

CENTRAL LIMIT THEOREMS FOR
D[0,1]-VALUED RANDOM VARIABLES

by

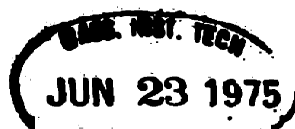
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ABSTRACT

Let $\{X(t) : t \in [0,1]\}$ be a stochastic process. Suppose $E(X(t)-X(s))^2 \leq f(|t-s|)$. We find conditions on f in order to establish the existence of a version of X with continuous sample paths. Furthermore, under these conditions X satisfies the central limit theorem in $C[0,1]$. Counterexamples are given to show that in a number of cases the results are best possible. For processes with only jump discontinuities satisfying weaker conditions on the second moments of increments, supplementary conditions are found to insure that the central limit theorem holds in $D[0,1]$. In particular, stochastically continuous independent increment processes satisfy the central limit theorem in $D[0,1]$. The case of processes with fixed discontinuities is treated separately.

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INTRODUCTION

Among the most striking features of the classical central limit theory for real-valued independent identically distributed random variables is the wide class of distributions for which weak convergence occurs. The central limit theorem asserts that there is weak convergence to the normal law if and only if the random variables have finite second moments. Because of the fundamental role of this theorem in probability and statistics, it is only natural to seek an analogue for stochastic processes. Since each stochastic process is a random variable in an appropriate function space, the natural setting for the problem is in terms of function space-valued random variables.

Examples of Strassen and Dudley (1969) and Dudley (1974) show that the classical assumption of finite second moments is not sufficient in general function spaces. The difficulty in trying to extend the classical technique of proof, which involves characteristic functions, is that there is no obvious analogue of the Lévy Continuity Theorem, even when characteristic functions can be defined.

Some of the first function space central limit theorems were proved for spaces in which an analogue of the Lévy Continuity Theorem could be found. In their work on G-spaces,

which include the L^p spaces for $2 \leq p < \infty$, Mourier and Fortet (1955) obtained a continuity theorem by taking advantage of the inequality

$$E ||X_1 + \dots + X_n||^2 \leq A \sum_{i=1}^n ||X_i||^2,$$

valid for X_i independent random variables with A constant. Also, using characteristic functions, Varadhan (1962) found necessary and sufficient conditions for the central limit theorem to hold in Hilbert space.

Because of the insufficiency of the second moment condition for the central limit theorem in general function spaces, it is important to classify those spaces in which it does suffice. Hoffmann-Jørgensen (1974) has recently obtained with Pisier such a classification for Banach spaces. A Banach space E is called type 2 if, whenever (ϵ_j) is a Rademacher sequence and (y_j) is a sequence in E such that $\sum ||y_j||^2 < \infty$, $\sum \epsilon_j y_j$ converges a.s. Hoffmann-Jørgensen and Pisier have shown that the central limit theorem holds for all sequences of independent identically distributed E -valued random variables with only the assumption that second moments exist if and only if E is of type 2. This result completes the work begun by Mourier and Fortet.

The spaces $C(S)$ of continuous functions on a compact metric space S need not be of type 2. For example, $C([0,1])$ is not of type 2. Since every separable Banach space is linearly isomorphic to a subspace of $C(S)$ for some S , it is natural that most recent work on central limit theorems has been for $C(S)$.

Strassen and Dudley (1969) introduced metric entropy considerations into the problem. They proved a central limit theorem for $C(S)$ -valued random variables with paths satisfying the Lipschitzian condition

$$|X(t,\omega) - X(s,\omega)| \leq M(\omega)\rho(s,t),$$

where $M \in L^\infty(\Omega)$ and ρ is a pseudo-metric satisfying a metric entropy condition. Giné (1974) extended these results to the case where $M \in L^2$; Dudley (1974) interpolated between the results for $p = 2$ and $p = \infty$. Culminating this line of investigation is the theorem of Jain and Marcus (1974), which improves the entropy condition.

Other lines of investigation have been followed by de Acosta (1970), who works on Banach spaces with a Schauder basis; Le Cam (1970), who establishes a concentration inequality which is valid in any locally convex space, and Araujo (1974) whose work includes a Lévy-Khintchine formula for separable Banach spaces.

One of the earliest central limit theorems in a function space was formulated by Donsker (1952) for the special case of empirical distributions. Donsker considered the space of all bounded real-valued functions on \mathbb{R} with the supremum norm. Dudley (1966) justified the weak convergence in this non-separable space. It had been noted earlier that since the empirical processes $Y_n(t) = \sqrt{n}(F_n(t, \omega) - F(t))$, where F_n is the empirical distribution function of F , have only jump discontinuities, the results could be made precise by introducing the space of functions with only jump discontinuities and the Skorohod topology. Gikhman and Skorohod (1969) have done precisely this by realizing the empirical processes, for $0 \leq t \leq 1$, as $D[0,1]$ -valued random variables. $D[0,1]$ is the space of real-valued functions on $[0,1]$ which are right continuous with left limits. For extension of the empirical processes to multi-dimensional parameters, see Dudley (1966), Straf (1970), and Bickel and Wichura (1971).

Processes with only jump discontinuities seem to be the logical object of investigation after continuous processes; and $D[0,1]$ is certainly the natural space in which to consider such processes on $[0,1]$. In this thesis we address ourselves to three main questions. First, when

do second moment conditions on the differences $X(t)-X(s)$, i.e., conditions of the form $E(X(t)-X(s))^2 \leq f(|t-s|)$, provide sufficient conditions for central limit theorems? Dudley (1974) shows that this type of condition is not sufficient in general. However, we show, in Chapters 5 and 7, that processes which satisfy the central limit theorem in D must have certain continuity properties in quadratic mean. Furthermore, we show that if f satisfies a simple integral condition then X has a separable version which is continuous a.s. and the central limit theorem is always satisfied for X . Examples are given in Chapter 4 to show that there are processes with no finite-valued separable version, and hence no continuous finite-valued version, for which the integral condition barely fails. Furthermore, examples are given to show that there are continuous processes which do not satisfy the central limit theorem and for which the integral condition barely fails.

Second, what additional assumptions on stochastically continuous processes yield good central limit theorems? Our main result is a condition on the second moments of the products of the increments $X(t)-X(s)$ and $X(t)-X(s)$ for $s \leq t \leq u$. The condition is natural in the sense that it is also a sufficient condition for sample

paths to be in D . As a consequence we can show that all stochastically continuous processes with independent increments and sample paths in D satisfy the central limit theorem. Another consequence is a central limit theorem for Markov processes from which we conclude that all finite state stochastically continuous Markov processes with sample paths in D satisfy the central limit theorem.

Third, how can processes with fixed discontinuities be handled? We show that if a process can be decomposed into the sum of two or more processes, each with finite second moments, where at most one of the processes has fixed discontinuities, then it suffices to show that each process in the sum separately satisfies the central limit theorem. Utilizing this method we show that the central limit theorem for processes which are stochastically continuous except for a finite number of fixed discontinuities can be reduced to the stochastically continuous case. The case of processes with a countable number of discontinuities, all fixed discontinuities, may be reduced to studying the central limit theorem for continuous functions on an appropriate compact subset of \mathbb{R} .

CHAPTER 1. PRELIMINARIES

This section includes definitions, properties of the space $D[0,1]$, and background material on the central limit theorem. Billingsley (1968) has given an excellent treatment of the subject of weak convergence of probability measures on the function spaces $C[0,1]$ and $D[0,1]$. Throughout this section all reference to Billingsley (1968) will be given by page number alone.

§1. THE SPACE D

Let $D \equiv D[0,1]$ denote the space of equivalence classes of all real-valued functions $x(t)$ on $[0,1]$ having right and left limits at every point, where two functions are considered equivalent if they coincide at all points of continuity. We adopt the convention of Billingsley that the representatives of each equivalence class are right-continuous with left limits:

$$(i) \quad \text{for } 0 \leq t < 1, \quad x(t+) = \lim_{s \downarrow t} x(s) \quad \text{exists and} \\ x(t) = x(t+);$$

$$(ii) \quad \text{for } 0 < t \leq 1, \quad x(t-) = \lim_{s \uparrow t} x(s) \quad \text{exists.}$$

This differs from both Parthasarathy (1967) and Gikhman and Skorohod (1969) who also require left continuity at $t = 1$. Henceforth we will tacitly assume that the space D

consists of those functions satisfying (i) and (ii).

This space is natural for studying stochastic processes with jump discontinuities. If a stochastic process X has almost all its sample paths in D , then X may be thought of as a D -valued random variable, i.e.

$X : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow D$ such that for each ω , $X(\omega) \in D$ and the value of $X(\omega)$ at t is $X(t, \omega)$. For example, separable stochastically continuous processes with independent increments (Gikhman and Skorohod (1969) p. 168), separable stochastically continuous sub-martingales (Doob (1953) p. 361), and Markov processes under extremely broad conditions (Kinney (1953)) all have versions with sample paths in D . Of course, $C \equiv C[0,1]$ is a subset of D .

Several important properties of functions in D are a consequence of the following lemma.

LEMMA 1.1 (p. 110) For each $x \in D$ and each positive ε , there exist points t_0, t_1, \dots, t_r such that $0 = t_0 < \dots < t_r = 1$ and

$$\sup\{|x(s) - x(t)| : s, t \in [t_{i-1}, t_i]\} < \varepsilon, \quad i = 1, \dots, r.$$

From this lemma it follows that $x \in D$ is bounded and has only countably many discontinuities (all jumps) of which there can be at most finitely many points t at

which the jump $|x(t) - x(t-)|$ exceeds a given positive number.

The uniform metric is too strong for many purposes since with it D becomes nonseparable. To avoid this difficulty Skorohod (1956) introduced the following topology on D . Let Λ denote the class of strictly increasing, continuous mappings of $[0,1]$ onto itself. For $\lambda \in \Lambda$, $\lambda(0) = 0$ and $\lambda(1) = 1$. The metric $d(x,y)$, for $x,y \in D$, is defined to be the infimum of those positive ϵ for which there exists $\lambda \in \Lambda$ such that

$$\sup_t |\lambda t - t| \leq \epsilon$$

$$\sup_t |x(t) - y(\lambda t)| \leq \epsilon.$$

A sequence of elements $\{x_n\}$ belonging to D converges to a limit x in the Skorohod topology if and only if there exist functions $\lambda_n \in \Lambda$ such that both $\lim_{n \rightarrow \infty} x_n(\lambda_n t) = x(t)$ and $\lim_{n \rightarrow \infty} \lambda_n t = t$ uniformly in t . When relativized to C , this topology gives the usual uniform topology. The physical interpretation of the Skorohod topology is that time can be measured no more accurately than position.

With the metric d , D is separable but not complete. However, there is another metric d_0 , generating the same Skorohod topology, under which D is a complete separable

metric space. $d_0(x,y)$ is defined to be the infimum of those positive ϵ for which there exists $\lambda \in \Lambda$ such that

$$\sup_{s \neq t} \left| \log \left| \frac{\lambda t - \lambda s}{t - s} \right| \right| \leq \epsilon$$

$$\sup_t |x(t) - y(\lambda t)| \leq \epsilon.$$

The following proposition shows the relationship between functions in D which have uniformly small jumps and the subspace C .

PROPOSITION 1.2 Let $C^\epsilon = \{x \in D : d_0(x,C) < \epsilon\}$. Then $x \in C^\epsilon$ implies that $\sup_t |x(t) - x(t-)| < 2\epsilon$. Also, if $x \in D$ is such that $\sup_t |x(t) - x(t-)| < \epsilon$ then $x \in C^\epsilon$.

PROOF. Suppose $x \in C^\epsilon$. Then there exists $y \in C$ such that $d_0(x,y) < \epsilon$ which implies $\sup_t |x(t) - y(\lambda t)| < \epsilon$ for some $\lambda \in \Lambda$. Since λ is continuous, $\lambda(t) = \lambda(t-)$ for all t . Thus $\sup_t |x(t) - x(t-)| \leq \sup_t \{ |x(t) - y(\lambda t)| + |y(\lambda t) - y(\lambda t-)| \} < 2\epsilon$.

Next suppose $x \in D$ is such that $\sup_t |x(t) - x(t-)| < \epsilon$. By Lemma 1.1, there exist t_0, \dots, t_r such that $0 = t_0 < \dots < t_r = 1$ and $\sup\{|x(t) - x(s)| : s, t \in [t_{i-1}, t_i]\} < \epsilon$ for $i = 1, \dots, r$. To construct a continuous function y

such that $\sup_t |x(t) - y(t)| < \epsilon$, proceed as follows:

Let $\epsilon_i = |x(t_i) - x(t_i^-)|$, $i = 1, \dots, r$. Each $\epsilon_i < \epsilon$.
Choose δ_i' such that $0 < t - t_i < \delta_i'$ implies

$$|x(t) - x(t_i)| < \min\left\{\frac{\epsilon - \epsilon_i}{2}, \frac{\epsilon_i}{2}\right\}. \text{ Let } \delta_i = \min\{\delta_i', t_{i+1} - t_i\}.$$

Define

$$y(0) = x(0)$$

$$y(t_i) = x(t_i^-) \quad i = 1, \dots, r$$

$$y(t_i + \delta_i/2) = x(t_i + \delta_i/2) \quad i = 1, \dots, r$$

Extend y to be linear in between.

$$\sup_{t \in [t_i, t_i + \delta_i/2)} |y(t) - x(t)|$$

$$\leq \sup_{t \in [t_i, t_i + \delta_i/2)} \{|y(t) - x(t_i)| + |x(t_i) - x(t)|\}$$

$$< \epsilon_i + (\epsilon - \epsilon_i)/2 < \epsilon \quad \text{for } i = 1, \dots, r.$$

$$\sup_{t \in [t_i + \delta_i/2, t_{i+1})} |y(t) - x(t)|$$

$$\leq \sup_{t \in [t_i + \delta_i/2, t_{i+1})} \max\{|x(t_{i+1}^-) - x(t)|, |x(t_i + \delta_i/2) - x(t)|\}$$

$$< \epsilon \quad i = 1, \dots, r.$$

Also, $\sup_{t \in [0, t_1)} |y(t) - x(t)| < \epsilon$ by the way t_1 was chosen.

Thus, $\sup_t |x(t) - y(t)| < \epsilon$ which implies $x \in C^E$. ///

For t_1, \dots, t_k in $[0,1]$ the natural projection $\pi_{t_1 \dots t_k}$ from D to \mathbb{R}^k is defined by

$$\pi_{t_1 \dots t_k}(x) = (x(t_1), \dots, x(t_k)).$$

π_0 and π_1 are everywhere continuous while π_t is continuous at x if and only if x is continuous at t .

However, for all t_1, \dots, t_k in $[0,1]$, $\pi_{t_1 \dots t_k}$ is measurable with respect to the σ -field \mathcal{D} of Borel sets for the Skorohod topology (p. 121). Since D is a complete separable metric space, (D, \mathcal{D}) is a standard Borel space.

Let T_0 be a subset of $[0,1]$ and define F_{T_0} to be the family of sets

$$F_{T_0} = \{\pi_{t_1 \dots t_k}^{-1} H : H \subset \mathbb{R}^k, t_1, \dots, t_k \in T_0, k \geq 1\}.$$

F_{T_0} generates \mathcal{D} provided T_0 contains 1 and is dense in $[0,1]$ (p. 121). Consequently, a probability measure P on (D, \mathcal{D}) is completely determined by its finite dimensional distributions, $P\pi_{t_1 \dots t_k}^{-1}$, for time points in T_0 .

We have already seen that a stochastic process with sample paths in D is a D -valued random variable. Conversely, if X is a D -valued random variable then for

each t , $X(t)$ is a real random variable so that X is actually a stochastic process. Throughout this thesis we will use both terms.

If $X(t)$ is a separable stochastic process and if there exists on (D, \mathcal{D}) a probability measure having the same finite-dimensional distributions as $\{X(t)\}$ then the sample paths of X need not be in D . However, they must be right continuous at $t = 0$, have left limits at $t = 1$ and have limits from both sides at all $t \in (0, 1)$, (p. 136). This emphasizes the fact that right continuity is only a convention. However, if the finite-dimensional distributions of a separable stochastic process can be realized as the finite-dimensional distributions of a probability measure on (C, \mathcal{C}) , then the sample paths are continuous with probability 1.

Whitt ((1974) p. 8) has shown that the restrictions $r_{ab} : D[0, 1] \rightarrow D[a, b]$ defined by $r_{ab} x(t) = x(t)$, $a \leq t \leq b$ for $[a, b] \subset [0, 1]$ are measurable.

There are three moduli on D which are used in characterizing the compact subsets of D .

(1) The uniform modulus,

$$\begin{aligned} \omega_x(\delta) &= \sup_{0 \leq t \leq 1-\delta} \sup\{|x(s) - x(u)| : s, u \in [t, t+\delta]\} \\ &\equiv \sup_{0 \leq t \leq 1-\delta} \omega_x[t, t+\delta], \end{aligned}$$

is the modulus of continuity for functions in C .

(2) The natural modulus on D is

$$\omega_x^i(\delta) = \inf_{\{t_i\}} \max_{0 < i \leq r} \sup\{|x(s) - x(u)| : s, u \in [t_{i-1}, t_i]\}$$

where the infimum extends over the finite sets of points $\{t_i\}$ satisfying

$$0 = t_0 < t_1 < \dots < t_r = 1$$

$$t_i - t_{i-1} > \delta \quad i = 1, \dots, r.$$

It is natural because it leads to a complete characterization of compact sets (p. 116) and a necessary and sufficient condition for x to lie in D is that $\lim_{\delta \rightarrow 0} \omega_x^i(\delta) = 0$. For continuous functions $\omega_x^i(\delta)$ is essentially the same as $\omega_x(\delta)$ (p. 111).

(3) The following modulus is often more convenient to work with than ω_x^i .

$$\omega_x^{\#}(\delta) = \sup \min\{|x(t) - x(t_1)|, |x(t_2) - x(t)|\}$$

where the supremum extends over t_1, t and t_2 satisfying $t_1 \leq t \leq t_2$, $t_2 - t_1 \leq \delta$. $\omega_x^{\#}(\delta) \leq \omega_x^i(\delta)$ and there can be no inequality in the other direction. However, compact sets can be characterized in terms of $\omega_x^{\#}(\delta)$ and the behavior of x near 0 and 1 (p. 119).

§2. THE CENTRAL LIMIT THEOREM

A sequence P_n of probability measures on D converges weakly to a probability measure P on D (written $P_n \Rightarrow P$) if, for every bounded, continuous function $F : D \rightarrow \mathbb{R}$,

$$\int_D F(x) dP_n(x) \rightarrow \int_D F(x) dP(x).$$

We will now discuss what it means to say that the central limit theorem (CLT) holds in D . Let X_1, X_2, \dots be independent D -valued random variables with the same distribution $L(X)$. Assume that they are defined on the same probability space $(\Omega, \mathcal{F}, \text{Pr})$. We suppose that for any $t \in [0, 1]$,

$$EX(t) = 0, \quad EX^2(t) < \infty.$$

Let $Z_n = n^{-1/2}(X_1 + \dots + X_n)$ and $P_n = L(Z_n)$. The sequence $\{X_i\}$ is said to satisfy the CLT if there exists a Gaussian process Z with sample paths in D and law $L(Z) = P$ such that $P_n \Rightarrow P$. By considering finite-dimensional distributions it is easy to see that if the CLT holds for one such sequence then it holds for all sequences with the same properties. Thus, we can unambiguously say that the CLT holds for X , or $L(X)$, if a sequence $\{X_i\}$ as above satisfies the CLT. We also

write $L(Z_n) \rightarrow L(Z)$.

We observe that the second moment structure of the limiting Gaussian process Z is identical to that of X .
If

(1) there exists a Gaussian process with sample paths in D which has the same covariance as X ; then, by Theorem 15.1, p. 124 of Billingsley (1968), in order to show that the CLT holds for X it suffices to verify

(2) Convergence of the finite-dimensional distributions

$$P_n \pi_{t_1 \dots t_k}^{-1} \Rightarrow P \pi_{t_1 \dots t_k}^{-1} \quad \text{for } t_i \in [0,1];$$

(3) Tightness of $\{P_n\}$.

If condition (1) holds so that P is a measure on D , then by the ordinary CLT, condition (2) always holds.

We thus turn our attention to the tightness of $\{P_n\}$, a condition which by Prohorov's Theorem (p. 37) is also a necessary condition.

A sequence $\{P_n\}$ of measures on D is tight if given $\epsilon > 0$, there exists a compact subset K of D such that, for all n , $P_n(K) > 1 - \epsilon$. Using the Arzela-Ascoli characterization of compactness, tightness can be broken down into two sets of conditions, uniform boundedness (UB) conditions and uniform equicontinuity (UEC) conditions.

By Theorems 15.2 and 15.3, p. 125 of Billingsley (1968), the sequence $\{P_n\}$ is tight if and only if the condition UB and either UEC1 or UEC2 below hold:

(UB) for each positive η , there exists a such that

$$P_n\{x : \sup_t |x(t)| > a\} \leq \eta, n \geq 1.$$

(UEC1) for each positive ε , there exists δ , $0 < \delta < 1$, and an integer n_0 such that

$$P_n\{x : \omega'_x(\delta) \geq \varepsilon\} \leq \eta, n \geq n_0$$

(UEC2) for each positive ε , there exists δ , $0 < \delta < 1$, and an integer n_0 such that

$$P_n\{x : \omega''_x(\delta) \geq \varepsilon\} \leq \eta, n \geq n_0$$

$$P_n\{x : \omega_x[0, \delta) \geq \varepsilon\} \leq \eta, n \geq n_0$$

$$P_n\{x : \omega_x[1-\delta, 1) \geq \varepsilon\} \leq \eta, n \geq n_0.$$

Furthermore, if ω'_x is replaced in UEC2 by ω_x giving a condition UEC3, then the condition

(UB') for each positive η there exists a such that

$$P_n\{x : |x(0)| > a\} < \eta, n \geq 1$$

together with UEC3 imply $\{P_n\}$ is tight, and, if P is the weak limit of a subsequence $\{P_{n_i}\}$, then $P(C) = 1$

(Theorem 15.5 p. 127).

Actually, convergence of the finite-dimensional distributions and any of the UEC conditions together imply the UB condition (p. 126). On the other hand, in finding counterexamples, the UB condition is the one we have found most useful to try to contradict.

The criteria most easily applied to verify weak convergence in the case of the CLT are given by the following theorem (Theorem 15.6, p. 128).

THEOREM 1.3 Let X_n and X be stochastic processes. Suppose that the finite-dimensional distributions $\pi_{t_1 \dots t_k}^{X_n} \rightarrow \pi_{t_1 \dots t_k}^X$ for all $t_1, \dots, t_k \in [0, 1]$; that $P\{X(1) \neq X(1-)\} = 0$; and that

$$(*) \quad P\{|X_n(t) - X_n(s)| \geq \lambda, |X_n(u) - X_n(t)| \geq \lambda\} \leq \lambda^{-2\gamma} [F(u) - F(s)]^{2\alpha}$$

for $s \leq t \leq u$ and $n \geq 1$, where $\gamma \geq 0$, $\alpha > 1/2$, and F is a nondecreasing, continuous function on $[0, 1]$. Then $L(X_n) \rightarrow L(X)$ on D .

When considering sample-continuous processes on a compact metric space S , rather than processes on $[0, 1]$ with sample paths in D , the CLT problem can be formulated

in an analogous manner, where this time the convergence is in the weak-star topology on the dual of $C(S)$ and the limiting Gaussian process must be sample-continuous. The best known results for $C(S)$ are due to Jain and Marcus (1974).

Since under certain circumstances the CLT problem for processes properly in D can be reduced to the CLT problem for continuous functions on an auxiliary compact metric space, we will now state Jain and Marcus's main result. The hypotheses of their theorem involve the concept of metric entropy.

If T is a metric space with metric (or pseudo-metric) ρ , let $N_\rho(T, \epsilon)$ denote the minimal number of balls of radius $\leq \epsilon$ which cover T . Let $H_\rho(T, \epsilon) = \log_e N_\rho(T, \epsilon)$. $H_\rho(T, \epsilon)$ is called the metric entropy of T with respect to ρ .

JAIN AND MARCUS CLT. Let (S, d) be a compact metric space. Let X be a $C(S)$ -valued random variable on (Ω, F, P) with $E(f(X)) = 0$, for $f \in C(S)^*$ and $\sup_t E|X(t)|^2 = 1$. Suppose there exist a non-negative random variable M , $EM^2 = 1$, and a metric ρ on S , which is continuous with respect to d , such that given $s, t \in S$, $\omega \in \Omega$,

$$|X(s, \omega) - X(t, \omega)| \leq M(\omega) \rho(s, t).$$

If

$$\int_0^1 H_p^{1/2}(S,u) du < \infty$$

then X satisfies the CLT in $C(S)$.

A stochastic process X is said to be stochastically continuous at a point $t_0 \in [0,1]$ if for each $\varepsilon > 0$, $P(|X(t) - X(t_0)| \geq \varepsilon) \rightarrow 0$ as $|t - t_0| \rightarrow 0$. If X has sample paths in D , stochastic continuity at t_0 implies that $P(|X(t_0) - X(t_0^-)| > 0) = 0$: it suffices to show that there exists $\delta_k \downarrow 0$ and $s_k \uparrow t_0$ such that $P(|X(t_0) - X(s_k)| > \delta_k \text{ i.o.}) = 0$. By stochastic continuity, $\lim_{s \rightarrow t_0} P(|X(t_0) - X(s)| > 1/k^2) \leq 1/k^2$; so let $\delta_k = 1/k^2$

and there exists s_k such that $P(|X(t) - X(s_k)| > 1/k^2) \leq 2/k^2$. But $\sum_{k=1}^{\infty} P(|X(t) - X(s_k)| > 1/k^2) \leq \sum_{k=1}^{\infty} 2/k^2 < \infty$.

Thus, by Borel-Cantelli, $P(|X(t) - X(s_k)| > 1/k^2 \text{ i.o.}) = 0$.

X is said to be stochastically continuous if it is stochastically continuous at each point.

If X has sample paths in D and is not stochastically continuous at a point t_1 , then with positive probability X has a jump discontinuity at t_1 . t_1 is said to be a fixed point of discontinuity.

Following Dudley ((1973) p. 68), we will say a stochastic process $\{X(t), t \in [0,1]\}$ is sample-continuous if there is a version of the process with continuous sample paths, i.e., there is a countably additive probability measure on $C[0,1]$ with the same joint distributions as $X(t)$ on finite subsets of $[0,1]$.

CHAPTER 2. FUNCTIONAL AND DECOMPOSITION CLT'S

In this section are grouped together a few of the facts concerning functional and decomposition central limit theorems which will prove useful throughout the rest of this thesis.

By a functional central limit theorem we mean a theorem giving sufficient conditions on a function f so that $f(X)$ satisfies the CLT whenever X does. The main problem is to establish suitable conditions on f .

The basic tool in proving functional limit theorems is the Continuous Mapping Theorem stated below:

Suppose $X_n, n \in \mathbb{N}$ and X are random variables with values in a separable metric space S with the Borel σ -field. Let $f_n, n \in \mathbb{N}$ and f be measurable functions from S into a separable metric space S' , also having the Borel σ -field. Let D_f be the set of discontinuities of f .

THEOREM 2.1 (Continuous Mapping Theorem)

(Billingsley (1968) p. 31 and 34)

(i) If $L(X_n) \rightarrow L(X)$ and $P\{X \in D_f\} = 0$, then $L(f(X_n)) \rightarrow L(f(X))$.

(ii) Let $E = \{x \in S : \text{there exists a sequence } x_n \rightarrow x \text{ such that } f_n(x_n) \not\rightarrow f(x)\}$. E is measurable;

and if $P(E) = 0$ and $L(X_n) \rightarrow L(X)$ then
 $L(f_n(X_n)) \rightarrow L(f(X))$.

The importance of determining which functions are measurable and/or continuous from D into D or $D \times D$ into D is now apparent. In his 1974 paper Whitt considers conditions for certain elementary functions to be continuous or otherwise preserve convergence at points belonging to appropriate subsets of D or $D \times D$.

We summarize the facts concerning those functions which will be important to us here.

2.2.1. Addition is measurable on $D \times D$ and continuous at those (x,y) for which $\text{Disc}(x) \cap \text{Disc}(y) = \emptyset$ where $\text{Disc}(x)$ is the set of discontinuity points of x in $[0,1]$. In particular, addition is continuous on $D \times C$ (Whitt p. 22).

2.2.2. Multiplication is measurable on $D \times D$ and continuous at those x,y for which $\text{Disc}(x) \cap \text{Disc}(y) = \emptyset$ (Whitt p. 24).

2.2.3. Let D_0 be the set of non-decreasing $[0,1]$ -valued functions in D . Let C_0 be the set of strictly increasing $[0,1]$ -valued functions in C . Composition on $D \times D_0$ is measurable and continuous at $(x,y) \in (C \times D_0) \cup (D \times C_0)$ (Whitt (1974) p. 14).

It is now possible to prove several useful functional central limit theorems.

LEMMA 2.3 Suppose $X(t)$ satisfies the CLT. Let $\psi(t)$ be a continuous function on $[0,1]$. Then $(\psi \cdot X)(t)$ satisfies the CLT. Furthermore, if the limiting Gaussian process is sample-continuous and $\psi \in D$ then the conclusion holds.

PROOF. Since ψ is not random, Billingsley's Theorem 4.4, p. 27, 2.2.2 above and the Continuous Mapping Theorem yield the desired results. ///

LEMMA 2.4 Suppose that $X(t)$ satisfies the CLT in D and $\eta \in C_0$ or that $X(t)$ satisfies the CLT with a continuous limiting Gaussian process and $\eta \in D_0$. Then $(X \circ \eta)(t)$ satisfies the CLT in D .

PROOF. Again η is non-random, so Billingsley's Theorem 4.4, 2.2.3 above, and the Continuous Mapping Theorem yield the result. ///

If $X(t)$ satisfies the CLT, then the time-reversed process $Y(t) = X((1-t)-)$ also satisfies the CLT.

The main purpose of the rest of this section is to prove a decomposition theorem which will justify to some extent our separate treatments of processes with only fixed discontinuities and of stochastically continuous processes. In order to prove the Decomposition CLT, it is necessary to prove a CLT for random vectors whose coordinates are $D[0,1]$ -valued random variables, i.e.

a CLT in $D^k = D^k[0,1]$ which is the product of k copies of D , with the product topology.

Conditions for weak convergence of vectors in D^k are given by the following theorem of Iglehart:

THEOREM 2.5 (Iglehart (1968) Theorem 5, p. 11)

Let $\{P_n\}$ and P be probability measures on (D^k, \mathcal{D}^k) . Then (i) and (ii) are necessary and sufficient conditions for $P_n \rightarrow P$:

(i) $P_n \pi_{t_1 \dots t_r}^{-1} \rightarrow P \pi_{t_1 \dots t_r}^{-1}$ whenever

$t_1, \dots, t_r \in T_P$ where

$T_P = \{t : P\{\underline{x} \in D^k : \underline{x}(t) \neq \underline{x}(t-)\} = 0\}$.

(ii) the families of marginal measures $\{P_n^i\}$ on (D, \mathcal{D}) for $i = 1, \dots, k$ are tight, where $P_n^i(A) = P_n(\mathcal{D} \times \dots \times \mathcal{D} \times A \times \mathcal{D} \times \dots \times \mathcal{D})$ with A in the i th place.

THEOREM 2.6 (CLT for D^k)

Let $\underline{X} = (X_1, \dots, X_k)$ be a D^k -valued random variable. Suppose that X_j , $j = 1, \dots, k$ satisfy the CLT in D with limiting Gaussian process $Z_{*,j}$. Then \underline{X} satisfies the CLT in D^k with limiting Gaussian process $\underline{Z}_* = (Z_{*,1}, \dots, Z_{*,k})$ where the joint distributions of the $Z_{*,j}$ are determined by the convergence of the

finite-dimensional distributions.

PROOF. In order to apply Theorem 2.5 we must verify conditions (i) and (ii). Let $\tilde{X}^{(i)}$ denote independent copies of \tilde{X} . Each component $X_j^{(i)}$ is an independent copy of X_j . Let $\tilde{Z}_n = n^{-1/2} \sum_{i=1}^n \tilde{X}^{(i)}$. $\pi_{t_1, \dots, t_r}^{\tilde{Z}_n} = (Z_n(t_1), \dots, Z_n(t_r))$
 $= (Z_{n,1}(t_1), \dots, Z_{n,k}(t_1), Z_{n,1}(t_2), \dots, Z_{n,1}(t_r), \dots, Z_{n,k}(t_r))$
 which is a vector in R^{rk} . So condition (i) follows by the CLT in R^{rk} .

$$P_1^j = L(X_j) \quad \text{and} \quad P_n^j = L(Z_{n,j}) \longrightarrow L(Z_{*,j}) = P^j$$

by the hypothesis that X_j satisfies the CLT in D . Thus, the marginals $\{P_n^j\}$ are tight.

So, by Theorem 2.5, \tilde{X} satisfies the CLT in D^k with limiting Gaussian process Z_* . ///

It is now easy to prove the Decomposition CLT.

THEOREM 2.7 (Decomposition CLT)

Let X be a D -valued random variable with the decomposition $X = X_1 + \dots + X_k$, where each X_i is a D -valued random variable satisfying the CLT with limiting Gaussian random variable Y_i . Furthermore, assume that all except possibly one of the Y_i 's are

C-valued random variables. Then X satisfies the CLT with limiting Gaussian random variable $Y = Y_1 + \dots + Y_k$ where the joint distributions of the Y_j are determined by convergence of the finite-dimensional distributions.

PROOF. We can assume that if there is one X_i whose limiting Gaussian random variable is not C-valued then it is X_1 . By Theorem 2.6, the random vector (X_1, \dots, X_k) satisfies the CLT with limiting Gaussian random variable (Y_1, \dots, Y_k) . Addition is continuous on $D \times C \times \dots \times C$, using 2.2.1 and induction. So, since $L(Y_1, \dots, Y_k)(D \times C \times \dots \times C) = 1$ by hypothesis, the Continuous Mapping Theorem applies; and thus $X = X_1 + \dots + X_k$ satisfies the CLT. ///

In Chapter 5 it will be shown that the limiting Gaussian process, corresponding to a stochastically continuous process with sample paths in D , must be sample-continuous.

CHAPTER 3. CONTINUITY AND THE CENTRAL LIMIT THEOREM

We begin our investigation of the CLT problem by considering conditions which imply sample-continuity. We are thus simultaneously considering the CLT problem in both C and D. There are two main results in this chapter. First, we show that if X is a separable process with $E|X(t)-X(s)|^r \leq f(|t-s|)$, where f satisfies a simple integral condition, then X is sample-continuous. Second, we prove that whenever the integral condition is satisfied with $r = 2$, then the CLT holds in C for X .

Dobrushin (see Loève (1963) p. 515) has given the following criterion for a general separable stochastic process not to have jump discontinuities:

$$\sup P[|X(t+h) - X(t)| \geq \epsilon] = o_{\epsilon}(h)$$

where the supremum is taken over all intervals $[t, t+h]$ in $[0, 1]$. A simple application of Chebyshev's inequality now shows that if $E|X(t+h)-X(t)|^r = o(h)$ for some $r > 0$, then X cannot have jump discontinuities. In particular, if X satisfies such a moment condition and also has sample paths in D a.s., then X must be sample-continuous. Observe that for $r = 2$ such a condition is best possible as shown by the Poisson process, which satisfies $E(X(t+h)-X(t))^2 = |h|$.

Kolmogorov (see Loève (1963) p. 519) showed that if $X(t)$ is a process (with or without sample paths in D) for which there exist α and $r > 0$ such that $E|X(t)-X(s)|^r \leq C|t-s|^{1+\alpha}$ for some constant C and all t and s , then $X(t)$ is sample-continuous. Delporte (1964) weakened Kolmogorov's condition to

$$E|X(t)-X(s)|^r \leq C \frac{|t-s|}{|\log|t-s||^{\alpha+\epsilon}}$$

where $\alpha = 1/r$ and $\epsilon > 0$. The counterexamples of Proposition 4.8 show that when $r = 2$, this result is best possible.

We include a proof of the following theorem of Delporte because the method is instructive and the integral condition of Theorem 3.5 will be derived from Delporte's theorem.

$$\text{Let } A_q(\omega) = \sup_{0 \leq s < 2^{q-1}} |X(\frac{s+1}{2^{q-1}}, \omega) - X(\frac{s}{2^{q-1}}, \omega)|, \quad s \in \mathbb{Z}.$$

We use $\|\cdot\|_r$ to denote the usual norm on $L^r(\Omega, \mathcal{P}, \mathbb{P})$.

THEOREM 3.1 (Delporte (1964) p. 179-80) Let $\phi(h)$ be a non-negative function on $[0,1]$ which is non-decreasing in h for h sufficiently small and such that $\phi(h) \rightarrow 0$ as $h \rightarrow 0$. If $X(t, \omega)$ is a separable process satisfying

$$\sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} \|A_q\|_r < \infty, \quad r \geq 1$$

then there exists a random variable $A \in L^1(\Omega, \mathbb{P})$ such that

$$|X(t, \omega) - X(s, \omega)| \leq A(\omega) \phi(|t-s|) \quad \text{a.s.}$$

for $|t-s|$ sufficiently small.

PROOF. (Essentially Delporte's proof)

$$\sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} E A_q \leq \sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} \|A_q\|_r < \infty$$

implies that $\sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} A_q(\omega) < \infty \quad \text{a.s.}$

Let δ be such that for $h < \delta \leq 1$, ϕ is non-decreasing in h . Given $t, s \in [0, 1]$ with $t > s$ and $t-s < \delta$, let $h = t-s$. There is a $q \geq 1$ such that $2^{-q-1} \leq h < 2^{-q}$. Let $r_q = p 2^{-q}$ and $b_q = r_q + 2^{-q}$ be such that $r_q \leq s < b_q$. Then there are two possibilities, either $r_q \leq s < t \leq b_q$ or $r_q \leq s < b_q \leq t < b_q + 2^{-q}$ since $s < t = s + h$.

Using the dyadic expansion for s ,

$s = r_q + a_1 2^{-q-1} + a_2 2^{-q-2} + \dots$, the separability of X , and repeated application of the triangle inequality we obtain

$$|X(s, \omega) - X(r_q, \omega)| \leq \sum_{j=q+2}^{\infty} A_j(\omega).$$

The same bound holds for t replacing s if $t < b_q$.

Otherwise,

$$|X(t, \omega) - X(b_q, \omega)| \leq \sum_{j=q+1}^{\infty} A_j(\omega)$$

by the same method. As a result, in either case

$$\begin{aligned} |X(t, \omega) - X(s, \omega)| &\leq 3 \sum_{j=q+1}^{\infty} A_j(\omega) \\ &\leq 3\phi(h) \sum_{j=q+1}^{\infty} [\phi(2^{-j-1})]^{-1} A_j(\omega) \end{aligned}$$

since $\phi(h) \geq \phi(2^{-q-i})$ for every $i \geq 1$. Let

$$A(\omega) = 3 \sum_{j=1}^{\infty} [\phi(2^{-j-1})]^{-1} A_j(\omega).$$

There remains to be shown only that $A \in L^{\mathbb{F}}(\Omega, \mathbb{P}_{\mathbb{R}})$.

For $n > m$,

$$\begin{aligned} & \left\| \sum_{j=1}^n [\phi(2^{-j-1})]^{-1} A_j - \sum_{j=1}^m [\phi(2^{-j-1})]^{-1} A_j \right\|_{\mathbb{R}} \\ &= \left\| \sum_{j=m+1}^n [\phi(2^{-j-1})]^{-1} A_j \right\|_{\mathbb{R}} \leq \sum_{j=m+1}^n [\phi(2^{-j-1})]^{-1} \|A_j\|_{\mathbb{R}} \rightarrow 0 \end{aligned}$$

as $m, n \rightarrow \infty$. Thus, the partial sums are Cauchy in the complete metric space $L^{\mathbb{F}}(\Omega, \mathbb{P}_{\mathbb{R}})$ which implies that $A \in L^{\mathbb{F}}(\Omega, \mathbb{P}_{\mathbb{R}})$. ///

DEFINITION. A function h from $[0, 1]$ into $[0, \infty]$ is called a modulus (of continuity) if and only if both the following hold:

- (i) h is continuous and $h(0) = 0$;

(ii) $h(x) \leq h(x+y) \leq h(x) + h(y)$ for all $x, y \geq 0$ with $x+y \leq 1$.

DEFINITION. Let $X(t)$, $t \in [0,1]$ be a stochastic process. h is called a sample modulus for X if and only if

(i) h is a modulus

(ii) for almost all ω there exist finite constants k_ω such that for all $s, t \in [0,1]$

$$|X(t, \omega) - X(s, \omega)| \leq k_\omega h(|t-s|).$$

If ϕ , in Theorem 3.1, is a non-decreasing function on $[0,1]$ then, as a consequence of the proof,

$$|X(t) - X(s)| \leq A(\omega) \phi(|t-s|) \text{ a.s. whenever } |t-s| < 1.$$

If $s = 0$ and $t = 1$, then $|X(1) - X(0)| = A_1(\omega)$

$$\leq \phi(1) [\phi(2^{-1})]^{-1} A_1(\omega) \leq \phi(1) A(\omega). \text{ Thus,}$$

$$|X(t) - X(s)| \leq A(\omega) \phi(|t-s|) \text{ a.s. for all } s, t. \text{ If}$$

in addition, ϕ is continuous and $\phi(x+y) \leq \phi(x) + \phi(y)$

for all $x, y \geq 0$, then ϕ is a sample modulus for $X(t)$.

COROLLARY 3.2 Let $X(t)$ be a separable stochastic process with $E|X(t) - X(s)|^r \leq f(|t-s|)$ for some $r \geq 1$ and all $s, t \in [0,1]$. If there exists a non-negative, non-decreasing function ϕ on $[0,1]$ such that $\phi(h) \rightarrow 0$ as $h \rightarrow 0$ and such that

$$\sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} 2^{q/r} (f(2^{-q+1}))^{1/r} < \infty$$

then $X(t)$ is sample-continuous and there exists a random variable $A \in L^r(\Omega, \mathcal{P}, \mathbb{P})$ such that

$$|X(t, \omega) - X(s, \omega)| \leq A(\omega) \phi(|t-s|) \text{ a.s.}$$

PROOF.

$$\|A_q\|_r = (E(\sup_{0 < s < 2^{q-1}} |X((s+1)2^{-q+1}) - X(s2^{-q+1})|^r))^{1/r}$$

$$\leq \left(\sum_{s=0}^{2^{q-1}-1} E|X((s+1)2^{-q+1}) - X(s2^{-q+1})|^r \right)^{1/r}$$

$$\leq \left(\sum_{s=0}^{2^{q-1}-1} f(2^{-q+1}) \right)^{1/r} = 2^{(q-1)/r} (f(2^{-q+1}))^{1/r}$$

Now,

$$\sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} \|A_q\|_r \leq \sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} 2^{(q-1)/r} (f(2^{-q+1}))^{1/r}$$

$< \infty$

which by Delporte's Theorem implies X is sample-continuous and furthermore

$$|X(t, \omega) - X(s, \omega)| \leq \phi(|t-s|) A(\omega)$$

where $A(\omega) = 3 \sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} A_q(\omega) \in L^r$. ///

LEMMA 3.3 There exists a non-decreasing, non-negative function ϕ on $[0,1]$ such that $\phi(x) \rightarrow 0$ as $x \rightarrow 0$ and such that

$$\sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} 2^{q/r} (f(2^{-q+1}))^{1/r} < \infty$$

if and only if

$$\sum_{q=1}^{\infty} 2^{q/r} (f(2^{-q+1}))^{1/r} < \infty.$$

PROOF. The only if part is obvious since $[\phi(2^{-q-1})]^{-1} \rightarrow \infty$ as $q \rightarrow \infty$.

So assume $\sum_{q=1}^{\infty} 2^{q/r} (f(2^{-q+1}))^{1/r} < \infty$. Then there exists an increasing sequence of positive real numbers c_q such that $\sum_{q=1}^{\infty} c_q 2^{q/r} (f(2^{-q+1}))^{1/r} < \infty$. Let $\phi(2^{-q-1}) = 1/c_q$ and extend linearly in between; ϕ has the desired properties. ///

LEMMA 3.4. If f is a non-negative function on $[0,1]$ which is non-decreasing in a neighborhood of 0, then, for some $r \geq 1$,

$$\sum_{q=1}^{\infty} 2^{q/r} [f(2^{-q+1})]^{1/r} < \infty \text{ iff } \int_0 y^{-(r+1)/r} (f(y))^{1/r} dy < \infty.$$

PROOF. Let δ be such that $f(x)$ is non-decreasing in $[0, \delta]$. Choose m such that $2^{-m+2} < \delta$.

$$\begin{aligned}
& \text{(If): } \sum_{q=m}^{\infty} 2^{(q-1)/r} [f(2^{-q+1})]^{1/r} \\
&= \sum_{q=m}^{\infty} 2^{q-1} [2^{-(q-1)(r-1)} f(2^{-q+1})]^{1/r} \\
&\leq \sum_{q=m}^{\infty} \int_{2^{q-2}}^{2^{q-1}} (x^{1-r} f(x^{-1}))^{1/r} dx \quad \text{since } f(x) \text{ is non-} \\
&\quad \text{decreasing on } [0, \delta]
\end{aligned}$$

$$\begin{aligned}
&= 2 \int_{2^{m-2}}^{\infty} (x^{1-r} f(x^{-1}))^{1/r} dx \\
&= 2 \int_0^{2^{-m+2}} y^{-(r+1)/r} (f(y))^{1/r} dy \quad \text{letting } y = x^{-1} \\
&< \infty.
\end{aligned}$$

$$\begin{aligned}
& \text{(Only if): } \int_0^{2^{-m+2}} y^{-(r+1)/r} (f(y))^{1/r} dy \\
&= \sum_{q=m}^{\infty} \int_{2^{q-2}}^{2^{q-1}} (x^{1-r} f(x^{-1}))^{1/r} dx \quad \text{by reversing the above} \\
&\quad \text{reasoning}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{q=m}^{\infty} 2^{(q-2)(1-r)/r} [f(2^{-q+2})]^{1/r} 2^{q-2} \\
&= \sum_{q=m-1}^{\infty} 2^{(q-1)/r} [f(2^{-q+1})]^{1/r} < \infty. \quad ///
\end{aligned}$$

THEOREM 3.5. Let f be a non-negative function on $[0, 1]$ which is non-decreasing in a neighborhood of 0.

Let $X(t)$ be a separable stochastic process such that for some $r \geq 1$, $E|X(t)-X(s)|^r \leq f(|t-s|)$. If

$$\int_0^\infty y^{-(r+1)/r} (f(y))^{1/r} < \infty$$

then $X(t)$ is sample-continuous.

PROOF. The theorem follows immediately from Corollary 3.2, Lemma 3.3 and Lemma 3.4. ///

Define $\log_k |x|$ inductively by $\log_0 |x| = |x|$, $\log_1 |x| = \log |x|$, the usual natural logarithm, and $\log_k |x| = \log |\log_{k-1} |x||$. Define $e_k(u)$ inductively by $e_1(u) = e^u$ and $e_k(u) = \exp(e_{k-1}(u))$. Let $E_k(u) = \exp(-e_{k-1}(u))$. We will suppress the 1 if $u = 1$.

For $k = 1, 2, \dots$ and $\epsilon > 0$ let

$$f_{k,\epsilon}(x) = \frac{x}{|\log |x||^2 \dots |\log_{k-1} |x||^2 |\log_k |x||^{2+\epsilon}}$$

Note that $f_{k,\epsilon}(x)$ decreases to 0 as $x \rightarrow 0$ for $x < E_k$. We now consider the special case of Theorem 3.5 when $r = 2$ and $f(x) = O(f_{k,\epsilon}(x))$ as $x \rightarrow 0$.

COROLLARY 3.6 Given $k \in \mathbb{N}$ and $\epsilon > 0$, let $X(t)$ be a separable stochastic process with $E(X(t)-X(s))^2 = O(f_{k,\epsilon}(|t-s|))$ as $|t-s| \rightarrow 0$. Then

$X(t)$ is sample-continuous. Furthermore if ϕ is a non-negative, non-decreasing function on $[0,1]$ with $\phi(u) = |\log_{k+1}|u||^{-\epsilon'/2}$ for $|u| < E_k$ and $\epsilon' < \epsilon$, then there exist $\delta > 0$ with $\delta < E_k$ and a random variable $A \in L^2(\Omega, \text{Pr})$ such that for $|t-s| < \delta$,

$$|X(t, \omega) - X(s, \omega)| \leq A(\omega) \phi(|t-s|) \text{ a.s.}$$

PROOF. Let $\gamma < E_k$ be such that $|t-s| < \gamma$ implies $E(X(t) - X(s))^2 \leq C f_{k, \epsilon}(|t-s|)$. We can set $f(|t-s|) = C f_{k, \epsilon}(|t-s|)$ for $|t-s| < \gamma$. As an immediate consequence of Theorem 3.5, since

$$\begin{aligned} & \int_0^\gamma y^{-3/2} (f(y))^{1/2} dy \\ &= C \int_0^\gamma y^{-1} (\log y \dots \log_{k-1} y)^{-1} (\log_k y)^{-1-\epsilon/2} dy < \infty, \end{aligned}$$

$X(t)$ is sample-continuous.

In order to obtain the desired Lipschitz condition we verify the hypotheses of Corollary 3.2. There exists a $\delta < E_k$ such that $f_{k, \epsilon}(x)$ is monotone on $[0, \delta]$. Choose m such that $2^{-m+1} < \delta$ and $m \geq (e_{k-1}/\log 2) + 1$. Then

$$\begin{aligned}
& \sum_{q=m}^{\infty} [\phi(2^{-q-1})]^{-1/2} (q-1)^{1/2} [f(2^{-q+1})]^{1/2} \\
& \leq C \sum_{q=m}^{\infty} \frac{\log_{k-1}((q+1)\log 2)^{\varepsilon'/2}}{(q-1)\log 2 |\log((q-1)\log 2)| \dots |\log_{k-1}((q-1)\log 2)|^{1+\varepsilon/2}} \\
& \leq C_{k,\varepsilon'} \sum_{q=m}^{\infty} \frac{\log_{k-1}((q-1)\log 2)^{\varepsilon'/2}}{(q-1)\log 2 |\log((q-1)\log 2)| \dots |\log_{k-1}((q-1)\log 2)|^{1+\varepsilon/2}} \\
& = C_{k,\varepsilon'} \sum_{q=m}^{\infty} ((q-1)\log 2 |\log((q-1)\log 2)| \dots |\log_{k-1}((q-1)\log 2)|^{1+\alpha})^{-1}
\end{aligned}$$

where $\alpha = (\varepsilon - \varepsilon')/2 > 0$.

This sum is finite by the integral test because

$$\begin{aligned}
& \int_{e_{k-1}/\log 2}^{\infty} (x \log 2 (\log(x \log 2)) \dots (\log_{k-1}(x \log 2))^{1+\alpha})^{-1} dx \\
& = - (\log_{k-1}(x \log 2))^{-\alpha} \Big|_{e_{k-1}/\log 2}^{\infty} < \infty.
\end{aligned}$$

Therefore, by Corollary 3.2, we again see that $X(t)$ is sample-continuous and in addition it satisfies the desired Lipschitz condition. ///

Delporte ((1964) p. 179) concluded the above result for $k = 1$. Without the hypothesis of separability, the proof of Theorem 3.1 shows that the process X is uniformly continuous on the dyadic rationals. If X is also stochastically continuous, it will have a unique continuous version.

Theorem 3.1, with the additional assumption of stochastic continuity, Theorem 3.5, and Corollaries 3.2 and 3.6

can thus be stated without the hypothesis of X being separable, if we replace X in the Lipschitz conditions by the continuous version \tilde{X} . In the future we will denote the extensions of these theorems by putting an asterisk (*) after the number.

It will be shown in Chapter 4 that Corollary 3.6* is best possible, in the sense that for each $K \in \mathbb{N}$ there is a process satisfying

$$E(X(t)-X(s))^2 = O(f_{K,0}(|t-s|)) \text{ as } |t-s| \rightarrow 0$$

which has no continuous version. Since the process discussed there has no separable version either, the question of whether Corollary 3.6 gives best possible results still remains.

Whenever $X(t)$ has sample paths in D a.s., $X(t)$ is separable since for $s \in Q = \text{rationals}$,

$$\liminf_{\delta \downarrow 0} \inf_{|t-s| < \delta} |X(s)| \leq X(t) \leq \limsup_{\delta \downarrow 0} \sup_{|t-s| < \delta} |X(s)| \text{ by right continuity.}$$

THEOREM 3.7 Let f be a non-negative function on $[0,1]$ which is non-decreasing near 0. Let $X(t)$ be a stochastic process with mean 0, finite second moments, and sample paths in D , satisfying

$$E(X(t)-X(s))^2 \leq f(|t-s|) \text{ for all } s, t \in [0,1]$$

and $\int_0^{\infty} y^{-3/2} (f(y))^{1/2} dy < \infty$.

Then X is sample-continuous and satisfies the CLT in C .

PROOF. According to Theorem 3.5, X is sample-continuous. Let $\{X_i\}$ be a sequence of independent, identically distributed C -valued random variables with law $L(X)$. Let $Z_n = n^{-1/2}(X_1 + \dots + X_n)$. Z_n are C -valued, hence separable. Moreover, $E(Z_n(t) - Z_n(s))^2 = E(X(t) - X(s))^2$ for all n ; so Lemmas 3.3 and 3.4 imply that all the Z_n satisfy the hypotheses of Corollary 3.2. Thus, there exist random variables $A^{(n)} \in L^2(\Omega, \text{Pr})$ and a non-negative, non-decreasing function ϕ such that

$$|Z_n(t, \omega) - Z_n(s, \omega)| \leq A^{(n)}(\omega) \phi(|t-s|).$$

As seen from the proof of Corollary 3.2,

$$\begin{aligned} \|A^{(n)}\|_2^2 &= \left\| 3 \sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} A_q^{(n)} \right\|_2^2 \\ &\leq 9 \left(\sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} \|A_q^{(n)}\|_2 \right)^2 \\ &\leq 9 \left(\sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} 2^{(q-1)/2} [f(2^{-q+1})]^{1/2} \right)^2 \\ &\equiv M \quad (\text{this defines } M) \\ &< \infty. \end{aligned}$$

In order to show that the CLT holds in C it suffices to show convergence of the finite-dimensional distributions,

existence of a sample-continuous Gaussian process with the covariance of X , and condition UEC3 of Chapter 1.

Given $\epsilon > 0$ and $\eta > 0$, choose δ such that $\phi(\delta) \leq \epsilon^2 \eta / M$. Then

$$\begin{aligned} P_n \{ \omega_X(\delta) \geq \epsilon \} &= P_n \left\{ \sup_{|t-s| < \delta} |Z_n(t) - Z_n(s)| \geq \epsilon \right\} \\ &\leq P_n \{ A^{(n)}(\omega) \phi(\delta) \geq \epsilon \} \leq \epsilon^{-2} E(A^{(n)} \phi(\delta))^2 \\ &= \phi(\delta) \epsilon^{-2} E(A^{(n)})^2 \leq M \epsilon^{-2} \phi(\delta) \leq \eta. \end{aligned}$$

Thus condition UEC3 is satisfied.

Let Z be a Gaussian process with the same covariance as X . Since $E(Z(t) - Z(s))^2 = E(X(t) - X(s))^2$, Z satisfies the hypotheses of Theorem 3.5*; hence, Z is sample-continuous. By the ordinary CLT, the finite-dimensional distributions of Z_n converge to those of Z . Hence, X satisfies the CLT in C . ///

Examples in the next chapter show that the CLT need not hold if the integral condition in Theorem 3.7 is not satisfied.

CHAPTER 4. COUNTEREXAMPLES

The following examples have several purposes. The examples of §1, Proposition 7.8, show that the integral and sum conditions for sample-continuity in Theorem 3.5 and Lemma 3.4 are sharp in the case of second moments. Together with Corollary 3.6*, these examples settle the problem, mentioned in Dudley ((1973) p. 93) of what is the best exponent of the logarithm. They furthermore show that the exponents for multiple logarithms found in Corollary 3.6* are best possible.

By modifying the examples of §1 which have both unbounded and discontinuous sample paths, hence are not even in D , we obtain continuous processes satisfying the same moment conditions. Proposition 4.12 of §2 shows that these continuous processes cannot satisfy the CLT in C . As a result, these examples strongly suggest that Lipschitz conditions on second moments of the increments alone imply that the CLT holds in C only if they also imply sample-continuity.

In the final section, §3, we extend some examples of Dudley (1974) to show that if X is C -valued with $E(X(t)-X(s))^2 \leq |t-s|^\alpha$ for all $s, t \in [0, 1]$, $\alpha < 1$, then even the additional assumption of $\sup_{t, \omega} |X(t, \omega)| \leq 1$ is not sufficient to imply that the CLT holds for X . Dudley's

examples work for $\alpha < \frac{1}{2}$, so this extension fills in the gap for $\frac{1}{2} \leq \alpha < 1$. Since by Kolmogorov's Theorem (Loève (1963) p. 519), $\alpha > 1$ implies sample-continuity and hence the CLT (Araujo (1974) p. 27), this leaves only the case $\alpha = 1$ undetermined.

For results in this chapter we suppress the chapter number 4.

§1. We begin by introducing a convenient notation and several service lemmas to be used in the proof of Proposition 1.

Notation: Let $\log|y|$ be the usual natural logarithm of $|y|$. Inductively we define $\log_k|y| = \log|\log_{k-1}|y||$. Let $e_0(x) = x$, $e(x) = e_1(x) = \exp(x)$. Inductively we define $e_k(x) = \exp(e_{k-1}(x))$. Let $E_j(x) = \exp(-e_{j-1}(x))$. For convenience, x will be suppressed if $x = 1$.

Approximation (*): if $0 \leq p < 1$ then

$$|\log(1-p)| = \sum_{j=1}^{\infty} \frac{p^j}{j} \leq \sum_{j=1}^{\infty} p^j = \frac{p}{1-p}.$$

LEMMA 1. Given $k \in \mathbb{N}$, there exists $\gamma(k)$ such that if $z \in [3B, E_k - B]$ and $B < \gamma(k)$ then for every $j = 2, \dots, k$

$$\log \frac{\log_{j-1}(z-B)}{\log_{j-1}(z+B)} \leq (1/2) \log_j(z-B).$$

PROOF. Since $\log \frac{\log_{j-1}(z-B)}{\log_{j-1}(z+B)} = \log_j(z-B) - \log_j(z+B)$

it suffices to show that $\log_j(z+B) \geq (1/2) \log_j(z-B)$.

Differentiating,

$$\frac{d}{dz} \left[\frac{\log_j(z+B)}{\log_j(z-B)} \right] = \left[(z+B) \log_j(z-B) \log_{j-1}(z+B) \dots \log(z+B) \right]^{-1} \\ - (\log_j(z+B)) \left[(z-B) (\log_j(z-B))^2 \log_{j-1}(z-B) \dots \log(z-B) \right]^{-1} = 0$$

implies that

$$\frac{\log_j(z+B)}{\log_j(z-B)} = \frac{\log_{j-1}(z-B)}{\log_{j-1}(z+B)} \dots \frac{\log(z-B)}{\log(z+B)} \frac{(z-B)}{(z+B)}.$$

Letting $y = z+B$, $y \in [4B, E_k]$ we obtain

$$\frac{\log_j y}{\log_j(y-2B)} = \frac{\log_{j-1}(y-2B)}{\log_{j-1} y} \dots \frac{\log(y-2B)}{\log y} \frac{y-2B}{y} \\ = \frac{\log_{j-1}(1/(y-2B))}{\log_{j-1}(1/y)} \dots \frac{\log(1/(y-2B))}{\log(1/y)} \left(1 - \frac{2B}{y}\right)$$

Since $\frac{1}{y-2B} > \frac{1}{y} \geq e_k$, $\log_r(1/(y-2B)) > \log_r(1/y)$ for

$r = 1, \dots, k$. Thus, $\frac{\log_j y}{\log_j(y-2B)} > \left(1 - \frac{2B}{y}\right) \geq \frac{1}{2}$.

The critical points found above might be maxima or inflection points rather than minima. We check the endpoints of the interval in which z is defined.

At $z = 3B$, $\frac{\log_j(z+B)}{\log_j(z-B)} = \frac{\log_j(1/4B)}{\log_j(1/2B)}$ which converges to 1

at $B \rightarrow 0$; so there exists δ_j such that $B < \delta_j$ implies that $\log_j(1/4B) \geq (1/2) \log_j(1/2B)$. Let

$\delta(k) = \min(\delta_1, \dots, \delta_k)$.

$$\text{At } z = E_k - B, \frac{\log_j(z+B)}{\log_j(z-B)} = \frac{\log_j E_k}{\log_j(E_k - 2B)}$$

which converges to 1 as $B \rightarrow 0$; so there exists η_j such

that $B < \eta_j$ implies $\log_j E_k \geq (\log_j(E_k - 2B))/2$.

Let $\eta(k) = \min(\eta_1, \dots, \eta_k)$.

Let $\gamma(k) = \min(\delta(k), \eta(k))$. Then for all $B < \gamma(k)$ the desired result holds. ///

LEMMA 2. $\lim_{x \rightarrow 0} (\log_{k+1} |E_k - x|)^2 |\log x|^{2k} = 0$ for every $k \in \mathbb{N}$.

PROOF.

$$\begin{aligned} \text{Using l'H\^opital's rule, } \lim_{x \rightarrow 0} \frac{\log_{k+1}(1/(E_k - x))}{(\log(1/x))^{-k}} \\ = \lim_{x \rightarrow 0} \frac{(\log_k(1/(E_k - x)) \dots \log(1/(E_k - x)) (E_k - x))^{-1}}{k(\log(1/x))^{-k-1} (1/x)}. \end{aligned}$$

The numerator converges to a constant c_k and the denominator goes to ∞ as $x \rightarrow 0$. Hence, the result follows. ///

LEMMA 3. For every $P > 0$,

$$\int_1^\infty (\log y)^2 E_k(Py) e_{k-1}(Py) \dots e(Py) dy \leq \frac{2E_k(P)}{P^3 (e_{k-1}(P))^2 \dots (e(P))^2}.$$

PROOF. Differentiating,

$$\frac{d}{dy}g(y) = \frac{e_2(y)}{y^3 e(y)} - 3 \frac{e_2(y)}{y^4 (e(y))^2} - 2 \frac{e_2(y)}{y^3 (e(y))^2} = 0 \text{ implies that}$$

$ye(y) = 3+2y$. The left side increases faster than the right side, $(y+1)e(y) > 2$ for $y \geq 1$. At $y = 1$, $e < 5$ and at $y = 2$, $2e(2) > 7$. Thus, the only solution is between 1 and 2. The solution is a minimum since $\lim_{y \rightarrow \infty} g(y) = \infty$ and $\lim_{y \rightarrow 0} g(y) = \infty$. ///

LEMMA 5. If $k \geq 3$, $f(y) = \frac{e_k(y)}{y^2 e(y) \dots e_{k-2}(y) (e_{k-1}(y))^3}$

does not have a maximum or a minimum inside $(1, \infty)$.

PROOF. Differentiating,

$$\begin{aligned} \frac{d}{dy}[f(y)] &= \frac{e_k(y)}{y^2 (e_{k-1}(y))^2} - 2 \frac{e_k(y)}{y^3 e(y) \dots e_{k-2}(y) (e_{k-1}(y))^3} \\ &- \sum_{j=1}^{k-2} \frac{e_k(y)}{y^2 e_j(y) \dots e_{k-2}(y) (e_{k-1}(y))^3} - 3 \frac{e_k(y)}{y^2 (e_{k-1}(y))^3} = 0 \end{aligned}$$

implies (**): $1 = \frac{2}{ye(y) \dots e_{k-1}(y)} + \sum_{j=1}^{k-2} \frac{1}{e_j(y) \dots e_{k-1}(y)}$

$+ \frac{3}{e_{k-1}(y)}$. Since the right side decreases as y increases,

$y \geq 1$, we obtain an upper bound by evaluation at $y = 1$:

$$\frac{2}{e \dots e_{k-1}} + \sum_{j=1}^{k-2} \frac{1}{e_j \dots e_{k-1}} + \frac{3}{e_{k-1}} \leq \frac{2}{e^{k-1}} + \frac{(k-2)}{e_{k-1}} + \frac{3}{e_{k-1}}.$$

Since $k \geq 3$, each term decreases as k increases so we obtain an upper bound when $k = 3$: $\frac{2}{e^2} + \frac{1}{e_2} + \frac{3}{e_2}$

$$= \frac{2}{e^2} + \frac{4}{e_2} < 1. \text{ Thus, (**)} \text{ has no solution in } [1, \infty), \text{ so}$$

no maxima or minima of $f(y)$ occur inside $(1, \infty)$. ///

LEMMA 6. If $k \geq 3$, $h(y) = \frac{e_k(y)}{y^3 (e_{k-1}(y) \dots e_1(y))^2}$ does

not have a maximum or minimum inside $(1, \infty)$.

PROOF. Differentiating,

$$\begin{aligned} \frac{d}{dy}[h(y)] &= \frac{e_k(y)}{y^3 e_{k-1}(y) \dots e_1(y)} - 3 \frac{e_k(y)}{y^4 (e_{k-1}(y) \dots e_1(y))^2} \\ &\quad - 2 \sum_{j=1}^{k-1} \frac{e_k(y)}{y^3 e_1(y) \dots e_{j-1}(y) (e_j(y) \dots e_{k-1}(y))^2} = 0 \end{aligned}$$

$$\text{implies (***)}: 1 = \frac{3}{y e_1(y) \dots e_{k-1}(y)} + 2 \sum_{j=1}^{k-1} \frac{1}{e_j(y) \dots e_{k-1}(y)}.$$

Since the right side decreases as y increases, evaluating at $y = 1$, gives

$$\frac{3}{e \dots e_{k-1}} + 2 \sum_{j=1}^{k-1} \frac{1}{e_j \dots e_{k-1}} < \frac{3}{e \dots e_{k-1}} + \frac{2(k-1)}{e_{k-1}}$$

which decreases as k increases; so at $k = 3$ we obtain

$$\frac{3}{e e_2} + \frac{4}{e_2} < 1.$$

Thus, (***) has no solution in $(1, \infty)$. ///

LEMMA 7. If $R \leq (1/4)\exp(-k^2)$,
 $\log(1/4R) > \log_2(1/4R) \dots \log_k(1/4R)$.

PROOF. Let $y = (1/4R)$. Since $y \geq \exp(k^2)$,
 $y > (\log y)^k \geq \log y \dots \log_{k-1} y$. Thus,
 $\log y > \log_2 y \dots \log_k y$ giving the desired result.

PROPOSITION 8. For any $k \in \mathbb{N}$ there is a process
 $X(t, \omega)$, $0 \leq t \leq 1$, with discontinuous sample paths which
satisfies the condition

$$E(X(t) - X(s))^2 \leq c \frac{|t-s|}{|\log|t-s||^2 |\log_2|t-s||^2 \dots |\log_k|t-s||^2}.$$

PROOF. Let $k \in \mathbb{N}$ be fixed. We define the process

$$X_k(t, \omega) \equiv X(t, \omega) = \begin{cases} \log_{k+1} |(1+2t)/4 - \omega| & \text{if } |(1+2t)/4 - \omega| \leq E_k \\ & \omega \neq (1+2t)/4 \\ 0 & \text{otherwise} \end{cases}$$

where $t \in [0, 1]$ and $\omega \in ([0, 1], \text{Lebesgue})$.

Each sample path of $X(t, \omega)$ is discontinuous at
 $(1+2t)/4 = \omega$ and is unbounded since

$$\lim_{t \uparrow 2\omega - (1/2)} X(t, \omega) = \lim_{t \uparrow 2\omega - (1/2)} X(t, \omega) = \infty.$$

(1) $X(t, \omega)$ has finite second moments:

$$\begin{aligned} \text{Let } v &= (1+2t)/4, \quad E(X(t))^2 = \int_0^1 X^2(t, \omega) d\omega \\ &= \int_{v-E_k}^{v+E_k} (\log_{k+1} |v-\omega|)^2 d\omega = \left| \int_{v-E_k}^v + \int_v^{v+E_k} \right| (\log_{k+1} |v-\omega|)^2 d\omega \\ &= 2 \int_{e_{k-1}}^{\infty} (\log_k s)^2 e^{-s} ds \quad \text{upon substituting } s = -\log(v-\omega) \end{aligned}$$

in the first integral and $s = -\log(\omega-v)$ in the second integral.

$$\text{If } k = 1, \quad EX^2(t) = 2 \int_1^{\infty} (\log s)^2 e^{-s} ds \equiv M_1 < \infty.$$

If $k \geq 2$, letting $u = \log_{k-1} s$ we see that

$$\begin{aligned} EX^2(t) &= 2 \int_1^{\infty} (\log u)^2 E_k(u) e_{k-1}(u) \dots e(u) du \\ &= 2 [-(\log u)^2 E_k(u) \Big|_1^{\infty} + 2 \int_1^{\infty} \frac{\log u}{u} E_k(u) du] \\ &= 4 \int_1^{\infty} \frac{\log u}{u} E_k(u) du \equiv M_k < \infty. \end{aligned}$$

Now we prove that $X(t, \omega)$ satisfies the stated Lipschitz condition. Let $\gamma(k)$ be as in Lemma 1. By Lemma 2, there exists $\alpha(k)$ such that $|x| < 2\alpha(k)$ implies $(\log_{k+1} |E_k - x|)^2 |\log x|^{2k} \leq 1$. Choose $r = \min(E_k, \gamma(k), \alpha(k), \exp(-k^2), 1/144)$.

If $r \leq |t-s| \leq 1$ then $E(X(t)-X(s))^2$
 $\leq E(|X(t)| + |X(s)|)^2 \leq 2[E(X(s))^2 + E(X(t))^2] \leq 4M_k$
 where M_k is defined as above.

$$\text{Thus, } E(X(t)-X(s))^2 \leq \frac{4M_k |\log r|^2 \dots |\log_k r|^2 |t-s|}{r |\log |t-s||^2 \dots |\log_k |t-s||^2}$$

$$\equiv C(r, k) \frac{|t-s|}{|\log |t-s||^2 \dots |\log_k |t-s||^2}.$$

Next assume $0 < |t-s| < r$ and $t > s$.

Let $v = (1+2t)/4$ and $u = (1+2s)/4$. Then $v - u < \frac{1}{2}r$.

$$E(X(t)-X(s))^2 = \int_{u-E_k}^{v-E_k} (\log_{k+1} |u-w|)^2 dw$$

$$+ \int_{v-E_k}^{u+E_k} \left(\log \left| \frac{\log_k |v-w|}{\log_k |u-w|} \right| \right)^2 dw + \int_{u+E_k}^{v+E_k} (\log_{k+1} |v-w|)^2 dw$$

$\equiv I_1 + I_2 + I_3$. If $k = 1$, use the maximum of the above bounds of integration and 0.

(2) Considering the first integral

$$I_1 = \int_{u-E_k}^{v-E_k} (\log_{k+1} |u-w|)^2 dw \leq (v-u) (\log_{k+1} |u-v+E_k|)^2$$

$$= \frac{(v-u)}{|\log(v-u)|^2 \dots |\log_k(v-u)|^2} (\log_{k+1} |E_k - (v-u)|)^2.$$

$$\leq \frac{(v-u)}{|\log(v-u)|^2 \dots |\log_k(v-u)|^2} (\log_{k+1} |E_k - (v-u)|)^2 |\log(v-u)|^{2k}$$

$$\leq \frac{(v-u)}{|\log(v-u)|^2 \dots |\log_k(v-u)|^2} \text{ by Lemma 2 and our choice of}$$

$r < a(k)$. Therefore,

$$I_1 \leq G_k \frac{|t-s|}{|\log|t-s||^2 \dots |\log_k|t-s||^2} \text{ for a constant } G_k$$

$$\text{since } v-u = \frac{t-s}{2}.$$

(3) For the third integral we note that

$$\begin{aligned} I_3 &= \int_{u+E_k}^{v+E_k} (\log_{k+1}|v-\omega|)^2 d\omega \leq (v-u) (\log_{k+1}|v-u-E_k|)^2 \\ &\leq G_k \frac{|t-s|}{|\log|t-s||^2 \dots |\log_k|t-s||^2} \text{ by the same reasoning as} \\ &\text{for } I_1. \end{aligned}$$

(4) The second integral, I_2 , is the hardest to estimate. It will be divided into a number of parts.

$$I_2 = \left[\int_{v-E_k}^u + \int_u^v + \int_v^{u+E_k} \right] \left(\log \left| \frac{\log_k|v-\omega|}{\log_k|u-\omega|} \right| \right)^2 d\omega$$

$$\equiv I_{2,1} + I_{2,2} + I_{2,3}$$

$$\text{Now } I_{2,1} + I_{2,3} = \left[\int_{v-E_k}^u + \int_v^{u+E_k} \right] \left(\log \left| \frac{\log_k|v-\omega|}{\log_k|u-\omega|} \right| \right)^2 d\omega$$

$$= 2 \int_{v-u}^{E_k} \left(\log \left| \frac{\log_k(y+u-v)}{\log_k y} \right| \right)^2 dy$$

upon letting $y = v-\omega$ in $I_{2,1}$ and $y = \omega-u$ in $I_{2,3}$.

Now letting $A = v-u = \frac{1}{2}|t-s|$ we find that

$$I_{2,1} + I_{2,3} = \int_A^{E_k} \left(\log \left| \frac{\log_k(y-A)}{\log_k y} \right| \right)^2 dy$$

$$= \left[\int_A^{2A} + \int_{2A}^{E_k} \right] \left(\log \left| \frac{\log_k(y-A)}{\log_k y} \right| \right)^2 dy$$

$$\equiv J_1 + J_2$$

$$(5) \quad J_1 = \int_A^{2A} \left(\log \left| \frac{\log_k(1/(y-A))}{\log_k(1/y)} \right| \right)^2 dy$$

$$\leq \int_A^{2A} \left(\log \frac{\log_k(1/(y-A))}{\log_k(1/2A)} \right)^2 dy \quad \text{since } (1/y) \geq (1/2A) > (1/r) \\ \geq e_k$$

$$\leq \int_{\log_k(1/A)}^{\infty} (\log(z/B))^2 E_k(z) e_{k-1}(z) \dots e(z) dz$$

upon letting $z = \log_k(1/(y-A))$ and $B = \log_k(1/2A)$

$$= B \int_{B^{-1} \log_k(1/A)}^{\infty} (\log y)^2 E_k(By) e_{k-1}(By) \dots e(By) dy$$

letting $y = z/B$

$$\leq B \int_1^{\infty} (\log y)^2 E_k(By) e_{k-1}(By) \dots e_1(By) dy$$

$$\leq \frac{2E_k(B)}{B^3 (e_{k-1}(B))^2 \dots (e(B))^2} \quad \text{by Lemma 3}$$

$$\leq \frac{4A}{(\log_k 2A)^2 \dots (\log 2A)^2} = \frac{2|t-s|}{|\log|t-s||^2 \dots |\log_k|t-s||^2}$$

$$\text{Thus, } J_1 \leq \frac{2|t-s|}{|\log|t-s||^2 \dots |\log_k|t-s||^2}.$$

$$(6) \quad J_2 = \int_{2A}^{E_k} \left(\log \left| \frac{\log_k(y-A)}{\log_k y} \right| \right)^2 dy$$

$$= \int_{3R}^{E_k - R} \left(\log \left| \frac{\log_k(z-R)}{\log_k(z+R)} \right| \right)^2 dz \quad \text{letting } z = y - (A/2) \quad \text{and}$$

$$R = A/2 = |t-s|/4$$

$$= \int_{3R}^{E_k - R} \left(\log \left[\left(1 - \log \frac{\log_{k-1}(z-R)}{\log_{k-1}(z+R)} \right) / \log_k(z-R) \right] \right)^2 dz$$

$$\leq 4 \int_{3R}^{E_k - R} \left(\left(\log \frac{\log_{k-1}(z-R)}{\log_{k-1}(z+R)} \right) / \log_k(z-R) \right)^2 dz$$

by Lemma 1 and Approximation (*) with $p \leq \frac{1}{2}$ so
 $1/(1-p) \leq 2$.

We now consider two cases, $k = 1$ and $k \geq 2$.

In what follows let $D(k) = \log_k(1/4R)$. We will suppress the 1 if $k = 1$.

Case 1 ($k=1$): By the above,

$$J_2 \leq 4 \int_{3R}^{E_k - R} \left(\frac{\log \left(\frac{z-R}{z+R} \right)}{\log(z-R)} \right)^2 dz = 4 \int_{3R}^{E_k - R} \left(\frac{\log \left(1 - \frac{2R}{z+R} \right)}{\log(z-R)} \right)^2 dz$$

$$\leq 4^2 \int_{3R}^{E_k - R} \left(2R / [(z+R) \log(z+R)] \right)^2 dz$$

by Approximation (*), again with $p \leq \frac{1}{2}$

$$= 4^3 R^2 \int_1^D e^y y^{-2} dy \quad \text{letting } y = -\log(z+R)$$

$$= 4^3 R^2 [e^y y^{-2} \Big|_1^D + 2 \int_1^D e^y y^{-3} dy]$$

$\leq 4^3 R^2 [(1/4R)D^{-2} + (1/2R)D^{-2} + 4e]$ since $e^y y^{-3}$ has a minimum at 3 and no other extreme points in $[1, D]$ we have approximated \int_3^D by the value at the upper limit times the length and we have approximated \int_1^3 by the value at the lower limit times 2.

$$\text{Hence, } J_2 \leq \frac{12|t-s|}{(\log|t-s|)^2} + 4^4 e R^2 \leq 780 \frac{|t-s|}{|\log|t-s||^2}.$$

Case 2 ($k \geq 2$):

$$\begin{aligned} J_2 &\leq 4 \int_{3R}^{E_k - R} \left(\frac{\log_{k-1}(z-R)}{\log_{k-1}(z+R)} \right) / \log_k(z-R) \Big)^2 dz \\ &= 4 \int_{3R}^{E_k - R} \left(\log \left[1 - \frac{\log_{k-2}(z-R)}{\log_{k-2}(z+R)} \right] / \log_{k-1}(z-R) \right)^2 \\ &\quad (\log_k(z-R))^{-2} dz \\ &\leq 4^2 \int_{3R}^{E_k - R} \left(\frac{\log_{k-2}(z-R)}{\log_{k-2}(z+R)} \right)^2 (\log_{k-1}(z-R) \log_k(z-R))^{-2} dz \end{aligned}$$

again by Lemma 1 and Approximation (*)

$$\begin{aligned} &\leq \dots \leq 4^k \int_{3R}^{E_k - R} \left(\log \left(\frac{z-R}{z+R} \right) \right)^2 (\log(z-R) \dots \log_k(z-R))^{-2} dz \\ &= 4^k \int_{3R}^{E_k - R} \left(\log \left(1 - \frac{2R}{z+R} \right) \right)^2 (\log(z-R) \dots \log_k(z-R))^{-2} dz \\ &\leq 4^{k+2} \int_{3R}^{E_k - R} R^2 ((z+R) \log(z-R) \dots \log_k(z-R))^{-2} dz \end{aligned}$$

by Approximation (*) with $p = (2R)/(z+R) \leq 1/2$

$$\leq 4^{k+2} \int_{3R}^{E_k - R} R^2 ((z+R) \log(z+R) \dots \log_k(z+R))^{-2} dz$$

$$= 4^{k+2} R^2 \int_1^{D_k} e_k(y) (ye_{k-1}(y) \dots e_0(y))^{-1} dy$$

letting $y = \log_k(1/(z+R))$

$$= 4^{k+2} R^2 [e_k(y) (e_{k-1}(y) \dots e_0(y))^{-2} \Big|_1^{D_k}$$

$$+ 2 \int_1^{D_k} e_k(y) y^{-1} (e_{k-1}(y) \dots e_0(y))^{-2} dy$$

$$+ 2 \sum_{j=1}^{k-1} \int_1^{D_k} e_k(y) (ye_0(y) \dots e_{j-1}(y))^{-1} (e_j(y) \dots e_{k-1}(y))^{-2} dy]$$

$$\leq 4^{k+2} R^2 [(1/4R) (D_1 \dots D_k)^{-2}$$

$$+ 2 \int_1^{D_k} e_k(y) y^{-1} (e_{k-1}(y) \dots e_0(y))^{-2} dy$$

$$+ 2(k-1) \int_1^{D_k} e_k(y) (ye_0(y) \dots e_{k-2}(y))^{-1} (e_{k-1}(y))^{-2} dy]$$

$$= 4^{k+2} R^2 [(1/4R) (D_1 \dots D_k)^{-2}$$

$$+ 2 \int_1^{D_k} e_k(y) y^{-1} (e_{k-1}(y) \dots e_0(y))^{-2} dy$$

$$+ 2(k-1) \{e_k(y) (e_0(y) \dots e_{k-2}(y))^{-2} (e_{k-1}(y))^{-3} \Big|_1^{D_k}$$

$$+ 2 \int_1^{D_k} e_k(y) y^{-1} (e_0(y) \dots e_{k-2}(y))^{-2} (e_{k-1}(y))^{-3} dy$$

$$\begin{aligned}
& + 2 \sum_{j=1}^{k-1} \int_1^{D_k} e_k(y) (ye_0(y) \dots e_{j-1}(y))^{-1} (e_j(y) \dots e_{k-2}(y))^{-2} \\
& \qquad \qquad \qquad (e_{k-1}(y))^{-3} dy \\
& + 3 \int_1^{D_k} e_k(y) y^{-1} (e_0(y) \dots e_{k-2}(y))^{-1} (e_{k-1}(y))^{-3} dy \} \\
\leq & 4^{k+2} R^2 \{ ((2k-1)/4R) (D_1 \dots D_k)^{-2} \\
& + (4k-2) \int_1^{D_k} e_k(y) y^{-1} (e_{k-1}(y) \dots e_0(y))^{-2} dy \\
& + 2(k-1)(2k+1) \int_1^{D_k} e_k(y) (ye_0(y) \dots e_{k-2}(y))^{-1} (e_{k-1}(y))^{-3} dy
\end{aligned}$$

We now treat $k \geq 3$ and $k = 2$ separately. If $k \geq 3$,

$$\begin{aligned}
J_2 \leq & 4^{k+2} R^2 \{ ((2k-1)/4R) (D_1 \dots D_k)^{-2} + (4k-2) \{ (1/4R) (D_k \dots D_1)^{-2} \\
& + D_k e_k (e \dots e_{k-1})^{-2} \} + 2(k-1)(2k+1) \{ (D_k \dots D_2)^{-1} D_1^{-3} (1/4R) \\
& + D_k e_k (e \dots e_{k-2})^{-1} (e_{k-1})^{-3} \} \} \text{ using Lemma 6 for the first} \\
& \text{integral and Lemma 5 for the second integral when } k \geq 3. \\
\text{Hence, } J_2 \leq & 4^{k+2} R^2 \{ ((6k-3+2(k-1)(2k+1))/4R) (D_1 \dots D_k)^{-2} \\
& + e_k ((4k-2+2(k-1)(2k+1))/4) R^{-1/2} (e \dots e_{k-1})^{-1} \} \text{ since}
\end{aligned}$$

$D_k = \log_k(1/4R) < (1/4)R^{-(1/2)}$ because $R \leq (1/144)$,
and by Lemma 7.

Therefore $J_2 \leq N_k \frac{|t-s|}{|\log|t-s||^2 \dots |\log_k|t-s||^2}$ if $k \geq 3$,

for some constant N_k .

When $k = 2$,

$$J_2 \leq 4^4 R^2 [(3/4R)(D_2 D_1)^{-2} + 12 \int_1^{D_2} e_2(y) y^{-3} (e(y))^{-2} dy]$$

$$\leq 4^4 R^2 [(3/4R)(D_2 D_1)^{-2} + (3/R)(D_2 D_1)^{-2} + 12e_2 e^{-2}]$$

using Lemma 4 to approximate $\int_a^{D_2}$ by the upper limit and \int_1^a by the lower limit where a is the minimum occurring between 1 and 2.

$$\text{Hence, } J_2 \leq \frac{4^2 \cdot 15 |t-s|}{|\log |t-s||^2 |\log_2 |t-s||^2} + 4^4 \cdot 12 e^R$$

$$\leq 5,000 \frac{|t-s|}{|\log |t-s||^2 |\log_2 |t-s||^2} \text{ if } k = 2.$$

Combining all three cases we find that

$$J_2 \leq L_k \frac{|t-s|}{|\log |t-s||^2 \dots |\log_k |t-s||^2}$$

where $L_k = \max(N_k, 5,000)$. So

$$I_{2,1} + I_{2,3} = J_1 + J_2 \leq (L_k + 2) \frac{|t-s|}{|\log |t-s||^2 \dots |\log_k |t-s||^2}$$

$$(7) \quad I_{2,2} = \int_u^v \left(\log \left| \frac{\log_k(v-w)}{\log_k(w-u)} \right| \right)^2 dw$$

$$= \left[\int_{\frac{v+u}{2}}^v + \int_u^{\frac{v+u}{2}} \right] \left(\log \left| \frac{\log_k(v-w)}{\log_k(w-u)} \right| \right)^2 dw$$

$$= 2 \int_0^{\frac{v-u}{2}} \left(\log \left| \frac{\log_k(\frac{v-u}{2} - z)}{\log_k(\frac{v-u}{2} + z)} \right| \right)^2 dz$$

upon letting $z = \omega - \frac{1}{2}(u+v)$ in the first integral and $z = \frac{1}{2}(v+u) - \omega$ in the second integral. Thus,

$$\begin{aligned} I_{2,2} &= 2 \int_0^R \left(\log \left| \frac{\log_k(R-z)}{\log_k(R+z)} \right| \right)^2 dz \\ &= 2 \int_0^R \left(\log \left(\frac{\log_k(1/y)}{\log_k(1/(2R-y))} \right) \right)^2 dy \quad \text{letting } y = R-z \\ &\leq 2 \int_0^R \left(\log \frac{\log_k(1/y)}{S} \right)^2 dy \end{aligned}$$

since $2e_k \leq 1/2R \leq 1/(2R-y) \leq 1/y$ for $y \in [0, R]$,
and setting $S = \log_k(1/2R)$

$$= 2 \int_{\log_k(1/R)}^{\infty} \left(\log \frac{z}{S} \right)^2 E_k(z) e_{k-1}(z) \dots e(z) dz$$

letting $z = \log_k(1/y)$

$$= 2S \int_{S^{-1} \log_k(1/R)}^{\infty} (\log x)^2 E_k(Sx) e_{k-1}(Sx) \dots e(Sx) dx, \quad x = \frac{z}{S}$$

$$\leq 2S \int_1^{\infty} (\log x)^2 E_k(Sx) e_{k-1}(Sx) \dots e(Sx) dx$$

$$\leq 4E_k(S) / (S e_{k-1}(S) \dots e(S))^2 \quad \text{by Lemma 3}$$

$$= 4 \frac{|v-u|}{|\log_k |v-u||^2 \dots |\log |v-u||^2} \leq B_k \frac{|t-s|}{|\log_k |t-s||^2 \dots |\log |t-s||^2}$$

for some constant B_k .

Combining everything we obtain

$$E(X(t)-X(s))^2 \leq C \frac{|t-s|}{|\log|t-s||^2 \dots |\log_k|t-s||^2}$$

where $C = \max\{C(r,k), 2G_k + L_k + 2 + B_k\}$. ///

§2. We will now modify the processes of Proposition 8 to obtain mean zero processes with continuous sample paths which satisfy the same Lipschitz conditions. Let $X(t, \omega)$ be as in Proposition 8 with $k \in \mathbb{N}$ fixed.

$$\text{Let } \tilde{X}(t, \omega) = \begin{cases} X(t, \omega) & \text{if } |(1+2t)/4 - \omega| \geq \delta_\omega, \omega \neq 3/4 \\ \log_{k+1} \delta_\omega^{-1} & \text{if } |(1+2t)/4 - \omega| \leq \delta_\omega, \omega \neq 3/4 \\ 0 & \text{if } \omega = 3/4 \end{cases}$$

where $\delta_\omega = E_{k+1}(8/(3-4\omega))$

$$\text{Let } Y(t, \underline{\omega}) = Y(t, \omega \times k) = \begin{cases} \tilde{X}(t, \omega) & \text{if } k = 0 \\ -\tilde{X}(t, \omega) & \text{if } k = 1 \end{cases}$$

where $\underline{\omega} \in ([0,1] \times \{0,1\}, \text{Lebesgue} \times (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)) \equiv \lambda \times \mu$

The following three lemmas will be used to show that $Y(t)$ does not satisfy the CLT, even though $Y(t)$ satisfies a tight Lipschitz condition on the second moments of its increments.

LEMMA 9.

$$E_{\lambda \times \mu} (Y(t) - Y(s))^2 \leq \frac{C|t-s|}{|\log|t-s||^2 \dots |\log_k|t-s||^2}.$$

PROOF. $E_{\lambda \times \mu} (Y(t) - Y(s))^2 = E_{\lambda} (\tilde{X}(t) - \tilde{X}(s))^2$. Now to show that for all ω, t, s where $\omega \neq (1+2t)/4$ or $(1+2s)/4$, $|\tilde{X}(t, \omega) - \tilde{X}(s, \omega)| \leq |X(t, \omega) - X(s, \omega)|$. We assume that $t > s$.

Case 1. $t, s \notin (\omega - \delta_{\omega}, \omega + \delta_{\omega})$, then equality holds.

Case 2. $t, s \in [\omega - \delta_{\omega}, \omega + \delta_{\omega}]$ implies $|\tilde{X}(t) - \tilde{X}(s)| = 0$.

Case 3. Only one of t and s is in $[\omega - \delta_{\omega}, \omega + \delta_{\omega}]$.

Suppose $s \in [\omega - \delta_{\omega}, \omega + \delta_{\omega}]$

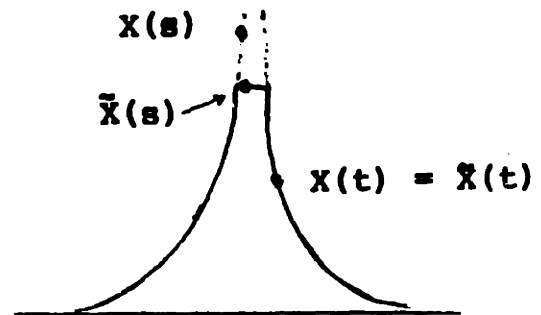
but t is not. Then

$$X(s) > \tilde{X}(s) > \tilde{X}(t) = X(t).$$

$$\text{So, } |\tilde{X}(t) - \tilde{X}(s)| = \tilde{X}(s) - \tilde{X}(t)$$

$$< X(s) - X(t) = |X(t) - X(s)|.$$

The other case is the same.



Thus, $E_{\lambda \times \mu} (Y(t) - Y(s))^2 \leq E_{\lambda} (X(t) - X(s))^2$ and we can

apply Proposition 8. ///

Let $Y^{(i)}(t)$, $i = 1, 2, \dots$ denote i.i.d. copies of

$$Y(t) \text{ and } z_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n Y^{(i)}(t).$$

LEMMA 10 Given $a > 0$, $\sup_t |(1/\sqrt{n})Y^{(n)}(t, \underline{\omega})| \geq 2a$

implies either $\sup_t |z_n(t, \underline{\omega})| \geq a$ or $\sup_t |z_{n-1}(t, \underline{\omega})| \geq a$.

PROOF. Assume not, then $\sup_t |Z_n(t, \underline{\omega})| < a$ and $\sup_t |Z_{n-1}(t, \underline{\omega})| < a$. However, $\sup_t |(1/\sqrt{n})Y^{(n)}(t, \underline{\omega})| = \sup_t |Z_n(t, \underline{\omega}) - (1-n^{-1})^{(1/2)}Z_{n-1}(t, \underline{\omega})| < 2a$, a contradiction. ///

LEMMA 11. For any $b \in R^+$, there exists N_b such that $n \geq N_b$ implies that

$$P\{ \max_{1 \leq i \leq n} \sup_t |(1/\sqrt{n})Y^{(i)}(t)| \geq 2b \} > 1/4.$$

PROOF. We will need three facts:

(1) $\sum_{k=m}^{2m} \binom{k}{m} 2^{-k} = 1$ since each term in the sum represents the probability of winning a best of $(2m+1)$ contest at trial number $k+1$,

$$(2) \quad 2^{-k-1} \binom{k+1}{m} \leq 2^{-k} \binom{k}{m} \quad \text{for } 2m \leq k \leq 3m,$$

$$(3) \quad \binom{k}{m} \leq \binom{k}{2m} \quad \text{for } 3m \leq k \leq 4m.$$

$$\text{For } \underline{\omega} \in [\frac{1}{4}, \frac{3}{4}) \times \{0, 1\},$$

$$\sup_t |(1/\sqrt{n})Y^{(i)}(t, \underline{\omega})| = (1/\sqrt{n}) \log_{k+1} (\delta_{\omega(i)}^{-1})$$

where we can assume the independent copies of Y are taken on a product space $Y^{(i)}(t, \underline{\omega}) = Y(t, \omega(i))$.

By the Strong Law of Large Numbers, as $n \rightarrow \infty$ approximately $n/2$ of the $\omega(i)$'s fall in $[\frac{1}{4}, \frac{3}{4})$. Thus, given δ , $0 < \delta < 1/4$, there exists $N(\delta)$ such that $n \geq N(\delta)$ implies

$$B_n \equiv P\{\underline{\omega} : \max_{1 \leq i \leq n} \sup_t (1/\sqrt{n}) |Y^{(i)}(t, \underline{\omega})| < 2b\}$$

$$\leq \delta + P\{\tilde{\omega} \in [0, 1]^n : 1 \leq i(1) < i(2) < \dots < i([n/4]) \leq n$$

with $\omega(i(j)) \in [\frac{1}{4}, \frac{3}{4}]$ for $j = 1, \dots, [n/4]$ and

$$\max_{1 \leq j \leq [n/4]} (1/\sqrt{n}) \log_{k+1} (\delta_{\omega(i(j))}^{-1}) < 2b\}$$

Let $A_{i(1), \dots, i([n/4])} = \{\tilde{\omega} \in [0, 1]^n : \omega(r) \in [\frac{1}{4}, \frac{3}{4}]$ for
 $r = i(1), \dots, i([n/4])$ and $\omega(r) \notin [\frac{1}{4}, \frac{3}{4}]$ for
 $r \in \{1, \dots, i([n/4])\} \setminus \{i(1), \dots, i([n/4])\}\}$.

$$\text{So } B_n \leq \delta + \Sigma P\{\tilde{\omega} \in A_{i(1), \dots, i([n/4])} : \max_{1 \leq j \leq [n/4]} (\frac{1}{\sqrt{n}}) \log_{k+1} \delta_{\omega(i(j))}^{-1}$$

where the sum is taken over all possible strictly increasing sequences $i(1), \dots, i([n/4])$ with $1 \leq i(j) \leq n$ for all j .

$$= \delta + \Sigma P\{\tilde{\omega} \in A_{i(1), \dots, i([n/4])} : \max_{1 \leq j \leq [n/4]} 8/([3-4\omega_{i(j)}] \sqrt{n}) < 2b\}$$

$$= \delta + \sum_{k=[n/4]}^n \binom{k}{[n/4]} (P\{\omega \notin [1/4, 3/4]\})^{k-[n/4]}.$$

$$(P\{\omega \in [1/4, 3/4] : 8/([3-4\omega] \sqrt{n}) < 2b\})^{[n/4]} \text{ by independence}$$

$$= \delta + \sum_{k=[n/4]}^n \binom{k}{[n/4]} 2^{-k} (1-2/\sqrt{nb})^{[n/4]} \text{ for } n \text{ large}$$

$$\leq \delta + e^{-c\sqrt{n}} \sum_{k=m}^{2m} \binom{k}{m} 2^{-k} + \sum_{k=3m}^{4m} \binom{k}{2m} 2^{-k} + \sum_{k=2m}^{3m} \binom{k}{m} 2^{-k} + 7 \binom{n}{[n/4]} 2^{-n}$$

setting $m = [n/4]$, using (3) and $e^{[n/4] \log(1-2/\sqrt{nb})} \leq M e^{-c\sqrt{n}}$

for n large where M and c are constants

$$\leq \delta + e^{-c\sqrt{n}} (2 + m \binom{2m}{m} 2^{-2m} + 7 \binom{n}{[n/4]} 2^{-n}) \text{ by (2)}$$

$\rightarrow \delta$ as $n \rightarrow \infty$ as can be seen by using Stirling's formula.

Choose $N_b \geq N(\delta)$ such that $n \geq N_b$ implies

$$(1-2/\sqrt{nb})^{[n/4]} < 1/2. \text{ Then for } n \geq N_b,$$

$$P\{\underline{\omega} : \max_{1 \leq i \leq n} \sup_t (1/\sqrt{n}) |Y^{(i)}(t, \underline{\omega})| < 2b\} < 3/4.$$

Hence,

$$P\{\underline{\omega} : \max_{1 \leq i \leq n} \sup_t (1/\sqrt{n}) |Y^{(i)}(t, \underline{\omega})| \geq 2b\} > 1/4. \quad ///$$

It is now easy to prove that $Y(t)$ does not satisfy the CLT in $C[0,1]$.

PROPOSITION 12. For any $k \in \mathbb{N}$, there exists a continuous stochastic process $Y(t, \omega)$ which does not satisfy the CLT in $C[0,1]$ but such that

$$E(Y(t) - Y(s))^2 \leq \frac{C|t-s|}{|\log|t-s||^2 \dots |\log_k|t-s||^2}$$

PROOF. Let $Y(t)$ be as defined above. $Y(t)$ is continuous and by Lemma 9 satisfies the required Lipschitz condition. In order to show that $Y(t)$ does not satisfy the CLT it suffices to show that there exists $\epsilon > 0$ such that given $a > 0$, there exists $n(a)$ for which $P\{\sup_t |Z_{n(a)}(t)| \geq a\} > \epsilon$.

Lemma 11 implies that given $a > 0$ and $\epsilon = 1/8$ there exists N_a such that if $n \geq N_a$ then

$$P\{\max_{1 \leq i \leq n} \sup_t (1/\sqrt{n}) |Y^{(i)}(t)| \geq 2a\} > 2\epsilon.$$

But this implies that if $n \geq N_a$

$$P\left\{\max_{1 \leq i \leq n} \sup_t (1/\sqrt{I}) |Y^{(i)}(t)| \geq 2a\right\} > 2\epsilon.$$

Thus, by Lemma 10

$$P\left\{\max_{1 \leq i \leq n} \sup_t |z_i(t)| \geq a\right\} > 2\epsilon.$$

Applying the Levy inequality for processes (see Dudley (1967) Lemma 4.4, p. 300 or Kahane (1968) Lemma 1, p. 12), we see that if $n \geq N_a$ then

$$P\left\{\sup_t |z_n(t)| \geq a\right\} \geq (1/2)P\left\{\max_{1 \leq i \leq n} \sup_t |z_i(t)| \geq a\right\} > \epsilon. \quad ///$$

§3. The examples in the previous section show that certain Lipschitz conditions on the second moments of differences $X(s)-X(t)$ do not imply that X satisfies the CLT. Thus, it is necessary to place additional assumptions on X . The following examples show that even the additional strong assumption of X being uniformly sample-bounded does not imply that the CLT holds if X is only known to satisfy a Lipschitz condition of order α , $\alpha < 1$. These examples are based on the same scheme as those in Strassen and Dudley ((1969) sec. 3) and Dudley (1974).

PROPOSITION 13. For any $\alpha < 1$ there is a process $X(t, \omega)$, $0 \leq t \leq 1$, with continuous sample paths, $|X(t, \omega)| \leq 1$ for all t and ω , and $E(X(s)-X(t))^2 \leq |s-t|^\alpha$ for all $s, t \in [0, 1]$, such that the CLT does

not hold for X .

PROOF. The following definition of a process X depending on a choice of constants $\{k_n, n = 1, 2, \dots\}$ is taken verbatim from Dudley ((1974) pp. 56-57). For each $n = 1, 2, \dots$, (we) shall decompose $[0, 1]$ into a set I_n of N_n equal subintervals, where $N_n = \prod_{s=1}^n 6k_s$, k_s integers. Thus each interval in I_{n-1} is decomposed into $6k_n$ equal subintervals to form I_n , where $I_0 = \{[0, 1]\}$.

For each n and each $j = 0, \dots, k_n - 1$, we define a piecewise linear continuous function g_{nj} as follows. Let

$$g_{nj}(x) = \begin{cases} 0 & \text{if } N_n x/3 \text{ is an integer} \\ 1 & \text{if } 6i+1 \leq N_n x \leq 6i+2 \\ -1 & \text{if } 6i+4 \leq N_n x \leq 6i+5 \end{cases}$$

where

$$i = j + rk_n, \quad r = 0, 1, \dots, N_{n-1} - 1,$$

and let g_{nj} be continuous and linear on those closed intervals for which it was previously defined only at the endpoints, namely $6i+u \leq N_n x \leq 6i+u+1$, $u = 0, 2, 3, 5$.

Note that for each j , inside every interval in I_{n-1} is an interval in I_n on which $g_{nj} = 1$ and

another on which $g_{nj} = -1$.

Let $p_n = cn^{-\beta}$ where $1 < \beta < 2$ and $c = 1 / \sum_{n=1}^{\infty} n^{-\beta}$.

To be definite, (we) take $\beta = 3/2$.

Now (we) define a probability measure μ on $C[0,1]$ by setting $\mu(\{g_{nj}\}) = \mu(\{-g_{nj}\}) = p_n/2k_n$ for $n = 1, 2, \dots$ and each $j = 0, \dots, k_n - 1$. Let X be a random variable with distribution μ . Then clearly $|X(t)| \leq 1$. Also for each t , $EX(t) = 0$ since X is symmetric and bounded.

Dudley proves that the CLT never holds for X with the given p_n , for any $k_n \geq 2$. What remains is for us to choose an appropriate sequence $\{k_n, n = 1, 2, \dots\}$ and then estimate the mean-square differences. Define k_n inductively by:

$$k_1 = 2 \quad N_1 = 6k_1 = 12$$

$$k_n = N_{n-1}^{\alpha/(1-\alpha)} \text{ for } n \geq 2 \text{ and } \alpha \text{ fixed } < 1.$$

Note that

$$(\#) \quad N_n = 6k_n N_{n-1} = 6N_{n-1}^{1/(1-\alpha)}, \quad n \geq 2.$$

Now we estimate the mean-square differences. Given $s, t \in [0,1]$, take n such that

$$1/N_{n+1} < |s-t| \leq 1/N_n, \text{ where } N_0 = 1.$$

As Dudley has shown (p. 58),

$$\begin{aligned} E(X(s)-X(t))^2 &\leq \sum_{m < n} 2p_m k_m^{-1} (6N_m |t-s|)^2 \\ &\quad + 2p_n k_n^{-1} N_n^2 |s-t|^2 + 8 \left(\sum_{m > n} p_m k_m^{-1} \right). \end{aligned}$$

For $n = 0$,

$$E(X(s)-X(t))^2 \leq 4 = 48N_1^{-1} \leq 48|s-t| \leq 48|s-t|^\alpha. \quad (\#)$$

For $n = 1$,

(1) The first term is vacuous

$$(2) \quad 2p_1 k_1^{-1} N_1^2 |s-t|^2 \leq N_1^2 |s-t|^2 \leq 144 |s-t|^2 \leq 144 |s-t|^\alpha$$

$$(3) \quad 8p_{n+1} k_{n+1}^{-1} \leq 8N_n^{-\alpha/(1-\alpha)} \leq 48N_{n+1}^{-\alpha} \leq 48|s-t|^\alpha \quad (\#)$$

$$(4) \quad 8 \sum_{m=n+2}^{\infty} p_m k_m^{-1} = 8 \sum_{m=n+2}^{\infty} p_m N_{m-1}^{-\alpha/(1-\alpha)}$$

$$\leq 8N_{n+1}^{-\alpha/(1-\alpha)} \sum_{m=n+2}^{\infty} p_m < 8N_{n+1}^{-\alpha/(1-\alpha)}$$

$$\leq 8|s-t|^{\alpha/(1-\alpha)} \leq 8|s-t|^\alpha.$$

$$\therefore E(X(s)-X(t))^2 \leq 200|s-t|^\alpha.$$

For $n \geq 2$,

$$(1') \quad 2p_1 k_1^{-1} (6N_1 |s-t|)^2 \leq (36 \cdot 144) |t-s|^2 \leq 5184 |t-s|^\alpha$$

$$(1'') \quad \sum_{1 < m < n} 2p_m k_m^{-1} 36N_m^2 |s-t|^2$$

$$= \sum_{1 < m < n} 72p_m N_m^{-\alpha/(1-\alpha)} N_m^2 |s-t|^2$$

$$\leq |s-t|^\alpha \sum_{1 < m < n} 432p_m N_m^{2-\alpha} |s-t|^{2-\alpha}$$

$$\leq |s-t|^\alpha 432 \sum_{1 < m < n} p_m \leq 432 |s-t|^\alpha$$

$$(2') \quad 2p_n k_n^{-1} N_n^2 |s-t|^2 \leq N_{n-1}^{-\alpha/(1-\alpha)} N_n^2 |s-t|^2$$

$$\leq 6N_n^{-\alpha} N_n^2 |s-t|^2 = 6N_n^{2-\alpha} |s-t|^2 \leq 6 |s-t|^\alpha$$

(3) and (4) are as for the case $n = 1$ above.

$$\therefore E(X(s)-X(t))^2 \leq 5678 |s-t|^\alpha.$$

Replace X by $X/76$ to get rid of the constant. ///

CHAPTER 5. STOCHASTICALLY CONTINUOUS PROCESSES

In Chapters 3 and 4 we have shown that certain assumptions on moments of the differences $X(t)-X(s)$ imply both sample-continuity and the CLT for X while weaker assumptions imply neither. Therefore, most moment conditions must be accompanied by additional hypotheses in order to provide good CLT's. The main purpose of this chapter is to show that for stochastically continuous processes, a good supplementary hypothesis is given by a condition on moments of products of the differences $X(t)-X(s)$ and $X(u)-X(t)$ for $s \leq t \leq u$. As a consequence, it will be shown that every stochastically continuous process with independent increments and sample paths in D satisfies the CLT.

Sufficient conditions for a process to have sample paths in D are much less stringent than conditions implying sample-continuity. For instance, if X is a separable, stochastically continuous process then the existence of a continuous non-decreasing function F and numbers $\varepsilon > 0$, $\gamma \geq 0$ such that

$$P\{|X(t)-X(s)| \geq \lambda, |X(u)-X(t)| \geq \lambda\} \leq \lambda^{-\gamma} [F(u)-F(s)]^{1+\varepsilon}$$

for all $0 \leq s \leq t \leq u \leq 1$ is sufficient to imply that

X has a version with sample paths in D (see Gikhman and Skorohod (1969) p. 161 or Billingsley (1968) p. 130).

By Chebyshev's inequality, a condition of the form $E(X(t)-X(s))^Y(X(u)-X(t))^Y \leq [F(u)-F(s)]^{1+\epsilon}$ with F as above, also implies the existence of a version with sample paths in D .

DEFINITION. A stochastic process $\{X(t)\}$ is continuous in quadratic mean (CQM) if and only if for all t ,

$$\lim_{s \rightarrow t} E|X(t)-X(s)|^2 = 0.$$

It is well-known that a Gaussian process is CQM if and only if it is continuous in probability. We now show that a necessary condition for a stochastically continuous process to satisfy the CLT is that it be CQM.

LEMMA 5.1 Let $\{X(t)\}$ be a stochastically continuous process which satisfies the CLT in D . Then X is CQM.

PROOF. Let Z be the limiting Gaussian process which has sample paths in D a.s. The process Z is right continuous a.s. so $Z(t) = \lim_{s \rightarrow t} Z(s)$ a.s. which implies that

$$\lim_{s \rightarrow t} E(Z(s)-Z(t))^2 = 0. \text{ But}$$

$\lim_{s \uparrow t} E(X(t) - X(s))^2 = \lim_{s \uparrow t} E(Z(t) - Z(s))^2 = 0$; therefore,

X is right CQM at t . Similarly, Z has left limits a.s., so $Z(t-) = \lim_{s \uparrow t} Z(s)$ a.s. and thus $\lim_{s \uparrow t} E(Z(t-) - Z(s))^2 = 0$.

By the stochastic continuity of X , $\lim_{s \uparrow t} E(X(s) - X(t))^2 = \lim_{s \uparrow t} E(X(s) - X(t-))^2 = \lim_{s \uparrow t} E(Z(s) - Z(t-))^2 = 0$. Hence,

X is also left CQM at t . Therefore, X is CQM at t .

Since t was arbitrary, X is CQM. ///

The following example shows that not every stochastically continuous process with finite second moments and sample paths in D is CQM.

EXAMPLE. For $t \in [0, 1]$ and $\omega \in ([0, 1], \text{Lebesgue})$ let

$$\begin{aligned} X(t, \omega) &= n \quad \text{if } t \in [\omega, \omega + n^{-2}) \text{ and } \omega \in [n^{-1} - n^{-2}, n^{-1}] \\ &= 0 \quad \text{otherwise} \\ X(0, \omega) &\equiv 0 \end{aligned}$$

X has sample paths in D . Since $\sum_{n=1}^{\infty} P\{X(n^{-1}, \omega) > 0\} =$

$\sum_{n=1}^{\infty} n^{-2} < \infty$, the Borel-Cantelli Theorem implies that

$P\{X(n^{-1}, \omega) > 0 \text{ i.o.}\} = 0$; thus, X is stochastically

continuous at 0. X is obviously stochastically con-

tinuous everywhere else. However, $EX^2(t) \leq 1$ for all t

and $EX^2(n^{-1}) = 1$ for all n . So $E(X(n^{-1}) - X(0))^2 = EX^2(n^{-1}) = 1 \not\rightarrow 0$ as $n \rightarrow \infty$. Thus, X is not CQM. ///

The following theorem of Ito and Nisio describes the kinds of discontinuities which Gaussian processes are allowed to have. From this theorem we will deduce that every stochastically continuous Gaussian process with sample paths in D is continuous a.s.

THEOREM (Ito and Nisio (1968) Theorem 2, p. 210)

Let Z be a separable, measurable, CQM Gaussian process on $[0,1]$. There exists a function $\alpha(t) \equiv \alpha_Z(t)$, $0 \leq t \leq 1$, such that for each t

$$P[\limsup_{s \rightarrow t} Z(s) = Z(t) + \frac{1}{2}\alpha(t), \liminf_{s \rightarrow t} Z(s) = Z(t) - \frac{1}{2}\alpha(t)] = 1.$$

α is called the oscillation function of the Gaussian process Z .

COROLLARY 5.2 Any stochastically continuous Gaussian process with sample paths in D has sample paths in C a.s.

PROOF. Z is separable and stochastically continuous, hence measurable (Doob (1953) Theorem 2.5 p. 60). By the Ito-Nisio Theorem, there exists a fixed function α such that for almost every sample path, $Z(t, \omega)$ satisfies, for

all t ,

$$\limsup_{u,v \rightarrow t} |Z(u,\omega) - Z(v,\omega)| = \alpha(t).$$

This implies that $|Z(t) - Z(t-)| = \alpha(t)$ a.s.

$$\text{Let } t_0 \in (0,1]. \quad Z(t_0) - Z(t_0-) = \lim_{s \uparrow t_0} (Z(t_0) - Z(s))$$

where $L(Z(s))$ is Gaussian, mean 0. Thus, $L(Z(t_0) - Z(t_0-))$, being the limit of Gaussian mean 0 random variables, is also Gaussian, mean 0. The Ito-Nisio result then implies that for almost all ω , $Z(t_0) - Z(t_0-)$ equals either $+\alpha(t_0)$ or $-\alpha(t_0)$, which is impossible for a Gaussian, mean 0 random variable unless $\alpha(t_0) = 0$. But $\alpha(t_0) = 0$ implies t_0 is a continuity point of Z . Z , having sample paths in D , is right continuous at $t_0 = 0$. Thus, for almost all ω , $Z(t,\omega)$ is continuous. ///

THEOREM 5.3 Let $X(t)$ be a stochastically continuous process with $EX(t) = 0$ and $EX^2(t) < \infty$ for all $t \in [0,1]$. If the CLT holds for X in D then the limiting Gaussian process is sample-continuous; hence $L(Z)(C) = 1$.

PROOF. The theorem follows immediately from Proposition 5.1 and Corollary 5.2. ///

Before proving the main theorem of this chapter we need a method for determining whether a Gaussian process

is sample-continuous.

Let H be a real, infinite-dimensional Hilbert space with inner product (\cdot, \cdot) . Following Dudley ((1973) p. 67), a Gaussian process L on H will be called isonormal if and only if L is a linear map from H into real Gaussian random variables with $EL(x) \equiv 0$ and $EL(x)L(y) \equiv (x,y)$ for all $x,y \in H$. A set $F \subset H$ is called a GC-set if and only if L restricted to F is sample-continuous.

If $\{X(t) : t \in [0,1]\}$ is a Gaussian process CQM with $EX(t) = 0$, then $X(t)$ is sample-continuous if and only if $E = \{X(t) : t \in [0,1]\} \subset L^2(\Omega, Pr) = H$ is a GC-set (Dudley (1973) p. 69).

Sufficient conditions for sample-continuity of Gaussian processes are usually deduced from metric entropy considerations or fairly weak moment conditions. The latter is sufficient for us here. If there exist $\delta > 0$ and $C < \infty$ such that $E|X(s)-X(t)|^2 \leq C/|\log|t-s||^{1+\delta}$ then $X(t)$ is sample-continuous (Delporte (1964) p. 180).

We are now ready to prove the main theorem.

THEOREM 5.4. Let $X(t)$ be a stochastically continuous process with sample paths in D . Assume $EX(t) = 0$ and $EX^2(t) < \infty$ for all $t \in [0,1]$. Also, assume

(1) there exist a non-decreasing, continuous function G on $[0,1]$ and a number $\alpha > 1/2$ such that for all $0 \leq t \leq u \leq 1$,

$$E(X(u)-X(t))^2 \leq [G(u)-G(t)]^\alpha;$$

(2) there exist a non-decreasing, continuous function F on $[0,1]$ and a number $\beta > 1$ such that for all $0 \leq s \leq t \leq u \leq 1$,

$$E(X(u)-X(t))^2(X(t)-X(s))^2 \leq [F(u)-F(s)]^\beta.$$

Then X satisfies the CLT in D and $L(Z)(C) = 1$.

PROOF. We will verify the hypotheses of Theorem 1.3. We may assume that $|F(t)| \leq 1$ and $|G(t)| \leq 1$ for all t since dividing X by the positive constant $F(1)G(1)$ does not affect the CLT. Let $\{X_i\}$ be independent, identically distributed D -valued random variables with law $L(X)$. Let $Z_n = n^{-1/2}(X_1 + \dots + X_n)$. By Chebyshev's inequality,

$$\begin{aligned} & \lambda^{-4} P\{|Z_n(t)-Z_n(s)| \geq \lambda, |Z_n(u)-Z_n(t)| \geq \lambda\} \\ & \leq E(Z_n(t)-Z_n(s))^2(Z_n(u)-Z_n(t))^2 \\ & = n^{-2} E\left(\left(\sum_{i=1}^n X_i(t)-X_i(s)\right)^2 \left(\sum_{i=1}^n X_i(u)-X_i(t)\right)^2\right) \end{aligned}$$

$$\begin{aligned}
&= n^{-2} \{ nE(X(t)-X(s))^2 (X(u)-X(t))^2 \\
&\quad + n(n-1)E(X(u)-X(t))^2 E(X(t)-X(s))^2 \\
&\quad + 2n(n-1) (E(X(u)-X(t))(X(t)-X(s)))^2 \}
\end{aligned}$$

using independence of the X_i and mean 0

$$\begin{aligned}
&\leq n^{-2} \{ nE(X(t)-X(s))^2 (X(u)-X(t))^2 \\
&\quad + 3n(n-1)E(X(u)-X(t))^2 E(X(t)-X(s))^2 \}
\end{aligned}$$

using the Cauchy-Schwartz inequality

$$\begin{aligned}
&\leq E(X(u)-X(t))^2 (X(t)-X(s))^2 \\
&\quad + 3E(X(u)-X(t))^2 E(X(t)-X(s))^2 \\
&\leq [F(u)-F(s)]^\beta + 3[G(u)-G(t)]^\alpha [G(t)-G(s)]^\alpha \\
&\leq [F(u)-F(s)]^\beta + 3[G(u)-G(s)]^{2\alpha} \\
&\leq [F(u)-F(s)]^{\beta \wedge 2\alpha} + 3[G(u)-G(s)]^{\beta \wedge 2\alpha} \\
&\leq 2[(F + 3^{1/(\beta \wedge 2\alpha)} G)(u) - (F + 3^{1/(\beta \wedge 2\alpha)} G)(s)]^{\beta \wedge 2\alpha}
\end{aligned}$$

using the fact that $x^\gamma + y^\gamma \leq 2(xy)^\gamma \leq 2(x+y)^\gamma$

for $x, y \geq 0$

$$\leq (H(u)-H(s))^{\beta \wedge 2\alpha}$$

where $H = 2^{1/(\beta \wedge 2\alpha)} (F + 3^{1/(\beta \wedge 2\alpha)} G)$ which is a continuous, non-decreasing function on $[0,1]$.

Let Z be a Gaussian process with the covariance of X . $E(Z(t)-Z(s))^2 \leq [F(t)-F(s)]^\beta$ for some $\beta > 1/2$.

Let $F^{-1}(t) = \inf\{s \in [0,1] : F(s) = t\}$. Since

$$F \circ F^{-1}(t) = t,$$

$$E(Z \circ F^{-1}(t) - Z \circ F^{-1}(s))^2 \leq |t-s|^\beta \leq C/|\log|t-s||^{1+\delta}$$

for some $C < \infty$ and $\delta > 0$. This condition implies $Z \circ F^{-1}$ is sample-continuous. Thus $\{Z \circ F^{-1}(t) : t \in [0,1]\}$ is a GC-set, since $Z \circ F^{-1}(t)$ is CQM. Now, the sets of random variables $\{Z \circ F^{-1}(t) : t \in [0,1]\}$ and $\{Z(t) : t \in [0,1]\}$ are identical. Therefore, $\{Z(t) : t \in [0,1]\}$ is also a GC-set which implies Z is sample-continuous.

By the ordinary CLT, the finite-dimensional distributions of Z_n converge to those of a sample-continuous Gaussian process. Therefore, Theorem 1.3 implies that the CLT holds in D and $L(Z)(C) = 1$. ///

Notice that condition (2) is not at all unnatural because, as shown at the beginning of the chapter, it is a sufficient condition for a process to have sample paths in D .

Hypothesis (2) can be changed slightly with the help of a modified version of Theorem 1.3; namely, condition (*) of Theorem 1.3 can be replaced by

$$P\{|X_n(t) - X_n(s)| \geq \lambda, |X_n(u) - X_n(t)| \geq \lambda\} \\ \leq \frac{1}{\lambda^{2\zeta}} [F(t) - F(s)]^\gamma [F(u) - F(t)]^\gamma$$

for $s \leq t \leq u$ and $n \geq 1$, where $\zeta \geq 0$, $\gamma > 1/2$, and F is a non-decreasing, right continuous function on $[0,1]$ (Billingsley (1968) p. 133).

COROLLARY 5.5. Let X satisfy the hypotheses of Theorem 5.4 and replace assumption (2) by

(2') there exist a non-decreasing, right continuous function F on $[0,1]$ and $\gamma > 1/2$ such that

$$E(X(u)-X(t))^2(X(t)-X(s))^2 \leq [F(u)-F(t)]^\gamma(F(t)-F(s))^\gamma.$$

Then X satisfies the CLT in D and $L(Z)(C) = 1$.

PROOF. As in Theorem 5.4, assuming $|F| \leq 1$, $|G| \leq 1$,

$$I_n \leq \lambda^{-4} P\{|Z_n(u)-Z_n(t)| \geq \lambda, |Z_n(t)-Z_n(s)| \geq \lambda\}$$

$$\leq E(X(u)-X(t))^2(X(t)-X(s))^2$$

$$+ 3E(X(u)-X(t))^2E(X(t)-X(s))^2$$

$$\leq (F(u)-F(t))^\gamma(F(t)-F(s))^\gamma$$

$$+ 3(G(u)-G(t))^\alpha(G(t)-G(s))^\alpha$$

$$\leq (F(u)-F(t))^{\gamma \wedge \alpha}(F(t)-F(s))^{\gamma \wedge \alpha}$$

$$+ (G(u)-G(t))^{\gamma \wedge \alpha}(G(t)-G(s))^{\gamma \wedge \alpha}$$

$$\text{where } G' = 3^{1/2}(\gamma \wedge \alpha)G$$

$$\leq ((F+G')(u)-(F+G')(t))^{\gamma \wedge \alpha}((F+G')(t)-(F+G')(s))^{\gamma \wedge \alpha}$$

using the fact that $x^\beta y^\beta + r^\beta s^\beta \leq$

$(x^\beta+r^\beta)(y^\beta+s^\beta)$ since all terms are positive

and this is $\leq 4(x+r)^\beta(y+s)^\beta$ as in Theorem 5.4.

Thus, letting $H(t) = 4^{1/(\gamma \wedge \alpha)} (F+G')(t)$, H is a non-decreasing, right continuous function. Then,

$$I_n \leq (H(u)-H(t))^{\gamma \wedge \alpha} (H(t)-H(s))^{\gamma \wedge \alpha}.$$

Existence of a sample-continuous Gaussian process with the correct covariance and convergence of the finite-dimensional distributions follow as in Theorem 5.4. ///

Let $X(t)$ be a stochastic process with orthogonal increments. Define $F(t) = E(X(t)-X(0))^2$. Then

$$\begin{aligned} E(X(t)-X(s))^2 &= E(X(t)-X(0)+X(0)-X(s))^2 \\ &= E(X(t)-X(0))^2 + E(X(s)-X(0))^2 - 2E(X(t)-X(0))(X(s)-X(0)) \\ &= E(X(t)-X(0))^2 - E(X(s)-X(0))^2 - 2E(X(t)-X(s))(X(s)-X(0)) \\ &= E(X(t)-X(0))^2 - E(X(s)-X(0))^2 \text{ because of the orthogonal} \\ &\quad \text{increments} \\ &= F(t)-F(s) \end{aligned}$$

Notice that F is monotone non-decreasing since

$$F(t)-F(s) = E(X(t)-X(s))^2 \geq 0.$$

Also, for any orthogonal increments process X there is a set T which is at most countable such that $X(t)$ is CQM on $[0,1] \setminus T$ and for $t \in T$ $X(t)$ has right and left limits in QM (Doob (1953) p. 425). If $X(t)$ is also stochastically continuous with sample paths in D then $X(t)$ is actually CQM because

$$\lim_{s \uparrow t} E(X(t) - X(s))^2 = \lim_{s \uparrow t} E(X(t-) - X(s))^2 = 0 \text{ and}$$

$$\lim_{s \downarrow t} E(X(t) - X(s))^2 = \lim_{s \downarrow t} E(X(t+) - X(s))^2 = 0. \text{ In this}$$

case F is actually continuous.

COROLLARY 5.6 Let $X(t)$ be a stochastic process with negatively correlated squared increments. Assume $EX(t) = 0$, $EX^2(t) < \infty$ for all $t \in [0,1]$ and that there exist a non-decreasing, continuous function G on $[0,1]$ and a number $\alpha > 1/2$ such that for all $0 \leq t \leq u \leq 1$,

$$E(X(u) - X(t))^2 \leq [G(u) - G(t)]^\alpha.$$

Then X satisfies the CLT.

PROOF. The proof follows immediately from Theorem 5.4 since for $0 \leq s \leq t \leq u \leq 1$

$$E(X(u) - X(t))^2 E(X(t) - X(s))^2 \leq E(X(u) - X(t))^2 E(X(t) - X(s))^2$$

$$\leq [G(u) - G(t)]^\alpha [G(t) - G(s)]^\alpha \leq [G(u) - G(s)]^{2\alpha}$$

and $2\alpha > 1$. ///

As an immediate consequence we obtain the following corollary.

COROLLARY 5.7 Every stochastically continuous process with orthogonal increments and negatively correlated squared

increments satisfies the CLT.

We now deduce that every stochastically continuous process with independent increments and sample paths in D satisfies the CLT. This is quite general because every separable stochastically continuous process with independent increments has a version with sample paths in D (see Gikhman and Skorohod (1969) Theorem 3 and Corollary p. 168).

COROLLARY 5.8 Let $X(t)$ be a stochastically continuous process with independent increments and sample paths in D . Assume $X(0) = 0$ a.s., $EX(t) = 0$, and $EX^2(t) < \infty$ for all t . Then X satisfies the CLT with limiting Gaussian process $Z(t) = W \circ F(t)$ where W is a standard Brownian motion and $F(t) = EX^2(t)$.

PROOF. Since a mean 0 process with independent increments has orthogonal increments and uncorrelated squared increments, the hypotheses of Corollary 5.7 are satisfied. In order to determine the limiting Gaussian process we consider the covariance of $X(t)$. For $t \geq s$

$$\begin{aligned} EX(t)X(s) &= \frac{1}{2}\{EX^2(t)+EX^2(s)-E(X(t)-X(s))^2\} \\ &= \frac{1}{2}\{F(t)+F(s)-(F(t)-F(s))\} \\ &= F(s) \\ &= F(s) \wedge F(t). \end{aligned}$$

Thus, the limiting Gaussian process is $W \circ F(t)$ where W is a standard Brownian motion. ///

By the Decomposition CLT, the assumption $X(0) = 0$ a.s. is really not a restriction.

DEFINITION. We say that $Z(t)$ is a generalized Wiener process if $Z(t) = W \circ \eta(t)$ where W is a Brownian motion and η is a continuous, non-decreasing function on $[0,1]$.

CHAPTER 6. MARKOV PROCESSES

In the last chapter we saw that one class of Markov processes, the stochastically continuous, independent increment processes, always satisfy the CLT. We proceed now to consider other Markov processes which have sample paths in D . Since the sum of Markov processes is not always Markov, as shown by Example 6.3, below, most of the criteria appearing in the current literature for convergence of sequences of Markov processes are inapplicable.

This section contains some of the CLT results which we have obtained for Markov processes. In particular, it will be shown that every stochastically continuous Markov process which has only a finite number of states, has stationary transitions, and has sample paths in D does satisfy the CLT.

We begin by investigating several properties of functions of Markov processes.

PROPOSITION 6.1. Let ψ be a function on $[0,1]$ which is non-zero on $(0,1)$. Let $X(t)$ be a Markov process. Then $\psi(t)X(t)$ is Markov.

PROOF. Let $0 \leq s_1 < \dots < s_k < t < u_1 < \dots < u_m \leq 1$. Let A_i ($i = 1, \dots, m$) and B_j ($j = 1, \dots, k$) be Borel

measurable subsets of \mathbb{R} . First, assume $\psi(t) \neq 0$ for every $t \in [0,1]$.

$$\Pr\{\psi(s_j)X(s_j) \in B_j \ (j = 1, \dots, k),$$

$$\psi(u_i)X(u_i) \in A_i \ (i = 1, \dots, m) \mid \psi(t)X(t)\}$$

$$= \Pr\{X(s_j) \in B_j' \ (j = 1, \dots, k), X(u_i) \in A_i' \ (i = 1, \dots, m) \mid X(t)\}$$

where $A_i' = A_i/\psi(u_i)$, etc.

$$= \Pr\{X(s_j) \in B_j' \ (j = 1, \dots, k) \mid X(t)\} \Pr\{X(u_i) \in A_i' \ (i=1, \dots, m) \mid X(t)\}$$

by the Markov property for X

$$= \Pr\{\psi(s_j)X(s_j) \in B_j \ (j = 1, \dots, k) \mid \psi(t)X(t)\}.$$

$$\Pr\{\psi(u_i)X(u_i) \in A_i \ (i = 1, \dots, m) \mid \psi(t)X(t)\}.$$

If $\psi(0) = 0$ and $s_1 = 0$ then

$$\Pr\{\psi(0)X(0) \in B_1, \psi(s_j)X(s_j) \in B_j \ (j = 2, \dots, k),$$

$$\psi(u_i)X(u_i) \in A_i \ (i = 1, \dots, m) \mid \psi(t)X(t)\}$$

$$= \begin{cases} \Pr\{\psi(s_j)X(s_j) \in B_j \ (j=2, \dots, k), \\ \psi(u_i)X(u_i) \in A_i \ (i = 1, \dots, m) \mid \psi(t)X(t)\} & \text{if } 0 \in B_1 \\ 0 & \text{if } 0 \notin B_1 \end{cases}$$

The rest follows as above.

The cases where $\psi(1) = 0$ or both $\psi(0) = 0$ and $\psi(1) = 0$ can be treated similarly. ///

Note that if $\psi(t) = 0$ for some $t \in (0,1)$ the conclusion of Proposition 6.1 is false. For example, let

$$\psi(t) = \begin{cases} 1 & \text{if } t \neq 1/2 \\ 0 & \text{if } t = 1/2 \end{cases}$$

and let $X(t)$ be a Markov process with the two states, 0 and 1, defined in the following way:

$$X(t) = \begin{cases} X(0) & \text{if } 0 \leq t < \frac{1}{2} \\ X(\frac{1}{2}) & \text{if } \frac{1}{2} \leq t < 1 \\ X(1) & \text{if } t = 1 \end{cases}$$

where the finite Markov chain $X(0), X(\frac{1}{2}), X(1)$ has transition probabilities

$$P(0,0) = P(1,1) = .99 \quad \text{and} \quad P(1,0) = P(0,1) = .01.$$

Obviously, in predicting $(\psi X)(1)$ given $(\psi X)(\frac{1}{2})$ it helps to know $(\psi X)(0)$.

We observe that if $X(t)$ is any stochastically continuous process (Markov or not) and $P(X(t) \neq 0) > 0$ for every t , in order that $\psi(t)X(t)$ be stochastically continuous, ψ must be continuous. However, if there exists a point t_0 such that $X(t_0) = 0$ a.s., $\psi(t_0)$ may be any finite value.

PROPOSITION 6.2. Let X be a process satisfying the CLT. Suppose that the limiting Gaussian process Z is Markov. Let ψ be a continuous function on $[0,1]$ which is non-zero on $(0,1)$. If $Y(t) = \psi(t)X(t)$, then Y satisfies the CLT with the limiting Gaussian Markov process $\psi \cdot Z$. The conclusion holds also if Z is continuous a.s. and $\psi \in D$.

PROOF. The conclusion follows immediately from Proposition 6.1 and Lemma 2.3. ///

The following example shows, in general, it is not true that the sum of two i.i.d. Markov processes is Markov.

EXAMPLE 6.3 Let Y be a finite Markov chain with parameter set $\{0, 1/3, 2/3, 1\}$, with state space $\{0, 1, 2\}$, with $Y(0) = 0$ a.s., and with transition probabilities $p(i, j) = 1/2$ for $i \neq j$, $i, j = 0, 1, 2$. Define $X(t)$ by

$$X(t) = \begin{cases} Y(0) & 0 \leq t < 1/3 \\ Y(1/3) & 1/3 \leq t < 2/3 \\ Y(2/3) & 2/3 \leq t < 1 \\ Y(1) & t = 1 \end{cases}$$

Let $X'(t)$ be an independent copy of $X(t)$. The state space of $S(t) = X(t) + X'(t)$ is $\{0, 1, 2, 3, 4\}$. $X(t)$ is Markov but we will show that $S(t)$ is not Markov.

$P\{S(1) = 1 | S(2/3) = 2, S(1/3) = 4\} = 0$ since $S(1/3) = 4$ implies $X(1/3) = X'(1/3) = 2$; $S(1/3) = 4$ and $S(2/3) = 2$ imply $X(2/3) = X'(2/3) = 1$; hence, $X(1) \neq 1$ and $X'(1) \neq 1$.

However, $P\{S(1) = 1 | S(2/3) = 2\} > 0$ since the path

$$\begin{array}{ccccccc} (X(0), X'(0)) & \rightarrow & (X(\frac{1}{3}), X'(\frac{1}{3})) & \rightarrow & (X(\frac{2}{3}), X'(\frac{2}{3})) & \rightarrow & (X(1), X'(1)) \\ " & & " & & " & & " \\ (0,0) & \rightarrow & (1,1) & \rightarrow & (2,0) & \rightarrow & (0,1) \end{array}$$

occurs with positive probability. ///

The above example is a three-state Markov process. For a smaller number of states, namely one or two, the sum of any number of i.i.d. Markov processes is always Markov. It is clear in the case of one state. So suppose the Markov process X has two states, c and d . Let

$$S_n(t) = \sum_{i=1}^n X_i(t). \quad \text{If } S_n(t) = k \text{ then there exist}$$

unique numbers $m, r \in \mathbb{N}$ such that $m+r = n$ and $mc+rd = k$. This specifies exactly how many of the independent $X_i(t)$'s are c and how many take the value d . Thus, $S_n(t')$ for $t' < t$ will supply no further information for predicting the future.

PROPOSITION 6.4. Let X be a stochastic process with $EX(t) = 0$, and $0 < EX^2(t) < \infty$ for every $t \in [0,1]$. If X satisfies the CLT with a continuous limiting Gaussian Markov process Z then $X(t) = \psi(t)Y(t)$ where

(i) $\psi(t)$ is a continuous function on $[0,1]$ which is strictly positive;

(ii) Y is a stochastic process which satisfies the CLT with a generalized Wiener-process as its limiting Gaussian process;

(iii) if X is Markov then so is Y .

PROOF. The structure theorem for Gaussian Markov processes on $(0,1)$ implies that $Z(t) = \psi(t)W \circ \eta(t)$

$$\text{where } \psi(t) = \begin{cases} \frac{EX^2(s_0)EX^2(t)}{EX(s_0)X(t)} & \text{if } t \leq s_0, s_0 \text{ fixed } \in (0,1) \\ EX(s_0)X(t) & \text{if } t \geq s_0 \end{cases}$$

$$\text{and } \eta(t) = \begin{cases} \frac{EX^2(t)}{(EX(s_0)X(t))^2} & \text{if } t \geq s_0 \\ \frac{(EX(s_0)X(t))^2}{(EX^2(s_0))^2 EX^2(t)} & \text{if } t \leq s_0 \end{cases}$$

for $t \in (0,1)$ and W is a Brownian motion process.

These definitions make sense because $EX(s_0)X(t) \neq 0$

(see Neveu (1968) p. 52-56).

$\psi(t)$ is a continuous, positive function on $(0,1)$ which can be extended to a continuous positive function on $[0,1]$ since for all t , $0 < EX^2(t) < \infty$. $\eta(t)$ is a continuous, non-decreasing, non-negative function on $(0,1)$ which can be extended to have the same properties on $[0,1]$ since for all t , $0 < EX^2(t) < \infty$. The proof in Neveu's book then extends to give $Z(t) = \psi(t)W \circ \eta(t)$ for all $t \in [0,1]$.

Let $Y(t) = X(t)/\psi(t)$; if X is Markov, so is Y by Proposition 6.1. Since

$$L(n^{-1/2} \sum_{i=1}^n \psi(t) Y_i(t)) = L(n^{-1/2} \sum_{i=1}^n X_i(t)) \rightarrow L(\psi(t)W \circ \eta(t)),$$

applying Lemma 2.3 we see that

$$L\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n Y^{(i)}(t)\right) \rightarrow L(W \circ \eta(t)). \quad ///$$

A necessary condition for any process X (Markov or not) to satisfy the CLT with a limiting Gaussian Markov process Z is that

$$(*) \quad EX(s)X(u)EX^2(t) = EX(t)X(s)EX(t)X(u).$$

The reason is that the covariance of a Gaussian Markov process always satisfies the above condition and the second moment structure of X is the same as that of Z .

PROPOSITION 6.5. Either one of the following conditions is sufficient for a Markov process with second moments to have a covariance satisfying (*):

$$(i) \text{ if } s < t, E(X(s) | X(t)) = a(s,t)X(t)$$

$$(ii) \text{ if } s < t, E(X(t) | X(s)) = a(s,t)X(s)$$

where the $a(s,t)$ are constants.

PROOF. Let $s < t < u$. Suppose (i) is satisfied.

$$\begin{aligned} & E((X(s) - a(s,t)X(t))X(u)) \\ &= E((X(s) - E(X(s) | X(t)))X(u)) \\ &= E(E((X(s) - E(X(s) | X(t)))X(u) | X(t))) \\ &= E(E((X(s) - E(X(s) | X(t))) | X(t))E(X(u) | X(t))) \\ &\quad \text{by the Markov property.} \\ &= 0. \end{aligned}$$

$$\begin{aligned} EX(s)X(t) &= E(E(X(s) | X(t))X(t)) = \\ E(a(s,t)X^2(t)) &= a(s,t)EX^2(t) \text{ which implies that} \\ a(s,t) &= EX(s)X(t)/EX^2(t). \end{aligned}$$

Using the above two facts we see that

$$\begin{aligned} 0 &= E((X(s) - a(s,t)X(t))X(u)) = EX(s)X(u) - a(s,t)EX(t)X(u) = \\ &EX(s)X(u) - EX(t)X(u)EX(s)X(t)/EX^2(t) \text{ which implies that} \\ &EX(s)X(u)EX^2(t) = EX(t)X(u)EX(s)X(t). \text{ This is just} \\ &\text{condition (*).} \end{aligned}$$

The proof using (ii) is similar. ///

Many familiar processes satisfy the conditions of Proposition 6.5. For instance, if X is a mean zero process with independent increments, second moments, and $X(0) = 0$ a.s. then X satisfies condition (ii) since $E(X(t)|X(s)) = X(s) + E(X(t)-X(s)|X(s)) = X(s) + E(X(t)-X(s)) = X(s)$ if $t > s$. An independent increment process run backwards will satisfy condition (i) for a similar reason. In these two examples we have learned nothing new since we know that the CLT holds and the limiting Gaussian process is a generalized Wiener process in the first case and such a process run backwards in the second case.

All two state mean 0 processes satisfy both conditions (i) and (ii) which may be seen as follows:

Let $r > 0$ and $-b < 0$ denote the two states.

$E(X(t)|X(s) = r) = c(t,s)$ and $E(X(t)|X(s) = -b) = d(t,s)$.

Thus, passing a line through two points,

$$E(X(t)|X(s)) = \alpha(t,s)X(s) + \beta(t,s)$$

for coefficients $\alpha(t,s)$ and $\beta(t,s)$.

$$0 = EX(t) = E[E(X(t)|X(s))] = \beta(t,s).$$

Therefore, $E(X(t)|X(s) = \alpha(t,s)X(s)$ as desired.

The main theorem of this section is the following CLT for Markov processes. Its major attribute is that

the partial sums $Z_n(t)$ are not required to be Markov. There is, however, the disadvantage that if $X(0) = 0$ a.s., $X(1) = 0$ a.s. and for some t_0 , $\text{ess sup}_\omega |X(t_0, \omega)| = \infty$ then the hypotheses cannot possibly be satisfied. Thus, Tied-down Brownian Motion does not satisfy the hypotheses of the theorem.

THEOREM 6.6 Let X be a Markov process and let F be a non-decreasing continuous function. Suppose that either

(i) for all $t \geq s$, there exists $\beta > 1/2$ such that

$$\text{ess sup}_\omega E[(X(t)-X(s))^2 | X(s)] \leq [F(t)-F(s)]^\beta$$

or

(ii) for all $s \leq t$, there exists $\beta > 1/2$ such that

$$\text{ess sup}_\omega E[(X(t)-X(s))^2 | X(t)] \leq [F(t)-F(s)]^\beta.$$

Then X satisfies the CLT.

PROOF. We will prove the theorem assuming (i); the proof assuming (ii) is similar. Let $s < t < u$. We will verify the hypotheses of Theorem 5.4. First assume that $EX(t) = 0$ for all t .

$$E(X(t)-X(s))^2 = \int E[(X(t)-X(s))^2 | X(s)] dP$$

$$\leq \text{ess sup}_\omega E[(X(t)-X(s))^2 | X(s)] \leq [F(t)-F(s)]^\beta.$$

$$E(X(u)-X(t))^2 (X(t)-X(s))^2$$

$$= \int E[(X(u)-X(t))^2 (X(t)-X(s))^2 | X(t)] dP$$

$$= \int E[(X(u)-X(t))^2 | X(t)] E(X(t)-X(s))^2 | X(t)] dP$$

by the Markov property

$$\leq \text{ess sup}_\omega E[(X(u)-X(t))^2 | X(t)] E(X(t)-X(s))^2$$

$$\leq [F(u)-F(t)]^\beta [F(t)-F(s)]^\beta \text{ by (i) and the above calculation}$$

$$\leq [F(u)-F(s)]^{2\beta} \text{ since } F \text{ is non-decreasing.}$$

If $EX(t) = m(t)$ then

$$\text{ess sup}_\omega E[(X(t)-X(s)-m(t)+m(s))^2 | X(s)]$$

$$= \text{ess sup}_\omega \{E[(X(t)-X(s))^2 | X(s)] + (m(t)-m(s))^2$$

$$\quad - 2(m(t)-m(s))E[X(t)-X(s) | X(s)]\}$$

$$\leq \text{ess sup}_\omega \{E[(X(t)-X(s))^2 | X(s)] + (m(t)-m(s))^2$$

$$\quad + 2|m(t)-m(s)|E[|X(t)-X(s)| | X(s)]\}$$

by the Conditional Jensen's inequality

$$\leq \text{ess sup}_\omega \{E[(X(t)-X(s))^2 | X(s)] + E(X(t)-X(s))^2$$

$$\quad + 2(E(X(t)-X(s))^2)^{1/2} (E[(X(t)-X(s))^2 | X(s)])^{1/2}\}$$

by the Cauchy-Schwartz and Conditional Jensen's inequalities

$$\leq 4 \text{ess sup}_\omega E[(X(t)-X(s))^2 | X(s)]$$

$$\leq 4[F(t)-F(s)]^\beta.$$

Hence, if $Y(t) = X(t) - m(t)$ then Y has mean 0 and by the case for mean 0,

$$E(Y(t) - Y(s))^2 \leq 4[F(t) - F(s)]^\beta,$$

$$E(Y(u) - Y(t))^2 (Y(t) - Y(s))^2 \leq 16[F(u) - F(s)]^{2\beta}.$$

Therefore, by Theorem 5.4, X satisfies the CLT. ///

As an application of this theorem we consider separable, stochastically continuous Markov processes with stationary transitions and only finitely many states. We will show that the conditions of Theorem 6.6 are always satisfied for these processes.

If $[p_{ij}(\cdot)]$ is a stationary Markov transition matrix function such that $\lim_{t \rightarrow 0} p_{ij}(t) = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$ then the limit

$$\lim_{t \rightarrow 0} \frac{1 - p_{ii}(t)}{t} = q_i$$

exists for all i , but may be infinite. In fact,

$$\frac{1 - p_{ii}(t)}{t} \leq q_i \quad \text{for all } t \text{ (Chung (1967) Theorem 4 p. 131}$$

and p. 135).

If $\{X(t)\}$ is a separable process determined by $[p_{ij}(\cdot)]$ together with an initial probability distribution, then

$$P\{X(t, \omega) = i, t_0 \leq t \leq t_0 + \alpha | X(t_0, \omega) = i\} = e^{-q_i \alpha}$$

(Chung (1967) Theorem 5 p. 152-153). Thus, if $q_i = 0$ the process upon reaching i , stays there forever; if $q_i = +\infty$, there is an instantaneous jump so the sample paths cannot be in D . A sufficient condition that almost all sample paths be step functions is that the set of q_i be bounded above (Chung (1967) Corollary 2 p. 260).

COROLLARY 6.7. Let X be a stochastically continuous Markov process with stationary transitions, only finitely many states, and sample paths in D . Then X satisfies the CLT.

PROOF. We will verify the conditions of Theorem 6.6.

Let F be the set of points in the state space of X .

Let $N = \max_{i \in F} |i|$ which is finite since F is finite.

Suppose $t > s$ then, $\sup_{i \in F} E((X(t) - X(s))^2 | X(s) = i)$

$= \sup_{i \in F} E((X(t-s) - X(0))^2 | X(0) = i)$ since X has stationary transitions

$= \sup_{i \in F} P\{X(t-s) \neq X(0) = i\} E((X(t-s) - X(0))^2 | X(t-s) \neq X(0) = i)$

$\leq 4N^2 \sup_{i \in F} P\{X(t-s) \neq X(0) = i\}$

$= 4N^2 \sup_{i \in F} (1 - p_{ii}(t-s)).$

Now $1-p_{ii}(t-s) \leq q_i(t-s)$, since X is stochastically continuous; and the q_i are finite, since X has sample paths in D . Therefore, $\sup_{i \in F} E((X(t)-X(s))^2 | X(s)=i) \leq$

$4N^2 \sup_{i \in F} q_i |t-s| = C|t-s|$. Hence, condition (ii) of

Theorem 6.6 is satisfied. ///

COROLLARY 6.8 Let X be a stochastically continuous Markov process with sample paths in D , stationary transitions, and a countable bounded state space. Furthermore, assume that X has a bounded set of q_i 's. Then the CLT holds for X .

PROOF. The result follows as in Corollary 6.7, except let $N = \sup_{t, \omega} |X(t, \omega)|$. ///

CHAPTER 7. PROCESSES WITH FIXED DISCONTINUITIES

In the previous sections our attention was restricted to stochastically continuous processes. Now we consider processes which also have points of fixed discontinuity. We show, using the Decomposition CLT, that the CLT for processes with only a finite number of fixed discontinuities can be reduced to proving a CLT for stochastically continuous processes. This is a consequence of the fact that processes having only a finite number of fixed discontinuities and whose sample paths are step functions a.s. always satisfy the CLT.

If there is a countable set T of fixed discontinuity points, we consider the case in which X is continuous a.s. at all points of $[0,1] \setminus T$. In this case we show that the problem can be reduced to considering the CLT in $C(J_T)$ where J_T is a certain compact metric space.

Finally, we discuss the existence of a Gaussian process with a given covariance and sample paths in D .

PROPOSITION 7.1 Let Y be a stochastic process with sample paths in D and let T denote the set of points of fixed discontinuity of Y . Assume $EY^2(t) < \infty$ for all $t \in [0,1]$, T is a finite set, and the sample paths of Y are a.s. step functions with jumps only on T . Then Y satisfies the CLT.

PROOF. Let $T = \{t_1, \dots, t_N\}$ be the set of fixed discontinuity points of Y . Let

$$A_T = \{x \in D : x \text{ is constant on } [0, t_1), \dots, [t_{N-1}, t_N), [t_N, 1]\}.$$

In order for the theorem to make sense, we must show that A_T is a measurable subset of D . $(A_T, \|\cdot\|_\infty)$ is a standard Borel space. The uniform topology on D is finer than the Skorohod topology, so every Skorohod open set is a uniform open set. Thus the injection $i : A_T \rightarrow D$ is continuous, hence Borel. Since the 1-1 Borel image of a standard space in a standard space is Borel measurable (Kuratowski (1966) Theorem 1 p. 489), A_T is Borel measurable. Thus it makes sense to say that the sample paths of Y are in A_T a.s.

Y is determined by the random variables $\{Y(0), Y(t_1), \dots, Y(t_N)\}$. Therefore, the CLT follows for Y by the finite-dimensional CLT. ///

THEOREM 7.2 Let X be a stochastic process with only a finite set T of fixed discontinuities. Then there exist a stochastically continuous process V and a process Y whose sample paths a.s. are step functions, with jumps only at points of T , such that $X = Y + V$. Furthermore, the CLT holds for X if the CLT holds for V .

PROOF. Let $Y(t, \omega) = \sum_{t_i^- < t} X(t_i, \omega) - X(t_i^-, \omega)$.

Y has a finite number of fixed discontinuity points, namely the set T . The sample paths of Y are step functions and $EY^2(t) \leq C \sum_{t_i^- < t} E(X(t_i) - X(t_i^-))^2 < \infty$.

Let $V(t, \omega) = X(t, \omega) - Y(t, \omega)$. Then $V(t, \omega)$ is stochastically continuous and has finite second moments since both $X(t)$ and $Y(t)$ do. By Proposition 7.1, Y satisfies the CLT. Since V is stochastically continuous, if V satisfies the CLT it will have a sample-continuous limiting Gaussian process, by Theorem 5.3. Hence $X(t) = Y(t) + V(t)$ will satisfy the CLT by the Decomposition CLT. ///

A process X which has a set T of fixed discontinuity points and whose sample paths almost surely are continuous at all $t \in [0, 1] \setminus T$ will be called a pure fixed discontinuity process (p.f.d.p.)

Let X be a p.f.d.p. with sample paths in D . The set T of fixed discontinuity points of X is a countable set (Billingsley (1968) p. 124). Let $Q^T = \{x \in D: \text{the only discontinuity points of } x \text{ occur at points of } T\}$. We will show that Q^T may be identified with the space of continuous functions on

a compact metric space J_T .

Let $J_T = [0,1] \cup T^-$ where $T^- = \{t_i^- : t_i \in T\}$.

Define a metric e on J_T by

$$e(s,t) = |s'-t'| + \sum_{\substack{t \wedge s < t_i^- < t \vee s \\ t_i \in T}} 2^{-i}$$

where $s' = s$ if $s \in [0,1]$ and $(t_j^-)' = t_j$ if $t_j \in T$. Here the order relation $<$ denotes the usual "less than" ordering between points of $[0,1]$, and is extended to $[0,1] \cup T^-$ by the convention that $t_i^- < t_i$, $s < t_i^-$ if both $s \notin T$ and $s < t_i$, and $t_i^- < t_j^-$ if $t_i < t_j$. e is clearly a metric.

LEMMA 7.3. (J_T, e) is a compact metric space.

PROOF. It suffices to show that every sequence in J_T has a convergent subsequence. Let $\{s_j : j = 1, 2, \dots\}$ be a sequence of points in J_T . Then there exists a subsequence $s_{j,k}$ and a point $s \in [0,1]$ such that $|s_{j,k}' - s| \rightarrow 0$. From $s_{j,k}$ it is possible to extract a subsequence $s_{j,k,m}$ which is either non-decreasing or non-increasing with respect to the ordering $<$. Suppose the former, $s_{j,k,m} \uparrow$. We claim that $e(s_{j,k,m}, \bar{s}) \rightarrow 0$, where $\bar{s} = s$ if $s \notin T$ and $\bar{s} = s^-$ if $s \in T$. Given

$\epsilon > 0$, there exists N such that $\sum_{i \geq N} 2^{-i} < \epsilon$. Choose M such that $m \geq M$ implies that $t_1, \dots, t_N \notin [s_{j,k,m}, s)$ and $|s'_{j,k,m} - s| < \epsilon$. Then $e(s_{j,k,m}, \bar{s}) < 2\epsilon$ for $m \geq M$. The argument for $s_{j,k,m} +$ follows similarly upon letting $\bar{s} = s$ for all s . ///

Let $f(s) \equiv e(0, s)$, $s \in J_T$. We now show that f is an isometry of J_T with $(\overline{f([0,1])}, |\cdot - \cdot|)$.

f is a strictly increasing function, hence 1-1.

$|f(t) - f(s)| = |e(0, t) - e(0, s)| = t' \vee s' - t' \wedge s' + \sum_{t \wedge s < t_i \leq t \vee s} 2^{-i} = e(t, s)$. Thus, f is an isometry with

its image, which must be compact. If

$t \in T$, $\lim_{s \uparrow t} f(s) = f(t^-)$. From this it follows that

$$f(J_T) = \overline{f([0,1])}.$$

LEMMA 7.4 If $y \in Q^T$ then y extends uniquely to a continuous function on (J_T, e) .

PROOF. Let $y(t_j^-) = \lim_{s < t_j} y(s)$ for $t_j \in T$. Let $u_0 \in J_T$. It is necessary to show that, given $\epsilon > 0$ there exists $\delta > 0$ such that $e(u, u_0) < \delta \Rightarrow |y(u) - y(u_0)| < \epsilon$. We prove the case in which $u_0 = t_k^-$; the others are similar. Suppose $u_0 = t_k^-$.

Let M be such that $j > M$ implies $|y(t_j) - y(t_j^-)| < \epsilon/2$.
 Choose $\delta_1 > 0$ such that $u \in (t_k - \delta_1, t_k)$ implies
 $|y(t_k^-) - y(u)| < \epsilon/2$. Let $\delta = \min(\gamma, \delta_1, 2^{-k-1})$ and
 suppose $e(t_k^-, u) < \delta$. Then since $e(t_k^-, u) < 2^{-k-1}$, $u' < t_k$.
 If $u = u'$, then $|y(u) - y(t_k^-)| < \epsilon/2$. If $u = t_j^-$, then
 $j > M$ and $|y(t_j^-) - y(t_k^-)| \leq |y(t_j^-) - y(t_j)| + |y(t_j) - y(t_k^-)| <$
 $\epsilon/2 + \epsilon/2 = \epsilon$. ///

Clearly, if $g \in C(J_T)$ then $g|_{[0,1]} \in Q^T$. So the
 spaces $(Q^T, \|\cdot\|_\infty)$ and $C(J_T)$ are homeomorphic under
 the mapping $h : (Q^T, \|\cdot\|_\infty) \rightarrow C(J_T)$ defined by $x \rightarrow x^*$
 where $x^*(t) = x(t)$ for $t \in [0,1]$ and
 $x^*(t^-) = \lim_{s \uparrow t} x(s)$ for $t \in T$. For $g \in C(J_T)$, $h^{-1}(g) =$
 $g|_{[0,1]}$.

For the next theorem we need to know that Q^T is a
 measurable subset of D . $(Q^T, \|\cdot\|_\infty)$, being homeomorphic
 to the standard space $C(J_T)$, is itself a standard space.
 As in Proposition 7.1, the injection $i : Q^T \rightarrow D$ is
 continuous, hence Borel. The Kuratowski theorem quoted
 in the proof of Proposition 7.1 now provides the conclusion
 desired.

THEOREM 7.5. Let X be a stochastic process such
 that $L(X)(Q^T) = 1$. Let (J_T, e) be the compact metric

space defined above. If $h(X)$ satisfies the CLT in $C(J_T)$ then X satisfies the CLT in D .

PROOF. If the CLT holds for $h(X)$ in $C(J)$ then by the Continuous Mapping Theorem the CLT holds for $X = h^{-1}h(X)$ in $(Q^T, \|\cdot\|_\infty)$. But weak convergence in the uniform topology on Q^T implies weak convergence in the Skorohod topology on Q^T . Consequently, since $L(X)(Q^T) = 1$, the CLT holds in D for X . ///

This theorem shows that it is sufficient to verify that $h(X)$ satisfies the CLT in $C(J_T)$; so, for instance, the conditions of the Jain-Marcus CLT are applicable.

As in the case of stochastically continuous processes, a process with fixed discontinuity points which satisfies the CLT must satisfy certain quadratic mean (QM) continuity properties.

LEMMA 7.6 Let X be a stochastic process which satisfies the CLT in D . Let $T = \{\text{fixed discontinuity points of } X\}$. Then X is CQM at all $t \notin T$ and X is right CQM with left limits in QM at $t \in T$.

PROOF. The proof is the same as in Lemma 5.1, except that X is not stochastically continuous at $t \in T$, so we can conclude only right CQM with left limits in QM for

As a consequence of this lemma, the limiting Gaussian process Z must be CQM except at points of T . At points of T it must be right CQM and have limits from the left in QM. Thus, if X has fixed discontinuities, Z will not be sample-continuous.

Note that if a Gaussian process with sample paths in D has a fixed discontinuity then that discontinuity occurs with probability 1 since $Z(t) - Z(t-)$ is Gaussian and thus equals 0 with either probability 0 or 1.

The CLT can not hold for X unless there exists a Gaussian process with sample paths in D and the covariance of X . It is therefore important to determine when such a Gaussian process exists. Given the process X , construct a Gaussian process Z with the same covariance. Assume that X is CQM except for a countable set of points T , at which it is right CQM with left limits in QM. Z will have the same properties. Let J_T be the same set as defined above for p.f.d.p.'s. (J_T, e) is a compact metric space on which $h(Z)$ is actually CQM. The Ito-Nisio Theorem extends to mean 0, CQM Gaussian processes on J_T . Consequently, the same argument as in Corollary 5.2 shows that a CQM Gaussian process with sample paths in D , when extended to J_T , must actually be continuous. Hence it suffices to determine when $h(Z)$ is sample-continuous on J_T , or

equivalently, to determine when Z a.s. has sample paths in Q^T . $h(Z)$ is sample-continuous on J_T if and only if the set $F = \{Z(t) : t \in [0,1] \cup T^-\}$ is a GC-set.

F is always a GC-set if $\int_0^1 H_T(J_T, x)^{1/2} dx < \infty$ where

$H_T(J_T, x)$ is the metric entropy of J_T with respect to τ and τ is the pseudo-metric defined by

$$\tau(s, t) = E(Z(t) - Z(s))^2 \quad (\text{Dudley (1973) Theorem 1 p. 71}).$$

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BIOGRAPHICAL SKETCH

Marjorie Hahn was born Marjorie Greene on December 30, 1948 in Salt Lake City, Utah where she attended public school. After graduation with a B.S. degree from Stanford University in 1971, Mrs. Hahn accepted a National Science Foundation Graduate Fellowship to study at M.I.T. In 1973 she married Peter Florin Hahn, himself an aspiring mathematician.