpha t_\beta$$

where $t_\alpha, t_\beta \in \{t_1, \ldots, t_r\}$, and $A_{\alpha,\beta}$ are constants.

$$g'(t_1, \ldots, t_r) = A'_0 + \sum_{p=1}^{r} A_p t'_p + \sum_{p=1}^{r} A'_p t_p + \sum_{\alpha,\beta \;\; \beta \geq \alpha} A_{\alpha,\beta} t'_\alpha t_\beta + \sum_{\alpha,\beta \;\; \beta \geq \alpha} A_{\alpha,\beta} t_\alpha t'_\beta \quad (2.17)$$

and :

$$g'(t_1,\ldots,t_r) = f + \sum_{j=1}^{n} d_j \frac{h'_j}{h_j} log(1-h_j) - \sum_{i=1}^{m} a_i \frac{L'_i}{L_i} \qquad (2.18)$$

Using the dependency relations (**), we obtain from (2.17) and (2.18) :

$$f - A'_0 - \sum_{p=1}^{r} A_p t'_p + \sum_{j=r+1}^{n} d_j \frac{h'_j}{h_j} R_j - \sum_{i=1}^{m} a_i \frac{L'_i}{L_i}$$

$$= \sum_{p=1}^{r} [-d_p \frac{h'_p}{h_p} - \sum_{j=r+1}^{n} c_{jp} d_j \frac{h'_j}{h_j} + 2A_{pp} t'_p + \sum_{\alpha \neq p} A_{\alpha p} t'_\alpha + A'_p] t_p$$

(where $A_{\alpha p} = A_{p\alpha}$ if $\alpha > p$).

From the above, we deduce that :

$$2A_{pp} t'_p + \sum_{\alpha \neq p} A_{\alpha p} t'_\alpha = d_p \frac{h'_p}{h_p} + \sum_{j=r+1}^{n} c_{jp} d_j \frac{h'_j}{h_j} - A'_p$$

and, by integration, we get :

$$A_{pp} t_p + \sum_{\alpha \neq p} \frac{1}{2} A_{\alpha p} t_\alpha = \frac{1}{2}[d_p log h_p + \sum_{j=r+1}^{n} c_{jp} d_j log h_j - A_p] + c_p \qquad (2.19)$$

where c_p is a constant.

Notice that we can write :

$$g(t_1,\ldots,t_r) = A_0 + \sum_{p=1}^{r} A_p t_p + \sum_{p=1}^{r} [A_{pp} t_p + \sum_{\alpha \neq p} \frac{1}{2} A_{\alpha p} t_\alpha] t_p$$

and, using (2.19) and (2.16), we get :

$$\int f = \sum_{j=1}^{n} d_j l_2(h_j) + A_0 + \sum_{p=1}^{r} A_p t_p - \frac{1}{2} \sum_{p=1}^{r} A_p t_p + \sum_{p=1}^{r} c_p t_p$$

$$+ \frac{1}{2} \sum_{p=1}^{r} [d_p log h_p + \sum_{j=r+1}^{n} c_{jp} d_j log h_j] t_p + \sum_{i=1}^{m} a_i log L_i$$

which gives :

$$\int f = \sum_{p=1}^{r} d_j[l_2(h_j) + \frac{1}{2}(log\ h_p)t_p] + \sum_{j=r+1}^{n} d_j[l_2(h_j) + \frac{1}{2}[\sum_{p=1}^{r} c_{jp}t_p]log\ h_j]$$

$$+A_0 + \frac{1}{2}\sum_{p=1}^{r} A_p t_p + \sum_{p=1}^{r} c_p t_p + \sum_{i=1}^{m} a_i log\ L_i$$

But we had :

$$\sum_{p=1}^{r} c_{jp}t_p = log(1 - h_j) - R_j$$

for $j \in \{r+1, \ldots, n\}$ and $t_p = log(1 - h_p)$. So :

$$\int f = \sum_{j=1}^{n} d_j D(h_j) - \frac{1}{2}\sum_{j=r+1}^{n} d_j R_j log\ h_j + A_0 + \frac{1}{2}\sum_{p=1}^{r} A_p log(1 - h_p)$$

$$+\sum_{p=1}^{r} c_p log(1 - h_p) + \sum_{i=1}^{m} a_i log\ L_i\ ,$$

$R_j \in k$, $A_0, A_p \in K$, $L_i \in K$ and, by lemma 1, $\int f = S$, where S is a simple elementary-dilogarithmic expression. This completes the proof of proposition 1. ∎

Chapter 3

The Functional Identities of the Dilogarithm

In this chapter, we exhibit and prove two identities of the dilogarithm that will be shown in Chapter 4, in addition to the identities among primitives and the identities among exponentials, to be capable of generating all the algebraic relations among dilogarithms and logarithms built up from the rational functions by taking transcendental exponentials, logarithms and dilogarithms.

For a differential field k and t dilogarithmic over k we observe the following fact : t is defined up to the addition of a constant multiple of a logarithmic or more precisely : if $t = -a'/a \ \psi$, where $\psi' = (1-a)'/(1-a)$, ψ is defined up to a constant. So, if $\psi_1' = (1-a)'/(1-a)$ we deduce that $\psi_1 = c + \psi$, where c is a constant and $t' = -(a'/a)(\psi_1 - c) = -(a'/a)\psi_1 + c \ a'/a$ so t is defined up to the addition of $c \ loga$.

Also, if ϕ is an element of $k - \{0, 1\}$ and $t = D(\phi)$ it follows that t is defined up to the addition of a linear combination of $log\phi$ and $log(1-\phi)$ with constant coefficients. Informally, t is equal to:

$$l_2(\phi) + \frac{1}{2} log\phi log(1-\phi)$$

This motivates considering the dilogarithm and the associated function D as defined mod the vector space generated by constant multiples of logarithms over k. We denote from now on this vector space by M_k for any differential field k. So, if $W \in M_k$, then there exist constants $c_1, \ldots, c_n$ and $u_1, \ldots, u_n$ such that u_i, $1 \leq i \leq n$, is logarithmic over k for all i, and :

$$W = \sum_{i=1}^{n} c_i u_i$$

The first identity satisfied by the dilogarithm is given by the following lemma which is relatively easy to prove.

Lemma 2: (See [2]) If k is a differential field of characteristic zero, then for all $f \in k - \{0, 1\}$:

$$D(1/f) \equiv -D(f) \qquad (\text{mod } M_k)$$

Proof:

$$D'(\frac{1}{f}) = \frac{1}{2}\frac{f'}{f}\,\phi + \frac{1}{2}\frac{(1-\frac{1}{f})'}{1-\frac{1}{f}}\,\theta$$

where :

$$\phi' = \frac{(1-\frac{1}{f})'}{1-\frac{1}{f}} \text{ and } \theta' = \frac{(\frac{1}{f})'}{(\frac{1}{f})} = -\frac{f'}{f}$$

So :

$$\phi' = \frac{(1-f)'}{(1-f)} - \frac{f'}{f}$$

$$\Rightarrow \quad D'(\frac{1}{f}) \equiv \frac{1}{2}\frac{f'}{f}(log(1-f) - logf) - \frac{1}{2}(\frac{(1-f)'}{(1-f)} - \frac{f'}{f})logf \qquad (\text{mod } M_k')$$

$$\Rightarrow \quad D'(\frac{1}{f}) \equiv \frac{1}{2}\frac{f'}{f}log(1-f) - \frac{1}{2}(\frac{(1-f)'}{(1-f)})logf = -D'(f) \qquad (\text{mod } M_k')$$

$$\Rightarrow \quad D(\frac{1}{f}) \equiv -D(f) \qquad (\text{mod } M_k)$$

(M_k' is the space of derivatives of M_k). ■

The second identity satisfied by the dilogarithm is one of the main discoveries of this thesis. It is given in the following proposition whose proof, although lengthy and involved, uses only standard techniques from differential algebra.

Proposition 2: (See [2]) Let k be a differential field of characteristic zero, and let θ be transcendental over k with $k(\theta)$ being a differential field having the same subfield of constants as k. Let $f(\theta) \in k(\theta)$ and K be the splitting field of $f(\theta)$ and $1-f(\theta)$. We define, if a is a zero or a pole of $f(\theta)$, $ord_a f(\theta)$ to be the multiplicity of $(\theta - a)$; this is positive if a is a zero of $f(\theta)$ and negative if a is a pole $f(\theta)$. Then, there exists $f_1 \in k$ such that :

$$D(f(\theta)) \equiv D(f_1) + \sum_{a,b \;\; a\neq b} ord_b(1-f) ord_a(f)\ D(\frac{\theta - b}{\theta - a}) \qquad (\text{mod } M_{K(\theta)}) \qquad (A)$$

where a runs over the zeros and poles of f, and b runs over the zeros and poles of $(1-f)$.

Remark: The splitting field of a rational function $S(\theta) = \frac{T(\theta)}{U(\theta)}$ were T and U are relatively prime is the splitting field of the polynomial $T(\theta)U(\theta)$.

Proof: Let $f(\theta) = f_0 P(\theta)/Q(\theta)$, where $f_0 \in k$, and $P(\theta)$, $Q(\theta)$ are relatively prime polynomials over k which are monic. We can also assume that $degP(\theta) \geq degQ(\theta)$, otherwise, using lemma 2, we replace f by $1/f$.

$$1 - f(\theta) = \frac{Q(\theta) - f_0 P(\theta)}{Q(\theta)} = g_0 \frac{R(\theta)}{Q(\theta)}$$

where $g_0 \in k$, and $R(\theta)$ is a monic polynomial relatively prime with both P and Q.

First step:

$$D'(f) = -\frac{1}{2}\frac{f'}{f} log(1-f) + \frac{1}{2}\frac{(1-f)'}{(1-f)} log f$$

is well-defined mod $M'_{K(\theta)}$. We can check easily that, if $a \neq b$ and $a,b \in K$, then :

$$D'(\frac{\theta - b}{\theta - a}) \equiv \frac{1}{2}(\frac{\theta' - b'}{\theta - b} - \frac{b' - a'}{b - a}) log(\theta - a) + \frac{1}{2}(\frac{b' - a'}{b - a} - \frac{\theta' - a'}{\theta - a}) log(\theta - b)$$

$$+ \frac{1}{2}(\frac{\theta' - a'}{\theta - a} - \frac{\theta' - b'}{\theta - b}) log(b - a) \qquad (\text{mod } M'_{K(\theta)})$$

(this is because $log\ (gh) = log\ g + log\ h + constant$, and $log\ 1/g = -log g + constant$.)

Second step: Consider the set $I_1 = \{(a,b) \mid \;\; a$ is a zero of P or of Q, b is a zero of R or of Q, but whenever one of a and b is a zero of Q the other is not $\}$. (So the set (a,b), a zero of Q and b zero of Q is excluded).

We have:

$$f_0\frac{P(\theta)}{Q(\theta)}+g_0\frac{R(\theta)}{Q(\theta)}=1 \tag{B}$$

$$\Longleftrightarrow \quad f_0P(\theta)+g_0R(\theta)=Q(\theta) \tag{C}$$

Let us compute :

$$-\frac{1}{2}[\sum_{(a,b)\in I_1} ord_a(f)\ ord_b(1-f)\ \frac{b'-a'}{b-a}\ log(\theta-a)]$$

$$+\frac{1}{2}[\sum_{(a,b)\in I_1} ord_a(f)\ ord_b(1-f)\ \frac{b'-a'}{b-a}\ log(\theta-b)] \quad modM'_{K(\theta)}$$

We call the above quantity or sum S_1 :

$$S_1=-\frac{1}{2}\sum_{a\ zero\ of\ P} ord_af\ [\sum_{b\ zero\ or\ pole\ of\ (1-f)} ord_b(1-f)\ \frac{b'-a'}{b-a}]\ log(\theta-a)$$

$$-\frac{1}{2}\sum_{a\ zero\ of\ Q} ord_af\ [\sum_{b\ zero\ of\ R} ord_b(1-f)\ \frac{b'-a'}{b-a}]\ log(\theta-a)$$

$$+\frac{1}{2}\sum_{b\ zero\ of\ R} ord_b(1-f)\ [\sum_{a\ zero\ or\ pole\ of\ f} ord_af\ \frac{b'-a'}{b-a}]\ log(\theta-b)$$

$$+\frac{1}{2}\sum_{b\ zero\ of\ Q} ord_b(1-f)\ [\sum_{a\ zero\ of\ P} ord_af\ \frac{b'-a'}{b-a}]\ log(\theta-b)$$

since $(a,b)\in I_1$.

Now, (B) above implies, if a is a zero of P, that:

$$g_0\frac{R(a)}{Q(a)}=1 \quad \Rightarrow \quad \frac{g_0'}{g_0}+\frac{R'(a)}{R(a)}-\frac{Q'(a)}{Q(a)}=0$$

but, as we can easily check :

$$\sum_{b\ zero\ or\ pole\ of\ (1-f)} ord_b(1-f)\ \frac{b'-a'}{b-a}=\frac{R'(a)}{R(a)}-\frac{Q'(a)}{Q(a)}=-\frac{g_0'}{g_0} \tag{3.1}$$

(where a is a zero of P).

Also, if b is a zero of R, we have, using (B) above :

$$f_0 \frac{P(b)}{Q(b)} = 1 \quad \Rightarrow \quad \frac{f_0'}{f_0} = -\frac{P'(b)}{P(b)} + \frac{Q'(b)}{Q(b)}$$

So we get :

$$\sum_{a \text{ zero or pole of } f} ord_a f \, \frac{b' - a'}{b - a} = \frac{P'(b)}{P(b)} - \frac{Q'(b)}{Q(b)} = -\frac{f_0'}{f_0} \tag{3.2}$$

(where, in the above, b is a zero of R).

Now, we look at the sum :

$$S_2 = -\frac{1}{2} \sum_{a \text{ zero of } Q} ord_a f \, [\sum_{b \text{ zero of } R} ord_b(1-f) \, \frac{b' - a'}{b - a}] \, log(\theta - a)$$

$$+\frac{1}{2} \sum_{b \text{ zero of } Q} ord_b(1-f) \, [\sum_{a \text{ zero of } P} ord_a f \, \frac{b' - a'}{b - a}] \, log(\theta - b)$$

$$\Rightarrow \quad S_2 = \frac{1}{2} \sum_{a \text{ zero of } Q} ord_a f \, [\sum_{b \text{ zero of } R} -ord_b(1-f) \, \frac{b' - a'}{b - a}$$

$$+ \sum_{b \text{ zero of } P} ord_b f \, \frac{b' - a'}{b - a} \,] \, log(\theta - a)$$

But the relation $f_0 P(\theta) + g_0 R(\theta) = Q(\theta)$ implies, if a is a zero of Q, that :

$$f_0 P(a) + g_0 R(a) = 0 \quad \Rightarrow \quad \frac{f_0'}{f_0} + \frac{P'(a)}{P(a)} \; = \; \frac{g_0'}{g_0} + \frac{R'(a)}{R(a)}$$

$$\Rightarrow \quad \frac{P'(a)}{P(a)} - \frac{R'(a)}{R(a)} = \frac{g_0'}{g_0} - \frac{f_0'}{f_0} \tag{3.3}$$

and :

$$- \sum_{b \text{ zero of } R} ord_b(1-f) \, \frac{b' - a'}{b - a} = -\frac{R'(a)}{R(a)}$$

$$\sum_{b \text{ zero of } P} ord_b(f) \, \frac{b' - a'}{b - a} = \frac{P'(a)}{P(a)}$$

(if a is a zero of Q).

(3.3) and the above imply that :

$$S_2 = \frac{1}{2} \sum_{a \text{ zero of } Q} ord_a f \, [\frac{g_0'}{g_0} - \frac{f_0'}{f_0}] \, log(\theta - a)$$

which is exactly :

$$S_2 = -\frac{1}{2} \sum_{b \text{ zero of } Q} ord_b(1-f) \, \frac{f_0'}{f_0} \, log(\theta - b) + \frac{1}{2} \sum_{a \text{ zero of } Q} ord_a f \, \frac{g_0'}{g_0} \, log(\theta - a)$$

(3.1) and (3.2) imply, respectively, that :

$$-\frac{1}{2} \sum_{a \text{ zero of } P} ord_a f \, [\sum_{b \text{ zero or pole of } (1-f)} ord_b(1-f) \, \frac{b' - a'}{b - a}] \, log(\theta - a)$$

$$= \frac{1}{2} \sum_{a \text{ zero of } P} ord_a(f) \, \frac{g_0'}{g_0} \, log(\theta - a)$$

This sum will be denoted by S_3.

$$\frac{1}{2} \sum_{b \text{ zero of } R} ord_b(1 - f) \left[\sum_{a \text{ zero or pole of } f} ord_a f \, \frac{b' - a'}{b - a} \right] log(\theta - b) =$$
$$- \frac{1}{2} \sum_{b \text{ zero of } P} ord_b(1-f) \, \frac{f_0'}{f_0} \, log(\theta - b)$$

This sum will be denoted by S_4.

Now, $S_1 = S_2 + S_3 + S_4$, and by regrouping the terms in S_2, S_3 and S_4 we deduce that :

$$S_1 = \frac{1}{2} \sum_{a \text{ zero or pole of } f} ord_a(f) \, \frac{g_0'}{g_0} \, log(\theta - a)$$
$$-\frac{1}{2} \sum_{b \text{ zero or pole of } (1-f)} ord_b(1-f) \, \frac{f_0'}{f_0} \, log(\theta - b) \tag{3.4}$$

Now, consider the four following sums :

$$\Sigma_3 = \frac{1}{2} \sum_{a \text{ zero of } P} ord_a f \, [\sum_{b \text{ zero or pole of } (1-f)} ord_b(1-f) \, log(b - a)] \, \frac{\theta' - a'}{\theta - a}$$

$$\Sigma_4 = -\frac{1}{2} \sum_{b\ zero\ of\ R} ord_b(1-f)\ [\sum_{a\ zero\ or\ pole\ of\ f} ord_a(f)\ log(b-a)]\ \frac{\theta' - b'}{\theta - b}$$

$$\Sigma_2 = \frac{1}{2} \sum_{a\ zero\ of\ Q} ord_a f\ [\sum_{b\ zero\ of\ R} ord_b(1-f)\ log(b-a)]\ \frac{\theta' - a'}{\theta - a}$$

$$-\frac{1}{2} \sum_{b\ zero\ of\ Q} ord_b(1-f)\ [\sum_{a\ zero\ of\ P} ord_a(f)\ log(b-a)]\ \frac{\theta' - b'}{\theta - b}$$

and : $\Sigma_1 = \Sigma_2 + \Sigma_3 + \Sigma_4$. It follows immediately that :

$$\Sigma_1 = \sum_{(a,b)\in I_1} \frac{1}{2} ord_a(f)\ ord_b(1-f)\ [\frac{\theta' - a'}{\theta - a} - \frac{\theta' - b'}{\theta - b}]\ log(b-a)$$

Now, and as before, integrating (3.1), (3.2), and (3.3), we deduce :

$$\sum_{b\ zero\ or\ pole\ of\ (1-f)} ord_b(1-f)\ log(b-a) = logR(a) - logQ(a) + constant$$

$$= -log\ g_0 + constant, \qquad \text{where a is a zero of } P \tag{3.1'}$$

$$\sum_{a\ zero\ or\ pole\ of\ f} ord_a f\ log(b-a) = logP(b) - logQ(b) + constant$$

$$= -log\ f_0 + constant, \qquad \text{where b is a zero of } R \tag{3.2'}$$

$$\sum_{b\ zero\ of\ R} -ord_b(1-f)\ log(b-a) + \sum_{b\ zero\ of\ P} ord_b(f)\ log(b-a)$$

$$= log\ g_0 - log\ f_0 + constant, \qquad \text{where a is a zero of } Q \tag{3.3'}$$

Plugging (3.1'), (3.2') and (3.3') in Σ_3, Σ_4 and Σ_2, respectively, and regrouping, as we have done for computing S_1, we obtain :

$$\Sigma_1 \equiv -\frac{1}{2} \sum_{a\ zero\ or\ pole\ of\ f} ord_a f\ \frac{\theta' - a'}{\theta - a}\ log\ g_0$$

$$+\frac{1}{2}\sum_{b\ zero\ or\ pole\ of\ (1-f)} ord_b(1-f)\ \frac{\theta'-b'}{\theta-b}\ log\ f_0 \qquad (\text{mod } M'_{K(\theta)}) \qquad (3.4')$$

(This is because we had constants in relations (3.1'), (3.2') and (3.3').)

Third step: We compute $D'(f(\theta))$ $mod\ M'_{K(\theta)}$, which can be immediately verified to be :

$$D'(f(\theta)) \equiv \frac{1}{2}[-\sum_{a,b} ord_a(f)\ ord_b(1-f)\ \frac{(\theta-a)'}{\theta-a}\ log(\theta-b)$$

$$+\sum_{a,b} ord_a(f)\ ord_b(1-f)\ \frac{(\theta-b)'}{\theta-b}\ log(\theta-a)\ ]$$

$$-\frac{1}{2}\sum_a ord_a(f)\ \frac{(\theta-a)'}{\theta-a}\ log\ g_0 + \frac{1}{2}\sum_b ord_b(1-f)\ \frac{(\theta-b)'}{\theta-b}\ log\ f_0$$

$$-\frac{1}{2}\sum_b ord_b(1-f)\ \frac{f_0'}{f_0} log(\theta-b) + \frac{1}{2}\sum_a ord_a(f)\ \frac{g_0'}{g_0}\ log(\theta-a)$$

$$-\frac{1}{2}\frac{f_0'}{f_0} log\ g_0 + \frac{1}{2}\frac{g_0'}{g_0} log\ f_0 \qquad (\text{mod } M'_{K(\theta)}) \qquad (3.5)$$

(where $\sum_{a,b}$ runs over all zeros and poles of f and $(1-f)$, respectively, $\sum_a$ runs over the zeros and poles of f, and $\sum_b$ runs over the zeros and poles of $(1-f)$).

The term :

$$\sum_{(a,b)\notin I_1,\ a\neq b} [-ord_a(f)\ ord_b(1-f)\ \frac{(\theta-a)'}{\theta-a}\ log(\theta-b)$$

$$+\ ord_a(f)\ ord_b(1-f)\ \frac{(\theta-b)'}{\theta-b}\ log(\theta-a)]$$

is zero since a and b run over the roots of Q.

So :

$$D'(f(\theta)) \equiv -\frac{1}{2}\frac{f_0'}{f_0}\ log\ g_0 + \frac{1}{2}\ \frac{g_0'}{g_0} log\ f_0$$

$$-\frac{1}{2}\sum_{(a,b)\in I_1} ord_a f\ ord_b(1-f)\ [\ \frac{(\theta-b)'}{\theta-b}\ log(\theta-a) - \frac{(\theta-a)'}{\theta-a}\ log(\theta-b)\ ]$$

$$+\Sigma_1 + S_1 \qquad (\text{mod } M'_{K(\theta)})$$

$$(3.5) \quad \Rightarrow \quad D'(f(\theta)) \equiv \Big[\sum_{(a,b)\in I_1} ord_a f \; ord_b(1-f) \; D(\frac{(\theta-b)}{\theta-a}) \Big]'$$

$$-\frac{1}{2}\frac{f_0'}{f_0} log \; g_0 + \frac{1}{2}\frac{g_0'}{g_0} \; log \; f_0 \qquad (\text{mod } M'_{K(\theta)})$$

Now, we distinguish three cases :

Case 1:

$$deg \; P \; > \; deg \; Q \quad (\text{strict inequality})$$

$$\Rightarrow \quad deg(Q(\theta) - f_0 P(\theta)) = deg \; P(\theta) \quad \Rightarrow \quad g_0 = -f_0$$

And :

$$log(-f_0) = log g_0 = log f_0 + constant$$

So :

$$-\frac{1}{2}\frac{f_0'}{f_0} \; log \; g_0 + \frac{1}{2}\frac{g_0'}{g_0} \; log \; f_0 \equiv 0 \qquad (\text{mod } M'_{K(\theta)})$$

and we take f_1 in proposition 2 to be a constant. So $D'(f_1) = 0$.

Case 2: If $deg \; P = deg \; Q$ (and $f_0 \neq 1$), then the leading coefficient of $Q(\theta) - f_0 P(\theta)$ is $1 - f_0 \;\Rightarrow\; g_0 = 1 - f_0$

$$\Rightarrow \quad -\frac{1}{2}\frac{f_0'}{f_0} \; log \; g_0 + \frac{1}{2}\frac{g_0'}{g_0} \; log \; f_0 = -\frac{1}{2}\frac{f_0'}{f_0} \; log(1-f_0) + \frac{1}{2}\frac{(1-f_0)'}{(1-f_0)} \; log \; f_0$$

and we take f_1 in proposition 2 to be f_0.

Case 3: $deg \; P = deg \; Q$ and $f_0 = 1$.

Let $I = \{(a,b) \mid a \text{ pole or zero of } f, b \text{ pole or zero of } (1-f)\}$. Then, $I - I_1 = \{(a,b) \mid a \text{ zero of } Q, b \text{ zero of } Q\}$. But :

$$D(\frac{\theta-b}{\theta-a}) \equiv -D(\frac{\theta-a}{\theta-b}) \qquad (\text{mod } M_{K(\theta)})$$

$$\Rightarrow \quad \sum_{(a,b)\in I-I_1} ord_a(f) \; ord_b(1-f) \; D(\frac{\theta-a}{\theta-b}) \equiv 0 \qquad (\text{mod } M_{K(\theta)})$$

So :

$$\sum_{(a,b)\in I_1} ord_a(f) \; ord_b(1-f) \; D(\frac{\theta-b}{\theta-a})$$

$$\equiv \sum_{(a,b)\in I} ord_a(f)\ ord_b(1-f)\ D(\frac{\theta-b}{\theta-a}) \pmod{M_{K(\theta)}} \tag{3.6}$$

Now, $deg\ P = deg\ Q$ and $f_0 = 1 \Rightarrow$

$$1 - f = \frac{Q(\theta) - P(\theta)}{Q(\theta)} \Rightarrow\ deg(Q(\theta) - P(\theta)) < degQ(\theta)$$

But, since :

$$D(f) \equiv -D(1-f) \equiv D(\frac{1}{1-f}) \pmod{M_{K(\theta)}}$$

and:

$$\sum_{(a,b)\in I_1} ord_a(f)\ ord_b(1-f)\ D(\frac{\theta-b}{\theta-a})$$

is unchanged if we replace f by $1/(1-f)$, we are again in case 1.

But, by the results of case 1 and case 2, and relations (3.5) and (3.6), proposition 2 is proved. ∎

We would like to end this chapter by giving one example that illustrates the power of these two identities in generating well known identities of the dilogarithm.

Example: Let $k = C(z)$, where z is transcendental over C and $z' = 1$, and C is the field of complex numbers. Applying Lemma 2 and Proposition 2 to $f(z) = z^2$, $f(z) = z$, and $f(z) = -z$, repectively, yields

$$\begin{aligned} D(z^2) &\equiv 2D(\frac{z-1}{z}) + 2D(\frac{z+1}{z}) \pmod{M_{C(z)}} \\ D(z) &\equiv D(\frac{z-1}{z}) \pmod{M_{C(z)}} \\ D(-z) &\equiv D(\frac{z+1}{z}) \pmod{M_{C(z)}} \end{aligned}$$

So,

$$D(z^2) \equiv 2D(z) + 2D(-z)$$

which implies that

$$l_2(z^2) \ + \ \frac{1}{2}log z^2 log(1-z^2)$$

$$\equiv \quad 2\left[l_2(z) + l_2(-z) + \frac{1}{2}logzlog(1-z) + \frac{1}{2}log(-z)log(1+z)\right] \qquad (\text{mod } M_{C(z}$$

and we obtain

$$l_2(z^2) \equiv 2l_2(z) + 2l_2(-z) \qquad (\text{mod } M_{C(z)}),$$

which is a well known identity of the dilogarithm.

Chapter 4

An Extension of Liouville's Theorem

In this chapter, we state and prove the major result of this thesis. Our result is a new theorem that generalizes Liouville's theorem on integration in finite terms. It allows dilogarithms to occur in the integrals in addition to elementary functions. The proof is based on the two identities of the Bloch-Wigner-Spence function given in Lemma 2 and Proposition 2 of the previous chapter. It also uses Proposition 1 of Chapter 2 in several places.

The statement of the theorem uses the following definition of a transcendental-dilogarithmic-elementary extension of a differential field:

Definition: A transcendental-dilogarithmic-elementary extension of a differential field k is a differential field extension K such that there is a tower of differential fields $k = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_N = K$ all having the same constant field and for each $i = 1, \ldots, N$ we have one of the following three cases :

(1") $K_i = K_{i-1}(\theta_i)$, where θ_i is logarithmic over K_{i-1}.

(2") $K_i = K_{i-1}(\theta_i)$, where θ_i is exponential over K_{i-1}. We also assume θ_i transcendental over K_{i-1} in this case.

(3") $K_i = K_{i-1}(\theta_i, \theta_i')$, where $\theta_i = l_2(a)$ for some $a \in K_{i-1}$.

The theorem reads as follows:

Theorem: (See [2]) Let k be a differential field of characteristic zero, which is a Liouville extension of its subfield of constants assumed algebraically closed. Let $f \in k$ and suppose that there exists a transcendental-dilogarithmic-elementary extension K of k such that :

$$\int f \in K$$

Then, the integral $\int f$ is a simple elementary-dilogarithmic expression over k. That is :

$$\int f = g + \sum_{i=1}^{m} s_i \, log \, v_i + \sum_{j=1}^{n} c_j \, D(h_j) \quad (n, m \text{ are positive integers})$$

where g, s_i, v_i, $h_j \in k$, and the c_j's are constants.

The rest of the chapter is devoted to the proof of this theorem.

We start by recalling a lemma due to Kolchin [4].

Lemma 3: Let k be a differential field of characteristic zero. Assume that $u_1, \ldots, u_n$ are logarithmic and algebraically independent over k , that v is exponential over k and that $k(v, u_1, \ldots, u_n)$ and k have the same field of constants. Then, if v is algebraic over $k(u_1, \ldots, u_n)$ there exists an integer $n \neq 0$ such that $v^n \in k$.

Corollary 3.1: Let k be a differential field of characteristic zero. Assume that $u_1, \ldots, u_m$ are logarithmic over k, that v is exponential and transcendental over k and that $k(v, u_1, \ldots, u_m)$ and k have the same field of constants. Then,v is transcendental over $k(u_1, \ldots, u_m)$.

Proof: We can assume using Ostrowski's theorem and without loss of generality that there exists $n \leq m$ such that $k(u_1, \ldots, u_m)$ is algebraic over $k(u_1, \ldots, u_n)$ where $u_1, \ldots, u_n$ are assumed to be algebraically independent. If v were algebraic over $k(u_1, \ldots, u_m)$ it would be algebraic over $k(u_1, \ldots, u_n)$, but by the previous lemma there exists an integer $n \neq 0$ such that $v^n \in k$ and this contradicts the fact that v is transcendental over k. ∎

Lemma 4: Let k be a differential field of characteristic zero. Assume that $u_1, \ldots, u_n$ are logarithmic over k, that t is primitive over k and that $k(t, u_1, \ldots, u_n)$

and k have the same field of constants. If t is algebraic over $k(u_1,\ldots,u_n)$ then there exist constants $c_1,\ldots,c_n$ and an element $s \in k$ such that :

$$t = \sum_{i=1}^{n} c_i u_i + s$$

Proof: This is Ostrowski's theorem , for a proof see [8]. ∎

Proposition 3: Let k be a differential field of characteristic zero, and let θ be primitive and transcendental over k. Let $\alpha_1,\ldots,\alpha_n \in k \quad (\alpha_i \neq \alpha_j,\ if \quad i \neq j)$, $u_1,\ldots,u_m \in k$ and assume the existence of constants $c_1,\ldots,c_n,d_1,\ldots,d_m$ such that :

$$\sum_{i=1}^{n} c_i\ log(\theta - \alpha_i) + \sum_{j=1}^{m} d_i\ logu_j \in k(\theta)$$

(where k and $k(\theta)(log(\theta-\alpha_1),\ldots,log(\theta-\alpha_n),logu_1,\ldots,logu_m)$ have the same field of constants). Then $c_1 = c_2 = \cdots = c_n = 0$.

Proof: There exists $s(\theta) \in k(\theta)$ such that :

$$\sum_{i=1}^{n} c_i\ log(\theta - \alpha_i) + \sum_{j=1}^{m} d_i\ logu_j + s(\theta) = 0$$

This implies that :

$$\sum_{i=1}^{n} c_i \frac{\theta' - \alpha_i'}{\theta - \alpha_i} + s'(\theta) = -\sum_{j=1}^{m} d_i \frac{u_j'}{u_j}$$

In a suitable finite normal algebraic extension field K of k $s(\theta)$ will split into linear factors so that we can write :

$$s(\theta) = \sum_{j,\nu} h_{\nu j}(\theta - \alpha_j)^{\nu} + \sum_{\alpha,i} l_{\alpha i}(\theta - \beta_i)^{\alpha} + (element\ of\ K[\theta])$$

where j ranges over the set $\{1,2,\ldots,n\}$, ν ranges over a finite set of negative integers, i ranges over a finite set of positive integers, α ranges over a finite set of negative integers and $h_{\nu,j}$, $l_{\alpha,i}$ and $\beta_i \in K \quad (\alpha_i \neq \beta_j, \forall i,j)$.

We work in the differential field $K(\theta)$ which is an extension of $k(\theta)$. By assumption we have :

$$(*) \qquad \sum_{i=1}^{n} c_i \, \frac{\theta' - \alpha_i'}{\theta - \alpha_i} + \sum_{i,\nu} (h_{\nu i} \, (\theta - \alpha_i)^{\nu})' + \sum_{\alpha,j} (l_{\alpha,j} \, (\theta - \beta_j)^{\alpha})' \ \in K[\theta]$$

The baisc idea of the proof is the following. When the various functions appearing in (*) are expressed as quotients of polynomials in θ we get no pole cancellation, and therefore all the c_i's and $h_{\nu i}$'s will vanish.

Since θ is primitive over k we have $\theta' = a$, where a belongs to k.

$$\frac{\theta' - \alpha_i'}{\theta - \alpha_i} = \frac{a - \alpha_i'}{\theta - \alpha_i}$$

We claim that $a - \alpha_i' \neq 0$ that is the previous fraction is in lowest terms.

If $a - \alpha_i' = 0$ then $(\theta - \alpha_i)' = \theta' - \alpha_i' = 0$ which implies that $\theta - \alpha_i$ is a constant in k and that contradicts the fact that θ is transcendental over k.

Now :

$$(h_{\nu i} \, (\theta - \alpha_i)^{\nu})' = h_{\nu i}' \, (\theta - \alpha_i)^{\nu} + \nu \, h_{\nu i} \, (\theta - \alpha_i)^{\nu - 1} \, (\theta' - \alpha_i')$$

We notice that since $\theta' - \alpha_i' \in K$ and is different from zero and since $-\nu + 1 > 1$ the various terms of the left-hand side of (*) would not cancel unless $h_{\nu i} = 0$ for all the ν's and the i's and this will imply that $c_i = 0$ for all $i \in \{1, 2, \ldots, n\}$ which is what we want to prove. ∎

Proposition 4: Let k be a differential field of characteristic zero, and let θ be exponential and transcendental over k. Let $\alpha_1, \ldots, \alpha_n \in k$ $(\alpha_i \neq \alpha_j$ if $i \neq j$ and $\alpha_i \neq 0$ for all $i)$, $u_1, \ldots, u_m \in k$, and assume the existence of constants $c_1, \ldots, c_n, d_1, \ldots, d_m$ such that :

$$\sum_{i=1}^{n} c_i \, log(\theta - \alpha_i) + \sum_{j=1}^{m} d_j \, log \, u_j \ \in k(\theta)$$

(where k and $k(\theta)(\, log(\theta - \alpha_1), \ldots, \, log(\theta - \alpha_n), log \, u_1, \ldots, \, log \, u_m)$ have the same field of constants). Then $c_1 = c_2 = \cdots = c_n = 0$.

Proof: There exist $s(\theta) \in k(\theta)$ such that :

$$\sum_{i=1}^{n} c_i \, log(\theta - \alpha_i) + \sum_{j=1}^{m} d_j \, log \, u_j + s(\theta) = 0$$

This implies that :

$$\sum_{i=1}^{n} c_i \, \frac{\theta' - \alpha_i'}{\theta - \alpha_i} + s'(\theta) = - \sum_{j=1}^{m} d_j \, \frac{u_j'}{u_j}$$

In a suitable finite normal algebraic extension field K of k $s(\theta)$ will split into linear factors so that we can write :

$$s(\theta) = \sum_{i,\nu} h_{\nu j}(\theta - \alpha_i)^{\nu} + \sum_{\alpha,j} l_{\alpha,j}(\theta - \beta_j)^{\alpha} + (element \; of \; K[\theta])$$

where i ranges over the set $\{1, 2, \ldots, n\}$, ν ranges over a finite set of negative integers, j ranges over a finite set of positive integers, α ranges over a finite set of negative integers and $h_{\nu i}$, $l_{\alpha,j}$ and $\beta_j \in K$ $(\alpha_i \neq \beta_j, \forall i, j)$

We work in the differential field $K(\theta)$ which is an extension of $k(\theta)$. By assumption we have :

$$(**) \qquad \sum_{i=1}^{n} c_i \, \frac{\theta' - \alpha_i'}{\theta - \alpha_i} + \sum_{i,\nu} (h_{\nu i} \, (\theta - \alpha_i)^{\nu})' + \sum_{\alpha,j} (l_{\alpha,j} \, (\theta - \beta_j)^{\alpha})' \;\; \in K[\theta]$$

The baisc idea of the proof is the following. When the various functions appearing in (**) are expressed as quotients of polynomials in θ we get no pole cancellation, and therefore all the c_i's and $h_{\nu i}$'s will vanish.

Since θ is exponential over k we have $\theta' = a'\theta$ where a belongs to k.

$$\frac{\theta' - \alpha_i'}{\theta - \alpha_i} = \frac{a'\theta - \alpha_i'}{\theta - \alpha_i}$$

We claim that the previous fraction is in lowest terms.

For if the fraction were not in lowest terms we would have $\alpha_i' = a'\alpha_i$ but since $\alpha_i \neq 0$ we get $\alpha_i'/\alpha_i = a' = \theta'/\theta$ which implies that $(\theta/\alpha_i)' = 0$ and this gives θ/α_i is a constant in k and that contradicts the fact that θ is transcendental over k.

Now :

$$(h_{\nu i}\ (\theta - \alpha_i)^{\nu})' = h'_{\nu i}\ (\theta - \alpha_i)^{\nu} + \nu\ h_{\nu i}\ (\theta - \alpha_i)^{\nu-1}\ (\theta' - \alpha'_i)$$

By what has been done and since $-\nu + 1\ >\ 1$ the various terms of the left-hand side of (**) would not cancel unless $h_{\nu i} = 0$ for all the ν's and the i's and this will imply that $c_i = 0$ for all $i \in \{1, 2, \dots, n\}$ which is what we wanted to prove. ■

Corollary 3.2: In the conditions of propositions 3 and 4, $log(\theta - \alpha_1), \dots, log(\theta - \alpha_n)$ (where $\alpha_i \neq 0$ for all i if θ is exponential) are algebraically independent over $k(\theta)(log\ u_1, \dots, log\ u_m)$.

Proof: If $log(\theta - \alpha_1), \dots, log(\theta - \alpha_n)$ were not algebraically independent and since $log(\theta - \alpha_1), \dots, log(\theta - \alpha_n), log\ u_1, \dots, log\ u_m$ are logarithmic over $k(\theta)$, we deduce by Ostrowski's theorem that there exist constants $c_1, \dots, c_n$ not all zero and constants $d_1, \dots, d_m$ such that :

$$\sum_{i=1}^{n} c_i\ log(\theta - \alpha_i) + \sum_{j=1}^{m} d_j\ log\ u_j\ \in k(\theta)$$

and the above implies by propositions 3 and 4 that $c_1 = c_2 = \cdots = c_n = 0$ which gives a contradiction. ■

Proposition 5: Let k be a differential field of characteristic zero. Let θ be transcendental over k where we assume that k and $k(\theta)$ have the same field of constants. Let $s(\theta) \in k(\theta)$ be such that $s'(\theta) \in k$. Then :

(1") If θ is primitive over k, $s(\theta) = c\ \theta + v$, where c is a constant and $v \in k$.

(2") If θ is exponential over k, $s(\theta) \in k$.

Proof: By partial fraction decomposition we can check that $s(\theta)$ is a polynomial in θ. In the case θ is primitive over k we write :

$$s(\theta) = \sum_{j=0}^{m} b_j\ \theta^j$$

where each $b_j \in k$ and assume, as we may, that $m > 0$, $b_m \neq 0$. Then :

$$s'(\theta) = b'_m\ \theta^m + (\ mb_m\theta' + b'_{m-1}\ )\ \theta^{m-1} + (\ elements\ of\ k[\theta]\ of\ degree\ < m-1\ )$$

Since $s'(\theta) \in k$, we get $b'_m = 0$, so b_m is a constant, and if $m > 1$ we get $mb_m\theta' + b'_{m-1} = 0$ that is $(mb_m\theta + b_{m-1})' = 0$ so that $mb_m\theta + b_{m-1} \in k$ contradicting the transcendency of θ over k; thus $m = 1$ and $s(\theta)$ is of the form $c\,\theta + v$ where c is a constant and $v \in k$.

In the case where θ is exponential over k then $\theta' = a'\,\theta$ for some $a \in k$ and if we write again :

$$s(\theta) = \sum_{j=0}^{m} b_j\ \theta^j$$

with each $b_j \in k$ and $b_m \neq 0$ we have :

$$s'(\theta) = \sum_{j=0}^{m}(\ b'_j + jb_ja'\ )\ \theta^j$$

If $m \neq 0$ we have $b'_m + mb_ma' = 0$ so that $\frac{b'_m}{b_m} + m\ \frac{\theta'}{\theta} = 0$ or $(b_m\ \theta^m)' = 0$ giving $b_m\ \theta^m \in k$ which is impossible , so $s(\theta) \in k$. ∎

Now we are ready to prove the main theorem in this thesis. We recall first the definition of a transcendental-dilogarithmic-elementary extension of a differential field k, which is a differential field K such that there is a tower of differential fields $k = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_N = K$ all having the same constant field and for each $i = 1, \ldots, N$ we have one of the following three cases :

(1") $K_i = K_{i-1}(\theta_i)$, where θ_i is logarithmic over K_{i-1}.

(2") $K_i = K_{i-1}(\theta_i)$, where θ_i is exponential over K_{i-1}. we also assume θ_i transcendental over K_{i-1} in this case.

(3") $K_i = K_{i-1}(\theta_i, \theta'_i)$, where $\theta_i = l_2(a)$ for some $a \in K_{i-1}$.

The number N will be called the length of K over k.

<u>**Theorem**</u>: Let k be a differential field of characteristic zero, which is a Liouville extension of its subfield of constants assumed algebraically closed. Let $f \in k$ and

suppose that there exists a transcendental-dilogarithmic-elementary extension K of k such that :

$$\int f \in K$$

Then, the integral $\int f$ is a simple elementary-dilogarithmic expression over k. That is :

$$\int f = g + \sum_{i=1}^{m} s_i \; log \; v_i + \sum_{j=1}^{n} c_j \; D(h_j) \quad (n, m \; are \; positive \; integers)$$

where g, s_i, v_i, $h_j \in k$, and the c_j's are constants.

<u>Proof</u>: It is by induction on N, the length of K over k.

If $N = 0$ then $\int f = g \in k$ and the theorem is proved.

If $N > 0$, we apply the induction hypothesis to $f \in K_1$ and the tower $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_N = K$, to obtain :

$$\int f = g + \sum_{i=1}^{m} s_i \; log \; v_i + \sum_{j=1}^{n} c_j \; D(h_j) \tag{4.1}$$

where g, s_i, v_i, $h_j \in K_1$, and the c_j's are constants.

We want to modify equation (4.1) in such a way that g, s_i, v_i,and h_j are in $k = K_0$.

For this we consider three major cases .

<u>Case 1</u>: $K_1 = k(\theta)$ and θ logarithmic over k : $\theta = log \; a$, $a \in k$. If θ is algebraic over k, then, by lemma 4, $\theta \in k$ and there is nothing to prove.

So, we assume θ transcendental, and factor v_i, $h_j, 1 - h_j$ over k. So we will be working over k^0, the splitting field of these quantities which we assume normal.

By proposition 2 :

$$D(h_j(\theta)) \equiv D(H_j) + \sum_{a,b} ord_a(h_j) \; ord_b(1 - h_j) \; D(\frac{\theta - b}{\theta - a}) \qquad (\text{mod } M_{k^0(\theta)})$$

where $H_j \in k$, $a, b \in k^0, a \neq b$ where a and b are the zeros and poles of h_j and $1 - h_j$ respectively.

So, (4.1) can be written as :

$$\int f \;\equiv\; g(\theta) + \sum_{i=1}^{n} S_i(\theta)\; log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta)\; log\; f_p$$

$$+ \sum_{j=1}^{m} d_j\; D(H_j) + \sum_{i,j} c_{ij}\; D(\;\frac{\theta - \alpha_i}{\theta - \alpha_j}) \qquad (\text{mod } M_{k^0(\theta)}) \tag{4.1'}$$

where in the last sum $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$, $i \neq j$ and $\alpha_i \neq \alpha_j$ if $i \neq j$. Also d_j, c_{ij} are constants , f_p, $H_j \in k$ for $p \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, m\}$ and $g(\theta)$, $S_i(\theta)$, $s_p(\theta) \in k(\theta)$ for $i \in \{1, \ldots, n\}$ and $p \in \{1, \ldots, r\}$.

We notice that the last sum can be written as :

$$\sum_{i,j} c_{ij}\; D(\;\frac{\theta - \alpha_i}{\theta - \alpha_j}\;)$$

$$\equiv \;\; d_{12}\; D(\;\frac{\theta - \alpha_1}{\theta - \alpha_2}) + d_{13}\; D(\;\frac{\theta - \alpha_1}{\theta - \alpha_3}) + \cdots + d_{1n}\; D(\;\frac{\theta - \alpha_1}{\theta - \alpha_n})$$

$$+ d_{23}\; D(\;\frac{\theta - \alpha_2}{\theta - \alpha_3}) + d_{24}\; D(\;\frac{\theta - \alpha_2}{\theta - \alpha_4}) + \cdots + d_{2n}\; D(\;\frac{\theta - \alpha_1}{\theta - \alpha_n})$$

$$\vdots$$

$$+ \sum_{j>i} d_{ij}\; D(\;\frac{\theta - \alpha_i}{\theta - \alpha_j})$$

$$\vdots$$

$$+ d_{n-1,n}\; D(\;\frac{\theta - \alpha_{n-1}}{\theta - \alpha_n})$$

$$+ \; constant \qquad (\text{mod } M_{k^0(\theta)}) \tag{4.2}$$

(This is possible because : $D(\;\frac{\theta - \alpha_i}{\theta - \alpha_j}) \equiv -D(\;\frac{\theta - \alpha_j}{\theta - \alpha_i}) \quad (\text{mod } M_{k^0(\theta)})$.)

We call the above expression reduced, that is, (4.2). For example :

$$d_1\; D(\;\frac{\theta - \alpha_1}{\theta - \alpha_2}) + d_2\; D(\;\frac{\theta - \alpha_1}{\theta - \alpha_3}) + d_3\; D(\;\frac{\theta - \alpha_2}{\theta - \alpha_3})$$

is reduced, while the expression :

$$d_1 \; D(\frac{\theta - \alpha_1}{\theta - \alpha_2}) + d_2 \; D(\frac{\theta - \alpha_1}{\theta - \alpha_3}) + d_3 \; D(\frac{\theta - \alpha_2}{\theta - \alpha_1})$$

is not reduced.

Without changing the notation $S_i(\theta)$, (4.1') becomes :

$$\int f = g(\theta) + \sum_{i=1}^{n} S_i(\theta) \; log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta) \; log \; f_p$$

$$+ \sum_{j_0=1}^{m} d_{j_0} \; D(H_{j_0}) + \sum_{i=1}^{n-1} \sum_{j>i} d_{ij} \; D(\frac{\theta - \alpha_i}{\theta - \alpha_j}) \tag{4.3}$$

(with $\alpha_i \neq \alpha_j$ for all $i \neq j$) and $(1 < j \leq n)$.

Now, we take the derivative of (4.3), to get :

$$f \;=\; g'(\theta) + \sum_{i=1}^{n} S_i(\theta) \; \frac{(\theta - \alpha_i)'}{(\theta - \alpha_i)} + \sum_{i=1}^{n} S_i'(\theta) \; log(\theta - \alpha_i)$$

$$+ \sum_{p=1}^{r} s_p'(\theta) \; \frac{f_p'}{f_p} + \sum_{p=1}^{r} s_p'(\theta) \; log \; f_p$$

$$+ \sum_{j_0=1}^{m} d_{j_0} \; [\, -\frac{1}{2} \, \frac{H_{j_0}'}{H_{j_0}} \; log(1 - H_{j_0}) + \frac{1}{2} \, \frac{(1 - H_{j_0})'}{(1 - H_{j_0})} \; log \; H_{j_0} \,]$$

$$+ \sum_{i=1}^{n-1} \sum_{j>i} d_{ij} \; [\, \frac{1}{2} \, (\frac{\theta' - \alpha_i'}{\theta - \alpha_i} - \frac{\alpha_i' - \alpha_j'}{\alpha_i - \alpha_j}) \; log(\theta - \alpha_j)$$

$$+ \frac{1}{2} \, (\frac{\alpha_i' - \alpha_j'}{\alpha_i - \alpha_j} - \frac{\theta' - \alpha_j'}{\theta - \alpha_j}) \; log(\theta - \alpha_i)$$

$$+ \; \frac{1}{2} \, (\frac{\theta' - \alpha_j'}{\theta - \alpha_j} - \frac{\theta' - \alpha_i'}{\theta - \alpha_i}) \; log(\alpha_i - \alpha_j) \,] \tag{4.4}$$

Identifying the term which multiplies $log(\theta - \alpha_1)$, we get :

$$S_1'(\theta) + \sum_{j>1} \frac{1}{2} \, (\frac{\alpha_1' - \alpha_j'}{\alpha_1 - \alpha_j} - \frac{\theta' - \alpha_j'}{\theta - \alpha_j}) \; d_{1j} = 0 \tag{4.5}$$

This is because the $log(\theta - \alpha_i)$ $(1 \leq i \leq n)$ are algebraically independent over

$k^0(\theta)(\ log \ H_{j_0} \ (1 \leq j_0 \leq m) \ , \ log(1-H_{j_0}) \ (1 \leq j_0 \leq m) \ , \ log \ f_p \ \ (1 \leq p \leq r) \ , \ log(\alpha_i - \alpha_j) \ \ (i < j) \ )$, by corollary (3.2).

Now, (4.5) implies :

$$S_1(\theta) + \sum_{j>1} \frac{1}{2} \ (\ log(\alpha_1 - \alpha_j) - \ log(\theta - \alpha_j) \ ) \ d_{1j} + constant = 0$$

which, by proposition 3, gives $d_{1j} = 0$ for $j > 1$, and $S_1(\theta) = s_1$ is a constant. By induction we prove easily that $d_{ij} = 0$ for all i,j and that $S_i(\theta) = s_i$ is a constant. So we get :

$$\int f = g(\theta) + \sum_{i=1}^{n} s_i \ log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta) \ log \ f_p + \sum_{j_0=1}^{m} d_{j_0} \ D(H_{j_0}) \qquad (4.6)$$

where d_{j_0},s_i are constants, $\alpha_i \in k^0$, and $f_p, H_{j_0} \in k$.

At this point, we distinguish two cases :

<u>Case 1-a</u> : θ is algebraic over $k^0(\ log \ H_{j_0} \ (1 \leq j_0 \leq m) \ , \ log(1-H_{j_0}) \ (1 \leq j_0 \leq m) \ , \ log \ f_p \ \ (1 \leq p \leq r) \ )$. So, by lemma 4, we get :

$$\theta = \sum_{p=1}^{r} c_p \ log \ f_p + \sum_{j_0=1}^{m} b_{j_0} \ log \ H_{j_0} + \sum_{j_0=1}^{m} a_{j_0} \ log(1 - H_{j_0}) + g$$

where c_p, b_{j_0}, a_{j_0} are constants, and $g \in k^0$.

So, $L_i = \theta - \alpha_i$ is a linear logarithmic expression over $F = k^0(\ log \ H_{j_0} \ (1 \leq j_0 \leq m) \ , \ log(1-H_{j_0}) \ (1 \leq j_0 \leq m) \ , \ log \ f_p \ \ (1 \leq p \leq r) \ )$, and (4.6) can be written as :

$$\int f - \sum_{j_0=1}^{m} d_{j_0} \ l_2(H_{j_0}) - \sum_{j_0=1}^{m} O.l_2(1 - H_{j_0})$$

$$- \sum_{p=1}^{r} O.l_2(1 - f_p) - \sum_{i=1}^{n} s_i \ log \ L_i \ \ \in F \qquad (4.7)$$

which implies, by proposition 1, that $\int f$ is a simple elementary-dilogarithmic expression over k and our theorem is proved in this case.

<u>Case 1-b</u> : θ is transcendental over $F = k^0(\ log \ H_{j_0} \ (1 \leq j_0 \leq m) \ , \ log(1 -$

H_{j_0}) $(1 \leq j_0 \leq m)$, $log\ f_p$ $(1 \leq p \leq r)$). (4.6) can be written as :

$$\int (f - [\sum_{j_0=1}^{m} d_{j_0}\ D(H_{j_0})]') = g(\theta) + \sum_{i=1}^{n} s_i\ log(\theta - \alpha_i)$$

$$+ \sum_{p=1}^{r} s_p(\theta)\ log\ f_p \tag{4.8}$$

From this, and as in the proof of Liouville's theorem, we deduce that $s_i = 0$ for all $1 \leq i \leq n$.

Also, by proposition 5, we deduce that there exists c, a constant, and $v \in F$ such that:

$$g(\theta) + \sum_{p=1}^{r} s_p(\theta)\ log\ f_p = c\ \theta + v \qquad (\theta = log\ a)$$

so :

$$\int f - \sum_{j_0=1}^{m} d_{j_0}\ l_2(H_{j_0}) - \sum_{j_0=1}^{m} O.l_2(1 - H_{j_0})$$

$$- \sum_{p=1}^{r} O.l_2(1 - f_p) - c\ log\ a \quad \in F$$

$\Rightarrow$ by proposition 1 that $\int f$ is a simple elementary-dilogarithmic expression over k, and the theorem is proved in case 1.

Case 2: $K_1 = k(\theta, \theta')$ and $\theta = l_2(a)$, where $a \in k$. Let $k_1 = k(\ log(1-a)\ )$. So, $\theta' \in k_1$. If θ is algebraic over k_1, then, by lemma 4, $\theta \in k_1$. So, writing (4.1) again, we have :

$$\int f = g + \sum_{i=1}^{m} s_i\ log\ v_i + \sum_{j=1}^{n} c_j\ D(h_j) \tag{4.9}$$

where g, s_i, v_i, $h_j \in k_1$.

Then, using case 1 (the logarithmic case), we deduce that $\int f$ is a simple elementary-dilogarithmic expression over k.

So, we consider the case θ transcendental over k_1. As in the previous case, (4.9) can be written :

$$\int f = g(\theta) + \sum_{i=1}^{n} S_i(\theta)\ log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta)\ log\ f_p$$

$$+\sum_{j_0=1}^{m} d_{j_0}\ D(H_{j_0}) + \sum_{i=1}^{n-1}\sum_{j>i} d_{ij}\ D(\frac{\theta-\alpha_i}{\theta-\alpha_j})$$

where $f_p, H_{j_0} \in k_1$, $\alpha_i \neq \alpha_j$, $i \neq j, 1 < j \leq n$, and $\alpha_i \in k_1^0$ a normal finite extension of k containing the roots of v_i, h_j, and $(1-h_j)$ for all i,j.

Now, we use the same argument as in case 1 $(\theta = log\ a)$ and proposition 3 to deduce :

$$\int f = g(\theta) + \sum_{i=1}^{n} s_i\ log(\theta-\alpha_i) + \sum_{p=1}^{r} s_p(\theta)\ log\ f_p$$

$$+\sum_{j_0=1}^{m} d_{j_0}\ D(H_{j_0}) \qquad (4.10)$$

where $f_p, H_{j_0} \in k_1$ $\alpha_i \in k_1^0$ and s_i, d_{j_0} are constants.

We also distinguish two cases :

<u>Case 2-a</u> : θ is algebraic over $F_1 = k_1^0(\ log\ H_{j_0}\ (1 \leq j_0 \leq m)\ ,\ log(1-H_{j_0})\ (1 \leq j_0 \leq m)\ ,\ log\ f_p\ \ (1 \leq p \leq r)\ )$.

We apply again the same argument as in case 1-a (using lemma 4), and obtain : $\int f$ is a simple elementary-dilogarithmic expression over k_1 $\Rightarrow$ by case 1 and since$f \in k$ that $\int f$ is a simple elementary-dilogarithmic expression over k.

<u>Case 2-b</u> : θ is transcendental over $F_1 = k_1^0(\ log\ H_{j_0}\ (1 \leq j_0 \leq m)\ ,\ log(1-H_{j_0})\ (1 \leq j_0 \leq m)\ ,\ log\ f_p\ \ (1 \leq p \leq r)\ )$

Then, from (4.10) and as in case 1-b $(\theta = log\ a)$, we deduce that : $s_i = 0$, for all $1 \leq i \leq n$

and that there exists c, a constant, and $v \in F_1$, such that :

$$g(\theta) + \sum_{p=1}^{r} s_p(\theta)\ log\ f_p = c\,\theta + v \quad (\theta = l_2(a))$$

$$(4.10) \ \Rightarrow$$

$$\int f - \sum_{j_0=1}^{m} d_{j_0}\ l_2(H_{j_0}) - \sum_{j_0=1}^{m} O.l_2(1-H_{j_0})$$

$$-\sum_{p=1}^{r} O.l_2(1-f_p) - c\ l_2(a) \quad \in F_1 = F_1(\ log(1-a)\ )$$

(since $log(1-a)\ \in k_1)$ $\Rightarrow$ by proposition 1 that $\int f$ is a simple elementary-dilogarithmic expression over k_1 $\Rightarrow$ by case 1 that $\int f$ is a simple elementary-dilogarithmic expression over k.

Case 3: $K_1 = k(\theta)$, $\theta = exp\ a$, $a \in k$, and θ transcendental over k. As seen before, we can write (4.1) as :

$$\int f = g(\theta) + \sum_{i=1}^{n-1} S_i(\theta)\ log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta)\ log\ f_p$$

$$\overline{}$$

$$(only\ (n-1)\ terms)$$

$$+ \sum_{j_0}^{m} d_{j_0}\ D(H_{j_0}) + \sum_{i=1}^{n-1} \sum_{j>i} d_{ij}\ D(\frac{\theta - \alpha_i}{\theta - \alpha_j}) \tag{4.11}$$

$(\alpha_i \in k^0), (1 < j \leq n)$ and k^0 is a finite normal extension of k..

In this case $log\ \theta \in k$, we assume that $\alpha_n = 0$, and that $\alpha_i \neq 0\ for\ i \neq n$.

The derivative of (4.11) is exactly (4.4), from which we extract the coefficient of $log(\theta - \alpha_1)$ and use corollary (3.2) to obtain :

$$S_1'(\theta) + \sum_{j>1, j\neq n} \frac{1}{2} \left(\frac{\alpha_1' - \alpha j'}{\alpha_1 - \alpha_j} - \frac{\theta' - \alpha j'}{\theta - \alpha_j} \right) d_{1j} + \frac{1}{2}\ d_{1n} \left(\frac{\alpha_1'}{\alpha_1} - \frac{\theta'}{\theta} \right) = 0$$

$$\Rightarrow\ S_1(\theta) + \sum_{j>1, j\neq n} \frac{1}{2} \left(log(\alpha_1 - \alpha_j) - log(\theta - \alpha_j) \right) d_{1j}$$

$$+ \frac{1}{2}\ d_{1n}\ log\ \alpha_1 - \frac{1}{2}\ d_{1n}\ a = \ constant$$

(since $log\ \theta = a \in k$) $\Rightarrow$ by proposition 4 :

$$d_{1j} = 0, \quad for\ all\ j\ >\ 1,\ j \neq n$$

and :

$$S_1'(\theta) = \frac{1}{2}\ d_{1n} \left(\frac{\theta'}{\theta} - \frac{\alpha_1'}{\alpha_1} \right)$$

By induction on i, we can now deduce that :

$$d_{ij} = 0, \quad for\ all\ i\ and \quad for\ all\ j\ >\ 1,\ j \neq n$$

and:

$$S_i'(\theta) = \frac{1}{2}\, d_{in}\, (\ \frac{\theta'}{\theta}\ -\ \frac{\alpha_i'}{\alpha_i}\ ) \quad (1 \leq i \leq n-1) \tag{4.12}$$

So, (4.11) becomes :

$$\begin{aligned}
\int f \ &=\ g(\theta) + \sum_{i=1}^{n-1} S_i(\theta)\ log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta)\ log\ f_p \\
&+ \sum_{j_0=1}^{m} d_{j_0}\ D(H_{j_0}) + \sum_{i=1}^{n-1} d_{in}\ D(\ \frac{\theta - \alpha_i}{\theta}\ ) \\
\Rightarrow\ \ f &= g'(\theta) + \sum_{i=1}^{n-1} S_i(\theta)\ \frac{(\theta - \alpha_i)'}{(\theta - \alpha_i)} + \sum_{i=1}^{n-1} S_i'(\theta)\ log(\theta - \alpha_i) \\
&+ \sum_{p=1}^{r} s_p(\theta)\ \frac{f_p'}{f_p} + \sum_{p=1}^{r} s_p'(\theta)\ log\ f_p \\
&+ \sum_{j_0=1}^{m} d_{j_0}\ [\ -\frac{1}{2}\ \frac{H_{j_0}'}{H_{j_0}}\ log(1 - H_{j_0}) + \frac{1}{2}\ \frac{(1 - H_{j_0})'}{(1 - H_{j_0})}\ log\ H_{j_0}\ ] \\
&+ \sum_{i=1}^{n-1} d_{in}\ [\ \frac{1}{2}\ (\ \frac{\theta' - \alpha_i'}{\theta - \alpha_i} - \frac{\alpha_i'}{\alpha_i}\ )\ (a + c) + \frac{1}{2}\ (\ \frac{\alpha_i'}{\alpha_i}\ -\ \frac{\theta'}{\theta}\ )\ log(\theta - \alpha_i) \\
&+ \frac{1}{2}\ (\ \frac{\theta'}{\theta} - \frac{\theta' - \alpha_i'}{\theta - \alpha_i}\ )\ log\ \alpha_i\ ]
\end{aligned} \tag{4.13}$$

(c is a constant such that $log\ \theta = a + c$). In the above expression, the coefficient of $log(\theta - \alpha_i)$ is zero, as we have seen before.

Now, by corollary 3.1, θ is transcendental over :

$$\begin{aligned}
F_0 = k^0(log\alpha_i(1 \leq i \leq n-1), logH_{j_0}(1 \leq\ \ j_0\ \ \leq m),& \\
log(1 - H_{j_0})(1 \leq\ \ j_0\ \ \leq m), logf_p(1 \leq p \leq r)).&
\end{aligned}$$

On the other hand, we choose the $log\ f_p$ $(1 \leq p \leq r)$ in such a way that they are linearly independent and transcendental over k^0. Then, by lemma 3 and corollary (3.1), they are algebraically independent over $k^0(\theta)$.

From (4.13), we deduce that there exist subsets J_p,I_p,T_p such that :

$$s'_p(\theta) + \sum_{j_0 \in J_p} (-\frac{1}{2} \frac{H'_{j_0}}{H_{j_0}} d_{j_0}) + \sum_{j_0 \in I_p} (\frac{1}{2} \frac{(1-H_{j_0})'}{(1-H_{j_0})} d_{j_0})$$

$$+ \sum_{i \in T_p} \frac{1}{2} d_{in} (\frac{\theta'}{\theta} - \frac{\theta' - \alpha'_i}{\theta - \alpha_i}) = 0 \tag{4.14}$$

(this is the coefficient of $log\ f_p$; J_p,I_p,T_p exist because $log\ \alpha_i$,$log\ H_{j_0}$, $log(1 - H_{j_0})$ could depend on $log\ f_p$).

By proposition 4, we deduce that $d_{in} = 0$ for all $i \in T_p$. So:

$s'_p(\theta) \in k \ \Rightarrow \ s_p(\theta) = s_p \in k$ by proposition 5 (for all p).

So, (4.13) becomes :

$$f = g'(\theta) + \sum_{i=1}^{n-1} S_i(\theta) \frac{(\theta - \alpha_i)'}{(\theta - \alpha_i)} + \sum_{p=1}^{r} s_p \frac{f'_p}{f_p} + \sum_{p=1}^{r} s'_p\ log\ f_p$$

$$+ \sum_{j_0=1}^{m} d_{j_0} [-\frac{1}{2} \frac{H'_{j_0}}{H_{j_0}} log(1 - H_{j_0}) + \frac{1}{2} \frac{(1-H_{j_0})'}{(1-H_{j_0})} log\ H_{j_0}]$$

$$+ \sum_{i=1}^{n-1} d_{in} [\frac{1}{2} (\frac{\theta' - \alpha'_i}{\theta - \alpha_i} - \frac{\alpha'_i}{\alpha_i}) (a + c) + \frac{1}{2} (\frac{\theta'}{\theta} - \frac{\theta' - \alpha'_i}{\theta - \alpha_i}) log\ \alpha_i] \tag{4.15}$$

But, from (4.12), we had :

$$S'_i(\theta) = \frac{1}{2} d_{in} (\frac{\theta'}{\theta} - \frac{\alpha'_i}{\alpha_i})$$

$$\Rightarrow \quad S_i(\theta) = \frac{1}{2} d_{in} (a - log\ \alpha_i) + c_i, \ \ c_i \ is\ a\ constant \tag{4.16}$$

So, $S_i \in F_0 = k^0(log\ \alpha_i \ (1 \le i \le n-1),\ log\ H_{j_0} \ (1 \le j_0 \le m),\ log(1 - H_{j_0}) \ (1 \le j_0 \le m),\ log\ f_p \ (1 \le p \le r))$.

Computing the coefficient of $(\theta - \alpha_i)'/(\theta - \alpha_i)$ in (4.15), we get :

$$g'(\theta) + \sum_{i=1}^{n-1} (S_i(\theta) + \frac{1}{2} d_{in} [(a + c) - log\ \alpha_i]) \frac{\theta' - \alpha'_i}{\theta - \alpha_i} \ \in F_0$$

Considering the partial fraction decomposition of $g(\theta)$, we can prove, as in the proof of Liouville's theorem, that (since $\alpha_i \neq 0$) :

$$S_i(\theta) + \frac{1}{2}\, d_{in}[\ (a+c) -\ log\ \alpha_i\ ]\ = 0, \quad for\ all\ i\ \leq\ n-1 \qquad (4.17)$$

Comparing with (4.16), we deduce that :

$$d_{in}\ [\ a - log\ \alpha_i\ ] = constant, \quad for\ all\ i\ \leq\ n-1 \qquad (4.18)$$

We claim that $d_{in} = 0$, otherwise we would have :

$$a' - \frac{\alpha_i'}{\alpha_i} = 0 \ \Rightarrow\ \frac{\theta'}{\theta} - \frac{\alpha_i'}{\alpha_i} = 0 \ \Rightarrow\ N_0\, \frac{\theta'}{\theta} - \frac{Norm(\alpha_i)'}{Norm(\alpha_i)} = 0 \qquad (4.19)$$

where $N_0 = [k^0 : k]$, and Norm is the usual norm from k^0 to k.

So, (4.19) implies :

$$(\ \theta^{-N_0}\ Norm(\alpha_i)\ )' = 0 \ \Rightarrow\ \theta^{N_0} \in k \ \Rightarrow\ contradiction$$

and :

$$d_{in} = 0, \quad for\ all\ 1 \leq\ i\ \leq\ n-1$$

which implies that $S_i'(\theta) = 0$ by (4.17) $\Rightarrow$ $S_i(\theta) =$ $constant$(that we denote S_i) $\Rightarrow$

$$f = g'(\theta) + \sum_{i=1}^{n-1} S_i\ \frac{(\theta-\alpha_i)'}{(\theta-\alpha_i)} + \sum_{p=1}^{r} s_p\ \frac{f_p'}{f_p} + \sum_{p=1}^{r} s_p'\ log\ f_p$$

$$+ \sum_{j_0=1}^{m} d_{j_0}\ [\ -\frac{1}{2}\, \frac{H_{j_0}'}{H_{j_0}}\ log(1-H_{j_0}) + \frac{1}{2}\, \frac{(1-H_{j_0})'}{(1-H_{j_0})}\ log\ H_{j_0}\ ]$$

Let $F_{00} = k^0(\ log\ f_p\ \ (1 \leq p \leq r),\ log\ H_{j_0}\ \ (1 \leq j_0 \leq m),\ log(1-H_{j_0})\ \ (1 \leq j_0 \leq m))$. θ is transcendental over F_{00} and , as in the proof of the Liouville's theorem, we get $S_i = 0$, for all i. Also, we get $g(\theta) = g\ \in F_{00}$, by proposition 5, so we get :

$$\int [\ f - [\sum_{j_0=1}^{m} d_{j_0}\ D(H_{j_0})\ ]'\ ] = g + \sum_{p=1}^{r} s_p\ log\ f_p$$

$$\Rightarrow \int f - \sum_{j_0=1}^{m} d_{j_0}\ l_2(H_{j_0}) - \sum_{j_0=1}^{m} O.l_2(1 - H_{j_0})$$

$$- \sum_{p=1}^{r} O.l_2(1 - f_p) \in F_{00}$$

$\Rightarrow$ $\int f$ is a simple elementary-dilogarithmic expression over k by proposition 1, so the theorem is proved. ■

We end this chapter by giving a nontrivial example that illustrates the fundamental concept behind our generalization of Liouville's theorem, which is that integration in finite terms is actually a simplification process.

In fact, what we have proved is:

Let k be a differential field of characteristic zero, which is a Liouville's extension of its subfield of constants assumed algebraically closed. Let f be an element in k and suppose that f has a transcendental-dilogarithmic-elementary integral. Then

$$\int f = g + \sum_{i=1}^{m} s_i w_i + \sum_{j=1}^{n} d_j v_j$$

where n and m are positive integers, $g \in k$, $s_i \in k$, for all i, $1 \leq i \leq m$, w_i is logarithmic for all i, $1 \leq i \leq m$, d_j is a constant for all j, $1 \leq j \leq n$, and $v_j = D(\phi_j)$, where $\phi_j \in k - \{0, 1\}$ for all j, $1 \leq j \leq n$. In our proof of the theorem, we observed that, although v'_j does not in general belong to k, it can even be transcendental over k, as is illustrated in the following example.

Example: Let k be any differential field of characteristic zero. Assume that θ is primitive and transcendental over k. Let $p(\theta)$ and $q(\theta)$ be two irreducible polynomials over k such that $\deg p > \deg q \neq 0$.

We consider the differential field $K = k(\theta)(\phi_1, \phi_2)$, where ϕ_1 and ϕ_2 are such that

$$\phi'_1 = \frac{p'(\theta)}{p(\theta)} \quad \text{and} \quad \phi'_2 = \frac{q'(\theta)}{q(\theta)}$$

It is immediate that ϕ_1 and ϕ_2 are algebraically indenpendent over $k(\theta)$. It is also clear that, if ϕ_3 is such that

$$\phi_3' = \frac{(p(\theta)+q(\theta))'}{p(\theta)+q(\theta)}$$

then ϕ_3 is transcendental over K. Consider the function:

$$f = \frac{1}{2}\left(\frac{q'}{q} - \frac{(p+q)'}{p+q}\right)\phi_1 - \frac{1}{2}\left(\frac{(p+q)'}{p+q} - \frac{p'}{p}\right)\phi_2 + \frac{1}{2}(\phi_1+\phi_2)\frac{(p+q)'}{p+q}$$

$f \in K$, and we can check that

$$\left[D(\frac{-p}{q}) + \frac{1}{2}(\phi_1+\phi_2)\phi_3\right] \equiv \int f \qquad (\bmod M_K)$$

but $(D(-p/q))'$ is transcendental over K since ϕ_3 is.

Chapter 5

Further Results

In this Chapter we report on two results that we have obtained but where we don't give the proofs because they are quite long. The first one concerns finding a decision procedure for integrating a function in terms of elementary functions and dilogarithms if such an integral exists. Formally this is stated as follows.

Theorem: Let $C(z)$ be a differential field of characteristic zero where z is transcendental over C, the field of complex numbers, and z is a solution to $z' = 1$.

Let $k = C(z, \theta_1, \theta_2, \ldots, \theta_n)$, $n \geq 0$, be a transcendental elementary extension of $C(z)$. Given $f \in k$ one can decide in a finite number of steps if f has a transcendental-dilogarithmic- elementary integral and if so determine g, v_i, s_i, c_j, and h_j, satisfying

$$\int f = g + \sum_{i=1}^{m} s_i \; log \; v_i + \sum_{j=1}^{n} c_j \; D(h_j) \quad (n, m \; are \; positive \; integers).$$

The second result extends the theorem we proved on integration in finite terms with elementary functions and dilogarithms to elementary functions, dilogarithms and trilogarithms.

The trilogarithm is defined as:

$$l_3(x) = \int_0^x \frac{l_2(t)}{t} dt.$$

Consider the function utilized by Kummer [13] defined as

$$\Lambda(x) = \int_0^x \frac{log^2(1-t)}{t} dt.$$

A simple integration by parts gives

$$\Lambda(x) = \log x log^2(1-x) + 2log(1-x) l_2(x) - 2l_3(1-x) + c$$

where c is a constant.

Definition: A transcendental-trilogarithmic-elementary extension of a differential field k of characteristic zero is a differential field extension K such that there is a tower of differential fields

$$k = K_0 \subset K_1 \subseteq \ldots \subseteq K_N = K$$

all having the same constant field and for each $i = 1, \ldots, N$ we have one of the following cases :

(1") $K_i = K_{i-1}(\theta_i)$, where θ_i is logarithmic over K_{i-1}.

(2") $K_i = K_{i-1}(\theta_i)$, where θ_i is exponential over K_{i-1}. we also assume θ_i transcendental over K_{i-1} in this case.

(3") $K_i = K_{i-1}(\theta_i, \theta_i')$, where $\theta_i = l_2(a)$ for some $a \in K_{i-1}$.

(4") $K_i = K_{i-1}(\theta_i, \theta_i', \phi)$ where $\theta_i' = (\frac{a'}{a})u$, $u = l_2(a)$, and $\phi' = \frac{(1-a)'}{(1-a)}$ for some $a \in K_{i-1} - \{0, 1\}$. In this case, we write $\theta_i = l_3(a)$.

We say that $f \in k$ has a transcendental-trilogarithmic-elementary integral if $\int f \in K$ where K is a transcendental-trilogarithmic-elementary extension of k.

We also give the following definition

Definition: A transcendental-Λ-elementary extension of a differential field k is a differential field extension K such that there is a tower of differential fields $k = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_N = K$ all having the same constant field and for each $i = 1, \ldots, N$ we have one of the following cases :

(1") $K_i = K_{i-1}(\theta_i)$, where θ_i is logarithmic over K_{i-1}.

(2") $K_i = K_{i-1}(\theta_i)$, where θ_i is exponential over K_{i-1}. we also assume θ_i transcendental over K_{i-1} in this case.

(3") $K_i = K_{i-1}(\theta_i, \theta_i')$, where $\theta_i = l_2(a)$ for some $a \in K_{i-1}$.

(4") $K_i = K_{i-1}(\theta_i, \theta_i')$ where $\theta_i' = (\frac{a'}{a}u^2)$ with $a \in K_{i-1} - \{0,1\}$, $u' = \frac{(1-a)'}{(1-a)}$. In this case, we write $\theta_i = \Lambda(a)$.

We say that $f \in K$ has a transcendental-Λ-elementary integral if $\int f \in K$ where K is a transcendental-Λ-elementary extension of k.

But since l_3 can be expressed using logarithms, dilogarithms, and the function Λ, for $f \in k$ (where k is a differential field of charactersistic zero) to have a transcendental-trilogarithmic-elementary integral, it is equivalent for f to have a transcendental-Λ-elementary integral.

It turns out as in the case of the dilogarithm that the function (see [13])

$$M(x) = \Lambda(x) - \frac{1}{3} log x log^2(1-x)$$

satisfies simpler identities than $\Lambda(x)$.

This motivates the following

Definition: If k is a differential field of characteristic zero, K a differential field extension of k such that $K = k(t, u, v)$, we say that $t = M(\phi)$, if ϕ is an element of $k - \{0,1\}$ and

$$t' = \frac{2}{3}\frac{\phi'}{\phi}u^2 - \frac{2}{3}\frac{(1-\phi)'}{(1-\phi)}uv$$

where $u' = \frac{(1-\phi)'}{(1-\phi)}$ and $v' = \frac{\phi'}{\phi}$.

From this definition it follows that t is defined up to the addition of a linear combination of $l_2(\phi)$, $log\phi$, $log(1-\phi)$, $log\phi log(1-\phi)$, and $log^2(1-\phi)$, with constant coefficients. Informally, t is equal to

$$\Lambda(\phi) - \frac{1}{3} log\phi \; log^2(1-\phi)$$

This motivates considering the function Λ and the associated function M as defined modulo the vector space V_k where $t \in V_k$ if and only if it can be written in the form

$$t = \sum_{i=1}^{n} c_i w_i + \sum_{j=1}^{m} d_j v_j + \sum_{r=1}^{N} \sum_{s=1}^{M} c_{rs} u_r u_s$$

where c_i, d_j, c_{rs} are constants, v_j, u_r, u_s are logarithmic over k, and w_i is dilogarithmic over k.

We can prove the following proposition:

Proposition: (i) If k is a differential field of characteristic zero, then for all $f \in k - \{0,1\}$

$$M\left(\frac{f}{f-1}\right) \equiv M(f) \pmod{V_k}$$
$$M(f) + M(\frac{1}{f}) + M(1-f) \equiv 0 \pmod{V_k}$$

(ii) If k is a differential field of characteristic zero, let θ be transcendental over k with $k(\theta)$ being a differential field having the same field of constants. Let $f(\theta) \in k(\theta) - \{0,1\}$ and K be the splitting field of $f(\theta)$ and $1 - f(\theta)$. If a is a zero or pole of $f(\theta)$, we define $ord_a f(\theta)$ to be the multiplicity of $(\theta - a)$; this is positive if a is a zero of $f(\theta)$ and negative if a is a pole of $f(\theta)$. Then there exists $f_1 \in k$ such that

$$\begin{aligned} M(f(\theta)) \equiv\ & M(f_1) + \frac{1}{2} \sum_{a,b,c,a\neq b,a\neq c,b\neq c} ord_a(1-f) ord_b(f) ord_c(1-f) M(\frac{\theta - b}{a-b}) \\ &+ \frac{1}{2} \sum_{a,b,c,a\neq b,a\neq c,b\neq c} ord_a(1-f) ord_b(f) ord_c(1-f) M(\frac{\theta - b}{c-b}) \\ &+ \frac{1}{2} \sum_{a,b,c,a\neq b,a\neq c,b\neq c} ord_a(1-f) ord_b(f) ord_c(1-f) M(\frac{a-c}{\theta - c}) \\ &- \frac{1}{2} \sum_{a,b,c,a\neq b,a\neq c,b\neq c} ord_a(1-f) ord_b(f) ord_c(1-f) M(\frac{(\theta - b)(a-c)}{(\theta - c)(a-b)}) \pmod{V_{K(\theta)}} \end{aligned}$$

where a runs over the zeros and poles of $1-f$, b runs over the zeros and poles of f, and c runs over the zeros and poles of $1-f$.

Using the above proposition and techniques similar to those used in proving the structure theorem for the dilogarithm, we can prove the following

Theorem: Let k be a differential field of charactreristic zero, which is a Liouvillian extension of its subfield of constants assumed algebraically closed. Let $f \in k$ and suppose that f has a transcendental-Λ-elementary integral, then

$$\int f = g + \sum_{r \in R} s_r log h_r + \sum_{i \in I, j \in J} t_{ij} log p_i log q_j + \sum_{m \in M} c_m l_2(g_m) + \sum_{l \in L} d_l M(H_l)$$

where R, I, J, M, and L are finite set of positive integers, and $g, s_r, h_r, t_{ij}, p_i, q_j, g_m, H_l \in k$, and c_m, d_l are constants.

We hope that the theorems we proved will spur others to continue this line of research.

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