Musical Variations from a Chaotic Mapping

by

Diana S. Dabby

Submitted to the Department of Electrical Engineering
and Computer Science
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Electrical Engineering and Computer Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 1995

Author . . . .
Department of Electrical Engineering
and Computer Science
September 1, 1995

Certified by . . . .
Kenneth N. Stevens
Professor
Thesis Supervisor

Accepted by . . . .
Frederic R. Morgenthaler
Chairman, Departmental Committee on Graduate Students

Musical Variations from a Chaotic Mapping

by

Diana S. Dabby

Submitted to the Department of Electrical Engineering
and Computer Science
on September 1, 1995, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Electrical Engineering and Computer Science

Abstract

A chaotic mapping provides a technique for generating musical variations of an original work. This technique employs two chaotic trajectories, each corresponding to a different set of initial conditions for the Lorenz system. These trajectories then map the pitch sequence of a musical score into a variation where the same set of pitches appear, but in a modified order.

The chaotic mapping is designed to provide two mechanisms — linking and tracking — to help the variation retain some of the flavor of the original piece. The linking aspect of the mapping ensures that no pitch event will occur in the variation that did not appear in the source. The tracking aspect allows pitches in the variation to occur exactly where they did in the original. Of course, when generating variations, too much tracking is undesirable. That is where the sensitive dependence of chaotic trajectories to initial conditions comes into play: the sensitive dependence property guarantees variability. The linking and tracking mechanisms tame that variability.

That the technique produces variations capable of being analyzed and used for musical means — despite the highly context-dependent nature of music — suggests the chaotic mapping might be applicable to other context-dependent sequences of symbols, e.g., DNA or protein sequences, pixel sequences from scanned art work, word sequences from prose or poetry or textural sequences requiring some intrinsic variation.

Thesis Supervisor: Kenneth N. Stevens
Title: Professor

Acknowledgments

Contents

List of Figures

Chapter 1

Introduction

This thesis consists of seven chapters and five appendices. Chapters 1-7 provide the context for the *The Variation Technique* (Chapter 1), explain the chaotic mapping (Chapter 2), provide musical analysis to evaluate the results (Chapter 3), offer the variation technique as an idea generator (Chapter 4), apply the technique to a contemporary work (Chapter 5), discuss technical issues pertinent to the chaotic mapping (Chapter 6), and conclude with a summary of the work (Chapter 7). Appendix A examines the Lorenz equations, used to simulate the chaotic trajectories. For those who wish to read further, Appendices B-D provide the motivational background for the technique, especially with regard to the question, "Why look to nonlinear dynamics and chaos for creating new musical forms and structures?" Appendix D also gives some background on the new science of *chaos*. A glossary of musical terms comprises Appendix E.

In recent years, it has been realized that chaos can sometimes be exploited for useful applications. This has been seen in the work of Pecora and Carroll on synchronization of chaotic systems [1]; Cuomo, Oppenheim and Strogatz, on chaotic circuits for private communications [2]; Ditto, Rauseo, and Spano, on experimental control of chaos [3]; Bradley on using chaos to broaden the capture range of phase-locked loops [4]; and Roy et al., on controlling chaotic lasers [5]. In this paper, chaos is harnessed to yield an application of a rather different sort: the creation of musical variations based on an original score.

The sensitive dependence property of chaotic trajectories offers a natural mechanism for variability. By affixing the pitch sequence of a musical work to a reference chaotic trajectory, it is possible to generate meaningful variations via a mapping between neighboring chaotic trajectories and the reference. The variations result from changes in the ordering of the pitch sequence. But two chaotic orbits started at nearly the same initial point in state space soon become uncorrelated. To counter this, the mapping was designed so that a nearby trajectory could often track the reference, thus tempering the extent of the separation. Tracking often results in pitches occurring in the variation exactly where they occurred in the original score. However, regardless of whether the two trajectories track, the mapping links the variation with the original by ensuring only those pitch events found in the source piece comprise the variation.

In this thesis, music is used to demonstrate the method, results and possible applications. The choice of music for illustration is deliberate. It is an application in which context, coherence and order are paramount. For instance, every pitch in a musical work is a consequence of the pitches that precede it and a foreshadowing of the pitches that follow. The technique's success with a highly context-dependent application such as music — i.e., its ability to generate variations that can be analyzed and used for musical means — indicates it may prove applicable to other sequences of context-dependent symbols, e.g., DNA or protein sequences, pixel sequences from scanned art work, word sequences from prose or poetry, textural sequences requiring some intrinsic variation, and so on.

The variation technique was not designed to alter music of the past. It is meant for music of our own time — for use in the creative process (as an idea generator) and as a springboard for a dynamic music where the written score changes from one hearing to the next. The analyses given in Chapters 4-5 demonstrate how a composer might use the technique as an idea generator, much in the same spirit as composers have taken the inversion*[1], retrograde* or retrograde inversion* of a motive, theme or section, in order to extend their original musical material. Sometimes an inversion is particularly pleasing or stimulating, yet the retrograde turns out blasé. Certainly, musicians are under no obligation to use any of these. This is also true with the variation technique. Any variation can be accepted, altered or rejected. The artist has choice.

Variations that are close to the original work, diverge from it substantially, or achieve degrees of variability in between these two extremes, can be created. Once an entire piece is varied, creating another version of it, the possibility exists for the work to change from one hearing to the next, from one concert to the next, and even within the same concert. The piece is still recognizable as the same piece from concert to concert, but changes have occurred in the score — changes prescribed by the composer. In a broad sense, the music has become dynamic — it changes with time much in the same way a river changes from day to day, season to season, yet is still recognized in its essence.

The application of mathematics to generate or reveal the underlying structure of music has a long history, from the explanation of the overtone series by Pythagoras to the use of numerology by J. S. Bach [6] and the Fibonacci series by Claude Debussy [7] and Bèla Bartok [8]. In 1954, Iannis Xenakis proposed a world of sound clouds, masses and galaxies all governed by new characteristics such as density and rate of change based on probability and stochastic theory [9]. In 1978 Voss and Clarke claimed that the spectral density of fluctuations in the audio power of musical selections ranging from Bach to Scott Joplin, varies as $1/f$ (approximately) down to a frequency of 5×10^{-4} Hz [10]. More recently, statistical methods have been used to analyze J. S. Bach's last fugue, Contrapunctus XIV from *The Art of Fugue*, in order to characterize the data set and postulate a data-driven (where features are learned from the data) approach to its completion [11].

Fractal and chaotic dynamics have inspired a number of algorithmic approaches

[1] Musical terms marked with an asterisk are briefly explained in the Glossary (Appendix E).

to music composition, where the output of a chaotic system is converted into notes, attack envelopes, loudness levels, texture, timbre, et cetera [12-19]. Chaos has also been used to explore sound synthesis, with the intent of creating new instruments and timbres [20-25]. Dynamical system tools such as phase portraits [26] and cusp-catastrophe diagrams [27] have been suggested for analyzing music and explaining paradigm shifts, respectively. Analogies between the language of dynamics, nonlinear dynamics and chaos to the musical language have been discussed, particularly in reference to whether there is anything inherently musical about the language of nonlinear dynamics and chaos. (See Appendices B, C and D.)

While much of the above work with algorithmic composition allows a chaotic system to free-run in order to generate musical ideas, the present work takes a different approach. A given musical piece becomes the source for any number of variations via a chaotic mapping. While these earlier approaches might have some difficulty accommodating disparate musical styles, the technique proposed here can take musical sequences of *any* style as input, and produce a virtually infinite set of variations. The stylistic flexibility is encoded in the method by allowing the chaotic mapping to tap the original sequence.

1.1 References

[1] L. Pecora and T. Carroll, "Synchronization in chaotic systems," Phys. Rev. Lett. **64**, 821-824 (1990).

[2] K. M. Cuomo, A. V. Oppenheim, and S. H. Strogatz, "Synchronization of Lorenz-based chaotic circuits with applications to communications," IEEE Trans. on Circuits and Systems II **40**, 626-633 (1993).

[3] W. L. Ditto, S. N. Rauseo, and M. L. Spano, "Experimental control of chaos," Phys. Rev. Lett. **65**, 3211-3214 (1990).

[4] E. Bradley, "Using chaos to extend the capture range of phase-locked loops," IEEE Trans. on Circuits and Systems I **40**, 808-818 (1993).

[5] R. Roy, T. W. Murphy, Jr., T. D. Maier, and Z. Gills, "Dynamic control of a chaotic laser: Experimental stabilization of a globally coupled system," Phys. Rev. Lett. **68**, 1259-1262 (1992).

[6] A. Newman, *Bach and the Baroque* (Pendragon, New York, 1985).

[7] R. Howat, *Debussy in Proportion* (Cambridge University Press, London, 1983).

[8] E. Lendvai, *The Workshop of Bartok and Kodaly* (Editio Musica, Budapest, 1983).

[9] I. Xennakis, *Formalized Music* (Indiana University Press, Bloominton, 1971).

[10] R. F. Voss and J. Clarke, "$1/f$ noise in music: Music from $1/f$ noise," *J. Acoust. Soc. Am.* **63**, 258-263 (1978).

[11] M. Dirst and A. S. Weigend. "Baroque Forecasting: On Completing J. S. Bach's Last Fugue," in *Time Series Prediction: Forecasting the Future and Understanding the Past*, Proceedings of the NATO Advanced Research Workshop on Comparative Time Series Analysis held in Santa Fe, New Mexico, May 14-17, 1992, edited by A. S. Weigend and N. A. Gershenfeld (Addison-Wesley, New York, 1994) 151.

[12] J. Harley, "Algorithms adapted from chaos theory: Compositional considerations," in *Proc. 1994 Int. Comp. Mus. Conf.* (ICMA, San Francisco, 1994) 209-212.

[13] M. Herman, "Deterministic chaos, iterative models, dynamical systems and their application in algorithmic composition," in *Proc. 1993 Int. Comp. Mus. Conf.* (ICMA, San Francisco, 1993) 194-197.

[14] P. Beyls, "Chaos and creativity: The dynamic systems approach to musical composition," *Leonardo* **1**, 31-36 (1991).

[15] J. Pressing, "Nonlinear maps as generators of musical design," *Computer Music J.* **12** (2), 35-46 (1988).

[16] Y. Nagashima, H. Katayose, S. Inoduchi, "PEGASUS-2: Real-time composing environment with chaotic interaction model," in *Proc. 1993 Int. Comp. Mus. Conf.* (ICMA, San Francisco, 1993) 378-380.

[17] R. Bidlack, "Chaotic Systems as Simple (but Complex) compositional algorithms," *Comp. Mus. J.* **16** (3), 33-47 (1992).

[18] A. DiScipio, "Composing by exploration of non-linear dynamical systems," in *Proc. 1990 Int. Comp. Mus. Conf.* (ICMA, San Francisco, 1990) 324-327.

[19] D. Little, "Composing with chaos; applications of a new science for music," Interface **22**, 23-51 (1993).

[20] B. Truax, "Chaotic non-linear systems and digital synthesis: An exploratory study," in *Proc. 1990 Int. Comp. Mus. Conf.*, (ICMA, San Francisco, 1990) 100-103.

[21] R. Waschka and A. Kurepa, "Using fractals in timbre construction: An exploratory study," in *Proc. 1989 Int. Comp. Mus. Conf.* (ICMA, San Francisco, 1989) pp. 332-335.

[22] G. Monro, "Synthesis from attractors," in *Proc. 1993 Int. Comp. Mus. Conf.* (ICMA, San Francisco, 1993) 390-392.

[23] G. Mayer-Kress, I. Choi, N. Weber, R. Bargar, A. Hübler, "Musical signals from Chua's circuit," *IEEE Trans. on Circuits and Systems*, **40** 688-695 (1993).

[24] X. Rodet, "Models of musical instruments from Chua's circuit with time delay," *IEEE Trans. on Circuits and Systems*, **40** 696-701 (1993).

[25] B. Degazio, "Towards a chaotic musical instrument," in *Proc. 1993 Int. Comp. Mus. Conf.* (ICMA, San Francisco, 1993) 393-395 (1993).

[26] J. P. Boon, A. Noullez and C. Mommen, "Complex dynamics and musical structure," *Interface* **19**, 3-14 (1990).

[27] C. Georgescu and M. Georgescu, "A system approach to music," *Interface* **19**, 15-52 (1990).

Chapter 2

The Chaotic Mapping

The chaotic mapping is the engine for the variation technique. It contributes two mechanisms — linking and tracking — for tempering the sensitivity of chaotic trajectories to initial conditions. Linking occurs in every application of the mapping. Whether tracking takes place depends on initial conditions, step size, length of the integration, etc.

Figure 1 illustrates the mapping that creates the variations. First, a chaotic trajectory with an initial condition (IC) of (1, 1, 1) is simulated using a fourth order Runge-Kutta implementation of the Lorenz equations [1],

$$\dot{x} = \sigma(y - x) \tag{2.1}$$

$$\dot{y} = rx - y - xz \tag{2.2}$$

$$\dot{z} = xy - bz, \tag{2.3}$$

with step size $h = .01$ and Lorenz parameters $r = 28$, $\sigma = 10$, and $b = 8/3$.[1] This chaotic trajectory serves as the reference trajectory. Let x_i denote the i^{th} x-value in the reference trajectory; the sequence of x-values, obtained after each time step, is plotted in Figure 1a. Each x_i is associated with a pitch p_i from the pitch sequence $\{p_i\}$ (Figure 1b) heard in the original work. For example, the first pitch p_1 of the piece is paired with x_1, the first x-value of the reference trajectory; p_2 is paired with x_2, and so on. The pairings continue until every p_i has been given an x_i (Figure 1c). (Non-musicians can think of the pitch sequence as a sequence of symbols.)

Next, a new trajectory is started at an IC differing from the reference (Figure 1d). For each new x-component x'_j, the chaotic mapping is applied

$$f(x'_j) = p_{g(j)}, \tag{2.4}$$

where $g(j)$ denotes the index i of the smallest x_i for which $x_i \geq x'_j$ (Figure 1e). In other words, given an x'_j, the smallest x_i is found such that $x_i \geq x'_j$. Then its corresponding pitch p_i is assigned to that x'_j. This defines the new pitch $f(x'_j)$. The new

[1]See Appendix A for an explanation of why these particular parameters are chosen.

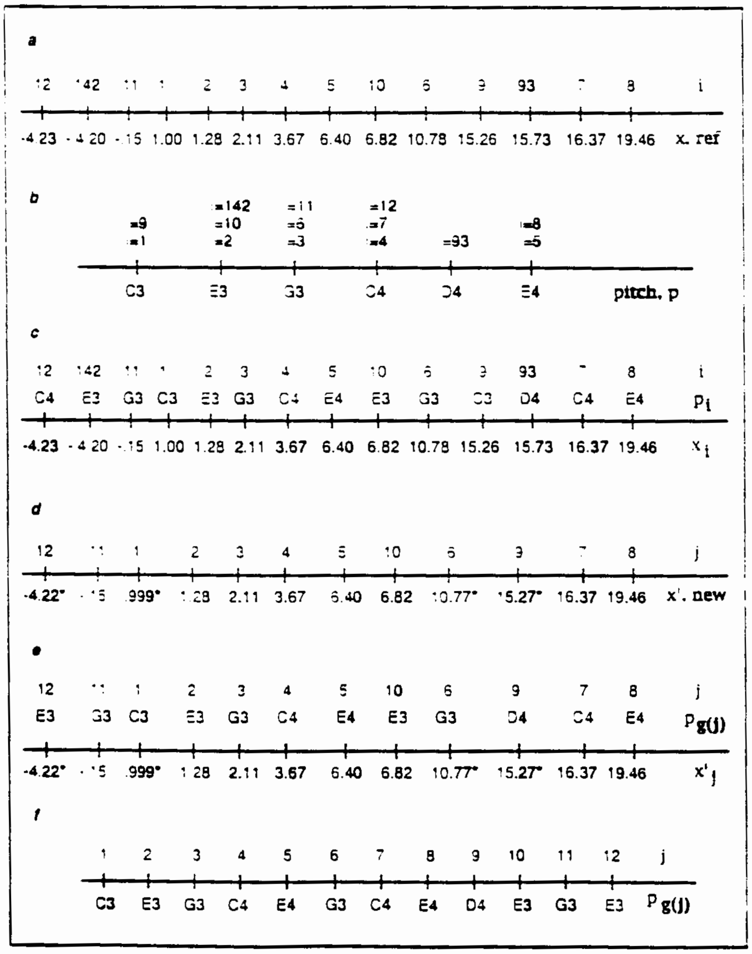


Figure 2-1: Generating the first 12 pitches of a variation. *(a)* The first 12 x-components $\{x_i\}$, $i = 1, \ldots, 12$, of the reference trajectory starting from the IC (1, 1, 1), are marked below the x-axis (not drawn to scale). Two more x-components, that will later prove significant, are indicated: $x_{93} = 15.73$ and $x_{142} = -4.20$. *(b)* The first 12 pitches of the Bach Prelude in C (WTC I) are marked below the pitch axis. The order in which they are heard is given by the index $i = 1, \ldots, 12$. The 93rd and 142nd pitches of the original Bach are also given. *(c)* Parts a and b combine to give an explicit pairing. *(d)* The first 12 x'-components of the new trajectory starting from the IC (.999, 1, 1) are marked below the x'-axis (not drawn to scale). Their sequential order is indicated by the index $j = 1, \ldots, 12$. Those $x'_j \neq x_i$, $i = j$, are starred. *(e)* For each x-component x'_j, apply the chaotic mapping. All pitches remain unchanged from the original until the ninth pitch. Because $x'_9 = 15.27 \leq x_{93} = 15.73$, x'_9 adopts the pitch $D4$ that was initially paired with x_{93}. The next two pitches of Variation 1 replicate the original Bach, but the twelfth pitch, $E3$, arises because $x'_{12} \leq x_{142} = -4.20 \longmapsto E3$. *(f)* The variation is heard by playing back $p_{g(j)}$ for $j = 1, ..., N$, where $N = 176$, the number of pitches in the first 11 measures of the Bach.

variation produced by the chaotic mapping is thus the pitch sequence $p_{g(1)}, p_{g(2)}, \ldots$ given in Figure 1*f*. Sometimes the new pitch agrees with the original pitch: at other times they differ. This is how a variation can be generated that may retain the flavor of the source.

Note that nearby trajectories do not have to track each other exactly to ensure that many pitches in the variation occur exactly where they did in the source. Rather, the x-components of the two orbits just have to fall within the same region on the x-axis for the two trajectories to effectively track each other in x. This tracking aspect of the chaotic mapping may or may not occur in a variation, depending on IC's, step size, length of the integration, etc. Yet, even when the trajectories do not track, the mapping ensures that only those pitch events occurring in the original sequence will appear in the variation, thus preserving a link between each variation and its source. This linking aspect of the mapping always occurs. The sensitivity of neighboring chaotic trajectories to initial conditions ensures that variability will occur, while the linking and tracking aspects of the chaotic mapping moderate the degree of variation.

The chaotic mapping may implement tracking in x between the reference and new trajectories, resulting in a new pitch agreeing with the original pitch, when $x_i - x'_i$ is greater than zero but sufficiently small. Another case results when $x'_i = x_i$. Then $x_i - x'_i = 0$, and the new pitch must agree with the original pitch (unless an x_i occurred more than once, in which case, the last pitch assigned to the repeated x_i is chosen). Therefore, for $x_i - x'_i \geq 0$, the chaotic mapping can help temper the built-in variability resulting from the sensitive dependence property.

On the other hand, the mapping — in tandem with the sensitivity of chaotic trajectories to initial conditions — is capable of generating a pitch different from the original pitch. Whenever $x_i - x'_i < 0$, the variation may not track the source.

A way to examine whether the new and reference trajectories may track in x is to plot the difference in x-values, i.e., $x_i - x'_i$, for the duration of the piece. By noting the number of positive, negative and zero excursions, as well as their magnitudes, one can see if the chaotic mapping has potential for enabling the reference and new trajectories to track in x. This is discussed again in Chapter 6.

The ability of the mapping to link the variation with the source is maintained regardless of whether the reference and new trajectories have a transient (such as orbits with IC's close to (1, 1, 1)) or whether the trajectories are (approximately) on the strange attractor. The tracking mechanism of the mapping may come into play, whether or not the chaotic trajectories are transient or on the attractor.

Finally, any composition may contain pitches simultaneously struck together to form chords. All or part of a chord can be associated with one or more x_i. But in this thesis, each chord is considered an indissoluble musical event occurring at a specific i in the sequence of N events. Therefore, every pitch or chord event is paired, in sequential order, with its corresponding x_i. Those chords appearing in the variation assume the dynamics (i.e., the loudness levels of the component notes) that each possessed in the original. If a single note appears in the variation, substituting for a chord, it adopts the dynamic level of the lowest note in the replaced chord. However, if any dynamic level occurs in the variation that is not desirable, the musician can change it. Otherwise, the tempo, rhythm, and dynamic levels heard in the variation are the same as the original.

2.1 References

[1] E. N. Lorenz, "Deterministic non-periodic flow," J. Atmos. Sci. **20**, 130-141 (1963).

Chapter 3

Results and Analysis

Chapter 3 uses musical analysis to evaluate several variations generated by the chaotic mapping. More analytical discussion follows in an Appendix at the end of the chapter. As remarked earlier, those musical terms marked by an asterisk are explained in the Glossary of Musical Terms affixed to the end of the thesis.

3.1 Variations on a Prelude by J. S. Bach

To demonstrate the results and determine whether they make musical sense, consider J. S. Bach's Prelude in *C* Major from *The Well-tempered Clavier, Book I* (WTC I) as the original work on which three variations are built.[1] A strong harmonic progression*, analogous to an arpeggioed 5-part Chorale, underlies the Bach Prelude, (Figure 3-1).

3.1.1 Variations 1 and 2, built on the first two phrases of the Bach

Variation 1 (Figure 3-2*a*) introduces extra melodic elements: the *D*4 appoggiatura*[2] on beat 3 of measure (*m.*) 1; the departure from triadic arpeggios within the first two measures; the introduction of a contrapuntal bass line (*A*2, *B*2, *C*3, *E*3) on the off-beat of *m.* 5; and the dominant seventh tone on *F*4 heard in *m.* 7. The above devices were familiar to composers of Bach's time, though they might not have used these melodic elements in quite the same way.[3]

[1]For interpreting the musical scores that demonstrate the results, non-musicians might imagine the horizontal lines and intervening spaces on the scores as a kind of graph paper. The notes could be considered points on the graph. Then, pattern-matching could alert the non-musician to changes between the variation and the original piece.

[2]The *D*4 in the soprano voice of *m.* 1 is prolonged, all the while creating tension, until its relaxation or resolution on *E*4. Though the prolongation is not literally written out, the *D*4 — clearly distinct from the lower voices (*E*3, *G*3, *E*3) — is heard as an accented unresolved dissonance until beat 4, when it resolves upwards by step.

[3]See Appendix for a discussion and further analysis of Variation 1.

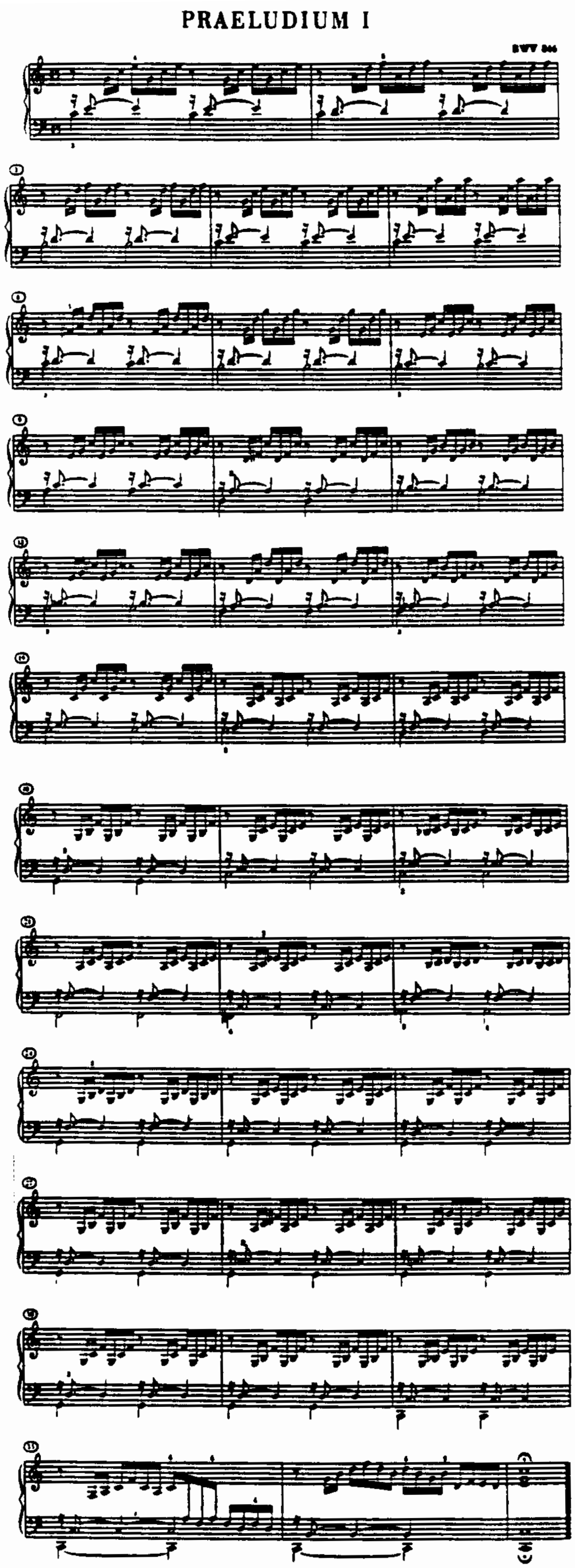


Figure 3-1: The pitch sequence of the original Bach Prelude.

Figure 3-2: The pitch sequences of Variation 1 and Variation 2 (all note durations omitted). The two variations are built upon the first 11 measures of the 35-measure Bach Prelude. The Runge-Kutta solutions for both trajectories encircle the attractor's left lobe 8 times and the right lobe 3 times. The simulations advance 1000 time steps with $h = .01$. They are sampled every 5 points ($5 = [1000/176]$, where $[\cdot]$ denotes integer truncation and $176 = N$). All computations are double precision; the x-values are then rounded to two decimal places before the mapping is applied. Though the differences between graphs of neighboring orbits may not be detectable to the eye, they are to the ear. *Top*, Variation 1, built from chaotic trajectories with new IC (.999, 1, 1) and reference IC (1, 1, 1). The chaotic mapping enabled the reference and new trajectories to track in x for 145 out of 176 x-values, resulting in 145 pitches of the variation occurring exactly where they did in the original. *Bottom*, Variation 2, built from chaotic trajectories with new IC (1.01, 1, 1) and reference IC (1, 1, 1). The chaotic trajectories were able to track in x for 98 of 176 x-values, so that 98 pitches in the variation replicate, in order, those of the original.

Variation 2 (Figure 3-2b) evokes the Prelude, but with some striking digressions: for instance, its key is obscured for the first half of the opening measure. Compared to Variation 1, Variation 2 departs further from the Bach. This is to be expected: The IC that produced Variation 2 is farther from the reference IC than the IC that produced Variation 1.

Like Variation 1, Variation 2 introduces musical elements not present in the source piece, e.g., the melodic turn*[4] ($F4$, ($G3$), $E4$, $F4$, $G4$, ($A3$), $F4$) heard through beats three and four of m. 3. In each of its measures, Variation 2 breaks the pattern of the Prelude — where the second half of each measure repeats the first half — by introducing melodic figuration and superimposed voices. For instance, note the bass motif of m. 6-8 ($B2$, $B2$, $C3$, $A2$, $D3$, $C3$, $B2$) and the soprano motif of m. 9-11 ($D4$, $A4$, $C4$, $D4$, $A4$, $G4$, $A4$, $B3$, $E4$, $B3$, $D4$). In the figure, each is indicated by double stems, i.e., two stems that rise (fall) from the note head.

3.1.2 Variation 3, built on the entire Bach Prelude

The original 35-measure Bach Prelude exhibits three prevailing time scales. The slowest is marked by the whole-note because the harmony changes only once per measure. Note that when the pitch sequence changes, the times at which the harmony changes is altered. The fastest time scale is given by the sixteenth-note which arpeggios or "samples" the harmony of the slowest time scale. The half-note time scale represents how often the bass is heard, i.e., the bass enters every half-note until the last three bars (m. 33-35), when it occurs on the downbeat only. Variation 3 (Figure 3-3) alters all three time scales to a greater extent than the previous variations.

The half-note time scale is first disturbed in m. 3, where the bass enters successively on the weakest parts of the sixteenth note groups, rather than on the much stronger first and third beats of each measure in the original. The whole-note time scale is broken, among other places, in mm. 28-29, which have the harmonic progression I_6 $VII^4_{\flat 3}$ II_6 VII^6_4/II. Each measure possesses two different harmonic chords, rather than the original's one harmony per measure, i.e., the harmonic rhythm* is in half-notes rather than whole notes. The fastest time scale is disrupted by melodic lines emerging from the sixteenth-note motion. They interfere with the sixteenth-note time scale because, as melodies, they possess a rhythm (or time scale) of their own. An example is indicated by slurs in mm. 4-6.

Of course, things do not always go perfectly when making these variations. For example, Variation 3 indicates what can occur if an x'_j exists for which there is no $x_i \geq x'_j$. Specifically, x'_{342} through x'_{350} of Variation 3 (m. 22) exceeded all $\{x_i\}$, resulting in no pitch assignment for these x-values. This is not a problem. When such an instance occurs, pitches can be inserted by hand to preserve musical continuity, or the pitches of the original piece can be substituted.

The last pitch event of the Bach Prelude is a 5-note C major chord, at $N = 545$. All or part of this chord could be associated with x_N. As stated in Chapter 2, any musical work that contains pitches simultaneously struck together, can generate vari-

[4]See Appendix for a discussion and more analysis of Variation 2.

Figure 3-3: The pitch sequence of Variation 3, with durations omitted. The mapping was applied to all $N = 545$ pitch and chord events of the Prelude, with trajectories having reference IC (1, 1, 1) and new IC (1, .9999, .9). The Runge-Kutta solutions for both trajectories encircle the attractor's right lobe once. The simulations advance 545 time steps with $h = .001$, and are sampled every step. All computations are double-precision, with x-values rounded to six decimal places before the mapping is applied. The chaotic mapping fostered tracking in x, for 41 out of 545 x-values, resulting in 41 pitch events appearing precisely in the variation where they occurred in the original.

ations via a pairing that associates any or all of the chord with one or more x_i. However, for all variations generated and discussed in this thesis, each chord is considered an unbroken musical event occurring at a specific i in the sequence of N events. Each pitch or chord event is paired, in sequential order, with its corresponding x_i.

3.2 Variations on Additional Musical Compositions

The design that implements the variation technique has been applied to other works by Bach, Beethoven, Chopin, Gershwin and Bartok. The point of doing so was to show that one design could accommodate a number of pieces spanning the major styles of Western music from 1700 into the twentieth century. In a series of concert/lectures given in Hong Kong, Chicago, New York and Boston, it became clear that the musicians in the audiences never agreed on which of these variations were most musical. Two concert pianists who specialize in Bach said the Bach variations were their favorites. Yet a principal percussionist of the Boston Symphony Orchestra disliked the Bach variations, and advised that the Gershwin variation should serve as the best example of this technique, since "it outdid the original." Other professional performers and composers chose variations on a Chopin ètude (*f* minor, Op. 10) and the first movement of a Beethoven sonata (*F* major, Op. 10) as most relevant to their musical view. Yet to some avid music lovers with no professional training, these variations were least engaging.

3.3 Appendix

The appoggiatura of *m.* 1 in Variation 1 resolves upward by whole step. This is not how Bach would typically have treated an appoggiatura. In his music, most appoggiaturas resolve downward by step or upward by half step. Still, there are instances where he employs an upward resolution by whole step, e.g., the *g*♯ minor Prelude, WTC II, *mm.* 2, 4, 17, 31 and 42 — but they are rare. Such an unusual departure from customary practice creates a need for further confirmation, which is absent in Variation 1. In other words, a good composer writing in the Baroque style would not merely state, and then abandon, such an appoggiatura. Rather, the unusual treatment of the appoggiatura and its resolution would be emphasized elsewhere in the piece, as in fact happens in the *g*♯ minor Prelude. Yet even there, though Bach resolves the appoggiatura upward by whole step in *m.* 2, the resolution occurs on the seventh tone of the V^6_5 chord, which itself wants to go down — and does so very shortly.

With respect to harmonic progression, Variation 1 follows the original Prelude quite closely. For example, *m.* 2 and *m.* 4 can be analyzed as II^4_2 and I, respectively, just as in the original. However, the harmony in *m.* 2 is colored by the $E4$, an unaccented lower neighbor note* with a delayed resolution to the $F4$ of beat 2. Similarly, the passing tone* on $D3$ in *m.* 4 (passing from $E3$ of beat 1 to the $C3$ of *m.* 5) shades *m.* 4 differently from the original.

The turn in *m.* 3 of Variation 2 is introduced after the initial sounding of the $F4$ in the third beat, and is considered an unaccented inverted turn involving the notes $E4$, $F4$, $G4$, $F4$. However, this turn is postponed and interrupted. It is postponed by $G3$ which is part of the dominant seventh chord. The $G3$ is prolonged until the interruption by $A3$, which acts as a neighbor note* to $G3$, returning to it in *m.* 4. Because the postponement and interruption occur in a lower voice, the ear is able to hear the effect of the turn in the upper voice.

The harmonic progression of Variation 2 retains the basic harmony of the Bach Prelude, while diverging from it in ways that make the Variation sound as if written considerably later than the original score. For instance, the first half of *m.* 1 can be interpreted in the tonic by analyzing the $B2$ on the downbeat as an accented lower neighbor (or appoggiatura) to the $C3$ on beat 3, the $F\sharp3$ as a lower neighbor to $G3$, and the $D3$'s as lower neighbors to $E3$. The first 6 sixteenths of *m.* 1 could also be interpreted as V_6 with the $F\sharp3$ tonicizing G and the $E3$ an appoggiatura (or accented upper neighbor) to $D3$. In contrast to this, the second half of the measure clearly establishes C major.

Variation 2 alters the harmonic progression of the original score in *mm.* 4-5 (where the Variation introduces the VI chord[5] a half measure early, prolonging this harmony in *m.* 5), *m.* 7 (where, in beat 4, the addition of the seventh tone creates V^6_5, a departure from the V_6 of the original), *m.* 11 (where the dominant chord is heard, not on the downbeat, but on the third through seventh sixteenths, followed by VII_7/V — the $G3$ of the third beat functions as an accented neighbor note to $F\sharp$ — with a return to the dominant on the fourth beat). The high A in *m.* 11 can be heard as an appoggiatura to an implied G, especially if the G is supplied in *m.* 12.

[5]The $D4$ in beat 3 of *m.* 4 is interpreted as a prolonged appoggiatura, resolving to $E4$ on the last sixteenth of the measure. The $G3$ is analyzed as a lower neighbor to $A3$.

Chapter 4

Variations as Idea Generators

To demonstrate how a composer might use the variation technique as an idea generator, two examples are presented. For each, the original piece is given, followed by a variation generated by the technique, and finally a variation which a composer might construct based on the raw variation produced by the technique. The works chosen for these examples include Bach's *Prelude in C, WTC I* and the first Prelude from Gershwin's *Three Preludes for Piano*. The variation technique applied to a contemporary score is discussed in Chapter 5.

4.1 Motivation

In the privacy of a musician's studio, the variation technique can serve as an idea generator. Such a musical tool is not unlike others of the past. Bach, and composers since, have often taken the inversion, retrograde or retrograde inversion of a motive, theme or section, in order to extend their original musical material. There is a difference, however, between taking the inversion, retrograde or retrograde-inversion of an original piece, or part thereof, and applying the variation technique to the same material: While there exists only one inversion, one retrograde and one retrograde-inversion (not counting transpositions) of the original, there are virtually unlimited variations possible via the variation technique. The musician cannot say with absolute confidence that the variation technique yields no acceptable variation because perhaps the right initial conditions, step size, number of time steps, or truncation has not yet been tried. The vast number of possible variations available to the composer will comfort some and disquiet others. However, the more one interacts with the technique, and the more understanding one has of how it works, the greater the musician's insight, intuition, and ability to predict.

4.2 A Bach Variation as an Idea Generator

As stated in the Introduction to the Thesis, variations produced by the chaotic mapping often suggest musical material that can be further developed by a composer.

Recall the original Bach Prelude in C (WTC I) of Figure 3-1 and Variation 3 (Figure 3-3). The latter is now reproduced in Figures 4-1.

Five musical ideas are introduced by Variation 3:

1. The "advance" of the bass. By often appearing a sixteenth note early, on the weakest beats of the measure, the bass acquires an upbeat quality. Four examples of this are apparent in *mm.* 1-2.

2. Superimposed lines or motives, e.g., the bass melody of *mm.* 4-6 mentioned previously in Chapter 3.

3. Repeated notes. Pairs of repeated notes are heard throughout the variation, starting with the $G3$'s and $G2$'s in *mm.* 11-12.

4. Harmonic sequences*. Measures 12-15 vary the half sequence* of the original and imply the following harmonic sequence: VII_2^4/II – II^6 – $VII_{\flat 2}^4$ – I_6. But the $E\flat 2$ – G – $E\flat 2$ in the bass of $m.$ 14 and the $F\sharp 3$, $A2$, and $F\natural$ of $m.$ 15 are extraneous to the harmonic progression. Measures 28-29 also suggest a harmonic sequence with the progression I_6 – $VII_{\flat 3}^4$ – II_6 – VII_3^4/II, provided a $B\flat$ is added in the second half of $m.$ 29. This sequence is the retrograde of the harmony in *mm.* 12-15. The retrograde occurred in the raw variation because the new trajectory returned to those regions of the x-axis which harbored the original Prelude's half sequence of *mm.* 12-15 (but from the opposite direction), triggering pitches on the x-axis in reverse order to the earlier sequence.

5. The BACH motif. A transposition (C, B, D, $C\sharp$) of the notes $B\flat$, A, C, $B\natural$ — the musical spelling of Bach's name — appear in the soprano voice of *mm.* 28-29 (Figure 4-1). The retrograde of the BACH motif occurs in *mm.* 12-15 of both Variation 3 and the original Prelude.

The five ideas presented by Variation 3 invite development, but not all of the alterations from the original that comprise the variation are desirable. The musician can intervene by re-writing any part of the variation. The composer especially wants to take the musical suggestions posed by the variation technique and follow through on them. They have consequences for the rest of the piece. The good ideas suggest elaboration, and it is here that the composer's art comes into focus.

Figure 4-2 displays a composed variation written by the author, which is based on the raw material of Variation 3. (From now on, I will distinguish those variations generated by the chaotic mapping as **raw variations**, and the modified versions that I created as **composed variations**.) The composed variation retains the five "ideas" introduced by Variation 3 and develops them further. On the other hand, some notes of the raw variation are completely left out, replaced by others considered more fitting to the context at hand. For instance, the $D2$ of $m.$ 4 of the raw variation (Figure 4-1) is supplanted by $C2$ in the composed variation.

What follows now is an analysis of how the composed variation takes what are considered the best ideas of the raw variation and develops them. Idea 1, the advance

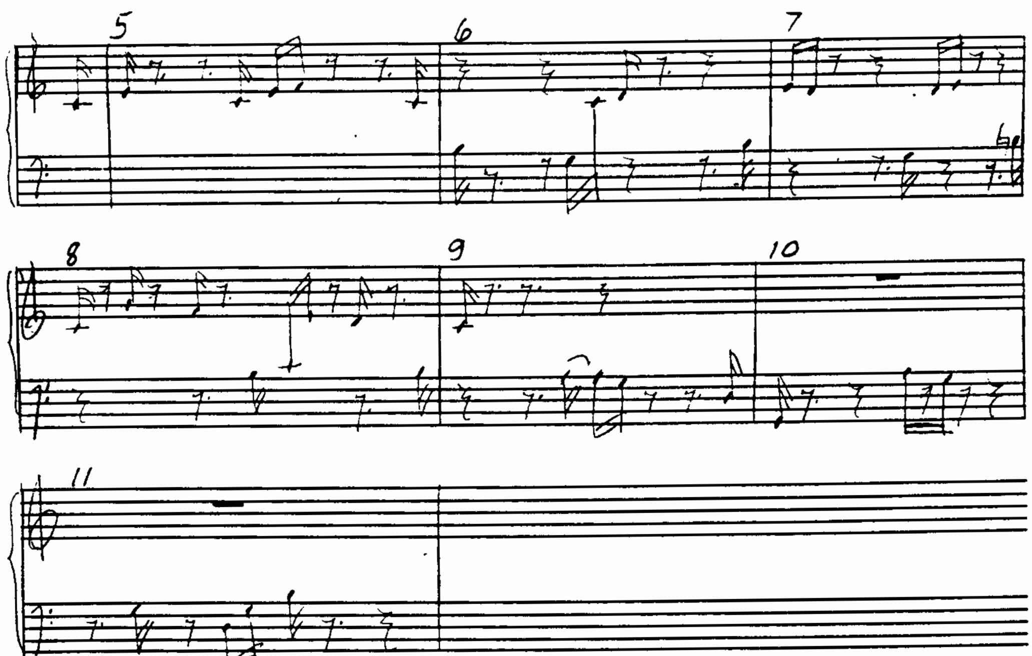
Figure 4-1: Variation 3 of the Bach Prelude, reproduced from Figure 3-3.

Figure 4-2: The composed variation of the Bach Prelude based on the raw variation of Figure 4-1.

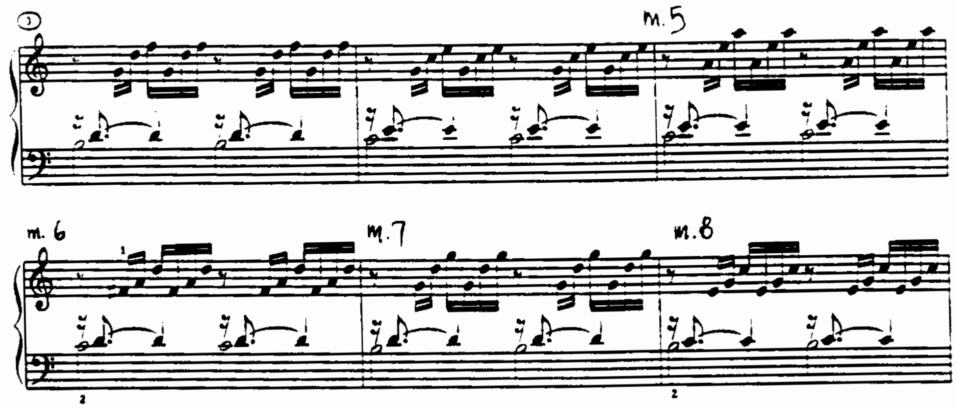
of the bass, first evident in m. 1, continues through m. 22 of the composed variation. It reappears twice more in the concluding measures of the composed variation — the last sixteenth ($D2$) of m. 33 leading to the downbeat of m. 34 and the final sixteenth ($C1$) of m. 34 to the ending chord of m. 35.

In both the raw and composed variations, the advance unfolds into a faster moving counter melody to the slower moving soprano melody, thus introducing Idea 2. But whereas this counter melody only lasts two measures in the raw variation,

it is extended to 7 measures in the composed variation, leading smoothly into the second phrase:



Another possibility for *mm.* 5-8 of the composed variation merges the sequence[1] of *mm.* 5-8 (in the original Bach),



with the countermelody suggested by the raw variation, to build a new sequence in *mm.* 5-8 of the composed variation:

Note that *mm.* 9-10 of the composed variation (Figure 4-2) do not sound congruous with the new sequence of *mm.* 5-8 given above. Consequenctly, *mm.* 9-10 were rewritten in the above example. Measure 11 duplicates the same bar of the composed variation (Figure 4-2) and is given for reference.

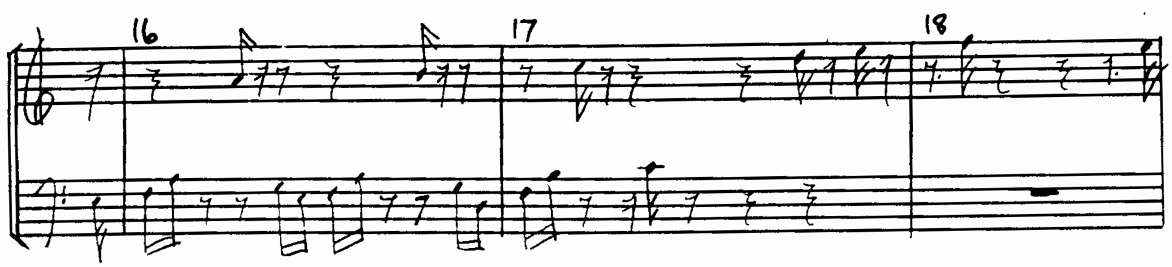
The inferred sequence of *mm.* 12-15 in Variation 3 (Figure 4-1) can be developed so that its two phrases are each set up with an upbeat in the bass, thus con-

[1]The term sequence has been designated to represent harmonic sequences.

tributing to their parallel structure:

The $Eb2 - G1 - Eb2$ and $F\sharp 3 - A2 - F3$, extraneous to the harmony in *mm.* 14-15 of Variation 3. are eliminated. Instead, the bass of *m.* 14 is a stepwise transposition of that found in *m.* 12, and the bass of *m.* 15 references the bass of *m.* 13 without transposing it exactly. In order to build what is almost a true half sequence, the second phrase systematically transposes the harmonic, melodic and rhythmic patterns of the first phrase until the second beat of *m.* 15 when the melodic pattern is disrupted in order to intensify the repeated note pair (Idea 3).

Idea 2 reappears in *mm.* 16-18, showing how the advance unfolds into another counter melody in the bass, this time starting with $E2$–$F2$–$A2$ on the upbeat to *m.* 16:



Note it has a similar intervallic structure to the earlier counter melody of *mm.* 5-10, enabling the listener to relate it to what was heard previously.

A second motif runs concurrently in the soprano. It outlines by step the sixth from $A3$ to $F4$. The sixth encompasses the upper three voices in 15 of the 35 measures comprising the original Bach.

Idea 3, the repeated note figure, first introduced in *m.* 11 of the raw variation, is now deferred to *m.* 12 of the composed variation so that it may characterize the sequence of *mm.* 12-15, where it is preserved in both halves of the sequence structure and expanded in *m.* 15 to give a local climax on $G4$. (The global climax of the composed variation occurs on the repeated $A4$'s of *m.* 31.) The repeated pair makes two more parallel statements — as a kind of denouement to the local "climax" of the second phrase. These occur in *m.* 17 ($C3$–$F3$–$F3$) and *m.* 19 ($C3$–$E3$–$E3$).

The "advance" bass idea bridges *m.* 19 to *m.* 20, thus spinning out the third (and final) phrase on the tail-end of the second. Two related counter melodies ensue

Figure 4-3: *Left*, The harmonic progression underlying the harmonic sequence in *mm.* 28-29 (composed variation). Note the BACH motif in the soprano which is transposed to C–B–D–$C\sharp$. *Right*, The harmonic sequence found in *mm.* 12-15 (composed variation) is the retrograde of the harmonic sequence in *mm.* 28-29 given above. The soprano gives the retrograde of the BACH motif (transposed).

which look back to the prior melodies. The first occurs in *mm.* 20-21,

while the second takes place in *mm.* 24-25,

Note that each of these melodies, though not containing as many different intervals as the earlier counter melodies of *mm.* 5-10 and *mm.* 16-18, share the same penchant for thirds and seconds. All four of these faster moving counter melodies are aurally reminiscent of each other due to their similar intervallic structure. They are also rhythmically alike in that the melodies mainly unfold in groups of one, two and three sixteenth notes.

The third phrase presents a sequence in *mm.* 28-29, the harmonic basis of which is given in Figure 4-3. Also shown in Figure 4-3 is the harmonic progression underlying the sequence of *mm.* 12-15, which commences the second phrase of the variation. The harmonic progression characterizing the sequence of *mm.* 28-29 is the retrograde diminution of the progression of *mm.* 12-15. That is, the harmony of the former — I_6 $VII^4_{\flat 3}$ II_6 VII^4_3/II — is the reverse of the latter and spans half as many measures. An exact sequence can be written for *mm.* 28-29 and substituted in both the raw and composed variations:

Recall that pairs of repeated notes helped build the musical character of the retrograde sequence given in *mm.* 12-15 of the composed variation. In the raw variation, though, the pairs of repeated notes sounded in *mm.* 11, 12 and 15 are not musically linked. They suggest a possible musical motive, but the idea remains unformed. By contrast, the repeated note pairs in *mm.* 29-31 (raw variation) are musically connected in their intensification towards the global climax of the variation. Both *mm.* 12-15 and *mm.* 28-31 contain sequences and a stretto-like treatment of the repeated note pair, leading to a local climax (*m.* 15) and a global climax (*m.* 31).

Idea 5 — the appearance of the BACH motif — occurs in the soprano of the harmonic sequence of *mm.* 28-29 (both raw and composed variations). It is transposed to *C–B–D–C*♯. The retrograde of this occurs in the soprano line of *mm.* 12-15 (both raw and composed variations).

Measures 32-33 of the composed variation make one last reference to the superimposed melodies heard in *mm.* 5-11, 16-18, 20-21, 24-26, with a brief motif — *D*4, *C*4, *B*♭3, *G*3, *F*3 — favoring the same intervals and rhythms heard in the earlier examples.

All of the musical examples so far discussed in this chapter — the original Bach Prelude (Figure 4-1*a*), a raw variation of it (Figure 4-1*b*), and the composed variation (Figure 4-2) — raise a commanding issue: that of *style.* The more divergent the variation technique, the more strain is placed on the delicate balance between harmony and counterpoint in Western tonal practice, vertical and horizontal structure in contemporary practice, foreground and background in Shenkerian analysis*, etc, in Bach's (or any other Western) idiom. However, these strains could just as easily arise if older "variation" techniques such as the inversion, retrograde, or retrograde inversion are applied to a melody, section, or entire piece.

The question arises: Which guidelines replace the old rules. Certainly, the composed variation of Figure 4-2 would not literally have to comply with Bach's practice. However, a composer writing a variation suggested by this technique must always listen for the more far-reaching implications of the raw variation, and decide whether — at least intuitively — the resulting language is self-consistent. For example, four notes are altered in *mm.* 28-31 so that a structural reduction (i.e., a middle ground Shenkerian analysis) of the soprano in the composed variation gives the line *C*, *B*, *D*, *C*♯, *E*, *F*♯, *G*, *A* rather than the meandering *C*, *B*, *D*, *C*♯, *C*♮, *D*, *G*, *A*. The advantage of the former is a greater thrust up to the high *A*, which is the climax of the composed variation. Similarly, the tenor line in *mm.* 8-10 of the raw variation (Figure 4-1*b*) — *B*2, *A*2, *G*2 — can be given more direction by the line — *B*2, *B*♭2, *A*2,

$G2$, $F2$, $E2$, $C2$, $D2$ — found in the composed variation (Figure 4-2).

4.3 A Gershwin Variation as an Idea Generator

George Gershwin wrote a set of three preludes for piano, published in 1927. Figure 4-4 gives the First Prelude. A variation of the entire Prelude is shown in Figure 4-5. As with the Bach variations, every chord and note in the original Gershwin is treated as a separate pitch event[2] and piggybacked onto the reference trajectory; rhythm (duration), dynamic levels and tempo are treated as discussed in Section 3.1.2, last paragraph.

Several ideas emerge from the raw variation, five of which are heard within the first 8 bars:

1. The descending fourth on the 2nd beat of *m.* 1.
2. The descending octave, followed by the ascending diminished fifth, of *m.* 2.
3. Alteration of the four-measure vamp* so that the original *m.* 3 does not repeat itself four times.
4. The melodic answer of *m.* 8 to *m.* 7.
5. The introduction of extra blue notes*, e.g., the $E\natural$ (enharmonic equivalent of $\flat 5$) in beat 2 of *m.* 8.
6. Octave lead-ins, acting as passing tones, to the downbeat of the next measure.

As with the Bach variations, a musician could take these six ideas and develop them further. For example, most of the blue notes appearing in the original also surface in the raw variation. But the raw variation adds extra blue notes in *m.* 8 ($E2$), *m.* 13 ($E4$), *m.* 30 ($A\flat 4$), *m.* 42 ($D\flat 3$) and *m.* 55 ($D\flat 3$). A musician might retain some of the blue notes from the raw variation, while contributing others, e.g., *m.* 12 ($A\flat 2$), *m.* 25 ($D\flat 3$), *m.* 33 ($D\flat 3$), *m.* 41 ($G\flat 2$), *m.* 45 ($C\flat 3$–$E2$–$A\flat 3$), *m.* 55 ($D\flat 4$), *m.* 62 ($D\flat 4$ and $A\flat 3$), in the composed variation (Figure 4-6).

A composer might take three of the six ideas given above and include them in the fortissimo return of the Prelude's main theme (*mm.* 50-53). Idea 2 — the descending octave followed by the ascending diminished fifth — is recalled in *m.* 51, with the diminished fifth interrupted by an octave blue note (Idea 5) on $E\natural$. The "answer" of *m.* 8 to the statement of *m.* 7 (Idea 4) is referenced in *m.* 53. (See composed variation, Figure 4-6.)

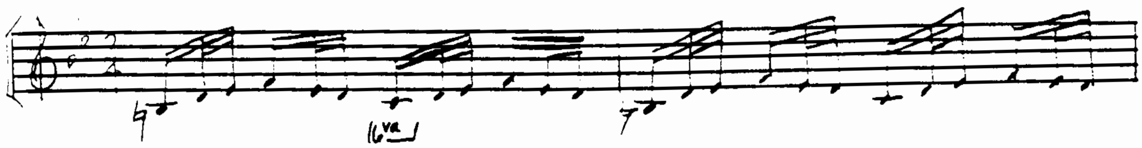
[2]The grace note on $E2$, found in *m.* 11 and *m.* 12, is treated as a simultaneous pitch event with the following $F2$.

Figure 4-4: Gershwin's First Prelude from *Three Preludes for Piano.*

Figure 4-5: A raw variation of Gershwin's First Prelude. The mapping was applied to all $N = 435$ pitch and chord events of the Prelude, with trajectories having reference IC (1, 1, 1) and new IC (1.0005, 1, 1). The methods are the same as in Figure 3-2, except that the simulations are sampled every 2 points ($2 = [1000/435]$, where $[\cdot]$ denotes integer truncation.) The chaotic trajectories tracked in x for 361 of 435 events, resulting in parts of the original pitch event sequence appearing unchanged in the variation.

Figure 4-6: A composed variation of the Gershwin Prelude based on the raw variation of Figure 4-5.

Melodically, the diminished fifth (included in Idea 2) was used to re-work quite a bit of the raw variation in order to make the composed variation. Measure 12 (composed variation) uses the descending diminished fifth twice to break the steadily descending scalar pattern adhered to by the original Prelude in *mm.* 11-13. The diminished fifth outlines the melodic structure for the right hand in *m.* 38-39 (composed variation):



In the parallel passage occurring earlier, the augmented fourth (enharmonic equivalent of the diminished fifth) interval outlines the "bounds" of the right hand passage in *m.* 33. The diminished fifth also outlines the last three notes of the left hand in *m.* 36 of the original prelude. A parallel measure to this is *m.* 41, where the *A*♭2 of the raw variation was changed to *G*♭2 in the composed variation. On the other hand, the melodic diminished fifth of *m.* 42 (*A*3–*D*♭3, beat 2, raw variation) was kept intact for the composed variation because it is consistent with previous treatment. Measure 54 of the composed variation preserves the essential descending scalar motion of the original (as did its parallel measure — *m.* 11), but here (as with *m.* 12), the pattern is disrupted, by making use of the diminished fifth, and other intervals, motivically.

The raw variation contains an instance of an octave passing tone in the bass (*m.* 19) on *G* which moves between *F* and the *A*♭ occurring on the downbeat of *m.* 20. This octave passing tone was kept in the composed variation. There are several places where such passing tones could have been inserted in the composed variation, but the most effective place musically, was *m.* 24, where the octave *A*'s serve as passing tones between the octave *G*'s and the *B*♭ appearing on the downbeat of *m.* 25. The *F*–*G*–*A*♭ move of *m.* 19-20 is linked to the *G*–*A*–*B*♭ motion of *mm.* 24-5, by interval, rhythm and function.

4.4 Summary

The variation technique can serve as an idea generator. This was shown with works by Bach and Gershwin. It will also be applied to a contemporary work in the next chapter. To use the variation technique as an idea generator, the musician first composes an original piece. Then, the variation technique takes that original piece as input and generates a raw variation. Next, identifying the best elements of the raw variation, the artist develops them into a composed variation of the original score.

Chapter 5

Towards a Dynamic Music

5.1 Variations to a Theme (1995, DS Dabby), a Contemporary Work

Variations to a Theme is, as its name implies, a set of variations which lead to a Theme — in this case, the Bach Prelude in C discussed earlier. Scored for piano, the piece begins out of nowhere, with only subtle references to the Bach. Gradually it telescopes to the Theme. The work is twelve minutes long, and during that time, various references are made to the Bach in its inversion, retrograde and retrograde inversion (IRRI). These are changed so that they do not occur in their original IRRI form. Every measure of the Bach appears in the piece, in at least one of its IRRI forms.

To convey some idea of the piece, excerpts from the score are given in Figures 5-1 and 5-2. Figure 5-1 shows the first variation of the piece and the opening measures of the second. Variation 1 (*mm.* 1-17) begins sparsely and emphasizes some of the intervallic structure of the Bach. For example, the descending sixth ($E6$–$G5$) of *m.* 1 refers to the same sixth that encompasses the right hand (upper voices) of *m.* 1 (and elsewhere) in the Bach Prelude. Variation 2 arises out of its predecessor and treats the left hand as a drum. By the penultimate variation, the "c minor" variation, the piece has verged much closer to the Theme. This can be heard in Figure 5-2.

Finally, the Cadenza (*mm.* 279-314), Chorale (*mm.* 315-321) and Prelude (starting *m.* 322) are given in Figure 5-3. The opening of the Cadenza (*mm.* 279-287) is loosely based on *mm.* 3-7 of the retrograde inversion of the Bach. Chordal figures and motifs from earlier sections of the score are alluded to throughout the Cadenza. For instance, *mm.* 301-302 recall the drum pattern of the left hand discussed earlier, while *mm.* 308-314 call forth the opening measures of the piece. The Chorale heard in *mm.* 315-321 emphasizes the key of F major, the subdominant of C major (the tonic of the Bach Prelude). In his works, Bach sometimes made a point of approaching the concluding key of a piece, by way of the subdominant. Here, the Prelude (the Theme) is prepared by a variation which starts in F and concludes with a high pedal point on the octave G's of *mm.* 320-21.

Figure 5-1: Variation 1 (*mm.* 1-17) of the continuous set of variations entitled *Variations to a Theme* (1995), followed by the first eight measures of Variation 2, which starts in *m.* 18.

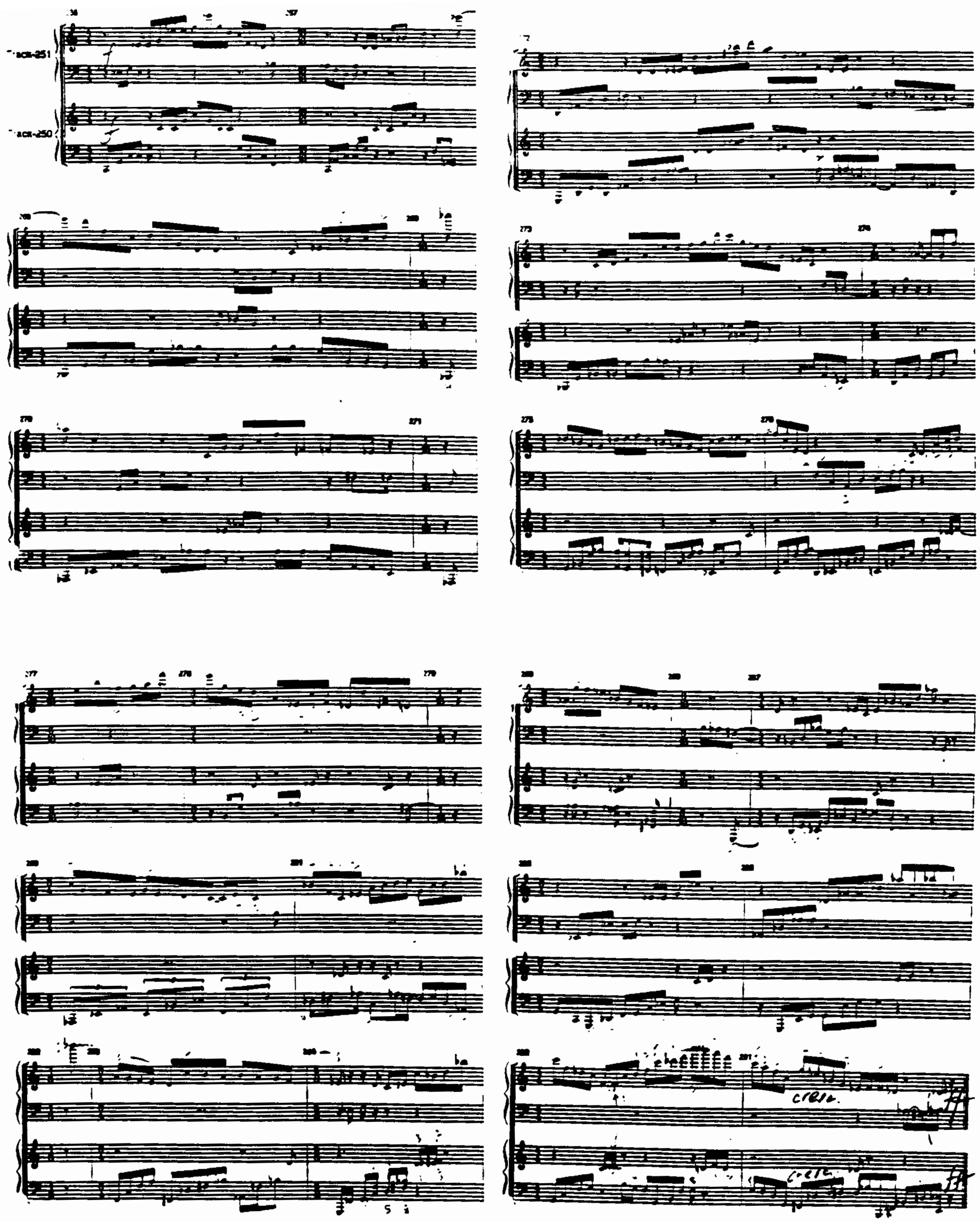


Figure 5-2: The penultimate variation of *Variations to a Theme* where the harmony of the Bach Prelude is elaborated freely in both minor and major. The individual staves for the right and left hands clue the performer that the independent character of the lines should be heard.

Figure 5-3: The Cadenza (*mm.* 279-314), Chorale (*mm.* 315-321), and Prelude (Theme) of *Variations to a Theme.* Only the first seven measures of the Theme (also a variation) are given, starting in *m.* 322, because the Theme presented here is reproduced in its entirety by Figure 4-1, the raw variation of the Bach Prelude. The raw variation of Figure 5-4 was generated from the above score, which includes the complete variation of Figure 4-1.

5.2 The Variation Technique as Idea Generator for the ending Theme of *Variations to a Theme*

After *Variations to a Theme* was composed, the variation technique was applied to it. The originating vision for the technique was to use it in tandem with a piece that was considered finished — a piece that made a statement, and hopefully, a compelling one. A musician could then go on a journey with her/his work, in that the variation technique might take the artist elsewhere, to some place new or unimagined.

One such excursion with the technique led to the raw variation given in Figure 5-4. The raw variation was generated by applying the technique to the following source material: the Cadenza and Chorale of Figure 5-3, with the Bach variation given by Figure 4-1. Six key elements of the raw variation (Figure 5-4) all take place within its first 11 bars. They are the inspiration behind much of the composed variation given by Figure 5-5: and are noted as follows:

1. The triadic nature of the original Prelude is retained by *mm.* 316-319, with the fourth beat of *m.* 317 and all of *m.* 318 constituting an almost exact retrograde of the first seven beats found in *mm.* 316-317.

2. The triadic nature of *mm.* 316-319 unfolds into a series of triadic chords (*mm.* 319-320), all deriving from the Chorale.

3. The opening of *Variations to a Theme* is hinted at in *m.* 321, where the high $E5$ and low bass notes commingle.

4. The tug between $A\natural$, $A\flat$ and G heard in *mm.* 321-325 refers to the free trill of *mm.* 307-310 (Cadenza).

5. The octaves on B, F, G and $A\flat$ in *mm.* 324-25 reference the octaves on $A\flat$–G–F–B–$E\flat$ heard in *mm.* 308-309 (Cadenza).

6. An allusion to the left hand drum is found in *m.* 326.

The composed variation keeps the triadic nature of the raw variation (*mm.* 316-19), where, like the raw variation of Figure 4-1, omission of the $C4$ causes an "advance" of the bass, enabling the bass notes to move ahead by one sixteenth and turning them into upbeats. The broken triads of the first seven beats reverse themselves exactly, starting on the fourth beat of *m.* 317. The exact retrograde of the first seven beats (raw variation) is carried further in the composed variation. Whereas the exact retrograde is maintained for four beats in the raw variation and stops two and a half beats short of the "chorale" chords (*m.* 319), the composed variation pursues the exact reversal for six beats and ceases just three notes shy (i.e., 3/4 beat) of the chordal entrance. This reversal sets up the introduction of the chords heard in *mm.* 319-320. The "chorale" chords are altered from the raw variation to make them more pianistic, since the tempo is not slow. But the basic flavor of the raw variation *mm.* 319-320 is kept intact. The introduction of these chords, which readily call back the Chorale,

Figure 5-4: The first 11 measures of a raw variation, which resulted from applying the variation technique to the Cadenza, Chorale and Prelude of *Variations to a Theme*. After the variation was generated, the measures pertaining to the Cadenza and Chorale were eliminated, leaving the newly varied Prelude. The IC's for the new and reference trajectories were (.9995, 1, 1) and (1, 1, 1), respectively. The simulations advanced 857 time steps with $h = .001$, and were sampled every step. All computations were double-precision, with x-values rounded to six decimal places before the mapping is applied. Six ideas are heard within its first 11 measures which strongly influenced the composed variation of Figure 5-5.

Figure 5-5: A composed variation of the concluding Theme from *Variations to a Theme*, based on the raw variation of the same Theme given by Figure 5-4.

Figure 5-6: Three motives from *Variations to a Theme* which appear fleetingly in *mm.* 332-334 of the composed Theme shown in Figure 5-5. The first and second motives are variants of one another.

provided the trigger for further development in the composed variation. Specifically, Idea 2 suggests nesting parts of the piece heard previously, within the composed Bach variation of Figure 4-2. For instance, the chords added in *mm.* 332-334 (Figure 5-5) recall three motives from *Variations to a Theme*, which are given in Figure 5-6, while the chords of *m.* 340 look back to the earlier statement of *mm.* 319-320. One of the reasons the Bach Prelude was chosen as the basis for Variations to a Theme was because one hand could play it. This happens in *mm.* 336-339 of the composed variation: The left hand plays a variant of *mm.* 21-25 of the composed Bach variation (Figure 4-2), while the right hand continues a melody first begun with the $A\flat3$ of the third beat of *m.* 334. That melody is heard roughly midway through *Variations to a Theme* and is given in Figure 5-7.

The Coda to the composed Theme (*mm.* 350-357, Figure 5-5) also alludes to a recurring motive in *Variations to a Theme.* The motive is based on the inversion of the first six measures of the original Bach (Figure 5-8). The Coda outlines the motive in *mm.* 350-352. It begins to evaporate in *m.* 353-354, reaching out to the highest and lowest C's on the piano. The sparseness and span between the voices, first heard at the beginning of the entire work, is re-introduced, only this time punctuated by

Figure 5-7: The melody which occurs approximately midway through *Variations to a Theme* is the basis for the right hand part in *mm.* 334-339 of the composed Theme given by Figure 5-5.

Figure 5-8: The inversion of the original Bach Prelude (*mm.* 1-6) is the basis for the Coda of the composed Theme given in Figure 5-5.

the fortissimo ringing of the octave-fourth in *m.* 355. Another "ringing" octave, this time on *G*4–*G*5, is introduced earlier, in *m.* 321. Again, Idea 3 of the raw variation — the reference to the opening measures of the entire score (*mm.* 1-17, Figure 5-1) — is followed through in the composed variation, whereas it is only hinted at in the raw version (*m.* 321 with its high *E*5 and low tones). Measures 320-321 of the composed variation bring back the low note–high note contrast of the opening, as well as refer to *mm.* 1-9 of Figure 5-1. But *mm.* 320-321 do this in faster time and much more compactly than originally heard in the in *mm.* 1-17 of *Variations to a Theme.*

Idea 4 is found in *mm.* 321-23 of the composed variation. These measures retain the tension among *A*♮, *A*♭ and *G* found in the raw variation (*mm.* 321-23), which arose from the free trill — based on *A*♭, *G*, *A*♭, *A*♮ — of the Cadenza.

Idea 5, as presented by the raw variation, is unformed. But it is reminiscent of the octaves building the motif *A*♭4–*G*4–*F*4–*B*4–*E*♭4 heard in *mm.* 308-9 of the Cadenza. Measures 324-25 of the composed variation highlight the inversion of this motif, transposed to start on *B*4, giving *B*4–*C*5–*D*5–*A*♭4–*E*5.

Finally, Idea 6 appears in the composed variation (*m.* 326), recalling the drum-like left hand heard earlier. It is now introduced on an upbeat, rather than on beat 2 (where it occurs in the raw variation, Figure 5-4).

5.3 Variations for a Dynamic Music

When *Variations to a Theme* is performed in concert, the pianist chooses whether the piece will end with the original Bach Prelude (Figure 3-1), the composed variation of Figure 4-2, or the composed variation of Figure 5-5. The written score has become dynamic: I have provided several variations of the concluding Prelude (Theme), all based on variations generated by the chaotic mapping.

At first, one might think such a performance reminiscent of the Classic piano concerto, where the pianist decides on a cadenza for the performance. There exist significant differences, though. In the Classic concerto form, the performer may play a cadenza written by the composer or by somebody else. Cadenzas may also be extemporized by the performer. Regardless of which is chosen, the cadenza is regarded as an elaboration of motivic material within the piece. It could never be heard as the foundation for the score. But the structure of *Variations to a Theme* is constructed so that it telescopes to the Theme. The concluding Prelude could not be construed as anything less than fundamental, since it provides the structural key to the whole score.

What is never left to chance, when using the variation technique, is the issue of style — with its considerations of structure on both foreground and background levels, as well as the interplay between vertical and horizontal constructs (or harmony and counterpoint) within the musical score. It is this stylistic issue that comes alive when creating composed variations based on the material suggested by the variation technique, as it would in the creation of any musical composition. In a performance of a Classic concerto, though, it is possible a pianist might choose a more "romantic" cadenza, i.e., one written later than the prevailing style of the Classic era.

Yet, there is a way in which *Variations to a Theme* (with a varying Theme supplied by the composer) is similar to the concerto cadenza. A knowledgeable audience will always anticipate a free cadenza of a concerto, knowing there are several possibilities available, wondering which one the artist might choose for that particular performance. The possibility of hearing the score change, no matter how well a listener knows the piece, is exciting.

In *Variations to a Theme*, only the concluding Theme is varied. More generally, an entire composition can be varied, creating another version of it. The possibility exists for the work to change from one hearing to the next, from one concert to the next, and even within the same concert. The piece is still recognizable as the same piece from concert to concert, but changes have occurred — changes prescribed by the composer. In essence, the written score itself has become dynamic — it changes with time much in the same way a river changes from day to day, season to season and is still recognized as the Charles or Hudson.

In a broad sense, a musical work is a dynamic entity in its own right. It moves in time. Even within a single hearing, a piece is changing from one event to another. It is this movement in time that may account for part of music's ability to draw strong emotions from listeners. Most people are not moved to tears by a painting hanging on a wall. A painting is fixed in time, static. Certainly, a number of moving "paintings" — the movies — cause audiences to cry all the time. Is it because music progresses in time, changes with time — that it strikes such a resonance with us all? By nature, it is dynamic, as are we. Yet each musical performance is based on a written score — something that is static (at least in Western Classical music.)

However, unlike the fixed scores of the past, the variation technique offers a dynamic musical score. And like those same past works, the composer creates every note of the variations comprising that score — only now, from the ideas generated by the technique, in tandem with the artist's imagination. The contemporary classical composer can create works that live and breathe from one performance to the next — not in random ways but in a musical space built by the artist with something compelling to say.

The variation technique offers composer and listener a journey through a dynamic musical world. The musician shapes this world — rules it — and then returns home to present that journey to her or his musical public. But now, unlike past times, one piece may take many journeys with its creator, and still be recognizable from one hearing to the next, as that piece. Astute listeners will recognize the original piece in the variation, just as attentive audiences of all times recognize the theme throughout a set of sectional variations.

The audience is asked to listen actively and is invited into a partnership with the composer. For one of the joys in hearing a dynamic, changing work is to listen on several levels — to hear how the variation relates to past variations and to hear the current variation as an entity onto itself.

5.4 Summary

In this chapter, the variation technique was applied to a contemporary work — after it was composed — as both idea generator and a means for a dynamic music. One such application of the technique led to the concluding Theme given in Figure 5-5. This composed Theme was based on six ideas generated by the technique which appeared in a variation of the Cadenza, Chorale and Prelude which conclude *Variations to a Theme.* Nested within the composed Theme are a number of motifs which occurred previously in the original work. It was also influenced by Variation 3 of the Bach Prelude (Figure 3-3).

When *Variations to a Theme* is performed in concert, the pianist chooses how the piece will conclude — with the original Bach Prelude (Figure 3-1), a composed variation based on Variation 3 (Figure 3-3) of the Bach Prelude, or the composed Theme (Figure 5-5).

The variation technique extends the variation form itself, by enabling an entire score to be varied according to the musical attributes of an original work and the sensibility of its composer. Once variations of an entire piece are available, the composition can change from one hearing to the next. The score becomes dynamic. By letting chaotic trajectories whiz through a musical landscape set up (and subsequently developed) by the composer — a living, breathing dynamic music can cast a revelrous spell, forever changing.

Chapter 6

Remarks

6.1 Extension of the Chaotic Mapping to the Y, Z Axes

By extending the mapping to the y, z axes, variations can be generated that differ in rhythm r (duration) and dynamic level d (loudness), as well as pitch. Accordingly, the variation technique takes the y-components $\{y_i\}$ of the reference trajectory and pairs them with the rhythmic sequence $\{r_i\}$ found in the original score. Each r_i is then marked on the y axis at the point designated by its y_i. The y axis becomes a rhythmic axis configured according to the sequence of rhythmic values found in the original composition. The new trajectory is started at an IC differing from the reference. For each new y-component y'_j, the chaotic mapping is applied

$$f(y'_j) = r_{h(j)}, \tag{6.1}$$

where $h(j)$ denotes the index i of the smallest y_i for which $y_i \geq y'_j$.

Just as it is possible for a composition to contain pitches simultaneously struck together (chords), a work can also include rhythms which are coincident with one another. Any or all of the simultaneously occurring rhythmic values can be paired with one or more y_i. But in this thesis, each group of coincident rhythms is considered an integral musical event occurring at a specific i in the sequence of N musical events. Note that the number of rhythmic events is therefore equal to the number of pitch events.

With respect to rhythmic variations, two cases must be taken into account. How one treats the cases can be quite subjective. So what follows are only suggestions, not hard and fast rules:

Case 1. Suppose the chaotic mapping assigns $r_{h(j)}$, which represents only a single rhythmic value, e.g., an eighth note, to y'_j.

- If the pitch event $p_{g(j)}$ — which is to receive $r_{h(j)}$ — consists of only a single pitch, then $p_{g(j)}$ assumes the single rhythmic value $r_{h(j)}$.

- If the pitch event $p_{g(j)}$ comprises a chord, then every pitch of the chord assumes the single rhythmic value $r_{h(j)}$.

Case 2. Suppose the chaotic mapping assigns $r_{h(j)}$, which encompasses k rhythmic values, e.g., a quarter note and two eighth notes, to y'_j.

- If the pitch event $p_{g(j)}$, which is to receive $r_{h(j)}$, consists of only a single pitch, then $p_{g(j)}$ assumes any of the k rhythmic values given by $r_{h(j)}$.
- If the pitch event $p_{g(j)}$ comprises a chord with l pitches, $l < k$, then the pitches of the chord assume the first l rhythmic values contained in $r_{h(j)}$.
- If the pitch event $p_{g(j)}$ is a chord with l pitches, $l \geq k$, then the first k pitches of the chord receive the k rhythmic values given by $r_{h(j)}$. The remaining $l - k$ pitches of the chord loop through the k rhythmic values of $r_{h(j)}$, until the last pitch of the chord has assumed a rhythm.

Variations in dynamic level can be handled similarly. All y's and r's can be replaced by z's and d's. Here, as elsewhere, it is important to keep in mind that any duration, dynamic level, or pitch event, that does not agree with the musician's sensibility can be changed.

6.2 Technical Factors which Influence the Output of the Chaotic Mapping

Factors affecting the nature and extent of variation are choice of the IC, step size, length of the integration, and the amount of truncation and round-off applied to the trajectories. For instance, if the step size is too big (e.g., $h = .1$), the x-values quickly separate from one another, effectively eliminating the potential tracking ability of the chaotic mapping, though the linking aspect of the mapping will still hold. If the step size is small ($h = .001$), the reference and new trajectories may be less likely to track in x due to little space between neighboring x-values on the x axis, especially if the phase portrait encircles the lobes of the attractor several times.

When initial conditions are chosen far apart for the reference and new trajectories, the variations sound like they have diverged considerably from the original. How acceptable this divergence is, depends on the individual piece. Assuming the same methods of Figure 3-2, the new trajectory IC (2, 2, 3) and reference trajectory IC (1, 1, 1) resulted in a Bach Prelude variation that was listenable, though not analyzable harmonically, while new IC (100, .7, 87) vs. reference IC (1, 1, 1) generated a Gershwin variation which caught several listeners' ears (assuming the same methods as Figure 4-5). But there are other instances where large disparities in IC's between reference and new trajectories produce undesirable results. For example, using the methods of Figure 3, a new IC of (-10, -10, -10) and reference IC (1, 1, 1), produced a Bach variation which commenced with 235 soundings of the last C major chord of the original, before proceeding to more interesting territory. In this case, the most negative value for all x_i was $x_{545} = -3.362097$, which was paired with the C major chord. But since the step size was small ($h = .001$) and $x'_0 = -10$, it took 235 time steps for the new trajectory x-values to reach a number that was greater than -3.362097. Until an x'_j became greater than x_{545}, the chaotic mapping assigned C major chords to the variation. At $x'_{236} = -3.312165$, a $D3$ was mapped to the variation, thus ending the repeated chords.

Variations 1 and 2 resulted from new and reference trajectories whose x-values were rounded to two decimal places before being paired with the pitch sequence of the original Bach. If the x-values are not rounded at all, then each x_i will be unique (for all practical purposes), and no x_i is associated with more that one pitch. But when the x_i's are rounded to one or two decimal places, it is possible for an x_i to repeat itself. For instance, the x-value -9.59 of the reference trajectory (1, 1, 1) occurs twice such that $x_{55} = x_{106} = -9.59$. Accordingly, x_{58} is associated with $p_{58} = E3$, but x_{106} is paired with $p_{106} = D3$. In the case of two or more pitch choices for a given x-value, the p_i with the largest i is chosen, e.g., the x-value -9.59 is paired with $p_{106} = D3$. This ensures a greater likelihood that a different note from the original will occur in the variation, e.g., $x'_{55} = -9.60 \longmapsto D3$ rather than the $E3$ of the original.

6.3 The Chaotic Mapping Applied to a Limit Cycle

Though the Lorenz system can exhibit periodic behavior, the mapping is most effective with chaotic trajectories. This is due to their infinite length, enabling sequences of any duration to be piggybacked onto them, and their extreme sensitivity to the IC. To see the drawback of limit cycle behavior, the same methods discussed in Figure 3-2 were applied to orbits near the limit cycle for $r = 350$ [1]. The IC (-8.032932, 44.000195, 330.336014) is on the cycle (approximately). In this case, however, if a trajectory starting at that IC serves as the reference for the mapping, a new trajectory, with its IC obtained by truncating the last digit of the reference IC, yields the original Prelude. That is, the IC (-8.03293, 44.00019, 330.33601) does not give a variation.

6.4 Infusing the Style of a Piece with Another

It is possible to influence the style of one piece, or part thereof, with that of another. For example, suppose Piece B is appended to Piece A. Their combined pitch sequence becomes the input for the variation technique. The output consists of a variation of AB, called $A'B'$. But then Piece B' can be cut away from A' and used independently. This was done in *Variations to a Theme* where the composed Theme of Figure 5-5 assumed elements not contained in the original Bach Prelude. The variation technique took the Cadenza (A), Chorale (B) and Prelude (C) as input and produced a variation $A'B'C'$. A' and B' were eliminated, leaving the varied Prelude C' influenced by the musical landscapes of the previous Cadenza and Chorale.

6.5 Generalizing the Chaotic Mapping to Any Sequence of Symbols

The technique introduced here, applied to the context-dependent works discussed in Chapters 3-5, generates variations that can be analyzed and used musically, suggesting that the method can be generalized to other sequences of context-dependent symbols $\{s_i\}$, $i = 1, \ldots, N$. Accordingly, the x-axis becomes a symbol axis (Figure 6-1) encompassing a finite number of regions — demarcated by symbols — rather than an infinite number of points. Each region returns one of N possible s_i. The chaotic mapping applied for each x'_j is given by

$$f(x'_j) = s_{g(j)}, \tag{6.2}$$

where $g(j)$ is defined as above.

6.6 The Variation Technique in conjunction with other Chaotic Systems

The chaotic mapping has also been applied to chaotic trajectories from another chaotic system proposed by Lorenz [2],

$$\begin{aligned} \dot{x} &= -y^2 - z^2 - ax + aF && (6.3)\\ \dot{y} &= xy - bxz - y + G && (6.4)\\ \dot{z} &= bxy + xz - z, && (6.5) \end{aligned}$$

where $a = 0.25$, $b = 4$, $F = 8$ and $G = 1$, as well as the Rössler system [3],

$$\begin{aligned} \dot{x} &= -y - z && (6.6)\\ \dot{y} &= x + ay && (6.7)\\ \dot{z} &= b + z(x - c), && (6.8) \end{aligned}$$

with chaotic parameters $a = b = 0.2$ and $c = 5$. For the same methods outlined in the caption to Figure 3-2, variations resulted which sustained the author's interest.

For each of the three chaotic systems described by Eqns. 2.1-3, Eqns. 6.2-4 and Eqns. 6.5-7, a graph is given in Figure 6-2 which plots the difference in x-values between the reference and new trajectories. These graphs suggest that the Lorenz Eqns. 2.1-3 may be better for variations than the other two systems, at least for close IC's, $h = .01$ and an integration length of about 1000 time steps — conditions used to produce the first two Bach variations and the Gershwin variation. Plotting the difference in x-values between reference and new trajectories, as simulated by Eqns. 2.1-3, displays a good balance between positive, negative and zero excursions. In particular, there are many alternations between the cases where $x_i - x'_i > 0$ and those where $x_i - x'_i < 0$. The repeated presence of positive (negative) excursions ensures

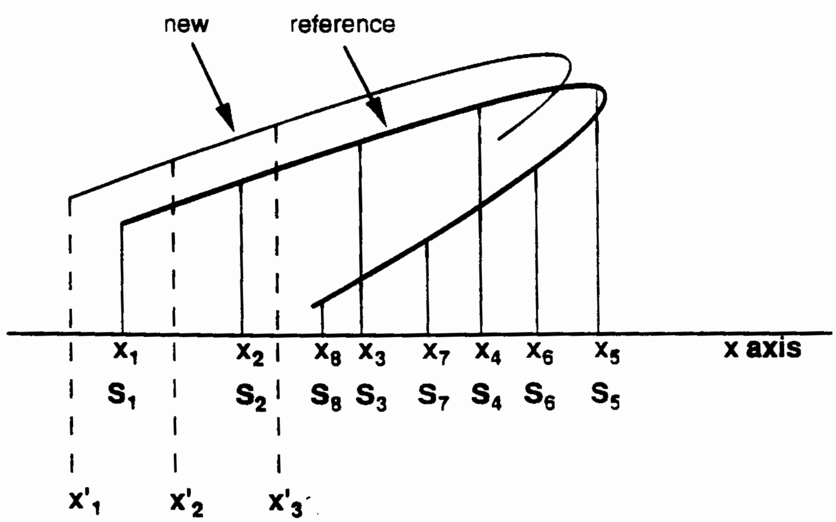


Figure 6-1: The chaotic mapping applied to a hypothetical sequence of symbols. The pairing between a sequence of symbols $\{s_i\}$, $i = 1, \ldots, 8$, and a partial sequence of x-components $\{x_i\}$, $i = 1, \ldots, 8$, from a chaotic trajectory (reference) is shown below the x axis. For each x'_j of a second chaotic trajectory (new), apply the chaotic mapping. For example, the mapping applied to x'_1, x'_2, and x'_3, yields s_1, s_2, and s_8.

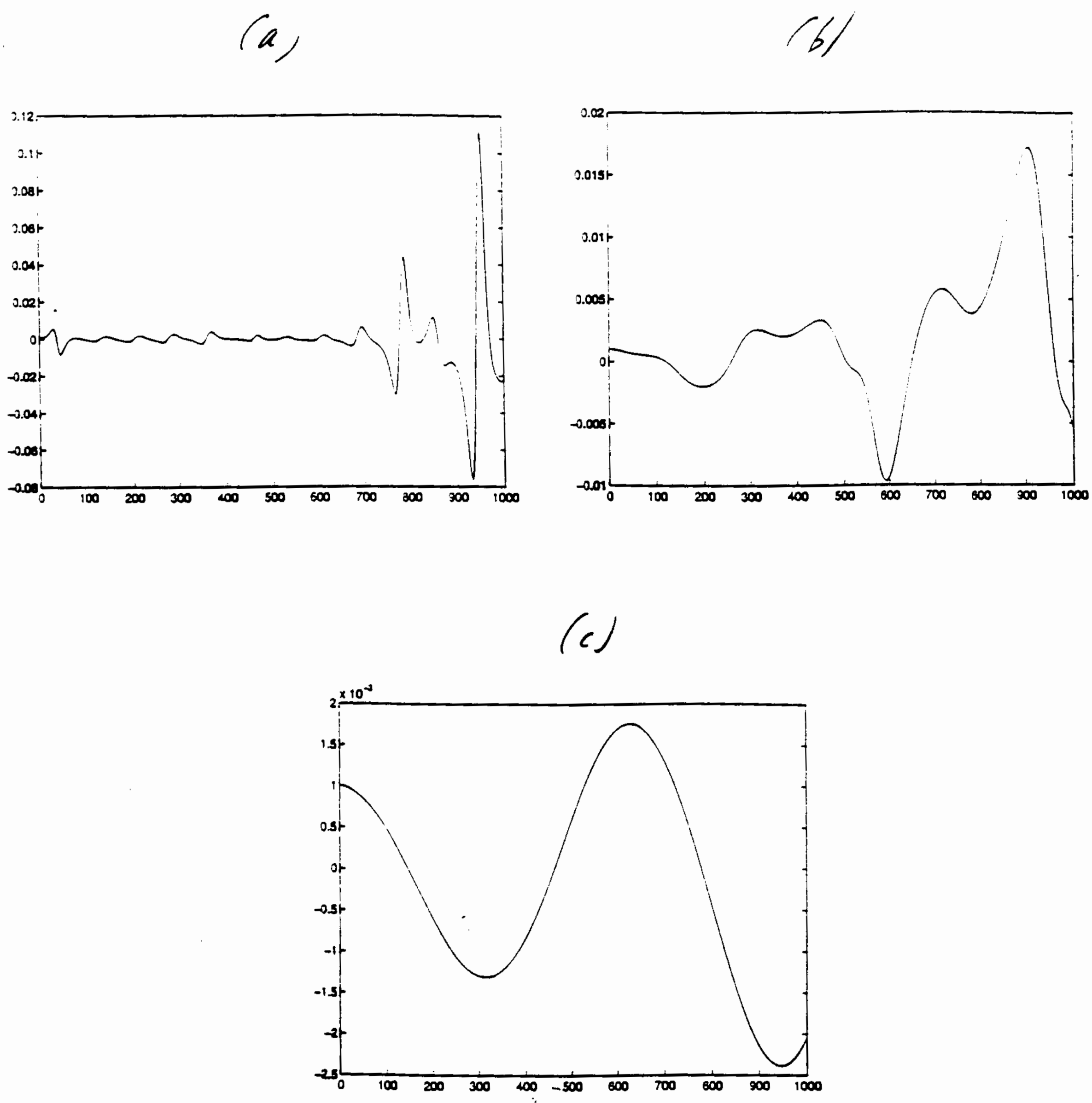


Figure 6-2: The difference in x-values (vertical axis) between chaotic trajectories with reference IC (1, 1, 1) and new IC (.999, 1, 1) vs. 1000 time steps of the integration (horizontal axis) for *(a)* Eqns. 2.1-3, *(b)* Eqns. 6.2-4, and *(c)* Eqns. 6.5-7. For all trajectories, the integrations are sampled every step with $h = .01$. All computations are double precision with x-values rounded to six decimal places.

that new trajectory x-values are at least falling to the left (right) of the reference x-values and therefore have a chance to trigger original (different) notes. The balance between positive and negative excursions, in conjunction with predominantly small differences in x-values, imply that the tracking mechanism of the chaotic mapping will preserve parts of the original pitch sequence in the variation. However, in the Rössler equations and the Lorenz Eqns. 6.5-7, there are far fewer alternations between these two cases, and in the latter case, the bounds for the differences in x-values are erratic. Though no definitive conclusions can be drawn on the basis of Figure 6-2, plotting the difference in x-values between new and reference trajectories suggests a way of evaluating a chaotic system for its variation potential.

The difference in x-values between reference and new trajectories will, for all three systems, eventually become quite significant. However, the rate at which they become pronounced will vary from system to system, assuming the same change in IC's between reference and new trajectories, step size, round-off, etc. For example, the graphs in Figure 6-3 show how the Rössler system can advance 3000 time steps and still show a maximum difference in $x_i - x'_j$ that is anywhere from one to three orders of magnitude LESS than the Lorenz system (Eqns. 2.1-3) at 800-2000 time steps.

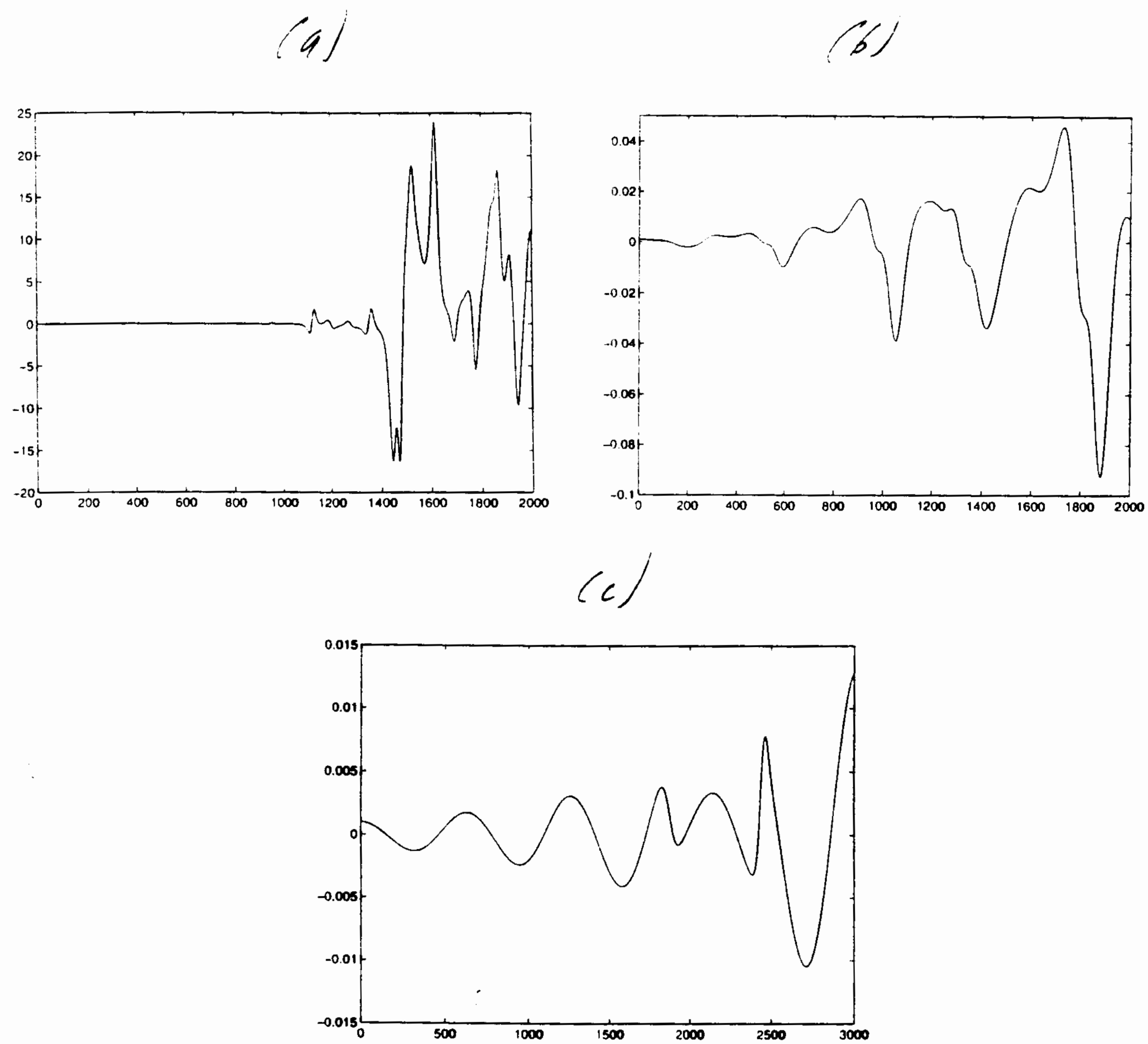


Figure 6-3: The difference in x-values (vertical axis) between chaotic trajectories with reference IC (1, 1, 1) and new IC (.999, 1, 1) vs. time steps of the integration (horizontal axis) for *(a)* Eqns. 2.1-3, *(b)* Eqns. 6.2-4, and *(c)* Eqns. 6.5-7. For all trajectories, the integrations are sampled every step with $h = .01$. All computations are double precision with x-values rounded to six decimal places.

The chaotic mapping in tandem with different chaotic systems will be examined more closely in later work.

6.7 References

[1] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors* (Springer, New York, 1982).

[2] E. N. Lorenz, "Deterministic and stochastic aspects of atmospheric dynamics," in *Irreversible Phenomena and Dynamical Systems Analysis in Geosciences* edited by C. Nicolis and G. Nicolis (D. Reidel, Dordrecht, The Netherlands) 159-179 (1987).

[3] O. E. Rössler, "An equation for continuous chaos," Phys. Lett. A **57**, 397-398 (1976).

Chapter 7

Conclusion

The variation technique was designed to take a highly context-dependent application (music) and generate variations — via a chaotic mapping — that may retain stylistic ties to the original or mutate beyond recognition, by appropriate choice of the IC, step size, length of the integration, etc. The technique works because two mechanisms — linking and tracking — were built into the design to help temper the sensitive dependence of chaotic trajectories to initial conditions. When nearby orbits track each other in x, pitches can be heard in the variation which occur exactly where they did in the source, thereby ensuring portions of the original recur in the variation. The sensitive dependence property of chaotic systems provides a natural mechanism for variability, making sure that the chaotic mapping returns a variation in which changes in the pitch sequence of the original have occurred. However, regardless of whether the mapping preserves or alters the original sequence, the linking aspect of the mapping guarantees that every pitch event of the variation can be found in the original piece. This intimate connection between variation and source always exists.

One way to predict whether a given pair of reference and new trajectories may yield a variation which retains more or less of the original sequence, is to plot the difference in x-values between the two orbits, $x_i - x'_i$. By observing the number of positive and zero excursions vs. negative digressions, as well as the relative amplitudes of these excursions, it is possible to evaluate the two trajectories according to what is desired in the variation. For example, if it is vital that the original piece be clearly recognized in the variation, then the number of negative digressions should not dwarf the number of positive or zero excursions. A small step size, however, with a length of integration resulting in more than one loop around a lobe of the Lorenz system, may effectively obliterate any hope for tracking. If a simulation of the reference trajectory ($h = .001$) is sampled every step, pitches piggybacked onto the reference can be extremely close to one another. When they are projected down onto the x-axis, little leeway is left between neighboring pitches, so that even if an x'_i falls slightly left of x_i, i.e., $x_i - x'_i > 0$, x'_i could easily trigger a different note from the original.

Once a number of reference and new trajectories have been found which produce acceptable variations for a given piece, there is no reason why they might not also apply to a completely different score. For instance, the same reference trajectory — with IC (1, 1, 1), $h = .01$, x-values rounded to two decimal places, and an integration length of 1000 time steps — was used for the first two Bach variations (Figure 3-2), the Gershwin variation discussed in Chapter 4 (Figure 4-5), as well as variations on scores by Bartok and Chopin. The new trajectories were also simulated with the same step size, number of time steps, and x-values rounded to two decimal places. but differed in IC's from the reference.

Similarly, the same reference trajectory with IC (1, 1, 1) and integration length of 1000 time steps, but with $h = .001$ and x-values rounded to six decimal places. produced Variation 3 of the Bach Prelude (Figure 3-3), the raw variation of the Theme which concludes *Variations to a Theme* (Figure 5-4), and variations on pieces by Beethoven and Bartok. Again, the new trajectories were simulated with the same step size, etc., but differed in IC's from the reference.

Sometimes the exact same reference and new trajectories used for one piece can be used for another. For instance, the same reference trajectory IC's (1, 1, 1), new trajectory IC's (1.0005, 1, 1), step size (.01), integration length (1000 time steps) and rounding (2 decimal places) that generated the Gershwin variation of Figure 4-5 were used to produce a variation on Chopin's *Étude in f minor*, Op. 10, that several professional musicians cited as memorable.

A musician's sensibility is helpful for understanding more fully which chaotic trajectories to use in the variation process. For example, even if he or she had little or no grasp of the technical details of the mapping, certain characteristics of the trajectories become apparent. Just by listening, a musician can quickly tell whether two trajectories are tracking in x. The ear can also identify if two orbits are the same or different in varying degrees. This is always a good check for the engineer's graphs, and vice versa. 198z Then there are characteristics of specific trajectories which the musician could identify, again by listening. For example, the new trajectory — with IC (1, .9999, .9), $h = .001$, x-values rounded to six places, and an integration length of 1000 time steps — triggers pitches on the x axis (planted by the reference with IC (1, 1, 1) and same step size, etc.) that can form the retrograde of an earlier string of pitch events, and occur in half the time. This was heard in the Bach variation given by Figure 4-3, where the harmonic sequence of *mm.* 28-29 of the variation turned out to be the retrograde diminution of the earlier sequence given in *mm.* 12-15. A musician would remember this particular trajectory as having the potential for a retrograde diminution of earlier material.

However, the musician would not understand why the retrograde diminution occurred unless he/she also understood the variation technique, from its pairing step to the chaotic mapping. Clearly, two heads are better than one, or perhaps two hats. As a musician works with the technique, though, a body of empirical findings is built, and from this repertoire, and knowledge of the original musical score, a composer can often zero in on a group of variations that resonate with her or his own artistic voice.

The variation technique is not limited to music, and indeed, one of the future goals of this work is to apply it to prose, poetry and computer art. However, the mapping can be generalized to any sequence of symbols. A few researchers in the life sciences have been intrigued by its success with a highly context-dependent application such as music, where variations are created which can be analyzed and used for musical means within a given context. As scientists, they often study numerous variants of a system and try to find the organizing thread that links them or. equivalently, a method that generates them. Whether the chaotic mapping can unlock some problems in biology or neuroscience is an open question.

What can be said at the moment is that this technique — designed for music of our own time — is both an idea generator and point of departure for a dynamic Classical music where the written score changes from one concert performance to the next. When used as an idea generator, the variation technique becomes a tool in the spirit of such recognized musical procedures as inversion and retrograde inversion. As with these, the musician composes a score, or any part thereof. This becomes the input for the variation technique. The chaotic mapping generates any number of variations which the composer can accept, alter, or reject. Identifying those elements of the variations that catch the ear and warrant further development, the artist creates a composed variation incorporating the best ideas that are consistent with his or her stylistic idiom.

However, if the artist chooses to make several composed variations of the same score, as triggered by the ideas generated by the technique, the possibility for different variations of the same work exists. The written score has become dynamic. Listeners hearing such a work for the first time will register its essence, and if heard subsequent times, will not only hear the composition as an entity onto itself, but also listen for its relation to other variations. The entire piece has become self-referencing: The music calls itself and listeners recursively recall prior variations. In this and all respects, the issue of style is paramount. For the background and foreground structure that is present in the original must always be taken into account when creating the variation. Equivalently, the vertical and horizontal structure of the composed variation, in tandem with its foreground and background layers, should present a consistent voice to the listener.

Finally, the variation technique does not generate music or any other kind of data as random events; rather, it creates a rich set of variations on the original that can be further developed and interpreted. Though the method will not flatter fools, it can lead explorers into landscapes where, amidst the familiar, variation and mutation allow wild things to grow.

Appendix A

The Lorenz Equations

The variation technique uses the chaotic regime of the Lorenz equations as a natural mechanism for variability. Some background on the Lorenz system is given here.

A.1 Introduction

The Lorenz equations represent a simplification of a system derived by Barry Saltzman [1] to study finite-amplitude convection. Convection is the transfer of heat or other atmospheric properties by massive motion within the atmosphere, especially by such motion directed upwards. The Lorenz equations [2] therefore describe a convective process and are given by

$$\dot{x} = \sigma(y - x) \tag{A.1}$$

$$\dot{y} = -rx + y + xz \tag{A.2}$$

$$\dot{z} = xy - bz, \tag{A.3}$$

where σ is the Prandtl number, r is the Rayleigh number, and b has no name, though in the convection problem, it is related to the height of the fluid layer [3]. The parameter values σ, r and b are always positive. In the chaotic regime, $a = 10$, $b = 8/3$ and $r = 28$. The variable x is proportional to the intensity of the convective motion; y is proportional to the temperature difference between the ascending and descending currents. Like signs of x and y mean that warm fluid is rising and cold fluid is descending. The variable z, proportional to the distortion of the vertical temperature profile from linearity [4], always remains positive for $r = 28$. This means that when the heat is being conducted without convection (i.e., the heat moves but the fluid does not), the vertical temperature profile is proportional to height: The temperature of the fluid decreases linearly with height, assuming the system is being heated from the bottom. But once convection starts (at $r = 1$), the temperature profile changes and z is a measure of the deviation from the linear profile. Positive values of z indicate that the strongest gradients occur near the boundaries. The equations are continuous since their right hand sides are composed of polynomial terms.

Lorenz discovered that over a wide range of parameters, this relatively simple deterministic system had solutions which never repeat exactly. In fact, almost all of the solutions are nonperiodic. Lorenz shows that all nonperiodic central trajectories are unstable. A central orbit is one that returns arbitrarily often, arbitrarily closely to any point through which it has previously passed. Lorenz also showed that noncentral[1] nonperiodic orbits are not uniformly stable, and if they are stable at all, their very stability is a transient property which tends to die out with time. Lorenz then concludes:

> In view of the impossibility of measuring initial conditions precisely, and thereby distinguishing between a central trajectory and a nearby noncentral trajectory, all nonperiodic trajectories are effectively unstable from the point of view of practical prediction [5].

For the chaotic regime — $\sigma = 10$, $b = 8/3$, $r = 28$ — all solutions are confined to a bounded region of state space and, when plotted in xyz space, eventually approach a complicated set of zero volume — the strange attractor. This attractor, what Lorenz referred to as a "limiting trajectory" in his paper, is neither a point, curve nor surface, but rather a fractal, with fractional dimension of approximately 2.05.

A.2 Properties of the Lorenz System

There exist only two nonlinearities in the Lorenz equations — the quadratic terms xz and xy. The equations are symmetric in x and y: if $x(t)$, $y(t)$, $z(t)$ is a solution, then $-x(t)$, $-y(t)$, and $z(t)$ is also a solution. The Lorenz system is dissipative such that volumes in state space shrink as the flow (or solution) evolves. To picture this, imagine an arbitrarily closed surface $S(t)$ enclosing a volume $V(t)$ in state space. All points on $S(t)$ are treated as initial conditions for solutions to the general 3-dimensional system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. These trajectories are allowed to evolve for an infinitesimal time dt. Then S assumes a new "skin", $S(t + dt)$, which encloses the volume $V(t + dt)$. If $\mathbf{n}$ denotes the outward normal on S, then $\mathbf{f} \cdot \mathbf{n}$ represents the outward normal component of the flow's velocity, since $\mathbf{f}$ is the instantaneous velocity of the points. In time dt, a patch of surface area dA sweeps out a volume $(\mathbf{f} \cdot \mathbf{n}\, dt)\, dA$, so that

$$V(t + dt) = V(t) + \int_S (\mathbf{f} \cdot \mathbf{n}\, dt)\, dA \quad \text{(A.4)}$$

$$\dot{V} = \frac{V(t + dt) - V(t)}{dt} = \int_S (\mathbf{f} \cdot \mathbf{n})\, dA. \quad \text{(A.5)}$$

By the divergence theorem, we get

$$\dot{V} = \int_V \nabla \cdot \mathbf{f}\, dV.$$

[1] A solution is called noncentral if it is not central.

For the Lorenz system,

$$\nabla \cdot \mathbf{f} = \frac{\delta}{\delta x}\sigma(y-x) + \frac{\delta}{\delta y}(rx - y - xz) + \frac{\delta}{\delta z}(xy - bz) \tag{A.6}$$

$$= -\sigma - 1 - b. \tag{A.7}$$

Therefore,

$$\dot{V} = (-\sigma - 1 - b)V \Longrightarrow V(t) = V(0)e^{-(\sigma+b+1)t}. \tag{A.8}$$

This implies that, for the Lorenz equations, volumes in state space contract exponentially fast [6]. Thus, points separated from one another in a given direction can come together very rapidly, and appear to merge [7]. This is why the left and right lobes of the strange attractor look like they intersect at the bottom of the attractor. The fact that a volume of initial conditions will eventually shrink to a set of zero volume means that all trajectories sooner or later end up on a limiting set which consists of fixed points, limit cycles, or for some parameter values, a strange attractor [8].

It turns out that volume contraction imposes strong constraints on the possible solutions of the Lorenz equations. For example, there are no quasiperiodic solutions of these equations. Any quasiperiodic solution would have to occur on a torus which would be invariant under the flow, i.e., given any initial condition on the torus, the resulting trajectory would stay on the torus forever. The points on the torus play a kind of musical chairs: Each point flows to a place previously occupied by some other point on the torus. As a set, the torus remains unchanged. The volume inside it stays fixed, even though points on the torus are flowing all the time. The constant volume of the torus contradicts the fact that all volumes must shrink exponentially fast [9].

Volume contraction also makes it impossible for the Lorenz system to have either repelling fixed points or repelling closed orbits. This follows intuitively since repellers are sources of volume, as can be imagined by encasing a repeller with a closed surface of initial conditions. After a short time, the surface will expand due to the trajectories being driven away. This contradicts the fact that all volumes of initial conditions contract in this system. As a result of eliminating quasiperiodic solutions and repellers, it's clear that all fixed points in the Lorenz system are stable nodes or saddle points and that closed orbits (if they exist) must be stable or saddle-like [10].

A.3 Fixed Points of the Lorenz System

There exist three fixed points associated with the Lorenz system:

$$(0,0,0),\ \ (+\sqrt{b(r-1)}, +\sqrt{b(r-1)}, r-1),\ \ (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1).$$

Lorenz called the second fixed point C^+ and the third C^-. These fixed points represent the steady state solutions, i.e., the states of no convection. The criterion for the onset of convection is $r = 1$, when the origin changes from a stable node to a saddle point with one outgoing and two incoming dirctions. Also at $r = 1$, the system undergoes a supercritical pitchfork bifurcation, resulting in the creation of C^+ and C^-. The outgoing eigendirection of the origin pushes the flow towards C^+ and C^-, which represent left- and right-turning convection rolls.

A.3.1 Stability of the Origin

Linearization about the fixed point at the origin results in the linearized equations:

$$\dot{x} = \sigma(y - x) \tag{A.9}$$

$$\dot{y} = rx - y \tag{A.10}$$

$$\dot{z} = -bz \Longrightarrow z(t) = z(0)e^{-bt}. \tag{A.11}$$

Note that the equation for $\dot{z}$ is decoupled from the equations for $\dot{x}$ and $\dot{y}$ and that $z(t) \to 0$ exponentially fast. Therefore, the stability at the origin is really determined by the system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma \\ r & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \tag{A.12}$$

The trace (τ) of the above system is $-\sigma - 1$ and the determinant (Δ) is $\sigma(1 - r)$. For $r > 1$, the determinant will always be negative which classifies the fixed point as a saddle point. But for $r < 1$, the determinant is positive so that, with the negative trace, the origin can be classified as either a stable node or stable spiral. Since $\tau^2 - 4\Delta > 0$, the origin is a stable node for $r < 1$. In fact, the origin is globally stable, i.e., every trajectory approaches (0, 0, 0) as $t \to \infty$.

A nice proof of the global stability of the origin for $r < 1$ exists which involves a Lyapunov function for the Lorenz system. But first, to understand the general concept, consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with a fixed point at $\mathbf{x}^*$. A Lyapunov function for this system is a continuously differentiable, real-valued function $V(\mathbf{x})$ with the following properties:

1. $V(\mathbf{x}) > 0 \;\; \forall \mathbf{x} \neq \mathbf{x}^*$ and $V(\mathbf{x}^*) = 0$, i.e., $V(\mathbf{x})$ is positive definite.

2. $\dot{V}(\mathbf{x}) < 0 \;\; \forall \mathbf{x} \neq \mathbf{x}^*$, i.e., all trajectories flow "downhill" toward $\mathbf{x}^*$.

If $V(\mathbf{x})$ exists, then $\mathbf{x}^*$ is globally asymptotically stable: For all initial conditions, $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$, and the system has no closed orbits. All trajectories slide monotonically down the graph of $V(\mathbf{x})$ toward $\mathbf{x}^*$ as seen in Figure A-1 [11].

By the second property of the Lyapunov function, $\dot{V} < 0$ everywhere except at $\mathbf{x}^*$. Therefore, orbits on V cannot stop anywhere except at $\mathbf{x}^*$ [12].

There is no way to systematically find a Lyapunov function for the Lorenz equations that will have the properties $V(\mathbf{x}) > 0$, $V(\mathbf{x}^*) = 0$, and $\dot{V}(\mathbf{x}) < 0$ so that the conclusion "$\mathbf{x}^*$ is globally asymptotic" is validated. However, it sometimes helps to define $V(\mathbf{x})$ as an expression involving sums of squares. One such expression de-

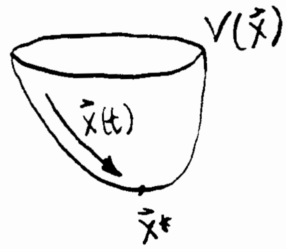


Figure A-1: A Lyapunov function showing all trajectories descending to the fixed point $\mathbf{x}^*$.

scribes surfaces of constant V which are concentric ellipsoids about the origin:

$$V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2. \tag{A.13}$$

Each ellipsoid has constant V, meaning that as x, y, z traverse a given ellipsoid, $V(x,y,z)$ does not change. V is similar to an energy function, only more abstract. The nested (or concentric) ellipsoids are arranged so that an ellipsoid with smaller V resides inside one with larger V.

With this Lyapunov function, we wish to show that if $r < 1$ and $(x,y,z) \neq (0,0,0)$, then $\dot{V} < 0$ along trajectories. This means that if V is measured at a particular x, y, z location, and then measured again later, V will be seen to have decreased. Trajectories keep moving to lower V, and therefore penetrate smaller and smaller ellipsoids as $t \to \infty$. But since V is bounded below by 0 (because $V(\mathbf{x}) > 0$) and since $\dot{V}(\mathbf{x})$ is decreasing, we can say that $V(\mathbf{x}(t)) \to 0$ and thus $\mathbf{x}(t) \to 0$, as required [13].

To show that $\dot{V} < 0$:

$$\begin{aligned}
\dot{V} &= 2\frac{1}{\sigma}x\dot{x} + 2y\dot{y} + 2z\dot{z} && \text{(A.14)}\\
\frac{1}{2}\dot{V} &= -x^2 - y^2 + xy(r+1) - bz^2 && \text{(A.15)}\\
&= -[x^2 + \frac{(r+1)^2}{4}y^2 - (r+1)xy] - bz^2 + \frac{(r+1)^2}{4}y^2 - y^2 && \text{(A.16)}\\
&= -[x - \frac{r+1}{2}y]^2 - bz^2 - y^2[1 - (\frac{r+1}{2})^2]. && \text{(A.17)}
\end{aligned}$$

If $r < 1$ and $(x,y,z) \neq (0,0,0)$, then $\dot{V} < 0$. It is possible that $\dot{V} = 0$ but only if the right hand side terms vanish. This requires $y = z = 0$. Then we're left with

$$\frac{1}{2}\dot{V} = -x^2,$$

which vanishes only if $x = 0$. Therefore, $\dot{V}(\mathbf{x}) = 0 \Longrightarrow (x,y,z) = (0,0,0)$. Else, $\dot{V} < 0 \Longrightarrow (x^*, y^*, z^*) = (0,0,0)$ is globally stable for $r < 1$ [14]. In summary, then, for $r > 1$, the origin is a saddle point with two stable eigenvectors and one unstable eigenvector. For $r < 1$, the origin is globally stable. For $r = 1$, a pitchfork bifurcation results in the creation of C^+ and C^-.

A.3.2 Stability of C^+ and C^-

Linearization about either C^+ or C^- results in the linearized equations:

$$\dot{u} = \sigma(v - u) \quad (A.18)$$
$$\dot{v} = u(r - z^*) - v - wx^* \quad (A.19)$$
$$\dot{w} = uy^* + vx^* - bw, \quad (A.20)$$

where $u = x - x^*$, $v = y - y^*$, and $w = z - z^*$. The Jacobian matrix is given by

$$\begin{bmatrix} -\sigma & \sigma & 0 \\ r - z^* & -1 & -x^* \\ y^* & x^* & -b \end{bmatrix}.$$

For either of the steady state solutions, the characteristic equation is

$$\lambda^3 + \lambda^2(\sigma + b + 1) + \lambda b(\sigma + r) + 2\sigma b(r - 1) = 0. \quad (A.21)$$

When $r > 1$, this equation possesses two complex conjugate roots and one real root. By substituting the value $\lambda = iw$ into the characteristic equation, it's possible to find the value of r at which a Hopf bifurcation will take place:

$$r = r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} = 24.74, \quad (A.22)$$

assuming $\sigma - b - 1 > 0$. The pure imaginary eigenvalues are

$$\lambda_{2,3} = \pm ib\sqrt{r + \sigma}.$$

The real eigenvalue (always negative) is

$$\lambda_1 = -(a + b + 1).$$

For $0 < r < r_H$, C^+ and C_- are linearly stable. But at $r = 24.74$, they lose their stability due to the Hopf bifurcation.

A Hopf bifurcation occurs whenever a complex conjugate pair of eigenvalues crosses from the left half plane (LHP) to the right half plane (RHP). By contrast, when a real eigenvalue crosses from the LHP to the RHP, a system undergoes a saddle node, transcritical or pitchfork bifurcation. As is the case with pitchfork bifurcations, Hopf bifurcations can be classified as either supercritical or subcritical. But whereas pitchfork bifurcations can occur in first order systems, Hopf bifurcations occur only in systems of order two or higher.

In a supercritical Hopf bifurcation, the fixed point loses its stability by expelling a stable periodic orbit, while in the subcritical Hopf, the fixed point loses its stability by absorbing an unstable periodic orbit. Clearly, the supercritical Hopf is less dangerous. since once the bifurcation has taken place, trajectories in the vicinity of the now unstable fixed point are attracted by the stable periodic orbit. The Lorenz system undergoes the more risky (from an engineering point of view) subcritical Hopf bifurcation at r_H. All trajectories within the vicinity of either C^+ or C^- are now repelled outwards from the fixed points. But to where? To a distant attractor, whether it be a stable node, stable limit cycle, torus or strange attractor. In the case of the Lorenz equations, the trajectories cannot veer out to infinity because it can be shown that all trajectories enter an ellipsoidal trapping region. They don't approach a torus because there are no quasiperiodic solutions for the Lorenz equations. Nor can the solutions end up at a fixed point since all fixed points of the system are saddle-like. As for the limit cycle, Lorenz gave an heuristic argument that there are no stable periodic orbits for $r = 28$, though a number of saddle cycles exist. These saddle cycles are saddle-like limit cycles and are explained more fully in the next section. If perturbed, they become chaotic.

Finally, to see where these solutions might go, Lorenz integrated his equations just past the Hopf bifurcation. He found the phase portrait which has adorned countless book covers ever since — the strange attractor.

A.3.3 Saddle Cycles, Homoclinic Orbits, and a Strange Invariant Set

From $r = 1$ until $r = 24.74$, C^+ and C^- are each surrounded by a saddle cycle. A saddle cycle consists of a two-dimensional unstable manifold (a plane) punctured by a two-dimensional cylindrical surface comprising a stable manifold. Where the cylindrical surface intersects the unstable plane, an unstable limit cycle known as a *saddle cycle* occurs. It is this saddle cycle that gradually tightens around C^+ (C^-), resulting in a narrower and narrower cylindrical surface until, at $r = r_H$, the cycle coalesces with the stable fixed point, changing it into a saddle point.

Now, if r is subsequently decreased from r_H, a pair of saddle cycles again encircle C^+ and C^-. As r decreases down to 13.926, these cycles grow in amplitude, until at $r = 13.926$, they touch the origin and create two homoclinic orbits via a homoclinic bifurcation. Homoclinic orbits start and end at the same fixed point.

Going the other way, as r approaches 13.926 from below, the same two saddle cycles appear, as well as an extremely complex invariant set (at $r = 13.926$). This set, composed of infinitely many nonperiodic orbits and saddle cycles, generates sensitive dependence on initial conditions within a small neighborhood surrounding it. But it is not an attractor because though trajectories can wander a long time within its coppice of nonperiodic orbits and saddle cycles, these trajectories will eventually exit the set and wind up at C^+ or C^-. As r increases towards 24.06, trajectories spend more and more time wandering around the invariant set. Transient chaos is said to occur between the values of $13.926 < r < 24.06$. One can visually identify this transient, "pre-turbulent" or "metastable" chaos by noticing whether the phase portrait first traces out the strange attractor before winding down to C^+ or C^-. Transient chaos does exhibit sensitive dependence on initial conditions so that a trajectory started at one point might eventually end up on C^+, while another started nearby may wind down to C^-. What is interesting about a deterministic system exhibiting transient chaos is the mix of unpredictable and predictable behavior: the final states of the system are known (e.g., the settling down to C^+ or C^-). But prior to that, the behavior is chaotic. Finally, at $r = 24.06$, the time spent wandering around the strange invariant set reaches infinity, and the invariant set has become an chaotic attractor.

Three attractors are present in the Lorenz system when r is between 24.06 and $24.74 = r_H$. These are the two stable nodes C^+, C^- and the strange attractor. It is possible to get hysteresis between equilibrium and chaos when slowly moving r past these two endpoints and back again. The phase portrait jumps from a stable fixed point to a strange attractor when $r = r_H = 24.74$. If r is increased a little further, the strange attractor remains. But if r diminishes to 24.74 and then slightly below, the strange attractor is still present. In fact, r must be reduced to 24.06 before a stable fixed point appears again. Then, going the other way, when r is increased slightly above 24.06, the fixed point remains. Only when r hits 24.74 will the strange attractor emerge.

There is a globally attracting limit cycle for all $r > 313$. Thus the system becomes dynamically simpler for these high values of r. However, for $28 < r < 313$, the dynamics are quite complicated. Small windows of periodic behavior are interspersed with chaotic regimes. For example, the three largest windows of periodicity occur for $99.524... < r < 100.795...$; $145 < r < 166$; and $r > 214.4$ [15].

A.4 References

[1] B. Saltzman, "Finite amplitude free convection as an initial value problem — I," J. Atmos. Sci. **19**, 329-341 (1962).

[2] E. N. Lorenz, "Deterministic nonperiodic flow," J. Atmos. Sci. **20**, 130-141 (1963).

[3] S. H. Strogatz, *Nonlinear Dynamics and Chaos* (Addison-Wesley, New York, 1994) 301.

[4] Lorenz, 135.

[5] Lorenz, 133.

[6] Strogatz, 312-313.

[7] Lorenz, 140.

[8] Strogatz, 313.

[9] Strogatz, 313.

[10] Strogatz, 314.

[11] Strogatz, 201.

[12] Strogatz, 201.

[13] Strogatz. 315.

[14] Strogatz, 315-316.

[15] Strogatz. 330-335.

Appendix B

A Dynamic System Approach to Western Musical Thought

It is possible to model a musical piece as a dynamic system where musical analogies are built upon such dynamic system concepts as state space, state variable, order, trajectory, autonomous/nonautonomous systems, and state equations. Models can range from the extremely precise to the metaphorical. They do not have to reflect reality perfectly. Rather, their purpose is to provide some insight into a real-world process or system. At some levels of the musico-dynamic model, less precision will be knowingly applied since, as might be expected, some of these analogies work better than others. For example, an extensive caveat is given for the state equation analogy.

The discussion opens with a simple projectile exercise which demonstrates the language of dynamic systems in a way that, hopefully, musicians might grasp. Musical analogs follow using a Chorale by J. S. Bach as the illustrative example.

B.1 Introduction

A musical work can be said to comprise a *dynamic system* where, in the broadest sense, a dynamic system is understood to mean any entity which changes with time. Classical dynamics is usually characterized by the concept of *state.* State represents everything that must be remembered from the past that is relevant to the future. The variables necessary to convey that information, according to the constraints of the model, are called *state variables* of the system. These are crucial time-dependent quantities whose future values are determined by substituting their present values into an array of equations governing the system. The *order* of a dynamic system is determined by the number of independent state variables present in the system. The dynamic system model for music presented here serves only to establish a link between the language of dynamics and that of music. The model is built to better understand which analogies bridge these two world views, and how robustly they can be sustained.

First, a simple rocket system is proposed to establish some working terminology in dynamics. By no means does this rocket system model a real-world rocket, e.g., it neglects friction, the uneven burn of the rocket motors, etc. Suppose the rocket is fired straight upwards with an initial velocity of 50 meters/second (m/s) under the constant negative acceleration of earth's gravity (g). There are two state variables in this model — velocity (v) and position (y). They are related by two *system equations* governing its motion:

$$v(k + \Delta k) = v(k) + a\Delta k \quad (B.1)$$

$$y(k + \Delta k) = y(k) + v(k)\Delta k, \quad (B.2)$$

where k represents time in discrete seconds, Δk is equivalent to 1 second, and a represents the constant acceleration of earth's gravity; i.e., $a = g = -10\ m/s^2$. Thus the two equations can be rewritten as

$$v(k + 1) = v(k) + a \quad (B.3)$$

$$y(k + 1) = y(k) + v(k). \quad (B.4)$$

They can also be described by the vector equation

$$\mathbf{z}(k + 1) = \mathbf{H}(\mathbf{z}(k)), \quad (B.5)$$

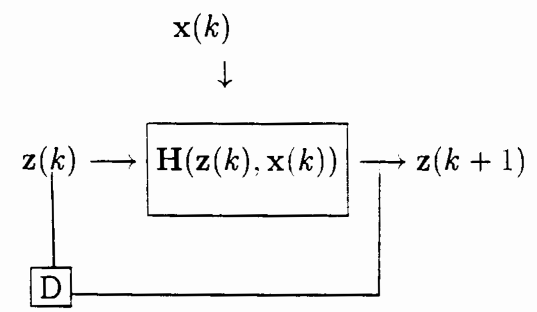
where $\mathbf{z}$ is known as the *state vector* because it denotes the *state* variables v and y, i.e., $\mathbf{z} = [v, y]$. The function or operator $\mathbf{H}$ generates the flow or trajectory of the system; in other words, for a given $\mathbf{z}(k)$, $\mathbf{H}$ determines the future values of the state variables, $v(k + 1)$ and $y(k + 1)$.[1]

If $\mathbf{H}$ were to depend explicitly not only on the present state $\mathbf{z}(k)$ but also on some input variables $\mathbf{x}(k)$, the the system equations could be written

$$\mathbf{z}(k + 1) = \mathbf{H}(\mathbf{z}(k), \mathbf{x}(k)), \quad (B.6)$$

and the system would be called *nonautonomous.* This means that the output of the system $\mathbf{z}(k+1)$ is governed by the "rule," "function,"or "mapping" $\mathbf{H}$, which, in turn, is a function of the current values of the state variables $\mathbf{z}(k)$ and various inputs $\mathbf{x}(k)$. These inputs are not specified as part of the system. A block diagram of the nonautonomous system can be given by:

[1]Note for musicians: The vector equation $\mathbf{z}(k + 1) = \mathbf{H}(\mathbf{z}(k))$ is just a short-hand way of writing Eqns. B.3 and B.4. It designates an array of equations, in this case, B.3 and B.4. The left hand side of the vector equation, represented by $\mathbf{z}(k+1)$, is just a more compact way of stipulating the left hand sides of Eqns. B.3-4. The right hand side, $\mathbf{H}(\mathbf{z}(k))$, stands for the right hand sides of Eqns. B.3-4.



(B.7)

If the rocket system had a motor on board, designated $F(k)$, then Eqns. B.3-4 would have to indicate the presence of the "drive" or input, making them into the following nonautonomous equations:

$$v(k+1) = v(k) + a + F(k) \quad (B.8)$$
$$y(k+1) = y(k) + v(k). \quad (B.9)$$

However, our hypothetical rocket system only depends *implicitly* on k due to its not having inputs. As a result, its output $\mathbf{z}(k+1)$ is governed only by the system function $\mathbf{H}$ and $\mathbf{z}(k)$: $\mathbf{H}$ generates the next values for the state variables v and y from their previous values. Thus, the system can be considered *autonomous*.

Now suppose that at $k = 0$, the rocket is fired with velocity $v(0) = 50\ m/s$ from position $y(0) = 0\ m$ with acceleration $a = -10\ m/s^2$. Substituting these initial values when $k = 0$ into Eqns. B.3-4 gives the values of the state variables at $k = 1$ second:

$$\begin{aligned} v(0+1) &= v(0) + a &\Longrightarrow\ v(1) &= 40\ m/s \\ y(0+1) &= y(0) + v(0) &\Longrightarrow\ y(1) &= 50\ m. \end{aligned} \quad (B.10)$$

To find the values of position, velocity and acceleration at $k = 2$ seconds, simply substitute the values already found for $y(1)$ and $v(1)$ into the equations to get:

$$\begin{aligned} v(1+1) &= 40 + (-10) &\Longrightarrow\ v(2) &= 30\ m/s \\ y(1+1) &= 50 + 40 &\Longrightarrow\ y(2) &= 90\ m. \end{aligned} \quad (B.11)$$

Since this system has two state variables, its *order* is TWO. This means that two coordinates are always necessary to specify its "state" at any time. These two coordinates — position and velocity — form the state vector. At each moment in time, this state vector can be plotted in a 2-dimensional space known as the system's *state space*. Each coordinate axis of the state space represents one of the state variables as shown in Figure B-1.

As the rocket shoots upward, its position and velocity coordinates change with time. Figure B-2 displays a plot of the first three points of the state vector which are taken at $k = 0$, $k + 1 = 1$, and $k + 1 = 2$ seconds:

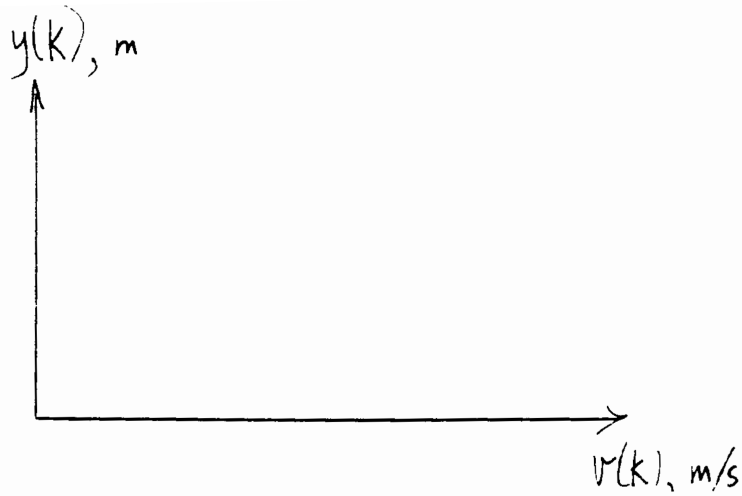


Figure B-1: The two coordinate axes — corresponding to position and velocity — comprising the two-dimensional state space of a hypothetical rocket system.

$$[v(0), y(0)], [v(1), y(1)], [v(2), y(2)]$$

or, more specifically,

$$[50, 0], [40, 50], [30, 90].$$

As time passes and the system evolves, more and more points representing the state vector can be plotted. This sequence of points traces a path in the state space. The path is known as the system *trajectory* or *orbit.* (See Figure B-3.) Each point on the trajectory captures the state of the system at a particular moment in time. Since each orbital point gives the values of v and y at a particular time t, the trajectory is often thought of as "the solution of the system," provided it starts from a designated starting place, that is, from a specified *initial condition.* In general, the system equations generate a family of trajectories, but it is the initial condition that determines which trajectory in the family is applicable. It is imperative to realize that "initial condition" is synonymous with "starting place." An initial condition does not have to start at $t = 0$; rather, the initial condition can be chosen to start at any particular time of interest.

B.2 A Musical Dynamic System

To determine whether a relatively straightforward piece, e.g., a short Chorale by J. S. Bach, could be modeled as a dynamic system, it is first necessary to answer the question: what can be considered a musical state variable? Recall that the state of the rocket system at each second in time was determined by the two state variables, position and velocity. Together, these traced out the system trajectory. What elements in music might depict the state of the "musical trajectory" at each moment or beat in time? Consider the Bach Chorale in G major, *Als Der Gütige Gott,* shown in Figure B-4. The soprano, alto, tenor, and bass lines each have their own melody

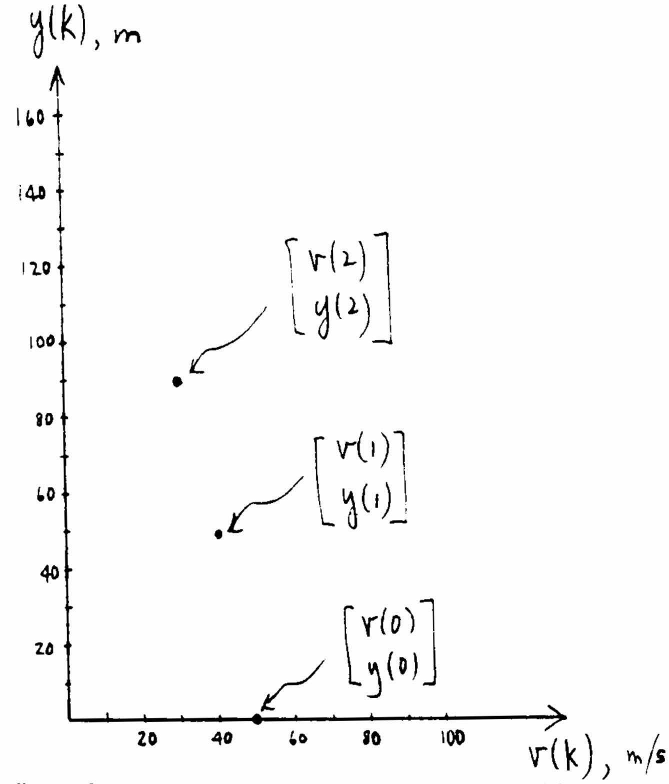


Figure B-2: The first three points of the state vector which correspond to $k = 0$, $k+1 = 1$, and $k+1 = 2$ seconds.

so that four melodies simultaneously sound throughout the Chorale. Each of these melodies consists of two fundamental elements:

1. a set of pitches which will be called the *pitch line*, and
2. a set of rhythmic values, designated the *rhythmic line*, which is attached to those pitches.

The pitch line has to have an attendant rhythmic line since, without duration, a pitch is inaudible.

In the Chorale, the pitch line of the soprano is given by

$$D_3 \mid G_3\ A_3\ B_3\ A_3 \mid G_3\ REST\ A_3 \mid B_3\ \ldots \mid \ldots\ C_4\ B_3\ A_3 \mid G_3\ ,$$

where numeric subscripts indicate the octave in which these pitches occur.[2] The rhythmic line is:

In the simplest dynamic system model that could be developed here, each pitch line represents a state variable whose state at any particular beat is given by the pitch

[2]E.g., C_3 signifies middle C.

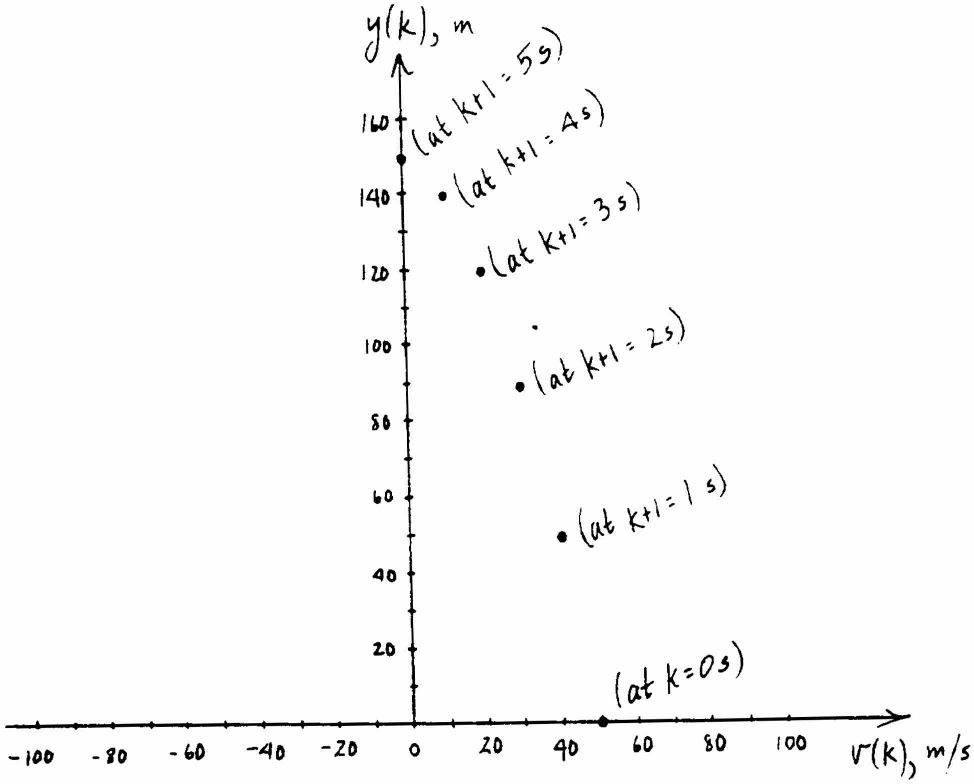


Figure B-3: The set of points comprising the state trajectory for the hypothetical rocket system while it ascends.

Figure B-4: The Bach Chorale in G major, *Als Der Gütige Gott.*

associated with that beat. Similarly, each rhythmic line comprises a state variable whose state at any one time is revealed by the note value at the beat in question. Therefore, eight state variables manifest themselves in the Chorale since each of the voices — soprano, alto, tenor, and bass (**SATB**) — possesses both a pitch line and a rhythmic line. These state variables only partially characterize the system in that they tell us nothing about how they are generated. Knowledge of the state equations is essential for predicting the future values of any state variables. This, and other topics such as autonomous/nonautonomous systems, will be discussed in the following sections.

The musico-dynamic system model being developed here only allows two types of state variables — pitch lines and rhythmic lines. Any number of these can exist, just as any number of capacitor voltages and inductor currents can comprise an n^{th}-order *RLC* circuit. Here, one could argue that tempo, instrumental color and other musical attributes should be named as state variables too. But these will be considered inputs to the system, not state variables in their own right. The pitch line and rhythmic line were chosen as the state variables for this model since, at least in Western Classical music, one and/or the other is always present in a musical work. Timbral shifts, harmonic practice, musical expression markings, tempo and other musical elements are not always present in a score.

Taken together, the state variables comprising the pitch and rhythmic lines of the Bach Chorale can be said to form a state vector. As it changes with time, the state vector defines a trajectory. This trajectory is a series of states resulting from the simultaneous sounding of the soprano, alto, tenor and bass lines, as the beat clicks along. One could hear the simultaneous sounding of the **SATB** lines as the *intersection* of these voices at each beat or subdivision of the beat. In a general dynamic system, the intersection of the state variables at subsequent moments in time traces out the system trajectory. No other symbolic apparatus exists to describe these intersections other than a series of points in the system's state space.

But in tonal music, a short hand method evolved which uses numerals to describe these intersections of simultaneous sounding pitches with their attendant rhythms. At every beat or fraction thereof, an intersection results in a harmonic chord. A sequence of these harmonic chords is known as a harmonic progession. The harmonic progression is encoded by a series of Roman and Arabic numerals which reveal how the "harmony progesses" from beat to beat. The harmonic function of each chord — whether it be tonic (I), supertonic (II), submediant (III), subdominant (IV), dominant (V), mediant (VI), or leading tone (VII) — is notated by a Roman numeral while the intervallic structure of each function is given by Arabic numerals. For example, the harmonic progression of the Bach G major Chorale is given by

$$I \mid vi \ V \ I \ V_7 \mid I \ V \mid I \ IV \ I_4^6 \ V_7 \mid I \mid I_6 \ I \ vii_6 \ I_6 \mid$$

$$V \ V_6/vi \mid vi \ IV_6 \ V_5^6 \ I \ V^7 \mid vi \ V \mid I \ IV_6 \ IV \ I_4^6 \ V_7 \mid I \ \|.$$

In the Chorale, at each beat in time (and sometimes in between beats),[3] the eight pitch and rhythmic state variables sung by the four voices spell out a harmonic chord. This chord represents the state of the Chorale at that particular beat. For instance, the initial state would be encompassed by the pitches of all the voices at the upbeat to measure (m.) 1 as well as the rhythms attached to these voices. So the state at the upbeat would be

D_3(♩); B_2(♩); G_2(♩); G_2(♩).

The intersection of these values gives the first "point" on the Chorale's musical journey or trajectory.

In music, the states are discrete and their number is finite. In the case of the Bach Chorale, these discrete states are "captured" by the harmonic progression. It must be noted here that the harmonic progression contains extra state information, other than the simultaneous sounding of the eight pitch and rhythmic lines. For example, if all that is known about the Chorale is its harmonic progression up to the subdominant (IV), the state vector cannot start there and continue to generate the rest of the progression unless there is a set of state equations capable of taking the current input and producing the future output. The concept of state encompasses everything the system has to remember from the past that is relevant to the future. This means there should be state variables that reveal every memorable aspect of the state, and n^{th}-order state equations that are functions of those n variables. A more precise model would in fact involve a number of state variables that would effectively characterize every aspect of Chorale writing. But then the model would have to be changed if a Sonata were being composed, or even a Chorale by a Baroque composer other than Bach. The model being developed here is one which might sustain different styles and a variety of composers. One way to do this is to require the state variables to be either pitch lines or rhythmic lines, since these are common to virutally all composers of Western Classical music.

Musical state variables are of course quite different from those given in physical systems. In a rocket system, for example, how the position state variable changes with time is completely captured by the velocity state variable, the derivative of position. But in the musical system of the Bach Chorale, how the pitch line (state variable) changes with time, though influenced by the rhythmic line (state variable), is not completely captured by the rhythm. Other factors — some quite intangible — contribute. In the Bach Chorale, the pitch and rhythmic lines cannot be recovered one from the other: the pitch line cannot be generated, given the rhythmic line, and vice versa. In the rocket example, however, this is easily done. The position and velocity state variables can be mathematically derived from each other via the operations of differentiation and integration.

[3] The occurrence of multiple states within the time frame of one beat is dealt with later (during the discussion of Bach's G major Fugue, *WTC II*, in the section on Autonomous/Nonautonomous systems, under the sub-heading, "An Apparent Inconsistency," Section B.3.1.)

B.2.1 State Equation Representations

Recall that state variables were defined as "crucial time-dependent quantities whose future values are determined by substituting their present values into an array of equations governing the system." What could be the "array of equations" governing a musical system? First consider the initial soprano's note as a musical starting place for the soprano melody. Then musical initial conditions for the soprano pitch (P) line and rhythmic (R) line could be designated by:

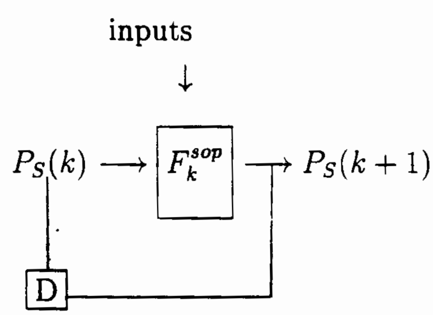
$$P_{soprano}(k=0) \quad = \quad D_3 \qquad \text{(B.12)}$$

$$R_{soprano}(k=0) \quad = \quad ♩\,, \qquad \text{(B.13)}$$

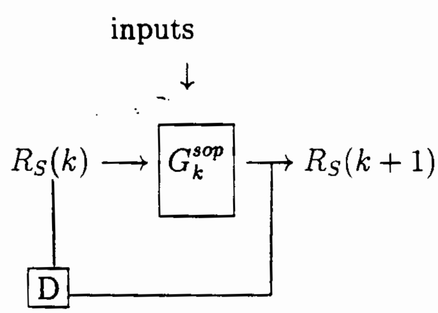
where k denotes time in discrete beats so that the $k = 0^{th}$ beat represents the initial beat of the Chorale.

What determines the second note of the soprano line? In other words, how does the composer select the pitch and rhythm comprising that note from among 12 pitch class possibilities and countless rhythmic choices? (Though many of Bach's Chorale melodies were written by others, this particular soprano melody was composed by him.)

Since each successive soprano note consists of a pitch and a rhythm — $P_S(k+1), R_S(k+1)$ — where P_S and R_S are state variables, we can use the following state equation representation for generating the next soprano note.



(B.14)



(B.15)

The various symbols and terms are now explained:

- $P_S(k)$ and $R_S(k)$ represent the present soprano note in terms of its pitch and rhythm, whereas $P_S(k+1)$ and $R_S(k+1)$ represent the next soprano note in terms of its pitch and rhythm.

- F_k^{sop} represents the musical rules or established principles applied at beat k to the pitch line of the soprano at beat k, e.g., the principles of good voice leading.

- F_k^{sop} is a function of the pitch line (state variable) and any number of inputs, such as style, inspiration, interpretation of the text, the context in which the Chorale appears, etc. Other inputs might be the entire past pitch and rhythmic lines of the bass, tenor and alto voices. However, for those inputs to be available, a much more complex model than the one being adopted here would have to generate all of those pitch and rhythmic lines. (See the Caveat of Section B.2.2.)

- G_k^{sop} represents the musical rules applied at beat k to the rhythmic line of the soprano at beat k. G_k^{sop} is also a function of any number of inputs, as well as a function of the rhythmic line (state variable).

- At beat 0, the upbeat of the Chorale, F_0^{sop} represents the rule applied at beat 0 to the pitch line at beat 0. G_0^{sop} represents the rule applied at beat 0 to the rhythmic line at beat 0.

In general, each F_k or "rule", in conjunction with the k^{th} pitch of the soprano generates the next pitch $P_S(k+1)$, just as each G_k generates the next rhythm. But note here, that no attempt is made to include *all* of the past pitches of the soprano. Just the most recent pitch is taken into account for the generation of the next. This is not how a real-world composer would create such a line: Memory of all past pitches would occur. But because musical memory of past states vastly increases the state space of the system, only first-order representations are given here. The next section deals with an expanding state space in greater detail.

In the Bach Chorale, each F_k^{sop} is different in some subtle and not so subtle ways from beat to beat. For example, the presence of the word "sand"[4] on the upbeat to *m.* 5 might have inspired the choice of the high D for the soprano on the downbeat of *m.* 5 (beat 18) — which also turns out to be the highest note in the Chorale. Yet it would be hard to imagine the word "ins",[5] which is found on the fourth beat of *m.* 8 (beat 33), as carrying much weight in the selection of the soprano's A. So the rules F_{18} and G_{18} might weigh more heavily "word painting", the practice of conveying the meaning of a word to the listener through musical sound, whereas the rules F_{33}, G_{33} might deem the desired harmonic progression more significant than the wording at beat 33, especially since the final cadence is being approached.

On the other hand, the awkward setting of "der gütige" for the alto might indicate that no special attention was given to the accentuation of the text (i.e., no word painting takes place), even though the overall meaning of the text was certainly considered.

---

[4]To send forth, in German.

[5]A contraction for "in das".

In summary, then, any F_k or G_k constitutes a rule. The rule is a function of the state variables and any number of inputs, some of which are more favored than others, depending on

1. where the beat (to which the rule is affixed) resides in the metrical structure of the measure. The metrical structure, for example, plays a significant role in the principles of good voice leading. These principles, treat harmonic practice differently at a downbeat than at a weaker beat.

2. the relative significance of the current beat within the structure of the piece as a whole, since some beats serve as focal points to which groups of beats gravitate.

B.2.2 A Caveat

The state equation representation just outlined for the soprano line of the Bach Chorale describes two first-order systems: one which generates the pitch line of the soprano and the other which generates the rhythmic line. In first-order systems, the next state is determined by the most recent state, not by any or all of the past states. Thus first-order state equation representations rely only on knowledge of the preceding pitch or rhythm — but not on all previous pitches and rhythms — to generate the next pitch or rhythm.

Consider the creation of a single rhythmic line. A first-order state equation representation for the composition of the next rhythm $R(k)$ would depend only on the previous rhythm $R(k-1)$ and the "rule" (with appropriate inputs) generating $R(k)$, given $R(k-1)$. Though this first-order representation would be a critical part of any compositional process, it by no means captures the entire procedure. A case could easily be made contending that every past rhythm of the rhythmic line (state variable) influences the future direction of the piece. One could also argue that every future rhythm can sometimes influence the past direction of the music, as when a composer writes the ending of a work before the beginning has been composed.

Still, if all prior rhythms (of our hypothetical single rhythmic line composition) were said to influence the choice of the present rhythm, it would be possible to expand the state equation representation in order to handle this. But at what cost? To answer this, consider a general discrete-time dynamic system whose present state $x(k)$ depends on its past two states, $x(k-1)$ and $x(k-2)$:

$$x(k) = ax(k-1) + bx(k-2) + c, \tag{B.16}$$

where a, b, and c are parameters. The system implies two first-order equations by the following method. Let

$$\begin{aligned} x_1(k) &= x(k) && \text{(B.17)} \\ x_2(k) &= x(k-1) = x_1(k-1). && \text{(B.18)} \end{aligned}$$

Then the two first-order equations are:

$$x_1(k) \;=\; ax_1(k-1) + bx_2(k-1) + c \tag{B.19}$$

$$x_2(k) = x_1(k-1), \tag{B.20}$$

with x_1 and x_2 designating new state variables, equivalent to the current and most recent state of x.

The implications for our simple rhythmic line are clear: as soon as the state equation representation takes into account the influence of the past two rhythms on the musical selection of the third, the number of first-order systems necessary for describing the process increases, as do the number of state variables. Once these increase, the order of the system becomes higher. Now imagine how many state equation representations would be required to describe the composition of the tenth rhythmic value of the piece — NINE! The dynamic system model would have to absorb the ever expanding state space required by systems with extensive memories because for any composer, all nine previous rhythms would influence selection of the tenth. Furthermore, as pointed out earlier, sometimes the conclusion of a musical work (or of a single phrase) is written before the body is completed, in which case the process or system could be called *anticipatory* — that is, the future behavior of the system influences earlier behavior. If this occurred during the composition of the piece, the number of state equation representations would grow yet again. In short, it is both impractical and probably impossible to completely codify a composer's thought process as s/he embarks upon a piece.

Yet of all the thought processes possible, it is a safe bet that knowledge of the preceding pitch enters into the determination of the next pitch, and that an awareness of the most recent rhythmic value influences the selection of the next rhythm. These are the two first-order processes which might hold from composer to composer, and therefore form a representation of the state process. The alternative is an ever expanding, cumbersome set of state representations which, even if one could design and implement all of them, would hold little credence from composer to composer. For each composer approaches the artistic act differently — not only with respect to her/his contemporaries, but also from piece to piece.

What is proposed, then, is an array of first-order state equation representations, similar in form to those articulated for the Bach Chorale, that captures one fundamental aspect of the compositional process, yet avoids the morass and probable folly of trying to account for every possible stroke underlying an infinitely creative and subjective art. The purpose of such a model is to reasonably capture a representation of reality. How "reasonably" depends on the proposed use for the model. More to the point, what is the purpose behind modeling a musical work as a dynamic system? As stated earlier, two questions have been posed: is the language of nonlinear dynamics and chaos at all inherently musical; and why look to nonlinear dynamics and chaos for new musical structures? These questions are examined more closely in a later appendix, *Musicality in the Language of Nonlinear Dynamics and Chaos*, but first it is necessary to establish that an appreciable set of musical analogs corresponds to dynamic system concepts, since nonlinear and chaotic systems are themselves dynamic systems.

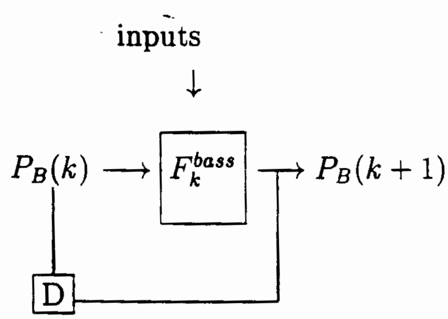
Thus, the model being developed here has to be good enough to establish a set of musical and dynamic system analogs that mesh with intuition, at least in spirit, if not always by letter. That they do not correspond in all cases exactly, has just been shown with the state equation representation for a musical work.

B.2.3 Back to Bach

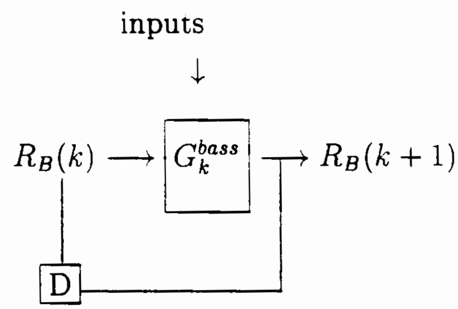
After writing the soprano line, Bach probably shifted his attention to composing a bass line that would support the soprano. The first note of the bass line is often the tonic note of the melody's key. However, in which octave that tonic is placed will depend on how close or far apart from each other the composer wishes the inner voices (the alto and tenor voices) to sound. Also, vocal range and the desired harmonic progression will influence the choice of octave. Selection of the second bass note is influenced by the previous bass pitch $P_B(k)$ and rhythm $R_B(k)$. The second bass note, and succeeding bass notes, are chosen according to a variety of inputs, as well as the rule F_k^{bass} which might include, among other guidelines, such well-established principles as:

- writing a bass line that provides good support to the melody since the bass line dictates the functional harmony;
- moving the bass line in contrary motion to the soprano as much as possible to enhance the independence of these two voices;
- maintaining the independence of the voices by avoiding parallel fifths and octaves with the soprano voice, as well as direct fifths and octaves;
- complementing the rhythmic line of the upper voice(s); and
- all of the principles F_k^{sop} which entered into writing the soprano melody. Some of the above guidelines may also necessitate other inputs. For example, in order to write a bass line that provides good support to the melody, the entire soprano line must be an input to F_k^{bass}, as well as to G_k^{bass}. Again, this would necessitate building a much more complex model capable of generating every pitch of the soprano line, given all past pitches.

The above are encompassed in the rules F_k^{bass} and G_k^{bass}. Together with the present states $P_B(k)$ and $R_B(k)$ and the various inputs, they give a first-order state equation representation which outputs the next bass pitch $P_B(k+1)$ and rhythm $R_B(k+1)$:



(B.21)



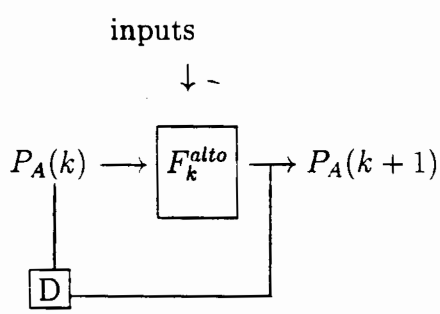
(B.22)

Finally, how does the composer choose the inner voices? The same principles which governed the composition of the soprano and bass also guide the choice of pitches and rhythms for the tenor and alto lines. These are the "filler" lines and usually are kept moving in a smooth way. A state equation representation might generate each successive pitch $P_A(k+1)$ and rhythm $R_A(k+1)$ contained in the alto line and each successive pitch $P_T(k+1)$ and rhythm $R_T(k+1)$ included in the tenor line, by adding the following principles to the system rule, in addition to those already given for generating the soprano line:

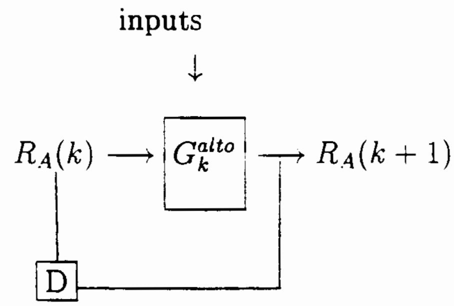
- have due regard for the desirability of having predominantly contrary motion in the texture;
- maintain the independence of the voices by avoiding parallel fifths and octaves with any other voice, as well as direct fifths and octaves;
- complement the pitch line of the other voices; and
- complement the rhythmic line of the other voices.

Again, the above can necessitate a number of extra inputs, such as knowledge of the entire pitch and rhythmic lines of the bass and soprano.

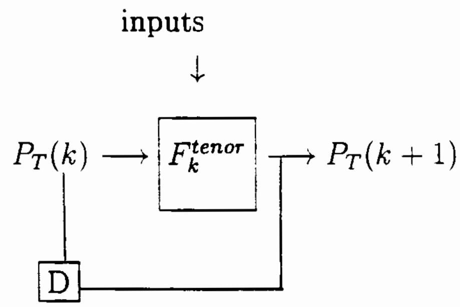
Here, then, are the state equation representations that generate the alto and tenor lines:



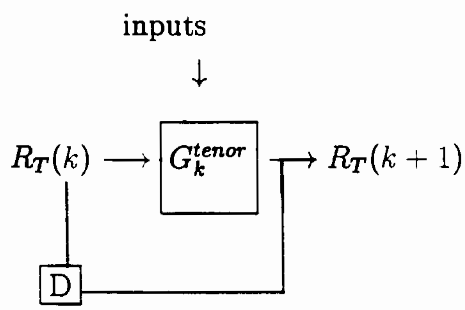
(B.23)



(B.24)



(B.25)



(B.26)

At first glance, it might appear that the principles embedded in F_k^{alto} and G_k^{alto} of B.23–B.24, are the same as those in F_k^{tenor} and G_k^{tenor} of B.25–B.26. However, vocal range and voice leading can vary depending on the voice. Also, the inputs given in the state representations for the alto and tenor voices are different, since various aesthetic concerns can be weighted differently from beat to beat, and also can be of lesser or greater significance from voice to voice.

All the considerations discussed above suggest that the equations propelling each state variable from one state to another in the rocket system, or any dynamic system, can be interpreted in the context of a musical dynamic system as a set of representations which include musical state variables, the rules F_k and G_k, and various inputs.

Fundamentally, each voice's pitch line and rhythmic line develop on the basis of what pitch and rhythm came before, according to the musical principles of the era included in F_k, G_k and the individual aesthetics of the composer — the inputs. The eight state equation representations B.14–B.15, B.21–B.22, and B.23–B.26, are first-order attempts to generate each of the four voices:

$$S\left\{\begin{array}{llll} p(0) = D_3\,; & p(1) = G_3\,; & p(2) = A_3\,; & p(3) = B_3 \;\ldots \\ r(0) = \text{♩}; & r(1) = \text{♩}; & r(2) = \text{♩}; & r(3) = \text{♩} \;\ldots \end{array}\right\},$$

$$A\left\{\begin{array}{llll} p(0) = B_2\,; & p(1) = E_3\,, G_3\,; & p(2) = G_3\,, F\sharp_3\,; & p(3) = G_3\,, F\sharp_3\,, E_3 \;\ldots \\ r(0) = \text{♩}\,; & r(1) = \text{♪}, \text{♪}; & r(2) = \text{♪}, \text{♪}; & r(3) = \text{♪}, \text{♪}, \text{♪} \ldots \end{array}\right\},$$

$$T\left\{\begin{array}{llll} p(0) = G_2\,; & p(1) = B_2\,; & p(2) = D_3\,; & p(3) = D_3 \;\ldots \\ r(0) = \text{♩}; & r(1) = \text{♩}\,; & r(2) = \text{♩}\,; & r(3) = \text{♩} \;\ldots \end{array}\right\},$$

$$B\left\{\begin{array}{llll} p(0) = G_2\,; & p(1) = E_2\,; & p(2) = D_2\,; & p(3) = G_2 \;\ldots \\ r(0) = \text{♩}; & r(1) = \text{♩}\,; & r(2) = \text{♩}\,; & r(3) = \text{♩} \;\ldots \end{array}\right\}.$$

The occurrence of multiple states within the time frame of one beat. are dealt with later (during the discussion of Bach's G major Fugue, *WTC II*, in the section on Autonomous/Nonautonomous systems, under the sub-heading, "An Apparent Inconsistency", Section B.3.1.)

Finally, the first four states of what can be considered a musical state vector can be depicted as:

At $k = 0$: At $k = 1$: At $k = 2$: At $k = 3$:

$$\begin{bmatrix} p(0) = D_3 \\ r(0) = \text{♩} \\ \\ p(0) = B_2 \\ r(0) = \text{♩} \\ \\ p(0) = G_2 \\ r(0) = \text{♩} \\ \\ p(0) = G_2 \\ r(0) = \text{♩} \end{bmatrix} \begin{bmatrix} p(1) = G_3 \\ r(1) = \text{♩} \\ \\ p(1) = E_3\,, G_3 \\ r(1) = \text{♪}, \text{♪} \\ \\ p(1) = B_2 \\ r(1) = \text{♩} \\ \\ p(1) = E_2 \\ r(1) = \text{♩} \end{bmatrix} \begin{bmatrix} p(2) = A_3 \\ r(2) = \text{♩} \\ \\ p(2) = G_3\,, F\sharp_3 \\ r(2) = \text{♪}, \text{♪} \\ \\ p(2) = D_3 \\ r(2) = \text{♩} \\ \\ p(2) = D_2 \\ r(2) = \text{♩} \end{bmatrix} \begin{bmatrix} p(3) = B_3 \\ r(3) = \text{♩} \\ \\ p(3) = G_3\,, F\sharp_3\,, E_3 \\ r(3) = \text{♪}, \text{♪}, \text{♪} \\ \\ p(3) = D_3 \\ r(3) = \text{♩} \\ \\ p(3) = G_2 \\ r(3) = \text{♩} \end{bmatrix},$$

with the ensuing states similarly displayed. It is this musical state vector that pinpoints the harmonic state at each beat, producing the harmonic progression. a musical analog of the system trajectory.

In the Bach Chorale, the number of unique states is finite, as it is in Western Classical music. Furthermore, each musical state occurs in a discrete amount of time within the context of the musical line. This is not the case with the rocket example, where each state is found to occur *at* a particular time k, rather than occupying an amount of time. In a musical work, each state has duration, as it must: A sound has to have duration in order to be perceived.

B.3 Autonomous (free-running) vs. Nonautonomous Musical Systems

One last consideration regarding the Bach Chorale: can the Bach Chorale be considered an autonomous or nonautonomous system? Recall that for the rocket system, the state variables, $y(k)$ and $v(k)$ were each functions of discrete time k. Thus, the velocity of the rocket at $k = 1$ was quite different from its value at $k = 5$ The same can be said for its position. State equations B.1–B.4 depend directly on the state variables, v and y, but only indirectly on k. The rule $\mathbf{H}$ which generates the next values of the state variables v and y is not a function of inputs, making the rocket example an autonomous system. However, in the Bach Chorale, as the beat clicks along — as $k = \textit{beat } i$, $(i = 1, 2, 3, ..., n)$ — the rule generating the next pitch and next rhythm is a function of various inputs, some of which are weighted more heavily than others, depending on which beat is at hand. Since inputs are present for each beat of the Chorale, *Als Der Gütige Gott* may be viewed as nonautonomous.

Virtually all musical works can be considered nonautonomous. The reason is apparent: With an autonomous system, if G_k generates a rhythm already played or sung, the rhythm must repeat itself. Why? Because the rule for generating any successive rhythm does not take into account any inputs. For example, suppose

$R(0) =$ ♩, $R(1) =$ ♪, $R(2) =$ ♩, $R(3) =$ ♪.

Assuming an autonomous, order one system such that $G_0 = G_1 = G_2 = \cdots = G_n$, a rule G_k generating this sequence of rhythms could be stated as

> G_k: *if the previous rhythm is a quarter note, make the present rhythm an eighth, and vice versa.*

Since $R(3) = R(1)$, and because $G_0 = G_1 = G_2 = \cdots = G_n$, then $R(4)$ must equal $R(0) =$ ♩ and $R(5) = R(3) = R(1) =$ ♪, with the rhythmic motive ♩♪ repeating itself until the final beat n. Another simple example of an autonomous musical system would be a steady drum beat.

However, with nonautonomous musical systems, the rule is a function of the inputs, so that if a prior state is reached, the piece does not necessarily follow a loop, but rather employs the rule with the applicable inputs to generate a future state — a state which may or may not replicate a prior state. One of many illustrations of this occurs in the second and sixth measures of Bach's *Fugue in G Major* from the *Well-tempered Clavier, Book II (WTC II)*:

Here. $P(4) = B_3$ where $P(4)$ is the downbeat of m. 2. The same state is reached on the downbeat of m. 6 where $P(16) = B_3$, but what ensues in m. 6 bears no resemblance to m. 2. If the system were autonomous, the pitches following $P(16)$ would be constrained to equal those following $P(4)$.

B.3.1 An Apparent Inconsistency

Multiple states appear in the Bach example above as well as in the Bach Chorale discussed earlier. Specifically, if we consider each discrete beat k equivalent to the beat indicated by the bottom number of the 3/8 time signature in the previous fugal subject, then two states can exist between beats four and five: B_3 is the state at beat four and G_3 is the state at beat five, but there exists another state, D_4, in between the two.[6] This is tantamount to saying that beat four exhibits two different states — something which is not allowed in dynamic system theory. Imagine what chaos would unfold if a rocket system displayed two different states at $k = 0$, e.g., (1) $v = 50\ m/s$, $y = 0\ m$ and (2) $v = 40\ m/s$, $y = 0\ m$.

In dynamic system theory, if an analyst found that two different states exist by the time the system passes from one discrete time interval Δk to another, the system analyst would reduce Δk by a suitable amount to capture each state separately and thereby preserve the integrity of the state equations. Certainly, we could do the same here: just stipulate that each k is a sixteenth note away from the previous k, thereby making Δk equivalent to a sixteenth. Then the downbeat of m. 2 would be the seventh beat of the piece and $P(7) = B_3$, $P(8) = D_4$, and $P(9) = G_3$, thus ensuring two states are not assigned to the same time interval in the state equation representation. Musically, however, this is not a good approach. It counters the natural inclination of musicians to perform, analyze and compose with the metrical structure of the bar always in mind. For this reason, the model allows two or more states per beat, unless otherwise stated, as seen in the musical state vector which closes Section B.2.3.

B.4 Summary

A dynamic model for music has been proposed which involves only two kinds of state variables — pitch lines and rhythmic lines. Thus, the model has a reduced state.

[6] As before, the subscripts indicate the octave in which these pitches occur.

This is because pitch lines and rhythmic lines are found in virtually every Western musical composition, whereas other musical elements such as instrumental color, musical expression markings, tempo indications, are not necessarily present. At least with regard to state variables, then, the model does not have to be re-worked for each new piece being considered. Those musical elements not included as state variables can be viewed as inputs to the piece (or system). Inputs would also include inspiration, style, context for the work, aesthetics, and so on.

State equation representations are given which only capture a small part of the process — generating the next note from the previous one — in conjunction with a rule (musical principles) and number of relevant inputs. The art of music-making is exactly that — an art. It cannot be codified. There are just too many possible inputs, coupled with reliance on past (and future) states. The use of inputs to characterize much of the artistic side of music-making captures the individual nature of each piece and suggests composition as a nonautonomous process.

Appendix C

The Dynamic System Concept of Order in the Context of a Musical Work

Species counterpoint provides the backdrop against which n^{th} order musical systems are constructed. From a dynamic system standpoint, order is equivalent to the number of independent state variables present. From a musical point of view, order might be said to depict the number of separate pitch or rhythmic lines, fragments, or accents, heard in a piece. In a broad sense, order is *dynamic*, i.e., it can change with time, as evident in the fugues of Bach, the symphonies and sonatas of Haydn, Mozart and Beethoven. *Dynamic order* is especially relevant to the piano literature and can be shown via a method for quantifying order — the *dynamic order discrete time (DODT)* graph. *DODT* graphs can be applied to a number of works from the Baroque, Classic and Romantic periods, as a way of quantifying their textural thinness and thickness.

C.1 Order in a Musical System

The order of a dynamic system is determined by the number of independent state variables. The musico-dynamic model developed in Appendix B specifies only two types of musical state variables — pitch lines and rhythmic lines. The model has a reduced *state*, intentionally, so that the model can apply to a number of pieces in the Western Classical tradition, where pitch and rhythmic lines are present in virtually all works, whereas other musical attributes, such as expression markings, are not. Recall the Bach Chorale analyzed in Section B.2 has eight musical state variables, consisting of four different pitch lines accompanied by four separate rhythmic lines. The ORDER of the piece is EIGHT. But the number of pitch and rhythmic lines varies from one work to another. Seventh century Gregorian chant, consists of a

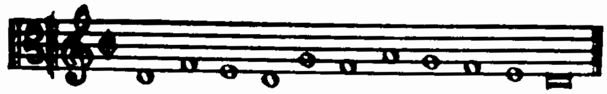


Figure C-1: An example of a cantus firmus from Johann Joseph Fux's *Gradus ad Parnassum.*

single melodic line representing only a pitch line and a rhythmic line.[1] Consequently, the order of Gregorian chant is TWO. The state of a Gregorian musical system at any point in time is equivalent to the intersection of the pitch line, i.e., equivalent to which pitch is currently being sung, and the rhythmic line, i.e., which rhythm is affixed to the current pitch. The state vector in this "second-order system" contains only two components — the pitch line and the rhythmic line.

In contrast, the state vector of any four-part Bach Chorale consists of eight state variables which define a harmonic trajectory (or progression) designating the state of the system (or state of the harmony) at any beat in time. In the Chorale, each state variable changes state as the system evolves, thereby running through its own set of values. For example, the soprano's pitch line changes state every time the soprano sings a new pitch; her rhythmic line changes state each time she sings a new rhythmic value. However, unlike the soprano line, the Chorale contains more than two state variables and its state is determined by the simultaneous effect of all the pitches and rhythms at each beat in time. Because of this, the harmonic progression of the Chorale is a natural way to depict its musical state.

The state space of a first-order system is one-dimensional: a line. The trajectory hops from rhythm to rhythm, or point to point on the line. A musical piece consisting of a series of rhythms for a wood block would constitute such a first-order piece.

A medieval *cantus firmus*[2] represents an example of a musical second-order system. Like the Gregorian chant it emulates, the cantus firmus relies on only two state variables — a pitch line and a rhythmic line. A simple cantus firmus from Johann Joseph Fux' *Gradus ad Parnassum (1725)* is given in Figure C-1.[3]

The previous cantus firmus example could be represented by a set of points in a plane, where the letter names along the x or pitch axis signify pitches, the note values along the y or rhythmic axis denote durations or rhythms, and the circled numbers above the points determine the actual sequence of notes. (See Figure C-2.) Because certain notes are repeated, yet proceed to other notes without looping, this cantus firmus is another example of a nonautonomous musical piece.

However, representing the line by a staff gives a far more concise picture. It is now possible to designate the sequence of the notes by simply assigning them a place

[1]Though the actual rhythm of that rhythmic line is debated by musicologists, the choice of rhythm for every performance must still be made since no music can exist without duration.

[2]A cantus firmus is a single melodic line, often based on the note sequence of a Gregorian chant.

[3]Mann, Alfred (tr. and ed.) with John Edmunds (1971) *The Study of Counterpoint from Johann Joseph Fux's Gradus ad Parnassum* (Norton, New York) 27.

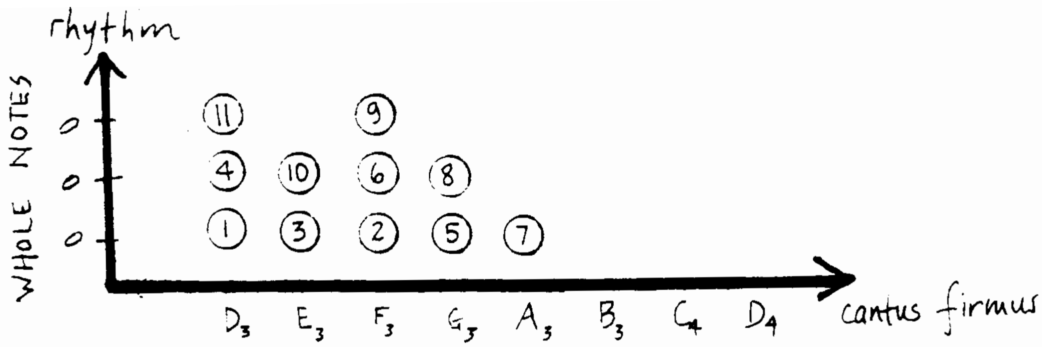


Figure C-2: A second-order musical state space represented by an x axis of pitches and a y axis of rhythms (here, whole notes, because the only rhythm in the given cantus firmus is the whole note). The circled numbers 1-11 indicate the sequential order of the notes.

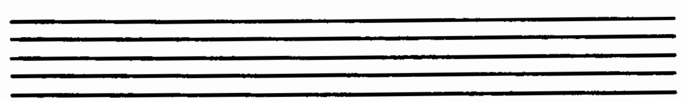


Figure C-3: A second-order musical state space can be thought of as a staff on which pitch and rhythm is stipulated.

on the staff. This automatically systematizes them, since the staff is read sequentially left to right, thus eliminating the need for the numbers used in Figure C-2 to convey temporal arrangement. The staff, then, can be viewed as a two-dimensional state space equivalent of the pitch-rhythm plane, as displayed in Figure C-3.

As a dynamic system example, the state of a second-order circuit could be described by two independent capacitor voltages, each of which functions as a state variable. An example of a second-order circuit is displayed in Figure C-4. The circuit's phase space consists of a plane spanned by the two state variables, V_{C1} and V_{C2}; its trajectory is given by the oval curve in Figure C-5.

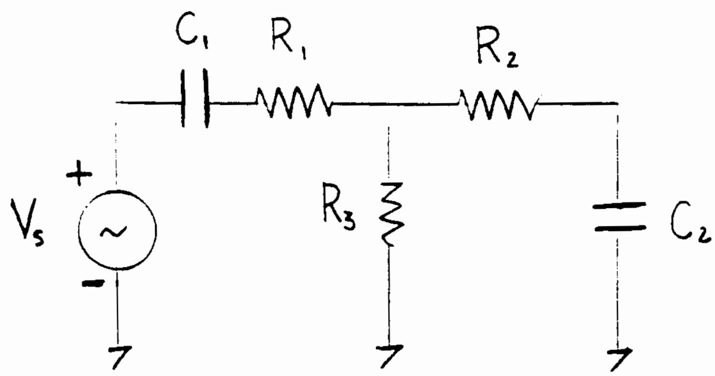


Figure C-4: A second-order circuit consisting of two capacitors, three resistors, and a voltage source.

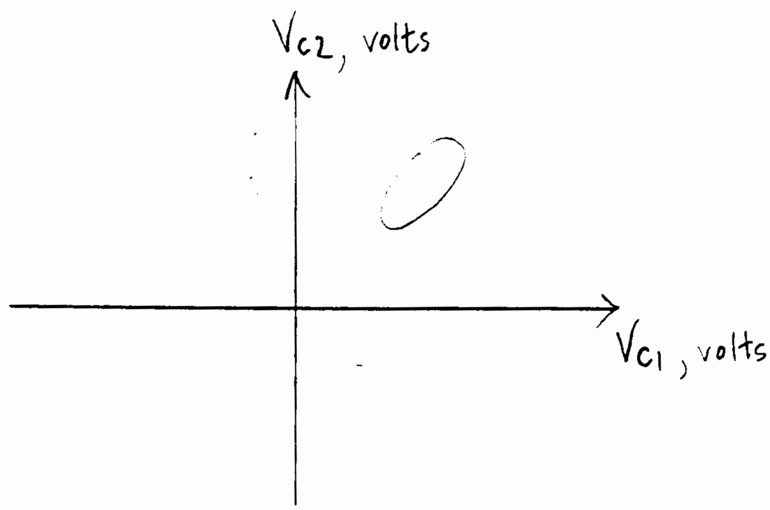


Figure C-5: A state space trajectory associated with a second-order circuit.

Figure C-6: First species counterpoint as an example of a fourth order musical system in which the four musical state variables comprise (1) the cantus firmus (c.f.) and (2) the melody composed to "counter" the cantus firmus. (Reprinted from Mann, p. 29.)

C.2 Order and Species Counterpoint

First species counterpoint, which plots a note-against-note melody to the cantus firmus, gives a fourth order system with two pitch lines and two rhythmic lines, as shown in Figure C-6. The "state" of the musical system can be thought of as the intervallic relationship between the two voices: fifth — third — third — fifth — third — fifth — third — third — sixth — sixth — octave. Therefore, the state of first species counterpoint is given by the number of pitch and rhythmic lines, each of which depicts a "musical state" variable.

With first species counterpoint, the "four"-dimensional musical state space forms two intersecting planes where the line of intersection is a rhythmic axis of whole notes. Because the two planes intersect along the same rhythmic axis (due to both the *cantus firmus* and the *counterpoint* sharing the same rhythmic value — the whole note), the musical space is actually three-dimensional. The horizontal axis represents the pitches of the cantus firmus and the vertical axis represents those of the composed melody which is to "counter" the cantus firmus. The numbers specify the sequence of "paired notes," as seen in Figure C-7. But the temporal order of first species counterpoint is more clearly revealed by the staff notation given in Figure C-8.

C.3 Dynamic Order and the Fugues of Bach

The differentiation of melodic lines which Fux tried to teach in his *Gradus ad Parnassum* culminated in the three-, four- and five-voice fugues of Bach. The minimal order for a three-voice fugue is six: three pitch line state variables coupled with three rhythmic line state variables. Four- and five-voice fugues would have minimal orders of eight and ten, respectively.

By distinguishing lines both rhythmically and with respect to pitch, composers increased the individuality of these melodies. Again, this can be shown via a state space representation using pitch on one axis and rhythm on another. Then the following implication, depicted in Figure C-9, arises: namely, the idea that any staff with pitch and an attendant rhythm represents a pitch-rhythmic "curve" in musical state space with minimal order of two. This is shown with the subject of Bach's *Fugue in*

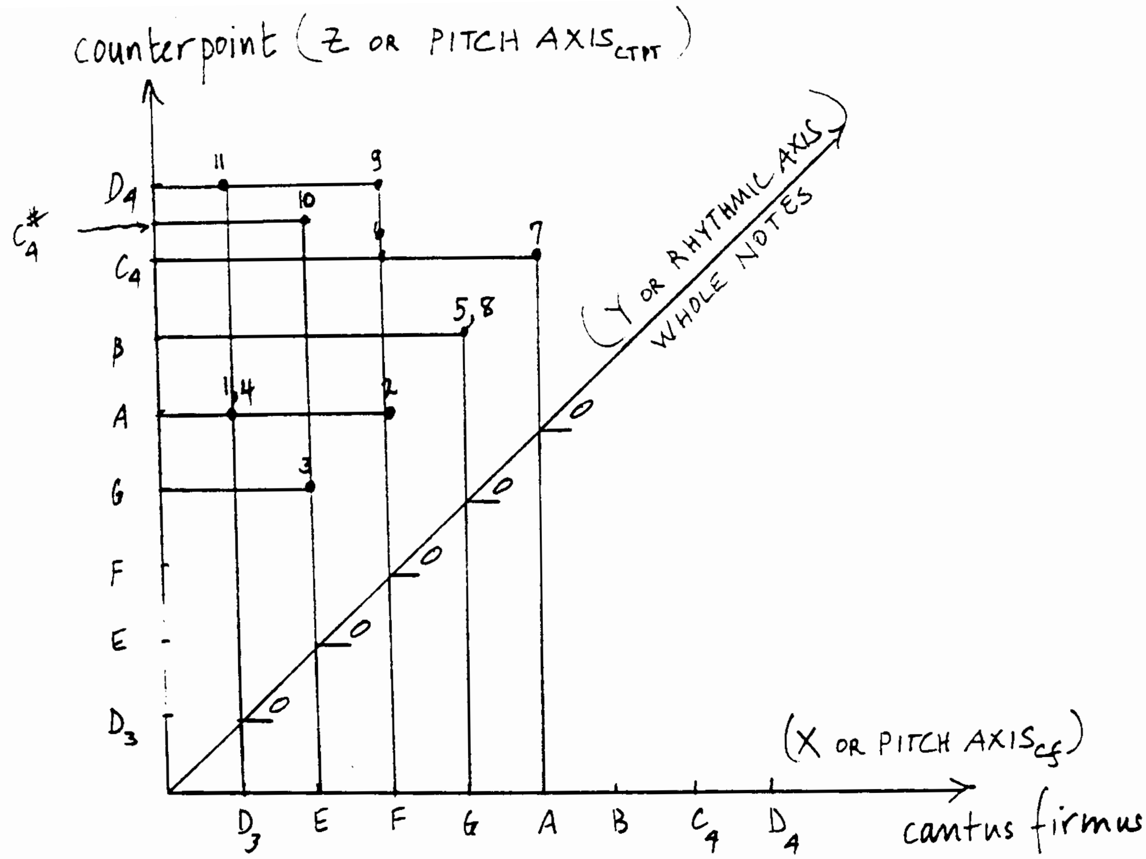


Figure C-7: A musical state space representation of first species counterpoint. Each point on the rhythmic axis denotes a whole note. The $x - y$ plane represents the *cantus firmus* and the $z - y$ plane depicts the *counterpoint*.

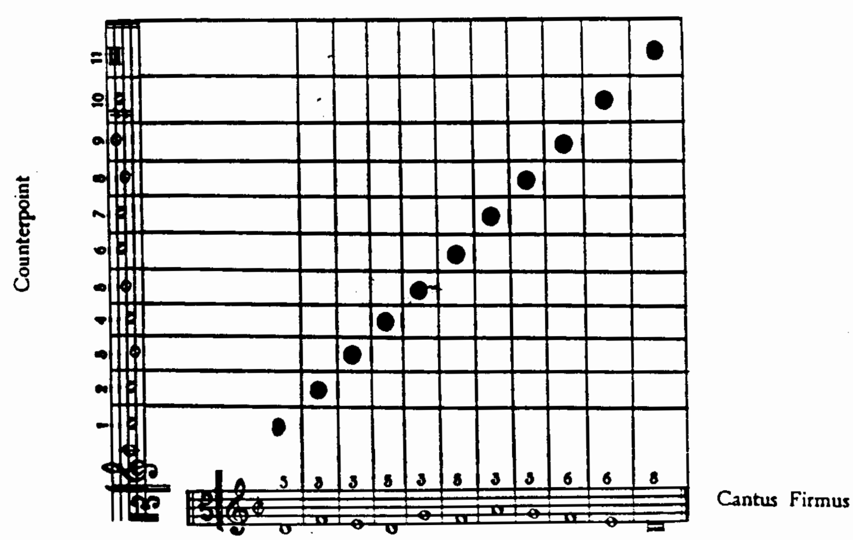


Figure C-8: A musical state space representation of first species counterpoint using musical staves to represent the two intersecting planes given in Figure C-7.

b♭ minor from the *Well-tempered Clavier, Book I (WTC I)*. (See Figure C-9.)

Note that a minimal order of two was designated for a staff conveying both pitch line and rhythmic line state variables. This order for the staff has no upper limit, at least theoretically, since two or more pitch/rhythmic lines could certainly occupy one staff. To cite but one example, consider the soprano introduction of the subject in Bach's *Fugue in f minor, WTC II* shown on just one staff in *mm.* 1-4 of Figure C-10. However, in *mm.* 4-9 the alto voice enters on the same staff as the soprano, increasing the order of the musical state space (i.e., the staff) to FOUR. Note that the musical state space for the soprano and alto is 4-dimensional. Since neither voice shares the same rhythm, their two planes do not intersect along a rhythmic axis as did the two planes given by the first species counterpoint example of Figure C-7.

Unlike the vast majority of Bach Chorales where order remains constant throughout the Chorale, the order in the Bach fugues is truly dynamic — it changes as time passes. Every fugue starts out with a minimal order of TWO, associated with the first statement of the subject. As each voice enters with the subject, the order of the fugue grows. After all voices have stated the subject, thus concluding what is called the *exposition* of the fugue, an *episode* ensues whose function is often to move the harmonic progression away from the tonic to a related key. Sometimes the episode "thins" out the fugal texture by incorporating fewer voices than the number present by the close of the exposition. Other times, it maintains the order set up by the end of the exposition, but leads to a section where the order drops. The important point is that the order of any Bach fugue will change with time, probably many times before the end of the work. The order stays dynamic by (1) adding voices, (2) simply omitting voices, (3) implying more than one voice, and/or (4) creating accents. All of these allow room for interpretation.

Recall the statement that "every fugue starts out with a minimal order of TWO." That is, the order of the opening fugal theme is at least TWO. Could it be FOUR? Consider the three-voice fugue which concludes Bach's *Toccata and Fugue in e minor*. The subject, first stated by the alto voice, is given below:

This subject, however, implies two melodic lines — given by the alternation of strong and weak sixteenth notes — each possessing its own pitch line and associated

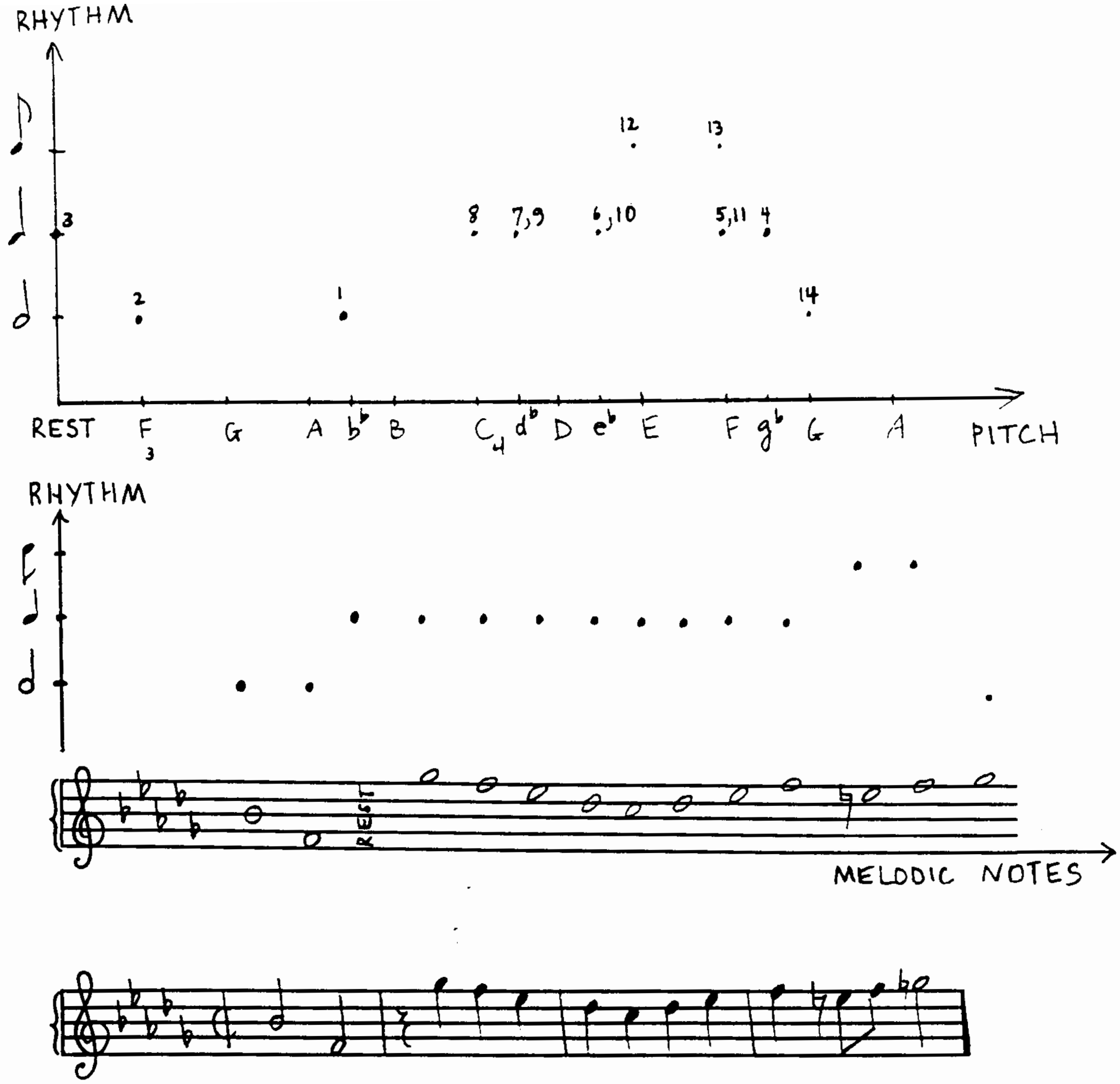
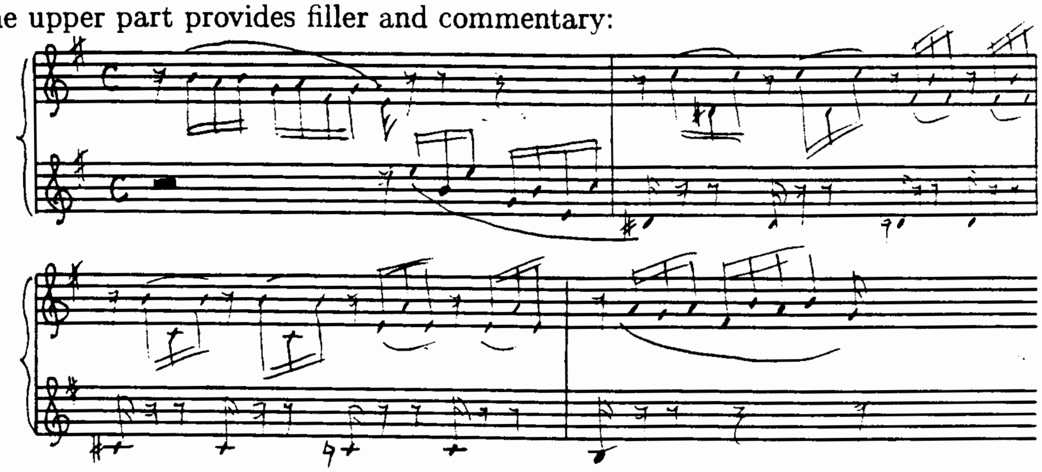


Figure C-9: (1) *Top*, State space depiction of a second-order musical theme with pitch and rhythm serving as the state variables and therefore forming the coordinate axes. (2) *Middle*, Replacing the horizontal pitch axis with staff notation shows the temporal arrangement of the notes. (3) *Bottom*, Finally, representing the second-order musical system by a staff with pitch and rhythm designated by the appropriate symbols, simplifies the notation.

Figure C-10: Any staff where notes and note values are specified has a minimal order of two. Here, *mm.* 4-9 of Bach's *Fugue in f minor, WTC II* show two voices, each of which possesses a pitch line and a rhythmic line, thus indicating measures with order FOUR.

rhythmic line. The strong sixteenths provide the structural line (which actually begins on the first weak one), while the other sixteenths are ornamental.

The theme could also be interpreted as a "question and answer" set of two brief phrases (*m.* 1). Then in *mm.* 2-4, the lower voice stresses the structural line, while the upper part provides filler and commentary:



In either case, the order of *mm.* $1-4$ is FOUR, although one could also hear the second example as having order EIGHT (*m.* 1), followed by SIX (*mm.* 2-4):

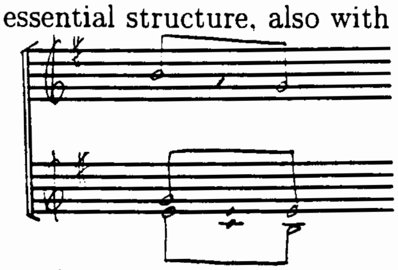
Finally, when the bass enters with the subject in *m.* 10, the fugue attains an order of EIGHT (or TWELVE, given the above example), since the alto and soprano each have order TWO and the bass contributes two (or four) more melodic lines, each of order TWO:

Each of the above three interpretations could be called a foreground* analysis of the theme, since each is quite close in notation to the actual musical score. The <u>absolute</u> foreground would be exactly notated as in the written score. There can be any number of background* (or middle ground*) levels for analysis, depending on the amount of information to be conveyed, the structural relationships inherent to the passage, and the amount of detail desired.[4] Heinrich Schenker*, the German theorist who first proposed layer analysis, referred to the background as the "strata underneath the surface [of the musical work in question]." In a revealing quote, he explains how his layer analysis was in a sense inevitable:

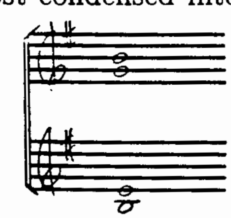
> All religious experience, all philosophy and science strives toward the most concise formula. It was a similar impulse that led me to conceive of a musical composition as emanating from the core of a fundamental structure, as the first composing-out of the basic triad (tonality). I *observed* the fundamental line (*Urlinie)*; I did not deduce it.[5]

For Schenker and a legion of musicians since, a layer analysis offers a means of "tracing foreground relationships to their origin in the background and middleground."[6] So, for example, a more middle ground analysis of the subject of the *Fugue in e minor* reveals yet another interpretation of the theme, this time having an order of SIX:

A background analysis results in still another interpretation of the theme, this time stripped down to its more essential structure, also with an order of SIX:



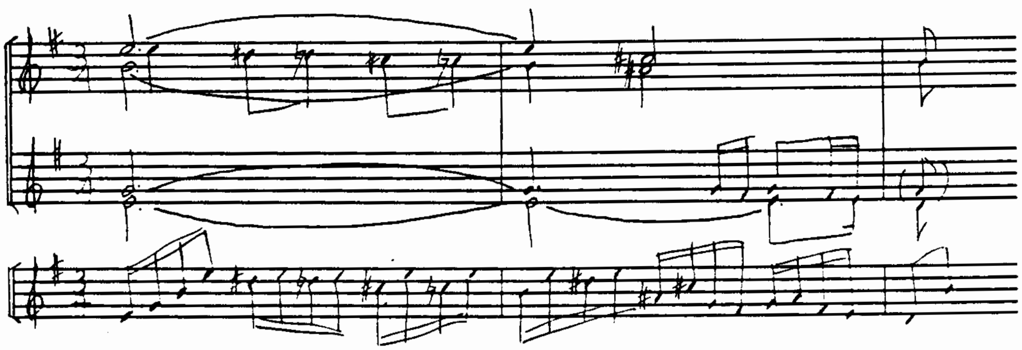
The absolute background[7] is the most condensed interpretation of the theme, without any further reduction possible:



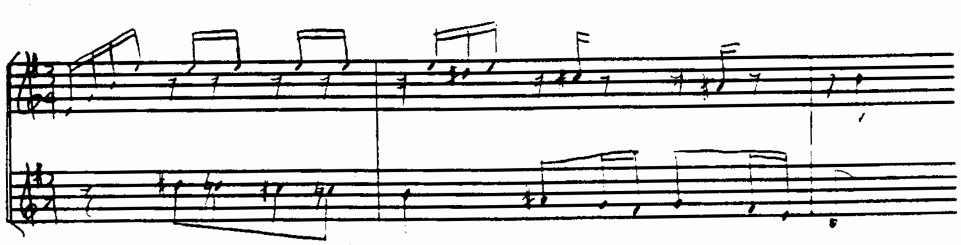
[4]Warfield, Gerald (1976) *Layer Analysis* (McKay Co., New York) 24.
[5]Jonas, Oswald; John Rothgeb, tr. (1937, 1982) *Schenkerian Analysis* (Longman, New York) 129.
[6]Jonas, p. 147.
[7]Warfield, p. 24.

Yet a more inlaid fugal subject opens the *Fugue in e minor, WTC I*:

Though this fugue is the only two-voiced fugue out of 48 in the *Well-tempered Clavier*, its subject hints at five separate melodic lines. The five implied voices written below arise from an analysis of the *e* minor fugue which would be termed middle ground. The upper $D\sharp$ is not an independent voice. It serves first as a passing tone from the high E to $D\natural$ in *m*. 1, then as an ornamental neighbor for the high E in *m*. 2. Perhaps Bach, sensing a challenge in the two-voiced fugue, sought to maximize the perceived texture by embedding five "voices" in the soprano's — and later the bass's — introduction of the theme. If he had treated the two-voiced fugue in more pedestrian fashion, without playing on the listener's ability to grasp the implied multiple lines, the minimal order of the subject would have been two. Instead, it is TEN in the middle ground analysis given below. (For comparison, the original fugal subject is written underneath the five implied voices.)



A more foreground analysis of the same theme shows an order of FOUR:



Recall that order has been defined as the number of independent musical lines (not counting doublings). Musical lines might include melodic lines, melodic fragments, or even accents, where accent is understood as any means of making a given musical event stand out. Accent is synonymous with contrast: it depends on, and is the result of contrast.[8] The order of a piece is not determined by simply counting pitch lines and rhythmic lines in the original score. Rather, the order of a musical passage (of any length) depends on the decisions made in producing an analysis. As shown above, a foreground analysis of the 2-voice Bach fugal theme led to an order number of FOUR, while a middle ground analysis suggested an order of TEN. This is because the middle ground analysis implies the simultaneity of voices by the filling in of "space." In each of these cases, decisions — interpretative decisions — are made. Some of these decisions are clear cut, while others require far more thought. Clearly, the methodology chosen for analysis must be stated — and consistently applied — for the order number to have any meaning. This is also crucial if the order of two different works is to be compared.

What these fugal examples show is that the order found in a musical work is not simply equal to two times the number of voices, though this is closer to being true for the part-writing of Renaissance composers who relied mainly on the human voice to convey their musical intent. Vocal writing doesn't imply that a Schenkerian analysis on the middle ground, similar to those given for the two *e* minor fugues, will not exist. Rather, a middle ground Schenkerian analysis, which often observes an underlying chordal structure, is just not as obvious for vocal writing as it is for instrumental writing. In instrumental writing, arpeggios and other chord-like figuration occur much more frequently.

The Baroque era was captivated with instrumental part-writing. Instruments were able to do things that the voice couldn't do as easily, such as arpeggios, scales and chords. For instance, both of the previously mentioned *e* minor fugal themes, though very difficult to sing, are easily played on a keyboard. Furthermore, each can give the illusion of two or more voices singing, rather than one. Though Bach extolled the *cantabile* or singing style in his Preface to the *Two-part Inventions*, he as well as his contemporaries realized how much more unfettered their imagination could be with instrumental writing. Not only could instruments imitate the singing, expressive quality of the voice, but also they displayed formidable range and virtuosity unattainable by even the greatest of singers. As a result, Bach's *Chaconne* for solo violin or his unaccompanied suites for cello take advantage of register shifts, double stops, and other devices not feasible for a singer, in order to convey more than a single melodic line. Of course, this is one of the fascinations with music for unaccompanied stringed instruments — just how does the composer give the illusion of several interweaving parts with only one instrument at his or her disposal? In these cases, the concept of order, as understood here, can be a far subtler aspect of music composed after 1700 than what was evidenced by Renaissance composers.

[8] Imbrie, Andrew (1973) "Extra" measures and metrical ambiguity in Beethoven. In Alan Tyson, ed. *Beethoven Studies, Vol. II* (Norton, New York) 52.

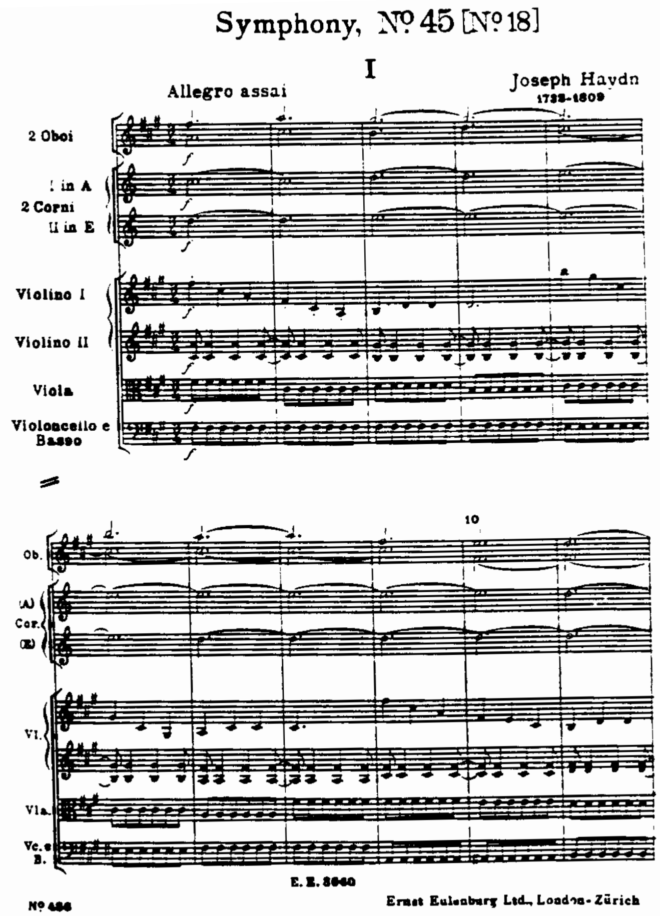


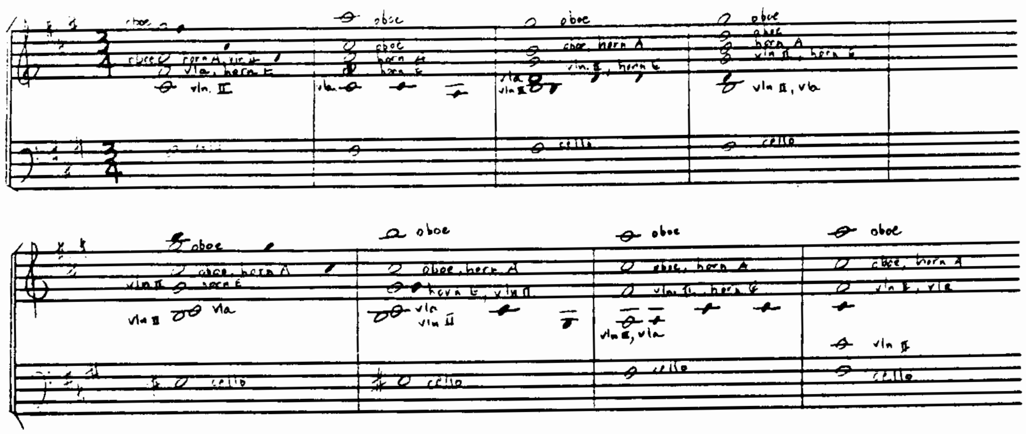
Figure C-11: The beginning of Haydn's *Farewell* Symphony shows the orchestration he employs. The order of the opening (assuming an absolute foreground analysis) is EIGHTEEN, not SIXTEEN, due to the sixths present in the second violin part.

C.4 Dynamic Order and the Classical Symphony

Carrying this idea of order to the symphonies of Mozart, Haydn and Beethoven, one might mistakenly deduce the order of, say, Haydn's *Symphony No. 45 in f♯ minor (Farewell)* by noting that it is scored for two oboes, two horns, first violins, second violins, viola, cello, doublebass, and bassoon. But the bassoon, which only plays in the fourth movement, is relegated to sharing the same line as the cello and doublebass, except for its final 4 measures. Consequently, only one pitch line and one associated rhythmic line exist for these three instruments together, at least until the very end. Assuming only single stops on the stringed instruments, the order of this symphony would appear to be SIXTEEN. But this is not the case, as the opening clearly shows in Figure C-11: the double stops of the second violin part increases the order of these first measures to EIGHTEEN, assuming an absolute foreground analysis.

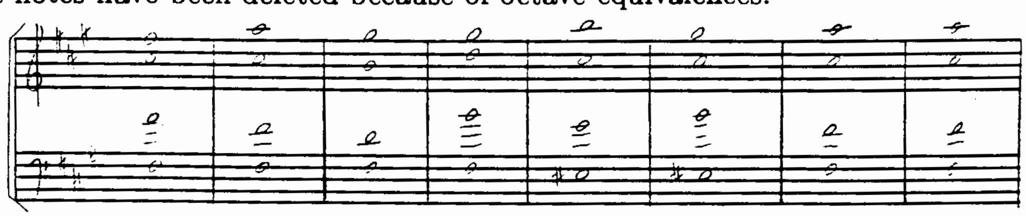
In fact, the maximum order, assuming an absolute foreground level, found in the *Farewell* symphony is order TWENTY-SIX! This occurs in *m.* 47 of the Finale, where quadruple and triple stops in the second violin and viola, respectively, combine with all available instrumental lines.

However, as with the *e* minor fugues, a distinction has to be made between literally sounding voices as performed by various instruments, and underlying structural contrapuntal parts. As shown in the example below, the first violin part (represented by the filled-in note heads) outlines five structural parts, which are reinforced by other instruments. The $F\sharp 4$ doubles the first oboe, the $C\sharp 4$ anticipates the second oboe of *m.* 2, thus filling in "space" for that $C\sharp 4$ in *m.* 1. The second oboe, horn in *A* and second violin all double the *A*3 of the first violin. This same *A*3 anticipates and fills in space for the tied *A*3 of the horn in *A* in *m.* 2. The first violin's $F\sharp 3$ (downbeat of *m.* 2) and the following $C\sharp 3$ are doubled by the horn in *E* and the viola, respectively. What is happening here is that the first violin part fills in the space necessary for the upper voices to sound the tonic chord over several registers. This happens again for the dominant chord in *mm.* 5-6.



Now, the order of the opening eight measures, according to the above middle ground analysis, actually varies from measure-to-measure, whereas in the absolute foreground, the order was consistently EIGHTEEN. Given the middle ground interpretation, *m.* 1 has order TWELVE: each pitch and rhythmic line of the sustained voices plus the pitch and rhythmic line of the descending first violin part. The second measure has order FOURTEEN, due to the addition of $C\sharp 4$, played by the second oboe. This $C\sharp 4$ would be an example of an accent — here, a color accent, where the space filled in by the first violin's $C\sharp 4$ of the previous measure is now taken over by the oboe. Measure 3 has order FOURTEEN, for similar reasons as those given for *m.* 1. Measure 4 has order FOURTEEN, because the second oboe adds a *D*4 to the ii^4_2 chord, creating timbral contrast, and therefore an accent.

A further reduction can be made, such as the background analysis given below, where notes have been deleted because of octave equivalences:



This reduction essentially gives the structural framework on which all the previous analyses hang. Yet it completely loses the sense of dynamic texture, i.e., the changing landscape of "thinness" and "thickness", conveyed by the earlier analyses. There is no way to tell from this reduction how the musical texture is conveyed: Foreground and middle ground analyses yield more textural information.

Throughout the *f♯ minor Symphony*, the order changes as the various pitch and rhythmic lines are silenced and then brought back again. This "thickening" of the score or "thinning" — not by instrumental doublings which affect orchestral color — but by the addition or removal of extra pitch/rhythmic lines, fragments and accents, contributes much to the dynamic, i.e., changing, character of the music. And of course, in the *Farewell* Symphony, the ultimate cessation of musical state variables takes place — the order decreases (on the absolute foreground level) from a high of TWENTY-SIX in the Finale to only FOUR by the close of the piece. As the background story goes, Prince Esterhazy's summer residence of Esterház could get quite crowded with invited guests. The last thing he wanted was to increase the palace pandemonium by including families of the hired musicians as his wards. So except for the families of Haydn, the first violinist Tomasini, and the singers Fribert and Dichtler, no other musicians could have their wives and children living with them while the Prince sojourned at Esterház. Unfortunately during the summer of 1772, the Prince prolonged his stay beyond summer and far into autumn. With that came even more longing on the part of his musicians for some meaningful contact with their families. So they approached Haydn, asking him to intercede on their behalf. Haydn, however, knew his Prince and realized that spoken intervention would not prompt the desired result. He had a better idea. His orchestra that summer consisted of two oboes, four horns, 6 violins, 1 viola, 1 cello, 1 doublebass, and 1 bassoon.[9] He wrote a four-movement symphony lasting about 25 minutes that captivated the mood of his musicians. The final movement couldn't maintain its impetus and exuberant mood, especially since Haydn injected an *Adagio* at the end. During this graceful, *dolce* episode, the Prince witnessed an unusual state of affairs. One of the two oboists and two of the horns ceased playing, quietly blew out their candles and left the stage. The others continued playing, but not for long before the bassoon departed. Seven bars later, the second oboe stopped playing, immediately followed by two more horns. Now only the strings remained. Soon, however, the doublebass, cello, and viola completed their parts and extinguished their candles. The violins played on, gradually diminishing in number until only Tomasini and a second violinist were left. They quietly concluded the movement. The last candles were snuffed, and Haydn followed in silence.

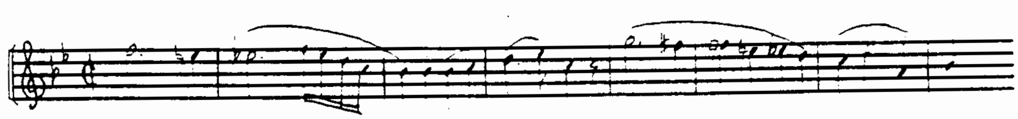
Figure C-12 shows the last 26 measures of the *Farewell* Symphony. Here, in one small snapshot can be seen the change in order, resulting from the gradual cessation of play by the musicians. It shows concisely just how dynamic the order of a symphony can be. It is possible to convey this regardless of whether the analysis proceeds on a foreground or middle ground level.

[9]Praetorius, Ernst (Ed.) (1936) *Symphony Nr. 45 in f♯ minor (Farewell)* by Joseph Haydn (Eulenburg, London) 1.

Figure C-12: The last 26 measures of Haydn's *Farewell* symphony show a clear reduction in order, as well as a clever way to get a Prince's attention.

C.4.1 Distinguishing Order from Orchestration

It is imperative to separate the concept of order from orchestration, though the latter certainly plays a role in discerning the former. But often, the two have nothing to do with one another. Consider the second theme of Mozart's *Symphony No. 40 in g minor, K. 550*, which first appears in *m.*44:



Mozart orchestrates the theme by assigning its opening measure with the ensuing downbeat to the first and second violins (doubled an octave apart) and to the violas and celli (primarily doubled at the unison.) The second measure is taken by the bassoon and clarinet, playing in unison until the following downbeat, at which point the strings re-enter and conclude the phrase:

Now, if orchestration were synonymous with order, *m.* 44 would be order EIGHT, followed by *mm.* 45 – 46 (order TWELVE, due to six instruments sounding on the downbeats), and *mm.* 47 – 50 (order EIGHT.) But since order is defined as the number of ***independent*** pitch and rhythmic lines, fragments or accents, *m.* 44 and *mm.* 48 – 50 have order SIX, not EIGHT, while the order of *mm.* 45 – 46 is EIGHT, followed by TEN, due to the $B\flat 3$ played by the clarinet and bassoon on the downbeat of *m.* 46. The word "independent" is used in the sense that doublings at the unison and/or octave do not count as separate lines. Rather, these doublings contribute to the orchestral color of the symphony, but do not increase the order because they do not pos-

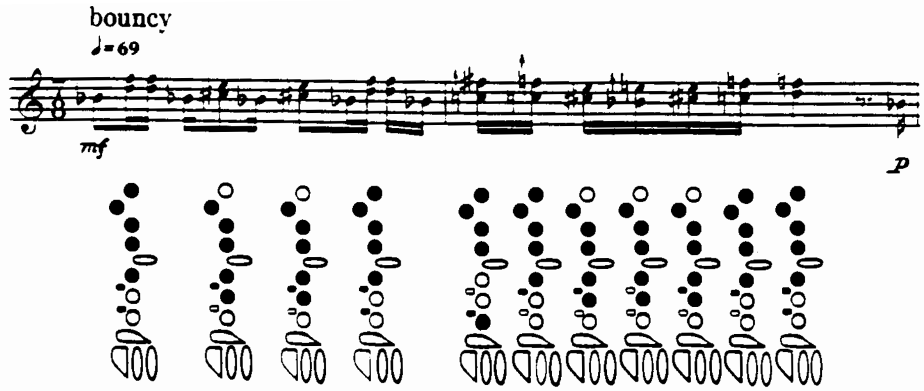


Figure C-13: Robert Dick example of multiphonics. The diagrams below the notes show the fingering patterns for the flute key bed.

sess pitches AND rhythms different from the doubled line.

C.5 Dynamic Order and the Piano Literature

Dynamic order — order that changes with time — exhibits itself in most music of the past 300 years. "Most" because a solo woodwind piece would more likely have a fixed order of two: the two musical state variables being the pitch line and its attendant rhythmic line, at least on a foreground level. In such a case, the order is not dynamic; it does not change with time. Actually, the order of much flute music is fixed at $n = 2$ until this century when flutists and composers became fascinated with "multiphonics," the means by which a flutist can sound more than one note simultaneously, thus opening the door further on multi-line compositions. Much of the pioneer work in multiphonics for the flute has been done by the flutist/composer Robert Dick. Figure C-13 shows the opening of the fifth etude from his *Flying Lessons*, a series of six contemporary etudes for solo flute.[10]

The piano, by comparison, from its inception engendered a repertoire of music whose minimal order, considering each piece as a whole, was FOUR, instead of the more usual TWO, typically evidenced by the human voice and early wind and brass instruments. That is, at a minimum, a meaningful piece for the piano entails a left hand part with a pitch line and attendant rhythmic line, as well as a right hand part with another pitch line and rhythmic line.

Furthermore, while the order evidenced in pre-twentieth century music for the human voice, brass and wind instruments often does not change, the order found in piano music changes frequently. This means a piece might start out with order TWO, like the first measure of Chopin's *Etude in c♯ minor, Op. 25 No. 7*, yet shift to order

[10]Dick, Robert (1984) *Flying Lessons: Six Contemporary Etudes for Solo Flute* (Multiple Breath Music Company, New York) 14.

Figure C-14: The Chopin *Etude in c♯ minor, Op. 25 No. 7* opens with order TWO (given by the melodic and rhythmic line of *m.* 1), but quickly changes to order SIX by the next measure, subsequently shifting to order EIGHT by *m.* 3.

SIX in the next measure and order EIGHT by *m.* 3, assuming a foreground or middle ground analysis. (See Figure C-14.)

It is this dynamic order that so distinguishes piano music from that of other instruments, making it sound as if the piano were indeed an orchestra. For the piano can glide from the wistfulness of a single flute-like line (order TWO) to an ensemble of parts (order FOUR and above), so that any good pianist will delineate and orchestrate what is being played by two (or fewer!) hands. The composers of the eighteenth and nineteenth centuries were undoubtedly aware of the piano's capability for what we have described as dynamic order, though there appears no such term. This is evidenced by their frequent transcriptions of chamber and symphonic works for solo piano. Some of them actually wrote out piano scores of their orchestral works and then orchestrated the piano scores. A case in point is the Brahms' *Two-Piano Sonata in d minor (1854)* which eventually became his first piano concerto, also in *d* minor.[11]

Perhaps the best way, then, to ascertain the dynamic order of Haydn, Mozart and Beethoven piano sonatas is to discern the pitch and rhythmic lines found in the piano score, particularly if one hand is playing more than one voice. For example, Figure C-15 shows the opening of the third movement from Beethoven's *Sonata in C Major, Op. 53*, where the left hand is responsible for both the movement's bass line as well as its melody, whereas the right hand provides an obbligato.

The left hand contributes four separate lines — the pitch and rhythmic lines of the bass, with sustaining pedal, in addition to the pitch and rhythmic lines of the melody:

By including the right hand obbligato, the order of the first bars is TWELVE since

[11]Musgrave, Michael (1985) *The Music of Brahms* (Routledge and Kegan Paul, London) 120.

Figure C-15: Third movement of Beethoven's *Sonata in C major, Op. 53*, where the left hand plays both the bass line and the melody while the right hand provides a harmonic obbligato.

the right hand actually outlines the following four parts, each possessing its own pitch line and associated rhythmic line:

Though the pianist might imagine the left hand bass line as a cello, the left hand melody line a clarinet, and the right hand obbligato a viola — the order given by the eight pitch/rhythmic lines arising from the right hand part would not be crystallized by imagining the orchestration. Furthermore, the effect of the pedal would be very hard to create using only a cello, viola and clarinet. These are but two reasons why the order of a musical work cannot be determined by how it might be orchestrated. Orchestral thinking, however, sometimes helps in discerning the order.

C.6 Dynamic Order Discrete Time Graphs

To generalize still further, assessing the order of a musical work can provide a blueprint for the amount of vertical and horizontal texture embedded in the piece, i.e., how thick, thin or delineated is the texture. One way to do this might lie in the construction of a discrete-time graph where the horizontal axis is marked off in measure numbers (or beats within measure numbers) and the vertical axis indicates the degree of order, i.e., whether a measure or beat is considered first order or higher. As a result, it is possible to plot the dynamic (or changing) order which occurs throughout a musical work. Furthermore, because these graphs are discrete-time graphs, they can be easily entered into a computer. Then programs can be written to sift and sort through a number of these dynamic order graphs in order to compare, for example, dynamic order in Haydn's symphonies with that found in his piano sonatas of the same period. It would also be possible to assess just how texturally composers wrote their respective piano sonatas and to establish whether there exists any correlation between how a composer matured as an orchestrator and how he or she developed as a composer of solo piano music, with respect to texture.

How might a *dynamic order discrete-time* (*DODT*) graph be employed? The first step is to decide whether order will be considered on a foreground or middle ground level. (The background level may be too far removed, or too distilled, to clarify the textural thinness or thickness.) The foreground and middle ground analyses will always be related in that each is either an elaboration (what Schenker referred to as a "prolongation") or reduction of the other.

C.6.1 *DODT* graphs based on foreground analysis.

The exposition of the Bach *Fugue in e minor, WTC I* can be given the foreground analysis of Figure C-16. The *DODT* graph for this two-voiced fugue, constructed according to the above foreground analysis, is shown in Figure C-17.

Figure C-16: Foreground analysis of the exposition of Bach's *Fugue in e minor, WTC I.*

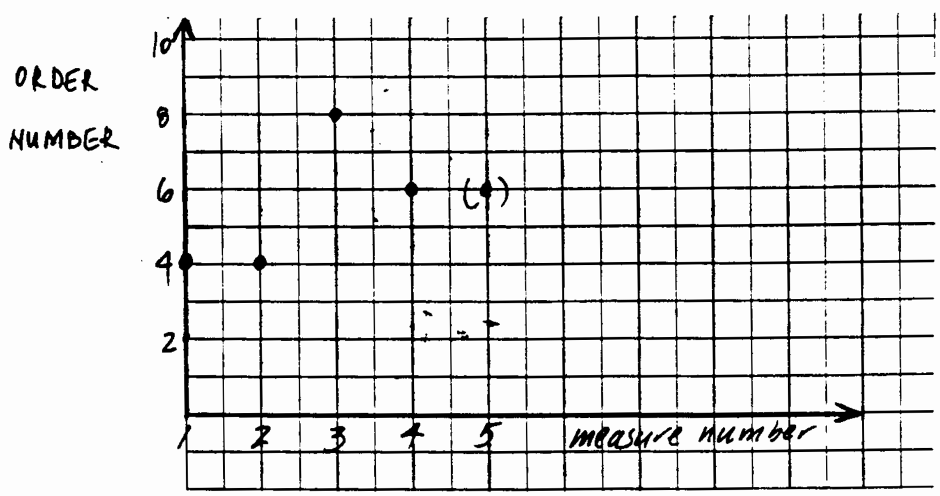
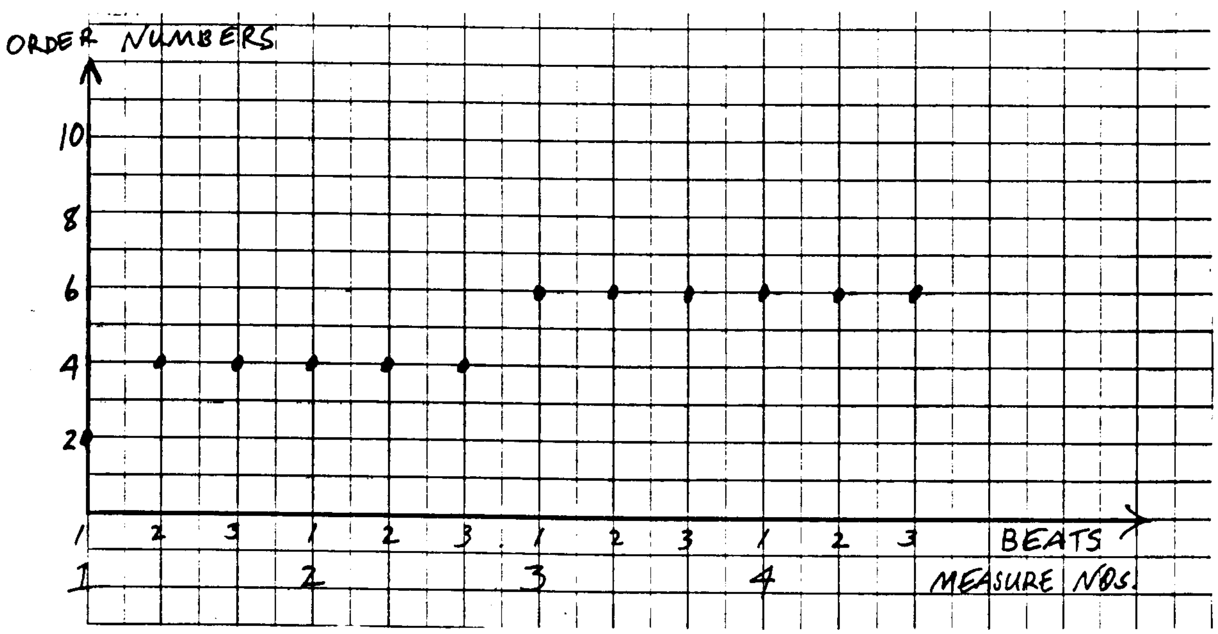


Figure C-17: *DODT* graph of the exposition of the Bach *Fugue in e minor, WTC I* based on the foreground analysis of Figure C-16.

For the exposition of the three-voice *Fugue in e minor* from the *Toccata and Fugue in e minor*, the foreground analysis of Figure C-18, similar in methodology to that constructed for the two-voice fugue, results in the *DODT* graph of Figure C-19.

Notice that by *m.* 10, when the bass enters, that the order has increased to TEN, from the previous EIGHT (downbeat of *m.* 9). Yet the same structural point — the entrance of the bass — in the two-voice fugue in *e* minor discussed earlier, has an order of TEN, showing that — at the entrance of the bass — a fugue of only two voices can imply as many textural lines as a three-voice fugue. Also, the jump from order FOUR (*mm.* 1-2) to TEN (*m.* 3) in the two-voice fugue provides a stronger contrast in polyphonic texture than does the corresponding moment in the three-voice fugue.

It is also possible to construct a *DODT* graph that quantifies the vertical and horizontal texture from beat-to-beat, rather than from measure-to-measure. This is shown with the two-voice fugal theme in *e* minor:



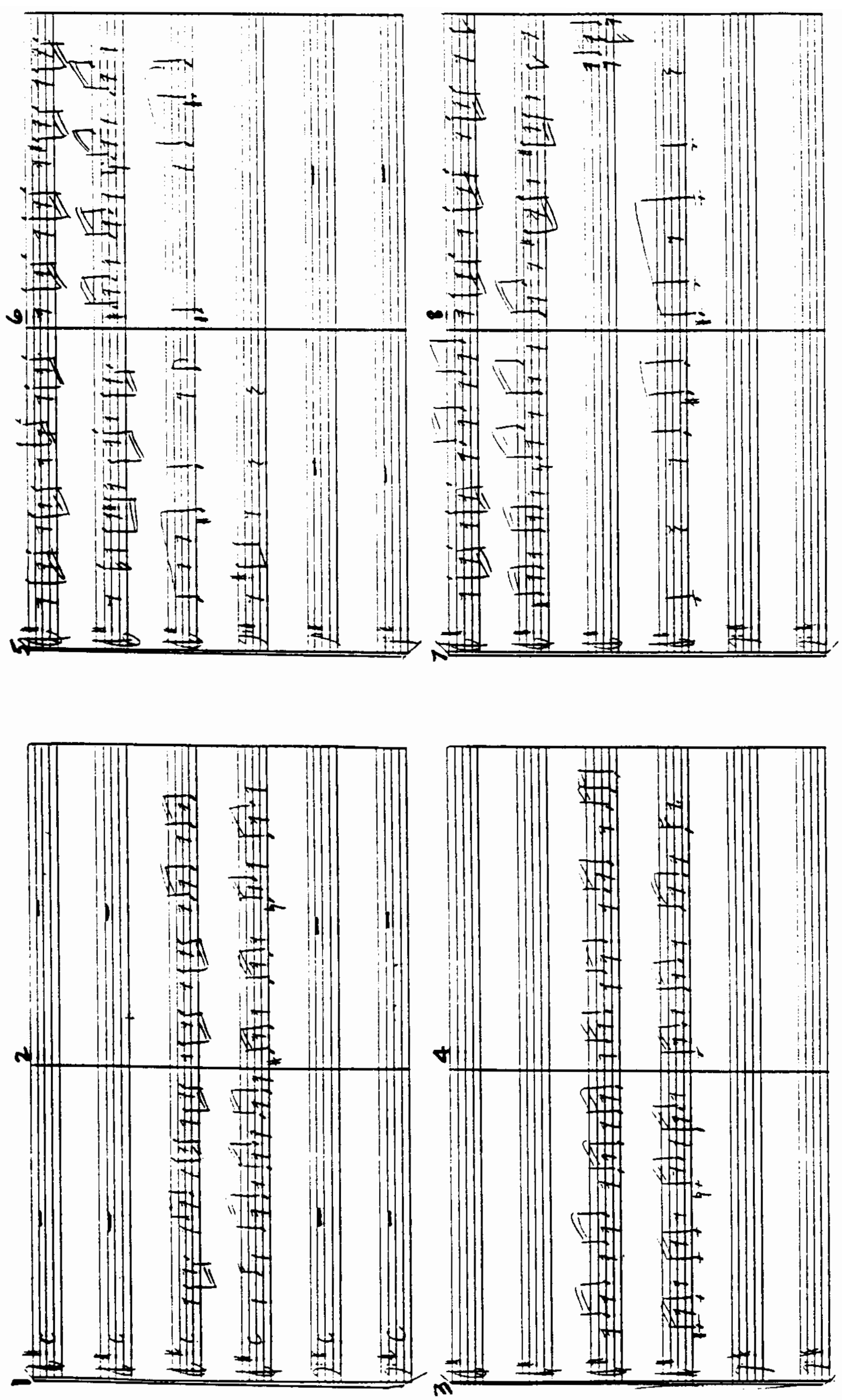
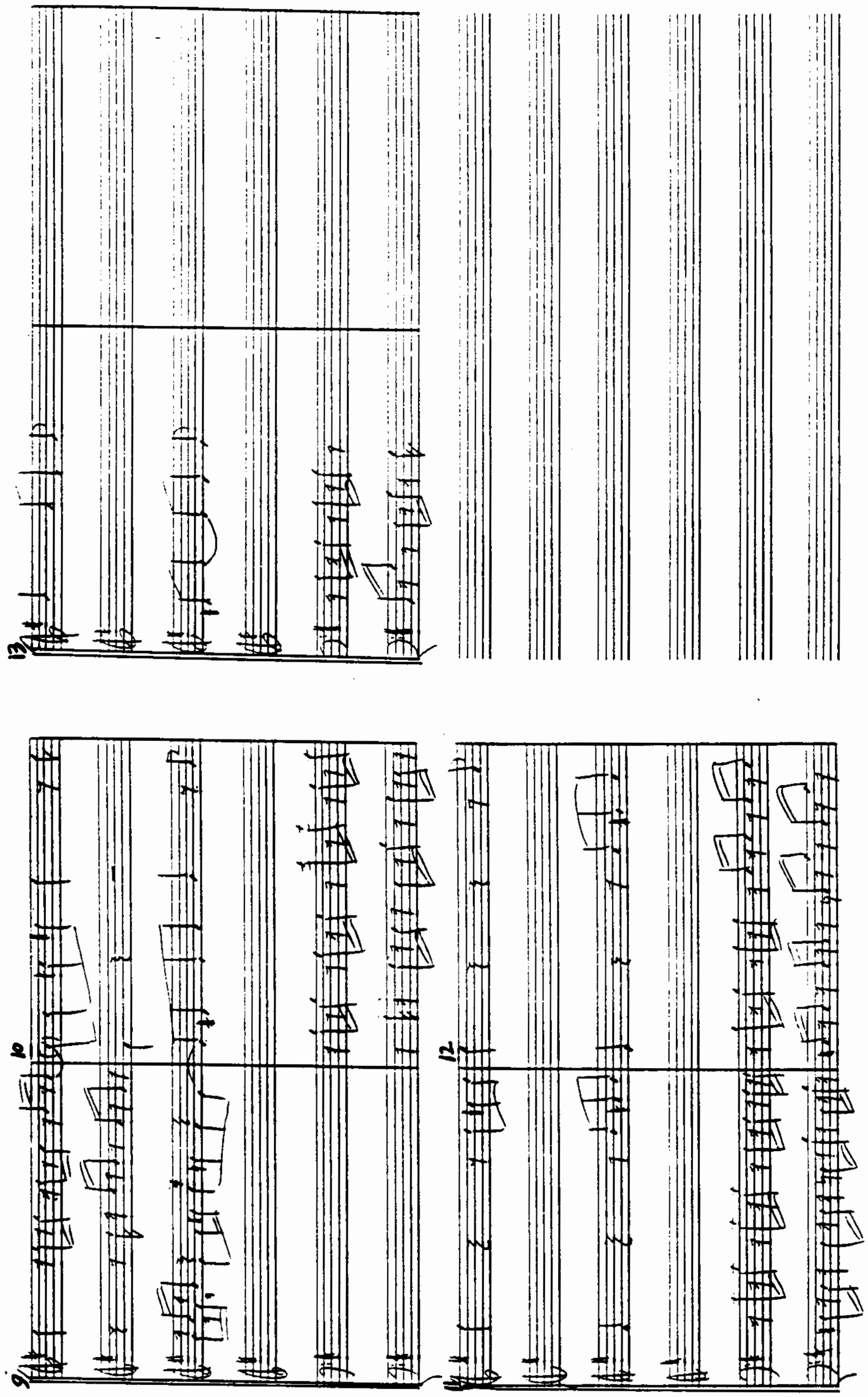


Figure C-18: Foreground analysis of the three-voice *Fugue in e minor* from the *Toccata and Fugue in e minor*. (Figure continued on the next page.)



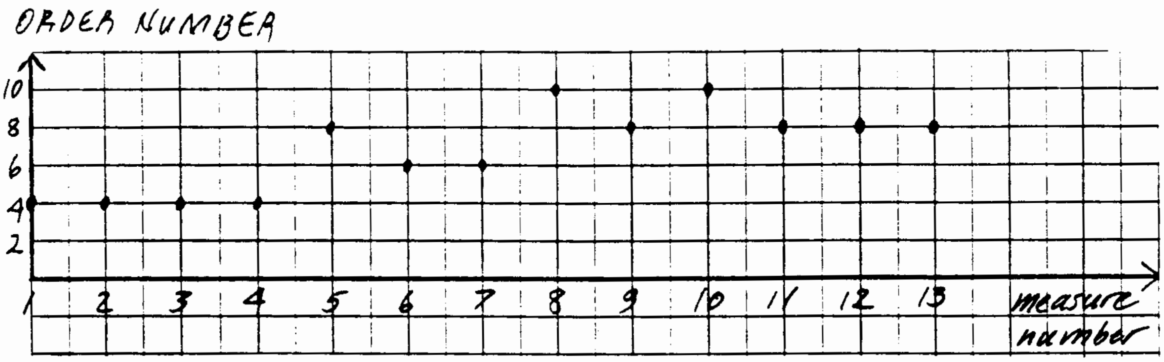


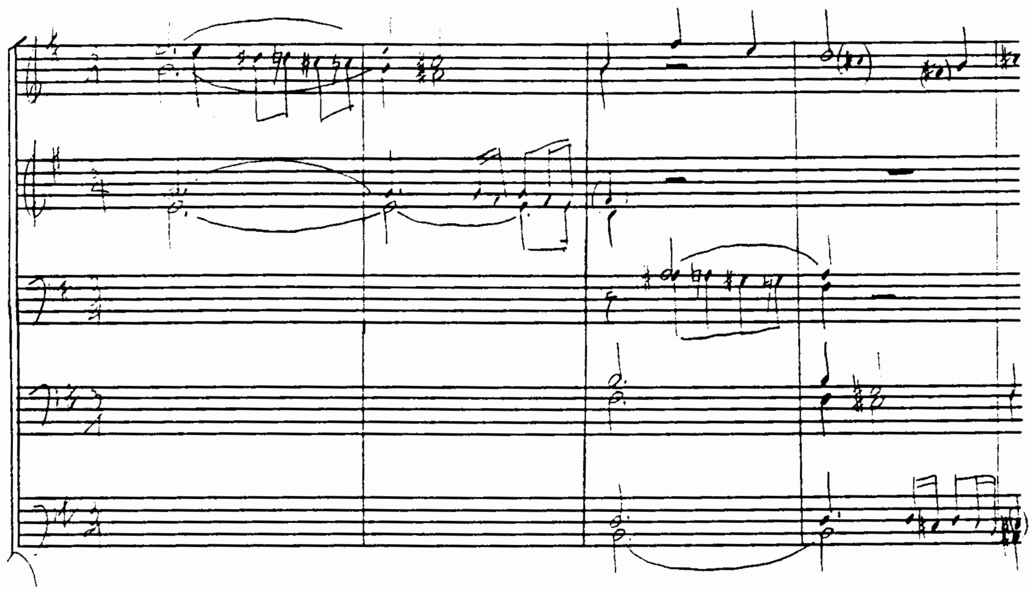
Figure C-19: *DODT* graph of the three-voice *Fugue in e minor* from the *Toccata and Fugue in e minor*, based on the foreground analysis given in Figure C-18.

C.6.2 *DODT* graphs based on middle ground analysis.

On a more middle ground level, the exposition of the two-voice *e* minor fugue can be analyzed as shown in Figure C-20, where its *DODT* graph is also provided. Using a similar methodology, the exposition of the three-voice *e* minor fugue is given in Figure C-21, along with its *DODT* graph. When comparing these middle ground analyses of the two *e* minor fugues and their resulting *DODT* graphs, one is struck (yet again) by how Bach managed to convey so much textural complexity with only two voices. The order number reaches TWENTY at the entrance of the bass in *m.* 3 of the two-voice fugue. At no place in the exposition of the three-voice fugue does the order rise above TEN.

Note that in *m.* 8 of the three-voice fugue, the order number was graphed as TEN, though the first half of *m.* 8 has order EIGHT. This is because the alto voice on beat 4, occurring in a higher register than previously, and with motivic material different from its prior quasi-cadential $A\sharp - B$, can be considered as an implied extra voice, thus increasing the order to TEN. When making *DODT* graphs on the measure level, the highest order attained in the measure has been marked on the graph. Clearly, if a more fine resolution of the textural thinness/thickness as revealed by melodic lines, fragments or accents is desired, then the *DODT* graph can be constructed on the beat level.

If one were to carry out the *DODT* analysis for the whole Bach fugue, on both the measure level and beat level, textural blueprints of the relative thinness and thickness of the fugue would exist for both foreground and middle ground analyses. These blueprints could then be compared to other *DODT* graphs of comparable fugues, e.g., those whose subjects suggest two or more voices in the foreground or middle ground by means of repeated notes (which can sustain the notes of one or more implied voices), register distinctions (which separate an implied upper voice from a lower one), arpeggios (which hint at two or more voices), and so on. It is hard to say right now where these graphs might lead a creative music theorist. Would it be insightful,



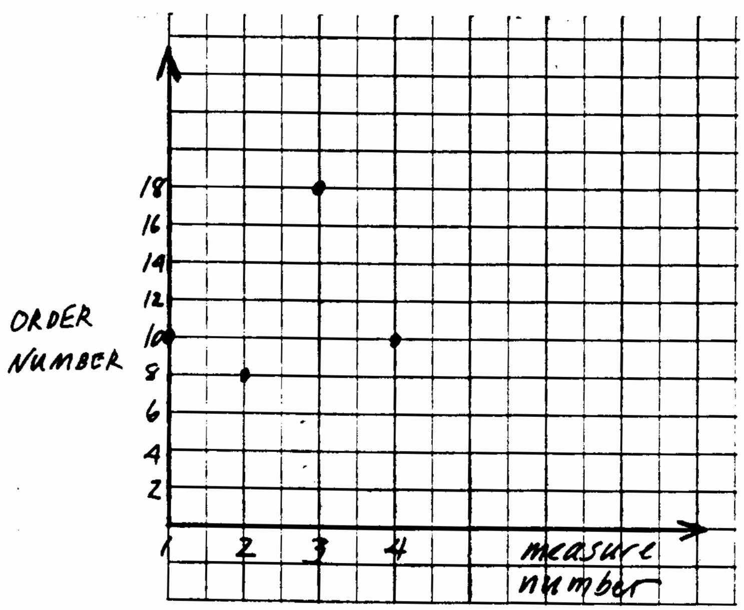


Figure C-20: *Top*, Middle ground analysis of the exposition of the two-voice *e* minor fugue. *Bottom*, *DODT* graph derived from the above analysis.

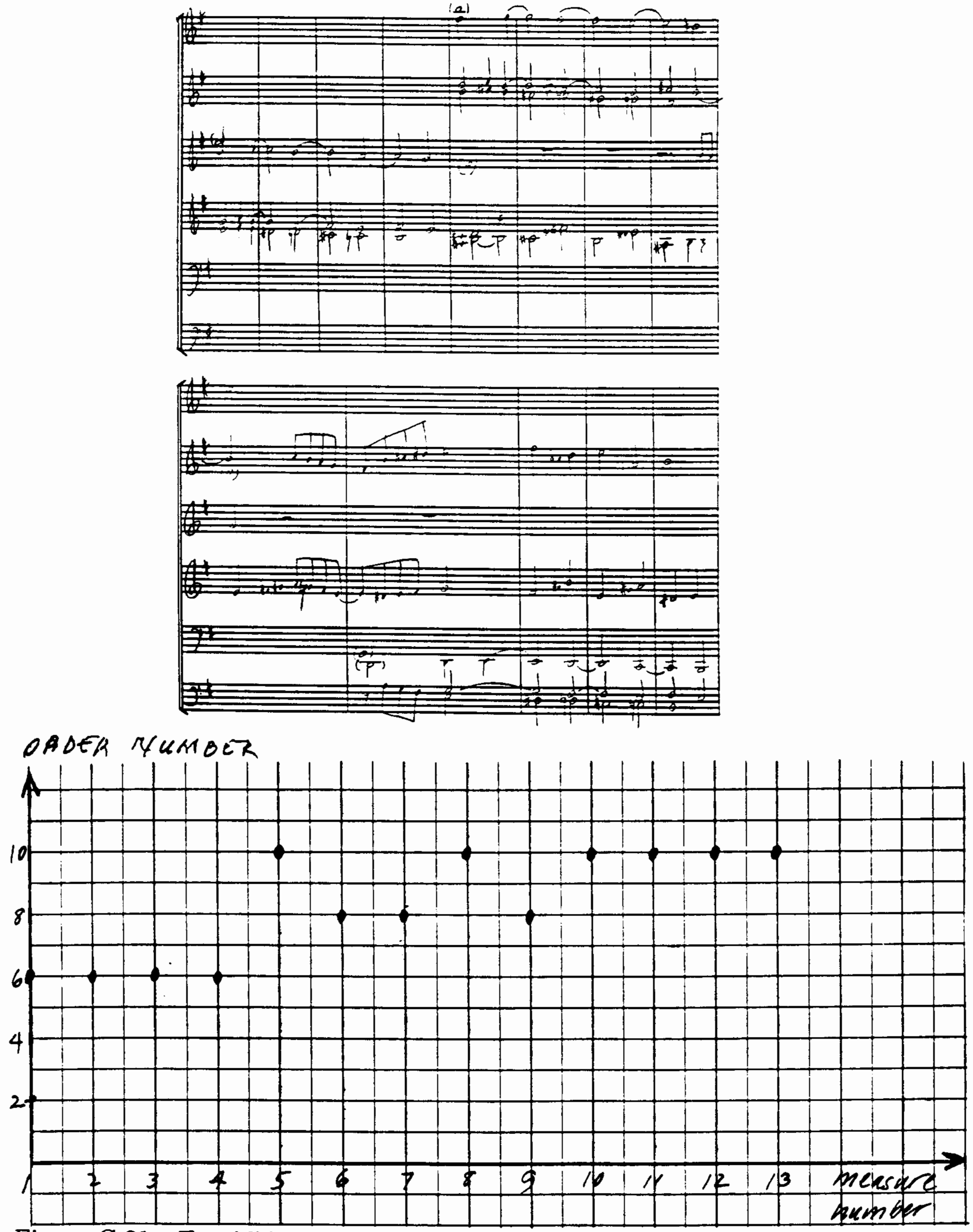


Figure C-21: *Top*, Middle ground analysis of the exposition of the three-voice *e* minor fugue. *Bottom*, the resulting *DODT* graph.

assuming a consistent methodology, to examine all the Bach fugues in light of graphs such as these? Would such an analysis reveal more of the "order" within the Bach fugues? Would it be possible to classify, or group together, various fugues on the basis of similarities in their *DODT* graphs? These questions, and others, might guide some future work.

With regard to the Classic piano sonata, some preliminary work with these graphs has been done. Specifically, a measure-to-measure *DODT* graph of the first movement of Haydn's *Sonata in D Major, Hob. XVI/24 (1773)* was constructed from a foreground analysis. It showed that Haydn, in this particular case, changed order most frequently at the beginnings and endings of phrases, perhaps to recapture the listener's attention after the inevitable cadence.

A measure-to-measure *DODT* graph (foreground level) of Mozart's *Sonata in D Major, K. 311 (1777)*, first movement, also shows higher degrees of order interspersed among the lower, with changes in order often occurring at cadence and new phrase points. In both the Haydn and the Mozart, more than half of the measures are order SIX alone. This means the predominant textural "thickness" consists of 3 pitch lines and 3 associated rhythmic lines. Contrast this with one of Beethoven's early works, the *Sonata in F Major, Op. 10 Nr. 2* started in 1796 and completed in 1798. In the Beethoven, approximately 50% of the measures span orders ranging from TWO through EIGHT, again based on a foreground *DODT*. The remaining half of the measures range from TEN to TWENTY-SIX (*m.* 41). This particular Beethoven example exhibits greater dynamic order than either the Mozart or Haydn.

This brings up an interesting question: is there an upper limit on the order (absolute foreground level) of piano sonatas, as opposed to an upper limit on the order (absolute foreground) of a symphony? While there would appear to be no limit on the order of a symphony, the order of a piano sonata has an anatomically determined limit of ≈TWENTY-FOUR, assuming each thumb straddles two notes, as is often the case in music of the nineteeth and early twentieth centuries. Of course, if each finger is capable of playing two notes simultaneously, the maximum would be 40, not a pleasant thought for a pianist! However, there is always the possibility of playing large clusters of notes with the forearm.

As might have been suspected, and what these three examples suggest, is that the textural "thickness" of the piano sonata increased as the Classic style progressed. One way to view the increase in order as the Classic style developed would be to calculate the percentage of measures of order TEN and above that occur in the Haydn, Mozart and Beethoven sonatas cited earlier. While the Haydn and Mozart have 10% and 16%, respectively, of their measures greater than or equal to order TEN, 49% of the measures — roughly half — contained in the Beethoven exhibit order TEN or more. By comparison, a measure-to-measure *DODT* analysis (foreground level) of Chopin's *Sonata No. 2 in b♭ minor, Op. 35 (1840)* reveals that 70% of the measures in the first movement are order TEN or above.

Of course, it seems obvious that order would increase dramatically in the music of the romantics over that of the classicists. The bolder harmonic treatment found in the romantic piano sonata would be partially responsible for the increase in order. Yet this is not always the case. Approximately forty years after the Beethoven sonata cited above. Schumann wrote his *Sonata No. 2 in g minor, Op. 22 (1835/8)*. A DODT graph of the first movement reveals that approximately half (48.3%) of its measures have order TEN or higher, compared to 48.5% of the much earlier Beethoven. So though forty years separates the two, the Schumann does not register significantly greater order than the Beethoven. Yet the Schumann sounds like a richer, more massive piece. It sounds thicker. Part of this is due to the effect of the sustaining pedal: Because the Schumann often maintains the same harmony over two or more measures, the pedal can be held longer, resulting in greater sonority, sweep and power, even though no new or extra voices are introduced in the underlying harmony. (Specifically, I'm thinking of the first 8 bars of the first movement.) The damper pedal would not be used as much in the Beethoven as in the Schumann. This is because the harmonic progression changes more often, resulting in frequent application of the dampers to the strings. However, it should be noted that there are places in the Beethoven where the harmony remains fairly constant over several measures, enabling more sonority through longer pedals. But these instances are just not as numerous as in the Schumann. The point here is that the richness or fullness conveyed by how a piece sounds does not necessarily infer an increase in order over an earlier period.

Yet, if one examines the foreground level of *mm*. 17-19 in Beethoven's *Sonata in f minor, op. 57 (1804-5)*, it is clear that the texture is much more massive in *mm*. 17-18 than the opening of the Schumann, irrespective of the pedal. This is because Beethoven prolongs the bass in the lowest possible register for an *f* minor chord and moves the right hand through all registers towards the goal of the high *f* minor arrival point. This part has order TWENTY-EIGHT because all the inversions ring out their parts, in a sense sustaining them in the listener's ear, as the pianist moves towards the downbeat of *m*. 18. The use of pedal through this passage further sustains all the "voices" until they resound together at the downbeat of *m*. 18. Then, the texture thins out to two lines, doubling one another and separated by two octaves. The order has dropped from TWENTY-EIGHT to TWO!

C.7 Summary of Appendices A and B

From Medieval monks to the Romantics, music as understood in a dynamic system context, has grown from the simplest SECOND-order chants of the Gregorians to the gargantuan, order ONE HUNDRED-PLUS, symphonies of Mahler. This expansion in order, a mirror of Western music history, reflects the progression through eras, forms, instrumental development and individual and collective artistry. As discussed in Appendix A, a musical work can be modeled as a dynamic system where the following analogies hold, in more or less degrees:

- The system's *state space* $\sim$ One or more musical staves.
- A *state variable* of the system $\sim$ A pitch line or rhythmic line.
- *Order* (=the number of state variables) $\sim$ The number of separate pitch lines and rhythmic lines, constituting melodic lines, fragments or accents.
- *State equations* $\sim$ First-order state representations which encompass musical principles, artistic and musical inputs, as well as the pitch and rhythmic line state variables. The musical principles guiding the composition are included in the system function (or rule), while the artistic issues (such as stylistic idiom) and other musical elements (such as tempo, orchestral color, expression markings) are designated inputs to the system function. However, these first-order state representations introduce a caveat. See Section B.2.2.
- *Autonomous* system $\sim$ A musical work, or movement, whose system function (or rule) is only a function of the state variables. No inputs are present.
- *Nonautonomous* system $\sim$ A musical work, or movement, whose system function or rule is a function of both the state variables and various inputs.
- The system *trajectory*:
    1. A trajectory of *order ONE* $\sim$ A non-pitched rhythmic line.
    2. A trajectory of *order TWO* $\sim$ At successive beats in time, the musical intersection (or simultaneous sounding) of (1) two non-pitched rhythmic lines or, (2) given a pitch line and a rhythmic line, the resultant melody, motivic fragment, or accent.
    3. A trajectory of *order THREE* $\sim$ At successive beats in time, the musical intersection of (1) three non-pitched rhythmic lines or, (2) one pitch line and one non-pitched rhythmic line.
    4. A trajectory of *order FOUR* $\sim$ At successive beats in time, the musical intersection of (1) two melodies or motives, each of which consists of a pitch line and a rhythmic line or, (2) four rhythmic lines or, (3) a pitch line and two non-pitched rhythmic lines.
    5. A trajectory of *order FIVE* $\sim$ At successive beats in time, the musical intersection of (1) five non-pitched rhythmic lines or, (2) two pitch lines and one non-pitched rhythmic line or, (3) one pitch line and three non-pitched rhythmic lines.

6. A trajectory of *order SIX* and *higher* ~ The harmonic progression which, for music of the 17th, 18th and 19th centuries, is a short-hand way of describing the music's state at any beat in time. More generally, the musical intersection of pitch and rhythmic lines at successive beats in time.

While the order of some musical works remains constant, others exhibit *dynamic order* — order that changes with time. Numerous examples can be cited such as the Haydn, Mozart and Beethoven symphonies, the fugues of Bach and the piano sonatas from their earliest manifestations to the present time. During the eighteenth and nineteenth centuries, the piano played a defining role in the increase of what we have termed "musical order" for a solo instrument. Due to its design, the piano allows its player to sound more than four pitch/rhythmic lines at any one time — and, with relative ease, when compared to other instruments — either loudly, softly or any gradation in between. Composers seized on these features to create a textural music (both horizontal and vertical) for the piano. This resulted in a body of work that through time stretched the order of keyboard instruments from FOUR upwards. This increase in order is plainly evident in the history of the fugue and sonata, forms that are particularly significant in the piano literature. *Dynamic order discrete time* graphs provide a quantifiable measure of order and implied texture in these and other musical forms for the piano. However, they also offer insights into chamber and orchestral music, revealing how and when a composer chooses to "thicken" or "thin" the textural counterpoint of a work. Thus, *DODT* graphs provide a looking glass into the horizontal and vertical textural thinking with which Classic/Romantic composers regarded music for solo and ensemble instruments. In light of this, one might ask the following questions:

- How, when and where does Beethoven vary the order (horizontal and vertical texture) of his piano sonatas?
- Is there a certain textural "style" that distinguishes Beethoven from, say, Mozart?
- Does Beethoven's treatment of order (horizontal and vertical texture) in the *Waldstein* sonata at all resemble that in the *Eroica* Symphony, both of which were written during the time frame 1803-4?

Questions like these might be better understood via a tool such as the *DODT* graph, which can be constructed so that different scores can be compared texturally, assuming the same methodology is used in preparing the graphs. Since they are easily stored in a computer, programs can be written to compare, contrast and classify these *DODT* graphs, providing an organized way to distill the dynamic order found in a number of works.

The practice of acquiring one discipline, in this case *dynamic systems*, and applying it to another — the language of *music* — has resulted in what may be a potentially useful tool (the *DODT*) for analyzing the horizontal and vertical texture of Western music, particularly music of the Baroque, Classic and Romantic periods. On the surface, such a move from the state space world of dynamics to the art of music holds little promise for revelation. But sometimes, a below-the-surface understanding of one field applied to another can provide insights.

Appendix D

Musicality in the Language of Chaos and Nonlinear Dynamics

There have been a number of attempts in recent years to apply *chaos* to contemporary music composition[1] But is there anything inherently musical about the language of nonlinear dynamics and chaos? If this language were shown to possess analogies with the musical language that shaped the harmonic practice of the Baroque, Classic, and Romantic periods, then the application of chaos to new music could be seen to rest on a foundation that links music of previous eras to a dynamic music for our own time.

D.1 Introduction

This Appendix presents the language of chaos and nonlinear dynamics, but with a musical twist. It establishes musical analogies for such chaotic dynamical properties as strange attractors, fractals, sensitive dependence on initial conditions, Lyapunov exponents, self-similarity, bifurcation, equilibrium points, and catastrophe. The previous two appendices argue that a musical work can be modeled as a dynamic system where a number of analogies hold. (See the Conclusion of Appendix C.) Much of the terminology underlying *Musicality in the Language of Chaos and Nonlinear Dynamics* stems from the reasoning presented to support these analogies. Whereas the previous appendices treat music from the standpoint of a *general* dynamic system, the present discussion concentrates on the language of nonlinear dynamics and chaos, both subsets of dynamic systems, and examines its musical analogs through applications to pre-twentieth century classical music.

[1]See References Chapter 1.

D.2 The Language of Nonlinear Dynamics and Chaos

D.2.1 Motivation

Dynamic systems can be linear or nonlinear. Linear systems, governed by linear differential equations, essentially constitute a make-believe world, albeit a very useful one. Their efficacy reveals itself every time a scientist models some natural (often nonlinear!) phenomenon based on a set of linear equations — linear in that no nonlinear terms such as x^2, $\sin x$ or e^x appear in the equations. Only linear terms such as x, βx or $\gamma^3 x$ are allowed, where β and γ are constants. Yet we live on a *nonlinear* planet, its destiny essentially ruled by a multitude of nonlinear differential equations. Why, then, would a scientist want to describe nonlinear events in terms of linear differential equations? Because linear behavior occurs in restricted regions of nonlinear phenomena. Also, there exists a large database of tried and true methods for solving linear differential equations whereas each nonlinear equation requires a separate method of solution. The historical difficulty of finding exact solutions to these problems eventually led to a more qualitative approach, first proposed by Poincaré. With that approach, an evocative language evolved to describe the inner workings of nonlinear dynamics. This language also carried over to the study of chaotic dynamical systems, nonlinearity being a necessary condition for chaos.

Still, in seeking to create new musical forms, styles, and sonorities from the world of nonlinear dynamics — and chaos in particular — it might be reassuring to find that some significant links exist between the language of nonlinear dynamics and the musical traditions of the past. Specifically, the purpose of this appendix is to illustrate that it is possible, and sometimes illuminating, to use the language of nonlinear and chaotic dynamics for discussion of 17th, 18th and 19th century music. While numerous examples illustrating the link between the language of nonlinear dynamics and musical analysis could be cited, the author tried to find one piece that would illustrate most of them — the *Sonata in f minor, op. 57, the "Appassionata"*, by Beethoven.

D.2.2 Historical Background

Delving into the literature, looking for a definition of chaos leads everywhere and nowhere. While there is no generally accepted mathematical definition of chaos, it is possible to see and hear it. The inventive Dutch engineer, Balthasar van der Pol, listened to chaos unsuspectingly in 1927. Writing about his experiments with an oscillating circuit, he noted:

> Often an irregular noise is heard in the telephone receivers before the frequency jumps to the next lower value. However, this is a subsidiary phenomenon, the main effect being the regular frequency demultiplication.[2]

[2] van der Pol, B. and van der Mark, J. (1927) Frequency Demultiplication *Nature* **20** *3019* 363.

Twenty-four years earlier, the French dynamicist/mathematician, Henri Poincaré, had not yet heard chaos, but he saw it, at least in his imagination. And he warned posterity:

> A very small cause which escapes our notice determines a considerable effect that we cannot fail to see, and then we say that the effect is due to chance. If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment.
>
> But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation *approximately.* If that enabled us to predict the succeeding situation with *the same approximation*, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so: it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon.[3]

Poincaré's "fortuitous phenomenon" was chaos. This was 1903. Sixty years later Edward Lorenz, a meteorologist and mathematician at MIT, wrote a seminal paper, "Deterministic Nonperiodic Flow," for which his computer, a dinosaur named the Royal McBee, had drawn a strange attractor, also known as a chaotic attractor. Lorenz did not call this trajectory in the phase plane "strange" or "chaotic". The word *chaos* is never mentioned in his paper. Rather he echoed, on the basis of experimental data, Poincaré's belief that for certain systems it is necessary to pinpoint initial conditions exactly in order to reliably predict future behavior. No close approximations would suffice:

> To verify the existence of deterministic nonperiodic flow, we have obtained numerical solutions of a system of three ordinary differential equations designed to represent a convective process. These equations possess three steady-state solutions and a denumerable infinite set of periodic solutions. All solutions, and in particular the periodic solutions, are found to be unstable. The remaining solutions therefore cannot in general approach the periodic solutions asymptotically, and so are nonperiodic.
>
> When our results concerning the instability of nonperiodic flow are applied to the atmosphere which is ostensibly nonperiodic, they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly. In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long-range forecasting would seem to be non-existent.[4]

[3]Crutchfield, J. P., Farmer, J. D., Packard, N. H., and Shaw, R. S. (1986) Chaos. *Sci. Am.*, **255** December, *6* 51-57.

[4]Lorenz, E. (1963) Deterministic nonperiodic flow *J. Atmos. Sci.* **20**, 130.

Both Poincaré and Lorenz had envisioned chaotic systems. They didn't call them "chaotic"; rather, they recognized the greatest telltale characteristic of such systems — the solution is highly dependent on its initial condition, yet the solution remains bounded. For deterministic, non-chaotic systems, this is not the case: Two stable solutions started at close initial conditions will remain close for all time; two unstable solutions, though displaying sensitive dependence on the initial condition, are never bounded.

In the last decade, striving to come up with an acceptable definition of chaos, mathematicians and engineers reveal individual and unmistakable styles of interpretation. For instance the mathematician, Robert Devaney, defines a discrete chaotic system in terms of a set:

> Let v be a set. $f : v \rightarrow v$ is said to be chaotic on v if
>
> 1. f has sensitive dependence on initial conditions.
> 2. f is topologically transitive.
> 3. Periodic points are dense in v.[5]

This definition is in stark contrast to that of Leon Chua, a professor of electrical engineering at UCBerkeley. Writing in the August 1987 *Proceedings of the IEEE,* an issue devoted to chaos, Chua addressed the question:

> But what is chaos? Roughly speaking, it is a more exotic form of *steady-state response.* Most electrical engineers have learned from basic circuits and systems courses that the response of all stable linear circuits and systems consists of a *transient* and a *steady-state* component. The steady-state response is what remains after the transient has decayed to zero, and can be either a constant (i.e., dc or equilibrium solution) or a periodic solution. This basic result has been so ingrained in the engineering curriculum that most engineers had subconsciously extrapolated it for nonlinear systems as well. We now know that when *nonlinearity* is present, there exists a wide range of parameters where the steady-state response is bounded, but *not* periodic. Instead, the response waveform becomes erratic with a broad continuous (rather than discrete as in the periodic case) frequency spectrum. Moreover, the response is so sensitive to initial conditions that unless a computer of infinite word length is used in the simulation, no long-term prediction of the precise solution waveform is possible.[6]

In his book *Nonlinear Dynamics and Chaos*, Steven Strogatz acknowledges that mathematicians do not agree on a definition of chaos. But any definition would have to include the following four points:

1. A chaotic system is nonlinear.

[5] Devaney, R. (1986) *Chaotic Dynamical Systems* (Benjamin/Cummings Publishing Co.) 50.
[6] Chua, L. (1987) Chaotic Systems *Proceedings of the IEEE.* **75** *8*, 979.

2. It is deterministic, i.e., it is governed by a set of n-dimensional equations, such that if the initial condition is known exactly, the future behavior of the system can be predicted.

3. However, the solution to a chaotic set of deterministic equations is highly dependent on the initial condition. As a result, nearby orbits will diverge.

4. A chaotic system exhibits nonperiodic long-term behavior, meaning that as $t \to \infty$, trajectories exist which can never be classified as periodic orbits, quasi-periodic orbits or fixed points.

In summary, then, chaos is nonperiodic long term behavior in a nonlinear, deterministic system, whose solution is highly dependent on initial conditions.[7]

D.2.3 Strange Attractors

Another term which is often associated with chaos is the *strange attractor*, which inhabits the state space of a dissipative chaotic system. First we define attractor as a closed set A for which the following can be said[8]:

- A is invariant, i.e., any trajectory starting in A remains in A for all time.
- A attracts an open set of initial conditions, such that if a trajectory starts within that open set containing A, the distance from the trajectory to A approaches 0 as $t \to \infty$. The largest such open set is known as the *basin of attraction* of A.
- A is minimal, meaning that there is no proper subset of A that satisfies the above properties.

It should be kept in mind, however, that the definition of attractor is still the focus of considerable debate.[9] There exist only four types of attractors:

1. stable node
2. torus
3. periodic attractor (an attracting limit cycle)
4. and the more recently named *strange attractor* (1971).

A strange attractor can be understood as an attractor whose basin of attraction exhibits sensitive dependence to initial conditions.

The mathematicians, David Ruelle and Floris Takens, had never seen a strange attractor, though they named it. By inventing the concept of a strange attractor,

---

[7]Strogatz, S. H. (1994) *Nonlinear Dynamics and Chaos* (Addison-Wesley, New York) 323.
[8]Strogatz, p. 324
[9]See Milnor, J. (1985) On the concept of attractor. *Commun. Math. Phys* **99** 177.

they were trying to describe a phase space trajectory which might possibly account for turbulence. In other words, they were seeking a signature for the onset of "turbulent" or "chaotic" behavior. Such a trajectory would have to possess certain properties. It would have to be bounded because, after all, turbulence doesn't get so out-of-hand that the world blows up along with it. Nonperiodicity was essential. The orbit would never be able to intersect itself because in so doing, the system would simply repeat a path it had traversed earlier. In short, confined to a bounded space, this attractor would have to be an infinitely long line, capable of capturing every possible "beat" of turbulent flow.[10] This infinitely long line would fold and stretch innumerable times in order to confine itself to a bounded space.[11] As a line, its dimension was one, but as a pattern occupying space, its dimension could be higher. Ruelle and Takens, familiar with Mandelbrot's new mathematics, deemed this strange attractor a fractal, an object of noninteger dimension.

Imagine the strange attractor, then, as an infinitely long line within a bounded region of the system's state space forever weaving in and around itself, spinning out variation after variation as it defines its entity in space without ever intersecting its past. The attractor must continually vary its path in order to avoid an intersection. Due to its attractor status, it possesses a basin of attraction. Now if a trajectory starts on the infinite line of the strange attractor, it will follow exactly the line of the attractor so that the strange attractor itself will be the system's solution. However, if the trajectory starts within the basin of attraction for the strange attractor, it will asymptotically approach, yet never actually touch the body of the attractor. Instead it mirrors very closely the set of the strange attractor. One example of a strange attractor is shown in Figure D-1.

A Musical Analog

Is it possible to see a musical parallel here? Consider the theme and variation form of the Andante movement of Beethoven's Op. 57. This movement can be viewed as a musical invariant set. (See definition of invariant set given above.) Such a set can be defined as one which includes all states comprising a piece such that if a musical "trajectory" were started at a state within the piece, it remains in the set, effectively calling forth the piece. On the other hand, if the musical path is started at a state *outside* the piece — at one which does not exactly duplicate some state within — then it cannot enter the invariant set of the piece in question.

Figure D-2 gives the 32-bar theme of the Andante movement upon which all subsequent variations are based. Now, a subset of the invariant set (composed of the Andante) is also an invariant set. Therefore, the 32-bar theme constitutes an invariant set too, where the musical state at each beat denotes a point on the set. Does this invariant set have a basin of attraction? Yes! Because the form of the second movement is theme and variation, each variation starts at a point, or state, near the

[10]Glieck, J. (1987) *Chaos* (MacMillan, New York) 138-9.

[11]This is due to the presence of both positive and negative Lyapunov exponents. More about Lyapunov exponents later.

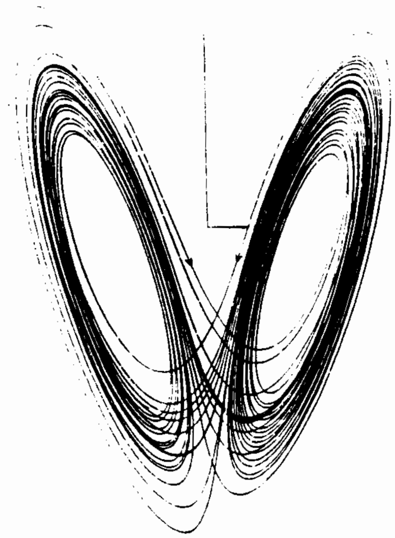


Figure D-1: The strange attractor, nicknamed the "Butterfly", which lives in the state space of the equations studied by Edward Lorenz.

Figure D-2: The 32-measure theme of the second movement of Beethoven's Op. 57.

invariant set of the theme. As a result, the initial state of each variation, as given by its starting pitch and rhythmic lines, is very close to a point of the theme's invariant set. Thus the opening state of each variation lies within a small "neighborhood" of the thematic invariant set. Furthermore, as time passes, the variation continues to "approach" the states encompassed by the theme, though never intersecting it and never replicating it exactly. The theme, then, is an invariant set which happens to possess a basin of attraction, making it a "musical attractor". But could it be likened to a strange attractor? Recall that the strange attractor, defined as a nonperiodic attractor, never intersects itself because in so doing, it would repeat its past behavior forever. Though there are repeat marks delineating the two 8-measure phrases of the Andante theme, it does not repeat itself forever. After the second time through the first 8-measure phrase, the music moves on to new material. In this sense, the repeats are not identical.

The strange attractor was also described as an infinitely long line forever weaving in and around itself within a bounded region of the system state space. The thematic attractor is also contained within a region of musical state space. However, unlike the strange attractor of Ruelle and Takens, it does not go on forever. Rather, it endures for only 32 beats (counting the 8-bar repetitions). This is a fundamental difference between the theoretical concept of the strange attractor and its musical analog. However, with simulations, the strange attractor is a finite line, evolving for a certain number of time steps

Yet another way to interpret the strange attractor within a musical context is to describe the entire Andante movement, or any piece, as such an attractor. The pitch event sequence of the piece is "bounded" or confined to a set of notes. In fact, this is exactly what is done in the variation technique explained in Chapter 2, where the first step is to pair each pitch event of an original work with a point on a strange attractor. Since there is no way to know whether a point is on (exactly) the strange attractor, a reference chaotic trajectory is used instead. Variations are made by starting another trajectory near the reference. The chaotic mapping ensures that the new trajectory triggers only those notes found in the source piece. Therefore, the variation is completely contained in the invariant set defined by the original musical score.

If the whole second movement is considered an invariant set, then its basin of attraction consists of all states in the first movement, since any musical path started at a point (or state) in the first movement must eventually reach the Andante movement. Because the second movement is an invariant set with a basin of attraction, it could be called an attractor. Furthermore, it could be considered "chaotic" in that the state trajectory never repeats itself; rather, it generates any number of variations within a bounded region of the musical state space. Though a mathematical strange attractor, in theory, weaves an infinite line through phase space, a simulated strange attractor will consist of a finite line stretching and folding in space. Similarly, a musical strange attractor can be conceived as a truncated fractal whose length is severed after a certain time, specifically 288 beats.

In any case, regardless of whether one considers the 32-bar theme, or the entire second movement, as the strange attractor, the concept of chaotic attractor provides an intriguing fit with the form of theme and variation.

D.2.4 Fractals and Self-Similar Structure

Fractals can be viewed as geometric objects which are much more sophisticated than the cones, spheres and other fixtures of Euclid's geometry. Because they exhibit a rich complexity irrespective of scaling, it is possible to take a magnifying glass to any part of the fractal — repeatedly — and still observe a plethora of fine structure, i.e., detail on arbitrarily small scales. Yet, this same approach applied to a geometric curve, cone or square — under repeated magnification — becomes a simple line, hardly a study for detail.

Though fractals are often easily identified, they lack a precise definition. It is difficult to define them much in the same way it is difficult to define the exact nature of a musical instrument. In the case of the piano, for example, a pianist might describe it as having an 88-note keyboard, three pedals, hammers, strings, felts, pin block, sound board, metal frame and wooden case. But while most pianos have 88 keys, some only have 85; a few have 94. Some pianos are nine feet long, others only 5'2". Many have a vertical sound board and only two pedals instead of three. Some work beautifully, others don't work at all. Yet despite the numbing number of variations from instrument to instrument, a piano is readily discerned because its essence (or Platonic ideal) comprises an identifying set of properties.

Fractals are also perhaps best defined by a set of properties, the most significant of which are illustrated by the middle-thirds Cantor set. The set, first postulated by the mathematician Georg Cantor, is formed by drawing a line segment and removing the middle third. Next discard the middle third of each of the two remaining segments. Continue to eliminate the middle third of each ensuing, but considerably shorter line, thus leading to a huge set of truncated line segments. The construction process is given in Figure D-3.

The infinite iterate of this scheme gives the Cantor set — an uncountable number of infinitesimal line segments separated by blank space. This set is *self-similar* which means that as one focuses on ever smaller regions within the set, it is possible to see the structure of the whole. The set also exhibits fine structure and has a non-integer dimension. There are many other properties of the Cantor set, but these three — fine structure, self-similarity and fractional dimension — are particular to fractals in general.

These properties can also be found in the Sierpinski carpet, a fractal conceived by the Polish mathematician Waclaw Sierpinski. It is constructed in the following way: take a black square, partition it into nine squares, remove the middle square; then, partition each of the remaining black squares into nine smaller squares, again removing each middle square, and so on. Figure D-3 demonstrates the process.

Recall that strange attractors are also fractals. For example, the Lorenz attractor of Figure D-1 displays its fractal character as an infinite number of sheets. The geometrical structure of the attractor is, for one circuit around both left and right lobes, a pair of surfaces that appear to merge at the bottom of the "butterfly." By the uniqueness theorem, however, these surfaces do not merge because the trajectory cannot intersect itself. For instance, the first circuit about the right hand lobe occurs on a surface, whereas the second circuit may occur on the same or a separate surface

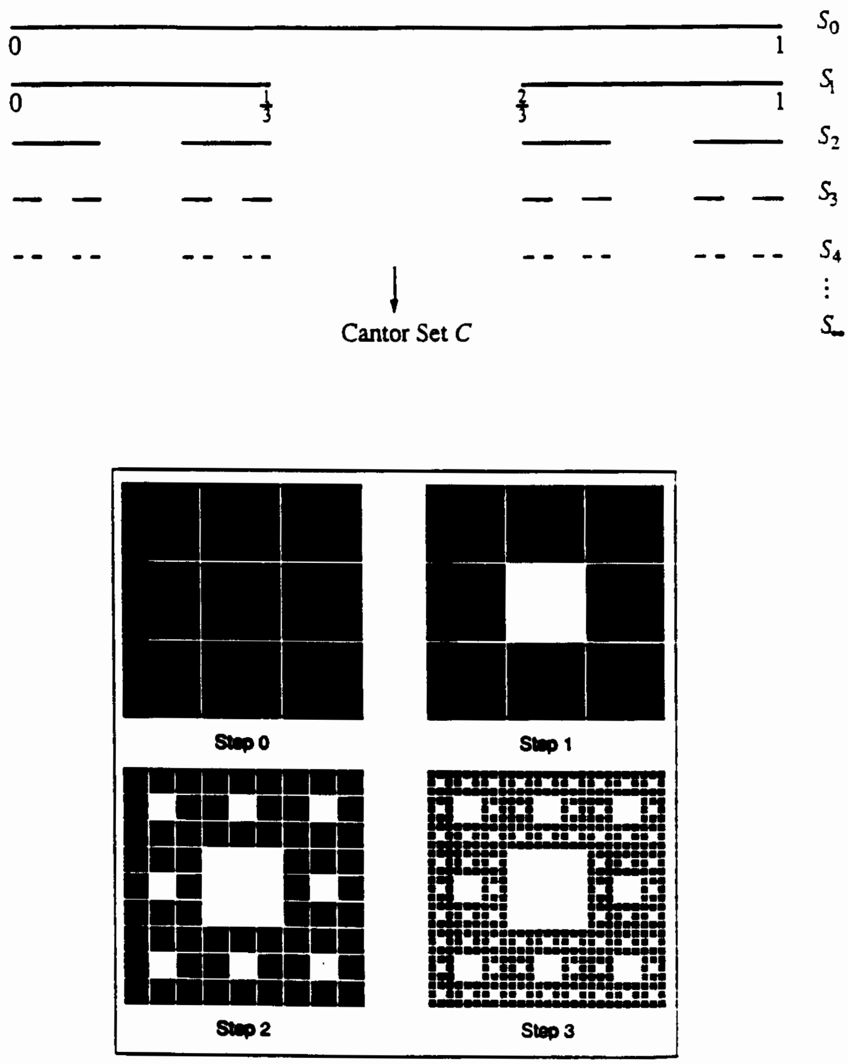


Figure D-3: How to construct the Cantor Set and the Sierpinski carpet.

Figure D-4: The fugal theme, displayed in the alto voice, of Bach's Fugue in c minor, *WTC II.*

which is extremely close to the first. Similarly, each circuit about the left hand lobe also occurs on another or previous surface. Thus, taken together, these sheets form an "infinite complex of surfaces" which would be called a fractal. The Lorenz attractor is therefore a set of points with zero volume but infinite surface area. Hence, it has a fractional dimension which has numerically been estimated at around 2.05. [12]

Finally, it is important to recognize that while simple fractals such as the Cantor set and Sierpinski carpet are strictly self-similar, fractals more typically display structure which is only approximately or statistically self-similar. This is especially true for real-world fractals, such as those which appear in nature. There, the small copies, while resembling the whole, have variations which convey a sense of statistical self-similarity.[13] For instance, the trunk of a tree is closely, though not exactly mirrored in each branch, which is reflected in each twig until the self-similar process ends with a leaf. Or does it...the main vein of the leaf, like the primordial trunk, branches into subveins which in turn duplicate themselves.

Musical Examples

Bach often used a form of musical self-similarity in his fugues by first stating the thematic subject and then recalling it later, but with all notes doubled in rhythmic value (known as augmentation) or halved in rhythmic value (diminution). Thus augmentation would be a scaled-up version of the original fugal theme, whereas diminution would be a scaled-down replica of the first subject. An example of augmentation can be found in his c minor fugue from the *Well-Tempered Clavier, Book II* (*WTC II*), where the one-measure subject is given by the alto voice in *m.* 1 of Figure D-4. It is later augmented rhythmically in the tenor voice of *mm.* 14-15 as illustrated in Figure D-5. The theme is readily recognizable regardless of the change in rhythmic scale.

Recall now the discussion concerning the second movement of Beethoven's Op. 57 where a case was made for considering either the theme or the entire movement as a strange attractor. Strange attractors, according to Ruelle and Takens' definition, must be fractals. As such, then, they should exhibit those fractal characteristics associated with the Cantor set: self-similarity, fine structure and non-integer dimension. Only the first two properties are considered; non-integer dimension has no meaningful musical analog, at least for now.

[12]Strogatz, S. H. (1994) *Nonlinear Dynamics and Chaos* (Addison-Wesley, New York) 320.
[13]Peitgen, Jürgens, and Saupe, p. 161.

Figure D-5: Augmentation of the *c* minor fugal theme in the tenor voice of measures 14-15.

Fine structure — detail at arbitrarily small scales — occurs throughout the Andante, particularly with regard to rhythmic and metrical scale. As an example, consider the theme, where every rhythmic and metrical grouping which is to occur in the movement, is first hinted at within those first 32 measures. No matter which scale is chosen for "magnification" or study, some foreshadowing of events or motivic significance can be found. For example, the 32nd note rhythm first introduced in *m.* 4 of the theme becomes the dominant rhythm of the third variation.

Does any semblance of self-similar structure appear in the second movement? This question can be addressed two ways: according to the phrase structure or via the metrical structure. Consider first the phrase structure of the theme. Its 32 measures break down into the two halves of the theme. Part A of the theme is represented by *mm.* 1-16, including the repetition, while Part B of the theme consists of *mm.* 17-32, and also includes the repeat sign. Part A includes TWO 2-groups of 8-measure phrases, again due to the repeat sign, and Part B also breaks down into TWO 2-groups of 8-measure phrases. For Part A, each 8-measure phrase is comprised of two coherent phrases; the first is $3\frac{1}{2}$ measures and the second is $4\frac{1}{2}$ measures long. They are indicated in Figure D-6 by the curved lines marked below the musical staff. Each approximately 4-measure phrase in turn consists of two coherent half-phrases, marked above the staff in Figure D-6. Notice that the (roughly) 4- and 2-measure phrases do not begin and end within the strict confines of the measure. Rather, they begin primarily on off-beats and conclude on downbeats.

Part B has a different type of phrase structure. On all levels of 2-groups, its phrasing conforms to the bar lines. Part B consists of TWO 2-groups of 8-measure phrases, again due to the repeat sign. Each 8-measure phrase is composed of two 4-measure phrases (*mm.* 9-12 and *mm.* 13-16), while each 4-measure phrase consists of two coherent 2-measure phrases. (See Figure 4-6.) Here, the (exactly) 4- and 2-

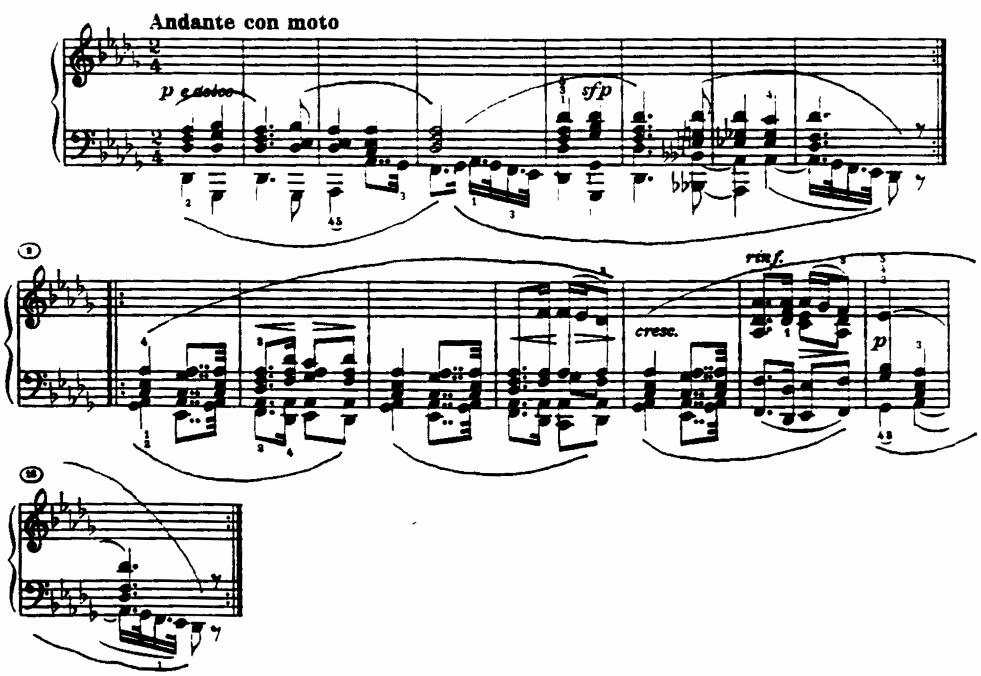


Figure D-6: The phrase structure of the Theme is given by the phrase markings above and below the staff. Those marked under the left hand part show how each 8-measure phrase is composed of two phrases. In Part A, the $3\frac{1}{2}$ measure phrase breaks down into two half-phrases, as does the $4\frac{1}{2}$ measure phrase. These half-phrases are shown above the right hand part. In Part B, the phrasing is more regular. Phrases start on downbeats and conclude at the end of a measure.

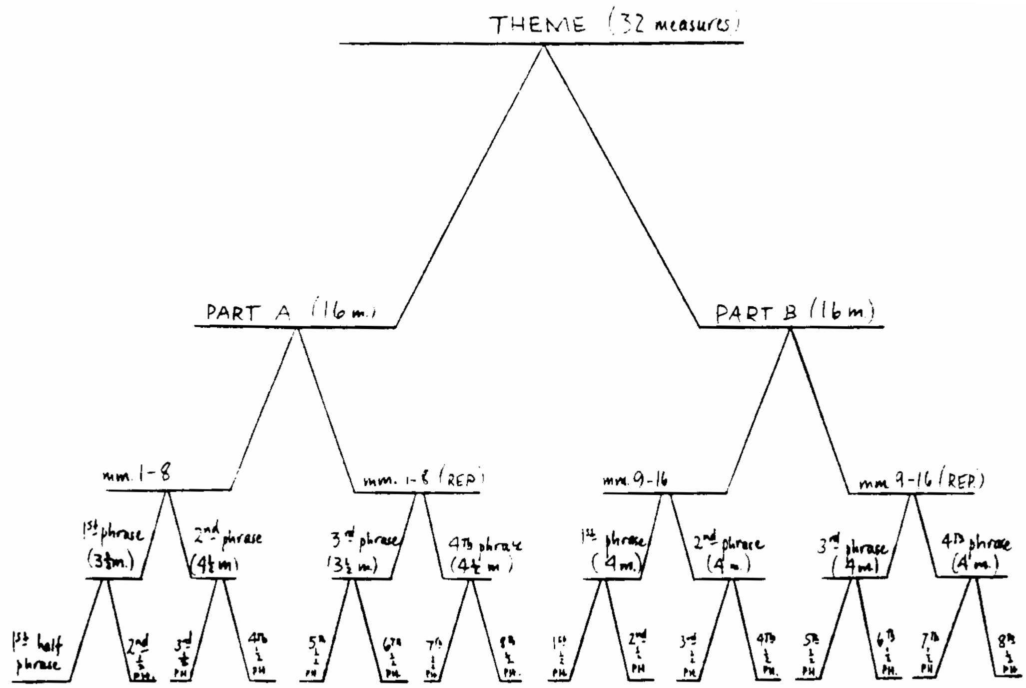


Figure D-7: The self-similarity of the phrase structure of the Theme is revealed by a nested structure of 2-groups.

measure phrases do begin and end within the confines of the measures.

The self-similar structure of the Theme is diagrammed in Figure D-7. It shows the recurring 2-groups on smaller and smaller scales. Unlike a mathematical fractal, the self-similarity displayed by the phrasing does not go on forever. It concludes at the level of the half-phrase.

Now consider the metrical structure of the theme. Meter is established in a listener's ear by those patterns of **accentual recurrence** that allow perception of a higher order. At larger levels of organization, the recurrence need not be exactly regular.[14] It is important to discern the difference between rhythm and meter. Rhythm is the proportional arrangement of sounds and silences with respect to their durations. Simply put, anything that has duration is rhythmic. Meter is the measurement of the distances between points in time. Distance and duration are not synonyms: Distance is the measure of duration. To see this, compare the distance between two points in the Cartesian plane with a line drawn between them.[15] The line might be analogous to a phrase. (A phrase has duration and is therefore un-

[14]Imbrie, Andrew (1973) "Extra" measures and metrical ambiguity in Beethoven. In Alan Tyson, ed. *Beethoven Studies, Vol. II* (Norton, New York) 52.

[15]Imbrie, p. 53.

derstood as a rhythmic event.) However, the distance between the points can describe whether this phrase is a 4-measure phrase, where the downbeat of each measure might be characterized as "strong", "weak", "medium", "weak", respectively, or as a 3-measure phrase, where each downbeat could be distinguished as "strong", "weak", "weak", respectively.

Accent is the specific agent responsible for pinpointing these instants in time. Accent therefore has no duration. Accent is established by the occurrence of contrast or change. For example, a harmonic change has no duration: only the component harmonies have it.[16] The weights — "strong", "medium", "weak" — given above have no duration: only the constituent pitches, note values and harmonies of these measures have duration. The weights are accents created by contrast or change.

A change in the dynamic markings (loudness/softness) has no duration if it is to create accent. The dynamic change must be instantaneous. Those dynamic changes involving duration such as *crescendo* or *diminuendo* defer the accent until some arrival point is reached. A melodic accent such as an agogic accent — when a pitch receives accent because it has a longer duration than its surrounding melodic notes — is also an instantaneous event. The listener's perception of the longer duration has the effect of causing the ear to retroactively place the accent at the pitch's point of attack.[17]

Recall that the occurrence of contrast or change will create accent, and accent is the agent for highlighting points in time. This can be seen by the existence of "strong" and "weak" measures in Beethoven's music. Beethoven sometimes indicated the existence of a higher metrical structure by marking "ritmo di tre battute" and "ritmo di quattro battute" in his symphonies.[18] Chopin was also clearly aware of higher units of metrical organization: Though his third Scherzo is marked in 3/4 time, the measures form groups of four which can be treated as a large scale 4/4 meter.

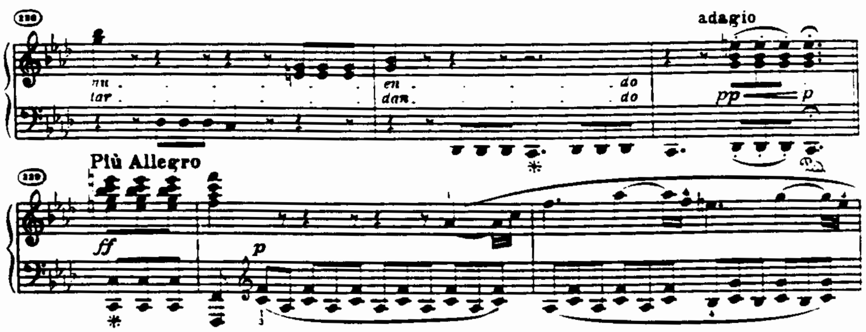
With respect to the Theme of the Andante movement of Op. 57, the metrical structure of Part A is given by 2 — 3 — 4 — 1 — 2 — 3 — 4 — 1 (Figure D-8). The numbers are indicative of the relative weighting of the eight downbeats comprising the theme, not counting the repeat sign. The number 1 indicates the strongest metrical weight; "2" suggests a weaker weight; "3" denotes a medium weight; and "4" represents the weakest weight. This means that the first 3 downbeats serve as a prolonged upbeat to the fourth downbeat; the downbeats of *mm.* 5-7 also function as a large-scale upbeat to *m.* 8. In Part A, significant downbeats occur in *m.* 4 and again in *m.* 8. On the other hand, Part B has a related, but different, metrical structure: 2 — 1 — 2 — 1 — 2 — 1 — 2 — 1 . It is consistent with the metrical structure of Part A by its alternation of relatively weak and strong beats, as well as by its starting with a weak weight. The metrical structure of Part B (2 downbeats per 4 measures) can be subsumed under a structure analogous to that of Part A (1 downbeat per 4 measures).

Interestingly, the overall metrical structure of the second movement reflects the metrical structure of Part A, which has prolonged upbeats to significant downbeats. The most significant metrical downbeat of any piece is its final resolving event. This

[16]Imbrie, p. 51-52.
[17]Imbrie, p. 52.
[18]Imbrie, p. 45.

Figure D-8: The metrical structure of Part A of the Theme.

is often the cadential chord of the work, though that is not the case with either the first or second movements of Op. 57. The opening to the Coda of the first movement, marked Piu Allegro, provides the most important downbeat ($m.$ 240) of the movement:



The rest of the Piu Allegro is an elaboration of that important downbeat. This downbeat also serves as the downbeat to the second movement, where the Theme, Variation 1, Variation 2, Variation 3, and return of the Theme could be assigned the following metrical structure:

2 | 3 | 4 | 5 |

Theme Var.1 Var.2 Var.3+Return of Theme.

Since the return of the theme enters on the heels of Variation 3, without any delineation by repeat signs, and because it contains a a transitory passage to the third movement, Variation 3 and the returning theme are together assigned the metrical function 5.

The numbers 2, 3, 4, and 5 indicate a prolonged upbeat on a very large scale to a downbeat. But to which downbeat? The downbeat in $m.$ 20 of the third movement, where the tonic key of f minor is firmly established.

In summary, the phrase structure of the theme is self-similar. It consists of a nested chain of 2-groups, where each "parent" in the tree produces two "children", as shown in Figure D-7. This diagram can be reproduced for each of the variations. The metrical structure of Part A of the Theme serves as the basis for the metrical structure of the entire Andante. The self-similarity is not exact since the metrical structure of Part A is 2 — 3 — 4 — 1, whereas the metrical structure of the whole movement is 2 | 3 | 4 | 5 | "1", where "1" is delayed to $m.$ 20 of the third movement.

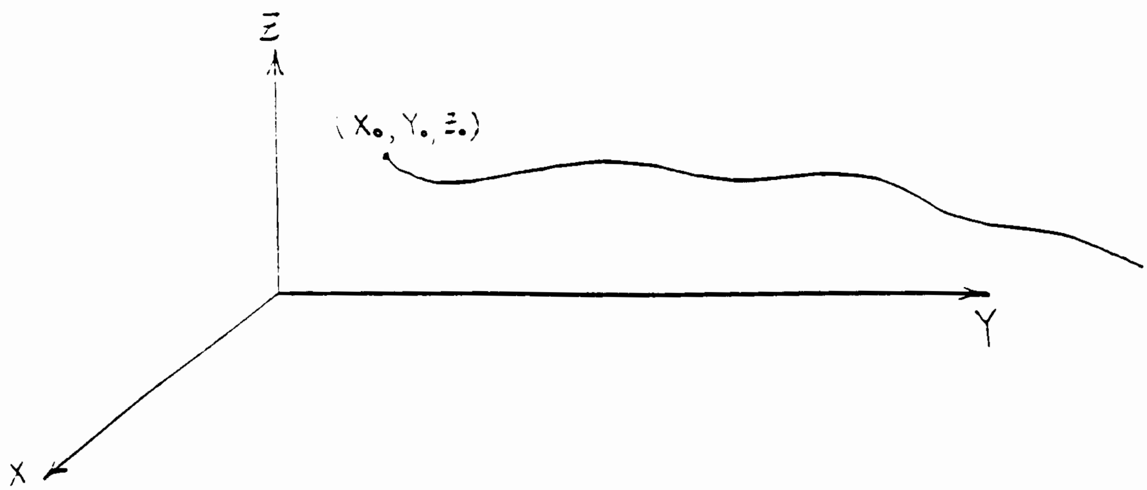


Figure D-9: System Trajectory in 3-space with Initial Conditions $x_0 = 1.11$, $y_0 = 2.22$, $z_0 = 3.33$.

D.2.5 Sensitive Dependence on the Initial State

Consider a third order autonomous system with three state variables, x, y, and z. Now suppose these have initial values

$$x_0 = 1.11 \quad (D.1)$$
$$y_0 = 2.22 \quad (D.2)$$
$$z_0 = 3.33. \quad (D.3)$$

In time the system traces out an orbit in 3-space shown in Figure D-9. This orbit is merely a graphical representation of the solution to the nonlinear equation governing the system.

In order to repeat the experiment let the initial conditions change by a fraction to

$$x_0' = 1.111 \quad (D.4)$$
$$y_0' = 2.221 \quad (D.5)$$
$$z_0' = 3.331. \quad (D.6)$$

At first the system might follow somewhat closely the previous orbit, but it soon diverges as seen in Figure D-10. So though two inital states can be chosen close to each other, their respective orbits lead to completely different states. This phenomenon is known as *sensitive dependence on the initial state.*

A Musical Example

It turns out that sensitive dependence on initial conditions can be related to *sonata form.* Sonata form, the quintessential musical form of the Classic/Romantic style,

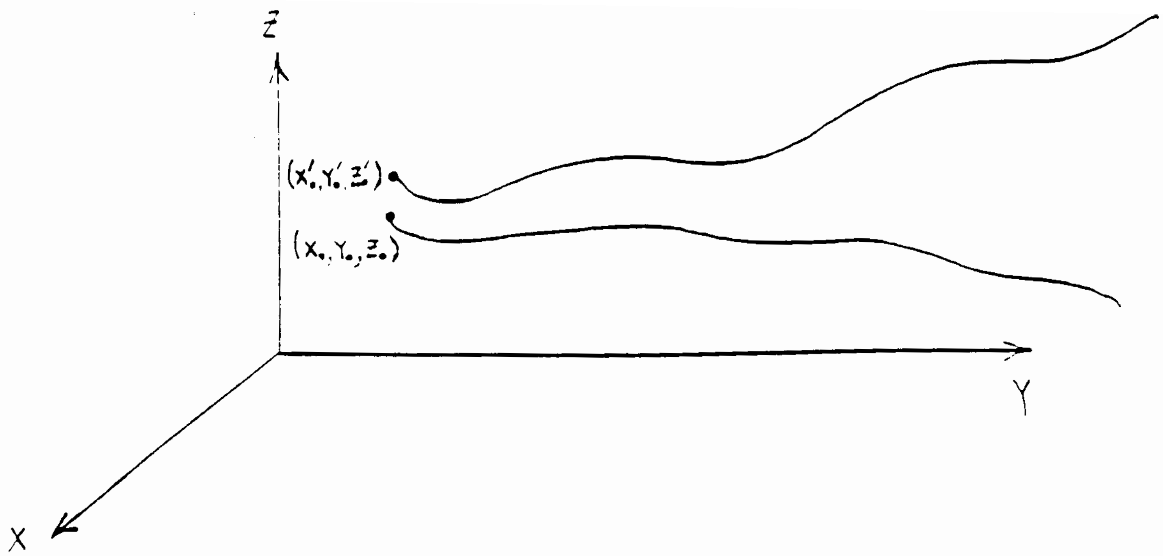


Figure D-10: System Trajectory in 3-space with Initial Conditions $x_0^{'} = 1.111$, $y_0^{'} = 2.221$, $z_0^{'} = 3.331$.

consists of (1) an exposition where thematic material is first introduced accompanied by a modulation away from the home key, (2) a development section which "develops" the material by modulating to keys, both near and far, and (3) a recapitulation which reiterates the material and maintains proximity to the home key or tonic. One of the structural lynchpins of this form is how the recapitulation often closely mirrors the exposition up to a certain point. While the exposition will deftly modulate from the tonic to a related key and firmly establish the new key by its closing note, the recapitulation will maintain a strong sense of the tonic key by refusing to migrate to the new key offered by the exposition. Indeed, a pianist's nightmare is missing the musical sleight-of-hand which maintains the recapitulation in the tonic key, only to find oneself back at the exposition, facing the prospect of performing the whole piece over again.

Now consider the first movement of Beethoven's Op. 57 Sonata. Note that the recapitulation, beginning at *m.* 134, mirrors the thematic material presented by the exposition as illustrated in Figure D-11. But by *m.* 151, the recapitulation journeys down a different path from that of the exposition. Specifically, compare *m.* 151 to *m.* 16, now shown in Figure D-12.

In the recapitulation, Beethoven inserts an *A*♮ where in the exposition he has written an *A*♭. This change of one note moves the recapitulation away from the keys of the exposition and instead to a new harmonic path or "trajectory." That is, the exposition opens in *f* minor and passes through *A*♭ major, concluding its section in *A*♭ minor. The recapitulation, on the other hand, also opens in *f* minor, but the *A*♮ in *m.* 151 results in the recapitulation modulating to *F* major, then *f* minor, the sonata's home key. Here, as elsewhere in this appendix, individual pitch lines and their attendant rhythmic lines constitute the *state variables*. The musical trajectory is determined by the pitch and rhythmic lines which, taken together, depict the harmonic progression. Starting values for the pitch and rhythmic line state variables

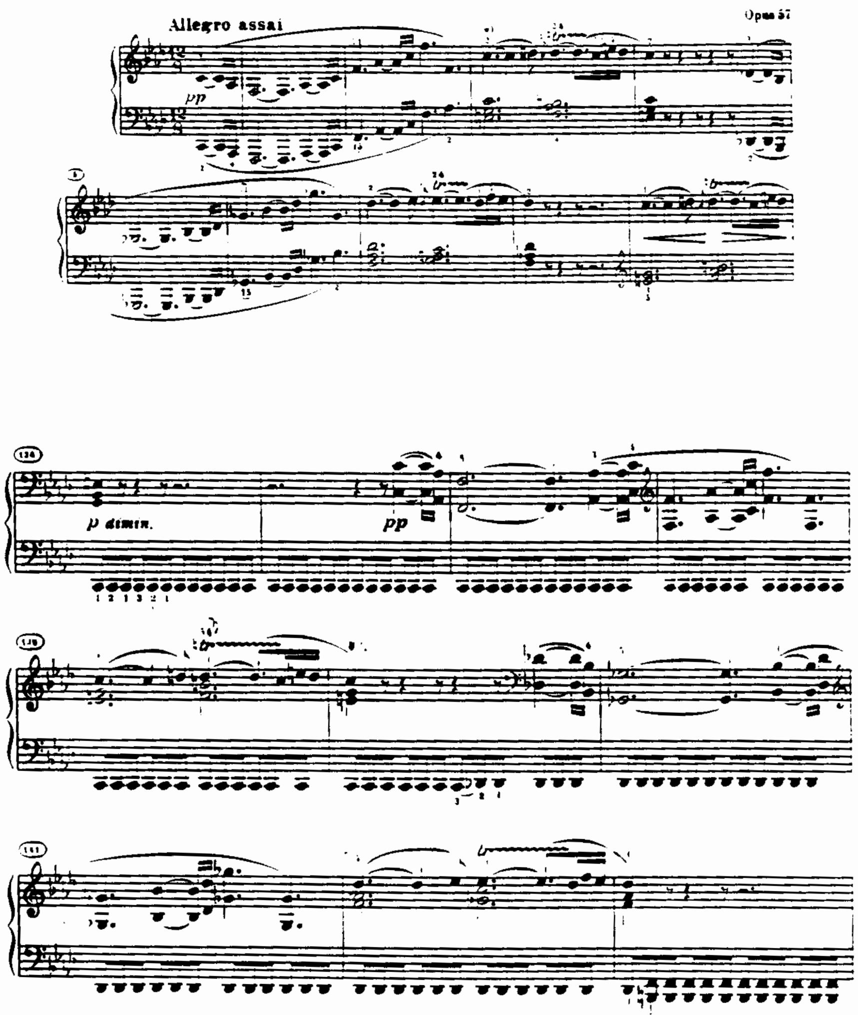


Figure D-11: *Top*, Opening theme of the exposition in the first movement of Beethoven's Op. 57. *Bottom*, Start of the recapitulation.

Figure D-12: Compare the pitch and rhythmic lines of m. 16 (*Top*) to the pitch and rhythmic lines of m. 151, (*Bottom*). Beat 604, located in m. 151 of the recapitulation, constitutes an "initial condition" which is very similar in state to m. 16 (beat 64) of the exposition. However, the pitch state at beat 604 ($C3$, $A\flat 2$), though differing by just one note from that of beat 64 (C, $A\natural$), results in the departure of the recapitulation from the harmonic path of the exposition. Thus two different harmonic paths arise from almost identical starting "values" for the pitch and rhythmic lines.

given on beat 4 of m. 16 (= beat 64 of the entire piece) could be designated as

$$RHPL(64) = C_3 \quad A\flat_2 \tag{D.7}$$
$$RHRL(64) = \text{♩‿𝅘𝅥𝅯 𝅘𝅥𝅯} \tag{D.8}$$
$$LHPL(64) = C_1 \quad A\flat_0 \tag{D.9}$$
$$LHRL(64) = \text{♩‿𝅘𝅥𝅯 𝅘𝅥𝅯} \tag{D.10}$$

where $RHPL(LHPL)$ and $RHRL(LHRL)$ are acronyms for right hand (left hand) pitch line or right hand (left hand) rhythmic line. Similarly, the starting values for the pitch line and rhythmic line state variables constituting beat 4 of m. 151 (= beat 604 of the whole piece) can be denoted as

$$RHPL(604) = C_3 \quad A\natural_2 \tag{D.11}$$
$$RHRL(604) = \text{♩‿𝅘𝅥𝅯 𝅘𝅥𝅯} \tag{D.12}$$
$$LHPL(604) = C_1 \quad A\natural_0 \tag{D.13}$$
$$LHRL(64) = \text{♩‿𝅘𝅥𝅯 𝅘𝅥𝅯} \tag{D.14}$$

Note that in dynamic system theory, an initial condition can be chosen to start anywhere. Measures 16 and 151 can be viewed as two closely related initial conditions, or starting states, within the invariant set consisting of the piece. Yet what ensues are two very different harmonic "trajectories."

At this point, it must be emphasized that a Beethoven sonata is a statement of **purposeful intent**, where every "change" in the piece is a planned-for event. This is not the case with a small change in, say, Lorenz's convection equations. Small disturbances in the atmosphere, produced by any number of uncoordinated natural or human factors, result in different trajectories. In the Beethoven sonata example, the different harmonic path taken by the recapitulation is intentional. Though a small change has occurred — the switch from $A\flat$ to $A\natural$, signifying the transformation from minor to major — another "disturbance", e.g., a wrong note struck by the performer would not lead to a new harmonic path for the recapitulation. In a weather system, though, each small change, whether planned or not, will result in a new trajectory. Once convection occurs in its chaotic regime, any number of small disturbances will result in variations. But in music, each change is a thought-out process. It is prepared musically, and its consequences are followed through.

In this thesis, however, *chaos* is used because it provides a natural mechanism for variability. The very fact that a new trajectury will produce a different path in state space is exactly what was considered valuable for creating a dynamic music, i.e., music that changes from one hearing to the next, as explained in Chapter 5. Because the chaotic mapping (fully explained in Chapter 2) taps into the pitch space of the original piece, variations often occur that sound musical, in part or whole. Some of the changes heard in the variations of the original work are worth pursuing further and the musician can at any point follow through on them. Alternatively, the musi-

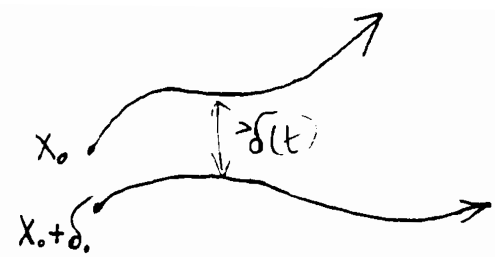


Figure D-13: The difference vector $\vec{\delta}(t)$ between two trajectories arising from two nearby initial conditions.

cian may choose to alter or eliminate any or all of the varied events.

D.2.6 Lyapunov Exponents

Lyapunov exponents are inexorably linked to the notion of sensitive dependence on initial conditions. Any unstable nonlinear system will exhibit sensitive dependence on the initial state vector. An invariant set possesses this property if the difference vector between two nearby trajectories $\vec{\delta}(t)$, grows exponentially with time. (See Figure D-13.) If it is found that numerically $\| \vec{\delta}(t)\| \approx \|\delta_0\|e^{\lambda t}$, then there exists sensitive dependence on the initial condition where $\lambda > 0$ is the system's most positive Lyapunov exponent. Note that an n-dimensional system will have n Lyapunov exponents.

Now, the feature that distinguishes sensitive dependence on initial conditions for an unstable nonlinear system from that exhibited by a chaotic system is the following: an unstable nonlinear system trajectory will veer off to infinity whereas a chaotic trajectory will remain within a bounded region of the system's state space. What restrains the chaotic trajectory?

If the system is chaotic and dissipative, its strange attractor must have at least one positive Lyapunov exponent and the sum of its n Lyapunov exponents must be negative. The required positive Lyapunov exponent accounts for the stretching or expansion of the fractal attractor, revealing the instability in the system. The negative sum of the Lyapunov exponents accounts for the folding over or contraction of the fractal attractor, ensuring it remains a bounded entity. This means that any trajectory starting at an initial condition within the strange attractor's basin of attraction will also remain within a bounded region of the system's state space since the trajectory is destined to forever asymptotically approach the strange attractor.

Furthermore, for two neighboring chaotic orbits, their separation, $\vec{\delta}(t)$, is an exponential function on average, and therefore not necessarily an exact exponential function. This parting of nearby paths makes it impossible to predict the system's future dynamics because in practical terms, it is very difficult to pinpoint exactly the initial condition. Often, all we can do is make a good estimate, but this is not good enough for accurate forecasting, as Lorenz saw.

Figure D-14: The four naturals in m. 67 cancel out the prior f minor key signature of four flats, marking the beginning of the development section.

A Musical Equivalent

Lyapunov exponents give a measure of how far the system has veered from its initial state, (or some other designated starting state). In musical terms, this is similar to asking how far the piece has strayed from its home key, (or any tonal center of interest). Interestingly, Shannon and Weaver pointed out in 1949 that the Lyapunov exponent may be interpreted via information theory as disclosing the rate of loss of information concerning the location of the initial condition.[19] For a musical interpretation, just substitute "tonic" for initial condition and "change in key" for Lyapunov exponent, so that "a change in key conveys a certain loss of information concerning the location of the tonic." Then look at development sections from any number of 18th and 19th century sonatas and symphonies. Specifically, consider the development section of the first movement of Op. 57. The exposition has concluded in $A\flat$ minor, the development opens with an E major broken triad which aurally obscures the key of f minor. Even visually, the old key has vanished, its four flats replaced by four naturals in m. 67 of Figure D-14.

The musical analog of positive and negative Lyapunov exponents can be described as "far and close digressions from the home key." Positive Lyapunov exponents would correspond to those harmonic digressions which are far removed from the home key, i.e., digressions which are distant from the home key. For example, a modulation from the key of f minor to its tritone (b minor) — or a modulation from f minor to (D major), the relative major of b minor — constitutes the farthest migration possible since the tritone is considered the most distant key from the tonic in Western harmonic practice.

One way to establish the *degree of digression (DOD)* from the home key is to use the circle of fifths as a barometer of how far removed the new key is from the home key. For example, consider C major the tonic key and make the following pairing:

[19]Drazin, P. G. (1992) *Nonlinear Systems* (Cambridge University Press, Cambridge) 141.

$$\begin{array}{ccccccccccccc} G\flat & D\flat & A\flat & E\flat & B\flat & F & C & G & D & A & E & B & F\sharp \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ e\flat & b\flat & f & c & g & d & a & e & b & f\sharp & c\sharp & g\sharp & d\sharp \end{array} \tag{D.15}$$

where upper case letters designate the major keys, lower case denote the relative minors, and numbers show the degree of digression from C major. Therefore the number 6 which is affiliated with the keys of $F\sharp$ and $G\flat$ major as well as their relative minors, $d\sharp$ and $e\flat$ minor, indicate that these keys are the farthest removed from the home key of C major. They also represent the greatest digression from a minor, the relative minor of C, since a minor and C major both share the same key signature.

Therefore, how distant the modulation moves from the tonic can be conveyed by the degree of digression, an integer between zero and six. If the $DOD = 0$, then the modulation has gone to the most closely related key which is the key which shares the same key signature as the tonic. Therefore, a modulation to a minor from the key of C major signifies a DOD of zero, since a minor and C major share the same key signature. Because of this, they are considered "relatives" — a minor is the relative minor of C major and C major is the relative major of a minor. However, if the $DOD = 6$, then the modulation has resulted in a key farthest away from the tonic — the tritone — something that would occur if C major migrated to $F\sharp(G\flat)$ major or $d\sharp(e\flat)$ minor.

Similarly, for the home key of Op. 57 — f minor — the following mapping illustrates the DOD numbers:

$$\begin{array}{ccccccccccccc} D & A & E & C\flat & G\flat & D\flat & A\flat & E\flat & B\flat & F & C & G & D \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ b & f\sharp & c\sharp & ab & eb & bb & f & c & g & d & a & e & b. \end{array} \tag{D.16}$$

Again, a DOD number of zero means that very little "distance" separates the key of f minor from its relative major, $A\flat$; i.e., a modulation to the key of $A\flat$ major from an initial harmonic state of f minor shows that the harmonic trajectory has not veered by much from its starting key of f. This corresponds nicely with the notion of the Lyapunov exponent since, the less positive the Lyapunov exponent, the more close the system trajectory is to its initial condition. But one caveat exists: Theoretically, Lyapunov exponents can be any positive or negative *real* number whereas DOD numbers would have to be confined to the integers between zero and six, at least for Western harmonic practice.

D.2.7 Bifurcation and Equilibrium Points

Bifurcation theory is almost as old as algebra itself. The motivating ideas behind it developed slowly and imperceptibly at first,[20] progressing from observations of simple quadratic equations to full-scale understanding of complex dynamical behavior. Es-

[20] Drazin, p. 3.

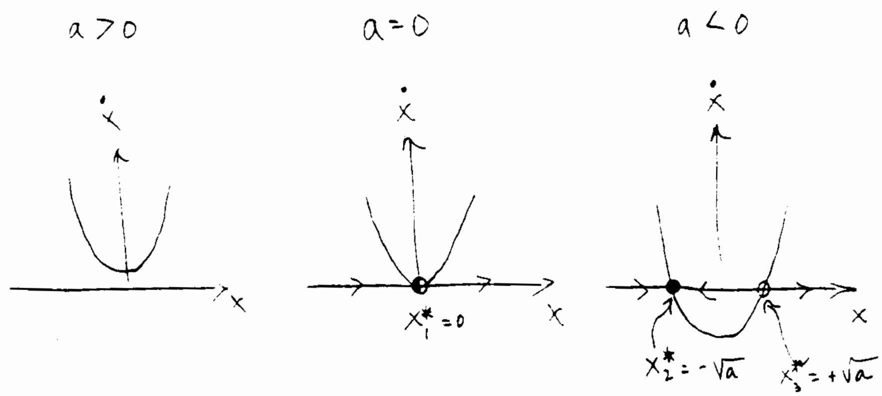


Figure D-15: Fixed points for the system $\frac{dx}{dt} = x^2 + a$ with $a > 0, a = 0, a < 0$. A blackened circle indicates a stable fixed point, an open circle designates an unstable fixed point, and a half-black/half-open circle shows a bistable fixed point.

sentially a bifurcation signals a change in the qualitative behavior of a system. It occurs when a system parameter, such as a variable resistor in a circuit, passes through a critical value known as a bifurcation point.

Consider the first order system

$$\frac{dx}{dt} = x^2 + a, \tag{D.17}$$

where a is a parameter of the system. Define *equilibrium point* or *fixed point* of a nonlinear system as a solution to the governing nonlinear differential equation for all time, meaning it is a steady state solution. To find the fixed points, set $\frac{dx}{dt} = 0$ and solve for x_1^*, x_2^*, x_3^*, the fixed points.

For $a < 0$, the equilibrium points are

$$x_{2,3}^* = \pm\sqrt{-a}. \tag{D.18}$$

For $a > 0$, no fixed points exist since they can't be imaginary. For $a = 0$, $x_1^* = 0$. These three cases are shown in Figure D-15.

Why would a change in the parameter a, which could result in a change in the number of equilibrium points, be so significant? Recall if a is positive, no fixed points exist. But if $a = 0$, then a *bistable* fixed point appears, with flow either towards or away from $x_1^* = 0$, depending on where one starts the initial condition. (Note that fixed points represent "resting place solutions" or "equilibrium solutions" since if $x(t) = x^*$ at $t = 0$, then the flow or trajectory $x(t)$ will remain at x^* for all time.[21]) Therefore, if the initial condition $x(0) < 0$, the flow will return to $x_1^* = 0$ so that if the system is started with a negative initial condition, it stays stable. If $x(0) = 0$, the flow

[21]Strogatz, p. 6.

completely stagnates at the fixed point: no movement takes place. But if $x(0) > 0$, the trajectory shoots off to $+\infty$, never to return to its initial position of $x_1^* = 0$.

What is perhaps even more common in system theory, though, is the following scenario. The system is indeed started at the equilibrium point $x(0) = x_1^* = 0$, but it accidentally gets "shaken", that is, there is a perturbation that jars the trajectory away from the equilibrium point. Now if that perturbation results in the flow moving left — to the stable side of x_1^* — then all is well and the system keeps chugging along. But if the unplanned disturbance moves $x(t)$ to the unstable side of equilibrium, then the system ceases operating as designed and its trajectory, or solution, "blows up": it reaches infinity in a finite amount of time.

This can be seen by solving the nonlinear differential equation (1) for $a = 0$. Its solution, verified by direct substitution, is given by

$$x(t) = \frac{x_0}{1 - x_0 t}, \tag{D.19}$$

where the initial condition $x(0)$ is specified by x_0. $\forall\ x_0 \leq 0$ as $t \to \infty$, $x(t) \to 0$. But $\forall\ x_0 > 0$ as $t \to \frac{1}{x_0}$, $x(t) \to \infty$. Therefore, for $a = 0$, the solution veers off to infinity if $x_0 > 0$ and if t approaches $\frac{1}{x_0}$ from the right or left.

Assessing the stability of fixed point solutions to nonlinear dynamical equations reveals the behavior of these systems as a parameter is varied. In this case, as a decreases in positive value and hits zero, the system goes from having no fixed points to one bistable fixed point. But what happens if a is further decreased in Eqn. D.17 and takes on negative values? To determine the stability of the fixed points $\pm\sqrt{a}$, let $\xi(t) = x(t) - x_{2,3}^*$ be a small perturbation away from $x_{2,3}^*$. To see whether the perturbation grows or decays, derive a differential equation for $\xi(t)$:

$$\dot{\xi}(t) = \frac{d}{dt}(x - x_{2,3}^*) = \dot{x} = \mathcal{F}(x) = \mathcal{F}(x_{2,3}^* + \xi) = x^2 + a. \tag{D.20}$$

The Taylor expansion gives

$$\mathcal{F}(x_{2,3}^* + \xi) = \mathcal{F}(x_{2,3}^*) + \xi\mathcal{F}\prime(x_{2,3}^*) + \mathcal{O}(\xi^2), \tag{D.21}$$

where $\mathcal{F}\prime$ signifies the first derivative of the function $\mathcal{F}$.

From the definition of fixed point, $\mathcal{F}(x_{2,3}^*) = 0$. The higher order terms can also be neglected, provided that $\xi\mathcal{F}\prime(x_{2,3}^*) \neq 0$ and ξ is very small. Then

$$\dot{\xi} = \xi\mathcal{F}\prime(x_{2,3}^*) \tag{D.22}$$

$$= (x - x_{2,3}^*)\mathcal{F}\prime(x_{2,3}^*). \tag{D.23}$$

If $\mathcal{F}\prime(x_{2,3}^*) > 0$, $x_{2,3}^*$ is unstable. Similarly, if $\mathcal{F}\prime(x_{2,3}^*) < 0$, $x_{2,3}^*$ is stable. From this it is clear that the equilibrium point $x_2^* = -\sqrt{a}$ is stable while the equilibrium point $x_3^* = \sqrt{a}$ is not.

A bifurcation diagram, given in Figure D-16, shows all stable and unstable fixed points as the parameter a is varied. For $a < 0$, the equilibrium points are $x_{2,3}^* = \pm\sqrt{a}$,

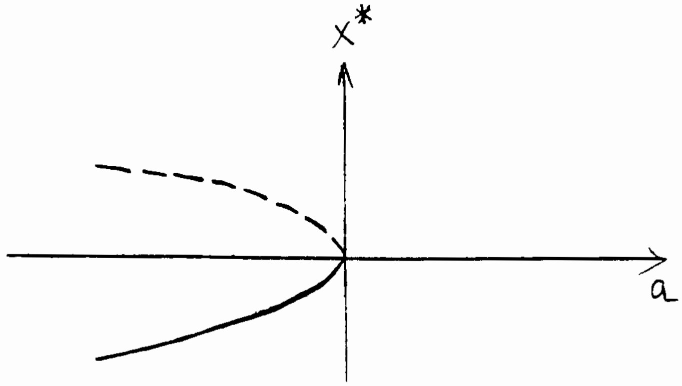


Figure D-16: Bifurcation diagram for the nonlinear dynamical system $\frac{dx}{dt} = x^2 + a$ as the parameter a is varied.

where $x_2^* = -\sqrt{a}$ is stable (indicated by the solid curve) and $x_3^* = \sqrt{a}$ is unstable (indicated by the dashed curve). For $a = 0$, only one equilibrium point exists, $x_1^* = 0$, which is bistable. For $a > 0$, no equilibrium points exist. Thus, as a decreases from positive values to 0, Eqn. D.17 acquires one bistable equilibrium point. Then, as a assumes negative values, perhaps due to aging in the system, two equilibrium points exist, one stable and the other unstable.

Musical Equilibrium Points

From a musical point of view, equilibrium points can be said to represent tonal centers in a piece. A tonal center is an established key present at the start of a piece or arrived at by means of a modulation. Some of these tonal centers are inherently unstable, especially those far removed from the home key. The ear discerns that the piece cannot linger forever in these far away places; eventually it must return to the home key, often by way of related keys, for a conclusion. An unstable tonal center is one in which the harmonic trajectory does not linger long. Rather, the unstable tonal center repels the musical path towards other tonal centers, which may be more or less stable. The rate at which different keys are visited and left can also convey instability and distance from the home key. Specifically, consider the E major/minor tonality of the opening 14 bars of the Op. 57 development section, first movement. The tonal center E/e is unstable, because the harmonic trajectory quickly moves away from E/e, never to return to this tonality.

By contrast, the stable equilibrium point (or stable tonal center) in the piece is the home key of f minor. If perturbed greatly or just a little from this tonic, the piece nevertheless returns to f, sooner or later, in every instance. Thus the key of f minor is globally stable, i.e., stable throughout the context of the whole first movement. In fact, the home key of virtually any work composed during the Baroque, Classic or Romantic periods can be considered globally stable in that the piece eventually returns to this key by its final bar.

Figure D-17: Modulation, starting in m. 23, from the tonic of f minor to $A\flat$, the relative major.

As seen earlier, some equilibrium points can be thought of as bistable (in first order systems) and neutrally stable in second order and higher dimensional systems. A bistable equilibrium point is "half-stable": it attracts flow from one side and repels flow from the other. Neutrally stable systems hold their fragile stability as if living in glass houses. Their lives, spent on the cusp of instability, lead many to designate them "not stable" systems. The bistable/neutrally stable parlance of nonlinear dynamics also describes those harmonic digressions that, though departed from the home key, are nonetheless related to it. One only has to look at the modulation over 12 measures from the home key of f minor to its relative major, $A\flat$, in the first movement of Op. 57, reproduced in Figure D-17. The listener knows $A\flat$ is smoothly related to f minor, yet she is also cognizant of the fact that the key of $A\flat$, though related to f minor, will time and time again assume the "not stable" role of a harmonic palette changer, while maintaining its "stable" role as significant relative to the home key.

Figure D-18: A "prepared" musical bifurcation is heard as the pianist drives for the *Presto* which marks the start of the coda. Here, the bifurcation point would be m. 308, where the *Presto* begins.

Musical Bifurcation

A dynamic system may possess several parameters, each with its own bifurcation point, so that its behavior is intricately entwined with the setting off of various bifurcations, both as solo events and in tandem with others. For example, inserting two variable resistors in Leon Chua's *Double Scroll* circuit unleashes a host of bifurcations triggered by the changing resistive values. In a musical system, such inputs as text, timbre, pedaling, accents, dynamic and expression markings vary — sometimes smoothly, sometimes abruptly. Not all changes in the musical inputs will result in a change in the musical character of the piece. But there are times when a critical point is reached and a powerful charge can infuse the music. A smooth transition to a fortissimo climax in a symphonic work might result from marshaling all the available musical parameters, varying them just the right amount, so that they simultaneously work to build a climactic change in the work's ambience, thus creating a musical bifurcation. A good example of this might be the vast C major section occurring midway through Bartok's One-Act Opera, *Bluebeard's Castle*, providing the arch structure which builds the whole piece. Another example would be the Maestoso section following the piano cadenza in the last movement of of Rachmaninov's *Second Piano Concerto*. Finally, the third movement of Beethoven's Op. 57 varies the inputs at hand — dynamic markings, tempo and implied use of pedal — as the music hurls itself, yet with all due preparation, into the *Presto* coda shown in Figure D-18. In this case, all the inputs are increasing in intensity. In another example, the inputs might be steadily decreasing in intensity. Still other instances of musical bifurcation might involve small variations in one input, while other inputs are varied up or down, in smaller or larger increments.

Figure D-19: An unprepared musical "bifurcation", i.e., a musical "catastrophe", occurs with the *forte* (**f**), *a tempo* dominant seventh chord which begins at the tail end of m. 13 and continues through m. 15, as well as in the *subito piano* of m. 16.

Musical Catastrophe

There exists a special nonlinear dynamical term for a bifurcation which occurs when the parameters have varied smoothly, yet an abrupt change in the response of the system results. This is known as a *catastrophe*. Composers have always been aware of the necessity for sudden shifts in timbre, accents, dynamic and expression markings, etc. Haydn, irritated that his aristocratic patrons continued to doze off during the slow movements of his symphonies, decided to insert a surprise — a sudden fortissimo chord rudely punctuating the most innocuous melody, jarring any sleepy count who had been lulled into a sense of false tranquility. The result — one of the more famous musical "catastrophes".

Several instances of "catastrophes" are heard in Beethoven's Op. 57, e.g., the *subito forte* (**f**) dominant seventh chord in m. 13 which startles the *morendo* feel of measures 10-13. Figure D-19 shows the thinning texture accompanied by the markings — ***pianissimo***, *poco ritardando* — such a contrast to the dramatic outburst that follows. Note that another musical "catastrophe" takes place in m. 16 with the sudden *piano* (**p**) and *pianissimo* (**pp**) lines. Whereas a catastrophe in a dynamic system is not a desired outcome — it results in the loss of planned-for behavior — a musical "catastrophe" is an intentional event. The *subito forte* and *subito piano* discussed above create accents[22] which have structural meaning.

However, perhaps the most arresting catastrophe is found in the "poised-to-pounce" transition between the second and third movements. As the Andante movement winds down as indicated by both dynamic and expression markings, suddenly a *fortissimo* (**ff**) diminished seventh chord punctures the calm with an *attacca l'Allegro* instruction urging the performer to proceed right into the fortissimo opening of the third movement. Here is a resounding *subito catastrofe*, occuring in m. 97 of Fig-

[22] Accent is synonymous with contrast.

Figure D-20: A musical "catastrophe" ignites the last measure of the second movement, thus setting the stage for the tumult of the third, and final, movement.

ure D-20.

What was quietly leading up to an expected cadence in *D*♭ major, the home key of the second movement, has now turned into a harmonic free-for-all. Beethoven could not have chosen a more unstable chord to close this movement than the diminished seventh, known for its tonal ambiguity in that it can lead to any one of 24 major and minor keys. Here, the **pp** diminished seventh of *m.* 96 followed by the **ff** diminished seventh chord of *m.* 97 create accent with implications beyond this movement. The expected cadence which would have concluded the second movement does not occur. The expected structural downbeat is deferred until *m.* 20 of the third movement.

D.3 Conclusion

Nonlinear dynamics and chaos provides a rich language which can be used, in part, to discuss music of the Baroque, Classic and Romantic eras. Such phenomena as Lyapunov exponents, bifurcation, equilibrium points, and catastrophe are all common to nonlinear dynamical systems, while sensitive dependence on initial conditions, strange attractors, and self-similarity are particular to chaotic dynamics. It is this cornucopia of "wild things" that spills over so bountifully into the combined art of musical description and musical listening.

Using the intent behind Lyapunov exponents to motivate a degree of digression (DOD) number can help quantify the distance between the tonal centers achieved at the close of the exposition and recapitulation, respectively. These tonal centers, or equilibrium points, can be stable or unstable, with the home key acting as a globally stable equilibrium point. As certain inputs such as text, timbre, pedaling, dynamic and expression marks are varied, the music can undergo a bifurcation — a change in its qualitative behavior. A simple example of this might be a *crescendo* from *piano* to a climactic *forte*. But sometimes, despite a smooth varying of the musical parameters, a sudden change arises in the music. This bifurcation is known as a catastrophe, e.g., a *subito fortissimo* passage immediately following a *sotto voce, morendo* line. But whereas a bifurcation in a dynamic system is not always a planned-for event, a musical "bifurcation" is intentional. Sudden changes in the musical dynamics create accents which can have much larger structural consequences, e.g., the subito ff cadence at the start of the Coda in the first movement of Op. 57. This cadence provides the most significant downbeat of the entire movement.

The nonlinear properties discussed above also apply to chaotic systems since they constitute a subset of nonlinear dynamics. But chaos introduces additional dynamic behavior. The sensitive dependence of chaotic trajectories to initial conditions can be linked to the musical sleight-of-hand (often a small change) that retains the recapitulation in the home key. Again, it must be kept in mind that the small changes that can propel a musical work to a new harmonic path, or counter the listener's expectation, are purposefully placed there. In a chaotic dynamic system, *any* small disturbance, whether planned or not, will lead to varied trajectories. But it is exactly this feature of a chaotic system that provides the engine for the variation technique, the main result of this thesis. It is shown in Chapter 2 how the chaotic mapping moderates such variability and generates musical variations based on the pitch space of the original piece. The variation technique can stimulate the musicality of the composer; the variations it produces are totally dependent on the piece provided by the musician; the musician determines how close/far the variations will remain to the original and also decides which variations are fuel for future thought and which are discarded.

Finally, there are some nice similarities between the musical form of *theme and variation* and a strange attractor. In particular, a case can be made for considering the Andante movement of Beethoven's Op. 57 as an attractor that could be called chaotic. The arguments presented in support of this are (1) the Andante can be considered a musical invariant set; (2) its basin of attraction can be said to comprise all musical states within the first movement; (3) the self-similarity evidenced by the 2-group phrase structure prevails throughout the movement; (4) the similarity between the large scale metrical structure of the entire second movement reflects the smaller scale metrical structure of the first half of the Theme; and (5) the fine structure which reveals itself at all levels of concentration.

Appendix E

Glossary of Musical Terms

An **appoggiatura** is an accented melodic dissonance which resolves by half or whole step, up or down, on a weaker beat or beat division[1]. The appoggiatura often leans towards a specific note of resolution and thus creates an expectation which is fulfilled when it resolves[2].

A **blue note** is a microtonal variant of a pitched note, usually flatted from the pure intonation of the note. It is most often associated with the third, fifth and seventh degrees of the scale[3].

A **harmonic progression** is a succession of root chords, represented by roman numerals indicating the scale degrees upon which the chords are built. These chords follow one another in a controlled and orderly way, according to principles of good voice leading[4]. The tonic chord of a piece is designated by I, its supertonic by II, the dominant by V, and the leading tone by VII. The tonic chord is built on the first degree of the scale, the supertonic on the second degree, and the dominant on the fifth and the leading tone on the seventh. Arabic numerals designate the inversion of the chord.

Harmonic rhythm is the rhythmic pattern provided by the changes of root harmony as they occur in a musical composition[5].

The **inversion** of a note sequence is found by changing each ascending interval into the corresponding descending interval, and vice versa[5].

The **neighbor note** or auxiliary is an unaccented tone one step above or below a harmonic tone, which returns immediately to the same tone[6]. The neighbor note is not necessarily a dissonant tone. An incomplete neighbor note results when the harmonic tone either does not precede or follow the neighbor note[4]. If the incomplete neighbor note is a step above (step below) the harmonic tone, it will be called an *upper neighbor* (*lower neighbor*) to that note.

Passing tones fill in a melodic skip on all intervening steps, either diatonic or chromatic[4].

To **retrograde** an ordering is to list it backwards, i.e., the ordering now begins with the last event and ends with the first[7].

The **retrograde inversion** is the reversal of the inverted note sequence.

A true **sequence** is the systematic transposition of a harmonic pattern and its associated melodic and rhythmic patterns to other degrees of the scale. Most theorists agree that a single transposition of such a pattern does not constitute a full sequence. the systematic transposition not having been established until the third occurrence of the initial pattern. Three separate appearances, involving two transpositions, are needed to show that the transposition interval is consistent. A **half sequence** is one in which only a single transposition of a harmonic pattern, and its associated melodic and rhythmic patterns. occurs[1]. A **harmonic sequence** occurs when a harmonic progression is immediately restated, starting on another degree of the scale[4].

Schenkerian analysis is the term applied to the analytic method proposed by Heinrich Schenker (1868-1935). Schenker used structural levels to provide a hierarchical differentiation of musical components. This hierarchical approach establishes a basis for describing and interpreting relations among the elements of any composition, from the moment-to-moment events at the surface of the music to longer range connections that ensure continuity and coherence over the scope of the entire piece[8]. A **foreground analysis** of a musical work represents only a partly reduced form of the piece and encompasses those musical components closest to the surface of the music. A **middleground analysis** presents an intermediate stage of reduction, while a **background analysis** gives a complete reduction of the piece, down to its bare harmonic essentials. There are any number of foreground, middle ground and background levels of analysis possible for a given composition because the analytic process is often subject to interpretation[9]. In Schenker's view, the total work at all levels, not only at the background layer, is the object of aesthetic perception and study. His last book, and final statement on this subject, is devoted to the consideration of foreground details and the meaning that they derive from the other structural layers[8].

A **turn** alternates upper and lower neighbor notes with the main note: upper neighbor, main note, lower neighbor, main note. An inverted turn follows the form: lower neighbor, main note, upper neighbor, main note[10].

A **vamp** is an accompanimental or transitional chord progression of indefinite duration, used as a filler until a soloist is ready to start or continue[3].

References

1 R. Donington, "Ornaments," in *New Grove Dictionary of Music and Musicians*, edited by Stanley Sadie (MacMillan, London, 1980) **13**, 828-834. See also W. Piston, *Counterpoint* (Norton, New York, 1947).

2 W. Drabkin, "Non-harmonic note," in *New Grove Dictionary of Music and Musicians*, edited by Stanley Sadie (MacMillan, London, 1980) **13**, 269 (1980).

3 G. Schuller, *Early Jazz* (Oxford University Press, New York, 1968).

4 W. Piston, *Harmony*, Fifth Edition, revised by M. DeVoto (Norton, New York, 1987).

5 W. Apel, *Harvard Dictionary of Music* (Belknap Press, Cambridge, Massachusetts, 1972).

[6] R. Sessions. *Harmonic Practice* (Harcourt, Brace, World, New York, 1951).

[7] J. Rahn, *Basic Atonal Theory* (Schirmer Books, New York, 1980).

[8] A. Forte, "Heinrich Schenker," in *New Grove Dictionary of Music and Musicians*, edited by Stanley Sadie (MacMillan, London, 1980) **16**, 628.

[9] I. D. Bent, "Analysis," in *New Grove Dictionary of Music and Musicians*, edited by Stanley Sadie (MacMillan, London, 1980) **1**, 361.

[10] Donington, 850-51.