

TEAM DECISION THEORY

by

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ABSTRACT

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This thesis explores four topics in the theory of teams: multiperiod team theory, sequential acquisition of information by a team, optimal iterative approximations of the team decision functions, and planning by constrained teams.

Essay One addresses the question, "How can the team best use its historic information?" In Bayesian decision theory information is accumulated when the decision maker updates his beliefs about the state of nature by replacing the prior distributions by the posterior distributions. The informational differences within the team prevents the construction of a team posterior distribution. To facilitate the accumulation of information "ex post communications" are added to the information structure to summarize current knowledge within the team. Optimality conditions are derived for multiperiod teams with either static or changing environments, and with either intertemporally separable or non-separable utility functions.

Second, the team may gather information one piece at a time and control the amount of information gathered. If each observation and resulting communication is costly, the team should only acquire information that increases the net expected payoff. Essay Two develops an optimal stopping rule for stopping the sequential accumulation of information where the evaluation of previous information is based on the ex post communications.

Third, the optimality conditions of team decision theory are complicated systems of integral equations. Essay Three explores approximate solutions of the team's problem. Drawing from the theory of gradient algorithms in optimal control, an iterative solution procedure is developed for the quadratic-normal team. When both information and computation are costly, optimal amounts of information and accuracy can be defined.

Fourth, constraints on decisions are particularly confounding in a theory that combines both informational differences and decentralized authority. Modifications of the team problem to incorporate joint constraints are catalogued in Essay Four. If the organization has internal differences in technological knowledge, differences that cannot be readily eliminated, then iterative planning mechanisms must extract pertinent data from the knowledgeable members. The properties of such decentralized procedures are studied in this essay.

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ABSTRACT

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The theory of team decision making developed by J. Marschak and R. Radner can be viewed as an extension of Bayesian statistical decision theory to an organization with many decision makers. Both theories study the use of information in mathematical optimization under uncertainty, but team theory by its multiperson nature also features informational differences, communication, interdependence and cooperation. This thesis explores four topics in the theory of teams: Multiperiod team theory, sequential acquisition of information by a team, optimal iterative approximations of the team decision functions, and planning by constrained teams.

Essay one addresses the question, "How can the team best use its historic information?" In Bayesian decision theory information is accumulated when the decision maker updates his beliefs about the state of nature by replacing the prior distributions by the posterior distributions. The informational differences within the team prevents the construction of a team posterior distribution. To facilitate the accumulation of information "ex post communications" are added to the information structure to summarize current knowledge within the team. Optimality conditions are derived for a multiperiod team with either static or changing environments and with either intertemporally separable or non-separable utility functions.

Second, the team may gather information one piece at a time and control the amount of information gathered. If each observation and resulting communication is costly, the team should only acquire information that increases the net expected payoff. Essay two develops an optimal rule for stopping the sequential accumulation of information where the evaluation of previous information is based on the ex post communications.

Third, the optimality conditions of team decision theory are complicated systems of integral equations. Essay three explores approximate solutions of the team's problem. Drawing from the theory of gradient algorithms in optimal control, an iterative solution procedure is developed for the quadratic-normal team. When both information and computation are costly, optimal amounts of information and accuracy can be defined.

Fourth, constraints on decisions are particularly confounding in a theory that combines both informational differences and decentralized authority. Modifications of the team problem to incorporate joint constraints are catalogued in essay four. If the organization has internal differences in technological knowledge, differences that cannot be readily eliminated, then iterative planning mechanisms must extract pertinent data from the knowledgeable members. The properties of such decentralized procedures are studied in this essay.

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ESSAY ONE

Multiperiod Team Decision Theory

MULTIPERIOD TEAM DECISION THEORY

I. BAYESIAN STATISTICAL DECISION THEORY
AND TEAM DECISION THEORY

a. Introduction The theory of team decision making developed by Jacob Marschak and Roy Radner⁽¹⁾ is closely related to Bayesian statistical decision theory. Both theories study the use of information in the maximization of expected utility, where utility is a function of the actions of the decision maker and the uncertain state of nature. Both theories specify optimal decision functions, which map sample information into actions. Team theory can be viewed as the extension of Bayesian decision theory to an interrelated group of decision makers.

What role does team theory play in economics? Marschak and Radner used it as the theoretical basis for the study of the optimal use of information within the group. The group most often cited is the "firm" and most examples are couched in terms of the business firm. Statistical decision theory is essentially a theory of one decision maker, the "statistician," while team theory studies the joint decision making of several individuals acting as a unit.

What is a team? Analogous to the common usage of the term, a team is a group of people who have identical tastes and beliefs: tastes referring to utility functions and beliefs referring to subjective probabilities of the random variables. The fundamental assumption of team theory is that each indiv-

idual maximizes the same group expected utility function where expected utility is a product of the utility function and the probability estimates. Team theory is a polar case of a game against nature, with the opposite pole being the zero-sum game of complete conflict. By moving to the extreme assumption of identical tastes and beliefs, the ambiguities of general non-zero-sum games disappear.

Each teammate controls a personal action variable and receives a personal sample statistic that is correlated with the unknown state of nature (hence providing information about the state of nature). Communication is allowed between teammates according to a prescribed rule or information structure. In fact this is the focus of the applications of team theory: how valuable is communication to the team? The team decision problem is a group problem not a set of individual decision problems because the interaction of the actions taken by different teammates affects the team utility; coordination is desirable because cross partial derivatives of utility with respect to different actions are non-zero. A joint choice of individual decision functions must be made. The teammate may implement his action personally, but he is not allowed to construct his own decision function ignoring the others. The economic theory of information solves the team decision problem for particular information structures (such as decentralized, centralized, partitioning into committees, or reporting only exceptional cases) and compare the corresponding optimal payoffs.

b. Statistical Decision Theory Bayesian statistical decision theory is constructed from the following basic elements,⁽²⁾

- Θ : random state of nature
 A : action variable
 $U(A, \Theta)$: utility function
 Y : observation from random sample
 $\alpha(Y)$: decision function
 $\phi(\Theta)$: prior probability density function of Θ
 $\phi(Y|\Theta)$: conditional prior p.d.f. of Y given Θ

The information provided by Y comes through the density function $\phi(Y|\Theta)$. In the following, ϕ 's will denote prior probabilities and f 's will denote posterior probabilities. Different densities will be denoted merely by the name of the random variable in the parenthesis. To facilitate the comparison of statistical decision theory and team decision theory the action variable will be taken as an N-vector $A = (a_1, a_2, \dots, a_N)'$. Hence the decision function is vector valued $\alpha(Y) = (\alpha_1(Y), \dots, \alpha_N(Y))'$. The statistician wants to select a decision function α to maximize expected utility:

$$(1) \quad W[\alpha] = E\{U(\alpha(Y), \Theta)\} = \int_{\Theta} \int_Y U(\alpha(Y), \Theta) \phi(Y|\Theta) \phi(\Theta) dY d\Theta.$$

Only first order conditions will be discussed here; problems of existence and uniqueness of global optimality are ignored and it is always assumed that utility is differentiable and concave.

The optimal decision functions $\hat{\alpha}_1, \dots, \hat{\alpha}_N$ must satisfy the following first order conditions for given Y :

$$(2) \quad 0 = \int_{\Theta} U_{a_i}(\hat{\alpha}(Y), \Theta) \phi(Y|\Theta) \phi(\Theta) d\Theta \quad i=1, 2, \dots, N.$$

Dividing by $\phi(Y)$, the prior probability of receiving the observation Y :

$$(3) \quad \phi(Y) = \int_{\Theta} \phi(Y|\theta) \phi(\theta) d\theta$$

and moving (3) within the integral, we can rewrite (2) as:

$$(4) \quad 0 = \int_{\Theta} u_{a_i}(\hat{a}(Y), \theta) f(\theta|Y) d\theta \quad i=1,2,\dots,N$$

where

$$(5) \quad f(\theta|Y) = \phi(Y|\theta)\phi(\theta) / \phi(Y).$$

$f(\theta|Y)$ is the posterior probability of θ given the observed value of Y . The first order conditions (4) can be interpreted as follows: select for each Y the actions so that the expected marginal utility with respect to a_i given the observed Y is zero. For any particular \bar{Y} the decision $\hat{a}(\bar{Y})$ can be found without computing the entire optimal decision function $\hat{a}(Y)$. It can simply be chosen as the value of A that optimizes against the posterior distribution $f(\theta|\bar{Y})$.

c. Team Decision Theory The team consists of N teammates indexed by $i=1,2,\dots,N$. Each teammate controls a personal action a_i and receives a personal information value y_i . The basic elements of the team problem are

Θ : random state of nature

$A=(a_1, a_2, \dots, a_N)'$: actions of teammates

$Y=(y_1, y_2, \dots, y_N)'$: information of teammates

$u(A, \theta)$: team utility function

$\phi(\theta)$: team prior p.d.f. of Θ

$\phi(Y|\theta)$: team conditional prior p.d.f. of Y given θ

$\hat{a}(Y) = (a_1(y_1), \dots, a_N(y_N))'$: team decision function

The individual information y_1 is a combination of personal observations and messages received by i^{th} teammate from other members; this will be discussed at length in the next subsection. The i^{th} teammates action can only depend on his own information y_1 . The i^{th} individual will never know what value of y_j the j^{th} teammate received. This critical restriction on the decision function is all that analytically differentiates team decision theory from statistical decision theory.

The individuals are all team players so they all desire to select decision functions to maximize the expected team utility.

$$(6) \quad W[\alpha] = E\{U(\alpha(Y), \theta)\} = \int_{\Theta} \int_{\mathcal{Y}} U(\alpha(Y), \theta) \phi(Y|\theta) \phi(\theta) dY d\theta.$$

Since the maximand is a functional and the instruments are functions, the technique for solving the problem is analogous to the calculus of variations. The first order conditions are actually the Euler equations of a specific calculus of variations problem. Again we will deal only with the first order conditions, will assume differentiable, concave utility, and will ignore existence and uniqueness questions.

Let $\hat{\alpha}(Y)$ be the optimal team decision function. All arbitrary decision functions can be written as $\alpha_i(y_i) = \hat{\alpha}_i(y_i) + \epsilon_i \gamma_i(y_i)$ where ϵ_i is a constant and γ_i is a function only of y_1 . The definition of optimality implies that $W[\hat{\alpha}] \geq W[\hat{\alpha} + E\gamma]$ for all arbitrary functions $\gamma(Y) = (\gamma_1(y_1), \dots, \gamma_N(y_N))'$, where E is a diagonal constant matrix. That is $W[\hat{\alpha} + E\gamma]$ treated as a function of E is maximized at $E=0$ for all arbitrary functions $\gamma(Y)$. The first order conditions are

$$(7) \quad \left. \frac{\partial W[\hat{\alpha} + E\gamma]}{\partial \epsilon_i} \right|_{E=0} = 0 \quad i=1, 2, \dots, N$$

where $\left. \frac{\partial W}{\partial \varepsilon_i} \right|_0 = \int_{\Theta} \int_{Y} u_{a_i}(\hat{\alpha}(Y), \theta) \chi_i(y_i) \phi(Y|\theta) \phi(\theta) dY d\theta$.

Reordering integration

$$(8) \quad 0 = \int_{y_i} \left[\int_{\Theta} \int_{Y(i)} u_{a_i}(\hat{\alpha}(Y), \theta) \phi(Y|\theta) \phi(\theta) dY(i) d\theta \right] \chi_i(y_i) dy_i$$

where $Y(i)$ denotes the vector $(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_N)$.

Use will be made of the following lemma.

Lemma: $\int_X f(x) g(x) dx = 0$ for all arbitrary $g(x)$ then $f(x) \equiv 0$.

In equation (8), $g(x)$ corresponds to $\chi_i(y_i)$ and $f(x)$ corresponds to the bracketed integral. Hence for each y_i the optimal decision functions $\hat{\alpha}$ must satisfy simultaneously the following:

$$(9) \quad 0 = \int_{\Theta} \int_{Y(i)} u_{a_i}(\hat{\alpha}(Y), \theta) \phi(Y|\theta) \phi(\theta) dY(i) d\theta \quad i=1, 2, \dots, N.$$

Dividing by $\phi_i(y_i)$, the prior probability of observing y_i ,

$$(10) \quad \phi_i(y_i) = \int_{\Theta} \int_{Y(i)} \phi(Y|\theta) \phi(\theta) dY(i) d\theta,$$

the first order conditions become:

$$(11) \quad 0 = \int_{\Theta} \int_{Y(i)} u_{a_i}(\hat{\alpha}(Y), \theta) f(Y(i), \theta | y_i) dY(i) d\theta \quad i=1, 2, \dots, N.$$

The density function

$$(12) \quad f(Y(i), \theta | y_i) = \phi(Y|\theta) \phi(\theta) / \phi_i(y_i)$$

is the joint posterior density of $Y(i)$, θ given the observed value of y_i . The conditions (11) are interpreted as selecting $\hat{\alpha}$ so that the expected marginal utility with respect to a_i given the personal information y_i is zero. We should compare the conditions (11) for team theory with the corresponding ones for statistical decision theory (4) and note the crucial differences the restrictions on α created. Unlike statistical decision theory, for a particular $\bar{Y} = (\bar{y}_1, \dots, \bar{y}_N)$ the team action $\hat{\alpha}(\bar{Y}) = (\hat{\alpha}_1(\bar{y}_1), \dots, \hat{\alpha}_N(\bar{y}_N))'$ can not be found without computing the entire optimal decision function $\hat{\alpha}(Y)$. This is because the

i^{th} teammate does not know the value of y_j , $j \neq i$, and hence it must be treated as a random variable.

d. Ex Ante Communication The team model of Marshack and Radner allows communication among teammates before actions must be selected. We assume that i runs a personal experiment which gives him a sample value z_i which is correlated with the true state of nature through a conditional p.d.f. $\phi_i(z_i|\theta)$. Communication of the results of these experiments is represented by a message matrix Y where y_{ij} is the message sent to i from j . A communication structure is represented by a matrix function $\eta(Z)$ where $\eta_{ij}(Z) = \eta_{ij}(z_j)$ is the message sent to teammate i from teammate j when he observes the value z_j .

$$(13) \quad Y = \eta(Z) = \begin{pmatrix} y_{11} & y_{12} & \dots & y_{1N} \\ y_{21} & y_{22} & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ y_{N1} & \cdot & \cdot & y_{NN} \end{pmatrix} = \begin{pmatrix} \eta_{11}(z_1) & \eta_{12}(z_2) & \dots & \eta_{1N}(z_N) \\ \eta_{21}(z_1) & \eta_{22}(z_2) & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \eta_{N1}(z_1) & \cdot & \cdot & \eta_{NN}(z_N) \end{pmatrix}$$

The i^{th} row of Y is the total message received by the i^{th} teammate and the j^{th} column of Y is the total message sent by j^{th} teammate. Typically the diagonal elements are of the form $\eta_{ii}(z_i) = z_i$.

If the personal sample experiments are independent so that $\phi(z|\theta) = \prod_{i=1}^N \phi_i(z_i|\theta)$, we can compute the induced conditional probability of the message matrix Y as follows:

$$(14) \quad g(Y|\theta) = \prod_{j=1}^N \left(\int_{z_j \in \bigcap_{i=1}^N \eta_{ij}^{-1}(y_{ij})} \phi_j(z_j|\theta) dz_j \right)$$

where $\bigcap_{i=1}^N \eta_{ij}^{-1}(y_{ij})$ is the set of all sample observations z_j that would cause teammate j to send the messages $(y_{1j}, \dots, y_{Nj})^*$.

The i^{th} row of Y is what we have been calling y_i and the p.d.f. $g(Y|\theta)$ is what has been called $\phi(Y|\theta)$. Because the message Y is sent before the team must select the action A , this communication is denoted "ex ante".

II. MULTIPERIOD TEAM DECISION THEORY
WITH A STATIC ENVIRONMENT

a. Multiperiod Statistical Decision Theory For simplicity of analysis I will make several assumptions about the multiperiod problems, none of which affect the major result.

Assumptions:

- (a) $T = 1, 2$: two periods, superscripts denote period
- (b) $\theta^1 = \theta^2 = \theta$: environment is constant
- (c) $u(A^1, A^2, \theta) = u(A^1, \theta) + u(A^2, \theta)$: additive identical utility
- (d) Perfect memory of past observations

The thrust of what follows is that the action in period two can be selected by the statistician in three identical ways: by constructing a decision function $\hat{a}^2(Y^1, Y^2)$ to maximize expected utility against the prior p.d.f. $\phi(\theta)$; by waiting for the observation $Y^1 = \bar{Y}^1$ and selecting a decision function $\hat{a}^2(Y^2)$ to maximize expected utility against the posterior p.d.f. $f(\theta | \bar{Y}^1)$; or by waiting for both observations \bar{Y}^1, \bar{Y}^2 and selecting an action \hat{A}^2 to maximize expected utility against posterior p.d. f. $f(\theta | \bar{Y}^1, \bar{Y}^2)$. Each of these procedures will result in the same action for a given set of observations \bar{Y}^1, \bar{Y}^2 .

The statistician must take two decisions, one in each period, based on two sample observations Y^1, Y^2 taken at the beginning of each period one the optimal decision function $\hat{a}^1(Y^1)$ is found by equation (4). The optimal decision for period two can be constructed in three ways, differentiated by what information

is assumed already known by the statistician. Just as identical actions result from selecting a decision function against the prior p.d.f. $\phi(\theta)$ or just a single action against the posterior p.d.f. $f(\theta|Y)$ in the single period case, each of these procedures leads to identical actions in a period two.

Before period one's observation is known a decision function for two, $\hat{A}^2(Y^1, Y^2)$, can be selected that maps (Y^1, Y^2) into the set of actions to maximize the expected utility:

$$(15) \int_{Y^1} \int_{Y^2} \int_{\Theta} U(\hat{A}^2(Y^1, Y^2), \theta) \phi(Y^1, Y^2 | \theta) \phi(\theta) d\theta dY^1 dY^2$$

Alternatively the statistician could wait until Y^1 is known, update his beliefs about θ , and then select a decision function $\hat{A}^2(Y^2)$ to maximize expected utility against the posterior p.d.f. of Y^2 ,

$$(16) \int_{Y^2} \int_{\Theta} U(\hat{A}^2(Y^2), \theta) f(Y^2, \theta | Y^1) d\theta dY^2$$

Finally the statistician could wait until both Y^1 and Y^2 are known update his beliefs about θ , and then select a single action \hat{A}^2 to maximize expected utility against the posterior p.d.f. of θ .

$$(17) \int_{\Theta} U(\hat{A}^2, \theta) f(\theta | Y^1, Y^2) d\theta.$$

In (15) A^2 is explicitly a function of both observations Y^1 and Y^2 . In (16) A^2 is explicitly a function of Y^2 and implicitly a function of Y^1 through the density $f(Y^2, \theta | Y^1)$. In (17) A^2 is implicitly a function of Y^1 and Y^2 through the density $f(\theta | Y^1, Y^2)$. In all three cases the explicit-implicit relationships between A^2 and Y^1, Y^2 are the same because the first order conditions for (15), (16), (17) are identical and are given by

$$(18) 0 = \int_{\Theta} U_{A^2}(\hat{A}^2, \theta) f(\theta | Y^1, Y^2) d\theta.$$

where

$$(19) f(\theta | Y^1, Y^2) = \frac{\phi(Y^1, Y^2, \theta)}{\phi(Y^1, Y^2)} = \frac{\phi(Y^1, Y^2 | \theta) \phi(\theta)}{\int_{\Theta} \phi(Y^1, Y^2 | \theta) \phi(\theta) d\theta}.$$

b. Multiperiod Team Decision Theory Four works on team theory have included a time dimension. Charles Kriebel⁽³⁾ specified a multiperiod problem but assumed away difficulties by making Θ different in each period and independent of all past Θ . Charles Ying⁽⁴⁾ developed what he called an "adaptive team". Marschak and Radner⁽⁵⁾ studied "dynamic teams" but focused on the problem of delayed information. Y.C. Ho and K.C. Chu⁽⁶⁾ explored team problems when present actions influence future information.

The above assumptions (a)-(d) will still hold. The team must make two decisions, one in each period, based on information provided at the beginning of each period. For period one the optimal team rules $\hat{\alpha}^1(Y^1)$ are defined by equation (11). After period one's information Y^1 is received, current team beliefs about Θ should be $f(\theta | Y^1)$ not $\phi(\theta)$ but each teammate only knows his own information y_1^1 and no one knows the value of the entire message matrix Y^1 . Not one teammate could compute $f(\theta | Y^1)$ and hence it cannot be used as period two's prior beliefs about Θ . Each teammate can compute $f(\theta | y_i^1)$ and could then maximize expected second period utility using $f(\theta | y_i^1)$ as the density of Θ , but if this was attempted we would not have a team problem in period two because of differences in beliefs. What should the team do in period two?

As long as the team model allows only ex ante communica-

tions about the individual experimental results, the following procedure is the optimal solution for two periods. The assumption of perfect memory for the team problem implies that the i^{th} teammate remembers exactly what message y_i^1 he received in period one. In period two he can make a decision based only on his two messages (y_i^1, y_i^2) . The joint prior density of (Y^1, Y^2, θ) for all teammates is defined by $\phi(Y^1, Y^2, \theta) = \phi(Y^1, Y^2 | \theta) \phi(\theta)$. The multiperiod maximand is the expected utility of both periods against this prior p.d.f. :

$$(20) \quad W^{1+2}(\alpha^1, \alpha^2) = E\{u(\alpha^1(Y^1), \theta) + u(\alpha^2(Y^1, Y^2), \theta)\} \\ = \int_{\Theta} \int_{y^2} \int_{y^1} [u(\alpha^1(Y^1), \theta) + u(\alpha^2(Y^1, Y^2), \theta)] \phi(Y^1, Y^2, \theta) dY^1 dY^2 d\theta.$$

The decision function for period α^2 has the restricted form:

$$(21) \quad \alpha^2(Y^1, Y^2) = (\alpha_1^2(y_1^1, y_1^2), \dots, \alpha_N^2(y_N^1, y_N^2))'$$

Unlike the statistician the team cannot substitute a posterior density of θ for the prior density of period two. The team must select a decision function $\hat{\alpha}^2$ at the same time it selects a function $\hat{\alpha}^1$. The assumption of additive utilities implies that the first order conditions of $\hat{\alpha}^1$ are disjoint from those of $\hat{\alpha}^2$, but $\hat{\alpha}^2$ is computed prior to period one as far as p.d.f.'s are concerned. $\hat{\alpha}^2$ is not changed between period one and two. The first order conditions for $\hat{\alpha}^2(Y^1, Y^2)$ are given by the following

$$(22) \quad 0 = \int_{\Theta} \int_{y^2(i)} u_{\alpha_i}(\hat{\alpha}^2(Y^1, Y^2), \theta) f(Y^1(i), Y^2(i), \theta | y_i^1, y_i^2) dY^1(i) dY^2(i) d\theta \\ i = 1, 2, \dots, N$$

where

$$(23) \quad f(Y^1(i), Y^2(i), \theta | y_i^1, y_i^2) = \phi(Y^1, Y^2 | \theta) \phi(\theta) / \phi_i(y_i^1, y_i^2)$$

is the conditional p.d.f. of θ and the non- i two period informa-

tion variables given of teammate 1's observations in both periods.

These first order conditions (22) are to be distinguished from the optimality conditions if the entire matrix Y^1 was known by all teammates prior to period two. In that case the posterior p.d.f. $f(\theta|Y^1)$ could be computed and the second period decision function $\hat{a}^2(y^2)$ would have to satisfy

$$(24) \quad 0 = \int_{\Theta} \int_{y^2(i)} u_{a_i}(\hat{a}^2(y^2), \theta) f(y^2(i), \theta | y_i^2, Y^1) dy^2(i) d\theta \quad i=1, 2, \dots, N$$

where

$$(25) \quad f(y^2(i), \theta | y_i^2, Y^1) = \phi(y^1, y^2 | \theta) \phi(\theta) / \phi(y_i^2, Y^1)$$

is the posterior probability of θ and the non-1 second period information variables given the known value of Y^1 and the i^{th} teammate's information y_i^2 .

c. Intertemporal Communication The above formulation of multi-period team theory excludes communication between periods other than individual "memory," which is a personal internal communication. A more general framework for intertemporal communication was suggested by Marschak and Radner in the "dynamic" team model. In the T^{th} period, messages are sent and received by teammates^{based} on the past history of individual observations and messages. In its most general form the message matrix Y^T is a function of the previous individual experimental outcomes $\bar{Z}(T) = (Z^1, Z^2, \dots, Z^T)$, an $N \times T$ matrix:

$$(26) \quad Y^T = \eta^T(\bar{Z}(T)).$$

When the team makes only ex ante communications based on a single communication function $\eta(Z)$ in each period but indivi-

dual memory is allowed, this is modelled by choosing an intertemporal communication structure η^T of the following form. η_{ij}^T is a T-vector valued function, each element of which corresponds to the message sent from j to i in one of the previous periods, i.e.,

$$(27) \quad \eta_{ij}^T = (\eta_{ij}(z_j^1), \eta_{ij}(z_j^2), \dots, \eta_{ij}(z_j^T)).$$

The i^{th} teammate makes his decision based on the information he receives in period T, which is the set of all past ex ante messages sent to him plus the current ex ante message about the current experimental observations Z^T . For the case that the individual experiments are independently drawn, the distribution of the message matrix Y^2 is given by

$$(28) \quad \phi^2(y^2|\theta) = \prod_{j=1}^N \int_{z_j^1 \in \prod_{i=1}^N \eta_{ij}^{-1}(y_{ij}^1)} \int_{z_j^2 \in \prod_{i=1}^N \eta_{ij}^{-1}(y_{ij}^2)} \phi_j(z_j^1|\theta) \phi_j(z_j^2|\theta) dz_j^1 dz_j^2.$$

One should note that even if the Z^T s come from independently drawn random samples, in general the Y^T s will not be independently drawn— $\phi(y^1, y^2|\theta) \neq \phi^1(y^1|\theta) \phi^2(y^2|\theta)$ — because the j^{th} column of Y^1 and Y^2 both depend on z_j^1 .

In its most general form the intertemporal communication makes multiperiod team theory analytically identical to single period team theory. However, interesting insights into important problems can be gained by studying various restrictions on the functional form of η^T . For example, Marschak and Radner study the effects of delayed information when θ is changing, by making information of the following form: $Y^T = \eta^T(\bar{Z}(T-d))$ where d is a delay. We will now study the restrictions that must be made to

allow the team to behave as an adapting organization; that is, one that modifies its beliefs about the random state of nature as its life progresses.

d. Ex Post Communication For a single period team, communication aimed at constructing a team posterior p.d.f. of θ , $f(\theta|Y')$, would be unnecessary since the only distributions needed in the optimality conditions (11) are computable from known information. But when we introduce multiple decisions over time, the fact that information for each teammate is different prevents the team from computing posterior densities. The team model can be modified to solve this problem by introducing a second set of communications between teammates. This communication should be such that each teammate gets the same information. Let us represent this communication by

$$(29) \quad x_i = X = \mu(Z) \quad i=1,2,\dots,N.$$

where x_i is the message received by teammate i . The messages are the same for all teammates and depend on the observed samples $Z=(z_1, z_2, \dots, z_N)$. Each message X has an induced probability, given the state of nature, defined by

$$(30) \quad h(X|\theta) = \int_{Z \in \mu^{-1}(X)} \phi(Z|\theta) dZ.$$

For every value of the message X , the posterior p.d.f. of θ is given by

$$(31) \quad f(\theta|X) = \frac{h(X|\theta) \phi(\theta)}{\int_{\Theta} h(X|\theta) \phi(\theta) d\theta}.$$

Because $x_1=x_2=\dots=x_N=X$ and the function $\mu(X)$ is the same for all teammates, the posterior distribution for θ given the message X can be constructed; it is the same for all teammates.

The communication X is not available to aid in the selection of an action variable but only to create a posterior p.d.f. of Θ . We shall refer to $\mu(Z)$ as an "ex post" communication structure, as though X occurs after the decisions are made. The entire team communication structure is represented by the two functions $\eta(Z)$ and $\mu(Z)$, the ex post and ex ante communication functions.

The signal X can be a vector or matrix, so μ can be a vector matrix valued function. The ex post communication structure defined by $\mu(Z) \equiv \eta(Z)$ implies that each teammate knows the entire ex ante message matrix Y . The ex post communication structure " $\mu(Z) = \text{constant}$ " provides no information since for all X , $\mu^{-1}(x) = \{\text{all possible } Z\text{'s}\}$. Notice that the null information structure implies that each member forgets his own personal sample value z_1 .

For given (η, μ) the two period problem optimal team procedure is as follows. For the first period the team selects a decision function $\alpha^1(y^1)$ to maximize

$$(32) \quad W^1[\alpha^1] = E\{U(\alpha^1(y^1), \Theta)\} = \int_{\Theta} \int_{y^1} U(\alpha^1(y^1), \Theta) \phi(y^1|\Theta) \phi(\Theta) dy^1 d\Theta.$$

Optimal conditions for α^1 are given by equation (11). After period one is complete, the team calculates a new density function for Θ by computing $f(\Theta|x)$ using equation (31), where X is the ex post message sent to all team members. The team selects a decision function $\alpha^2(y^2)$ for period two to maximize

$$(33) \quad W^2[\alpha^2] = E\{U(\alpha^2(y^2), \Theta) | X\} = \int_{\Theta} \int_{y^2} U(\alpha^2(y^2), \Theta) f(y^2, \Theta | X) dy^2 d\Theta.$$

The first order conditions for $\hat{\alpha}^2(y^2)$ are

$$(34) \quad 0 = \int_{\Theta} \int_{y^2(i)} U_{q_i}(\hat{\alpha}^2(y^2), \Theta) f(y^2(i), \Theta | y_i^2, X) dy^2(i) d\Theta \quad i=1, 2, \dots, N$$

where

$$(35) \quad f(Y^2(i), \theta | y_1^2, X) = \frac{f(Y^2, \theta | X)}{\int_{\Theta} \int_{Y^2(i)} f(Y^2, \theta | X) dY^2(i) d\theta}$$

is the posterior joint probability of $Y^2(i), \theta$ given the observed ex post message X and the observed value of y_1^2 .

Implicitly, the value of A^2 depends on X and hence on the observed value of Z^1 . For $\mu(Z^1) \equiv \eta(Z^1)$, the optimal second period conditions (33) are equivalent to those given by equation (24). For null ex post communication, $\mu = \text{constant}$, the conditions specified by (33) are not equivalent to those in equation (22) because null information implies loss of memory about the value of z_1^1 .

Ex post communication can be expressed in Marschak and Radner's intertemporal communication notation as follows:

$$(36) \quad \tilde{Y}^2 = (Y^2, X) = \begin{pmatrix} (y_{11}^2, X) & \dots & (y_{1N}^2, X) \\ \vdots & & \vdots \\ (y_{N1}^2, X) & \dots & (y_{NN}^2, X) \end{pmatrix} = (\eta(Z^2), \mu(Z^1)).$$

The decision function for period two is a function of Y^2 and is selected to maximize the expected utility

$$(37) \quad W[\alpha^2] = E\{U(\alpha^2(Y^2), \theta)\} = \int_{Y^2} \int_X \int_{\Theta} U(\alpha^2(Y^2, X), \theta) \phi(Y^2, X, \theta) d\theta dX dY^2$$

The first order conditions are given by

$$(38) \quad 0 = \int_{Y^2(i)} \int_{\Theta} U_{q_i}(\hat{\alpha}^2(Y^2, X), \theta) f(Y^2(i), \theta | y_1^2, X) d\theta dY^2(i), 1=1, 2, \dots, N.$$

which is identical to equation (34) hence both formulations

lead to identical decisions for given Y^2, X .

III. A TWO PERIOD TEAM EXAMPLE

a. Joint Production Under Price Uncertainty The organization which will be modeled here as a team is a business firm which produces on a day-to-day basis two commodities, q_1 and q_2 . The firm is divided into two production departments, each specializing in the production of the goods. Decision making is decentralized in the sense that department one chooses the daily output level of q_1 without being directed by a central authority.

The firm does not know the prices p_1 and p_2 it will receive for its products when they are sold. Through past experiences the two departments have identical subjective beliefs about the probability that the market will set prices at any particular levels. The departmental decisions about production levels must be made without knowledge of the exact price because the goods are not sold prior to production. The firm has a two day work-week followed by a market day when the preceding days' outputs are sold at the going market prices. On neither the first nor the second work day will the firm know exactly what prices will result on the market day.

This does not mean that the departments must decide on production levels based only on prior beliefs about prices. As each work day begins, the individual departments read trade newspapers, talk to prospective buyers and fellow businessmen, etc. to gather information about the "market conditions" of

their respective commodities. The information is summarized in a single statistic which will be called the "price forecast". Using this forecast as a guide to prices, the departments will make ^{decisions} on production levels.

The decision of each department cannot be made ignoring the possible decisions of the other department. The firm has some resources such as floor space, machines or tools that are used in the production of both commodities. As a result of these factors the joint cost of producing at levels q_1 and q_2 is such that

$\partial^2 \text{cost} / \partial q_1 \partial q_2 \neq 0$. That is, cost cannot be additively decomposed into two components $\text{cost}(q_1) + \text{cost}(q_2)$.

The firm is a team and each department desires to select an output level to maximize expected profits where profits are

$$\Pi = p_1 q_1 + p_2 q_2 - C(q_1, q_2) = P'Q - C(Q) \quad ;$$

$$P = (p_1 \ p_2)'; Q = (q_1 \ q_2)'; C(\cdot) = \text{cost}.$$

The firm operates on a day-to-day basis, each day attempting to maximize that day's profit $\Pi^t = P'Q^t - C(Q^t)$, $t=1,2$. It should be noted that the profit is not realized until the products $Q^1 + Q^2$ are sold at the prices P on market day. Cost functions are identical for each day and interest charges are neglected by assuming zero interest rates. The total profits are the sum of the two daily profits although each component occurs at different times in the week $\Pi^{1+2} = \Pi^1 + \Pi^2 = P'(Q^1 + Q^2) - C(Q^1) - C(Q^2)$.

We have yet to introduce communication to the firm's decision procedure. Each department observes a daily forecast price z_1^t which is correlated with the unknown price, P_1 . Commu-

nication is defined by a function η which maps $Z=(z_1, z_2)$ into a message matrix

$$Y, \quad Y = \eta(Z) = \begin{pmatrix} \eta_{11}(z_1) & \eta_{12}(z_2) \\ \eta_{21}(z_1) & \eta_{22}(z_2) \end{pmatrix}$$

Each department makes its decision on production level based on its messages to maximize that day's expected profits.

Specific assumptions about the functions and variables must be made so that the optimal decision functions can be calculated for particular communication structures. Let cost be a quadratic function of output levels.

$$(39) \quad C(Q) = c_{11} q_1^2 + 2c_{12} q_1 q_2 + c_{22} q_2^2 = Q' C Q; \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

The utility function is defined as the profits given this cost function.

$$(40) \quad U(Q, P) = P'Q - Q' C Q$$

Both department have the same prior joint p.d.f. for p_1 and p_2 and it is jointly normal with zero means and a variance-covariance matrix $\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$, so that

$$(41) \quad \phi(p_1, p_2) = (2\pi)^{-1/2} (1-r^2)^{-1/2} \exp\left[-\frac{1}{2} \frac{1}{1-r^2} (p_1^2 - 2rp_1 p_2 + p_2^2)\right].$$

It has been pointed that normality assumption about prices are unrealistic because there would always be a positive probability that prices are negative, an impossible event. In particular, with zero means the probability of negative price is exactly one half. We will ignore this serious objection because we want to demonstrate the optimality conditions for multiperiod

teams and normality with zero mean greatly simplifies calculations.

Each day the 1th department observes a forecast price z_1 which is distributed normally with variance=1 and mean= p_1 .

$$(42) \quad \phi_i(z_i | P) = \phi_i(\bar{z}_i | p_i) = (2\pi)^{-1/2} \exp(-\frac{1}{2} (\bar{z}_i - p_i)^2), \quad i=1,2.$$

We will compare two ex ante communication structures — no communication and complete communication — defined by the following communication functions.

$$\begin{aligned} \text{no communication:} \quad Y &= \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} = \eta(Z) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \\ \text{complete communication:} \quad Y &= \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \eta(Z) = \begin{pmatrix} z_1 & z_2 \\ z_1 & z_2 \end{pmatrix} \end{aligned}$$

The information y_1 for no communication is distributed normal with variance=1 and mean= p_1 . The information (y_{11}, y_{12}) is distributed jointly normal with variance-covariance $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and mean (p_1, p_2) :

$$(43) \quad \phi_i(y_{i1}, y_{i2} | P) = (2\pi)^{-1} \exp\left[-\frac{1}{2} (y_{i1} - p_i)^2 - \frac{1}{2} (y_{i2} - p_i)^2\right], \quad i=1,2.$$

b. Quadratic-Normal Teams Before continuing the two period example we should state some theorems developed by Radner⁽⁷⁾ concerning the optimal decision rules for a team maximizing a quadratic utility function when random variables are distributed normal.

Theorem 1: If a two member team's utility function is

$$U(A, \theta) = \lambda + 2\mu_1 \theta_1 a_1 + 2\mu_2 \theta_2 a_2 - v_{11} a_1^2 - 2v_{12} a_1 a_2 - v_{22} a_2^2$$

then the optimal decision functions $\hat{\alpha}_1(y_1)$ and $\hat{\alpha}_2(y_2)$

must for all (y_1, y_2) satisfy the following conditions:

$$\hat{\alpha}_1(y_1) = \frac{1}{v_{11}} (\mu_1 E\{\theta_1 | y_1\} - v_{12} E\{\hat{\alpha}_2(y_2) | y_1\})$$

$$\hat{\alpha}_2(y_2) = \frac{1}{v_{22}} (\mu_2 E\{\theta_2 | y_2\} - v_{12} E\{\hat{\alpha}_1(y_1) | y_2\}).$$

Theorem 2: If the utility function of a two member team is that of theorem 1 and the random variables $\theta_1, \theta_2, y_1, y_2$ are normally distributed with $E\{\theta_1\} = E\{\theta_2\} = 0, \text{var-cov}(\theta_1, \theta_2) = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}$, $E\{y_1 | \theta_1\} = \theta_1, E\{y_2 | \theta_2\} = \theta_2, \text{var}(y_1 | \theta_1) = \text{var}(y_2 | \theta_2) = 1$, then the optimal decision functions are linear in the information:

$$\hat{\alpha}_1(y_1) = \hat{\delta}_1 y_1, \quad \hat{\delta}_1 \text{ constant}$$

$$\hat{\alpha}_2(y_2) = \hat{\delta}_2 y_2, \quad \hat{\delta}_2 \text{ constant,}$$

where $\hat{\delta}_1$ and $\hat{\delta}_2$ are the solutions of the linear equation

$$\hat{\delta}_1 = \frac{1}{v_{11}} (\mu_1 - v_{12} \hat{\delta}_2)$$

$$\hat{\delta}_2 = \frac{1}{v_{22}} (\mu_2 - v_{12} \hat{\delta}_1).$$

Theorem 3: If the utility function of a two member team is that of theorem 1 then the maximum expected utility is $W[\hat{\alpha}] = \lambda + \mu_1 E\{\theta_1 \hat{\alpha}_1(y_1)\} + \mu_2 E\{\theta_2 \hat{\alpha}_2(y_2)\}$.

Corollary 1: If the utility function of two member team is that of theorem 1 and the random variables are distributed as in theorem 2, the maximum expected utility is $W[\hat{\alpha}] = \lambda + \mu_1 \hat{\delta}_1 + \mu_2 \hat{\delta}_2$.

In the example of the profit maximizing two product firm, we have a quadratic payoff function with $v_{1j} = \sigma_{1j}, \mu_i = \frac{1}{2}$ and $\lambda = 0$; also the random variables $\theta_1, \theta_2, y_1, y_2$ are distributed jointly normal; hence the optimal decision functions are linear in the information.

c. Optimal Decision On Day 1 On the first day the firm wants to select a decision function $Q^1 = \sigma^1(y^1) = (\sigma_1^1(y_1^1), \sigma_2^1(y_2^1))'$ to maximize expected first day profits $\Pi^1 = P^1 Q^1 - Q^1 C^1 Q^1$. When there is no communication the message y_1^1 is the individual sample observation z_1^1 ; therefore y_1^1 has a p.d.f. like (43). The optimal decision function will be linear:

$$(44a) \quad \hat{\sigma}_1^1(y_1^1) = \hat{\beta}_1^1 y_1^1$$

$$(44b) \quad \hat{\sigma}_2^1(y_2^1) = \hat{\beta}_2^1 y_2^1.$$

The coefficients $\hat{\beta}_1^1$ and $\hat{\beta}_2^1$ are defined by

$$(45a) \quad \hat{\beta}_1^1 = \frac{1}{c_{11}} (1/4 - 1/2 c_{12} \hat{\beta}_2^1)$$

$$(45b) \quad \hat{\beta}_2^1 = \frac{1}{c_{22}} (1/4 - 1/2 c_{12} \hat{\beta}_1^1)$$

or solving for the actual values:

$$(46a) \quad \hat{\beta}_1^1 = \frac{1}{2} (2 c_{22} - r c_{12}) / (4 c_{11} c_{22} - r^2 c_{12}^2)$$

$$(46b) \quad \hat{\beta}_2^1 = \frac{1}{2} (2 c_{11} - r c_{12}) / (4 c_{11} c_{22} - r^2 c_{12}^2).$$

When ex ante communication is "complete" both departments get the message vector (y_1^1, y_2^1) which has a p.d.f.

$$(47) \quad \phi(y_1^1, y_2^1 | P) = (2\pi)^{-1} \exp[-\frac{1}{2} (y_1^1 - P_1)^2 - \frac{1}{2} (y_2^1 - P_2)^2].$$

The optimal decision functions will be linear in the information and of the following form:

$$(48a) \quad \hat{\sigma}_1^1(y_1^1, y_2^1) = \hat{\beta}_{11}^1 y_1^1 + \hat{\beta}_{12}^1 y_2^1$$

$$(48b) \quad \hat{\sigma}_2^1(y_1^1, y_2^1) = \hat{\beta}_{21}^1 y_1^1 + \hat{\beta}_{22}^1 y_2^1.$$

The coefficients $\hat{\beta}_{ij}^1$ are defined by

$$(49a) \quad \hat{\beta}_{11}^1 = \frac{1}{c_{11}} (\frac{1}{2} (2 - r^2) / (4 - r^2) - c_{12} \hat{\beta}_{21}^1)$$

$$(49b) \quad \hat{\beta}_{12}^1 = \frac{1}{c_{11}} (\frac{1}{2} r / (4 - r^2) - c_{12} \hat{\beta}_{22}^1)$$

$$(49c) \quad \hat{\beta}_{21}^1 = \frac{1}{c_{22}} (\frac{1}{2} r / (4 - r^2) - c_{12} \hat{\beta}_{11}^1)$$

$$(49d) \quad \hat{\beta}_{22}^1 = \frac{1}{c_{22}} (\frac{1}{2} (2 - r^2) / (4 - r^2) - c_{12} \hat{\beta}_{12}^1).$$

or solving for the actual values

$$(50a) \quad \hat{S}_{11}^1 = \frac{1}{2} \frac{1}{4-r^2} \frac{C_{22}(2-r^2) - C_{12}r}{C_{11}C_{22} - C_{12}^2}$$

$$(50b) \quad \hat{S}_{12}^1 = \frac{1}{2} \frac{1}{4-r^2} \frac{C_{22}r - C_{12}(2-r^2)}{C_{11}C_{22} - C_{12}^2}$$

$$(50c) \quad \hat{S}_{21}^1 = \frac{1}{2} \frac{1}{4-r^2} \frac{C_{11}r - C_{12}(2-r^2)}{C_{11}C_{22} - C_{12}^2}$$

$$(50d) \quad \hat{S}_{22}^1 = \frac{1}{2} \frac{1}{4-r^2} \frac{C_{11}(2-r^2) - C_{12}r}{C_{11}C_{22} - C_{12}^2}$$

The maximum expected profits for no communication is given by

$$(51) \quad \overline{\pi}_{nc}^1[\hat{G}^1] = \frac{1}{2} \hat{S}_1^1 + \frac{1}{2} \hat{S}_2^1 = \frac{1}{2} \frac{C_{11} + C_{22} - r C_{12}}{4C_{11}C_{22} - r^2 C_{12}^2}.$$

The maximum expected profits for complete communication is given by

$$(52) \quad \overline{\pi}_{cc}^1[\hat{G}^1] = \frac{1}{2} (\hat{S}_{11}^1 + \hat{S}_{22}^1) + \frac{r}{2} (\hat{S}_{12}^1 + \hat{S}_{21}^1) \\ = \frac{1}{2} \frac{C_{11} + C_{22} - r C_{12} - r(2-r^2) C_{12}}{4C_{11}C_{22} - r^2 C_{12}^2 - (r^2 C_{11}C_{22} + 2(2-r^2) C_{12}^2)}.$$

If the correlation between p_1 and p_2 is zero ($r=0$) then knowledge of one sample statistic provides no information about the other commodity's price. In this case $\overline{\pi}_{cc}^1 > \overline{\pi}_{nc}^1$. If the correlation between p_1 and p_2 is perfect ($r=1$) knowledge of one sample statistic does provide information about the opposite price and we cannot say that $\overline{\pi}_{cc}^1 > \overline{\pi}_{nc}^1$ for all possible cost functions.

d. Optimal Decision For Day 2 Without Ex Post Communication At the beginning of the first work day each department gathers infor-

mation about the conditions of its respective market. The departments pass on this information to the other part of the firm according to the ex ante communication function. The daily decision on production levels is made and production is carried out. Suppose that at the end of the first business day the department heads simply go home without discussing "market conditions" any further. The next day each department gets a new report on the probable price that it can sell its product. This is used as the basis for the ex ante communication in day two and nothing is said about yesterday's information. On the second day the departments make their production decisions based on the message received on work day one (which they remember) and the message received on day two. The decision functions are of the form $Q^2 = \sigma^2(Y^1, Y^2) = (\sigma_1^2(y_1^1, y_1^2), \sigma_2^2(y_2^1, y_2^2))'$. The maximand is the expected profits for period two $\Pi^2 = P^1 Q^2 - Q^2 C Q^2$. The optimal decision functions must satisfy

$$(53a) \quad \hat{\sigma}_1^2(y_1^1, y_1^2) = \frac{1}{c_{11}} \left(\frac{1}{2} E \{ P_1 | y_1^1, y_1^2 \} - c_{12} E \{ \hat{\sigma}_2^2(y_2^1, y_2^2) | y_1^1, y_1^2 \} \right)$$

$$(53b) \quad \hat{\sigma}_2^2(y_2^1, y_2^2) = \frac{1}{c_{22}} \left(\frac{1}{2} E \{ P_2 | y_2^1, y_2^2 \} - c_{12} E \{ \hat{\sigma}_1^2(y_1^1, y_1^2) | y_2^1, y_2^2 \} \right).$$

Y^1 and Y^2 come from independent random samples with the same distributions:

$$(54) \quad \phi(Y^1, Y^2 | P) = \phi_1(y_1^1 | P_1) \phi_2(y_2^1 | P_2) \phi_1(y_1^2 | P_1) \phi_2(y_2^2 | P_2).$$

For the ex ante communication structure "no communication", y_1^t is distributed normal with variance=1 and mean= p_1 , $t=1,2$

$$(55) \quad \phi_i(y_i^t | P_i) = (\sqrt{2\pi})^{-1/2} \exp(-\frac{1}{2} (y_i^t - P_i)^2).$$

The optimal decision function for no communication will be linear and because of the identical distribution of y_1^1 and y_1^2 they will

be of the following form:

$$(56a) \quad \hat{\sigma}_1^2(y_1^1, y_1^2) = \hat{S}_1^2(y_1^1 + y_1^2)$$

$$(56b) \quad \hat{\sigma}_2^2(y_2^1, y_2^2) = \hat{S}_2^2(y_2^1 + y_2^2),$$

where the coefficients are

$$(57a) \quad \hat{S}_1^2 = \frac{1}{6} \frac{C_{22} - \frac{2}{3} r C_{12}}{C_{11}C_{22} - (\frac{2}{3} r C_{12})^2}$$

$$(57b) \quad \hat{S}_2^2 = \frac{1}{6} \frac{C_{11} - \frac{2}{3} r C_{12}}{C_{11}C_{22} - (\frac{2}{3} r C_{12})^2}.$$

The maximum expected second period profits are computed as

$$(58) \quad \overline{\Pi}_{nc}^2[\hat{\sigma}^2] = \hat{S}_1^2 + \hat{S}_2^2 = \frac{1}{6} \frac{C_{11} + C_{22} - \frac{4}{3} r C_{12}}{C_{11}C_{22} - (\frac{2}{3} r C_{12})^2}.$$

For ex ante "complete communication," $y_1^t = (y_1^t, y_2^t)$, the optimal solution is of the form

$$(59a) \quad \hat{\sigma}_1^2(Y^1, Y^2) = \hat{S}_{11}^2(y_1^1 + y_1^2) + \hat{S}_{12}^2(y_2^1 + y_2^2)$$

$$(59b) \quad \hat{\sigma}_2^2(Y^1, Y^2) = \hat{S}_{21}^2(y_1^1 + y_1^2) + \hat{S}_{22}^2(y_2^1 + y_2^2),$$

where the coefficients \hat{S}_{ij}^2 are the solutions of the simultaneous equations:

$$(60a) \quad \hat{S}_{11}^2 = \frac{1}{C_{11}} \left(\frac{1}{2} \frac{3-2r^2}{9-4r^2} - C_{12} \hat{S}_{21}^2 \right)$$

$$(60b) \quad \hat{S}_{12}^2 = \frac{1}{C_{11}} \left(\frac{1}{2} \frac{r}{9-4r^2} - C_{12} \hat{S}_{22}^2 \right)$$

$$(60c) \quad \hat{S}_{21}^2 = \frac{1}{C_{22}} \left(\frac{1}{2} \frac{r}{9-4r^2} - C_{12} \hat{S}_{11}^2 \right)$$

$$(60d) \quad \hat{S}_{22}^2 = \frac{1}{C_{22}} \left(\frac{1}{2} \frac{3-2r^2}{9-4r^2} - C_{12} \hat{S}_{12}^2 \right).$$

Solving these equations we get the values

$$(61a) \quad \hat{S}_{11}^2 = \frac{1}{2} \frac{1}{9-4r^2} \frac{C_{22}(3-2r^2) - C_{12}r}{C_{11}C_{22} - C_{12}^2}$$

$$(61b) \quad \hat{S}_{12}^2 = \frac{1}{2} \frac{1}{9-4r^2} \frac{C_{22}r - C_{12}(3-2r^2)}{C_{11}C_{22} - C_{12}^2}$$

$$(61c) \quad \hat{s}_{21}^2 = \frac{1}{2} \frac{1}{q-4v^2} \frac{c_{11}v - c_{12}(3-2v^2)}{c_{11}c_{22} - c_{12}^2}$$

$$(61d) \quad \hat{s}_{22}^2 = \frac{1}{2} \frac{1}{q-4v^2} \frac{c_{11}(3-2v^2) - c_{12}v}{c_{11}c_{22} - c_{12}^2}$$

The maximum expected profit for period two with complete communication is

$$(62) \quad \overline{\Pi}_{cc}^2 = \hat{s}_{11}^2 + \hat{s}_{22}^2 + v(\hat{s}_{12}^2 + \hat{s}_{21}^2) \\ = \frac{1}{2} \frac{1}{q-4v^2} \frac{(c_{11} + c_{22})(3-v^2) - v(5-2v^2)c_{12}}{c_{11}c_{22} - c_{12}^2}$$

e. Ex Post Communication Structures The above solution for optimal output decision functions assumed that at the end of the first day the departments did no more communication to come to a consensus about the probabilities of future prices. Suppose we now allow ex post communication. Two particular ex post communication structures will be studied here: "complete" ex post communication by only the team's "captain." If at the end of the first day the decision-makers of each department play a round of golf together and in the process give their colleagues the complete picture of market conditions that they observed that morning, then the firm's ex post communication is "complete". Suppose that at the end of the day the decision-maker for department one puts a message on the bulletin board near the exit summarizing the market conditions that department 1 observed that morning. All decision-makers in the other department read this

message and this is the only information that they remember the next day. The ex ante communication will be "no communication" in this example. The two ex post communication structures are represented formally by

$$\mu(z) = (z_1, z_2) \quad \text{"complete" ex post communication}$$

$$\mu(z) = (z_1) \quad \text{"captain's message" ex post communication}$$

To find the optimal decision function for day two when the ex post communication was "complete", the following equations must be solved.

$$(63a) \quad \hat{\sigma}_1^2(y_1^2) = \frac{1}{c_{11}} \left(\frac{1}{2} E\{P_1 | y_1^2, Y^1\} - c_{12} E\{\hat{\sigma}_2^2(y_2^2) | y_1^2, Y^1\} \right)$$

$$(63b) \quad \hat{\sigma}_2^2(y_2^2) = \frac{1}{c_{22}} \left(\frac{1}{2} E\{P_2 | y_2^2, Y^1\} - c_{12} E\{\hat{\sigma}_1^2(y_1^2) | y_2^2, Y^1\} \right).$$

After the beliefs are updated by the ex post information, the variables continue to be distributed normally but the means of (p_1, p_2) are no longer zero. Radner shows that in this case the optimal decision functions are still linear in the information but with the addition of a constant term,

$$(64a) \quad \hat{\sigma}_1^2(y_1^2) = \hat{s}_1^2 y_1^2 + \hat{v}_1^2$$

$$(64b) \quad \hat{\sigma}_2^2(y_2^2) = \hat{s}_2^2 y_2^2 + \hat{v}_2^2.$$

The optimal coefficients \hat{s}_1^2, \hat{s}_2^2 must satisfy

$$(65a) \quad c_{11} \hat{s}_1^2 + c_{12} \frac{r}{c-2r^2} \hat{s}_2^2 = \frac{1}{2} \frac{2-r^2}{c-2r^2}$$

$$(65b) \quad c_{12} \frac{r}{c-2r^2} \hat{s}_1^2 + c_{22} \hat{s}_2^2 = \frac{1}{2} \frac{2-r^2}{c-2r^2}$$

Solving these equations

$$(66a) \quad \hat{s}_1^2 = \frac{1}{2} (2-r^2) \frac{(c-2r^2) c_{22} - r c_{12}}{c_{11} c_{22} - r^2 c_{12}^2}$$

$$(66b) \quad \hat{s}_2^2 = \frac{1}{2} (2-r^2) \frac{(c-2r^2) c_{11} - r c_{12}}{c_{11} c_{22} - r^2 c_{12}^2}.$$

Given the above values of \hat{S}_1^2 and \hat{S}_2^2 , the constant terms \hat{V}_1^2 and \hat{V}_2^2 must satisfy

$$(67a) \quad c_{11} \hat{V}_1^2 + c_{12} \hat{V}_2^2 = \frac{1}{2} \frac{1}{6-2r^2} ((2-r^2)y_1^1 + ry_2^1) - \frac{1}{6-2r^2} c_{12} \hat{S}_2^2 (ry_1^1 + (3-2r^2)y_2^1)$$

$$(67b) \quad c_{12} \hat{V}_1^2 + c_{22} \hat{V}_2^2 = \frac{1}{2} \frac{1}{6-2r^2} (ry_1^1 + (2-r^2)y_2^1) - \frac{1}{6-2r^2} c_{12} \hat{S}_1^2 ((3-2r^2)y_1^1 + ry_2^1).$$

It is clear that the constant terms depend on the value of Y^1 .

This is the implicit relationship between Y^1 and Q^2 that was mentioned above.

When ex post information is the "team captain's" message,

the optimal solution is again of the form

$$(68a) \quad \hat{\sigma}_1^2(y_1^2) = \hat{S}_1^2 y_1^2 + \hat{V}_1^2$$

$$(68b) \quad \hat{\sigma}_2^2(y_2^2) = \hat{S}_2^2 y_2^2 + \hat{V}_2^2.$$

The coefficients \hat{S}_1^2 and \hat{S}_2^2 must satisfy the first order conditions

$$(69a) \quad c_{11} \hat{S}_1^2 + \frac{r}{3} c_{12} \hat{S}_2^2 = \frac{1}{6}$$

$$(69b) \quad \frac{r}{4-r^2} c_{12} \hat{S}_1^2 + c_{22} \hat{S}_2^2 = \frac{1}{2} \frac{(3-2r^2)}{9-4r^2}.$$

Solving these equations we get

$$(70a) \quad \hat{S}_1^2 = \frac{1}{2} \frac{c_{22} - \frac{r(3-2r^2)}{9-4r^2} c_{12}}{3c_{11}c_{22} - \frac{r^2}{4-r^2} c_{12}^2}$$

$$(70b) \quad \hat{S}_2^2 = \frac{1}{2} \frac{c_{11} \frac{3(3-2r^2)}{9-4r^2} - c_{12} \frac{r}{4-r^2}}{3c_{11}c_{22} - \frac{r^2}{4-r^2} c_{12}^2}.$$

Given these values, the constant terms \hat{V}_1^2 and \hat{V}_2^2 must satisfy

$$(71a) \quad c_{11} \hat{V}_1^2 + c_{12} \hat{V}_2^2 = \left(\frac{1}{6} - \frac{r}{3} c_{12} \hat{S}_2^2 \right) y_1^1$$

$$(71b) \quad c_{12} \hat{V}_1^2 + c_{22} \hat{V}_2^2 = \left(\frac{r}{2} (9-4r^2) - c_{12} \hat{S}_1^2 \frac{2-r^2}{4-r^2} \right) y_1^1$$

Again the constant terms depend on the ex post message y_1^1 .

IV. MULTIPERIOD TEAM DECISION THEORY
WITH A DYNAMIC ENVIRONMENT

a. Introduction In the previous section the environment was static in the sense that the unknown state of nature did not change in time. Only information and actions changed as time passed. The next step is to study the multiperiod team decision problem as the unknown random state of nature takes on different values in each period.

We will begin with a two period problem although results are easily extended to T periods. In each period a new state of nature is believed to occur, first θ^1 then θ^2 . The team has a set of "beliefs" about the unknown states of nature summarized in the joint prior p.d.f. $\phi(\theta^1, \theta^2)$. Notice that if θ^1 and θ^2 are believed to be statistically independent, $\phi(\theta^1, \theta^2) = \phi_1(\theta^1) \phi_2(\theta^2)$, then the two period team problem is dichotimized into single period team problems.

Again assume total utility is additively separable in time with identical single period utility functions,

$$(72) \quad u(A^1, A^2; \theta^1, \theta^2) = u(A^1, \theta^1) + u(A^2, \theta^2).$$

In each period information is available to teammates through an ex ante observation-communication system as specified above.

Decision functions for both periods are again of the restricted form:

$$(73) \quad \alpha(Y) = (\alpha_1(y_1), \alpha_2(y_2), \dots, \alpha_n(y_n))'$$

The team desires to select decision functions α^1 and α^2 to maximize

total expected utility:

$$(74) \quad W^{1+2}[\alpha^1, \alpha^2] = E\{u(\alpha^1, \theta^1) + u(\alpha^2, \theta^2)\}.$$

Before first order optimality conditions are derived, specific assumptions must be made about the information available to each teammate in both periods.

b. Individual Memory Assume that the only intertemporal communication allowed is individual memory of past messages; that is, in period two the i^{th} teammate knows only the values of (y_i^1, y_i^2) .

The second period team decision function is then defined by

$$(75) \quad \alpha^2(Y^1, Y^2) = (\alpha_i^2(y_i^1, y_i^2), \dots, \alpha_N^2(y_N^1, y_N^2))'$$

The two information variables Y^1 and Y^2 are postulated to have come from independent random samples so that the prior conditional joint density is

$$(76) \quad \phi(Y^1, Y^2 | \theta^1, \theta^2) = \phi^1(Y^1 | \theta^1) \phi^2(Y^2 | \theta^2).$$

Because θ^1 and θ^2 are not independent, knowledge of Y^1 will provide information about the unknown θ^2 . (If θ^1 and θ^2 are independent then the posterior p.d.f. of θ^2 given Y^1 and Y^2 is only a function of Y^2 ; $f(\theta^2 | Y^1, Y^2) = f(\theta^2 | Y^2)$.)

With these hypotheses about intertemporal communication in mind, the two period team problem is to select $\alpha^1(Y^1)$ and $\alpha^2(Y^1, Y^2)$ to maximize total expected utility:

$$(77) \quad W^{1+2} = \int_{\theta^1} \int_{\theta^2} \int_{y^1} \int_{y^2} (u(\alpha^1(Y^1), \theta^1) + u(\alpha^2(Y^1, Y^2), \theta^2)) \phi^1(Y^1 | \theta^1) \phi^2(Y^2 | \theta^2) \phi(\theta^1, \theta^2) dY^1 dY^2 d\theta^1 d\theta^2.$$

Let $\hat{\alpha}^1$ and $\hat{\alpha}^2$ denote the optimal team decision functions. Because of the additivity of utility, first order conditions for $\hat{\alpha}^1$ are

disjoint from those of $\hat{\alpha}^2$. The first period team decision function $\hat{\alpha}^1(\gamma^1)$ must satisfy simultaneously for all y_1^1 :

$$(78) \quad 0 = \int_{\Theta^1} \int_{Y^1(i)} u_{q_i}(\hat{\alpha}^1(\gamma^1), \theta^1) f(\gamma^1(i), \theta^1 | y_i^1) d\gamma^1(i) d\theta^1 \quad i=1, 2, \dots, N.$$

The density function

$$(79) \quad f(\gamma^1(i), \theta^1 | y_i^1) = \frac{\phi^1(\gamma^1 | \theta^1) \phi^1(\theta^1)}{\phi_i^1(y_i^1)} \\ = \frac{\phi^1(\gamma^1 | \theta^1) \int_{\Theta^2} \phi(\theta^1, \theta^2) d\theta^2}{\int_{y_i^1(i)} \int_{\Theta^1} \int_{\Theta^2} \phi^1(\gamma^1 | \theta^1) \phi(\theta^1, \theta^2) d\theta^1 d\theta^2 d\gamma^1(i)},$$

is the joint posterior density of $\gamma^1(i)$ and θ^1 given the observed value of y_i^1 .

The second period's decision function $\hat{\alpha}^2(\gamma^1, \gamma^2)$ must for all (y_1^1, y_1^2) satisfy simultaneously:

$$(80) \quad 0 = \int_{Y^1(i)} \int_{Y^2(i)} \int_{\Theta^2} u_{q_i}(\hat{\alpha}^2(\gamma^1, \gamma^2), \theta^2) f(\gamma^1(i), \gamma^2(i), \theta^2 | y_i^1, y_i^2) d\theta^2 d\gamma^2(i) d\gamma^1(i)$$

for $i=1, 2, \dots, N$ where the density function is the joint posterior density of $\gamma^1(i), \gamma^2(i)$ and θ^2 given the values of the i^{th} teammate's observation in both periods — y_i^1 and y_i^2 .

$$(81) \quad f(\gamma^1(i), \gamma^2(i), \theta^2 | y_i^1, y_i^2) = \frac{\phi^2(\gamma^2 | \theta^2) \phi(\gamma^1, \theta^2)}{\phi_i^2(y_i^1, y_i^2)} \\ = \frac{\phi^2(\gamma^2 | \theta^2) \int_{\Theta^1} \phi^1(\gamma^1 | \theta^1) \phi(\theta^1, \theta^2) d\theta^1}{\int_{y_i^1(i)} \int_{y_i^2(i)} \int_{\Theta^1} \int_{\Theta^2} \phi^1(\gamma^1 | \theta^1) \phi^2(\gamma^2 | \theta^2) \phi(\theta^1, \theta^2) d\theta^1 d\theta^2 d\gamma^1(i) d\gamma^2(i)}.$$

Suppose there are T periods and ^{the} only intertemporal communication is individual memory of the previous observation. The team decision function for the t^{th} period is of the restricted form:

$$(82) \quad \alpha^t(\cdot; \cdot, \gamma^{t-1}) = (\alpha_1^t(y_1^t, y_1^{t-1}), \dots, \alpha_N^t(y_N^t, y_N^{t-1}))'.$$

If $\phi(\theta^1, \theta^2, \dots, \theta^T)$ is the joint prior density of all T states of nature, the joint prior density of (θ^t, θ^{t-1}) is given by

$$(83) \quad \phi^t(\theta^t, \theta^{t-1}) = \int_{\theta^1} \int_{\theta^2} \dots \int_{\theta^{t-2}} \int_{\theta^{t-1}} \dots \int_{\theta^t} \phi(\theta^1, \theta^2, \dots, \theta^t) d\theta^1 \dots d\theta^{t-2} d\theta^{t-1} \dots d\theta^t.$$

The joint prior density of Y^t, Y^{t-1} and θ^t is given by

$$(84) \quad \phi^t(Y^t, Y^{t-1}, \theta^t) = \phi^t(Y^t | \theta^t) \int_{\theta^{t-1}} \phi^{t-1}(Y^{t-1} | \theta^{t-1}) \phi^t(\theta^t, \theta^{t-1}) d\theta^{t-1}.$$

With additive utility, the desires to select α^t to maximize expected utility:

$$(85) \quad W^t[\alpha^t] = \int_{Y^{t-1}} \int_{Y^t} \int_{\theta^t} u(\alpha^t(Y^t, Y^{t-1}), \theta^t) \phi^t(Y^t | \theta^t) \phi(Y^{t-1}, \theta^t) d\theta^t dY^t dY^{t-1}.$$

The first order conditions for $\hat{\alpha}^t(Y^t, Y^{t-1})$ are exactly analogous to those of the two period problem.

$$(86) \quad 0 = \int_{\theta^t} \int_{Y^t(i)} \int_{Y^{t-1}(i)} u_{\alpha_i}(\hat{\alpha}^t(Y^t, Y^{t-1}), \theta^t) f(Y^t(i), Y^{t-1}(i), \theta^t | Y_i^t, Y_i^{t-1}) dY^{t-1}(i) dY^t(i) d\theta^t,$$

where

$$(87) \quad f(Y^t(i), Y^{t-1}(i), \theta^t | Y_i^t, Y_i^{t-1}) = \frac{\phi^t(Y^t | \theta^t) \phi(Y^{t-1}, \theta^t)}{\phi_i^t(Y_i^t, Y_i^{t-1})}$$

The reader can easily extend this T period problem to allow the individual memory of the past τ message for $1 \leq \tau \leq t-1$.

c. Ex Post Communication As long as θ^1 and θ^2 are not independent, information concerned with θ^1 is indirectly information about θ^2 . The team would like to update its beliefs about the second period's random variables based on its information Y^1 . As noted above this is impossible because no team knows the entire value of Y^1 . Ex post communication was introduced to allow the team to adapt its beliefs according to its information. The ex post message $X^1 = \mu(Z^1)$ received by all teammates is statistically distributed by the p.d.f.

$$(88) \quad h(x^1 | \theta^1) = \int_{z^1} e^{\mu^{-1}(x^1)} \phi(z^1 | \theta^1) dz^1$$

Given the value of x^1 all teammates update their beliefs by computing the conditional joint density of Y^2 and θ^2 given x^1 :

$$(89) \quad f(y^2, \theta^2 | x^1) = \frac{\phi^2(y^2 | \theta^2) \phi(x^1, \theta^2)}{\phi(x^1)}$$

$$= \frac{\phi^2(y^2 | \theta^2) \int_{\theta^1} h(x^1 | \theta^1) \phi(\theta^1, \theta^2) d\theta^1}{\int_{\theta^1} \int_{\theta^2} h(x^1 | \theta^1) \phi(\theta^1, \theta^2) d\theta^1 d\theta^2}$$

The team selects a second period decision function $\alpha^2(y^2)$ to maximize expected utility against the posterior p.d.f. $f(y^2, \theta^2 | x^1)$:

$$(90) \quad W^2[\alpha^2] = \int_{y^2} \int_{\theta^2} u(\alpha^2(y^2), \theta^2) f(y^2, \theta^2 | x^1) dy^2 d\theta^2$$

The first order conditions that the optimal decision function $\hat{\alpha}^2(y^2)$ must satisfy are:

$$(91) \quad 0 = \int_{y^2(i)} \int_{\theta^2} u_{\alpha}(\hat{\alpha}^2(y^2), \theta^2) f(y^2(i), \theta^2 | x^1, y_1^2) dy^2(i) d\theta^2$$

The density function defined by

$$(92) \quad f(y^2(i), \theta^2 | x^1, y_1^2) = \frac{f(y^2, \theta^2 | x^1)}{f_i(y_1^2 | x^1)}$$

$$= \frac{\phi(y^2, \theta^2, x^1)}{\phi_i(y_1^2, x^1)}$$

is the joint posterior density of $Y^2(i)$ and θ^2 given the ex post message x^1 and the ex ante individual message y_1^2 .

APPENDIX: UTILITY AND THE RATE OF CHANGES OF ACTIONS

Previously total utility was additively separable in time, i.e.,

$$(93) \quad U(A^1, A^2, \dots, A^T, \theta^1, \theta^2, \dots, \theta^T) = \sum_{t=1}^T U^t(A^t, \theta^t).$$

In addition single period utility functions, U^t were the same for all $t=1, 2, \dots, T$. This latter assumption can be dropped without changing the nature of the optimality conditions. For example, we could introduce a rate of time preference ρ and define the t^{th} period's utility by

$$(94) \quad U^t(A^t, \theta^t) = (1+\rho)^{-t} U(A^t, \theta^t)$$

In this case the optimality conditions would be identical to those derived above. Although the additive form of utility could be dropped, a special type of additive utility has been found useful. Many intertemporal economic problems (most notably "optimal economic growth") assume that total utility is additive in time but also assume that the single period's utility is a function not only of the policy instruments but of the rate of change of the instruments. In the neoclassical optimal growth literature today's utility is a function not only of the capital stock but also the rate of change of the capital stock.

This new definition of total utility is

$$(95) \quad U(A^1, \dots, A^T, \theta^1, \dots, \theta^T) = \sum_{t=1}^T (1+\rho)^{-t} U^*(A^t, A^t - A^{t-1}, \theta^t) \\ = \sum_{t=1}^T (1+\rho)^{-t} U(A^t, A^{t-1}, \theta^t)$$

The first difference $A^t - A^{t-1}$ is the discrete analogue to the rate

of change of actions. A rate of time preference has been added and will play a role in the optimality conditions.

Let us begin with a static environment: $\Theta = \Theta^1 = \Theta^2 = \dots = \Theta^T$. Let us also begin with an assumption not yet explored: all teammates forget the values of all their past message. The decision for period t is based only on the single information Y^t and no intertemporal communications are made. The decision functions $\alpha^t(Y^t) = (\alpha_1^t(y_1^t), \dots, \alpha_N^t(y_N^t))'$, $t=1, 2, \dots, T$, are selected to maximize total expected utility:

$$(96) \quad W(\alpha^1, \dots, \alpha^T) = E \left\{ \sum_{t=1}^T (1+\rho)^{-t} U(\alpha^t(Y^t), \alpha^{t-1}(Y^{t-1}), \Theta) \right\} \\ = \sum_{t=1}^T (1+\rho)^{-t} E \left\{ U(\alpha^t(Y^t), \alpha^{t-1}(Y^{t-1}), \Theta) \right\}$$

with $A^0 = \alpha^0(Y^0)$ a given constant action

Denote the optimal decision functions by $(\hat{\alpha}^1, \dots, \hat{\alpha}^T)$. All arbitrary decision functions can be written as $(\hat{\alpha}^1 + \Delta^1 \gamma^1, \dots, \hat{\alpha}^T + \Delta^T \gamma^T)$ where Δ^t is a constant diagonal matrix and γ^t is an arbitrary team function of the same form as $\hat{\alpha}^t$. $W(\alpha^1, \dots, \alpha^T)$ treated as a function of Δ^t must be maximized at $\Delta^t = 0$, $t=1, 2, \dots, T$, by the definition of the optimality of $\hat{\alpha}^t$. The first order conditions are: for all arbitrary γ^t

$$(97) \quad \left. \frac{\partial W}{\partial \delta_i^t} \right|_0 = 0 \quad i=1, 2, \dots, N \quad t=1, 2, \dots, T$$

$$(98) \quad \left. \frac{\partial W}{\partial \delta_i^t} \right|_0 = (1+\rho)^{-t} E \left\{ U_i(\hat{\alpha}^t(Y^t), \hat{\alpha}^{t-1}(Y^{t-1}), \Theta) \gamma_i^t(y_i^t) \right\} + \\ (1+\rho)^{-t-1} E \left\{ U_{N+i}(\hat{\alpha}^{t+1}(Y^{t+1}), \hat{\alpha}^t(Y^t), \Theta) \gamma_i^t(y_i^t) \right\}.$$

where U_i denotes the partial derivative with respect to the i^{th} component of the current action and U_{N+i} denotes the partial derivative with respect to the i^{th} component of the previous

action:

$$(99) \quad U_i(A^t, A^{t-1}, \theta) = \frac{\partial}{\partial a_i^t} U(A^t, A^{t-1}, \theta)$$

$$(100) \quad U_{N+i}(A^t, A^{t-1}, \theta) = \frac{\partial}{\partial a_i^{t-1}} U(A^t, A^{t-1}, \theta).$$

By reordering integrations and applying the lemma " $\int f(x)g(y)dx = 0$ for all $g(x)$ implies $f(x)=0$ " and finally dividing by the marginal prior density of y_1^t the first order conditions for optimal team decision functions for all y_1^t the optimal decision functions must simultaneously satisfy the following equations:

$$(101) \quad 0 = \int_{y^t(i)} \int_{y^{t-1}} \int_{\theta} U_i(\hat{a}^t(y^t), \hat{a}^{t-1}(y^{t-1}), \theta) f(y^t(i), y^{t-1}, \theta) y_1^t d y^t(i) d y^{t-1} d \theta \\ + (1+\rho)^{-1} \int_{y^{t+1}} \int_{y^t(i)} \int_{\theta} U_{N+i}(\hat{a}^{t+1}(y^{t+1}), \hat{a}^t(y^t), \theta) f(y^{t+1}, y^t(i), \theta | y_1^t) \\ d y^{t+1} d y^t(i) d \theta \\ = E \{ U_i(\hat{a}^t, \hat{a}^{t-1}, \theta) | y_1^t \} + (1+\rho)^{-1} E \{ U_{N+i}(\hat{a}^{t+1}, \hat{a}^t, \theta) | y_1^t \}$$

for $i=1, 2, \dots, N$; $t=1, 2, \dots, T$.

Certainly the complete loss of memory of all past observations is an unrealistic assumption. Instead, suppose that each teammate remembers the immediately prior message y_1^{t-1} but no other form of intertemporal communication is allowed. The decision functions will be of the form $\alpha^t(y^t, y^{t-1}) = (\alpha_1^t(y_1^t, y_1^{t-1}), \dots, \alpha_N^t(y_N^t, y_N^{t-1}))'$. The optimality conditions for the decision functions $1, \dots, T$ are

$$(102) \quad 0 = \int_{y^t(i)} \int_{y^{t-1}(i)} \int_{y^{t-2}} \int_{\theta} U_i(\hat{a}^t(y^t, y^{t-1}), \hat{a}^{t-1}(y^{t-1}, y^{t-2}), \theta) \\ f(y^t(i), y^{t-1}(i), y^{t-2}, \theta | y_1^t, y_1^{t-1}) d y^t(i) d y^{t-1}(i) d y^{t-2} d \theta \\ + (1+\rho)^{-1} \int_{y^{t+1}} \int_{y^t(i)} \int_{y^{t-1}(i)} \int_{\theta} U_{N+i}(\hat{a}^{t+1}(y^{t+1}, y^t), \hat{a}^t(y^t, y^{t-1}), \theta) \\ f(y^{t+1}, y^t(i), y^{t-1}(i), \theta | y_1^t, y_1^{t-1}) d y^{t+1} d y^t(i) d y^{t-1}(i) d \theta \\ = E \{ U_i(\hat{a}^t, \hat{a}^{t-1}, \theta) | y_1^t, y_1^{t-1} \} + (1+\rho)^{-1} E \{ U_{N+i}(\hat{a}^{t+1}, \hat{a}^t, \theta) | y_1^t, y_1^{t-1} \}$$

for $i=1,2,\dots,N$; $t=1,2,\dots,T$.

This can be generalized to the individual memory of the past τ messages $y^{t-\tau}, y^{t-\tau+1}, \dots, y^t$ for $1 \leq \tau \leq t-1$. For example if the teammates can remember all their past messages ($\tau = t-1$) then the optimal decision functions must satisfy

$$(103) \quad 0 = E \{ u_i(a^t, a^{t-1}, \theta | y_i^1, \dots, y_i^t) \} + (1+r)^{-1} E \{ u_{N+i}(a^{t+1}, a^t, \theta | y_i^1, \dots, y_i^t) \}$$

for $i=1,2,\dots,N$; $t=1,2,\dots,T$.

Similar equations can be constructed when intertemporal communication is not individual memory but ex post communication. There is also no problem in extending the theory to a dynamic environment where the state of nature changes each period.

FOOTNOTES

- (1) J. Marschak and R. Radner (1972).
- (2) See any of the following: DeGroot(1970), Pratt, Raiffa, and Schlaifer(1965), Raiffa(1968), Raiffa and Schlaifer(1961) or Savage(1972).
- (3) C. Kriebel(1968).
- (4) C. Ying(1969).
- (5) Marschak and Radner(1972), Chapter 7.
- (6) Ho and Chu(1972).
- (7) R. Radner(1962) or Marschak and Radner(1972), page 168.

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ESSAY TWO

Sequential Team Decision Theory
And Optimal Stopping Rules

SEQUENTIAL TEAM DECISION THEORY
AND OPTIMAL STOPPING RULES

I. SEQUENTIAL SAMPLING

a. Decision Rules and Stopping Rules In the introductory discussion of statistical decision theory the information variable was treated as a scalar, although it could have been a vector of sample observations. Many statistical problems involve a fixed number of observations, m , represented by the vector $Y=(y_1, y_2, \dots, y_m)'$ where the number of observations is a given parameter. However, there are other statistical decision problems where the number of observations is not fixed but is determined by the statistician.

One class of such problems is called sequential sampling.⁽¹⁾ In these problems the statistician makes his observations one at a time; after each observation he evaluates his total information and decides either to continue gathering information or to stop sampling and make a decision using only the previous observations. Each observation provides information about the unknown state of nature and increases expected utility. Typically the cost of each observation is a constant, C . Net expected utility is computed as the numerical difference between expected utility and the disutility of the cost of information. Many studies⁽²⁾ have pointed out the special nature of this additive utility assumption, but it greatly simplifies these

complex decision problems. A sequential decision problem requires the selection of two rules: the decision rule and the stopping rule. If the sample terminates after m observations, the decision rule determines which action is to be taken for each possible vector of observations:

$$(1) A = \alpha_m(Y) = \alpha_m(y_1, y_2, \dots, y_m).$$

The stopping rule specifies after m observations $Y = (y_1, y_2, \dots, y_m)$ whether sampling should be terminated and an action be chosen or whether another observation y_{m+1} should be drawn. If the observations come from a set \mathcal{Y} , the stopping rule generates a sequence of subsets $S_m \subset \mathcal{Y}^m = \mathcal{Y} \times \mathcal{Y} \times \dots \times \mathcal{Y}$ called stopping sets. If $(y_1, \dots, y_m) \in S_m$ then sampling terminates. If $(y_1, \dots, y_m) \notin S_m$ then another sample will be drawn. These stopping sets can be used to construct another sequence of sets called termination-at- m sets which define the observations which will cause sampling to stop after m observations but not before:

$$(2) T_m = \bar{S}_1 \cap \bar{S}_2 \cap \dots \cap \bar{S}_{m-1} \cap S_m$$

where \bar{S}_k is the compliment of S_k .

The following is the total net expected utility of a sequential decision procedure with a maximum of B observations (possibly infinite), where $\alpha = (\alpha_1, \dots, \alpha_m)$ is the vector of decision functions and $T = (T_1, \dots, T_m)$ is the collection of termination-at- m sets:

$$(3) W[\alpha, T] = E\{U(\alpha_m(y_1, \dots, y_m), \theta) - m U(c)\} \\ = \sum_{m=1}^B \int_{T_m} \int_{\Theta} U(\alpha_m(y_1, \dots, y_m), \theta) \phi(y_1|\theta) \dots \phi(y_m|\theta) \phi(\theta) d\theta dy_1 \dots dy_m \\ + \sum_{m=2}^B m U(c) \text{Prob}((y_1, \dots, y_m) \in T_m).$$

An optimal sequential decision procedure is a stopping

rule and a set of decision functions that together maximize total net expected utility, $W[\alpha, \tau]$. A fundamental property of the optimal sequential decision procedure is that the decision function must be optimal for each possible m -vector of observations. That is if sampling stops after the m observations (y_1, \dots, y_m) , the decision for each possible (y_1, \dots, y_m) must maximize expected utility against the posterior p.d.f. of Θ given (y_1, \dots, y_m) . Because of this property the discussion of optimal sequential decision procedures focuses on the optimal stopping rule, implicitly assuming that the decision functions make optimal use of information.

b. Backward Induction If there is a finite limit on the number of observations, the optimal stopping rule may be developed by a procedure of backward induction.⁽³⁾ The rationale for this technique of solving the general sequential sampling problem is given and will be formalized in the next subsection.

Suppose the utility is a function of the unknown state of nature, Θ , and the statistician's action A . The information variables y_1 are independent and have the same conditional p.d.f. given Θ , $\phi(y_i | \Theta)$. The statistician is allowed to take a sequential sample of up to B observations. Each observation has a fixed disutility $U(C)$. The decision rule, α_m , maps a m -vector (y_1, \dots, y_m) into an action A . The prior p.d.f. of Θ is $\phi(\Theta)$.

The initial question the statistician must answer is : should the first observation be taken or should a decision be made based only on the prior beliefs about Θ ? The answer depends

on how the information in y_1 is used. If y_1 is observed, will another observation y_2 be drawn or will a decision be made using just y_1 ? If observations are limited to B , the final question in this sequence of questions will be: if y_1, \dots, y_{B-1} have been observed, will the last sample y_B be taken or will the decision be made without this additional information? The statistician can usually find the stopping subset of \mathcal{Y}^{B-1} for which the final observation should not be taken. Moving backwards a similar procedure can be set up for y_1, \dots, y_{B-2} and so forth back to y_1 . Thus "for each possible value of y_1 , the optimal continuation throughout the remaining stages"⁽⁴⁾ is known and the original question resolved.

c. Bounded Stopping Rules Optimal stopping rules require the construction of the posterior p.d.f. of θ after the m observations (y_1, y_2, \dots, y_m) are drawn.

$$(4) f(\theta | y_1, \dots, y_m) = \frac{\prod_{i=1}^m \phi(y_i | \theta) \phi(\theta)}{\int_{\Theta} \prod_{i=1}^m \phi(y_i | \theta) \phi(\theta) d\theta}.$$

The expected utility after sampling has stopped with m observations is denoted

$$(5) W_0(f(\theta | y_1, \dots, y_m)) = \max_A \int_{\Theta} u(A, \theta) f(\theta | y_1, \dots, y_m) d\theta.$$

Implicitly the optimal action is a function of the observations.

Suppose the statistician has observed $B-1$ values y_1, \dots, y_{B-1} and wants to know if the final variable should be observed. He continues sampling if the expected utility of deciding based on y_1, \dots, y_{B-1} is less than net expected utility with the final obser-

vation, i.e.,

$$(6) \quad W_0(f(\theta|y_1, \dots, y_{B-1})) < E\{W_0(f(\theta|y_1, \dots, y_B)) | y_1, \dots, y_{B-1}\} - u(c).$$

Define the optimal expected utility when the current beliefs about θ are $f(\theta|y_1, \dots, y_{B-1})$ and when only one additional sample may be taken by

$$(7) \quad W_1(f(\theta|y_1, \dots, y_{B-1})) = \max [W_0(f(\theta|y_1, \dots, y_{B-1})), E\{W_0(f(\theta|y_1, \dots, y_B)) | y_1, \dots, y_{B-1}\} - u(c)].$$

Backing up to the previous stage, the computed posterior p.d.f. for θ is $f(\theta|y_1, \dots, y_{B-2})$. The optimum expected utility given the first $B-2$ observations, but not the last two, is

$$(8) \quad W_0(f(\theta|y_1, \dots, y_{B-2})) = \max_A \int_{\Theta} U(A, \theta) f(\theta|y_1, \dots, y_{B-2}) d\theta.$$

The expected net utility of taking another observation is

$$(9) \quad E\{W_1(f(\theta|y_1, \dots, y_{B-1})) | y_1, \dots, y_{B-2}\} - u(c).$$

The optimum expected utility for the remainder of the procedure when the current beliefs about θ are $f(\theta|y_1, \dots, y_{B-2})$ and when at most two more samples can be drawn is

$$(10) \quad W_2(f(\theta|y_1, \dots, y_{B-2})) = \max [W_0(f(\theta|y_1, \dots, y_{B-2})), E\{W_1(f(\theta|y_1, \dots, y_{B-1})) | y_1, \dots, y_{B-2}\} - u(c)].$$

By induction the functions $W_B(\phi(\theta))$, $W_{B-1}(f(\theta|y_1))$, ..., $W_0(f(\theta|y_1, \dots, y_B))$ can be defined using the relationship

$$(11) \quad W_{B-m}(f(\theta|y_1, \dots, y_m)) = \max [W_0(f(\theta|y_1, \dots, y_m)), E\{W_{B-m-1}(f(\theta|y_1, \dots, y_{m+1})) | y_1, \dots, y_m\} - u(c)].$$

This provides the statistician with all the needed information to establish a stopping rule.

Proposition 2-1: If the prior p.d.f. of θ is $\phi(\theta)$ then $W_B(\phi(\theta))$ is the total expected utility from the optimal sequential

decision procedure in which not more than B observation can be taken. Furthermore, for $m=1,2,\dots,B-1$ after the values y_1, y_2, \dots, y_m have been observed and the posterior p.d.f. of θ becomes $f(\theta|y_1, \dots, y_m)$, the expected net utility from the optimal continuation is $W_{B-m}(f(\theta|y_1, \dots, y_m))$.

DeGroot also states the optimal stopping rule as follows:

Proposition 2-2: Among all sequential decision procedures in which not more than B observations can be taken, the following procedure is optimal. If $W_0(\phi(\theta)) \geq W_B(\phi(\theta))$ a decision is chosen immediately without any observations. Otherwise, y_1 is observed. Furthermore, for $m=1,2,\dots,B-1$ suppose the values y_1, y_2, \dots, y_m have been observed. If $W_0(f(\theta|y_1, \dots, y_m)) \geq W_{B-m}(f(\theta|y_1, \dots, y_m))$ a decision is chosen without further observations. Otherwise y_{m+1} is observed. If sampling has not been terminated earlier, it must be terminated after y_B has been drawn.

d. Example: Maximum of Two Observations Suppose the statistician make at most two observations. The stopping rule must specify the conditions under which (1) no samples are made, (2) only one sample is taken and (3) both samples are observed. The analysis begins by computing the maximum expected utility if both observation are made. The action A^2 is selected to maximize expected utility against the posterior p.d.f. $f(\theta|y_1, y_2)$.

This defines

$$(12) W_0(f(\theta|y_1, y_2)) = \max_{A^2} \int_{\Theta} u(A^2, \theta) f(\theta|y_1, y_2) d\theta.$$

Under what conditions should the statistician observe the second sample when he had already observed the value y_1 ? If the statistician does not observe the second sample then he must select an action that maximizes expected utility against the posterior p.d.f. $f(\theta|y_1)$.

$$(13) W_0(f(\theta|y_1)) = \max_{A^1} \int_{\Theta} u(A^1, \theta) f(\theta|y_1) d\theta.$$

Expected net utility if y_2 is observed at this stage is

$$(14) E\{W_0(f(\theta|y_1, y_2))|y_1\} - u(c) = \max_{A^2} \int_{y_2} \int_{\Theta} u(A^2(y_2), \theta) f(\theta, y_2|y_1) dy_2 d\theta - u(c).$$

The stopping set is then defined as

$$(15) S_1 = \{y_1 | W_0(f(\theta|y_1)) > E\{W_0(f(\theta|y_1, y_2))|y_1\} - u(c)\}.$$

If y_1 falls in the stopping set, no further observations are made. The maximum net expected utility for the remainder of the procedure when y_1 is known and no more than one additional observation can be drawn is

$$(16) W_1(f(\theta|y_1)) = \max [W_0(f(\theta|y_1)), E\{W_0(f(\theta|y_1, y_2))|y_1\} - u(c)].$$

Under what conditions should the statisticians even make one observation? If no observations are made, an action must be selected to maximize expected utility against the prior p.d.f. $\phi(\theta)$; this defines

$$(17) W_0(\phi(\theta)) = \max_A \int_{\Theta} u(A, \theta) \phi(\theta) d\theta.$$

If a sample is taken it will either be in the stopping set or it will not. The expected utility of the optimal continuation is

$$(18) E\{W_1(f(\theta|y_1))\} - u(c) = \int_{y_1 \in S_1} \int_{\Theta} u(\hat{A}^1(y_1), \theta) \phi(y_1|\theta) \phi(\theta) d\theta dy_1 + \int_{y_1 \notin S_1} \int_{y_2} \int_{\Theta} u(\hat{A}^2(y_2; y_1), \theta) \phi(y_1|\theta) \phi(y_2|\theta) \phi(\theta) d\theta dy_2 dy_1 - u(c) - u(c) \text{Prob}(y_1 \notin S_1).$$

The optimal stopping rule is thus defined by the following sequential decisions:

(a) If $W_0(\phi(\theta)) > E\{W_1(f(\theta|y_1))\} - U(c)$ then make no samples.

Otherwise make the first observation y_1 .

(b) If the first observation is in the stopping set S_1 then do not take another observation. If y_1 is not in S_1 then take exactly one more sample before making a decision on the best action.

II. TEAM DECISIONS WITH SEQUENTIAL COMMUNICATION

a. Sequential Observations and Intertemporal Communication In the chapter on multiperiod teams a decision making procedure was studied which consisted of three phases: (1) the teammates observed the environment, (2) the teammates communicated the results of their observations and (3) the teammates decided on their optimal actions. Each period these phases were repeated. This team model was general enough to include a fixed number of observations or communications; the observations z_j could be vectors and the communication functions $\gamma_{ij}(z_j)$ could be vector-valued. Let us call an observation and its corresponding communication an information gathering operation, abbreviated IGO. If the number of IGOs allowed before actions are chosen is an element of the decision of problem and not a fixed parameter, the theory of teams must be modified.

Suppose the team makes IGOs one at a time; after each IGO the team evaluates its total information and decides either to continue gathering information or to stop and make a team decision using only the current knowledge. This sequential team decision problem has two components: a series of decision functions to determine what actions are taken for the given information and a stopping rule which determines how much information shall be gathered.

Care must be taken to differentiate sequential team problems from non-sequential, multiple observation team problems.

thus each teammate will independently give the same answer.

b. Interim Actions and Opportunity Losses In the theory of team decisions all communications and computations were accomplished instantaneously. This could also be assumed for "sequential" team theory but several interesting situations can be studied only if communication becomes a time consuming operation. Suppose that each IGO takes a finite, positive length of time to complete. At the end of the IGO an optimal action is selected and implemented, but while the information was being gathered an opportunity may have been lost because either no action was taken or the action that was suboptimal. If the number of IGOs is fixed this opportunity loss is unavoidable because the team cannot change the length of time it spends gathering information. However the sequential team can determine how much time is spent gathering information because it can select the stopping rule. The sequential team should recognize such opportunity losses when it picks its stopping rule.

There are various ways that opportunity losses can occur in sequential decision problems. Three specific schemes will be studied in this chapter. First, it can be assumed that utility is realized only at the end of the period instead of continuously throughout the period. No opportunities are lost while information is gathered. Second, a known interim action \tilde{A} can be effective until a final decision is reached. As information is collected tentative actions are proposed but not implemented until the team

stops gathering information. Third, at each stage of information gathering the tentative actions based on the current knowledge can be temporarily implemented. The effective actions change as information accumulates. Once information gathering stops the final actions remain effective for the rest of the period.

Two decision making procedures not studied are: (1) Suppose that a change in effective action creates a cost which lowers utility. At the beginning of the period a historical interim action is in force. As information is accumulated the team must decide not only what is the optimal action and whether or not more information should be gathered, but it must also decide whether or not to replace the interim action with the current tentative action. (2) Suppose the team can not only choose how many IGOs it makes but also how many communications it makes based on each observation. The teammates can not only communicate the results of their observations but can also communicate the result of their individual ex ante messages.

c. Assumptions The sequential team decision problem requires the introduction of several new components to team theory. For convenience all assumptions about the old and the new components of team theory will be presented here.

Assumption 2-1: The length of the period of operation is a fixed time λ .

Assumption 2-2: the environment takes on an unknown value Θ at

time $t=0$ and is unchange for all time $0 \leq t \leq \lambda$. The team's prior p.d.f. for θ is $\phi(\theta)$.

Assumption 2-3: The ex ante communication function $Y = \eta(z)$ is the same for all IGOs. The only intertemporal communication is the ex post message $X = \mu(z)$; the ex post communication function $\mu(z)$ is the same for all IGOs.

Assumption 2-4 The sequence of team observation Z^1, Z^2, \dots , come from independent, identically distributed random samples. This, along with Assumption 2-3, implies that the distribution of ex ante messages are independent and identically distributed for each IGO with conditional p.d.f. $\phi(Y|\theta)$. Similarly all ex post messages are independent with the same conditional p.d.f. $h(X|\theta)$.

Assumption 2-5: Team actions implemented only after the ex post message has been received.

Assumption 2-6: The cost of IGO is independent of the actual messages sent and depends only on the functional forms $\eta(\cdot)$ and $\mu(\cdot)$. This, along with assumption 2-3, implies that each IGO cost a fixed amount $C = C(\eta, \mu)$.

Assumption 2-7: The time required to complete an IGO is independent of the actual messages and depends only on the functional forms $\eta(\cdot)$ and $\mu(\cdot)$. (Note: computations of decision functions and stopping rules are instantaneous.) This, along with Assumption 2-3, implies that each IGO takes a fixed time $K = K(\eta, \mu)$.

Assumption 2-8: Utility is a function of the team action A , the

unknown state of nature and the total cost of gathering information. The utility will be additively separable as follows $U = U(A, \Theta) - U(C_{\text{total}})$.

From assumption 2-1 and 2-7 an absolute upper limit on the number of IGOs is defined. Let m denote the number of IGOs; it must satisfy the restriction $0 \leq m \leq \lambda/\kappa$.

d. Stopping Rules Without Interim Actions A sequential team makes a series of IGOs and after each one proposes a tentative action. Exactly how does the team select these tentative actions? When the team is about to make the m^{th} IGO, it wants to maximize expected utility given its accumulated information. All teammates have observed and remembered the $m-1$ past ex post messages x^1, x^2, \dots, x^{m-1} and calculated the joint posterior p.d.f. of the ex ante message Y^m and the state of nature Θ :

$$(19) f(Y^m, \Theta | x^1, \dots, x^{m-1}) = \phi(Y^m | \Theta) \prod_{i=1}^{m-1} [\phi(x^i | \Theta)] \phi(\Theta) / \int_{\Theta} \prod_{i=1}^{m-1} [\phi(x^i | \Theta)] \phi(\Theta) d\Theta.$$

The team picks a decision function $\alpha^m(Y^m) = (\alpha_1^m(y_1^m), \dots, \alpha_N^m(y_N^m))'$ to maximize expected utility against the posterior p.d.f. $f(Y^m, \Theta | x^1, \dots, x^{m-1})$.

The first order conditions the optimal decision functions must satisfy are

$$(20) 0 = \int_{\Theta} \int_{y_i^m} u_{q_i}(\alpha^m(Y^m), \Theta) f(Y^m(i), \Theta | y_i^m, x^1, \dots, x^{m-1}) dY^m(i) d\Theta, \quad i=1, \dots, N.$$

It should be noted that each individual decision function is implicitly a function of the ex post message x^1, \dots, x^{m-1} . This could be introduced explicitly by writing the decision function as $\alpha^m(Y^m, x^{m-1}, \dots, x^1)$ and maximizing

$$(21) \int_{\Theta} \int_{y^m} \int_{x^{m-1}} \dots \int_{x^1} u(\alpha^m(y^m, x^{m-1}, \dots, x^1), \theta) \phi(y^m | \theta) \prod_{i=1}^{m-1} [\phi(x^i | \theta)] \phi(\theta) dx^1 \dots dx^{m-1} dy^m d\theta.$$

The two procedure lead to identical decisions for given values of y^m, x^m, \dots, x^1 . Also notice that the m^{th} decisions does not depend on the m^{th} ex post message x^m because x^m is received after decisions are made. However x^m is used in all subsequent decisions.

At what point does the team decide whether or not to stop the information gathering process and what information is used to make such a decision? Suppose the team had made its m^{th} observation and corresponding ex ante communication and each teammate has selected a tentative action $\hat{a}_i^m = \hat{\alpha}_i^m(y_i^m, x^{m-1}, \dots, x^1)$. After the team receives its ex post message x^m it must decide whether to stop and implement the tentative $\hat{A}^m = (\hat{a}_1^m, \dots, \hat{a}_N^m)$ or to continue gathering information. At this point in time all teammates know the value of the ex post message x^1, x^2, \dots, x^m (including x^m) and the entire vector \hat{A}^m of a tentative actions. (5)

The team establishes an optimal stopping rule by backward induction beginning from the final IGO. Suppose that all B IGOs have been made and a tentative action $\hat{A}^B = \hat{\alpha}^B(y^B, x^{B-1}, \dots, x^1)$ has been selected. Because there cannot be another IGO this action must be implemented and has a net expected utility

$$(22) W_0^B(\hat{A}^B, x^B, \dots, x^1) = \int_{\Theta} u(\hat{A}^B, \theta) f(\theta | x^1, \dots, x^B) d\theta - u(Bc).$$

Back up to the $B-1^{\text{th}}$ IGO. The team has a choice of implementing its tentative action $\hat{A}^{B-1} = \hat{\alpha}^{B-1}(y^{B-1}, x^{B-2}, \dots, x^1)$ or making one more IGO. If \hat{A}^{B-1} is implemented, the expected net utility is

$$(23) \int_{\Theta} u(\hat{A}^{B-1}, \theta) f(\theta | x^1, \dots, x^{B-1}) d\theta - u((B-1)c).$$

If the information gathering is continued the expected net utili-

ty of continuation is

$$(24) E \{ W_0^B | X^{B-1}, \dots, X^1 \}.$$

Define the maximum expected net utility after B-1 IGO and with at most one more IGO by

$$(25) W_1^{B-1}(\hat{A}^{B-1}, X^{B-1}, \dots, X^1) = \max [\int_{\Theta} U(\hat{A}^{B-1}, \theta) f(\theta | X^1, \dots, X^{B-1}) d\theta - U((B-1)c), \\ E \{ W_0^B | X^{B-1}, \dots, X^1 \}].$$

Back up to B-2th information gathering sequence. The team has a choice of implementing its tentative action $\hat{A}^{B-2} = \hat{\alpha}^{B-2}(Y^{B-2}, X^{B-3}, \dots, X^1)$ or taking at least one more IGO. If \hat{A}^{B-2} is implemented the expected net utility is

$$(26) \int_{\Theta} U(\hat{A}^{B-2}, \theta) f(\theta | X^1, \dots, X^{B-2}) d\theta - U((B-2)c).$$

If information gathering is continued the expected net utility of continuation is

$$(27) E \{ W_1^{B-1} | X^{B-2}, \dots, X^1 \}.$$

Define the maximum expected net utility after B-2 IGOs and with no more than 2 more IGOs by

$$(28) W_2^{B-2}(\hat{A}^{B-2}, X^{B-2}, \dots, X^1) = \max [\int_{\Theta} U(\hat{A}^{B-2}, \theta) f(\theta | X^1, \dots, X^{B-2}) d\theta - U((B-2)c), \\ E \{ W_1^{B-1} | X^{B-2}, \dots, X^1 \}].$$

By induction we can define the maximum expected net utility after m IGO with most B-m additional IGOs by

$$(29) W_{B-m}^m(\hat{A}^m, X^m, \dots, X^1) = \max [\int_{\Theta} U(\hat{A}^m, \theta) f(\theta | X^1, \dots, X^m) d\theta - U(mc), \\ E \{ W_{B-m-1}^{m+1} | X^m, \dots, X^1 \}].$$

Finally let \hat{A}^0 be the vector of team actions that maximizes expected utility against the prior density $\phi(\theta)$; i.e., \hat{A}^0 satisfies $\int_{\Theta} U_{a_i}(\hat{A}^0, \theta) \phi(\theta) d\theta = 0$, $i=1, 2, \dots, N$. Define the maximum expected utility if no more than B IGOs are made by

$$(30) W_B^0 = \max \left[\int_{\Theta} U(\hat{A}^0, \theta) \phi(\theta) d\theta, E\{W_{B-1}^1\} \right].$$

These $B+1$ expected net utilities can be used to specify the optimal team stopping rule without interim actions.

Proposition 2-3: If the team models satisfies Assumptions 2-1

through 2-8 and utility depends only on the final action selected, the following stopping rule is optimal.

(0) If $\int_{\Theta} U(\hat{A}^0, \theta) \phi(\theta) d\theta > E\{W_{B-1}^1\}$ the decision \hat{A}^0 is implemented without any information gathering. Otherwise the first IGO is made.

(1) If $\int_{\Theta} U(\hat{A}^1, \theta) f(\theta | x^1) d\theta - U(c) > E\{W_{B-2}^2 | x^1\}$ the action $\hat{A}^1 = \hat{\alpha}^1(\gamma^1)$ is implemented without further information gathering. Otherwise the second IGO is made.

⋮

(m) If $\int_{\Theta} U(\hat{A}^m, \theta) f(\theta | x^1, \dots, x^m) d\theta - U(mc) > E\{W_{B-m-1}^{m+1} | x^1, \dots, x^m\}$ the action $\hat{A}^m = \hat{\alpha}^m(\gamma^m, x^{m-1}, \dots, x^1)$ is implemented without further information gathering. Otherwise the $m+1^{\text{th}}$ IGO is made.

⋮

(B) The action $\hat{A}^B = \hat{\alpha}^B(\gamma^B, x^{B-1}, \dots, x^1)$ is implemented if the B^{th} IGO was made.

e. Stopping Rules With Interim Actions When the environment of the team takes on a new value at the beginning of the period the teammates begin looking for new values for their action variables.

Current actions may be totally suboptimal with the respect to the new state of nature. It is now assumed that the team receives utility continuously throughout the period. As a result each IGO creates two costs: (1) there is a cost $C = C(\eta, \mu)$ associated with the transmission of messages and (2) while information is being accumulated the interim action \tilde{A} may be suboptimal; hence an opportunity loss is incurred. (6)

If exactly m IGOs are made and then an action \hat{A}^m implemented for the rest of the period, the total utility for a given θ is

$$(31) \quad U(\tilde{A}, \hat{A}, \theta) = U(\tilde{A}, \theta)K_m + U(\hat{A}^m, \theta)(\lambda - K_m) - U(mC).$$

The team establishes an optimal stopping rule by backward induction from the final IGO. Suppose that all B IGOs have been made and a tentative action $\hat{A}^B = \hat{\alpha}^B(\gamma^B, X^{B-1}, \dots, X^1)$ has been selected. By definition of optimality \hat{A}^B is more valuable than \tilde{A} , so \hat{A}^B is implemented for the remainder of the period. The additional net expected utility to be gained is

$$(32) \quad W_0^B(\hat{A}^B, X^B, \dots, X^1) = (\lambda - BK) \int_{\Theta} U(\hat{A}^B, \theta) f(\theta | X^1, \dots, X^B) d\theta - U(BC).$$

Back up to the $B-1$ th IGO. The team has a choice of implementing its tentative action $\hat{A}^{B-1} = \hat{\alpha}^{B-1}(\gamma^{B-1}, X^{B-2}, \dots, X^1)$ or continuing with \tilde{A} until the final IGO is completed. If \hat{A}^{B-1} is implemented, the expected net utility for the remainder of the period is

$$(33) \quad (\lambda - (B-1)K) \int_{\Theta} U(\hat{A}^{B-1}, \theta) f(\theta | X^1, \dots, X^{B-1}) d\theta - U((B-1)C).$$

If the information gathering is continued the expected net utility of continuation is

$$(34) \quad K \int_{\Theta} U(\tilde{A}, \theta) f(\theta | X^1, \dots, X^{B-1}) d\theta + E\{W_0^B | X^{B-1}, \dots, X^1\}.$$

Define the maximum expected net utility after B-1 IGOs with at most one additional IGO by

$$(35) W_{B-1}^{B-1}(\hat{A}^{B-1}, x^{B-1}, \dots, x^1) = \max \left[(\lambda - (B-1)K) \int_{\Theta} u(\hat{A}^{B-1}, \theta) f(\theta | x^1, \dots, x^{B-1}) d\theta - u((B-1)c), K \int_{\Theta} u(\tilde{A}, \theta) f(\theta | x^1, \dots, x^{B-1}) d\theta + E \{ W_0^B | x^{B-1}, \dots, x^1 \} \right].$$

Back up to the B-2th IGO. The team has a choice of implementing its tentative action $\hat{A}^{B-2} = \hat{\alpha}^{B-2}(x^{B-2}, x^{B-3}, \dots, x^1)$ or allowing \tilde{A} to be in effect at least one more IGO. If \hat{A}^{B-2} is implemented the expected net utility for the remainder of the period is

$$(36) (\lambda - (B-2)K) \int_{\Theta} u(\hat{A}^{B-2}, \theta) f(\theta | x^1, \dots, x^{B-2}) d\theta - u((B-2)c).$$

If information gathering is continued the expected net utility of continuation is

$$(37) K \int_{\Theta} u(\tilde{A}, \theta) f(\theta | x^1, \dots, x^{B-2}) d\theta + E \{ W_{B-1}^{B-1} | x^{B-2}, \dots, x^1 \}.$$

Define the maximum expected net utility after B-2 IGOs with at most two additional IGOs by

$$(38) W_{B-2}^{B-2}(\hat{A}^{B-2}, x^{B-2}, \dots, x^1) = \max \left[(\lambda - (B-2)K) \int_{\Theta} u(\hat{A}^{B-2}, \theta) f(\theta | x^1, \dots, x^{B-2}) d\theta - u((B-2)c), K \int_{\Theta} u(\tilde{A}, \theta) f(\theta | x^1, \dots, x^{B-2}) d\theta + E \{ W_{B-1}^{B-1} | x^{B-2}, \dots, x^1 \} \right].$$

By induction define the maximum expected net utility after m IGOs with no more than B-m additional IGOs by

$$(39) W_{B-m}^m(\hat{A}^m, x^m, \dots, x^1) = \max \left[(\lambda - mK) \int_{\Theta} u(\hat{A}^m, \theta) f(\theta | x^1, \dots, x^m) d\theta - u(mc), K \int_{\Theta} u(\tilde{A}, \theta) f(\theta | x^1, \dots, x^m) d\theta + E \{ W_{B-m-1}^{m+1} | x^m, \dots, x^1 \} \right].$$

Finally let \hat{A}^0 be the vector of actions that maximizes expected utility against the prior density $\phi(\theta)$. Define the maximum expected utility when no information is known and with at most B IGOs by

$$(40) W_B^0 = \max \left[\lambda \int_{\Theta} u(\hat{A}^0, \theta) \phi(\theta) d\theta, \kappa \int_{\Theta} u(\tilde{A}, \theta) \phi(\theta) d\theta + E\{W_{B-1}^1\} \right].$$

These $B+1$ expected net utilities can be used to specify the optimal team stopping rule with interim actions.

Proposition 2-4: If the team model satisfies Assumptions 2-1

through 2-8 and \tilde{A} is the interim action, the following stopping rule is optimal

- (0) If $\lambda \int_{\Theta} u(\hat{A}^0, \theta) \phi(\theta) d\theta > \kappa \int_{\Theta} u(\tilde{A}, \theta) \phi(\theta) d\theta + E\{W_{B-1}^1\}$ then the action \hat{A}^0 is implemented immediately without gathering any information. Otherwise the first IGO made.
- (1) If $(\lambda - \kappa) \int_{\Theta} u(\hat{A}^1, \theta) f(\theta | x^1) d\theta - u(c) > \kappa \int_{\Theta} u(\tilde{A}, \theta) f(\theta | x^1) d\theta + E\{W_{B-2}^2 | x^1\}$ then the action $\hat{A}^1 = \hat{A}^1(\psi^1)$ is implemented without gathering more information. Otherwise the second IGO is made.
- ⋮
- (m) If $(\lambda - m\kappa) \int_{\Theta} u(\hat{A}^m, \theta) f(\theta | x^1, \dots, x^m) d\theta - u(mc) > \kappa \int_{\Theta} u(\tilde{A}, \theta) f(\theta | x^1, \dots, x^m) d\theta + E\{W_{B-m-1}^{m+1} | x^1, \dots, x^m\}$ then the action $\hat{A}^m = \hat{A}^m(\psi^m, x^{m-1}, \dots, x^1)$ is implemented without further information. Otherwise the $m+1^{\text{th}}$ IGO is made.
- ⋮
- (B) The action $\hat{A}^B = \hat{A}^B(\psi^B, x^{B-1}, \dots, x^1)$ is implemented if the B^{th} IGO is made.

f. Stopping Rules With Adapting Interim Actions Again, each IGO creates both a cost due to transmission and an opportunity loss due to the time required to complete the communication. In this

decision procedure the action in effect while information is gathered is not a fixed interim action, \tilde{A} , but is the tentative action of the previous IGO, \hat{A}^{m-1} . This is very close to the multi-period team model with a static environment in which the team adjusted its decision according to its information; now the team must decide when enough information has been accumulated.

The team establishes an optimal stopping rule by backward induction from the final IGO. Suppose that all B IGOs have been made and a tentative action $\hat{A}^B = \hat{\alpha}^B(\gamma^B, \chi^{B-1}, \dots, \chi^1)$ has been selected. \hat{A}^B is implemented immediately for the remainder of the period, netting an expected utility

$$(41) W_0^B(\hat{A}^B, \chi^B, \dots, \chi^1) = (\lambda - Bk) \int_{\Theta} u(\hat{A}^B, \theta) f(\theta | \chi^1, \dots, \chi^B) d\theta - u(Bc).$$

Back up to the B-1th IGO. The team has a choice of implementing its tentative action $\hat{A}^{B-1} = \hat{\alpha}^{B-1}(\gamma^{B-1}, \chi^{B-2}, \dots, \chi^1)$ for the remainder of the period or allowing it to be effective only until the final IGO is completed. If \hat{A}^{B-1} is implemented for the rest of the period, the expected net utility is

$$(42) (\lambda - (B-1)k) \int_{\Theta} u(\hat{A}^{B-1}, \theta) f(\theta | \chi^1, \dots, \chi^{B-1}) d\theta - u((B-1)c).$$

If information gathering is continued the expected net utility of continuation is

$$(43) k \int_{\Theta} u(\hat{A}^{B-1}, \theta) f(\theta | \chi^1, \dots, \chi^{B-1}) d\theta + E\{W_0^B | \chi^{B-1}, \dots, \chi^1\}.$$

Define the maximum expected net utility after B-1 IGOs with at most one additional IGO by

$$(44) W_1^{B-1}(\hat{A}^{B-1}, \chi^{B-1}, \dots, \chi^1) = k \int_{\Theta} u(\hat{A}^{B-1}, \theta) f(\theta | \chi^1, \dots, \chi^{B-1}) d\theta + \max_x [(\lambda - Bk) \int_{\Theta} u(\hat{A}^{B-1}, \theta) f(\theta | \chi^1, \dots, \chi^{B-1}) d\theta - u((B-1)c), E\{W_0^B | \chi^{B-1}, \dots, \chi^1\}].$$

By induction define the maximum expected net utility after m IGOs with at most $B-m$ additional IGOs by

$$(45) \quad W_{B-m}^m(\hat{A}^m, x^m, \dots, x^1) = K \int_{\Theta} u(\hat{A}^m, \theta) f(\theta | x^1, \dots, x^m) d\theta + \\ \max [(\lambda - (m+1)K) \int_{\Theta} u(\hat{A}^m, \theta) f(\theta | x^1, \dots, x^m) d\theta - u(mC), \\ E \{ W_{B-m-1}^{m+1} | x^m, \dots, x^1 \}].$$

Finally let \hat{A}^0 be the vector of actions that maximizes expected utility against the prior density $\phi(\theta)$. Define the maximum expected utility when no information is known and with at most B IGOs by

$$(46) \quad W_B^0 = K \int_{\Theta} u(\hat{A}^0, \theta) \phi(\theta) d\theta + \max [(\lambda - K) \int_{\Theta} u(\hat{A}^0, \theta) \phi(\theta) d\theta, E \{ W_{B-1}^1 \}].$$

These $B+1$ expected net utilities can be used to specify the optimal team stopping rule with adapting interim actions.

Proposition 2-5: If the team Assumption 2-1 through 2-8 and actions are adjusted, the following stopping rule is optimal.

- (0) If $(\lambda - K) \int_{\Theta} u(\hat{A}^0, \theta) \phi(\theta) d\theta > E \{ W_{B-1}^1 \}$ then the decision \hat{A}^0 is implemented for the entire period. Otherwise the first IGO is made.
- (1) If $(\lambda - 2K) \int_{\Theta} u(\hat{A}^1, \theta) f(\theta | x^1) d\theta - u(C) > E \{ W_{B-2}^2 | x^1 \}$ then the action $\hat{A}^1 = \hat{a}^1(y^1)$ is implemented for the remainder of the period without gathering more information. Otherwise the second IGO is made.
- ⋮
- (m) If $(\lambda - (m+1)K) \int_{\Theta} u(\hat{A}^m, \theta) f(\theta | x^1, \dots, x^m) d\theta - u(mC) > E \{ W_{B-m-1}^{m+1} | x^m, \dots, x^1 \}$ then the action $\hat{A}^m = \hat{a}^m(y^m, x^{m-1}, \dots, x^1)$ is implemented for the remainder of the period without gathering more information. Otherwise the $m+1^{\text{th}}$ IGO is made.

\vdots
 \vdots
 (B) The action $\hat{A}^B = \hat{\alpha}^B(\gamma^B, \chi^{B-1}, \dots, \chi^1)$ is implemented if the B^{th} IGO was made.

8. Sequential Teams With A Maximum of Two Observations This is an example of a sequential team which can make at most two IGOs. The team can do three things: (1) make a decision based only on prior beliefs, (2) make a decision based on an ex ante message Y^1 or (3) make a decision an ex ante post message X^1 and an ex ante message Y^2 . The optimal stopping rules of propositions 2-3, 2-4 and 2-5 will be discussed with $B=2$ and stopping sets will be defined for those three sequential decision procedures.

Take first the scheme without interim actions or opportunity losses. If the first IGO has been taken and the team decides to continue gathering information, the expected value of continuation is

$$(47) \quad E\{W_0^2 | X^1\} = \int_{Y^2} \int_{\Theta} U(\hat{\alpha}^2(\gamma^2; X^1), \theta) f(\gamma^2, \theta | X^1) d\theta dY^2 - U(c).$$

If the second IGO is not made, the expected value of implementing the tentative action $\hat{A}^1 = \hat{\alpha}^1(\gamma^1)$ is

$$(48) \quad \int_{\Theta} U(\hat{A}^1, \theta) f(\theta | X^1) d\theta - U(c).$$

A stopping set for the first stage is the set of all (\hat{A}^1, X^1) that satisfy

$$(49) \quad \int_{\Theta} U(\hat{A}^1, \theta) f(\theta | X^1) d\theta - U(c) > \int_{Y^2} \int_{\Theta} U(\hat{\alpha}^2(\gamma^2; X^1), \theta) f(\gamma^2, \theta | X^1) d\theta dY^2 - U(c).$$

Maximum expected net utility for the rest of the decision procedure is

$$(50) W_1^1 = \max [\int_{\theta} u(\hat{A}^1, \theta) f(\theta | X^1) d\theta - u(c), E\{W_0^2 | X^1\}].$$

Define a conditional stopping subset of the ex post message for each possible action by

$$(51) S^1(\hat{A}^1) = \{ X^1 : \int_{\theta} u(\hat{A}^1, \theta) f(\theta | X^1) d\theta - u(c) > E\{W_0^2 | X^1\} \}.$$

If the team has a choice of implementing \hat{A}^1 or making another IGO, the decision depends entirely on whether or not the ex post message X^1 fall in $S^1(\hat{A}^1)$.

Backing up to the stage 0, the team has a choice of making the decision \hat{A}^0 and receiving expected utility

$$(52) \int_{\theta} u(\hat{A}^0, \theta) \phi(\theta) d\theta$$

or making the first IGO. At this time the expected utility of taking at least the first IGO depends on $S^1(\hat{A}^1)$. The decision function $\hat{\alpha}^1(\gamma^1)$ is known but not tentative action \hat{A}^1 because is not yet known. The conditional subset S^1 are thus a function of the observed value of γ^1 and this is denoted $S^1(\gamma^1) = S^1(\hat{\alpha}^1(\gamma^1))$.

The expected net utility continuation is

$$(53) E\{W_1^1\} = \int_{\theta} \int_{\gamma^1} \int_{X^1 \in S^1(\gamma^1)} (u(\hat{\alpha}^1(\gamma^1), \theta) - u(c)) \phi(\gamma^1, X^1 | \theta) \phi(\theta) dX^1 d\gamma^1 d\theta + \\ \int_{\gamma^1} \int_{\theta} \int_{\gamma^2} \int_{X^1 \notin S^1(\gamma^1)} (u(\hat{\alpha}^2(\gamma^2, X^1) - u(2c)) \phi(X^1, \gamma^1 | \theta) \phi(\gamma^2 | \theta) \phi(\theta) \\ dX^1 d\gamma^1 d\gamma^2 d\theta.$$

If $\int_{\theta} u(\hat{A}^0, \theta) \phi(\theta) d\theta > E\{W_1^1\}$ then no information will be gathered.

Otherwise at least one IGO is made

Secondly, let us analyze a team with an opportunity loss due to the suboptimality of the interim action \tilde{A} . If the first IGO has been taken and the team decides to continue gathering informa-

tion, the expected value of continuation is

$$(54) E\{\omega^z | X^1\} = K \int_{\Theta} u(\tilde{A}, \theta) f(\theta | X^1) d\theta + (\lambda - 2K) \int_{\Theta} \int_{Y^z} u(\hat{a}^z(\gamma^z; X^1), \theta) f(\gamma^z, \theta | X^1) d\gamma^z d\theta - u(zc).$$

If the second IGO is not made the expected utility of implementing the tentative action \hat{A}^1 for the rest of the period is

$$(55) (\lambda - K) \int_{\Theta} u(\hat{A}^1, \theta) f(\theta | X^1) d\theta - u(c).$$

The maximum expected net utility for the rest of the decision procedure is

$$(56) W_1^1 = \max [(\lambda - K) \int_{\Theta} u(\hat{A}^1, \theta) f(\theta | X^1) d\theta - u(c), E\{\omega^z | X^1\}].$$

A stopping set for the first stage is the set of all (\hat{A}^1, X^1) that satisfy

$$(57) (\lambda - K) \int_{\Theta} u(\hat{A}^1, \theta) f(\theta | X^1) d\theta - u(c) > K \int_{\Theta} u(\tilde{A}, \theta) f(\theta | X^1) d\theta + (\lambda - 2K) \int_{\Theta} \int_{Y^z} u(\hat{a}^z(\gamma^z; X^1), \theta) f(\gamma^z, \theta | X^1) d\gamma^z d\theta - u(zc).$$

Define a conditional stopping set for each \hat{A}^1 by

$$(58) S^1(\hat{A}^1) = \{X^1 : (\lambda - K) \int_{\Theta} u(\hat{A}^1, \theta) f(\theta | X^1) d\theta - u(c) > K \int_{\Theta} u(\tilde{A}, \theta) f(\theta | X^1) d\theta + (\lambda - 2K) \int_{\Theta} \int_{Y^z} u(\hat{a}^z(\gamma^z; X^1), \theta) f(\gamma^z, \theta | X^1) d\gamma^z d\theta - u(zc).\}$$

If a team has a choice of implementing \hat{A}^1 or allowing \tilde{A} to be effective while more information is gathered, the decision depends on whether or not the ex post message X^1 falls in $S^1(\hat{A}^1)$.

Backing up to the previous stage, the team has a choice of making the decision \hat{A}^0 effective for the entire period and receiving net expected utility

$$(59) \lambda \int_{\Theta} u(\hat{A}^0, \theta) \phi(\theta) d\theta$$

or allowing \tilde{A} to be effective while at least one IGO is made.

The expected utility of continuation is

$$(60) K \int_{\Theta} u(\tilde{A}, \theta) \phi(\theta) d\theta + E\{W_1^1\}$$

where

$$(61) E\{\omega_1^1\} = \int_{\theta} \int_{y^1} \int_{x^1 \in S^1(y^1)} ((\lambda - \kappa) u(\hat{a}^1(y^1), \theta) - u(c)) \phi(y^1, x^1 | \theta) \phi(\theta) dx^1 dy^1 d\theta \\ + \int_{\theta} \int_{y^2} \int_{y^1} \int_{x^1 \notin S^1(y^1)} (\kappa u(\hat{A}, \theta) + (\lambda - 2\kappa) u(\hat{a}^2(y^2, x^1), \theta) - u(zc)) \\ \phi(y^1, x^1 | \theta) \phi(y^2 | \theta) \phi(\theta) dx^1 dy^1 dy^2 d\theta.$$

If $\lambda \int_{\theta} u(\hat{A}^0, \theta) \phi(\theta) d\theta > \kappa \int_{\theta} u(\hat{A}, \theta) \phi(\theta) d\theta + E\{\omega_1^1\}$ then no information is gathered. Otherwise at least one IGO is made.

Thirdly, suppose the sequential team can take as the interim action the tentative action of the previous IGO. If the first IGO has been taken and the team decides to continue gathering information, the expected net utility of continuation is

$$(62) \kappa \int_{\theta} u(\hat{A}^1, \theta) f(\theta | x^1) d\theta + (\lambda - 2\kappa) \int_{\theta} \int_{y^2} u(\hat{a}^2(y^2; x^1), \theta) f(y^2, \theta | x^1) dy^2 d\theta - u(zc).$$

If the second information gathering operation is not made, the expected utility of implementing \hat{A}^1 for the rest of the period is

$$(63) (\lambda - \kappa) \int_{\theta} u(\hat{A}^1, \theta) f(\theta | x^1) d\theta - u(c).$$

The maximum expected utility at this stage with at most one additional IGO is

$$(64) W_1^1 = \kappa \int_{\theta} u(\hat{A}^1, \theta) f(\theta | x^1) d\theta + \max [(\lambda - 2\kappa) \int_{\theta} u(\hat{A}^1, \theta) f(\theta | x^1) d\theta - u(c), \\ (\lambda - 2\kappa) \int_{\theta} \int_{y^2} u(\hat{a}^2(y^2; x^1), \theta) f(y^2, \theta | x^1) dy^2 d\theta - u(zc)].$$

A stopping set for the first stage is the set of all (\hat{A}^1, x^1) that satisfy

$$(65) (\lambda - 2\kappa) \int_{\theta} u(\hat{A}^1, \theta) f(\theta | x^1) d\theta - u(c) > (\lambda - 2\kappa) \int_{\theta} \int_{y^2} u(\hat{a}^2(y^2; x^1), \theta) \\ f(y^2, \theta | x^1) dy^2 d\theta - u(zc).$$

Define a conditional stopping set given \hat{A}^1 by

$$(66) S^1(\hat{A}^1) = \{x^1 : (\lambda - 2\kappa) \int_{\theta} u(\hat{A}^1, \theta) f(\theta | x^1) d\theta - u(c) > (\lambda - 2\kappa) \\ \int_{\theta} \int_{y^2} u(\hat{a}^2(y^2; x^1), \theta) f(y^2, \theta | x^1) dy^2 d\theta - u(zc)\}.$$

If a team has a choice of implementing \hat{A}^1 for the rest of the

period or allowing \hat{A}^1 to be effective only while the final IGO is made, the stopping rule depends only on whether or not the ex post message x^1 falls in $S^1(\hat{A}^1)$.

Backing up to the previous stage, the team has a choice of making \hat{A}^0 effective for the entire period and receiving net expected utility

$$(67) \quad \lambda \int_{\Theta} u(\hat{A}^0, \theta) \phi(\theta) d\theta$$

or allowing \hat{A}^0 to be effective only while an IGO is made. The expected utility of continuation is

$$(68) \quad \kappa \int_{\Theta} u(\hat{A}^0, \theta) \phi(\theta) d\theta + E\{\omega_1\},$$

where

$$(69) \quad E\{\omega_1\} = \int_{\Theta} \int_{y^1} \int_{x^1 \in S^1(y^1)} [(\lambda - \kappa) u(\hat{a}^1(y^1), \theta) - u(z)] \phi(y^1, x^1 | \theta) \phi(\theta) dx^1 dy^1 d\theta \\ + \int_{\Theta} \int_{y^1} \int_{y^2} \int_{x^1 \notin S^1(y^1)} [\kappa u(\hat{a}^1(y^1), \theta) + (\lambda - \kappa) u(\hat{a}^2(y^2; x^1), \theta) - u(z)] \\ \phi(y^1, x^1 | \theta) \phi(y^2 | \theta) \phi(\theta) dx^1 dy^2 dy^1 d\theta.$$

If $(\lambda - \kappa) \int_{\Theta} u(\hat{A}^0, \theta) \phi(\theta) d\theta > E\{\omega_1\}$ then no information will be gathered. Otherwise at least one IGO is made.

III. AN EXAMPLE OF SEQUENTIAL TEAM DECISION MAKING

a. Joint Production With Quadratic Costs The organization modeled in this example is a business firm producing two commodities, $Q=(q_1, q_2)'$, when their prices, $P=(p_1, p_2)'$, are unknown. (7) Costs are quadratic $C(Q)=Q'CQ=c_{11}q_1^2+2c_{12}q_1q_2+c_{22}q_2^2$ which implies profits are also quadratic:

$$(70) \quad \Pi(Q, P) = P'Q - Q'CQ$$

The firm is divided into two departments, each specializing in the production of one of the goods and each deciding on the output level of their commodity. Both departments have identical prior subjective beliefs about the probabilities of particular prices and both departments are "team players" interested only in maximizing total expected profits.

Each department i observes a price forecast z_i which is correlated with the unknown price p_i . The firm makes ex ante communications defined by the function η :

$$(71) \quad Y = \eta(Z) = \begin{pmatrix} \eta_{11}(z_1) & \eta_{12}(z_2) \\ \eta_{21}(z_1) & \eta_{22}(z_2) \end{pmatrix}.$$

Only one particular ex ante communication function will be studied here — no ex ante communication:

$$(72) \quad Y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} = \eta(Z) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}.$$

That is, the departments keep their information to themselves. An ex post communication is made based on the function μ :

$$(73) \quad X = \mu(z).$$

Only one particular ex ante communication will be studied here — complete ex post communication:

$$(74) \quad X = (x_1, x_2) = \mu(z) = (z_1, z_2).$$

Thus each department knows both price forecasts ex post.

b. Normal Variables and Posterior Distributions The random variables P and Z are distributed normally. This contradicts the real world fact that prices must be non-negative, but normality (along with quadratic profits) implies linear decision functions which makes computation simpler.

The joint prior p.d.f. for p_1 and p_2 is a binormal density with zero means and a variance-covariance matrix $\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$:

$$(75) \quad \phi(p_1, p_2) = (2\pi)^{-1} (1-r^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{1}{1-r^2} (p_1^2 - 2rp_1p_2 + p_2^2)\right).$$

The i^{th} department observes a forecast z_i which has a normal distribution with mean= p_i and variance=1.

$$(76) \quad \phi_i(z_i|P) = \phi_i(z_i|p_i) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} (z_i - p_i)^2\right).$$

Hence the ex ante information y_i is also normal with mean= p_i and variance=1.

$$(77) \quad \phi_i(y_i|P) = \phi_i(y_i|p_i) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} (y_i - p_i)^2\right).$$

The ex post message (x_1, x_2) is distributed jointly normal with mean= (p_1, p_2) and variance-covariance= $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

$$(78) \quad \phi(x_1, x_2|P) = (2\pi)^{-1} \exp\left(-\frac{1}{2} (x_1 - p_1)^2 - \frac{1}{2} (x_2 - p_2)^2\right).$$

The following posterior p.d.f. are used in calculating optimal decision functions and stopping rules for the firm.

$$(79) f(p_1 | y_1) = (2\pi)^{-\frac{1}{2}} \sqrt{2} \exp\left(-\frac{1}{2} \left(p_1 - \frac{y_1}{2}\right)^2\right)$$

$$(80) f(p_2 | y_2) = (2\pi)^{-\frac{1}{2}} \sqrt{2} \exp\left(-\frac{1}{2} \left(p_2 - \frac{y_2}{2}\right)^2\right)$$

$$(81) f(y_2 | y_1) = (2\pi)^{-\frac{1}{2}} \sqrt{\frac{2}{4-r^2}} \exp\left(-\frac{1}{2} \frac{2}{4-r^2} \left(y_2 - \frac{r}{2} y_1\right)^2\right)$$

$$(82) f(y_1 | y_2) = (2\pi)^{-\frac{1}{2}} \sqrt{\frac{2}{4-r^2}} \exp\left(-\frac{1}{2} \frac{2}{4-r^2} \left(y_1 - \frac{r}{2} y_2\right)^2\right).$$

Let superscripts denote different IGOs.

$$(83) f(P | X_1^1, X_2^1) = (2\pi)^{-1} \sqrt{\frac{4-r^2}{1-r^2}} \exp\left(-\frac{1}{2} [(P_1 - \bar{P}_1), (P_2 - \bar{P}_2)]' \Sigma^{-1} [(P_1 - \bar{P}_1), (P_2 - \bar{P}_2)]\right)$$

$$\Sigma = \begin{pmatrix} \frac{3-2r^2}{9-4r^2} & \frac{r}{9-4r^2} \\ \frac{r}{9-4r^2} & \frac{3-2r^2}{9-4r^2} \end{pmatrix} \quad ; \text{ var-cov matrix}$$

$$\bar{P}_1 = \frac{3-2r^2}{9-4r^2} X_1^1 + \frac{r}{9-4r^2} X_2^1 \quad ; \text{ mean of } P_1$$

$$\bar{P}_2 = \frac{r}{9-4r^2} X_1^1 + \frac{3-2r^2}{9-4r^2} X_2^1 \quad ; \text{ mean of } P_2$$

$$(84) f(P | y_1^2, X_1^1, X_2^1) = (2\pi)^{-1} \sqrt{\frac{6-2r^2}{1-r^2}} \exp\left(-\frac{1}{2} [(P_1 - \bar{P}_1), (P_2 - \bar{P}_2)]' \Sigma_1^{-1} [(P_1 - \bar{P}_1), (P_2 - \bar{P}_2)]\right)$$

$$\Sigma_1 = \begin{pmatrix} \frac{2-r^2}{6-2r^2} & \frac{r}{6-2r^2} \\ \frac{r}{6-2r^2} & \frac{3-2r^2}{6-2r^2} \end{pmatrix} \quad ; \text{ var-cov matrix}$$

$$\bar{P}_1 = \frac{2-r^2}{6-2r^2} (y_1^2 + X_1^1) + \frac{r}{6-2r^2} X_2^1 \quad ; \text{ mean of } P_1$$

$$\bar{P}_2 = \frac{r}{6-2r^2} (y_1^2 + X_1^1) + \frac{3-2r^2}{6-2r^2} X_2^1 \quad ; \text{ mean of } P_2$$

$$(85) f(P | y_2^2, X_1^1, X_2^1) = (2\pi)^{-1} \sqrt{\frac{6-2r^2}{1-r^2}} \exp\left(-\frac{1}{2} [(P_1 - \bar{P}_1), (P_2 - \bar{P}_2)]' \Sigma_2^{-1} [(P_1 - \bar{P}_1), (P_2 - \bar{P}_2)]\right)$$

$$\Sigma_2 = \begin{pmatrix} \frac{3-2r^2}{6-2r^2} & \frac{r}{6-2r^2} \\ \frac{r}{6-2r^2} & \frac{2-r^2}{6-2r^2} \end{pmatrix} \quad ; \text{ var-cov matrix}$$

$$\bar{p}_1 = \frac{3-2r^2}{6-2r^2} X_1' + \frac{r}{6-2r^2} (y_2^2 + X_2') \quad ; \text{ mean of } p_1$$

$$\bar{p}_2 = \frac{r}{6-2r^2} X_1' + \frac{2-r^2}{6-2r^2} (y_2^2 + X_2') \quad ; \text{ mean of } p_2$$

$$(86) f(y_2^2 | y_1^2, X_1', X_2') = (2\pi)^{-\frac{1}{2}} \sqrt{\frac{6-2r^2}{9-4r^2}} \exp\left(-\frac{1}{2} \frac{6-2r^2}{9-4r^2} (y_2^2 - \bar{y}_2^2)^2\right)$$

$$\bar{y}_2^2 = \frac{r}{6-2r^2} (y_1^2 + X_1') + \frac{3-2r^2}{6-2r^2} X_2' \quad ; \text{ mean of } y_2^2$$

$$(87) f(y_1^2 | y_2^2, X_1', X_2') = (2\pi)^{-\frac{1}{2}} \sqrt{\frac{6-2r^2}{9-4r^2}} \exp\left(-\frac{1}{2} \frac{6-2r^2}{9-4r^2} (y_1^2 - \bar{y}_1^2)^2\right)$$

$$\bar{y}_1^2 = \frac{3-2r^2}{6-2r^2} X_1' + \frac{r}{6-2r^2} (y_2^2 + X_2') \quad ; \text{ mean of } y_1^2$$

c. Cost and Time of Communication It now remains to quantify the cost and time of communication by picking the functions $C(\eta, \mu)$ and $K(\eta, \mu)$. Define a matrix $\chi = (\chi_{ij})$ such that $\chi_{ii} = 0$, and for $i \neq j$:

$$\chi_{ij} = \begin{cases} 1 & \text{if department } i \text{ sends an ex ante message to department } j \\ 0 & \text{if department } i \text{ does not send an ex ante message to department } j \end{cases}$$

Define a vector $\Psi = (\psi_i)$ such that

$$\psi_i = \begin{cases} 1 & \text{if } \partial \mu / \partial z_i \neq 0 \text{ ; i.e., the ex post message depend on } z_i \\ 0 & \text{if } \partial \mu / \partial z_i = 0. \end{cases}$$

Assume that all departmental observations, z_1 , take place simultaneously and last an equal length of time τ_0 . Each department can make only one communication at a time. Suppose that each ex ante message, regardless of its content, takes an equal length of time to transmit τ_1 . Suppose that an ex post message takes a length of time to transmit that depends linearly on $\sum_{i=1}^N \psi_i$. If sending and receiving cannot be done simultaneously, define a function $K(\eta, \mu)$ as the time that will allow all departments to complete their communications:

$$(88) \quad K(\eta, \mu) = \tau_0 + \tau_1 \max_i \left[\sum_{j=1}^N \chi_{ij} + \sum_{j=1}^N \chi_{ji} \right] + \tau_2 \sum_{i=1}^N \psi_i.$$

Suppose that each departmental observation cost C_0 and each ex ante message cost an equal amount C_1 . Suppose also that the cost of an ex post message depend linearly on how many observations influence its value. The cost of an IGO is

$$(89) \quad C(\eta, \mu) = N C_0 + C_1 \sum_{i=1}^N \sum_{j=1}^N \chi_{ij} + C_2 \sum_{i=1}^N \psi_i.$$

In the particular case that $N=2$ and there is no ex ante communication but complete ex post communication, we have the following time and cost functions for an IGO:

$$(90) \quad K(\eta, \mu) = \tau_0 + 2 \tau_2$$

$$(91) \quad C(\eta, \mu) = 2 C_0 + 2 C_2.$$

To complete the specification of the model, the disutility function of the cost of communication must be specified. Let it be linear in total cost with a unit coefficient: $-U(C_{\text{total}}) = -C_{\text{total}}$

The total utility function is then equal to net profits.

$$(92) \quad \Pi_{\text{net}} = P'Q - Q'CQ - C_{\text{total}}.$$

In this example the length of period is such that at most only two IGOs can be made; i.e.,

$$(93) \quad 2(\tau_0 + 2\tau_2) < \lambda < 3(\tau_0 + 2\tau_2).$$

d. Optimal Decision Functions If the firm makes only one IGO and then decides on output levels, what is the optimal relationship between ex ante messages and output levels? As derived in the previous chapter they are linear functions of the ex ante messages.

$$(94a) \quad \hat{q}_1^1 = \hat{\sigma}_1^1(y_1^1) = \hat{s}_1^1 y_1^1$$

$$(94b) \quad \hat{q}_2^1 = \hat{\sigma}_2^1(y_2^1) = \hat{s}_2^1 y_2^1$$

where the coefficients are

$$(95a) \quad \hat{s}_1^1 = \frac{1}{2} \frac{2c_{22} - v c_{12}}{4c_{11}c_{22} - r^2 c_{12}^2}$$

$$(95b) \quad \hat{s}_2^1 = \frac{1}{2} \frac{2c_{11} - r c_{12}}{4c_{11}c_{22} - r^2 c_{12}^2}.$$

If both IGOs are made, the optimal decision functions are linear in the ex ante messages and the ex post information.

$$(96a) \quad \hat{q}_1^2 = \hat{\sigma}_1^2(y_1^2, X_1^1, X_2^1) = \hat{s}_1^2 y_1^2 + \hat{d}_{11} X_1^1 + \hat{d}_{12} X_2^1$$

$$(96b) \quad \hat{q}_2^2 = \hat{\sigma}_2^2(y_2^2, X_1^1, X_2^1) = \hat{s}_2^2 y_2^2 + \hat{d}_{21} X_1^1 + \hat{d}_{22} X_2^1$$

where the coefficients \hat{s}_1^2 and \hat{s}_2^2 are

$$(97a) \quad \hat{s}_1^2 = \frac{1}{2} (2 - v^2) \frac{(6 - 2v^2)c_{22} - v c_{12}}{(6 - 2v^2)^2 c_{11}c_{22} - v^2 c_{12}^2}$$

$$(97b) \quad \hat{s}_2^2 = \frac{1}{2} (2 - v^2) \frac{(6 - 2v^2)c_{11} - r c_{12}}{(6 - 2v^2)^2 c_{11}c_{22} - v^2 c_{12}^2}.$$

Given the values of \hat{S}_1^2 and \hat{S}_2^2 the values of \hat{d}_{11} and \hat{d}_{21} are found by solving the simultaneous linear equations:

$$(98a) \quad c_{11} \hat{d}_{11} + c_{12} \hat{d}_{21} = c_{11} \hat{S}_1^2$$

$$(98b) \quad c_{12} \hat{d}_{11} + c_{22} \hat{d}_{21} = \frac{1}{2} \frac{r}{6-2r^2} - c_{12} \frac{3-2r^2}{6-2r^2} \hat{S}_1^2.$$

The values of \hat{d}_{22} and \hat{d}_{12} are found by solving the simultaneous linear equations:

$$(99a) \quad c_{11} \hat{d}_{12} + c_{12} \hat{d}_{22} = \frac{1}{2} \frac{r}{6-2r^2} - c_{12} \frac{3-2r^2}{6-2r^2} \hat{S}_2^2$$

$$(99b) \quad c_{12} \hat{d}_{12} + c_{22} \hat{d}_{22} = c_{22} \hat{S}_2^2.$$

e. Optimal Stopping Rules Without Interim Actions Suppose that the firm must make a single production run, producing both goods q_1 and q_2 , and then it sell the commodities at the prices established on the market day. Before beginning the production process the firm's departments can observe industrial reports, newspapers, etc. The time constraints are such that this information gathering can be done twice at most before actual production must begin. This is a sequential team problem without opportunity losses. The communication system has been described above.

If the first IGO has been made and the second one will be made, the expected net profits of continuation is

$$(100) \quad E \{ \Pi_0^2 | x_1', x_2' \} = E \{ P' \hat{\sigma}^2 - \hat{\sigma}^2' C \hat{\sigma}^2 | x_1', x_2' \} - 4(C_0 + C_2) \\ = \frac{1}{2} (k_{11} x_1'^2 + k_{12} x_1' x_2' + k_{22} x_2'^2) + \frac{1}{2} (\hat{S}_1^2 + \hat{S}_2^2) \\ - 4(C_0 + C_2)$$

where k_{11}, k_{12}, k_{22} are constants that depend only on the parameter c_{11}, c_{12}, c_{22} and r . See the appendix for the specific values of all

coefficient in these examples.

If the second IGO is not made, the expected net profit of implementing the tentative output levels $\hat{Q}^1 = \hat{Q}^1(\gamma^1)$ is

$$(101) E \{ P^1 \hat{Q}^1 - \hat{Q}^1 C \hat{Q}^1 | X_1^1, X_2^1 \} - 2(C_0 + C_2) = l_1 X_1^1 + l_2 X_2^1 - \hat{Q}^1 C \hat{Q}^1 - 2(C_0 + C_2)$$

where l_1 and l_2 are constants that depends only on \hat{Q}^1 and r . It is unprofitable to continue gathering information if

$$(102) E \{ P^1 \hat{Q}^1 - \hat{Q}^1 C \hat{Q}^1 | X_1^1, X_2^1 \} - 2(C_0 + C_2) > E \{ \Pi_0^2 | X_1^1, X_2^1 \}.$$

The stopping set for each tentative action Q^1 is defined by

$$(103) S^1(\hat{Q}^1) = \{ X_1^1, X_2^1 : D(X_1^1, X_2^1) > 0 \},$$

where $D(X_1^1, X_2^1)$ is a quadratic function of the ex post messages.

$$(104) D(X_1^1, X_2^1) = u_{11} X_1^1{}^2 + u_{12} X_1^1 X_2^1 + u_{22} X_2^1{}^2 + u_1 X_1^1 + u_2 X_2^1 + u_0.$$

If the firm has a choice of implementing \hat{Q}^1 or continuing the gathering of information, it need only look to see if its ex post forecast makes $D(X_1^1, X_2^1)$ positive.

Should the firm make even one IGO? If it does not, it must pick output levels \hat{Q}^0 that maximize net profits against the prior beliefs. The expected net profits of doing this is

$$(105) E \{ P^1 \hat{Q}^0 - \hat{Q}^0 C \hat{Q}^0 \} = \hat{Q}^0 C \hat{Q}^0.$$

Because the prior mean of both prices is zero, the first order conditions for \hat{Q}^0 requires that $\hat{Q}^0 C \hat{Q}^0 = 0$. Hence net profits of making no observations is zero. If the net profits of making at least one IGO is non-negative then the first observation should be made.

There is an exact correspondence between ex ante and ex post messages in this problem: $\gamma^1 = \eta(z^1) = \mu(z^1) = X^1$. The decision of department one depends only on y_1^1 , but, ex post, it knows the

value of y_2^1 . Looking ahead before any observations made, the decision to make one or two IGOs depends solely on the observed values of y_1^1 and y_2^1 . Given the γ^1 , \hat{Q}^1 is determined exactly via the decision function $\hat{\sigma}^1(\gamma^1)$, and the ex post message X^1 is determined exactly by $X^1 = \gamma^1$. The stopping rule depends only on γ^1 and can be formulated as follows: the coefficients in the function D are such

$$(106a) \quad u_1 = h_{11} y_1^1 + h_{12} y_2^1$$

$$(106b) \quad u_2 = h_{21} y_1^1 + h_{22} y_2^1$$

$$(106c) \quad u_0 = h_{11}^0 y_1^{1^2} + h_{12}^0 y_1^1 y_2^1 + h_{22}^0 y_2^{1^2} + h_0^0$$

The function D can be rewritten, using $X^1 = \gamma^1$, as

$$(107) \quad F(\gamma^1) = \omega_{11} y_1^{1^2} + \omega_{12} y_1^1 y_2^1 + \omega_{22} y_2^{1^2} + \omega_0.$$

For a given ex ante price forecast γ^1 the firm will make exactly one IGO if $F(\gamma^1) > 0$. The firm will make two IGOs if $F(\gamma^1) \leq 0$. The net profits of making at least one IGO and not more than two IGOs is

$$(108) \quad E\{\Pi_1^1\} = \int_P \int_{F(\gamma^1) > 0} (P' \hat{\sigma}^1(\gamma^1) - \hat{\sigma}^1(\gamma^1)' C \hat{\sigma}^1(\gamma^1) - 2(C_0 + C_2)) \phi(P, \gamma^1) d\gamma^1 dP \\ + \int_P \int_{y_2} \int_{F(\gamma^1) \leq 0} (P' \hat{\sigma}^2(\gamma^2; \gamma^1) - \hat{\sigma}^2(\gamma^2; \gamma^1)' C \hat{\sigma}^2(\gamma^2; \gamma^1) - 4(C_0 + C_2)) \\ \phi(P, \gamma^1, \gamma^2) d\gamma^1 d\gamma^2 dP.$$

If $E\{\Pi_1^1\} \geq 0$ then the first IGO should be made.

f. Optimal Stopping Rule With Adapting Interim Actions Suppose that the firm can produce both goods continuously over the time interval $0 \leq t \leq \lambda$ but can make at most two IGOs. At the end of the production period all produced goods are sold at the same price. If the firm decides to produce Q goods per unit time for length of time τ , the total profits for that period are $\tau(P'Q - Q'CQ)$. This is a sequential team with opportunity losses and adapting

interim actions.

If the first IGO has been made and the second one will be made, the expected net profits of continuation is

$$(109) (\tau_0 + 2\tau_2) E\{P'\hat{Q}^1 - \hat{Q}^1'C\hat{Q}^1 | X_1^1, X_2^1\} + E\{\pi_0^2 | X_1^1, X_2^1\} = \\ (\tau_0 + 2\tau_2)(\ell_1 X_1^1 + \ell_2 X_2^1 - \hat{Q}^1'C\hat{Q}^1) + (\lambda - 2\tau_0 - 4\tau_2)^{\frac{1}{2}}(R_{11}X_1^1{}^2 + R_{12}X_1^1X_2^1 \\ + R_{22}X_2^1{}^2 + \hat{S}_1^2 + \hat{S}_2^2) - 4(C_0 + C_2).$$

If the second IGO is not made, the expected profits of implementing \hat{Q}^1 for the remainder of the period is

$$(110) (\lambda - \tau_0 - 2\tau_2)(\ell_1 X_1^1 + \ell_2 X_2^1 - \hat{Q}^1'C\hat{Q}^1) - 2(C_0 + C_2).$$

It is unprofitable to continue gathering information if (110) >

(109). The stopping set for each tentative action \hat{Q}^1 is defined as

$$(111) S^1(\hat{Q}^1) = \{X_1^1, X_2^1 : G(X_1^1, X_2^1) > 0\}$$

where $G(X_1^1, X_2^1)$ is a quadratic function of the ex post messages:

$$(112) G(X_1^1, X_2^1) = v_{11}X_1^1{}^2 + v_{12}X_1^1X_2^1 + v_{22}X_2^1{}^2 + v_1X_1^1 + v_2X_2^1 + v_0.$$

If the firm has a choice of fixing outputs at Q^1 for the rest of the period or gathering more information, it need only look to see if its ex post price forecast makes $G(X_1^1, X_2^1)$ positive.

Should even one IGO be made? If no information is gathered, the expected profits for the entire period is zero. If the firm

begins gathering information the expected profits of continuation is

$$(113) E\{\pi_1^1\} = \int_P \int_{H(Y^1) > 0} ((\lambda - \tau_0 - 2\tau_2)(P'\hat{\sigma}^1 - \hat{\sigma}^1'C\hat{\sigma}^1) - 2(C_0 + C_2)) \phi(P, Y^1) dY^1 dP \\ + \int_P \int_{H(Y^1) \leq 0} ((\tau_0 + 2\tau_2)(P'\hat{\sigma}^1 - \hat{\sigma}^1'C\hat{\sigma}^1) + (\lambda - 2\tau_0 - 4\tau_2) \\ (P'\hat{\sigma}^2 - \hat{\sigma}^2'C\hat{\sigma}^2) - 4(C_0 + C_2)) \phi(P, Y^1, Y^2) dY^1 dY^2 dP$$

where

$$(114) H(Y^1) = \xi_{11}Y_1^1{}^2 + \xi_{12}Y_1^1Y_2^1 + \xi_{22}Y_2^1{}^2 + \xi_0.$$

If $E\{\pi_1^1\} \geq 0$ then the first IGO should be made.

Appendix : Coefficients in Stopping Rules

$$R_{11} = \hat{s}_1^2 \left(\frac{2-v^2}{4-v^2} \right)^2 + \hat{s}_2^2 \left(\frac{v}{4-v^2} \right)^2 + \hat{d}_{11} \frac{2-v^2}{4-v^2} + \hat{d}_{21} \frac{v}{4-v^2}$$

$$R_{12} = (\hat{s}_1^2 + \hat{s}_2^2) \frac{2v(2-v^2)}{(4-v^2)^2} + (\hat{d}_{11} + \hat{d}_{22}) \frac{v}{4-v^2} + (\hat{d}_{12} + \hat{d}_{21}) \frac{2-v^2}{4-v^2}$$

$$R_{22} = \hat{s}_1^2 \left(\frac{v}{4-v^2} \right)^2 + \hat{s}_2^2 \left(\frac{2-v^2}{4-v^2} \right)^2 + \hat{d}_{12} \frac{v}{4-v^2} + \hat{d}_{22} \frac{2-v^2}{4-v^2}$$

$$l_1 = \hat{\beta}_1^1 \frac{2-v^2}{4-v^2} + \hat{\beta}_2^1 \frac{v}{4-v^2}$$

$$l_2 = \hat{\beta}_1^1 \frac{v}{4-v^2} + \hat{\beta}_2^1 \frac{2-v^2}{4-v^2}$$

$$u_{11} = -1/2 R_{11}$$

$$u_{12} = -1/2 R_{12}$$

$$u_{22} = -1/2 R_{22}$$

$$u_1 = l_1$$

$$u_2 = l_2$$

$$u_0 = 2(c_0 + c_2) - \hat{q}^1 \cdot c \hat{q}^1 - 1/2(\hat{s}_2^2 + \hat{s}_1^2)$$

$$h_{11} = \hat{s}_1^1 \frac{2-v^2}{4-v^2}$$

$$h_{12} = \hat{s}_2^1 \frac{v}{4-v^2}$$

$$h_{21} = \hat{s}_1^1 \frac{v}{4-v^2}$$

$$h_{22} = \hat{s}_2^1 \frac{2-v^2}{4-v^2}$$

$$h_{11}^0 = -c_{11} (\hat{s}_1^1)^2$$

$$h_{12}^0 = -2c_{12} \hat{s}_1^1 \hat{s}_2^1$$

$$h_{22}^0 = -c_{22} (\hat{s}_2^1)^2$$

$$h_0^0 = 2(c_0 + c_2) - 1/2(\hat{s}_1^2 + \hat{s}_2^2)$$

$$w_{11} = u_{11} + h_{11} + h_{11}^0$$

$$w_{12} = u_{12} + h_{12} + h_{21} + h_{12}^0$$

$$w_{22} = u_{22} + h_{22} + h_{22}^0$$

$$w_0 = h_0^0$$

$$v_{11} = -(\lambda - 2\tau_0 - 4\tau_2) \frac{1}{2} R_{11}$$

$$v_{12} = -(\lambda - 2\tau_0 - 4\tau_2) \frac{1}{2} R_{12}$$

$$v_{22} = -(\lambda - 2\tau_0 - 4\tau_2) \frac{1}{2} R_{22}$$

$$v_1 = +(\lambda - 2\tau_0 - 4\tau_2) l_1$$

$$v_2 = +(\lambda - 2\tau_0 - 4\tau_2) l_2$$

$$v_0 = -(\lambda - 2\tau_0 - 4\tau_2) (\hat{\phi}'^1 c \hat{\phi}^1 + \frac{1}{2} (\hat{s}_1^2 + \hat{s}_2^2)) + z(c_0 + c_2)$$

$$\xi_{11} = (\lambda - 2\tau_0 - 4\tau_2) \omega_{11}$$

$$\xi_{12} = (\lambda - 2\tau_0 - 4\tau_2) \omega_{12}$$

$$\xi_{22} = (\lambda - 2\tau_0 - 4\tau_2) \omega_{22}$$

$$\xi_0 = z(c_0 + c_2) - \frac{1}{2} (\lambda - 2\tau_0 - 4\tau_2) (\hat{s}_1^2 + \hat{s}_2^2)$$

FOOTNOTES

- (1) See Wald(1947), Blackwell and Girshick(1954) and Jackson(1960).
- (2) Raiffa and Schlaifer(1961), Marschak and Radner(1972) and LaValle(1968).
- (3) This argument is drawn from DeGroot(1970).
- (4) DeGroot,(1970), pp: 277-78.
- (5) The knowledge of other teammate's tentative actions could be used to guess what ex ante messages they received and hence, could provide indirect information about θ . Just as ex post messages are not used in current decision making, it is assumed that the team does not readjust its actions to make use of this indirect information.
- (6) This formulation is derived from T.Marschak(1968).
- (7) This organization is also studied in the essay on multiperiod teams.

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ESAY THREE

ITERATIVE TEAM DECISION THEORY

ITERATIVE TEAM DECISION THEORY

I. INTRODUCTION

This essay examines one way the team might solve its decision problem. As noted in Essay One the optimization problem involving information is a infinite dimensional problem in general. The objective function, expected utility, is a functional and the instrument of optimization is a function defined on a set which generally has an infinite number of elements. In statistical decision theory the complexity of the problem can be reduced significantly by waiting until the information is known and then solving a scalar posterior problem. This procedure cannot be used in team decision theory because at no time before actions are selected will the teammates have identical information; therefore a posterior density cannot be computed and the problem cannot be reduce to a N-dimensional optimization. The differences in information force the optimality conditions to take on a "Nash equilibrium" character; the i^{th} teammate makes the optimal use of his information given the optimal decision rules of all other teammates. But how do the other teammates know their optimal decision rules if the i^{th} teammate is still computing his rule? The team's person-by-person optimality conditions are generally a system of N integral equations in N functions which must be solved simultaneously to get the best decision functions. How is this system solved?

Imagine that each teammate knows all the features of the team's problem -- utility function, communication function, conditional distribution of all observations and prior distribution of the state of nature. Each teammate can therefore express the integral equations that define the person-by-person optimal decision rules of other teammates in order to select his optimal action, he can compute all N optimal decision rules by solving the system of integral equations. Since there are assumed to be no differences in tastes and beliefs, each teammate's solution of the person-by-person conditions will be the same decision rules derived by any other teammate. However, the solution of the person-by-person optimality conditions may be very difficult. Even the easiest team model with quadratic utility and normal random variables, the solution of the person-by-person optimality conditions requires the inversion of a N-by-N matrix. In more complicated problems the integral equations may have no analytic solution. Therefore, the strong assumption that all teammates know all features of the problem still does not make the general problem tractable.

Suppose that the teammates are not quite as knowledgeable; perhaps the teammate 1 knows only those portions of the team problem that enter in the 1th person-by-person optimality condition. If the utility function can be written

$$U(A, \theta) = \sum_{i=1}^N U^i(a_i, \theta) + \sum_{i=1}^N \sum_{j=1}^N U^{ij}(a_i, a_j, \theta) + U^0(A, \theta)$$

it might be true that the 1th teammate only knows the functions $U^1, U^{11}, U^{12}, \dots, U^{1N}$ and U^0 . In this case no teammate can express

all the integral equations which define the decision functions, and therefore direct solution of the team problem is impossible. What prevents the team from pooling all their knowledge and then solving the system of integral equations? Unless the utility functions can be expressed in a small number of parameters, it may be very expensive to pool the information. In addition, the utility functions may be only implicitly known by the teammates; in the theory of the firm, a firm may know its technology but not its production function, which requires a previous maximization

These are two reasons that an iterative solution to the team problem may be required: direct solution may be cumbersome and costly or direct solution may be impossible due to technological decentralization. This essay analyses one iterative solution procedure for a quadratic-normal team problem. Since this procedure is based on the gradient algorithm, section II outlines the gradient procedure for functional problems. Section III shows how the procedure is formulated in a decision context with a single decision maker. Finally, section IV studies the iterative solution of the team problem.

II. OPTIMIZATION BY STEEPEST ASCENT

a. Gradient Methods in Calculus Consider the problem of finding an unconstrained maximum of a function $f(x_1, x_2, \dots, x_n)$. A necessary condition for a point $(\bar{x}_1, \dots, \bar{x}_n)$ to be a local maximum is that the partial derivatives vanish at that point as follows:

$$(1) \quad \frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \quad \dots, \quad \frac{\partial f}{\partial x_n} = 0.$$

If $f(x_1, \dots, x_n)$ is a concave function these conditions are both necessary and sufficient for $(\bar{x}_1, \dots, \bar{x}_n)$ to be a maximum.

How do we find this point $(\bar{x}_1, \dots, \bar{x}_n)$? We have n equations that can be solved simultaneously for the values of the n variables $\bar{x}_1, \dots, \bar{x}_n$. However in many cases the functional forms of the partial derivatives of $f(x_1, \dots, x_n)$ make such a solution computationally difficult and expensive. As a result numerical methods are used to determine the optimal values of x_1, \dots, x_n . The most straightforward numerical method is that of gradient or steepest ascent.

Gradient methods are iterative algorithms that begin with an approximate solution $x_1^0, x_2^0, \dots, x_n^0$ and move successively closer to satisfying the optimality conditions (1). Each coordinate is varied separately in such a way as to increase $f(x_1, \dots, x_n)$. If τ is the iteration number, the gradient method requires

$$(2) \quad x_i^{\tau+1} - x_i^\tau \quad \text{have the same sign as} \quad \frac{\partial f}{\partial x_i}(x_1^\tau, \dots, x_n^\tau) \quad i=1, 2, \dots, n$$

where the vector $(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n})$ evaluated at τ is the gradient of f . If adjustments take place continuously, the requirement

(2) is replaced by the differential equations:

$$(3) \quad \frac{dx_i^\tau}{d\tau} = h_i \frac{\partial f}{\partial x_i}(x_1^\tau, \dots, x_n^\tau)$$

where k_i is the adjustment speed.

There are many variations of this hill climbing technique. At each step the change in x is the direction of the steepest slope at that point on the hill. The value of k_1, k_2, \dots, k_n determines the magnitudes of the change. If $f(x_1, \dots, x_n)$ is strictly concave then the gradient process (3) converges to the maximum point $\bar{x}_1, \dots, \bar{x}_n$, although in general the convergence becomes slower the nearer the process gets to the optimum. There are other numerical methods, such as Newton's method, that involve higher order derivatives which converge faster near the optimum but may not have stability properties like the gradient method. This chapter will deal only with generalized gradient processes.

b. Gradient Methods of Solution in Optimal Control There is a wide literature on the generalization of gradient algorithms to problems of optimal control.⁽¹⁾ The decision and team theory problems can be looked at as special examples of optimal control theory and, hence, a summary of the gradient methods for optimal control are presented here.

Consider a system of differential equations

$$(4) \quad \dot{x} = f(x, u, t)$$

where x is the n -vector of state variables and $u(t)$ is the m -vector control function. The optimal control problem is to choose the control $u(t)$ to transfer the initial state $x_0 = x(t_0)$ in accordance with (4) to a final value at time T in such a way as to maximize the functional

$$(5) \quad J[u] = M(x(T), T) + \int_{t_0}^T L(x, u, t) dt.$$

The optimal control must satisfy the necessary conditions of Pontryagin:

Pontryagin's Maximum Principle: Let $u^*(t)$ be an optimal control and $x^*(t)$ its corresponding state trajectory. Then there exist adjoint functions $p_1(t), \dots, p_n(t)$ such that

(a) The functions $x^*(t)$ and $p(t)$ satisfy the Hamiltonian system of differential equations $\dot{x} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial x}$

where H , the Hamiltonian, is $H(x, p, u, t) = L + p'f$

(b) $u^*(t)$ maximizes $H(x^*, p, u, t)$; that is for all t

$$\frac{\partial H}{\partial u}(x^*, p, u^*, t) = 0$$

(c) At the terminal point $(T, x^*(T))$ the transversality conditions hold— $p(T) = \frac{\partial M}{\partial x^*}$, $H^*(T) = -\frac{\partial M}{\partial T}$.

The necessary conditions reduce to the following equations

$$(5) \quad \dot{x} = \frac{\partial H}{\partial p} = f(x, u, t), \quad x(t_0) = x_0$$

$$(6) \quad \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \frac{\partial f}{\partial x}' p, \quad p(T) = \frac{\partial M}{\partial x}$$

$$(7) \quad \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \frac{\partial f}{\partial u}' p = 0$$

The gradient procedure begins with an arbitrary control $\hat{u}(t)$ and corrects the non-optimality by adding a function $\delta u(t)$ to the control $\hat{u}(t)$ such that

$$(8) \quad J[\hat{u} + \delta u] > J[\hat{u}].$$

Let \hat{x} be the state trajectory found by solving (5) with $u = \hat{u}$. Let $\hat{x} + \Delta x$ denote a similar solution of (5) with $u = \hat{u} + \delta u$.

$$(9) \quad (\hat{x} + \Delta x) = f(\hat{x} + \Delta x, \hat{u} + \delta u, t), \quad \hat{x}(t_0) + \Delta x(t_0) = x_0.$$

It follows that

$$(10) \quad \Delta \dot{x} = f(\hat{x} + \Delta x, \hat{u} + \delta u, t) - f(\hat{x}, \hat{u}, t), \quad \Delta x(t_0) = 0.$$

Making a Taylor series approximation of the right-hand side of (10) we linearize the differential equation to find

$$(11) \quad \dot{\delta x} \cong \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial u} \delta u, \quad \delta x(t_0) = 0.$$

How does the value of the objection change when \hat{u} is modified?

A linear approximation is

$$(12) \quad \delta J = J[\hat{u} + \delta u] - J[\hat{u}] \cong \frac{\partial M}{\partial x} \delta x(T) + \int_{t_0}^T \left[\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial u} \delta u \right] dt.$$

Making use of substitutions from (6), (7) and (11) and remembering

$$P(T) = \frac{\partial M}{\partial x} \quad \text{and} \quad \delta x(t_0) = 0, \quad \text{equation (12) becomes}$$

$$(13) \quad \delta J = \int_{t_0}^T \frac{\partial H}{\partial u} \delta u dt$$

where $\frac{\partial H}{\partial u}$ is the gradient of H with respect to u . If the change

in control, δu , is chosen so that δJ is positive then we can increase the value of the objective functional. One way of doing this is to choose δu such that

$$(14) \quad \delta u = k \frac{\partial H}{\partial u}, \quad k > 0,$$

where k is analogous to the speed of adjustment. The change in performance is given by

$$(15) \quad \delta J = \int_{t_0}^T k \frac{\partial H}{\partial u} \frac{\partial H}{\partial u} dt \geq 0.$$

These first-order gradient methods usually move the value of the objective functional up rapidly the first few iterations, but as the optimal control is approached, the speed of convergence slows dramatically. Second-order algorithms are available analogous to Newton's method that converge more rapidly but which are computationally more difficult and which can diverge if the initial control is not near the optimum

c. Variational Derivatives in the Calculus of Variations The calculus of variations can be considered a subproblem in the the-

ory of optimal control. Indeed, it can be easily shown that Pontryagin's maximum principle implies the Euler and Legendre necessary conditions of the calculus of variations. Because the previous derivation of optimal team decisions was analogous to the calculus of variations, gradient methods of the calculus of variations. Hence, the gradient method in the notation of the calculus of variations will be presented rather than depend on the formulation given above for optimal control theory.

Suppose we desire to select the function $x(t)$ to maximize

$$(16) \quad J[x] = \int_0^T I(x, \dot{x}, t) dt.$$

Assume $x(t)$ is incremented by $h(t)$, which is different from zero only in the neighborhood of a point \bar{t} . Dividing the increment of the functional, $J[x+h] - J[x]$, by the area $\Delta \mathcal{A}$ lying between the curve $h(t)$ and the t -axis, we obtain the ratio

$$(17) \quad \frac{J[x+h] - J[x]}{\Delta \mathcal{A}}$$

Let $\Delta \mathcal{A}$ go to zero in such a way that both $\max |h(t)|$ and the length of the interval in which $h(t)$ is non-vanishing go to zero. If the ratio (17) converges to a limit, this limit is called the variational derivative of the functional $J[x]$ at the point \bar{t} and is denoted by $\frac{\delta J}{\delta x} \Big|_{t=\bar{t}}$. (2) The principal linear part of the variational derivative is

$$(18) \quad \begin{aligned} \frac{\delta J}{\delta x} \Big|_{t=\bar{t}} &= I_x(x, \dot{x}, \bar{t}) - \frac{d}{dt} (I_{\dot{x}}(x, \dot{x}, \bar{t})) \\ &= F(\ddot{x}(\bar{t}), \dot{x}(\bar{t}), x(\bar{t}), \bar{t}). \end{aligned}$$

The Euler necessary conditions of the calculus of variations require the extremal to satisfy

$$(19) \quad \frac{\delta J}{\delta x} \Big|_{t=\bar{t}} = 0 \quad \text{for all } 0 \leq \bar{t} \leq T.$$

If the function x is vector-valued, $x(t) = (x^1(t), \dots, x^n(t))$, a concept analogous to the gradient can be introduced. In addition, the functional can be a multiple integral. The variational gradient when X is an n -vector and t is a m -vector can be defined as follows.

We want to select functions $x^1(t_1, \dots, t_m)$, $x^2(t_1, \dots, t_m)$, \dots , $x^n(t_1, \dots, t_m)$ to maximize the functional

$$(20) \quad J[x^1, \dots, x^n] = \int_0^{T_1} \dots \int_0^{T_m} I(x^1, \dots, x^n, \nabla x^1, \dots, \nabla x^n, t_1, \dots, t_m) dt_1 \dots dt_m$$

where $\nabla x^i = (\frac{\partial}{\partial t_1} x^i(t_1, \dots, t_m), \dots, \frac{\partial}{\partial t_m} x^i(t_1, \dots, t_m))$.

Suppose we give $x^i(t_1, \dots, t_m)$ an increment $h^i(t_1, \dots, t_m)$. The change in the functional is

$$(21) \quad \Delta J = J[x^1+h^1, \dots, x^n+h^n] - J[x^1, \dots, x^n] \\ = \int_0^{T_1} \dots \int_0^{T_m} \{ I(x^1+h^1, \dots, x^n+h^n, \nabla(x^1+h^1), \dots, \nabla(x^n+h^n), t_1, \dots, t_m) \\ - I(x^1, \dots, x^n, \nabla x^1, \dots, \nabla x^n, t_1, \dots, t_m) \} dt_1 \dots dt_m.$$

A linear approximation of ΔJ around $x^1, \dots, x^n, \nabla x^1, \dots, \nabla x^n$ is

$$(22) \quad \Delta J \cong \int_0^{T_1} \dots \int_0^{T_m} \left[\sum_{i=1}^n (I_{x^i} h^i + \sum_{j=1}^m \frac{\partial I}{\partial x_{t_j}^i} \frac{\partial h^i}{\partial t_j}) \right] dt_1 \dots dt_m$$

Let us make the notation $\frac{\partial x^i}{\partial t_j} \equiv x_{t_j}^i$. Noting that $\frac{\partial}{\partial t_j} (\frac{\partial I}{\partial x_{t_j}^i} h^i) = \frac{\partial^2 I}{\partial x_{t_j}^i \partial t_j} h^i + \frac{\partial I}{\partial x_{t_j}^i} \frac{\partial h^i}{\partial t_j}$, we rewrite equation (22) as

$$(23) \quad \Delta J = \int_0^{T_1} \dots \int_0^{T_m} \left(\sum_{i=1}^n (I_{x^i} - \sum_{j=1}^m \frac{\partial^2 I}{\partial x_{t_j}^i \partial t_j}) h^i \right) dt_1 \dots dt_m + \\ \sum_{i=1}^n \int_0^{T_1} \dots \int_0^{T_m} \left(\sum_{j=1}^m \frac{\partial}{\partial t_j} (\frac{\partial I}{\partial x_{t_j}^i} h^i) \right) dt_1 \dots dt_m$$

The term $\sum_{i=1}^n \frac{\partial}{\partial t_j} (\frac{\partial I}{\partial x_{t_j}^i} h^i)$ is the divergence of $\frac{\partial I}{\partial x_{t_j}^i} h^i$ and the divergence theorem states that if Γ is a boundary of the region of integration then

$$(24) \quad \int_0^{T_1} \dots \int_0^{T_m} \text{Div}(\frac{\partial I}{\partial x_{t_j}^i} h^i) dt_1 \dots dt_m = \int_{\Gamma} h^i (\frac{\partial I}{\partial x_{t_1}^i}, \dots, \frac{\partial I}{\partial x_{t_m}^i}) \cdot d\sigma.$$

The arbitrary $h^i(t_1, \dots, t_m)$ can be selected to vanish on this boundary Γ so that

$$(25) \int_0^{T_1} \dots \int_0^{T_m} \text{Div} \left(\frac{\partial \mathcal{I}}{\partial x_i^j} h^i \right) dt_1 \dots dt_m = 0.$$

Hence the change in the functional is

$$\Delta \mathcal{J} = \sum_{i=1}^n \int_0^{T_1} \dots \int_0^{T_m} \left(\mathcal{I}_{x^i} - \sum_{j=1}^m \frac{\partial^2 \mathcal{I}}{\partial x_i^j \partial t_j} \right) h^i(t_1, \dots, t_m) dt_1 \dots dt_m.$$

The generalized Euler conditions are

$$\mathcal{I}_{x^i} - \sum_{j=1}^m \frac{\partial^2 \mathcal{I}}{\partial x_i^j \partial t_j} = 0 \quad i=1, 2, \dots, n.$$

The principle linear part of the variational partial derivative of a multiple integral functional is

$$(26) \frac{\delta \mathcal{J}}{\delta x^i} = \mathcal{I}_{x^i} - \sum_{j=1}^m \frac{\partial^2 \mathcal{I}}{\partial x_i^j \partial t_j} \quad i=1, 2, \dots, n.$$

Suppose the number of functions equals the number of independent integrands, $n=m$, and suppose the functions are restricted so that the i^{th} integrand, $x^i(t_1, \dots, t_m) = x^i(t_i)$.

Then the variational partial derivative is

$$(27) \frac{\delta \mathcal{J}}{\delta x^i} = \int_0^{T_1} \dots \int_0^{T_m} \left(\mathcal{I}_{x^i} - \sum_{j=1}^m \frac{\partial^2 \mathcal{I}}{\partial x_i^j \partial t_j} \right) dt_1 \dots dt_m$$

where the notation "(i)" implies the i^{th} integral does not appear.

d. Gradient Solution in the Calculus of Variations As in the gradient procedure of optimal control theory, the gradient procedure for the calculus of variations begin with an arbitrary function $\hat{x}(t)$ and corrects it by adding a function $\delta x(t)$ that increases the objective functional. If $\left. \frac{\delta \mathcal{J}}{\delta x} \right|_{\bar{t}}$ is positive then increasing $\hat{x}(\bar{t})$ will increase \mathcal{J} , and if $\left. \frac{\delta \mathcal{J}}{\delta x} \right|_{\bar{t}}$ is negative then decreasing $\hat{x}(\bar{t})$ will increase \mathcal{J} . One way to do this is to specify the $\delta x(t)$ function as follows:

$$(28) \delta x(t) = k \left. \frac{\delta \mathcal{J}}{\delta x} \right|_t \quad k > 0 \quad \text{for all } 0 \leq t \leq T$$

where

$$(29) \left. \frac{\delta \mathcal{J}}{\delta x} \right|_t = \mathcal{I}_x(\hat{x}(t), \dot{\hat{x}}(t), t) - \frac{d}{dt} \mathcal{I}_{\dot{x}}(\hat{x}(t), \dot{\hat{x}}(t), t) \\ = F(\hat{x}(t), \dot{\hat{x}}(t), \hat{x}(t), t)$$

and h is analogous to the speed of adjustment.

If we denote the iteration by τ , the change in the τ^{th} iteration is defined by

$$(30) \quad x^{\tau+1} - x^{\tau} = h F(\ddot{x}(t), \dot{x}(t), x(t), t)$$

or if the adjustment is continuous

$$(31) \quad \frac{dx^{\tau}(t)}{d\tau} = h F(\ddot{x}(t), \dot{x}(t), x(t), t).$$

If \mathbb{I} is strictly concave in x and \dot{x} then it appears that the gradient process is convergent.⁽³⁾ The process converges to an extremal that satisfies the Euler equations. Again we should notice that as we get closer to the optimum, the magnitude of $\frac{d\mathbb{I}}{dx}$ decreases and hence the speed of approach decreases. A second-order method could prevent this slowdown.

III. ITERATIVE SOLUTIONS IN STATISTICAL DECISION THEORY

a. Gradient Solutions with Quadratic Utility and Normal Densities

Statistical decision theory is related to the calculus of variations in that the objective is a functional and the instrument of optimization is a function. It differs from the traditional calculus of variations problem in that the felicity does not depend on rate of change of the decision function and the functional is a double integral where one integrand does not affect the decision function. The optimality conditions of decision theory are analogous but not identical to the Euler equations of the calculus of variations. Variational derivatives differ from those developed above. We will express the variational derivative in its general form below, but the remainder of the section will focus on the special case where utility is quadratic and random variables are normally distributed.

The statistical decision theory problem is to select a decision function $\alpha(y)$ to maximize expected utility:

$$(32) \quad W[\alpha] = \int_y \int_{\theta} u(\alpha(y), \theta) \phi(y|\theta) \phi(\theta) d\theta dy$$

where y is a sample value distributed conditionally by $\phi(y|\theta)$ and θ is the unknown state of nature distributed $\phi(\theta)$. The expected utility can be rewritten to look like

$$(33) \quad W[\alpha] = \int_y \left[\int_{\theta} u(\alpha(y), \theta) \phi(y|\theta) \phi(\theta) d\theta \right] dy \\ = \int_y I(\alpha(y), y) dy.$$

The variational derivative of $W[\alpha]$ is defined for each y as

$$(34) \quad \frac{\delta W[\alpha]}{\delta \alpha} = \frac{\partial I}{\partial \alpha} - \frac{d}{dy} \left(\frac{\partial I}{\partial \dot{\alpha}} \right) \\ = \frac{\partial I}{\partial \alpha} = \int_{\theta} u_{\alpha}(\alpha(y), \theta) \phi(y|\theta) \phi(\theta) d\theta.$$

The optimality conditions that α must satisfy are equivalent to $\frac{\delta W}{\delta \alpha} \Big|_y = 0$ for all y .

We will now make the assumptions that utility is quadratic and random variables are normally distributed.

$$(35) \quad U(A, \theta) = 2\mu A\theta - g A^2$$

$$(36) \quad \phi(y|\theta) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}(y-\theta)^2)$$

$$(37) \quad \phi(\theta) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}\theta^2).$$

Under these assumptions optimal decision rules will be linear in the information variable

$$(38) \quad \hat{\alpha}(y) = \hat{a} y.$$

Hence the only functional forms that need appear in the gradient process are linear and we can adjust them by adjusting the slope coefficient. The decision function of the τ iteration is denoted

$$(39) \quad \alpha^\tau(y) = a^\tau y.$$

The variational derivative for each y is then defined by

$$\begin{aligned} (40) \quad \frac{\delta W}{\delta \alpha} \Big|_y &= \int_{\theta} (2\mu\theta - 2g a^\tau y) \phi(y|\theta) \phi(\theta) d\theta \\ &= \int_{\theta} (2\mu\theta - 2g a^\tau y) f(\theta|y) d\theta \cdot \phi(y) \\ &= 2(\mu E\{\theta|y\} - g a^\tau y) \phi(y) \\ &= 2(\mu m_{\theta} y - g a^\tau y) \phi(y) \\ &= 2y \phi(y) (\mu m_{\theta} - g a^\tau) \end{aligned}$$

where m_{θ} is the constant such that $E\{\theta|y\} = m_{\theta} y$; in this example $m_{\theta} = \frac{1}{2}$.

If $\frac{\delta W}{\delta \alpha} > 0$ then we want to increase the value of $\alpha^\tau(y) = a^\tau y$. If $\frac{\delta W}{\delta \alpha} < 0$ then we want to decrease the value of $\alpha^\tau(y) = a^\tau y$. For a particular y , the sign of $\frac{\delta W}{\delta \alpha}$ depends only on the four coefficients μ, m_{θ}, g and a^τ . The gradient adjustment process can be

defined by the difference equation

$$(41) \quad a^{\tau+1} - a^{\tau} = \varepsilon(\mu m_{\theta} - \rho a^{\tau})$$

where $\varepsilon > 0$ is a parameter analogous to the speed of adjustment.

The solution of this difference equation if the initial decision function is $a^0(y) = a^0 y$ is

$$(42) \quad a^{\tau} = a^0 (1 - \varepsilon \rho)^{\tau} + \frac{\mu m_{\theta}}{\rho} (1 - (1 - \varepsilon \rho)^{\tau}).$$

If $a^0 = 0$ and $m_{\theta} = \frac{1}{2}$ then the solution is

$$(43) \quad a^{\tau} = \frac{\mu}{2\rho} (1 - (1 - \varepsilon \rho)^{\tau}).$$

If $\rho > 0$ and $\varepsilon \rho < 1$ then $\lim_{\tau \rightarrow \infty} a^{\tau} = \hat{a}$, the value of a that maximizes expected utility. Also note that

$$(44) \quad \lim_{\tau \rightarrow \infty} a^{\tau+1} - a^{\tau} = \lim_{\tau \rightarrow \infty} \varepsilon \mu m_{\theta} (1 - \varepsilon \rho)^{\tau} = 0.$$

That is, the adjustment process slows down as it gets closer to the optimum. Second-order methods could improve the speed of convergence.

If we treat the gradient adjustment process as a continuous procedure then the action coefficient should be adjusted according to the differential equation

$$(45) \quad \frac{da^{\tau}}{d\tau} = \varepsilon(\mu m_{\theta} - \rho a^{\tau}).$$

The solution of this first order differential equation with $a(0) = a^0$ is

$$(46) \quad a^{\tau} = a^0 e^{-\varepsilon \rho \tau} + \frac{\mu m_{\theta}}{\rho} (1 - e^{-\varepsilon \rho \tau}).$$

If $a^0 = 0$ and $m_{\theta} = \frac{1}{2}$ then the solution is

$$(47) \quad a^{\tau} = \frac{\mu}{2\rho} (1 - e^{-\varepsilon \rho \tau}).$$

If $\rho > 0$ then $\lim_{\tau \rightarrow \infty} a^{\tau} = \hat{a}$, the value of a that maximizes expected utility. Also notice that

$$(48) \quad \lim_{\tau \rightarrow \infty} \frac{da^{\tau}}{d\tau} = \lim_{\tau \rightarrow \infty} \frac{\varepsilon \mu}{2} e^{-\varepsilon \rho \tau} = 0.$$

The adjustment process slows down as it approaches the optimum.

b. Optimum Number of Iterations With the above formulation of the gradient solution of the statistical decision problem, let us look at the problem of selecting the number of iterations to complete before fixing the decision function. Suppose each successive approximation of the decision function costs C units of utility. The decision maker will select an optimum number of iterations of the adjustment process to maximize net expected utility.

$$(49) \text{ NET } W[\alpha^\tau] = W[\alpha^\tau] - C\tau.$$

The value of τ^* picked should satisfy

$$(50) W[\alpha^{\tau^*+1}] - W[\alpha^{\tau^*}] \leq C \leq W[\alpha^{\tau^*}] - W[\alpha^{\tau^*-1}].$$

If we imagine that the adjustment process is continuous, an approximate solution to this problem is to take τ^* to be the nearest integer to the number τ that satisfies

$$(51) \frac{dW[\alpha^\tau]}{d\tau} = C.$$

This states that the marginal increase in expected utility from refining the decision function should equal the marginal cost of refining the decision function.

How do we express $W[\alpha^\tau]$ as a function of τ ? If utility is quadratic and random variables are normal with $E\{\theta\} = 0$, then $W[\alpha^\tau]$ will depend only on the variance and covariances of θ and y . If $a^0 = 0$ then

$$\begin{aligned} (52) \quad W[\alpha^\tau] &= E\{2\mu a^\tau y \theta - \frac{1}{2} (a^\tau)^2 y^2\} \\ &= \frac{\mu^2}{2\delta} (2 \text{cov}(y, \theta) (1 - e^{-\delta \tau}) - \frac{1}{2} \text{var}(y) (1 - e^{-\delta \tau})^2) \\ &= \frac{\mu^2}{2\delta} (2(1 - e^{-\delta \tau}) - (1 - e^{-\delta \tau})^2). \end{aligned}$$

The marginal expected utility of more adjustment in the decision function is

$$(53) \quad \frac{dW(a^\tau)}{d\tau} = \frac{1}{2} \epsilon \mu^2 e^{-2\epsilon \eta \tau}.$$

The optimal number of iterations is the value of τ that satisfies

$$(54) \quad \frac{1}{2} \epsilon \mu^2 e^{-2\epsilon \eta \tau} = c.$$

This number is

$$(55) \quad \tau = \ln \left(\frac{\epsilon \mu^2}{2c} \right)^{\frac{1}{2\epsilon \eta}}.$$

Noting the fact that $\epsilon \eta c > 0$ we see that increasing the marginal cost of adjusting the decision function lowers the optimal number of iterations since $\frac{\partial \tau}{\partial c} = \frac{-1}{2\epsilon \eta c}$. Increasing the impact of the state of nature on utility (increasing μ) increases the number of iterations because this increases the distance between $a^0 = 0$ and the optimal decision coefficient $\hat{a} = \frac{\mu}{2\eta}$.

c. The Joint Selection of the Amount of Information and the Accuracy of the Decision Function

Suppose the statistician can not only select the number of iterations the gradient approximating process makes but can also select (non-sequentially) the number of observations. These two decisions cannot be made independently because they both affect expected utility in complex ways. Increasing the number of samples may or may not lead the statistician to less refined decision functions. Such tradeoffs will be studied here.

The statistician can draw some number of random samples, S , where each observation y is identically distributed

$$(56) \quad \phi(y|\theta) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(y-\theta)^2\right).$$

The state of nature is again distributed normally according to equation (37). Let $Y^s = (y^1, \dots, y^s)$ be the vector of observed samples.

Then the joint p.d.f. of Y^s and θ is

$$(57) \quad \phi(\theta, Y^s) = (2\pi)^{-\frac{s+1}{2}} \exp\left(-\frac{1}{2} [\theta \ Y^s]' \Sigma^{-1} [\theta \ Y^s]\right)$$

where

$$\Sigma = \begin{matrix} & \theta & y^1 & \dots & y^s \\ \begin{matrix} \theta \\ y^1 \\ \vdots \\ y^s \end{matrix} & \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 1 \\ & & \ddots & \vdots \\ 1 & 1 & \dots & 2 \end{bmatrix} & & & \end{matrix} \quad ; \text{ var-cov } (\theta, Y^s).$$

An important element in the following calculations is the expected value of θ given Y^s :

$$(58) \quad E\{\theta | y^1, \dots, y^s\} = \frac{1}{s+1} (y^1 + y^2 + \dots + y^s).$$

The optimal decision function will again be linear in the information:

$$(59) \quad \hat{\alpha}(Y^s) = \hat{a}^1 y^1 + \hat{a}^2 y^2 + \dots + \hat{a}^s y^s.$$

Because of the symmetry of the y^i 's each coefficient \hat{a}^s is the same as all others, so

$$(60) \quad \hat{\alpha}(Y^s) = \hat{a} (y^1 + y^2 + \dots + y^s).$$

Suppose that we have an approximate decision function $\alpha(Y^s) = a(y^1 + \dots + y^s)$; what is the expected utility of such a decision function?

$$\begin{aligned} (61) \quad W[\alpha(Y^s)] &= E\left\{ 2\mu a (y^1 + \dots + y^s) \theta - \rho a^2 (y^1 + \dots + y^s)^2 \right\} \\ &= 2\mu a \sum_{i=1}^s \text{cov}(\theta, y^i) - \rho a^2 \sum_{i=1}^s \sum_{j=1}^s \text{cov}(y^i, y^j) \\ &= 2\mu a s - \rho a^2 (s^2 + s). \end{aligned}$$

What is the optimal number of observations for a particular decision function $\alpha(Y^s) = a(y^1 + \dots + y^s)$ assuming observations are costless?

It is the integer \hat{s} that satisfies

$$(62) \quad W[\alpha(Y^{\hat{s}})] \geq W[\alpha(Y^s)] \quad \text{for all } s.$$

As an approximation this could be the nearest integer to the number that satisfies

$$(63) \quad \frac{dW[\alpha(Y^s)]}{ds} = 2\mu a - \rho a^2(2s+1) = 0.$$

This number is

$$(64) \quad s = \frac{\mu}{\rho a} - \frac{1}{2}.$$

Obviously the optimal choice of the decision function depends on the number of observations. If the optimal value of a is

$$(65) \quad \hat{a}(s) = \frac{\mu}{(s+1)\rho}$$

then

$$(66) \quad \frac{dW[\hat{a}(Y^s)]}{ds} = \frac{\mu^2}{\rho} \frac{1}{(s+1)^2} > 0.$$

If the observations are costless then an infinite number should be drawn. If each observation has a cost of K utility units then the net utility of a decision function $\alpha(Y^s) = a(y^1 + \dots + y^s)$ is

$$(67) \quad \text{NET } W[\alpha(Y^s)] = W[\alpha(Y^s)] - sK.$$

The optimal number of observations for a fixed value of a is the integer \hat{s} that satisfies

$$(68) \quad W[\alpha(Y^{\hat{s}+1})] - W[\alpha(Y^{\hat{s}})] \leq K \leq W[\alpha(Y^{\hat{s}})] - W[\alpha(Y^{\hat{s}-1})].$$

We might approximate this result by taking \hat{s} as the nearest integer to the number s^* that satisfies

$$(69) \quad \frac{dW[\alpha(Y^{s^*})]}{ds} = K.$$

This number is

$$(70) \quad s^* = \frac{\mu}{\rho a} - \frac{K}{2\rho a^2} - \frac{1}{2}$$

Again if \hat{a} is adjusted to take account of the change in s then

(70) is incorrect. The correct condition is

$$(71) \quad \frac{dW[\hat{a}(Y^{s^*})]}{ds} = \frac{\mu^2}{\rho} \frac{1}{(s^*+1)^2} = K$$

which yields a number

$$(72) \quad S^* = \frac{\mu}{\delta \rho K} - 1.$$

Now let us introduce gradient solutions of the S -observation statistical decision problem. There is no conceptual difference from the formulation of the previous subsection. The variational derivative from each Y^s is defined by

$$(73) \quad \left. \frac{\delta W[\alpha^\tau(Y^s)]}{\delta \alpha} \right|_{Y^s} = \int_{\Theta} (2\mu\theta - 2\rho a^\tau(y^1 + \dots + y^s)) f(\theta|Y^s) d\theta \phi(Y^s) \\ = 2(\mu E\{\theta|Y^s\} - \rho a^\tau(y^1 + \dots + y^s)) \phi(Y^s) \\ = 2\left(\frac{\mu}{s+1} - \rho a^\tau\right) (y^1 + \dots + y^s) \phi(Y^s)$$

The gradient adjustment process can be defined by the differential equation

$$(74) \quad \frac{da^\tau}{d\tau} = \varepsilon \left(\frac{\mu}{s+1} - \rho a^\tau \right).$$

If the initial value of a^0 is zero then the solution of this equation is

$$(75) \quad a^\tau = \frac{\mu}{(s+1)\rho} (1 - e^{-\varepsilon \rho \tau}).$$

If each iteration in the gradient process costs a fixed amount C utility units, then the net utility is

$$(76) \quad \text{NET } W[\alpha^\tau(Y^s)] = W[\alpha^\tau(Y^s)] - C\tau - SK \\ = \frac{\mu^2}{\rho} \frac{s}{s+1} (2(1 - e^{-\varepsilon \rho \tau}) - (1 - e^{-\varepsilon \rho \tau})^2) - C\tau - SK.$$

We can approximate the optimal integer solution by selecting the nearest integers to the numbers S^* and τ^* that satisfy the marginal conditions

$$(77a) \quad \frac{\partial W[\alpha^{\tau^*}(Y^{S^*})]}{\partial \tau} = C$$

$$(77b) \quad \frac{\partial W[\alpha^{\tau^*}(Y^{S^*})]}{\partial S} = K.$$

The Hessian of $W[\alpha^\tau(Y^s)]$ evaluated at τ^* , S^* is

$$(78) \quad \begin{pmatrix} \frac{\partial^2 W}{\partial \tau^2} & \frac{\partial^2 W}{\partial \tau \partial S} \\ \frac{\partial^2 W}{\partial S \partial \tau} & \frac{\partial^2 W}{\partial S^2} \end{pmatrix} = \begin{pmatrix} -2\varepsilon \rho C & \frac{C}{S^*(S^*+1)} \\ \frac{C}{S^*(S^*+1)} & \frac{-K}{S^*+1} \end{pmatrix}$$

which is a negative semidefinite if

$$(79) \quad 2(s^*)^2(s^*+1) \varepsilon \eta K > c.$$

We will assume that this holds at the point (s^*, τ^*) and hence this is a local maximum.

If the price of additional information is changed, the direction of the response is determined by

$$(80) \quad \text{SIGN} \frac{\partial s^*}{\partial K} = \text{SIGN} \frac{\partial^2 W}{\partial \tau^2} \Big|_{\tau^*, s^*} \leq 0$$

$$(81) \quad \text{SIGN} \frac{\partial \tau^*}{\partial K} = \text{SIGN} \frac{\partial^2 W}{\partial \tau \partial s} \Big|_{\tau^*, s^*} \geq 0.$$

This is not a surprising result; increasing the marginal cost of information lowers the amount of information acquired and increases the attention paid to that information. If the marginal cost of refining the decision function increases, the response is

$$(82) \quad \text{SIGN} \frac{\partial s^*}{\partial c} = \text{SIGN} \frac{\partial^2 W}{\partial \tau \partial s} \Big|_{\tau^*, s^*} \geq 0$$

$$(83) \quad \text{SIGN} \frac{\partial \tau^*}{\partial c} = \text{SIGN} \frac{\partial^2 W}{\partial s^2} \Big|_{\tau^*, s^*} \leq 0.$$

Again, these results corresponds to expectations.

IV. ITERATIVE TEAM DECISION THEORY

a. Gradient Solutions with Quadratic Utility and Normal Densities

The team decision problem is to select a team decision function $\alpha(Y) = (\alpha_1(y_1), \alpha_2(y_2), \dots, \alpha_N(y_N))'$ to maximize the team's expected utility

$$(84) \quad W[\alpha] = \int_Y \int_{\Theta} u(\alpha(Y), \theta) \phi(Y|\theta) \phi(\theta) d\theta dY$$

where Y is the information variable distributed conditionally by $\phi(Y|\theta)$ and θ is the unknown state of nature distributed $\phi(\theta)$.

The variational partial derivative of $W[\alpha]$ with respect to $\alpha_i(y_i)$ is

$$(85) \quad \left. \frac{\delta W[\alpha]}{\delta \alpha_i} \right|_{y_i} = \int_{Y(i)} \int_{\Theta} u_{\alpha_i}(\alpha(Y), \theta) \phi(Y|\theta) \phi(\theta) d\theta dY(i)$$

where $Y(i) = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_N)'$. The optimality conditions that $\alpha(Y)$ must satisfy are equivalent to

$$(86) \quad \left. \frac{\delta W[\alpha]}{\delta \alpha_i} \right|_{y_i} = 0 \text{ for each possible } y_i, i=1, 2, \dots, N.$$

We will make the assumptions that utility is quadratic and random variables are normally distributed. The ex ante communication system is completely decentralized, so no teammate communicates the results of his personal sample. Other information structures could be used in the study of iterative teams but the calculations become complicated.

$$(87) \quad u(A, \theta) = 2\mu'A\theta - A'QA \quad Q: \text{ positive definite}$$

$$(88) \quad \phi(y_i|\theta) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}(y_i - \theta)^2)$$

$$(89) \quad \phi(Y|\theta) = \prod_{i=1}^N \phi(y_i|\theta)$$

$$(90) \quad \phi(\theta) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}\theta^2)$$

Under these assumptions optimal decision rules are linear in the information variables.

$$(91) \quad \hat{\alpha}_i(y_i) = \hat{a}_i y_i \quad i=1, 2, \dots, N.$$

Thus the only functional forms that need to be explored in a gradient process are linear and the only parameters that need to be adjusted are the slope coefficients (a_1, \dots, a_N) . After τ iterations of the adjustment process the decision functions are denoted

$$(92) \quad \alpha_i^\tau(y_i) = a_i^\tau y_i \quad i=1,2,\dots,N.$$

The variational partial derivative for each y_i in the quadratic-normal problem is

$$\begin{aligned} (93) \quad \frac{\delta W[\alpha^\tau]}{\delta \alpha_i} \Big|_{y_i} &= \int_{y_i(i)} \int_{\theta} (2\mu_i \theta - 2\varphi_i \alpha^\tau(\gamma)) f(\theta, \gamma(i) | y_i) d\theta d\gamma(i) \phi(y_i) \\ &= 2(\mu_i E\{\theta | y_i\} - \varphi_{ii} a_i^\tau y_i - \sum_{j \neq i} \varphi_{ij} a_j^\tau E\{y_j | y_i\}) \phi(y_i) \\ &= 2(\mu_i m_{\theta i} y_i - \varphi_{ii} a_i^\tau y_i - \sum_{j \neq i} \varphi_{ij} a_j^\tau m_{j i} y_i) \phi(y_i) \\ &= (\mu_i m_{\theta i} - \varphi_{ii} a_i^\tau - \sum_{j \neq i} \varphi_{ij} m_{j i} a_j^\tau) 2 y_i \phi(y_i) \end{aligned}$$

where $m_{\theta i}$ and $m_{j i}$ are the constants such that $E\{\theta | y_i\} = m_{\theta i} y_i$ and $E\{y_j | y_i\} = m_{j i} y_i$. In this example $m_{\theta i} = \frac{1}{2}$ and $m_{j i} = \frac{1}{2}$.

If $\frac{\delta W}{\delta \alpha_i} > 0$ then we want to increase the value of $\alpha_i^\tau(y_i) = a_i^\tau y_i$. Similarly, if $\frac{\delta W}{\delta \alpha_i} < 0$ then we want to decrease the value of $\alpha_i^\tau(y_i) = a_i^\tau y_i$. For a particular y_i , the sign of $\frac{\delta W}{\delta \alpha_i}$ depends only on the coefficients $\mu_i, \varphi_{i1}, \dots, \varphi_{iN}, m_{\theta i}, m_{1i}, \dots, m_{Ni}$ and $a_1^\tau, \dots, a_N^\tau$. The gradient adjustment process can be defined by the difference equations

$$(94) \quad a_i^{\tau+1} - a_i^\tau = \epsilon_i (\mu_i m_{\theta i} - \varphi_{ii} a_i^\tau - \sum_{j \neq i} \varphi_{ij} m_{j i} a_j^\tau) \quad i=1,2,\dots,N$$

where $\epsilon_i > 0$ is the speed of adjustment. It should be noted that if each individual i computes his own change in his decision function using (94) then he does not need to know the values of the parameters $\mu_k, \varphi_{k1}, \dots, \varphi_{kN}, m_{\theta k}, m_{1k}, \dots, m_{Nk}$ $k \neq i$. The teammate could actually be ignorant to the team's objective function as long as he knew the coefficients related to himself.

We express the system of difference equations in (94) in matrix notation as follows

$$(95) \quad A^{\tau+1} - A^{\tau} = V - GA^{\tau} \quad \text{or} \quad A^{\tau+1} = (I - G)A^{\tau} + V$$

where

$$(96) \quad V = (\varepsilon_1 \mu_1 m_{01}, \varepsilon_2 \mu_2 m_{02}, \dots, \varepsilon_N \mu_N m_{0N})'$$

$$(97) \quad G = \begin{pmatrix} \varepsilon_1 g_{11} & \varepsilon_1 g_{12} m_{21} & \dots & \varepsilon_N g_{1N} m_{N1} \\ \varepsilon_2 g_{21} m_{12} & \varepsilon_2 g_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ \varepsilon_N g_{N1} m_{1N} & \dots & \dots & \varepsilon_N g_{NN} \end{pmatrix}$$

If the initial decision functions are

$$(98) \quad \alpha_i^0(y_i) = a_i^0 y_i; \quad i = 1, 2, \dots, N$$

then the solution to (95) is

$$(99) \quad A^{\tau} = (I - G)^{\tau} A^0 + (I - (I - G)^{\tau}) G^{-1} V.$$

If $A^0 = 0$, $\varepsilon_1 = \dots = \varepsilon_N = \varepsilon$, $m_{0R} = m_{jR} = \frac{1}{2}$ then the solution is

$$(100) \quad A^{\tau} = (I - (I - \frac{\varepsilon}{2} H)^{\tau}) H^{-1} \mu$$

where

$$(101) \quad H = \begin{pmatrix} 2g_{11} & g_{12} & \dots & g_{1N} \\ g_{21} & 2g_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ g_{N1} & \dots & \dots & 2g_{NN} \end{pmatrix} = Q + \text{DIAG } Q; \quad \text{DIAG } Q = \begin{pmatrix} g_{11} & 0 & \dots & 0 \\ 0 & g_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & g_{NN} \end{pmatrix}$$

The stability of (100) depends on whether the largest eigenvalue

of $I - \frac{\varepsilon}{2} H$ is less than unity. If non-diagonal elements of H

are negative and $\varepsilon < \frac{1}{g_{ii}}$ for all i , then $I - \frac{\varepsilon}{2} H$ is a positive matrix.

If, in addition, H has a dominant diagonal or

$$(102) \quad 2g_{ii} + \sum_{j \neq i} g_{ij} > 0, \quad i = 1, 2, \dots, N$$

then the solution (100) is stable and converges to the optimal value

$$(103) \quad \hat{A} = H^{-1} \mu.$$

Irrespective of the sign patterns of H , if ε is small enough so that

$$(104) \quad g_{jj} + \frac{1}{2} \sum_{i \neq j} |g_{ij}| < \frac{1}{\varepsilon} \quad j=1,2,\dots,N$$

then the solution (100) is stable and converges to the optimal value defined in (103). The largest eigenvalue of $I - \frac{\varepsilon}{2} H$ cannot exceed the column sum of $I - \frac{\varepsilon}{2} H$ or (4)

$$(105) \quad |\lambda| \leq 1 - \varepsilon g_{jj} - \frac{\varepsilon}{2} \sum_{i \neq j} |g_{ij}|.$$

Inequality (104) implies that the maximum sum is less than unity.

Suppose the gradient adjustment process is continuous rather than discrete; the coefficients are adjusted according to the system of first-order differential equations

$$(106) \quad \frac{dA^T}{d\tau} = V - GA^T.$$

Under the assumptions $\varepsilon_1 = \dots = \varepsilon_N = \varepsilon$ and $m_{\theta R} = m_{jR} = \frac{1}{2}$ this system of equations is

$$(107) \quad \frac{dA^T}{d\tau} = \frac{\varepsilon}{2} M - \frac{\varepsilon}{2} H A^T.$$

The matrix H is the sum of Q and the diagonal matrix with the same elements as the diagonal of Q . Since Q is positive definite, H is positive definite. The general solution to the system of differential equations is of the form

$$(108) \quad A^T = \sum_{i=1}^N e^{\lambda_i \tau} M_{\lambda_i}$$

where λ_i is a eigenvalue of $-\frac{\varepsilon}{2} H$ and M_{λ_i} is its corresponding eigenvector. The convergence of the solution depends on the signs of the eigenvalues. If H is positive definite then its eigenvalues are positive. The eigenvalues of $-\frac{\varepsilon}{2} H$ are $-\frac{\varepsilon}{2}$ times the eigenvalues of H and hence $\lambda_i < 0$, $i=1,2,\dots,N$; convergence is assured.

Let $M = [M_{\lambda_1} \dots M_{\lambda_N}]$ be a N -by- N matrix with the i^{th} column equal to

M_{λ_i} . Define a vector of exponential functions $e^{\lambda \tau} = (e^{\lambda_1 \tau}, \dots, e^{\lambda_N \tau})'$.

A particular solution to the system (107) is

$$(109) \quad \bar{A} = H^{-1} \mu.$$

The total solution to the system (107) is therefore

$$(110) \quad A^\tau = H^{-1} \mu + M e^{\lambda \tau}.$$

Notice that the system converges to $H^{-1} \mu$ which are the coefficients which satisfy the person-by-person optimality conditions. The decision function after τ iteration is

$$(111) \quad \alpha^\tau(y) = Y A^\tau = Y H^{-1} \mu + Y M e^{\lambda \tau}$$

where we have put the vector of information variables along the diagonal of a diagonal matrix Y .

$$(112) \quad Y = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & & \cdot \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & y_N \end{pmatrix}$$

b. Optimum Number of Iterations Suppose the iterative process of approximating the optimal team decision function is a costly process. If each iteration lowers utility by C utility units, the team would like to select an optimum number of iterations, $\hat{\tau}$, to maximize net expected team utility

$$(113) \quad \text{NET } W[\alpha^\tau] = W[\alpha^\tau] - C \tau.$$

The $\hat{\tau}$ selected should satisfy

$$(114) \quad W[\alpha^{\hat{\tau}+1}] - W[\alpha^{\hat{\tau}}] \leq C \leq W[\alpha^{\hat{\tau}}] - W[\alpha^{\hat{\tau}-1}].$$

If the adjustments take place continuously as in (107), we can make an approximate solution to this problem by taking $\hat{\tau}$ to be

the nearest integer to the number τ^* that satisfies

$$(115) \quad \frac{dW[\alpha^{\tau^*}]}{d\tau} = c.$$

At this point the marginal increase in expected utility from improving the team decision function should just equal the marginal cost of improving the decision function.

How do we express $W[\alpha^{\tau}]$ as a function of τ ? If utility is quadratic and random variables are normal as in (87)-(90) then $W[\alpha^{\tau}]$ will depend only on the variance and covariances of Y and Θ :

$$(116) \quad \begin{aligned} W[\alpha^{\tau}] &= E\{z\mu' \alpha^{\tau}(Y)\Theta - \alpha^{\tau}(Y)'Q\alpha^{\tau}(Y)\} \\ &= E\{z\mu'YA^{\tau}\Theta - A^{\tau}'YQYA^{\tau}\} \\ &= z\mu'A^{\tau} - A^{\tau}'HA^{\tau} \end{aligned}$$

where H is defined in equation (101). The rate of change of $W[\alpha^{\tau}]$ with respect to τ is computed as

$$(117) \quad \begin{aligned} \frac{dW[\alpha^{\tau}]}{d\tau} &= z\mu' \frac{dA^{\tau}}{d\tau} - zA^{\tau}'H \frac{dA^{\tau}}{d\tau} \\ &= \varepsilon(\mu'\mu - z\mu'HA^{\tau} + A^{\tau}'H^2A^{\tau}) \end{aligned}$$

using the definition of the adjustment process (107). The solution path is

$$(118) \quad A^{\tau} = H^{-1}\mu + Me^{\lambda\tau}$$

and (117) reduces as follows

$$(119) \quad \begin{aligned} \frac{dW}{d\tau} &= \varepsilon(\mu'\mu - z\mu'H(H^{-1}\mu + Me^{\lambda\tau}) + (H^{-1}\mu + Me^{\lambda\tau})'H^2(H^{-1}\mu + Me^{\lambda\tau})) \\ &= \varepsilon(Me^{\lambda\tau})'H^2(Me^{\lambda\tau}) \\ &= \varepsilon\left(\frac{-z}{\varepsilon}\Lambda Me^{\lambda\tau}\right)' \left(\frac{-z}{\varepsilon}\Lambda Me^{\lambda\tau}\right) \\ &= \frac{4}{\varepsilon}(Me^{\lambda\tau})'\Lambda^2(Me^{\lambda\tau}) \end{aligned}$$

where $\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \lambda_n \end{pmatrix}$ is a diagonal matrix of eigenvalues.

The rate of change of $W[\alpha^\tau]$ is of the form

$$(120) \quad \frac{dW[\alpha^\tau]}{d\tau} = \frac{1}{\varepsilon} (d_1 e^{-r_1 \tau} + d_2 e^{-r_2 \tau} + \dots + d_m e^{-r_m \tau}).$$

The optimal number of iterations is then the root of the equation

$$(121) \quad d_1 e^{-r_1 \tau} + d_2 e^{-r_2 \tau} + \dots + d_m e^{-r_m \tau} = \varepsilon c.$$

The root of this equation will define a maximum because $W[\alpha^\tau]$ is concave in τ ; concavity can be proved as follows

$$(122) \quad \frac{d^2 W}{d\tau^2} = 2\varepsilon \left(A^\tau H^2 \frac{dA^\tau}{d\tau} - \mu' H \frac{dA^\tau}{d\tau} \right) \\ = \frac{\theta}{\varepsilon} (Me^{\lambda\tau})' \Lambda^3 (Me^{\lambda\tau}).$$

Define a vector X as

$$(123) \quad X = (Me^{\lambda\tau})$$

Then $\frac{d^2 W}{d\tau^2}$ is a quadratic form in X :

$$(124) \quad \frac{d^2 W}{d\tau^2} = \frac{\theta}{\varepsilon} X' \Lambda^3 X.$$

If λ_i is a eigenvalue of $-\frac{\varepsilon}{2}H$ then $(\lambda_i)^3$ is an eigenvalue of $(-\frac{\varepsilon}{2}H)^3$.

If H is positive definite then $\lambda_i < 0$ which implies $(\lambda_i)^3 < 0$ and

hence H is positive definite. The quadratic form (124) will thus

be negative no matter what value X takes and $\frac{d^2 W}{d\tau^2} < 0$, which implies W is concave.

If the cost of adjusting the team's decision function changes, the response of the optimal number of iterations will be in the opposite direction;

$$(125) \quad \frac{\partial \tau^*}{\partial c} = \varepsilon / \frac{d^2 W}{d\tau^2} < 0.$$

c. Opportunity Losses and the Optimal Number of Iterations Suppose the iterative process of approximating the optimal team decision function is time consuming. While the team is computing its

τ^* so that

$$(127) \quad \frac{dW[\tilde{A}, \alpha^{\tau^*}]}{d\tau} = E\{U(\tilde{A}, \theta)\} D + \frac{dW[\alpha^{\tau^*}]}{d\tau} (L - D\tau^*) - W[\alpha^{\tau^*}] D - C = 0$$

This implies a first-order condition for τ^*

$$(128) \quad \frac{2}{\varepsilon} (Me^{\lambda\tau})' [2(L - D\tau^*)\Lambda^2 - D\Lambda] (Me^{\lambda\tau^*}) = C + D \tilde{A}' \Phi \tilde{A} + D \mu' H^{-1} \mu$$

which is of the form

$$(129) \quad (d_{11} + d_{21}\tau^*) e^{-v_1\tau^*} + \dots + (d_{1m} + d_{2m}\tau^*) e^{-v_m\tau^*} = \text{constant} > 0$$

where $v_i > 0$. (We assume the upper limit on τ^* , $\tau^* \leq L/D$, plays no role in finding the root to equation (129).)

The first-order condition (127) is a maximum if $W[\tilde{A}, \alpha^{\tau}]$ is concave in τ . This concavity can be proved as follows.

$$(130) \quad \frac{d^2W[\tilde{A}, \alpha^{\tau}]}{d\tau^2} = (L - D\tau) \frac{d^2W[\alpha^{\tau}]}{d\tau^2} - 2D \frac{dW[\alpha^{\tau}]}{d\tau}$$

The components of the right-hand side of (130) are signed by:

$$(131) \quad \frac{d^2W[\alpha^{\tau}]}{d\tau^2} \leq 0 \quad \text{by equation (124).}$$

H^2 is positive definite by its relation to the positive definite matrix Φ ; therefore

$$(132) \quad \frac{dW[\alpha^{\tau}]}{d\tau} = \frac{4}{\varepsilon} (Me^{\lambda\tau})' \Lambda^2 (Me^{\lambda\tau}) > 0.$$

Hence

$$(133) \quad \frac{d^2W[\tilde{A}, \alpha^{\tau}]}{d\tau^2} \leq 0.$$

Comparative static results can also be established. If the marginal cost of iteration of the adjustment process increases then the optimal number of iteration decreases. If the speed of iteration is changed the response of τ^* is

$$(134) \quad \frac{\partial \tau^*}{\partial D} = (\tilde{A}' \Phi \tilde{A} + \mu' H^{-1} \mu) / \frac{d^2W[\tilde{A}, \alpha^{\tau^*}]}{d\tau^2} \leq 0$$

since Φ and H^{-1} are positive definite. If the length of the period that θ is unchanged is increased then the optimal number of iterations increases since

$$(135) \frac{\partial \tau^*}{\partial L} = - \frac{dW[\alpha^\tau]}{d\tau} / \left(\frac{d^2W[\alpha^\tau]}{d\tau^2} (L-D\tau) - 2D \frac{dW}{d\tau} \right) \geq 0.$$

Finally, since $E\{u(\tilde{A}, \theta)\} = -\tilde{A}'\theta\tilde{A}$ if the interim action is selected to have a higher payoff, then the number of iterations increases:

$$(136) \frac{d\tau^*}{dE\{u(\tilde{A}, \theta)\}} = -D / \frac{d^2W[\tilde{A}, \alpha^\tau]}{d\tau^2} \geq 0.$$

Now looking at the opportunity loss of the second specification, suppose that the tentative team decision function $\alpha^\tau(\gamma)$ is used to select a temporary action. The total net expected utility is

$$(137) W[\alpha^s, \alpha^\tau] = \int_0^{D\tau} W[\alpha^s] ds + (L-D\tau) W[\alpha^\tau] - c\tau.$$

The first order condition that τ^* must satisfy are

$$(138) \frac{dW[\alpha^s, \alpha^\tau]}{d\tau} = W[\alpha^{\tau^*}]D + (L-D\tau^*) \frac{dW[\alpha^{\tau^*}]}{d\tau} - DW[\alpha^{\tau^*}] - c = 0 \\ = (L-D\tau^*) \frac{dW[\alpha^{\tau^*}]}{d\tau} - c = 0.$$

The first order conditions is of form

$$(139) (L-D\tau^*) d_1 e^{-r_1 \tau^*} + \dots + (L-D\tau^*) d_m e^{-r_m \tau^*} = c\epsilon.$$

The total net expected utility is concave in τ since

$$(140) \frac{d^2W[\alpha^s, \alpha^\tau]}{d\tau^2} = (L-D\tau) \frac{d^2W[\alpha^\tau]}{d\tau^2} - D \frac{dW[\alpha^\tau]}{d\tau} \leq 0.$$

Comparative static results that follow are

$$(141) \frac{\partial \tau^*}{\partial c} = 1 / \frac{d^2W[\alpha^s, \alpha^\tau]}{d\tau^2} \leq 0$$

$$(142) \frac{\partial \tau^*}{\partial L} = - \frac{dW[\alpha^\tau]}{d\tau} / \frac{d^2W[\alpha^s, \alpha^\tau]}{d\tau^2} \geq 0$$

$$(143) \frac{\partial \tau^*}{\partial D} = \tau^* \frac{dW[\alpha^\tau]}{d\tau} / \frac{d^2W[\alpha^s, \alpha^\tau]}{d\tau^2} \leq 0.$$

d. The Joint Selection of the Amount of Information and the Team Decision Function Suppose that the team can not only select the number of iterations in a gradient process of approximating the optimal team decision function but can also select (non-sequentially) the number of information gathering operations it makes before an action is implemented. The two decisions must be made

jointly because of the complicated structure of the problem.

The teammates can observe some number of random samples, S and receiving the S corresponding ex ante communication. The probability distribution of each observation and ex ante communication function is assumed to be identical for each IGO. As a result the individual messages are identically distributed according to

$$(144) \quad \phi(y_{iR} | \theta) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2} (y_{iR} - \theta)^2) \quad i=1,2,\dots,N; R=1,2,\dots,S.$$

The state of nature is again distributed normal with mean=zero and variance=one. The important values in the iterative process are

$$(145) \quad E\{\theta | y_{i1}, \dots, y_{iS}\} = \frac{1}{S+1} (y_{i1} + \dots + y_{iS})$$

$$(146) \quad E\{y_{jR} | y_{i1}, \dots, y_{iS}\} = \frac{1}{S+1} (y_{i1} + \dots + y_{iS})$$

$$(147) \quad \text{cov}(\theta, y_{iR}) = 1$$

$$(148) \quad \text{var}(y_{iR}) = 2$$

$$(149) \quad \text{cov}(y_{iR}, y_{jR}) = 1.$$

The optimal individual decision functions will again be linear in the individuals information and because of the symmetry of the sample

$$(150) \quad \alpha_i(y_{i1}, \dots, y_{iS}) = \bar{a}_{is} (y_{i1} + \dots + y_{iS}) \quad i=1,2,\dots,N.$$

The tentative decision functions should thus be linear in $\sum_{R=1}^S y_{iR}$ i.e. $\alpha_i(y_{i1}, \dots, y_{iS}) = a_{is} \sum_{R=1}^S y_{iR}$. If each IGO costs an equal amount K in utility units, the net expected utility with S IGO's and decision function $\alpha(Y_S)$ is

$$(151) \quad \text{NET } W[\alpha(Y_S)] = W[\alpha(Y_S)] - SK = E\{2\mu' Y_S A_S \theta - A_S' Y_S Q Y_S A_S\} - SK$$

where $A_S = (a_{1S}, \dots, a_{NS})'$

$$(152) \quad Y_S = \begin{pmatrix} \sum_{R=1}^S y_{1R} & 0 & \dots & 0 \\ 0 & \sum_{R=1}^S y_{2R} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \sum_{R=1}^S y_{NR} \end{pmatrix}$$

$$(153) \text{ NET } W[\alpha(Y_s)] = 2s\mu'A_s - sA_s'FA_s - sK$$

$$(154) F = sQ + \text{DIAG } Q.$$

The optimal number of IGO's is the integer \hat{s} that satisfies

$$(155) \text{ NET } W[\alpha(Y_{\hat{s}})] \geq \text{ NET } W[\alpha(Y_s)] \text{ for all } s.$$

This might be approximated by picking the nearest integer to the number s^* that satisfies

$$(156) \frac{dW[\alpha(Y_{s^*})]}{ds} = 2\mu'A_s - 2s^*A_s'QA_s - A_s'\text{DIAG } QA_s = K.$$

This number is

$$(157) s^* = (2\mu'A_s - A_s'\text{DIAG } QA_s - K) / 2A_s'QA_s.$$

It should be pointed out that A_s is just an arbitrary vector of coefficients in the above analysis; the optimal vector of coefficients \hat{A}_s is a function of the number of observations: $\hat{A}_s = F^{-1}\mu$ where F is a matrix that is a function of s and Q . This dependence was not accounted for in (156).

Now let us introduce gradient solutions to this multiple observation team theory. The gradient of $W[\alpha(Y_s)]$ is defined by

$$(158) \frac{dW}{d\alpha_i} = 2(\mu_i E\{\theta_i | y_{i1}, \dots, y_{is}\} - g_{ii} a_{is} \sum_R y_{iR} - \sum_{j \neq i} g_{ij} a_{js} E\{\theta_j | y_{i1}, \dots, y_{is}\}) \\ = 2(\mu_i \frac{1}{s+1} \sum_R y_{iR} - g_{ii} a_{is} \sum_R y_{iR} - \sum_{j \neq i} g_{ij} a_{js} (\frac{s}{s+1}) \sum_R y_{iR}) \\ = (\mu_i \frac{1}{s+1} - g_{ii} a_{is} - \sum_{j \neq i} g_{ij} a_{js} \frac{s}{s+1}) 2 \sum_R y_{iR}.$$

The adjustment process to be studied is

$$(159) \frac{dA_s^T}{ds} = \varepsilon \left(\frac{1}{s+1} \mu - RA_s^T \right)$$

where

$$(160) R = \begin{pmatrix} g_{11} & \frac{s}{s+1} g_{12} & \dots & \frac{s}{s+1} g_{1N} \\ \frac{s}{s+1} g_{21} & g_{22} & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & \cdot \\ \frac{s}{s+1} g_{N1} & \cdot & \cdot & g_{NN} \end{pmatrix} = \frac{1}{s+1} F.$$

The system of differential equations is rewritten as

$$(161) \quad \frac{dA_s^\tau}{d\tau} = \frac{c}{s+1} \mu - \frac{c}{s+1} F A_s^\tau.$$

If the eigenvalues of $\frac{c}{s+1} F$ are denoted $(\beta_1, \dots, \beta_N) = \beta$ then the solution to (161) is

$$(162) \quad \frac{dA_s^\tau}{d\tau} = F^{-1} \mu + M e^{\beta \tau}$$

where $e^{\beta \tau}$ is a vector of exponential functions $(e^{\beta_1 \tau}, \dots, e^{\beta_N \tau})$ and M is the matrix of corresponding eigenvectors. The particular solution $F^{-1} \mu$ is the coefficient vector which satisfies the person-by-person optimality conditions.

The team decision function after τ iterations is

$$(163) \quad \alpha^\tau(Y_s) = Y_s A_s^\tau = Y_s F^{-1} \mu + Y_s M e^{\beta \tau}$$

If each unit of iterations in the gradient process cost a fixed amount of C utility units and each IGO costs an equal K units, the net utility of taking S observations and running τ iterations is

$$(164) \quad \begin{aligned} \text{NET } W[\alpha^\tau(Y_s)] &= W[\alpha^\tau(Y_s)] - sK - c\tau \\ &= E\{2s\mu' Y_s A_s^\tau \theta - A_s^\tau' Y_s \theta Y_s A_s^\tau\} - sK - c\tau \\ &= 2s\mu' A_s^\tau - s A_s^\tau' F A_s^\tau - sK - c\tau. \end{aligned}$$

We can approximate the optimal integer solution by selecting the nearest integers to the numbers s^* and τ^* that satisfies the marginal conditions

$$(165) \quad \frac{\partial W[\alpha^{\tau^*}(Y_{s^*})]}{\partial \tau} = c$$

$$(166) \quad \frac{\partial W[\alpha^{\tau^*}(Y_{s^*})]}{\partial s} = K.$$

In this general formulation, comparative static results are hard to derive because of the complex form of $W[\alpha^\tau(Y_s)]$ in equation (165).

FOOTNOTES

- (1) A good introduction is Mufti(1970), also see Bryson and Denham(1962), Bryson and Ho(1969), Kelley(1969), Lasdon, Mitter and Warren(1967), Luenberger(1969), chapter 10, Luenberger(1972), Mitter(1969) and Stein(1953).
- (2) Gelfand and Fomin(1963), pp. 27-29. No particular distinction is made between the competing technical definitions of the variational derivatives, Gateaux and Fréchet, since we are not using them to develop analytic results. See Luenberger (1969) chapter 7.
- (3) The differential equations (31) is defined not on a finite dimensional space but some finite dimensional Banach space. Difficulties arise in such an extension; see the appendix to Essay Four.
- (4) I would like to thank Professor Solow for this result.
 Let $M=(M_{ij})$ and let λ be any eigenvalue of M . The corresponding eigenvector X must satisfy $\lambda x_i = \sum_j m_{ij} x_j$. Let X_1 be the component of X with largest absolute value; then

$$|\lambda x_i| = |\lambda| |x_i| = \left| \sum_j m_{ij} x_j \right| \leq \sum_j |m_{ij}| |x_j| \leq |x_i| \sum_j |m_{ij}|;$$
 thus $|\lambda| \leq \sum_j |m_{ij}|$.
- (5) These are analogous to those developed in the chapter on sequential teams and optimal stopping rules.

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ESSAY FOUR

PLANNING BY A CONSTRAINED TEAM

PLANNING BY A CONSTRAINED TEAM

I. CONSTRAINED TEAM DECISION MAKING

a. Solution Difficulties with Joint Constraints on Actions Team theory studies the use of information in making decisions in a multi-member organization that faces a risky environment. The basic elements of team decision theory are identical to those of Bayesian statistical decision theory with the critical exception that members base their individual actions on different information. I will outline the theory of team theory noting the similarities and differences between it and Bayesian decision theory. In this presentation single agent constraints will be introduced and shown to be easily handled by team theory. However, when joint constraints are introduced that bind the actions of decision makers, difficulties occur which require modification of the team problem. The remainder of this section catalogues a few of the most obvious modifications that will allow a team to solve its constrained maximization problem.

The team consist of N teammates, indexed by $i=1,2,\dots,N$. Each teammate must select an action, a_i , which may be a vector. The unknown state of nature is represented by the variable Θ and it has a team priori probability density $\phi(\Theta)$. Utility depends on all teammates' actions and the state of nature, $U=U(a_1,a_2,\dots,a_N,\Theta)$. Each teammate i independently observes a separate statistic, z_i .

which is correlated with the unknown state of nature via a conditional density function $\phi_i(z_i|\theta)$. Notice that in Marschak and Radner's⁽¹⁾ terms, this information may be "noisy"; that is, more than one value of z_i is possible for a particular value of the state of nature θ . These observations are the basis for communication since teammate i does not know the value z_j observed by teammate j . A communication structure is defined as a matrix of communication functions, $\eta(z)$,

$$(1) \quad Y = \begin{bmatrix} y_{11} & y_{12} & \dots & y_{1N} \\ y_{21} & y_{22} & & \cdot \\ \vdots & & \ddots & \cdot \\ y_{N1} & \dots & \dots & y_{NN} \end{bmatrix} = \eta(z) = \begin{bmatrix} \eta_{11}(z_1) & \eta_{12}(z_2) & \dots & \eta_{1N}(z_N) \\ \eta_{21}(z_1) & \eta_{22}(z_2) & & \cdot \\ \vdots & & \ddots & \cdot \\ \eta_{N1}(z_1) & \dots & \dots & \eta_{NN}(z_N) \end{bmatrix}$$

where $y_{ij} = \eta_{ij}(z) = \eta_{ij}(z_j)$ is a message sent to i from j when j observes the sample value z_j . This message matrix Y is random with conditional probability

$$(2) \quad \phi(Y|\theta) = \prod_{j=1}^N \left(\int_{z_j \in \bigcap_{i=1}^N \eta_{ij}^{-1}(y_{ij})} \phi_j(z_j|\theta) dz_j \right)$$

where $\bigcap_{i=1}^N \eta_{ij}^{-1}(y_{ij})$ is the set of all sample values z_j that would cause the j^{th} teammate to send the messages $(y_{1j}, y_{2j}, \dots, y_{Nj})$.

Each teammate uses his messages $y_1 = (y_{11}, \dots, y_{1N})'$ to select his action a_1 . This is represented by the individual decision functions $\alpha_i = \alpha_i(y_i)$. It is important to notice that the action a_1 does not depend on the information y_j for $j \neq 1$. This is all that analytically differentiates team decision theory and Bayesian statistical decision theory. Denote the vector of decision functions $\alpha(Y) = (\alpha_1(y_1), \dots, \alpha_N(y_N))'$. The teammates want to select their individual decision

functions for a given communication structure to maximize the team's expected utility:

$$(3) E\{U(\alpha(Y), \theta)\} = \int_{\Theta} \int_Y U(\alpha(Y), \theta) \phi(Y|\theta) \phi(\theta) dY d\theta.$$

Radner's person-by-person necessary conditions require the optimal decision functions $\hat{\alpha}(Y)$ to satisfy at each y_1 the following:

$$(4) 0 = \int_{\Theta} \int_{Y(i)} U_{a_1}(\hat{\alpha}(Y), \theta) f(\theta, Y(i)|y_1) dY(i) d\theta$$

where $Y(i)$ is the message matrix with the i^{th} row deleted and the posterior density is $f(\theta, Y(i)|y_1) = \phi(Y|\theta)\phi(\theta)/\phi_1(y_1)$. These necessary conditions are interpreted as selecting $\hat{\alpha}$ so that the expected marginal utility with respect to a_1 , given the personal message y_1 , is zero for each possible message. One should notice the "Nash-equilibrium" nature of the person-by-person conditions; The i^{th} teammate must know α_j in order to select his optimal action. Therefore, the team must select optimal decision functions ex ante and cannot just select optimal actions ex post.

In the above presentation we have assumed differentiability of the utility function and assumed a_1 can be any real number. Suppose however, that the action a_1 had to belong to some subset of the real line:

$$(5) \quad g_1(a_1) \leq 0.$$

How would the team select decision functions to maximize team expected utility while simultaneously satisfying the single agent constraints on actions? Define a vector of multiplier functions

$$(6) \quad \beta(Y) = (\beta_1(y_1), \dots, \beta_N(y_N))$$

and then find a saddlepoint of the Lagrangian

$$(7) \quad L[\alpha, \beta] = E\{U(\alpha(Y), \theta)\} - E\{\beta(Y)' G(\alpha(Y))\}$$

where $G(\alpha(\gamma)) = (g_1(\alpha_1(y_1)), \dots, g_N(\alpha_N(y_N)))'$. Notice that L is a functional and the instruments of optimization are functions. The person-by-person saddlepoint Kuhn-Tucker conditions are

$$(8) \frac{\delta L}{\delta \alpha_i} = E \{ u_{a_i}(\hat{\alpha}(\gamma), \theta) | y_i \} - \hat{\beta}_i(y_i) \dot{g}_i(\hat{\alpha}_i(y_i)) = 0$$

$$(9) \frac{\delta L}{\delta \beta_i} = -g_i(\hat{\alpha}_i(y_i)) \geq 0, \quad \hat{\beta}_i(y_i)$$

$$(10) \frac{\delta L}{\delta \beta_i} \beta_i = -\hat{\beta}_i(y_i) g_i(\hat{\alpha}_i(y_i)) = 0$$

for each $y_i \in \mathcal{Y}_i$, $i=1, 2, \dots, N$.

If U is strictly concave and differentiable and g is convex and differentiable, the above conditions are both necessary and sufficient for $\hat{\alpha}, \hat{\beta}$ to be a saddlepoint. We see that there is no conceptual difficulty in the introduction of single agent constraints to the team problem.

The team may find that the vector of actions of its members is constrained because of limited resources or technological transformations. A general formulation of such constraints would be $g(a_1, \dots, a_N, \theta) \leq 0$, but in this chapter we will deal with the particular constraint

$$(11) b'A = \sum_{i=1}^N b_i a_i \leq c$$

where c is a fixed scalar denoting resource endowment and b is a vector of "resource use" coefficients. Notice that this linear constraint is independent of the state of nature θ ; that is, the state of nature neither influences the endowment nor the technical coefficients. The important feature of this constraint is that it binds the joint team action A .

The constrained team problem for a particular communication structure is to select the N individual decision functions to maximize expected team utility while satisfying the joint cons-

straint with certainty

$$(12) \max_{\alpha} E\{U(\alpha(Y), \theta)\}$$

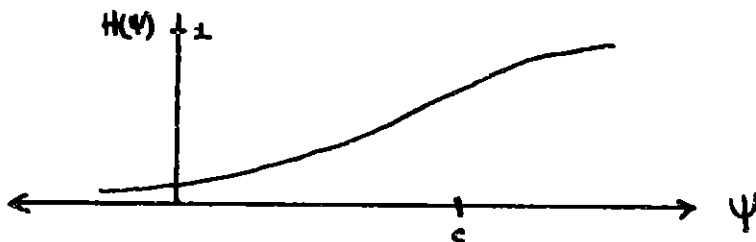
subject to $b'\alpha(Y) \leq c$ for all $Y \in \mathcal{Y}$.

The major question is how to guarantee that the constraint is satisfied for all possible information variables. ⁽²⁾

For particular set of N individual decision functions, $\alpha_1(y_1), \dots, \alpha_N(y_N)$, the random variable Ψ , defined by

$$(13) \Psi = b'\alpha(Y) = \sum_{i=1}^N b_i \alpha_i(y_i),$$

has a probability density function that can be computed given the joint density $\phi(y_1, \dots, y_N)$. Denote the p.d.f. of Ψ by $h(\Psi)$ and the corresponding cumulative distribution by $H(\Psi)$, where $h(\Psi) = H'(\Psi)$. If the joint density of y_1, \dots, y_N has "infinite tails" which only asymptotically approach zero then the cumulative distribution of Ψ might look like this



The important feature is that $H(\Psi)$ only asymptotically approaches 1. For any value of c , $H(c)$ is the probability that $b'\alpha(Y) \leq c$ or verbally, $H(c)$ is the probability that this decision function would use less of the resource than the team's endowment. What should be clear is that, in general, the probability of constraint violation, $1 - H(c)$, is positive for all c . If an unlikely event should occur (a statistical outlier is observed) then the teammates will select actions that are incompatible with the limited endowment of resource.

In previous chapters "team" decision theory and Bayesian "statistical" decision theory have been presented side-by-side to emphasize their analytic similarities and differences. Do the above difficulties arise in "constrained" statistical decision theory? No! There is only one decision-maker in statistical decision theory and hence there can be no difference in information or problem of coordination. In statistical decision theory the only type constraint possible is "single-agent" because there is only one agent. Any constrained problem can be solved using the above Lagrangian methods.

Viewed from another perspective, suppose the i^{th} teammate wants to select his action a_i so that his action is not that final "straw that breaks the camel's back." How much of the resource is still available for his personal use? The remainder of the endowment when the other teammates take their share is

$$(14) \quad c - \sum_{j \neq i} b_j \alpha_j(y_j).$$

Because y_j is not known by teammate 1, from his viewpoint the remaining resource is a random variable and no matter how small he selects a_i , he still might provide that the final "straw".

As an example, suppose there is a two member team with a utility function

$$(15) \quad U(a_1, a_2, \theta) = \mu_1 \theta a_1 + \mu_2 \theta a_2 - \frac{1}{2} f_{11} a_1^2 - \frac{1}{2} f_{22} a_2^2 - f_{12} a_1 a_2$$

and jointly normal random variables where

$$(16a) \quad E\{\theta | y_1\} = \bar{\theta}^1 y_1, \quad E\{\theta | y_2\} = \bar{\theta}^2 y_2$$

$$(16b) \quad E\{y_2 | y_1\} = \bar{y}_2^1 y_1, \quad E\{y_1 | y_2\} = \bar{y}_1^2 y_2$$

with $\bar{\theta}_1, \bar{\theta}_2, \bar{y}_1, \bar{y}_2$ given scalar parameters, and the joint constraint

$$(17) \quad a_1 + a_2 = 0$$

The constrained team problem is to

$$(18) \quad \max E \left\{ z\mu_1 \theta \alpha_1(y_1) + z\mu_2 \theta \alpha_2(y_2) - g_{11} \alpha_1(y_1)^2 - 2g_{12} \alpha_1(y_1) \alpha_2(y_2) - g_{22} \alpha_2(y_2)^2 \right\}$$

subject to $\alpha_1(y_1) + \alpha_2(y_2) = 0$ for all y_1, y_2 .

Form the Lagrangian

$$(19) \quad L = E \left\{ z\mu_1 \theta \alpha_1(y_1) + z\mu_2 \theta \alpha_2(y_2) - g_{11} \alpha_1(y_1)^2 - 2g_{12} \alpha_1(y_1) \alpha_2(y_2) - g_{22} \alpha_2(y_2)^2 \right\} - E \left\{ \lambda(y_1, y_2) (\alpha_1(y_1) + \alpha_2(y_2)) \right\}.$$

Because of the quadratic-normal assumptions, the optimal decision functions are linear, and so are the Lagrange multipliers:

$$(20) \quad \hat{\alpha}_1(y_1) = \hat{a}_1 y_1; \quad \hat{\alpha}_2(y_2) = \hat{a}_2 y_2; \quad \lambda(y_1, y_2) = \hat{\lambda}_1 y_1 + \hat{\lambda}_2 y_2.$$

The first order necessary conditions are

$$(21) \quad 0 = \mu_1 \bar{\theta}^1 - g_{11} \hat{a}_1 - g_{12} \bar{y}_2^1 \hat{a}_2 - \hat{\lambda}_1 - \hat{\lambda}_2 \bar{y}_2^1$$

$$(22) \quad 0 = \mu_2 \bar{\theta}^2 - g_{22} \hat{a}_2 - g_{12} \bar{y}_1^2 \hat{a}_1 - \hat{\lambda}_1 \bar{y}_1^2 - \hat{\lambda}_2$$

$$(23) \quad 0 = \hat{a}_1 y_1 + \hat{a}_2 y_2 \quad \text{for all } y_1 \text{ and } y_2.$$

The only solution to these conditions are $\hat{a}_1 = 0$ and $\hat{a}_2 = 0$. In other words, the only way to satisfy the constraint with certainty is to make the action independent of the information (in this case the constant action is zero).

In summary, three aspects of the jointly constrained team problem seem to be incompatible, (1) independent selection of actions by the different teammates, (2) different random information variables for each teammate and (3) an inflexible joint constraint on teammates' actions that must be satisfied with certainty. The remainder of this section looks at how the constrained team

problem can be modified by varying each of these three components to make solution well defined. At each step, many variations could be used and this is not an attempt to catalogue all interesting modifications of the constrained team problem. For particular problems, other equal interesting modifications might suggest themselves.

b. Quotas and Joint Constraints The team is not run by some omnipotent central authority; individual teammates choose the actions that they think are best and, because of the "team" assumptions of identical utility and probability functions, these independent decisions are optimal from the organizations viewpoint. But independent decision making makes coordination difficult. We begin by looking at one way the independence of the decision makers can be reduced so that the joint constraints can be satisfied with certainty.

Suppose that the actions of the teammates must be chosen from single-agent constraint sets Q_i that are bounded from above by M_i . Further suppose that the corresponding upper bound on $b_i A$ is less than or equal to the resource endowment, $\sum_i b_i M_i \leq C$. Then no teammate will select an action that pushes the resource use beyond the resource endowment. Or from teammate i 's viewpoint, the minimum amount of the resource remaining for his use, $C - \sum_{j \neq i} b_j M_j$ is greater than the maximum amount of the resource he could use and still remain feasible, $b_i M_i$. If this situation occurs then the joint constraint on the actions can be effectively ignored by all teammates (the shadow price of the resource will be zero). However, this upper bound M_i may not be a part of the original team problem;

Q_i be the set of all real numbers, an unbounded set. Hence, we might have to construct artificial upper bounds which we will call quotas. (3)

Definition: We call $q=(q_1, \dots, q_N)'$ a quota vector if

$$b'g \leq c.$$

Rather than allowing the individual teammate to select any action, we will add the single-agent restrictions

$$(24) \quad a_i \leq q_i \quad i=1, 2, \dots, N$$

where q is a quota vector. This quota vector is a discretionary variable and should be selected in conjunction with the decision functions to optimize the constrained team problem:

$$(25) \quad \begin{aligned} &\text{maximize } E \{ U(\alpha(Y), \theta) \} \\ &\text{subject to } \alpha(Y) \leq g \quad \text{for each } Y \in \mathcal{Y} \\ &\text{and } b'g \leq c. \end{aligned}$$

Assuming concavity and differentiability of U , we can characterize the optimum by the saddlepoint of a Lagrangian defined using multiplier functions $\lambda(Y) = (\lambda_1(y_1), \dots, \lambda_N(y_N))'$ and scalar multiplier γ :

$$(26) \quad L[\alpha, g, \lambda, \gamma] = E \{ U(\alpha(Y), \theta) \} + E \{ \lambda(Y)'(g - \alpha(Y)) \} + \gamma(c - b'g).$$

The Kuhn-Tucker conditions are:

$$(27) \quad \frac{\partial L}{\partial \alpha_i} \Big|_{y_i} = E \{ U_{\alpha_i}(\alpha(Y), \theta) | y_i \} - \lambda_i(y_i) = 0 \quad \text{for each } y_i \in \mathcal{Y}_i$$

$$(28) \quad \frac{\partial L}{\partial g_i} = E \{ \lambda_i(y_i) \} - \gamma b_i = 0$$

$$(29) \quad \frac{\partial L}{\partial \lambda_i} \Big|_{y_i} = g_i - \alpha_i(y_i) \geq 0, \quad \lambda_i(y_i) \geq 0 \quad \text{for each } y_i \in \mathcal{Y}_i$$

$$(30) \quad (g_i - \alpha_i(y_i)) \lambda_i(y_i) = 0 \quad \text{for each } y_i \in \mathcal{Y}_i$$

$$(31) \quad \frac{\partial L}{\partial \gamma} = c - b'g \geq 0, \quad \gamma \geq 0 \quad \text{and } (c - b'g)\gamma = 0.$$

These conditions have the following interpretations

(a) For each possible message y_1 , set the posterior expected

marginal utility with respect to a_1 equal to the shadow price of the i^{th} quota given y_1 .

- (b) Set the expected shadow price of the i^{th} quota equal to the resource cost per unit of the i^{th} quota.
- (c) The action must satisfy the quota and if the action is strictly less than the quota, the quota's shadow price is zero.
- (d) The quotas must satisfy the resource constraint. We will assume this constraint "bites", so $\lambda > 0$.

It should be noted that for particular y_1, y_j we can have $\alpha_i(y_i) < g_i$ and $\alpha_j(y_j) = g_j$, yet the i^{th} teammate cannot announce that he has excess resources and shift the resource to teammate j . In other words it is quite possible that the actual observations of the information will lead to $\lambda_i(y_i) = 0$ and $\lambda_j(y_j) > 0$. In a sense, this means the decisions maybe inefficient; that is, useful resources maybe wasted. That is the price the team must pay if it wants at least partially independent decision making based on different information to satisfy joint constraints with certainty.

As an example of how quotas might be used, imagine a multi-product firm that produces two goods a_1 and a_2 for sale next month at prices μ_1 and μ_2 . However for various reasons the national economic picture might change in the interim; if a depression occurs, no goods can be sold but if a upswing occurs, any number of goods can be sold. The firms revenue is thus

$$(32) (\mu_1 a_1 + \mu_2 a_2) \Theta$$

where $\Theta = 0$ means a depression

and $\Theta = 1$ means a upswing.

The firm's two decision makers, one for each good, agree that boom or bust are equally likely: $\phi(\theta=0) = \phi(\theta=1) = \frac{1}{2}$. The two decision makers are not the best judges of macro economic events, so each decision maker independently consults a macro forecast: teammate 1 looks at the St. Louis equation and teammate 2 looks at the Fed-MIT-Penn model. Naturally with such ideologically divergent macro models, the two teammates refuse to talk to each other; so there is no communication about macro forecasts. To be fair to both model, we will assume that both models are equally good at predicting

$$\begin{aligned} \phi(y_i = \text{bust} | \theta = 0) &= \frac{3}{4} & \phi(y_i = \text{boom} | \theta = 0) &= \frac{1}{4} \\ \phi(y_i = \text{bust} | \theta = 1) &= \frac{1}{4} & \phi(y_i = \text{boom} | \theta = 1) &= \frac{3}{4} \end{aligned}$$

for $i=1,2$.

We also assume the two predictions are independent so that

$$(33) \quad \phi(y_1, y_2 | \theta) = \begin{array}{l} [y_1 = \text{bust}, y_2 = \text{bust}] : \\ [y_1 = \text{bust}, y_2 = \text{boom}] : \\ [y_1 = \text{boom}, y_2 = \text{bust}] : \\ [y_1 = \text{boom}, y_2 = \text{boom}] : \end{array} \begin{array}{cc} \theta=0 & \theta=1 \\ \left(\begin{array}{cc} \frac{9}{16} & \frac{1}{16} \\ \frac{3}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{3}{16} \\ \frac{1}{16} & \frac{9}{16} \end{array} \right) \end{array}$$

We will assume the joint cost of producing a_1 and a_2 is quadratic:

$$(34) \quad C(a_1, a_2) = a_1^2 + a_1 a_2 + a_2^2$$

We will assume both prices equal one, $\mu_1 = 1 = \mu_2$. Finally we shall assume that there is only one machine-day available and that it takes two units of machine-time for one unit of either good (the goods are infinitely divisible). There is a joint constraint

$$(35) \quad 2a_1 + 2a_2 \leq 1$$

If we denote the decision rules as follows $a_{11} = \alpha_1(y_1 = \text{bust})$, $a_{12} = \alpha_1(y_1 = \text{boom})$, $a_{21} = \alpha_2(y_2 = \text{bust})$, $a_{22} = \alpha_2(y_2 = \text{boom})$,

then the problem is one of selecting a vector of actions, not a continuous function. If there was no constraint on actions, the optimal decisions would be

$$(36) \quad \begin{array}{ll} \hat{a}_{11} = \frac{1}{18} & \text{because of high probability of bust} \\ \hat{a}_{12} = \frac{5}{18} & \text{because of high probability of boom} \\ \hat{a}_{21} = \frac{1}{18} & \text{because of high probability of bust} \\ \hat{a}_{22} = \frac{5}{18} & \text{because of high probability of boom.} \end{array}$$

In the quota constrained optimum, both departments are constrained to $1/4$ units of their goods (notice the symmetry of the problem) and the optimum decisions are

$$(37) \quad \begin{array}{ll} \hat{a}_{11} = \frac{5}{84} > \frac{1}{18} \\ \hat{a}_{12} = \frac{1}{4} < \frac{5}{18} \\ \hat{a}_{21} = \frac{5}{84} > \frac{1}{18} \\ \hat{a}_{22} = \frac{1}{4} < \frac{5}{18}. \end{array}$$

The constraint binds the decisions when the macro forecasts say "boom" and, as a result, optimum actions for the opposite forecast increase. In the unconstrained case the maximum expected profits are $\frac{1}{9}$, while in the quota constrained case, maximum expected profits are $\frac{71}{672}$, a drop of about 5%. The shadow price of machinery is $\frac{1}{112}$, so an additional unit of machine-time would increase expected profit by about 12%. Notice that $\max_{l,m} 2a_{1l} + 2a_{1m} = 1$ but also note that $\min_{l,m} 2a_{1l} + 2a_{1m} = \frac{5}{21} < 1$. That is, while the constraint on machine time is satisfied with certainty, we can clearly have extra machine time which could be put to profitable use.

c. Decision Making with Identical Information The team's general inability to coordinate actions to satisfy an inflexible joint constraint is a result of the independent individual decision making based on different information. The previous subsection on quotas restricted the independence of decision makers and this subsection now drops the assumption of different information. J.S. Jordan⁽⁴⁾ has studied the necessary conditions for an information structure to be "constraint adequate," that is, information sufficient to guarantee that the constraint will not be violated by a decision rule. As an example, if information variables are "noiseless" (there exist functions $y_i = \eta_i(\theta)$ such that the partition of Θ induced by η_i is identical for all i) then the information structure will be constraint adequate. As we will modify the information structure so that all the actions which are bound by a joint constraint are selected using identical information. Some teammates must ignore potentially useful information and other teammates must receive additional information to equalize knowledge.

Suppose each teammate i controls several action variables, a_{i1}, \dots, a_{im_i} , and receives a message vector (y_{i1}, \dots, y_{iN}) . Without any loss of generality, let us assume that the first element in each member's action vector refers to the use of a limited resource. These actions are jointly constrained by

$$(38) \quad \sum_i b_i a_{i1} = b' A^1 \leq c.$$

If there was no problem of coordination, the i^{th} teammate would select each of his m_i actions using all his information; that

is, his decision functions would be of the form:

$$(39) \quad a_{i1} = \alpha_{i1}(y_{i1}, \dots, y_{iN}), \dots, a_{im_i} = \alpha_{im_i}(y_{i1}, \dots, y_{iN}).$$

However, the first component of each teammate's vector of actions is bound by the inflexible constraint, so let us postulate that there is an information variable y^* which is known to all teammates. This variable may be a subset of the individual message vectors or a specially constructed message. It is assumed that each teammate uses only this information y^* when selecting the jointly constrained action:

$$(40) \quad a_{i1} = \alpha_{i1}(y^*) \quad i = 1, \dots, N.$$

If we denote the vector of functions $(\alpha_{11}, \dots, \alpha_{N1})$ by α^1 then the constrained team problem is to select individual decision functions to maximize the team's expected utility subject to constraint on resources:

$$(41) \quad \max E \{ U(\alpha^1(y^*), \alpha_{12}(y_1), \dots, \alpha_{Nm_N}(y_N), \theta) \}$$

subject to $b' \alpha^1(y^*) \leq c$ for all y^* .

We can define a Lagrangian for this problem with a multiplier function $\lambda(y^*)$, $L = E \{ U(\alpha^1(y^*), \dots, \alpha_{Nm_N}(y_N), \theta) \} + E \{ \lambda(y^*) (c - b' \alpha^1(y^*)) \}$.

The Kuhn-Tucker conditions are then expressed as

$$(42) \quad \frac{\partial L}{\partial \alpha_{i1}} \Big|_{y^*} = E \{ U_{\alpha_{i1}}(\hat{\alpha}, \theta) | y^* \} - \hat{\lambda}(y^*) b_i = 0 \quad \text{for each } y^*$$

$$(43) \quad \frac{\partial L}{\partial \alpha_{iA}} \Big|_{y_i} = E \{ U_{\alpha_{iA}}(\hat{\alpha}, \theta) | y_i \} = 0 \quad \text{for each } y_i, A = 2, \dots, m_i$$

$$(44) \quad \frac{\partial L}{\partial \lambda} \Big|_{y^*} = c - b' \hat{\alpha}^1(y^*) \geq 0, \quad \hat{\lambda}(y^*) \geq 0 \quad \text{for each } y^*$$

$$(45) \quad (c - b' \hat{\alpha}^1(y^*)) \hat{\lambda}(y^*) = 0 \quad \text{for each } y^*$$

If U is differentiable and concave in actions then these conditions are both necessary and sufficient for optimality of the constrained problem.

As an example suppose there is a firm that produces three goods and is divided into two production departments. Department one independently selects the output levels of goods a_{11} and a_{12} while department two picks the output level of good a_2 . The goods are produced for sale next month and while relative prices are known (we will assume all goods have equal prices) the aggregate price level may inflate or deflate randomly. If we let Θ denote the price level next month, the firm's revenue from producing quantities a_{11}, a_{12}, a_2 is

$$(46) (a_{11} + a_{12} + a_2) \Theta.$$

The firm's two decision makers agree that the price level is distributed Gaussian-normal with an expected value of zero and variance of one. Department one, because it controls several variables, consults two price forecasts, the St. Louis equation and the F-M-P model, before it selects its output levels. Department two only observes the F-M-P forecast of price level. Both independent forecasts give unbiased Gaussian-normal forecasts of Θ with variance equal to 1.

$$(47) \phi(\theta) = N(0, 1)$$

$$(48) \phi(z_1 | \theta) = N(\theta, 1) \quad z_1 : \text{forecast of F-M-P model}$$

$$(49) \phi(z_2 | \theta) = N(\theta, 1) \quad z_2 : \text{forecast of St. Louis equation}$$

The joint cost of producing a_{11}, a_{12}, a_2 is quadratic

$$(50) C(a_{11}, a_{12}, a_2) = a_{11}^2 + a_{12}^2 + a_2^2 + a_{11}a_2 + a_{12}a_2$$

Finally, assume that goods a_{11} and a_2 use machinery while a_{12} does not and there is only one machine-day available. Both goods a_{11} and a_{12} require one machine-day for one unit of output, so the firm

faces the constraint $a_{11} + a_2 \leq 1$.

Because of the constraint on machine time department one agrees to select the output of a_{11} using only its F-M-P forecast variable z_1 but will use both z_1 and z_2 to pick a_{12} . In the linear-quadratic normal team problem optimal decision functions are linear in the information variables and in this particular constrained problem the optimal output rules are

$$(51) \quad \begin{aligned} \hat{\alpha}_{11}(z_1) &= -\frac{1}{6} z_1 + \frac{3}{5} \\ \hat{\alpha}_{12}(z_1, z_2) &= \frac{1}{4} z_1 + \frac{1}{6} z_2 + \frac{1}{5} \\ \hat{\alpha}_2(z_1) &= \frac{1}{6} z_1 + \frac{2}{5}. \end{aligned}$$

Notice that for all forecast from the F-M-P model, the firm does not violate its resource endowment of machinery. The expected profits are $-\frac{14.2}{24}$. If the first department made its decision on a_{11} based only on z_1 but there was no constraint on machine-time, expected utility from optimal decision rules would be $\frac{7}{24}$. If there was no constraint and the first department used all its information in selecting both a_{11} and a_{12} then the expected utility from optimal decision rules would be $\frac{8}{24}$. This loss in utility due to restricting the information structure for constrained variables is analogous to the loss in utility when quotas reduce the efficiency of the decision making, seen in the previous subsection.

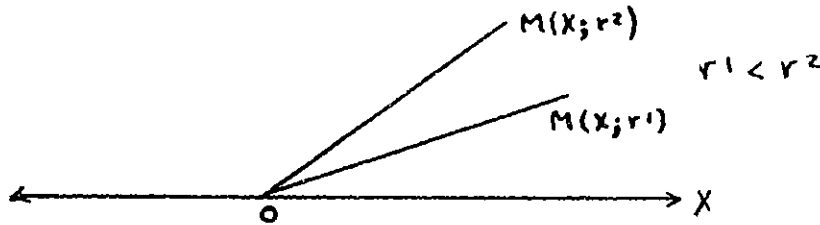
One final comment on the procedure of selecting all jointly constrained actions using identical information: this has appeared in the team literature in a slightly modified form. Groves and Radner⁽⁵⁾, in their study of the allocation of resource in a team, introduce a "resource manager" to the constrained team problem

and give him the power to select how much of a scarce resource each teammate will get. The resource manager acquires information about the environment in the same way that other team members do and uses that information to select the optimum allocation of the resource. Because the members of the team have identical tastes and beliefs, giving them the same information as the resource manager and requiring them to use only that information in selecting their portion of resources will lead formally to an identical resource allocation as the resource manager's allocation.

d. Penalty Functions and Joint Constraints What are the realities underlying our mathematical formulation of joint constraints on actions? The constraint $b'A \leq c$ has represented an immutable fact of nature for the team, such as the fixed volume in a warehouse. The team can only store a finite number of goods in a warehouse and thus the warehouse volume constraint must be satisfied with certainty. This example was selected deliberately to strain the meaning of the term "immutable". Almost all immutable constraints are slightly deceptive because of the unmentioned time period under consideration. Given suitable time almost all inflexible constraints on resources can be flexed; more warehouses may be built or additions made to old ones. In order to "flex" constraints, time and effort must be expended, usually at a progressively higher cost the more the constraint is flexed or the more rapidly it must be done. Additional resources can be purchased at a premium if the organization needs more than it was originally endowed. This sub-

section will study the constrained team problem when decision makers have complete independence and different information but when the constraint may be flexed by paying a penalty for extra resources.

Let $M(b'A-c;r)$ may be a family of penalty functions indexed by r which tells how big penalty (in utility measure) the team must pay for excess demand for the limited resource. Ideally we would like $M(x;r)$ to be positive and increasing for positive excess demand, x , and if x is non-negative then $M(x;r)=0$. Thus, the more resource demand exceeds the endowment, the bigger is the penalty, and if resource demand is less than endowment then no penalty is charged. The index r is an index of sterner punishment if $\frac{\partial M}{\partial r} > 0$ for all $x > 0$. A piecewise linear example is seen below:



The team wants to maximize the unconstrained difference between expected utility and expected penalty, rather than the jointly constrained expected utility.

$$(52) \max_{\alpha} E\{U(\alpha(Y), \theta)\} - E\{M(b'\alpha(Y) - c; r)\}.$$

The person-by-person necessary conditions that optimal decision functions $\hat{\alpha}_i$ must satisfy are

$$(53) 0 = E\{U_{\alpha_i}(\hat{\alpha}(Y), \theta) | y_i\} - b_i E\{M_x(b'\hat{\alpha}(Y) - c; r) | y_i\}$$

for each $y_i \in \mathcal{Y}_i \quad i=1, \dots, n.$

This states that for each message, the i^{th} member must select an

action that sets posterior expected marginal utility given the message equal to the posterior marginal penalty per unit of the i^{th} action given the message.

In the above formulation of M it was said that "ideally" we would like M to be zero for $x \leq 0$ and positive thereafter. This is an inconvenient restriction computationally (and analytically if M is not differentiable at the origin) and in examples we will only approximate these properties. For example, two convenient penalty functions are linear, $M(x;r) = rx$, and linear-quadratic, $M(x;r) = 2rx + r^2x^2$. These two will be used later because they do not complicate the quadratic-normal team problem which is a useful example. To see this let us use both linear and linear-quadratic penalty functions to solve a constrained quadratic-normal team problem.

Suppose we have a two member team with a linear-quadratic utility function

$$u(a_1, a_2, \theta) = 2\mu_1\theta a_1 + 2\mu_2\theta a_2 - \beta_{11}a_1^2 - 2\beta_{12}a_1a_2 - \beta_{22}a_2^2.$$

Assume the information variables and state of nature are Gaussian-normal with posterior expected values

$$(54) \quad E\{\theta | y_1\} = \bar{\theta}^1 y_1, \quad E\{\theta | y_2\} = \bar{\theta}^2 y_2 \\ E\{y_2 | y_1\} = \bar{y}_2^1 y_1, \quad E\{y_1 | y_2\} = \bar{y}_1^2 y_2.$$

The constraint is $b'A \leq c$.

Begin with a linear penalty function, $M(b'A - c; r) = 2r(b'A - c)$. The optimal decision rules will be linear in the information variables: $\hat{\alpha}_1(y_1) = \hat{a}_1 y_1 + \hat{s}_1$, $\hat{\alpha}_2(y_2) = \hat{a}_2 y_2 + \hat{s}_2$. The person-by-person optimality conditions require $\hat{\alpha}_1$ and $\hat{\alpha}_2$

to satisfy

$$(55) \quad 0 = \mu_1 E\{\theta(y_1)\} - f_{11}(\hat{a}_1 y_1 + \hat{s}_1) - f_{12}(\hat{a}_2 E\{y_2|y_1\} + \hat{s}_2) - r b_1$$

$$(56) \quad 0 = \mu_2 E\{\theta(y_2)\} - f_{22}(\hat{a}_2 y_2 + \hat{s}_2) - f_{12}(\hat{a}_1 E\{y_1|y_2\} + \hat{s}_1) - v b_2.$$

The optimal values of the slope and intercept coefficients of the decision functions are therefore:

$$(57) \quad \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} \bar{y}_2 \\ f_{12} \bar{y}_1 & f_{22} \end{pmatrix}^{-1} \begin{pmatrix} \mu_1 \bar{\theta}^1 \\ \mu_2 \bar{\theta}^2 \end{pmatrix}$$

$$(58) \quad \begin{pmatrix} \hat{s}_1 \\ \hat{s}_2 \end{pmatrix} = -r \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Notice that the constraint only affects the intercepts of the decision functions and not the slopes. The endowment of the resource, c , does not affect the decision at all.

Now suppose the penalty function is linear-quadratic, $M(x;r) = 2rx + r^2 x^2$. The person-by-person optimality conditions require the linear decision functions \hat{a}_1 and \hat{a}_2 satisfy

$$(59) \quad 0 = \mu_1 E\{\theta(y_1)\} - f_{11}(\hat{a}_1 y_1 + \hat{s}_1) - f_{12}(\hat{a}_2 E\{y_2|y_1\} + \hat{s}_2) \\ - b_1(r + r^2 b_1(\hat{a}_1 y_1 + \hat{s}_1) + r^2 b_2(\hat{a}_2 E\{y_2|y_1\} + \hat{s}_2) + v^2 c)$$

$$(60) \quad 0 = \mu_2 E\{\theta(y_2)\} - f_{22}(\hat{a}_2 y_2 + \hat{s}_2) - f_{12}(\hat{a}_1 E\{y_1|y_2\} + \hat{s}_1) \\ - b_2(r + r^2 b_2(\hat{a}_2 y_2 + \hat{s}_2) + r^2 b_1(\hat{a}_1 E\{y_1|y_2\} + \hat{s}_1) + r^2 c).$$

The optimal values of the coefficients are

$$(61) \quad \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \begin{pmatrix} f_{11} + r^2 b_1^2 & (f_{12} + r^2 b_1 b_2) \bar{y}_2 \\ (f_{12} + r^2 b_1 b_2) \bar{y}_1 & f_{22} + r^2 b_2^2 \end{pmatrix}^{-1} \begin{pmatrix} \mu_1 \bar{\theta}^1 \\ \mu_2 \bar{\theta}^2 \end{pmatrix}$$

$$(62) \quad \begin{pmatrix} \hat{s}_1 \\ \hat{s}_2 \end{pmatrix} = -(r + v^2 c) \begin{pmatrix} f_{11} + r^2 b_1^2 & f_{12} + r^2 b_1 b_2 \\ f_{12} + r^2 b_1 b_2 & f_{22} + r^2 b_2^2 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

With linear-quadratic penalty functions the slopes of the decision functions depend on the constraint and the amount of the resource endowed, c , helps determine optimal actions.

II. PLANNING PROCEDURES

a. The Planning Problem In this chapter it will be assumed that the organization's task is to solve a constrained maximization problem. While other alternatives have been studied, such as goal attainment or satisficing, the approach of constrained maximization is consistent with the basic team decision problem which is the focus of this chapter.

Suppose there are $N+1$ agents: N firms and the planning board. The firms are indexed $i=1,2,\dots,N$. There are m commodities indexed by the subscript $k=1,\dots,m$. Final consumption of good k is denoted c_k while net output of good k by firm i is denoted x_{ki} . If the firm produces good k then x_{ki} is positive and negative if it consumes the good. Net demand for good k is designated d_k , where

$$(63) \quad d_k = c_k - \sum_{i=1}^N x_{ki}$$

Vectors of the variables are defined as

$$d = (d_1, \dots, d_m)', \quad c_k = (c_1, \dots, c_m)', \quad x_i = (x_{i1}, \dots, x_{im})'$$

The vector of net outputs by firm i must represent technically feasible transformation of inputs into outputs which is represented by a set X_i , the set of all technically feasible outputs available to firm i . It is assumed that there are initially w_k units of good k available to the organization, and net demand must not exceed the initial resource, $d_k \leq w_k$. Preferences between final demands for goods are represented by a utility function $U(c)$.

The planning problem is to select a set of values of the vectors d, c, x_1, \dots, x_N to

(64) Max $U(c)$

subject to $d = c - \sum_{i=1}^N x_i \leq w$

$x_i \in X_i; \quad i = 1, \dots, N.$

A planning procedure is a iterative process for the solution of the organization's constrained maximization problem. Given an arbitrary initial approximating solution, the planning procedure specifies how it must be modified to produce a better approximation. The planning board transmits "prospective indices" to the firms: these prospective indices, denoted P^τ (where τ is an index of the stages of procedure), summarize the current approximating solution. The firms use these indices to compute an answer or proposal. The sequence of prospective indices-proposals ultimately terminated and the final approximating solution to the constrained maximization problem becomes the organization's "plan".

The size of the organization's constrained maximization problem, measured by the number of commodities or the complexity of the technology, is generally very large. This makes it impossible for the planning board to gather all the information needed to solve the problem by itself. As a result planning procedures are typically informationally decentralized; that is, the planning board does not pool all the information and solve the constrained maximization by itself, but it delegates responsibility for parts of the problem to the firms. Generally the planning board knows only the objective function and the organization-wide constraints, while the firms know only their own production possibilities.

The remainder of this section presents several of the better

known iterative, decentralized planning procedure for constrained maximization problems. These procedures will be used later as the basis for planning procedures by decentralized teams.

b. Lange-Arrow-Hurwicz Tatonnement Procedure The main conclusion of the economic theory of socialism is that optimal plan of the organization should satisfy the same marginal equalities as those resulting from equilibrium in a perfectly competitive situation. The traditional model of how markets reach their equilibrium positions postulates that prices "gro e" toward equilibrium by adjusting according to excess demand. The first formulation of a planning procedure, by Lange⁽¹⁾, was based on Walras's "tatonnement" process. The essence of the tatonnement process is that an auctioneer prices for all goods, receives supply and demand bids from firms and consumers based on the quoted prices, and then adjust prices by increasing the price of goods which had an excess demand and vice versa. Lange's procedure was studied in a rigorous manner by Arrow and Hurwicz⁽²⁾ and their work provides the basis for this subsection on tatonnement planning procedures.

The planning board replaces both the consumer and the auctioneer. The plan deals only with the production side of the organization and the planning board's utility function is a surrogate for the individual preferences of the consumers. The prospective indices of the planning board are the prices for each of the commodities, $P = (P_1, \dots, P_m)'$. The firms' proposals are their net production vectors, $x_i = (x_{i1}, \dots, x_{mi})$. The initial prices P^0 are

arbitrary and may be those resulting from some previous plan. The procedure from stage τ is defined by

(a) At stage τ , the i^{th} firm maximizes its profits, $P^\tau x_i^\tau$ subject to x_i^τ being feasible, $x_i^\tau \in X_i$; and reports net output x_i^τ to the planning board.

(b) The planning board finds the final consumption that maximizes the difference between utility and cost of consumption, $U(C^\tau) - P^\tau C^\tau$.

(c) The vector of net demands for all goods is $d^\tau = C^\tau - \sum_{i=1}^N x_i^\tau$.

The planning board increases the price of goods whose net demand exceeds resource availability:

$$P^{\tau+1} = \max(0, P^\tau + K(d^\tau - w))$$

where K is a speed of adjustment coefficient. These prices are announced to the firms and the procedure begins again.

As an example of a tatonnement planning procedure let us study an activity analysis model of production. Suppose we denote the i^{th} firm's sale of operation by x_i and the amount of good k produced by firm i at scale x_i by $g_{ki}(x_i)$. The net output of good k is thus $\sum_{i=1}^N g_{ki}(x_i)$. The resource allocation problem is stated as

$$(65) \text{ Max } U(c_1, \dots, c_m) \text{ subject to } c_k - \sum_{i=1}^N g_{ki}(x_i) \leq w_k \quad k=1, 2, \dots, m.$$

The Lagrangian for this problem is

$$(66) L = U(c_1, \dots, c_m) + \sum_{k=1}^m \lambda_k (w_k + \sum_{i=1}^N g_{ki}(x_i) - c_k)$$

where λ_k is the shadow price of the k good. The optimal conditions are specified as

$$(67) \frac{\partial L}{\partial c_k} - \lambda_k \leq 0 \quad , \text{ equal to } 0 \text{ if } c_k > 0$$

$$(68) \sum_{k=1}^m \lambda_k \frac{\partial g_{ki}}{\partial x_i} \leq 0, \quad \text{equal to 0 if } x_i > 0$$

$$(69) c_k - \sum_{i=1}^N g_{ki}(x_i) \leq w_k, \quad \text{equality if } \lambda_k > 0.$$

A tatonnement planning procedure would be specified by the following sequence, beginning at stage τ with shadow prices λ_k^τ $k=1, \dots, m$.

(a) Firms maximize profits at prices λ_k^τ , profit = $\sum_{k=1}^m \lambda_k^\tau g_{ki}(x_i)$, subject to $x_i^\tau \geq 0$ this leads to first order conditions

$$\sum_{k=1}^m \lambda_k^\tau \frac{\partial g_{ki}}{\partial x_i} \leq 0, \quad \text{with equality if } x_i^\tau > 0.$$

(b) The planning board finds final consumptions that maximizes the difference between utility and the cost of consumption at price λ_k^τ , $U(c_1^\tau, \dots, c_m^\tau) - \sum_{k=1}^m \lambda_k^\tau c_k^\tau$.

This leads to conditions $\frac{\partial U}{\partial c_k^\tau} - \lambda_k^\tau \leq 0$, with equality if $c_k^\tau > 0$.

(c) The planning board adjusts prices according to net demand for goods

$$\frac{d\lambda_k^\tau}{d\tau} = \begin{cases} 0 & \text{if } \lambda_k^\tau = 0 \quad \text{and} \quad c_k^\tau - \sum_{i=1}^N g_{ki}(x_i^\tau) > w_k \\ \kappa (c_k^\tau - \sum_{i=1}^N g_{ki}(x_i^\tau)) & \text{otherwise.} \end{cases}$$

c. Dantzig-Wolfe-Malinvaud Decomposition Linear programming has been the precursor of many developments in theoretical economics. A technique for solving large linear programs developed by Dantzig and Wolfe⁽³⁾, was used in a more general context by Malinvaud⁽⁴⁾ to study iterative planning procedures. The linear programming technique is called "decomposition" and will be presented here, followed by Malinvaud's planning process.

Suppose we want to maximize a linear function which can be

dichotomized into a function of two sets of variables $P_1'X + P_2'Y$ where $X = (x_1, \dots, x_m)'$, $P_1 = (P_{11}, \dots, P_{1m})'$, $Y = (y_1, \dots, y_n)'$ and $P_2 = (P_{21}, \dots, P_{2j}, \dots, P_{2n})'$. The linear constraints can be divided into three sets, (1) constraints on both X and Y , (2) constraints only on X , and (3) constraints only on Y .

$$(70) \quad \bar{A}_1 X + \bar{A}_2 Y \leq \bar{b} \quad : \quad s \quad \text{constraints}$$

$$(71) \quad A_1 X \leq b_1 \quad : \quad t \quad \text{constraints}$$

$$(72) \quad A_2 Y \leq b_2 \quad : \quad u \quad \text{constraints}$$

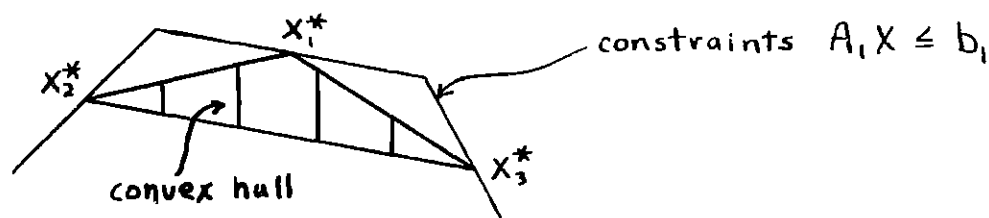
There will be provisional shadow prices for each constraint denoted $\pi_1, \dots, \pi_s, \pi_{s+1}, \dots, \pi_{s+t}, \pi_{s+t+1}, \dots, \pi_{s+t+u}$.

Now suppose that the problem is too large computationally to be solved all at once. If there is a sequence of activity vectors known to satisfy the last $t + u$ constraints, $X_1^*, \dots, X_L^*; Y_1^*, \dots, Y_L^*$, then any linear combination of those vectors will satisfy the last $t + u$ constraints. Denote a particular linear combination by

$$(73) \quad X_{\bar{z}}^{\text{ave}} = \sum_{k=1}^L \mu_k X_k^*, \quad \mu_k \geq 0, \quad \sum_{k=1}^L \mu_k = 1$$

$$(74) \quad Y_{\bar{z}}^{\text{ave}} = \sum_{k=1}^L \gamma_k Y_k^*, \quad \gamma_k \geq 0, \quad \sum_{k=1}^L \gamma_k = 1.$$

These vectors can lie anywhere in the convex hull of the corresponding sequence of vectors



A smaller linear program will now be solved: select weights μ_k, γ_k to

$$(75) \quad \max P_1' X_{\bar{z}}^{\text{ave}} + P_2' Y_{\bar{z}}^{\text{ave}}$$

$$\text{subject to } \bar{A}_1 X_{\tau}^{\text{ave}} + \bar{A}_2 Y_{\tau}^{\text{ave}} \leq \bar{b}$$

$$X_{\tau}^{\text{ave}} = \sum_{k=1}^{\tau} \mu_k X_k^*, \quad \sum_k \mu_k = 1, \mu_k \geq 0$$

$$Y_{\tau}^{\text{ave}} = \sum_{k=1}^{\tau} \gamma_k Y_k^*, \quad \sum_k \gamma_k = 1, \gamma_k \geq 0.$$

When this problem is solved, shadow prices will exist for the joint constraints, denoted $\pi^{\tau} = (\pi_1^{\tau}, \dots, \pi_s^{\tau})$. These are used to modify the objective functions of two smaller subsidiary problem. The two subsidiary problems to be solved are

$$(76) \quad \max (P_1 - \bar{A}_1 \pi^{\tau}) X_{\tau+1} \\ \text{s.t.} \quad A_1 X_{\tau+1} \leq b_1$$

and

$$(77) \quad \max (P_2 - \bar{A}_2 \pi^{\tau}) Y_{\tau+1} \\ \text{s.t.} \quad A_2 Y_{\tau+1} \leq b_2.$$

These two problems will result in vectors $X_{\tau+1}^*$ and $Y_{\tau+1}^*$ that are added to the sequence of activity vectors and a new joint linear program is solved. The process iterates until $X_{\tau}^{\text{ave}} = X_{\tau+1}^{\text{ave}}$ and $Y_{\tau}^{\text{ave}} = Y_{\tau+1}^{\text{ave}}$. Because of the finite number of extreme points and convex nature of the constraints, the procedure stops after a finite number of iterations.

The Malinvaud planning procedure generalizes the Dantzig-Wolfe decomposition problem by making the objective function concave, denoted $U(c_1, \dots, c_m)$, and the individual constraints convex. The problem is stated as

$$(78) \quad \max U(c_1, \dots, c_m) \\ \text{s.t.} \quad c - \sum_{i=1}^N x_i \leq w \\ x_i \in X_i \quad i=1, 2, \dots, N$$

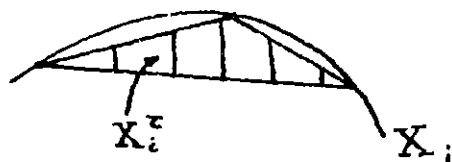
where U is a concave and X_i is convex. The prospective indices at stage τ are the prices of the commodities, $P^\tau = (P_1^\tau, \dots, P_n^\tau)$.

The firm's proposal at stage τ is a vector of net outputs that belong to its technology. The planning is defined at stage τ by the following rules:

(a) The i^{th} firm maximizes profit at the prices P^τ , profit = $P^\tau X_i$ subject to X_i belong to the technology X_i . This optimal vector is X_i^τ ;

(b) The planning board treats the individual technology as the convex hull of the past sequence of net outputs :

$$X_i^\tau = \{ X_i^{\text{ave}} \mid X_i^{\text{ave}} = \sum_{j=1}^{\tau} M_{ij} X_i^j, \sum_j M_{ij} = 1, M_{ij} \geq 0 \}$$



The planning board solves the subsidiary problem

$$(79) \quad \begin{aligned} &\max U(c) \\ &\text{s.t. } c - \sum X_i^{\text{ave}} \leq \omega \\ &X_i^{\text{ave}} \in X_i^\tau \quad i=1, 2, \dots, N; \end{aligned}$$

(c) From the subsidiary maximization problem, the planning board has shadow prices for each good which are used as the prospective indices in the next iteration, $P^{\tau+1}$

d. Weitzman's Dual Decomposition The Dantzig-Wolfe-Malinvand planning procedure builds up an approximation of the firms technology from the inside of the set of feasible net outputs. A dual procedure developed by Weitzman⁽⁵⁾ builds an approximation of the

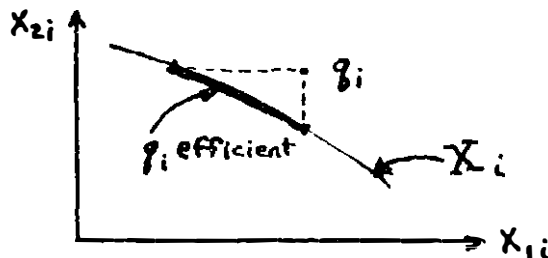
technology from the outside. The interesting distinction between the two is that Malinvaud's procedure is price guided in the sense that the prospective indices are tentative prices for the commodities while the Weitzman procedure is guided by output targets suggested tentatively by the planning board.

Suppose that at stage τ the planning board has an estimated set of technologically feasible net outputs for firm 1, denoted X_i^τ , such that the true technology is contained in X_i^τ , i.e. $X_i \subseteq X_i^\tau$. If the planning board learns a new feasible point $x_i^{\tau+1}$ on the boundary of X_i and shadow prices $(\pi_1^{\tau+1}, \dots, \pi_m^{\tau+1})$ that represent the marginal products at that point on the boundary, then the planning board can get a better estimate of X_i by the set

$$(80) \quad X_i^{\tau+1} \equiv X_i^\tau \cap \{x \mid \pi^{\tau+1} \cdot x \leq \pi^{\tau+1} \cdot x_i^{\tau+1}\}.$$

The construction of approximating technologies using this technique is the basis for Weitzman's planning procedure.

Before specifying the formal planning procedure we must define a useful concept. We say that a production point \hat{x}_i is "efficient with respect to a target quota $q_i = (q_{i1}, \dots, q_{im})$ " if $\hat{x}_i \in X_i$ and $\hat{x}_i \leq q_i$ and there exist a positive vector P_i such that $P_i \cdot x_i \leq P_i \cdot \hat{x}_i$ for all $x_i \in X_i$, $x_i \leq q_i$.



The prospective indices at stage τ of the planning board will be target quotas to each firm, q_i^τ . The firm's proposals

at stage τ of the planning procedure will be a q_i^τ - efficient net output $\lambda_i^{\tau+1}$ and the corresponding prices $\pi_i^{\tau+1}$ that specify the marginal product at $\lambda_i^{\tau+1}$. The planning procedure follows these rules

- (a) At stage τ if target quota q_i^τ is not producible then the i^{th} firm reports any q_i^τ - efficient point $\lambda_i^{\tau+1}$ and price vector $\pi_i^{\tau+1}$;
- (b) The planning board updates its set of feasible production possibilities $X_i^{\tau+1} = X_i^\tau \cap \{X \mid \pi^{\tau+1} X \leq \pi^{\tau+1} \lambda_i^{\tau+1}\}$;
- (c) The planning board solves the following problem for optimal target quotas:

$$(81) \max U(C^{\tau+1})$$

$$\text{s.t. } C^{\tau+1} - \sum_{i=1}^N q_i^{\tau+1} \leq W$$

$$\text{and } q_i^{\tau+1} \in X_i^{\tau+1} \quad i=1, 2, \dots, N.$$

The optimal values for $q_i^{\tau+1}$ are proposed in the next stage as target quotas.

While the Malinvaud process builds up a polyhedral approximation of the technology everywhere interior to the true technology, Weitzman's process creates an approximation of X_i by the building an envelope of supporting hyperplanes.

c. Heal's Quantity Target Gradient Procedure In practice the indices of planning boards are quantitative input-output targets rather the prices suggested by Lange, Arrow, Hurwicz, and Malinvaud. Weitzman's procedure is one procedure whose basic indices are quantitative targets. Heal has also developed a planning procedure, based on the gradient method, which uses quantitative indices

rather than prices.⁽⁶⁾ Heal's procedure has two desirable properties that do not hold for the gradient procedure of Lange- Arrow-Hurwicz; namely: (1) at any stage of planning, the tentative plan is feasible and (2) the objective function increases monotonically at each iteration.

Heal's planning method begins when the planning board proposes an allocation of inputs between firms. At these inputs the firms report the maximum output from these inputs and the marginal productivity of the inputs, given this information, the planning board reallocates inputs, shifting resources toward uses where they are more productive. The new allocation is used to begin another round of planning

The essence of the procedure can be described by an economy where there are no intermediate products; i.e., all firms produce consumption goods from primary resources. Firms are indexed by $i = 1, 2, \dots, N$, while resources are indexed by $j = 1, \dots, m$. The production of the i^{th} commodity by the i^{th} firm is denoted by the production function

$$(82) \quad C_i = f_i(X_{i1}, \dots, X_{im})$$

where X_{ij} is the amount of resource j used by firm i . Recognizing the use of slack variables, the constraint on resource j is represented by the equality

$$(83) \quad \sum_{i=1}^N X_{ij} = \omega_j \quad j = 1, \dots, m.$$

The planning board proposes an arbitrary allocation of resources amongst firms at stage τ , (X_{ij}^{τ}) , which satisfies the constraints (83). The firms use their technical knowledge to report the maximum output from their inputs

$$(84) \quad c_i^{\tau} = f_i(x_{i1}^{\tau}, \dots, x_{im}^{\tau})$$

and the marginal productivity of its input

$$(85) \quad f_{ij}^{\tau} = \frac{\partial}{\partial x_{ij}} f_i(x_{i1}^{\tau}, \dots, x_{im}^{\tau}).$$

The planning board lowers the allocation of a resource to a firm whose marginal social product is below the average marginal social product for all uses of the resource, or formally

$$(86) \quad \frac{d x_{ij}^{\tau}}{d \tau} = \varepsilon \left(\frac{\partial u}{\partial c_i^{\tau}} \cdot f_{ij}^{\tau} - \frac{1}{N} \sum_{k=1}^N \frac{\partial u}{\partial c_k^{\tau}} f_{kj}^{\tau} \right)$$

The procedure is defined so that

$$(87) \quad \frac{d}{d \tau} (x_{1j}^{\tau} + x_{2j}^{\tau} + \dots + x_{Nj}^{\tau}) \equiv 0$$

which means that the total use of a resource as inputs does not exceed the amount of the resource available.

When the output of the firms can be used as inputs by other firms, the planning procedure can be modified to include these intermediate products. The only complication is that average marginal social productivities, which guide the reallocation procedure, become computationally more difficult for the planning board.

The non-price guided gradient procedure of Heal has properties that the price guided gradient procedure of Arrow-Hurwicz does not: feasibility and monotonic increasing payoff. In order to gain these advantages the procedure requires the planning board and firm to exchange more information and creates a computation problem for the planning board.

III. PLANNING BY A CONSTRAINED TEAM

a. The Team Planning Problem This section explores procedures that a large team might use to derive its optimal strategies when actions are constrained. The size and complexity of some teams prevents any teammate or group of teammates from pooling all the useful information and finding the best decision functions. The team must have some program to elicit technological data from the individual members and to modify proposed solutions based on that information. Several planning procedures developed in the literature on centrally planned economies will be modified to fit the structure of the particular forms of the jointly constrained team decision problem.

The solution procedure outlined here all involve exchange of data between the individual teammates and a "planner". After some number of iterations decision rules are selected that determine the relationship between incoming signals about the uncertain environment and the actions taken by the team members. In some cases the final selection of an individual decision rule is made by the teammate and in others it is dictated by the planner. Once the final decision function is fixed the teammate acts only according to his rule and is responsible only for taking the indicated action for his given signal.

The important features of the organization studied here are (1) complete agreement between the decision makers on a single objective function (2) interdependence of individual actions

through a non-additive objective function (3) decentralized authority in decision making (4) differences among decision makers in the signals about the uncertain environment and (5) joint constraints on the actions of different members. The organization is a "team" with non-additive utility and joint constraints. As discussed previously, three components of the jointly constrained team problem are incompatible: independent decision making, different random signals and inflexible, joint constraints. As a result the constrained team problem must be modified to guarantee its consistency. A few such modifications were discussed above. Using these modified constrained team problems as a basis, several planning algorithms will be defined and analyzed.

An essential part of the theory of team decisions is the communication between teammates related to the unknown state of nature. Signals are observed by individual teammate and intraorganizational messages are sent in order to reduce the team's uncertainty about the state of nature. These messages are sent after decision rules are known and before actual actions must be selected and implemented. In the iterative planning process, which determines the rules relating signals-messages to actions, information is also exchanged. This information does not reduce the uncertainty of the state of nature; instead it is related to technological parameters which define the team's problem. Such parameters are known by at least some members of the team and are not considered part of the state of nature. Communications about the state of nature and communications about technological

parameters differ another way: The former are generally direct member-to-member transmissions while the latter are between the planner and the individual teammates. We will refer to communications dealing with the state of nature as "messages" and communication dealing with technological parameters as "prospective indices", if from the planner, or "proposals", if from individual members.

The iterative planning process might be thought of as a computational device by which the team problem is solved by distributing technological data to those who need it. Why doesn't the planner just gather all the information at one time, solve the team problem and announce the optimal decision rules? Often technological information, while known by various teammates, is hard to summarize or tabulate for use by other team members. Also, such summary information may still be too voluminous and costly to be justified. The planning procedure economizes on technological communication by requiring the transmission of only "relevant" data.

The team assumption that $U(A, \theta)$ is the common utility function of all teammates implies that $U(A, \theta)$ is "known" to all team members. However, judging from the person-by-person optimality conditions, the i^{th} team member needs to know only his marginal impact on utility $U_{a_i}(A, \theta)$ in order to select his optimal actions given the optimal decision rules of the other teammates. Knowledge of U_{a_i} does not necessarily imply knowledge of U . In the extreme case of additively decomposable utility, $U(A, \theta) = \sum_i U^i(a_i, \theta)$, all the i^{th} teammate must know in order to solve his part of the

problem is $U^i(a_i, \theta)$.

If the team's utility function can be written as
 (88) $U(A, \theta) = \sum_i U^i(a_i, \theta) + \sum_i \sum_j U^{ij}(a_i, a_j, \theta) + U^0(A, \theta)$
 where $U^{ij} \equiv U^{ji}$, and if the i^{th} teammate knows only the functions
 $U^i, U^{ii}, U^{i2}, \dots, U^{iN}$ and U^0 , then the teammates have enough knowl-
 edge to solve their parts of the problem, but no single member has
 enough knowledge to generate a complete solution.

In some cases the i^{th} teammate may only know the above
 functions implicitly. An example is a firm which knows when any
 particular tentative "activity" is feasible but cannot translate
 this into a production function detailing the maximum feasible
 output from given inputs. Even if the functions were known explic-
 itly, they may be too complex to be summarized in a small number
 of parameters. The transmission of the entire set of functions
 $U^i, U^{ii}, \dots, U^{iN}$ may simply be too costly for consideration. How
 does the team solve its problem? The team must have some proce-
 dure for exchanging relevant information about the utility func-
 tion. Many similar procedures exist in the planning literature
 but they exclude either signalling about the uncertain environ-
 ment or externalities in the payoff functions. We will modify
 some of the procedures to handle both the signals and externa-
 lites which characterize a team decision problem.

b. Price Guided Planning by a Team with Identical Information

The team consist of N members, indexed by i . Each team controls
 a personal action a_i and receives a personal information variable

y_1 . Team utility is $U(a_1, \dots, a_N, \theta)$ where θ , the state of nature has a team probability density function $\phi(\theta)$. The team faces a joint constraint on the actions $\sum_{i=1}^N b_i a_i \leq c$, where c is the endowment of the resource and b_i measures the amount of resource used per unit of the i^{th} action. There are no conceptual difficulties in introducing several joint constraints or allowing vector actions $a_i = (a_{i1}, a_{i2}, \dots, a_{im_i})$. For notational simplicity it will be assumed a_1 is a scalar and there is only one joint constraint.

As elaborated above the classical team assumptions of independent decision making by teammates and different information are generally incompatible with inflexible joint constraint $b'A \leq c$. One way around this problem is to have the teammates use the same information when they select actions that are jointly constrained. This modification will be the form studied here. We will assume $y_1 = y_j = y$ is the common information variable with a conditional probability density function $\phi(y|\theta)$ which is used by all teammates to select their actions. The team's problem is to select decision functions $\alpha(y) = (\alpha_1(y), \dots, \alpha_N(y))'$ to

(89) maximize $W = E\{U(\alpha(y), \theta)\} = \int_{\Theta} \int_{\mathcal{Y}} U(\alpha(y), \theta) \phi(y|\theta) \phi(\theta) dy d\theta$
 subject to $b'\alpha(y) \leq c$ for all $y \in \mathcal{Y}$.

The optimality conditions require the introduction of a Lagrange multiplier function $\lambda(y)$ and the definition of a Lagrangian:

$$(90) L = E\{U(\alpha(y), \theta)\} + E\{\lambda(y)(c - b'\alpha(y))\}.$$

If U is differentiable and concave in A , the following Kuhn-Tucker person-by-person conditions are both necessary and sufficient for

optimality.

Theorem: $\hat{\alpha}(y)$ are optimal if and only if there exist a multiplier function $\hat{\lambda}(y)$ such that

$$(91) \quad E\{u_{a_i}(\hat{\alpha}(y), \theta) | y\} = b_i \hat{\lambda}(y) \quad \text{for each } y \in \mathcal{Y} \quad i=1, 2, \dots, M$$

$$(92) \quad b' \hat{\alpha}(y) \leq c, \quad \hat{\lambda}(y) \geq 0 \quad \text{for each } y \in \mathcal{Y}$$

$$(93) \quad (b' \hat{\alpha}(y) - c) \hat{\lambda}(y) = 0 \quad \text{for each } y \in \mathcal{Y}.$$

The interpretation of these conditions are as follows: select decision functions so the posterior expected marginal utility with respect to a_1 given y equals the marginal cost in utility units of the resource used by a_1 . The resource demand must never exceed the endowment for all possible information and if for some information the endowment is not completely used, the shadow price of the resource in utility terms must equal zero.

Suppose the conditions (91) - (93) are very complicated to solve analytically or suppose, as discussed above, the technical knowledge of $U(A, \theta)$ is decentralized so that no team member has enough knowledge to solve (91) - (93). Some iterative procedure for exchanging technical knowledge must be used to elicit the optimal decision rule. The method to be explored now is based on the price-guided gradient procedure of Arrow and Hurwicz. (1a)

Suppose at stage τ , the i^{th} teammate has a decision function $\alpha_i^\tau(y)$ and the planner has a shadow price function $\lambda^\tau(y)$. Typically these functions will not satisfy the Kuhn-Tucker person-by-person. What the team would like to do is adjust the functions to get higher up the expected utility "hill" and this can be done by modifying each function by its corresponding variational par-

tial derivative of the Lagrangian. Define a team gradient procedure at stage τ by

$$(94) \quad \frac{d\alpha_i^\tau(y)}{d\tau} = E \{ U_{q_i}(\alpha^\tau(y), \theta) | y \} - b_i \lambda^\tau(y) \quad \text{for all } y, i=1,2,\dots,N$$

$$(95) \quad \frac{d\lambda^\tau(y)}{d\tau} = \begin{cases} 0 & \text{if } \lambda^\tau(y) = 0 \text{ and } b' \alpha^\tau(y) < c \\ b' \alpha^\tau(y) - c & \text{otherwise} \end{cases} \quad \text{For all } y.$$

How exactly does the team generate such a gradient solution processes? At stage τ the planner announces a shadow price function for the resource $\lambda^\tau(y)$. If the utility function was additively decomposable, this would be all the information the teammates would need to know to adjust their decision functions. However, in the general case with externalities in the utility the teammates must know the proposed decision functions of other teammates. So we must also allow the planner to disseminate the decision functions to teammates that require them. Given the shadow price of the resource and the decision functions of other teammates, the i^{th} teammate modifies his decision function by the posterior marginal net utility and reports this back to the planner. The planner modifies the shadow price function by the excess demand function for resources, taking account that it must never be negative. The planner announces this shadow price function and proposed decision functions and the process begins again. The process will terminate when the planner finds the current decision functions and shadow prices that satisfy (91) - (93).

It should be pointed out that while we have just define a gradient process that adjusts decision functions, this could be reduced to just adjusting an action vector. If the total problem only involves decision functions that depend on y , there is no

reason the team could not wait until the information y is known, say $y = \tilde{y}$, and then maximize posterior expected team utility given \tilde{y} by selecting a vector of actions $(\tilde{\alpha}_1, \dots, \tilde{\alpha}_N) = (\tilde{\alpha}_1(\tilde{y}), \dots, \tilde{\alpha}_N(\tilde{y}))$. The gradient solution of this posterior problem would be exactly like (94) and (95) except instead of adjusting at all $y \in \mathcal{Y}$, it would adjust only at $y = \tilde{y}$. The reason this simplification was not done above is that in most cases the team will have other actions not bound by a joint constraint and not selected based on the common information y . In such a case the entire decision function $\alpha_i(y)$ must be calculated because other decision makers will not know y . (1b)

A question that ought to be answered about any solution procedure is "Does the solution process converge to the optimum?" In gradient procedures this is usually answered by treating (94) and (95) as a system of first order, non-linear differential equations in \mathcal{T} and evaluating the stability of any initial value solution. Stability of differential equations is typically analyze using Lyapunov's second method. However, it should be noted that (94) and (95) is not a finite system of differential equations in the general case. Notably if \mathcal{Y} is a subset of the real numbers that is countably or uncountably infinite, bounded or unbounded, the differential equations will not be discrete and finite in number. Suppose for example $\mathcal{Y} = [0, 1]$, the closed unit interval, then (94) and (95) are actually a continuum of differential equations. The applicability of Lyapunov's method as well as existence and uniqueness results will not necessarily

carry over to these more general cases. The issues involved are discussed in the appendix on "Convergence of Gradient Methods in Abstract Spaces".

Suppose that \mathcal{Y} is a discrete, finite set of real numbers $\mathcal{Y} = \{y_1, \dots, y_r\}$, and let $\phi(y_e | \theta)$ be the probability that $y = y_e$, $1 \leq e \leq r$, given the state of nature. Then the team's problem is to select, not functions, but vectors of actions for each teammate; let us denote the decision function by the r -vector

$$(a_{i1}, \dots, a_{ir}) = (a_i(y_1), \dots, a_i(y_r))$$

The team problem is to select these vector of actions to

$$(96) \quad \max W = \sum_{e=1}^r \int_{\Theta} u(a_{1e}, \dots, a_{ne}, \theta) \phi(y_e | \theta) \phi(\theta) d\theta$$

subject to $\sum_i b_i a_{ie} \leq c \quad e = 1, 2, \dots, r.$

We must now introduce a Lagrange multiplier vector $\lambda = (\lambda_1, \dots, \lambda_r)'$ with each component corresponding to the identically numbered element of \mathcal{Y} . If at some stage τ in the solution procedure the teammates have approximate solutions $a_{ie}(\tau)$ and the planner has approximate shadow prices $\lambda_e(\tau)$ then the team adjusts its decisions and prices according to the following process

$$(97) \quad \frac{da_{ie}(\tau)}{d\tau} = \int_{\Theta} u_{a_i}(a_{1e}(\tau), \dots, a_{ne}(\tau), \theta) f(\theta | y_e) d\theta - b_i \lambda_e(\tau)$$

$$(98) \quad \frac{d\lambda_e(\tau)}{d\tau} = \begin{cases} 0 & \text{if } \lambda_e(\tau) = 0 \text{ and } b' A_e < c \\ \sum_i b_i a_{ie}(\tau) - c & \text{otherwise} \end{cases}$$

For each $e = 1, \dots, r$, where $f(\theta | y_e)$ is the posterior p.d.f. of θ given $y = y_e$.

In this problem we can give the following stability theorem

Theorem 1: If $u(a_1, \dots, a_n, \theta)$ is strictly concave and twice differentiable in a_1, \dots, a_n for all θ and $\hat{a}_{ie}, \hat{\lambda}_e \quad i=1, \dots, N; e=1, \dots, r$ is

a saddle point for the problem (96) then

- (a) the decisions \hat{a}_{je} $j=1, \dots, N$; $e=1, \dots, r$ are unique
- (b) the solution of the gradient algorithm (97) and (98) for any initial condition exists and is unique
- (c) the solution of (97) and (98) for any initial conditions converges to the saddle point.

Proof: Strict concavity of U implies there is at most one "peak" of W and if \hat{a}_{je} is a "peak" then it must be unique.

Since the Lagrangian is twice differentiable in both a_{je} and λ_e , the gradient process must satisfy Lipschitz conditions and will thus have a unique solution to (97) and (98) for any initial decision functions and shadow prices. Global convergence will follow if a Lyapunov function can be found.

Let

$$(99) \quad A = (a_{11}, \dots, a_{1r}, a_{21}, \dots, a_{1e_1}, \dots, a_{Nr})'$$

and then look at the function

$$(100) \quad D(A(\tau), \lambda(\tau)) = \frac{1}{2} \|A(\tau) - \hat{A}\|^2 + \frac{1}{2} \|\lambda(\tau) - \hat{\lambda}\|^2$$

The rate of change of D with respect to τ is

$$(101) \quad \dot{D}(\tau) = \dot{A}(\tau)' (A(\tau) - \hat{A}) + \dot{\lambda}(\tau)' (\lambda(\tau) - \hat{\lambda}).$$

If $L(A, \lambda)$ is the Lagrangian for this problem, the strict concavity of U implies

$$(102) \quad L(\hat{A}, \lambda) - L(A, \lambda) < \nabla_A L'(\hat{A} - A)$$

and linearity of L in λ implies

$$(103) \quad L(A, \hat{\lambda}) - L(A, \lambda) = \nabla_\lambda L'(\hat{\lambda} - \lambda).$$

Since a saddle point is defined as

$$(104) \quad L(A, \hat{\lambda}) \leq L(\hat{A}, \hat{\lambda}) \leq L(\hat{A}, \lambda)$$

it is easily seen that

$$(105) \quad 0 \leq L(\hat{A}, \lambda) - L(A, \hat{\lambda}) < \nabla_A L'(\hat{A} - A) - \nabla_\lambda L'(\hat{\lambda} - \lambda).$$

Since $-\nabla_\lambda L'(\lambda - \hat{\lambda}) \equiv (b'A - c)'(\lambda - \hat{\lambda})$ is non negative

if $\gamma = 0$ and $b'a - c < 0$, it can be shown that

$$(106) \quad -\nabla_\lambda L'(\lambda(\tau) - \hat{\lambda}) \geq \dot{\lambda}(\tau) (\lambda(\tau) - \hat{\lambda}).$$

By definition

$$(107) \quad \nabla_A L = \dot{A}(\tau) - \lambda(\tau) b$$

therefore $D(\tau)$ is monotonically decreasing in τ :

$$(108) \quad \dot{D}(\tau) \leq \nabla_A L'(A(\tau) - \hat{A}) - \nabla_\lambda L'(\lambda(\tau) - \hat{\lambda}) < 0.$$

Obviously $D(\tau)$ is bounded below zero and equals zero if

and only if $A(\tau) = \hat{A}$, $\lambda(\tau) = \hat{\lambda}$. Hence D is a Lyapunov function

and the saddle point is globally stable. Q.E.D.

As an example of how this price guided gradient procedure might occur in practice, imagine a firm that produces N goods a_1, a_2, \dots, a_N . The firm is divided into production departments, one for each good, which independently select the output level of their good. Today's price of the i^{th} good, p_i , is known by department i but the goods will not be sold until tomorrow. In the meanwhile general prices levels will have changed by some random amount so that the revenue from a_i units of good i will be $p_i q_i \Theta$ where Θ is the general price level. Decision about input levels must be made without exact knowledge of Θ , but macro forecasts are available to predict the new price level. All departments consult a company-wide price forecast Z before selecting outputs. The cost of the joint product $A = (a_1, \dots, a_N)$ is

$$(109) \quad C(A) = \frac{1}{2} A'QA$$

where Q is a positive definite matrix. Departments specialize in their own technology so that they know only the elements q_{11}, \dots, q_{1N} in the cost matrix. In addition to this, the departments all use machinery to produce their goods and a fixed amount of machinery is available for use. The departments face a joint constraint

$$(110) \quad b'A \leq c$$

where b_1 is machinery used per unit output of a_1 and c is the total machinery available to the firm. Again technical knowledge is decentralized so that b_1 is known only to the 1th department. The total machinery available is known only to the corporate planner.

The firm's objective is to maximize the firms expected profits by selecting output decision functions $A = \alpha(z) = (\alpha_1(z), \alpha_2(z), \dots, \alpha_N(z))'$ to maximize

$$(111) \quad E \left\{ \mu' \alpha(z) \theta - \frac{1}{2} \alpha(z)' Q \alpha(z) \right\}$$

subject to

$$b' \alpha(z) \leq c, \text{ for all forecasts } z.$$

There is no individual in the organization who knows all the parameters needed to solve this problem: μ , Q , b and c . Instead the corporate planner and departments engage in an iterative search for the optimal output decision rules. If $E\{\theta|z\} \geq \frac{c}{b'Q^{-1}\mu}$ then at the optimum all machinery will be occupied no matter what forecast is made and the optimal output decision rules will be linear in the posterior expected price level, $E\{\theta|z\}$. Therefore the firm knows it need only look at linear decision rules and shadow prices in its search for the optimum.

The planner begins announcing an arbitrary price function for machinery that is linear in $E\{\theta|\bar{z}\}$:

$$(112) \quad \lambda^\tau(\bar{z}) = v^\tau E\{\theta|\bar{z}\} + u^\tau$$

and arbitrary output decision functions that are linear in $E\{\theta|\bar{z}\}$:

$$(113) \quad \alpha^\tau(\bar{z}) = A^\tau E\{\theta|\bar{z}\} + s^\tau$$

The department modify their output decision rule by the difference between posterior marginal profits and marginal cost of machinery

$$(114) \quad \frac{d\alpha_i^\tau(\bar{z})}{d\bar{z}} = \mu_i E\{\theta|\bar{z}\} - \sum_{j=1}^N g_{ij} \alpha_j^\tau(\bar{z}) - b_i \lambda^\tau(\bar{z})$$

or

$$(115a) \quad \frac{da_i^\tau}{d\bar{z}} = \mu_i - \sum_{j=1}^N g_{ij} a_j^\tau(\bar{z}) - b_i \lambda^\tau(\bar{z})$$

$$(115b) \quad \frac{ds_i^\tau}{d\bar{z}} = - \sum_{j=1}^N g_{ij} s_j^\tau - b_i u^\tau$$

Notice that the i^{th} department needs to know only parameters it ought to know: $\mu_i, g_{i1}, \dots, g_{iN}, b_i$. It is not required to know μ_j, g_{jR}, b_j or $c_j, j \neq i \neq R$. The department reports back to the planner the new slope and intercept coefficients of its output decision rule. (If b_1 's are to be known only to the department and not the planner then a machinery demand function must also be reported to the planner; since demand for machinery is $b_i \alpha_i(\bar{z})$, this would involve reporting only two parameters, $b_i a_i^\tau$ and $b_i s_i^\tau$. We will assume that the planner already knows b_1 so he can construct the resource demand function.)

The planner uses the coefficients of the individual output decision functions to modify the shadow price function for machinery. He adjust the shadow price function by the excess demand for machinery

$$\frac{d\lambda^\tau(\bar{z})}{d\bar{z}} = b' \alpha^\tau(\bar{z}) - c,$$

or

$$(116) \quad \frac{dV^T}{dT} = b' A^T - c$$

$$(117) \quad \frac{dU^T}{dT} = b' S^T$$

The new output and price functions' slope and intercept coefficients are announced and the process begins again. Notice that the planner does not need to know \mathcal{M} or Q (or b as explained above) in order to adjust the price of machinery. His only required knowledge is of resource demand coefficients, $b' A^T$ and $b' S^T$, and the endowment of machines, c .

This adjustment process is decentralized in the sense that it does not require that any individual reveal his technical parameters directly. It is not in general monotonic, either in the adjusting coefficients or expected utility. It is convergent to the optimal decision rules although the rate of convergence is a decreasing function of T . To see how we derive this properties, let us write the adjustment equations as a system of first order, linear differential equations with constant coefficients:

$$(118) \quad \begin{pmatrix} \dot{A}^T \\ \dot{V}^T \end{pmatrix} = - \begin{pmatrix} Q & b \\ -b & 0 \end{pmatrix} \begin{pmatrix} A^T \\ V^T \end{pmatrix} + \begin{pmatrix} \mathcal{M} \\ 0 \end{pmatrix}$$

$$(119) \quad \begin{pmatrix} \dot{S}^T \\ \dot{U}^T \end{pmatrix} = - \begin{pmatrix} Q & b \\ -b & 0 \end{pmatrix} \begin{pmatrix} S^T \\ U^T \end{pmatrix} - \begin{pmatrix} 0 \\ c \end{pmatrix}$$

The solutions of these two independent sets of equations both depend on the eigenvalues of the $N+1 \times N+1$ matrix

$$(120) \quad B = - \begin{pmatrix} Q & b \\ -b & 0 \end{pmatrix}$$

Since B is not a positive matrix, it can have imaginary roots

and the solution may overshoot the singularity. However, if Q is positive definite then B is negative definite if and only if $|B| < 0$. The determinate of B can be shown to be equal to $-b'Q^{-1}b|Q|$ and hence B is negative definite. If B is negative definite then the real parts of its eigenvalues of B are negative and the solutions of (116) and (117) will converge. Simple calculation will show that $\begin{pmatrix} A \\ V \end{pmatrix} = -B^{-1} \begin{pmatrix} M \\ 0 \end{pmatrix}$ and $\begin{pmatrix} S \\ u \end{pmatrix} = B^{-1} \begin{pmatrix} 0 \\ c \end{pmatrix}$ satisfy the Kuhn-Tucker person-by-person optimality conditions.

c. Price Guided Planning by a Quota Team A N -member team faces an inflexible joint constraint on actions $b'A \leq c$ and has decided to introduce a system of quotas so that the constraint is satisfied with certainty. The team wants to select a vector of decision functions $\alpha(Y) = (\alpha_1(y_1), \dots, \alpha_N(y_N))'$ and a quota vector $q = (q_1, \dots, q_N)'$ to solve the following problem.

$$(121) \text{ Maximize } W = E \{ u(\alpha(Y), \theta) \}$$

subject to

$$\begin{aligned} \alpha(Y) &\leq g && \text{for each } Y \in \mathcal{Y} \\ b'g &= c. \end{aligned}$$

It is assumed that the joint constraint is binding so the inequality has been replaced by an equality in $b'g = c$. If $\lambda(Y) = (\lambda_1(y_1), \dots, \lambda_N(y_N))'$ are multiplier functions corresponding to $\alpha(Y) \leq g$ and γ is a scalar multiplier corresponding to $b'g = c$, a Lagrangian expression is defined by

$$(122) L[\alpha, g, \lambda, \gamma] = E \{ u(\alpha(Y), \theta) \} + E \{ \lambda(Y)'(g - \alpha(Y)) \} + \gamma(c - b'g).$$

If U is differentiable and strictly concave, the following Kuhn-

Tucker person-by-person rules are necessary and sufficient for optimality of $\hat{\alpha}, \hat{g}, \hat{\lambda}, \gamma$; For $i = 1, \dots, N$

$$(123) \frac{\partial L}{\partial \alpha_i} = E\{U_{\alpha_i}(\hat{\alpha}(y), \theta) | y_i\} - \hat{\lambda}_i(y_i) = 0 \quad \text{for each } y_i \in Y_i$$

$$(124) \frac{\partial L}{\partial g_i} = E\{\hat{\lambda}_i(y_i)\} - \gamma b_i = 0$$

$$(125) \frac{\partial L}{\partial \lambda_i} = \hat{g}_i - \hat{\alpha}_i(y_i) \geq 0, \quad \hat{\lambda}_i(y_i) \geq 0 \quad \text{for each } y_i \in Y_i$$

$$(126) (\hat{g}_i - \hat{\alpha}_i(y_i)) \hat{\lambda}_i(y_i) = 0 \quad \text{for each } y_i \in Y_i$$

$$(127) c - b' \hat{g} = 0.$$

These conditions are interpretable if we think of $\hat{\lambda}_i(y_i)$ as the shadow price of the i^{th} action and γ as the shadow price of the resource, both in utility units. The team must set posterior expected marginal utilities equal to the shadow price of the action, set expected shadow prices of actions equal to the marginal cost of the resource, select actions that satisfy the quota, set shadow prices equal to zero if the quota is not binding and select quotas that completely use up the resource endowment.

If the solution of these conditions is either difficult because of the complexity of the utility function and probability densities or impossible because the technological knowledge is decentralized within the team, an iterative exchange of knowledge must be used to find optimal decision rules and quotas. We will again study a modified form of the price guided gradient procedure developed by Arrow and Hurwicz.

Suppose at iteration τ the decision rules are $\alpha_i^\tau(y_i)$, the quotas are g_i^τ , the shadow price of actions $\lambda_i^\tau(y_i)$ and the shadow price of the resource is γ^τ . Since these functions and values typically will not be optimal, they

should be adjusted to get closer to the optimal. The gradient method of hill climbing suggests a step should be taken to get closer to the saddle point by following the direction of steepest ascent. This translates into the following adjustment equations for $i = 1, 2, \dots, N$:

$$(128) \quad \frac{d\alpha_i^T(y_i)}{d\tau} = E\{U_{q_i}(\alpha^T(\gamma), \theta) | y_i\} - \lambda_i(y_i) \quad \text{for all } y_i \in Y_i$$

$$(129) \quad \frac{dg_i^T}{d\tau} = E\{\lambda_i^T(y_i)\} - \gamma^T b_i$$

$$(130) \quad \frac{d\lambda_i^T(y_i)}{d\tau} = \begin{cases} 0 & \text{if } \lambda_i^T(y_i) = 0 \text{ and } \alpha_i^T(y_i) < g_i^T \\ \alpha_i^T(y_i) - g_i^T & \text{otherwise} \end{cases}$$

$$(131) \quad \frac{d\gamma^T}{d\tau} = b'g^T - c.$$

How do these equations mirror real world actions and informational exchanges? The planner is responsible for the quotas and all shadow prices; the teammate is responsible only for his decision function. The planner informs each teammate what price he must pay for a decision function, $\lambda_i^T(y_i)$, and the current decision rules of the other teammates (remember utility is non-additive). The teammate i knowing other members tentative decision functions try to maximize his expected net utility, paying for his actions at the given rate $\lambda_i^T(y_i)$ by adjusting his tentative decision function by the expected posterior marginal net utility $E\{U_{q_i} | y_i\} - \lambda_i^T(y_i)$. The teammate reports his new decision function back to the planner, who now must make three kinds of adjustment. First, the planner adjusts the amount of resource assigned to teammate i , $b_i g_i^T$ by adjusting g_i^T according to the expected net utility of per unit action a_i , $E\{\lambda_i(y_i)\} - \gamma^T b_i$. Second, the planner adjusts the shadow price of the i th decision function by

its excess demand on the quota, $\alpha_i(y_i) - g_i$. Third, the planner adjusts the shadow price of resource according to the excess demand made on it by the quota system, $b'g - c$. The modified shadow prices and decision rules are redistributed to the teammates and the process begins anew.

Stability of the procedure again depends on the concavity of the utility function. We have previously mentioned that if the ranges of information variables are not discrete and finite, the differential equation system must be treated in more abstract spaces than Euclidian m -space and not all finite dimensional result carry over. As a result we will state a convergence theorem only with discrete, finite Y_i .

Suppose the random variables can take on only a finite number of values:

$$(132) \quad \Theta = \{ \theta_1, \dots, \theta_{r_0} \}$$

$$(133) \quad Y_i = \{ y_{i1}, \dots, y_{ir_i} \}.$$

Since each teammate must select only r_1 values for his action, each one corresponding to an element of Y_i , let us denote the decision function by the r_1 -vector

$$(134) \quad (a_{i1}, \dots, a_{ir_i}) = (\alpha_i(y_{i1}), \dots, \alpha_i(y_{ir_i})).$$

Similarly the shadow price function can be written as r_1 -vector

$$(135) \quad (\lambda_{i1}, \dots, \lambda_{ir_i}) = (\lambda_i(y_{i1}), \dots, \lambda_i(y_{ir_i})).$$

If at iteration τ there are approximate solutions $a_{ie_i}(\tau)$, $g_i(\tau)$, $\lambda_{ie_i}(\tau)$, $\gamma(\tau)$ for $i=1, \dots, N$, $e_i=1, \dots, r_1$, the gradient process is defined by the following finite system of first order non-linear differential equations

$$(136) \frac{da_{ie_i}(\tau)}{d\tau} = \sum_{e_\theta=1}^{r_\theta} \sum_{e_i=1}^{r_i} \dots (i) \dots \sum_{e_N=1}^{r_N} u_{q_i}(q_{ie_i}(\tau), \dots, q_{Ne_N}(\tau), \theta_{e_\theta})$$

$$(137) \frac{dg_i(\tau)}{d\tau} = \sum_{e_i=1}^{r_i} \lambda_{ie_i}(\tau) \phi_i(y_{ie_i}) - \gamma(\tau) b_i \quad i=1, \dots, N, \quad e_i=1, \dots, r_i$$

$$(138) \frac{d\lambda_{ie_i}(\tau)}{d\tau} = \begin{cases} 0 & \text{if } \lambda_{ie_i}(\tau) = 0 \text{ and } a_{ie_i}(\tau) < g_i(\tau) \\ a_{ie_i}(\tau) - g_i(\tau) & \text{otherwise} \end{cases} \quad i=1, \dots, N; \quad e_i=1, \dots, r_i$$

$$(139) \frac{d\gamma(\tau)}{d\tau} = b'g(\tau) - c.$$

The following stability theorem can be made for the quota team problem with discrete, finite random variables.

Theorem: If $u(q_1, \dots, q_N, \theta)$ is twice differentiable and strictly concave in a_1, \dots, a_N for all θ and $\hat{a}_{ie_i}, \hat{g}_i, \hat{\lambda}_{ie_i}, \hat{\gamma}$ for $i=1, 2, \dots, N$; $e_i=1, 2, \dots, r_i$ is a saddle point for the above problem then

- (a) the solution $\hat{a}_{ie_i}, \hat{g}_i, \hat{\lambda}_{ie_i}, \hat{\gamma}$ is unique
- (b) for any initial values the solution of (136) - (139) exists and is unique, and
- (c) the saddle point is globally asymptotically stable.

Proof: The proof follows the previous stability theorem in almost all respects, so details will not be presented.

As an example suppose a firm produces two goods a_1 and a_2 . The goods are produced by two departments specializing in the corresponding good and are sold later in the week. The price of goods are fixed at μ_1 and μ_2 but there is uncertainty about the general economy. If a depression occurs before the goods are sold, the markets disintegrate and no goods can be sold (prices don't adjust downward). If the economy remains healthy, the firm can sell all its output at the given prices, the state of economy is

represented by $\theta \in \{\theta_1, \theta_2\} = \{0, 1\}$. The first department gets an economic forecasts $y_1 \in \{y_{11}, y_{12}\}$ where y_{11} is a prediction of "bust" and y_{12} is a prediction of "boom". Similarly the second gets a different economic forecast $y_2 \in \{y_{21}, y_{22}\}$. The two forecast have a joint conditional probability

$$(140) \quad \phi(y_{1l}, y_{2m} | \theta_k) = \pi_{lm}^k \quad l=1,2; m=1,2; k=1,2$$

or

$$(141) \quad \phi(y|\theta) = \pi = \begin{matrix} & \theta_1 & \theta_2 \\ \begin{matrix} [y_{11}, y_{21}] : \\ [y_{11}, y_{22}] : \\ [y_{12}, y_{21}] : \\ [y_{12}, y_{22}] : \end{matrix} & \begin{bmatrix} \pi_{11}^1 & \pi_{11}^2 \\ \pi_{12}^1 & \pi_{12}^2 \\ \pi_{21}^1 & \pi_{21}^2 \\ \pi_{22}^1 & \pi_{22}^2 \end{bmatrix} \end{matrix}$$

The team's subjective prior distribution of θ is $\phi(\theta_k) = p_k, k=1,2$ or written along the diagonal of a matrix

$$(142) \quad \phi(\theta) = P = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}.$$

The joint probability of y_1, y_2, θ is therefore $\phi(y, \theta) = \pi P$

The team produces goods (a_1, a_2) at a total cost f

$$(143) \quad C(a_1, a_2) = \frac{1}{2} c_{11} a_1^2 + c_{12} a_1 a_2 + \frac{1}{2} c_{22} a_2^2.$$

The firm faces a constraint on machinery

$$(144) \quad b_1 a_1 + b_2 a_2 \leq C,$$

and is forced to introduce a quota system

$$(145) \quad a_1 \leq f_1 \quad \text{and} \quad a_2 \leq f_2$$

in order to guarantee the machinery constraint is not violated.

The department's decision function is a vector of actions corresponding to the discrete information values

$$(a_{11}, a_{12}) = (\alpha_1(y_{11}), \alpha_1(y_{12})) \quad \text{and} \quad (a_{21}, a_{22}) = (\alpha_2(y_{21}), \alpha_2(y_{22})).$$

The teams problem is to select $a_{11}, a_{12}, a_{21}, a_{22}$ and (q_1, q_2) to

maximize expected profits

$$(146) \quad \pi = \sum_{k=1,2} \sum_{l=1,2} \sum_{m=1,2} (\mu_1 a_{1l} \theta_{1k} + \mu_2 a_{2m} \theta_{2k} - \frac{1}{2} c_{11} a_{1l}^2 - c_{12} a_{1l} a_{2m} - \frac{1}{2} c_{22} a_{2m}^2) \pi_{2m}^k P_k$$

subject to $a_{1l} \leq g_1$, $l=1,2$; $a_{2m} \leq g_2$, $m=1,2$

and $b_1 g_1 + b_2 g_2 = c$.

Suppose teammate 1 knows only the technical coefficients c_{11}, c_{12} , b_1 and price μ_1 . A price guided solution procedure is introduced with a resource planner responsible for generating shadow prices for the decisions, $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}$, and for the machinery, δ . A iterative solution to the constrained-quota profit maximizing team problem is made, which is defined by the differential equations⁽²⁾

$$(147) \quad \frac{da_{1l}(\tau)}{d\tau} = \frac{\partial \pi}{\partial a_{1l}} - \lambda_{1l}(\tau) \phi_1(y_{1l}) \quad l=1,2$$

$$(148) \quad \frac{da_{2m}(\tau)}{d\tau} = \frac{\partial \pi}{\partial a_{2m}} - \lambda_{2m}(\tau) \phi_2(y_{2m}) \quad m=1,2$$

$$(149) \quad \frac{d\delta(\tau)}{d\tau} = \lambda_{11}(\tau) \phi_1(y_{11}) + \lambda_{12}(\tau) \phi_1(y_{12}) - \delta(\tau) b_1 \quad i=1,2$$

$$(150) \quad \frac{d\lambda_{1l}(\tau)}{d\tau} = \begin{cases} 0 & \text{if } \lambda_{1l}(\tau) = 0 \text{ and } a_{1l}(\tau) < g_1(\tau) \\ a_{1l}(\tau) - g_1(\tau) & \text{otherwise} \end{cases} \quad l=1,2$$

$$(151) \quad \frac{d\lambda_{2m}(\tau)}{d\tau} = \begin{cases} 0 & \text{if } \lambda_{2m}(\tau) = 0 \text{ and } a_{2m}(\tau) < g_2(\tau) \\ a_{2m}(\tau) - g_2(\tau) & \text{otherwise} \end{cases} \quad m=1,2$$

$$(152) \quad \frac{d\delta(\tau)}{d\tau} = b_1 g_1(\tau) + b_2 g_2(\tau) - c.$$

The exchange of information is analogous to the more general formulation given above.

d. Decomposition Planning by a Team A traditional team assumption is that the utility function cannot be additively decomposed into functions that depend only on single member actions; i.e.

$U(A, \theta)$ cannot be expressed as $\sum_{i=1}^N u^i(a_i, \theta)$ or identically, $\frac{\partial^2 U}{\partial a_i \partial a_j} \neq 0$,

for all $i, j \neq i$. The reason this non-separability assumption appears is that the team problem could otherwise be polychotimized into N unrelated decision problems, one for each of the team members. This polychotomy cannot occur in the jointly constrained team problem even if utilities are additive; the joint constraint requires coordination of decisions no matter what form the utility function takes. In the following we will explicitly assume that utility is additive and study only the interrelationships of actions through a joint constraint. In addition we will make the assumption that the "planner" knows all components of utility function $U^i(q_i, \theta)$.

The team faces a joint constraint on actions, which we will write as

$$(153) \quad g_1(q_1) + \dots + g_N(q_N) \leq c.$$

The "planner" knows these functions $g_i(q_i)$ as well as the resource endowment c . The team handles this inflexible joint constraint by introducing quotas on the use of the scarce resource; that is a vector $\tilde{q} = (\tilde{q}_1, \dots, \tilde{q}_N)$ is selected so that

$$(154) \quad \sum_{i=1}^N \tilde{q}_i \leq c$$

and the additional constraints

$$(155) \quad g_i(q_i) \leq \tilde{q}_i$$

are added to the original problem. If this was the final specification of the team's problem, planning would not be required because the planner has all necessary technological information needed to solve the problem. He would merely have to distribute the optimal decision functions to the team members. But this will not generally be the final form of the constrained team problem;

single-agent constraints have not be specified. The i^{th} teammate must select his action from a set of feasible individual actions which will be denoted by the vector of inequalities

$$(156) \quad h_i(a_i) \equiv (h_{i1}(a_i), \dots, h_{it_i}(a_i))' \leq d_i \equiv (d_{i1}, \dots, d_{it_i})'$$

The planner does not know the functions $h_1(a_1)$ and therefore cannot solve the problem initially.

The final element of the theory problem is the probability distributions. We will assume that the state of nature can take only one of r_θ values: $\Theta = \{\theta_1, \dots, \theta_{r_\theta}\}$. The information variable of the i^{th} teammate can also take on only one of r_i values, $y_i = \{y_{i1}, \dots, y_{ir_i}\}$. Because of the separable utility functions, the only probabilities needed are $\phi_i(y_i|\theta)$ and $\phi(\theta)$. The i^{th} teammate knows both the conditional density of y_i given θ and the density of θ , but not the conditional densities of other teammates. The planner knows all the probability functions.

In summary, the team's problem is to select the individual decision vectors $(a_{i1}, \dots, a_{ir_i}) = (\alpha_i(y_{i1}), \dots, \alpha_i(y_{ir_i}))$ and quotas $\tilde{q}_i = (\tilde{q}_1, \dots, \tilde{q}_N)$ to maximize expected utility subject the quota constraints on actions, the constraint on the jointly used resource and single-agent constraints.

$$(157) \quad \max W = \sum_{e_1=1}^{r_1} \sum_{e_\theta=1}^{r_\theta} u^1(a_{1e_1}, \theta_{e_\theta}) \phi_1(y_{1e_1} | \theta_{e_\theta}) \phi(\theta_{e_\theta}) + \dots \\ \dots + \sum_{e_N=1}^{r_N} \sum_{e_\theta=1}^{r_\theta} u^N(a_{Ne_N}, \theta_{e_\theta}) \phi_N(y_{Ne_N} | \theta_{e_\theta}) \phi(\theta_{e_\theta})$$

subject to

$$\begin{aligned} q_i(a_{ie_i}) &\leq \tilde{q}_i & i=1, 2, \dots, N; e_i=1, 2, \dots, r_i \\ \tilde{q}_1 + \dots + \tilde{q}_N &\leq c \\ h_i(a_{ie_i}) &\leq d_i & i=1, 2, \dots, N; e_i=1, 2, \dots, r_i \end{aligned}$$

The technological knowledge of each agent is as follows: the plan-

ner knows the functions $U^1, \dots, U^N, \phi_1, \dots, \phi_N, g_1, \dots, g_N$ and the endowment c ; the i^{th} teammate knows the functions U^i, ϕ_i, g_i, h_i and his endowment d_i . Conversely, the technological ignorance of each agent is as follows: the planner does not know any of the functions h_1, \dots, h_N nor the individual endowments d_i ; the i^{th} teammate does not know the $U^j, \phi_j, g_j, h_j, d_j$ for $j \neq i$ nor the joint endowment c .

The solution procedure which will now be developed is based on an algorithm which is called "vertical decomposition":⁽³⁾ The Dantzig-Wolfe price guided linear decomposition was extended to non-linear objectives and constraints by Malinvaud. Weitzman developed the quantity target guided "dual" decomposition procedure. The vertical decomposition algorithm is most closely related to Weitzman's work because the planner's "indices" are quantitative and the individual agent's "proposals" are shadow prices. The vertical decomposition procedure will be specified in detail in its general form and then the team's particular problem will be analyzed in the framework just constructed.

Suppose there is an organization with two units, the "planner" and the "periphery". The planner controls a variable y (not to be confused with its previous use). The organizations problem is to select x and y to

$$\begin{aligned} \text{(I) maximize } & U(x) + V(y) \\ \text{subject to } & g(x) + h(y) \leq d \\ & x, y \geq 0. \end{aligned}$$

The planner knows U, g and d but not V and h , which are known only

to the periphery. If the planner knew

$$(II) \quad \psi(z) \equiv \max V(y) \text{ s.t. } h(y) \leq z, \quad y \geq 0$$

then he could solve the equivalent problem

$$(III) \quad \max U(x) + \psi(z) \\ \text{s.t. } g(x) + z \leq d \\ x \geq 0.$$

Theorem: If (\hat{x}, \hat{z}) solves problem (III) and \hat{y} solves (II) for $z = \hat{z}$ then (\hat{x}, \hat{y}) solves (I).

Proof: (\hat{x}, \hat{y}) is feasible for (I) by construction. The Kuhn-Tucker conditions for (III) are

$$(158) \quad \nabla_x U(\hat{x}) = \nabla_x g(\hat{x})' \hat{\pi}$$

$$(159) \quad \nabla_z \psi(\hat{z}) = \hat{\pi}$$

where $\hat{\pi}$ is the shadow price. The Kuhn-Tucker conditions for (II) are

$$(160) \quad \nabla_y V(\hat{y}) = \nabla_y h(\hat{y})' \hat{\beta}$$

where $\hat{\beta}$ is the shadow price. But by definition of $\psi(z)$, $\hat{\beta} = \nabla_z \psi(z)$.

Hence $\hat{\beta} = \hat{\pi}$ by (159). The conditions (158) and (160) are thus written as

$$(161) \quad \nabla_x U(\hat{x}) = \nabla_x g(\hat{x})' \hat{\pi}$$

$$(162) \quad \nabla_y V(\hat{y}) = \nabla_y h(\hat{y})' \hat{\pi}$$

But these are exactly the Kuhn-Tucker conditions for problem (I).

So (\hat{x}, \hat{y}) solves problem (I) with shadow price $\hat{\pi}$. Q.E.D.

The planner does not know the function $\psi(z)$ so he cannot directly solve problem (III) and then allow the periphery to maximize its utility subject to the quota \hat{z} . The vertical decomposition procedure builds an approximation to the function $\psi(z)$ so that the planner never has to know the functions $V(y), h(y)$.

Suppose the planner has at stage s an approximation of $\Psi(z)$, denoted $\Psi_s(z)$, with the property that $\Psi_s(z) \geq \Psi(z)$, for all z . The planner solves the psuedo-master problem

$$(163) \quad \begin{aligned} \max \quad & U(x) + \Psi_s(z) \\ \text{s.t.} \quad & g(x) + z \leq d, \quad x \geq 0. \end{aligned}$$

Let us denote the optimal values at stage s by (x^s, z^s) . The planner tells the periphery that he must not use more than z^s units of the resource, so the periphery solves

$$(164) \quad \begin{aligned} \max \quad & V(y) \\ \text{s.t.} \quad & h(y) \leq z^s \quad : \quad \pi^s \\ & y \geq 0 \end{aligned}$$

with optimal value y^s and shadow price π^s . At what stage is the approximation accurate enough to generate optimal actions?

Theorem: If $V(y^s) = \Psi_s(z^s)$ then (x^s, y^s) is optimal for the original problem.

Proof: $V(y^s) = \Psi(z^s)$ by definition of $\Psi(z)$. The approximation is everywhere greater than or equal to $\Psi(z)$. Suppose that (x^s, z^s) is not optimal, that is $U(x) + \Psi(z) > U(x^s) + \Psi(z^s)$ for some (x, z) such that $g(x) + z \leq d$. But then $U(x) + \Psi_s(z) \geq U(x) + \Psi(z) > U(x^s) + \Psi(z^s) = U(x^s) + \Psi_s(z^s)$ which contradicts definition of optimal (x^s, z^s) for the psuedo-master problem. Q.E.D.

If the approximation does not provide an accurate enough picture of $\Psi(z)$, how is $\Psi_s(z)$ modified by the planner? The periphery returns the shadow price π^s and its optimal value of utility $V(y^s) = \Psi(z^s)$. The planner uses this information to sharpen its approximation by creating a new $\Psi_{s+1}(z)$ defined by

$$(165) \psi_{s+1}(z) \equiv \min [\psi_s(z), \pi^s z + \psi(z^s) - \pi^s z^s].$$

We will state without proof that if $V(y)$ is concave and $h(y)$ is convex then $\psi(z)$ is a concave function. The term $\pi^s z + \psi(z^s) - \pi^s z^s$ is then a supporting hyperplane through the point $(\psi(z^s), z^s)$. Will the new approximation still be everywhere greater or equal to $\psi(z)$?

Theorem: $\psi(z) \leq \psi_{s+1}(z)$ for all z .

Proof: By concavity of $\psi(z)$

$$(166) \psi(z) - \psi(z^s) \leq \nabla_z \psi(z^s)(z - z^s)$$

The definition of π^s as the shadow price of z^s implies

$$(167) \psi(z) \leq \pi^s z + \psi(z^s) - \pi^s z^s.$$

Since by assumption $\psi(z) \leq \psi_s(z)$, $\psi(z)$ is less than or equal to both terms in the definition of $\psi_{s+1}(z)$ and hence is less than or equal to the minimum of the two. Q.E.D.

We now have a solution algorithm that tells us how the planner generates new approximations. We have not specified how the planner and periphery solve their constrained concave maximization problems at each stage, but we can imagine that they use some procedure such as "steepest ascent".

This vertical decomposition algorithm would not be very useful if it did not converge to the optimal results \hat{x} , \hat{y} , \hat{z} and $\hat{\pi}$.

Theorem: As s gets very large, z^s approaches \hat{z} and x^s approaches \hat{x}

if U and V are strictly concave functions while g and h are strictly convex functions.

Proof: The sequence $\{x^s, z^s\}$ must have a limit point since it is

bounded. Suppose that a limit point is (\bar{x}, \bar{z}) . Then $\lim_{s \rightarrow \infty} \psi_s(z) = \bar{\psi}(z)$,

the limiting form of the approximation, must satisfy $\bar{\psi}(\bar{z}) = \psi(\bar{z})$, and let us denote $\nabla\psi(\bar{z})$ by $\bar{\pi}$. Assume that (\bar{x}, \bar{z}) is different from (\hat{x}, \hat{z}) . The definition of (x^s, z^s) as the solution of the pseudo-master problem implies

$$(168) \quad u(x) + \bar{\psi}(z) + \bar{\pi}(d-g(x)-z) \leq u(\bar{x}) + \psi(z) + \bar{\pi}(d-g(\bar{x})-\bar{z}) \\ \leq u(\bar{x}) + \psi(\bar{z}) + \bar{\pi}(d-g(\bar{x})-\bar{z}) \quad \text{for all } x, z, \pi$$

The definition of (\hat{x}, \hat{z}) as solution of the master problem implies

$$(169) \quad u(x) + \psi(z) + \hat{\pi}(d-g(x)-z) \leq u(\hat{x}) + \psi(\hat{z}) + \hat{\pi}(d-g(\hat{x})-\hat{z}) \\ \leq u(\hat{x}) + \psi(\hat{z}) + \bar{\pi}(d-g(\hat{x})-\hat{z}).$$

Since $\psi_s(z) \geq \psi(z)$ it follows that $\bar{\psi}(z) \geq \psi(z)$ so

$$(170) \quad u(x) + \psi(z) + \bar{\pi}(d-g(x)-z) \leq u(x) + \bar{\psi}(z) + \bar{\pi}(d-g(x)-z).$$

Because the first inequality of (168) must hold for all x, z, π , it must hold for $\hat{x}, \hat{z}, \hat{\pi}$. Combining with (170) we have

$$(171) \quad u(\hat{x}) + \psi(\hat{z}) + \bar{\pi}(d-g(\hat{x})-\hat{z}) \leq u(\bar{x}) + \psi(\bar{z}) + \bar{\pi}(d-g(\bar{x})-\bar{z})$$

but $d-g(\hat{x})-\hat{z} \geq 0$, $\bar{\pi} \geq 0$ and $\bar{\pi}(d-g(\bar{x})-\bar{z}) = 0$, which implies

$$(172) \quad u(\hat{x}) + \psi(\hat{z}) \leq u(\bar{x}) + \psi(\bar{z})$$

Using the first inequality of (169) in a symmetric manner with $(x, z, \pi) = (\bar{x}, \bar{z}, \bar{\pi})$, we can show that

$$(173) \quad u(\bar{x}) + \psi(\bar{z}) \leq u(\hat{x}) + \psi(\hat{z})$$

which implies

$$(174) \quad u(\bar{x}) + \psi(\bar{z}) = u(\hat{x}) + \psi(\hat{z}).$$

That is, (\bar{x}, \bar{z}) and (\hat{x}, \hat{z}) are on the same indifference curve.

The second inequalities of (168) and (169) imply by first setting $\pi = \hat{\pi}$ then $\pi = \bar{\pi}$:

$$(175) \quad 0 = \bar{\pi}(d-g(\bar{x})-\bar{z}) \leq \hat{\pi}(d-g(\bar{x})-\bar{z})$$

$$(176) \quad 0 = \hat{\pi}(d-g(\hat{x})-\hat{z}) \leq \bar{\pi}(d-g(\hat{x})-\hat{z}).$$

But since (\bar{x}, \bar{z}) and (\hat{x}, \hat{z}) are on the same indifference curve, the first inequalities of (168) and (169) imply

$$(177) \quad \bar{\pi} (d - g(\hat{x}) - \hat{z}) \leq \bar{\pi} (d - g(\bar{x}) - \bar{z}) = 0$$

$$(178) \quad \hat{\pi} (d - g(\bar{x}) - \bar{z}) \leq \hat{\pi} (d - g(\hat{x}) - \hat{z}) = 0.$$

Therefore

$$(179) \quad \bar{\pi} (d - g(\hat{x}) - \hat{z}) = \hat{\pi} (d - g(\hat{x}) - \hat{z}) = 0$$

$$(180) \quad \hat{\pi} (d - g(\bar{x}) - \bar{z}) = \bar{\pi} (d - g(\bar{x}) - \bar{z}) = 0.$$

The conclusion is that $\bar{\pi} = \hat{\pi}$, or the slope of the indifference curve at (\bar{x}, \bar{z}) equals the slope at (\hat{x}, \hat{z}) . But since U and ψ are strictly concave, this implies $(\bar{x}, \bar{z}) = (\hat{x}, \hat{z})$ contradicting the assumption. Q.E.D.

It should be pointed out that if $v(y)$ and $h(y)$ are linear (so that $\psi(z)$ is piecewise linear) then convergence will occur in a finite number of steps. Each step introduces a new "flat" section of $\psi(z)$ and the number of flat sections is bounded.

The above vertical decomposition problem is the simplest version. Certainly there is no difficulty in adding more "peripheries" so that the objective is to maximize $u(x) + \sum_{i=1}^N v^i(y_i)$. Equally clearly, there could be single-agent constraints in addition to the joint constraint. If the problem is to maximize

$$(181) \quad u(x) + \sum_{i=1}^N v^i(y_i)$$

$$\text{s.t. } g(x) + h_i(y_i) \leq \omega_i \quad i=1, \dots, N$$

$$f_i(y_i) \leq b_i \quad i=1, \dots, N$$

$$k(x) \leq b$$

$$x, y_1, \dots, y_N \geq 0.$$

The planner then approximates n functions defined as

$$(182) \quad \Psi_j(\bar{z}_j) \equiv \max V^j(y_j) \text{ s.t. } h_j(y_j) \leq \bar{z}_j, f_j(y_j) \leq b_j, y_j \geq 0.$$

by collecting supporting hyperplanes tangent to this surface.

In this more general formulation of the vertical decomposition problem we see the analogue of the quota team problem (157). The variable x corresponds to the quotas \tilde{q} , the variable y_1 corresponds to the decision vector $(a_{11}, \dots, a_{1r_1})'$, the endowments w_1 are zero in the team problem, while b_1 plays the role of d_1 and b plays the role of c . In the team problem $u(x) \equiv 0$, $V^i(y_i) = E \{ u^i(\alpha_i, \theta) \}$, $g(x) = -\tilde{q}_i$, $h_i(y_i) = g_i(a_{ie_i})$, $f_i(y_i) = h_i(a_{ie_i})$ and $f(x) = \tilde{q}_1 + \dots + \tilde{q}_N$.

The team's planner does not know the functions

$$(183) \quad \Psi_i(\tilde{q}_i) \equiv \max \sum_{e_i=1}^{r_i} \sum_{\theta_{e_i}=1}^{\theta_{e_i}} u^i(a_{ie_i}, \theta_{e_i}) \phi(y_{ie_i} | \theta_{e_i}) \phi(\theta_{e_i})$$

subject to $h_i(a_{ie_i}) \leq d_i \quad e_i = 1, 2, \dots, r_i$
 $g_i(a_{ie_i}) \leq \tilde{q}_i \quad e_i = 1, 2, \dots, r_i.$

If he did, he could select \tilde{q} to maximize

$$(184) \quad \sum_{i=1}^N \Psi_i(\tilde{q}_i)$$

subject to $\tilde{q}_1 + \dots + \tilde{q}_N \leq c$

and then impose the optimal quotas on the teammates, who would then select their own optimal decision vectors. If the planner at stage τ has an approximation of $\Psi_i(\tilde{q}_i)$ such that $\Psi_i^\tau(\tilde{q}_i) \geq \Psi_i(\tilde{q}_i)$ for all \tilde{q}_i , then he selects tentative quotas to maximize

$$(185) \quad \sum_{i=1}^N \Psi_i^\tau(\tilde{q}_i)$$

subject to $\tilde{q}_1 + \dots + \tilde{q}_N \leq c.$

These optimal quotas $\tilde{q}^\tau = (\tilde{q}_1^\tau, \dots, \tilde{q}_N^\tau)$ are sent to members, who maximize expected utility subject to all its constraints plus the quota.

The member returns the maximum expected utility that it could

obtain given $\tilde{g}_i^z (= \Psi_i(\tilde{g}_i^z))$ and the expected shadow price of the quota, π_i^z . If $\Psi_i(\tilde{g}_i^z) = \Psi_i^z(\tilde{g}_i^z)$ for all i , then the planning problem is solved. Otherwise the approximation is improved:

$$(186) \quad \Psi_i^{z+1}(\tilde{g}_i) = \min [\Psi_i^z(\tilde{g}_i), \pi_i^z \tilde{g}_i + \Psi_i(\tilde{g}_i^z) - \pi_i^z \tilde{g}_i^z]$$

The procedure will converge if utilities are strictly convex functions, and in a finite number of steps in the linear case.

This section began with an argument that non-separable utility is not important when the team faces joint constraints. Therefore the decomposition algorithm was stated with additively separable utility. This section ends with the note that separable utility is not absolutely necessary for a well behaved algorithm. Decomposition procedures of the price guided type exist for non-additive utility functions of the form

$$(187) \quad u(a_1, \dots, a_N, \theta) = \sum_i u^i(a_i, \theta) + \sum_i \sum_j u^{ij}(a_i, a_j, \theta). \quad (4)$$

In these price guided decomposition algorithms the planner must not supply a price at each stage for the jointly constrained resources but also a tentative decision on a_1, \dots, a_N . The individual agents need to know approximately what actions the other members may take in order to coordinate their own actions to maximize

$u^i(a_i, \theta) + \sum_j u^{ij}(a_i, a_j, \theta)$ in the subsidiary problems. In the vertical decomposition algorithm, in addition to imposing tentative quotas,

the planner would need to announce tentative decisions so that peripheries could coordinate their proposals. The planner would then approximate the functions $\Psi_i(\tilde{g}_i, a_1, \dots, (i), \dots, a_N)$ defined by

$$(188) \quad \Psi_i(\tilde{g}_i, a_1, \dots, (i), \dots, a_N) \equiv \max E \{ u^i(a_i, \theta) + \sum_j u^{ij}(a_i, a_j, \theta) \}$$

subject to $h_i(a_i) \leq d_i$
 $g_i(a_i) \leq \tilde{g}_i$

By looking at supporting hyperplanes tangent to its surface.

e. Quantity Guided Gradient Planning by a Quota Team In the price guided gradient procedure, the planner announced the tentative price of the resource. In a team problem the "price" was a price function, $\lambda_i^{\bar{c}}(y_i)$, which told the i^{th} teammate how much he would have to pay for his action if he observed information y_1 . The member modified his current decision function proportional to the difference between his posterior expected marginal payoff given y_1 and the marginal cost, $\lambda_i^{\bar{c}}(y_i)$, and returned this proposed decision function to the planner. The price functions played the role of guiding "indices" while the decision function were the adjusting "proposals" of the individual members. Two modifications of this scheme will be made in this subsection.

First, the planner will guide the search for optimal decision rules by telling the individuals how much of the resource they will be allowed to use. The teammates will then try to make the best use of their information without using more than their allotment of the resource. The teammates answer the planner's quota allocation with the price at which they would have purchased that quantity of the resource. Roles have been switched. The planner's indices are now quantity quotas and the teammates proposals are shadow prices.

Second, the teammate does not merely make a small adjustment in the direction of steepest ascent. He immediately selects optimal decision rules in a quantum jump. In effect shadow prices

have an infinite speed of adjustment. A similar kind of assumption, could be made for the teammate's behavior in the price guided gradient; given the shadow price of resources, the teammate would then select actions to equalize the posterior expected marginal utility and the shadow price, rather than just decrease their difference. It has been made clear by Arrow and Hurwicz⁽⁵⁾ that instantaneous adjustment can lead to undefined solutions (in fact, it usually will in a purely linear case). However, if the initial position is close enough to the equilibrium this difficulty will not be important.

The team wants to select a vector of decision functions $\alpha(\gamma) = (\alpha_1(\gamma_1), \dots, \alpha_N(\gamma_N))'$ and a quota vector $q = (q_1, \dots, q_N)'$ to solve the following problem

$$(189) \quad \max W = E \{ u(\alpha(\gamma), \theta) \}$$

$$\text{subject to } \alpha(\gamma) \leq q \quad \text{for each } \gamma \in \mathcal{Y}$$

$$b'q = c$$

If $\lambda(\gamma) = (\lambda_1(\gamma_1), \dots, \lambda_N(\gamma_N))$ are multiplier functions corresponding to $\alpha(\gamma) \leq q$ and δ is a scalar multiplier corresponding to $b'q = c$, the Kuhn-Tucker person-by-person rules for optimal $\hat{\alpha}, \hat{q}, \hat{\lambda}, \hat{\delta}$ are

$$(190) \quad 0 = E \{ u_{q_i}(\hat{\alpha}(\gamma), \theta | \gamma_i) \} - \hat{\lambda}_i(\gamma_i) \quad \text{for each } \gamma_i \in \mathcal{Y}_i$$

$$(191) \quad 0 = E \{ \hat{\lambda}_i(\gamma_i) \} - \hat{\delta} b_i$$

$$(192) \quad 0 \leq \hat{\delta}_i - \hat{\alpha}_i(\gamma_i); \quad \hat{\lambda}_i(\gamma_i) \geq 0 \quad \text{for each } \gamma_i \in \mathcal{Y}_i$$

$$(193) \quad 0 = \hat{\lambda}_i(\gamma_i) (\hat{\delta}_i - \hat{\alpha}_i(\gamma_i)) \quad \text{for each } \gamma_i \in \mathcal{Y}_i$$

$$(194) \quad 0 = c - b' \hat{q}.$$

Suppose at iteration τ the decision rules are $\alpha_i^\tau(\gamma_i)$, the

quotas are g_i^τ and the shadow price of the resource is λ^τ . The quantity guided gradient procedure is defined by the following information flows and corrective actions.

(a) The planner announces the tentative decision functions for each teammate, $\alpha_i^\tau(y_i)$ and informs the teammate of his quota, g_i^τ .

(b) The i^{th} teammate takes the decision functions of the other teammates as fixed but ignores the tentative decision function $\alpha_i^\tau(y_i)$. Instead the teammate selects a new decision function $\alpha_i^{\tau+d\tau}(y_i)$ to maximize the team's expected utility given the other decision functions $\alpha_j^\tau(y_j)$ subject to the quota. That is he selects $\alpha_i^{\tau+d\tau}$ to

$$(195) \max E \{ U(\alpha_1^\tau(y_1), \dots, \alpha_i^{\tau+d\tau}(y_i), \dots, \alpha_N^\tau(y_N), \theta) \}$$

subject to $\alpha_i^{\tau+d\tau}(y_i) \leq g_i$ for each $y_i \in Y_i$

(c) In solving his constrained maximization problem, the i^{th} teammate must compute a shadow price function $\lambda_i^{\tau+d\tau}(y_i)$ corresponding to the quota constraint. The i^{th} teammate's proposal to the planner is the expected increase in team utility if an extra unit of the resource was given to the i^{th} teammate,

$$(196) E \{ \lambda_i^{\tau+d\tau}(y_i) \} / b_i = P_i^{\tau+d\tau} \text{ and his decision function } \alpha_i^{\tau+d\tau}(y_i)$$

(d) The planner uses the shadow prices of the individual quotas to adjust the allocation of resources according to

$$(197) b_i \frac{dg_i^\tau}{d\tau} = (P_i^{\tau+d\tau} - \frac{1}{N} \sum_{j=1}^N P_j^{\tau+d\tau}). \quad (6)$$

(e) The planner announces the new decision functions $\alpha_i^{\tau+d\tau}$ and new quotas $b_i g_i^{\tau+d\tau}$ and the procedure returns to (b) and repeats.

The quota guided gradient procedure can be expressed as the following set of Kuhn-Tucker person-by-person conditions for $\alpha_i^T(y_i)$ and system of differential equations

$$(198) \quad 0 = E\{u_{q_i}(\alpha^T(\gamma), \theta) | y_i\} - \lambda_i^T(y_i)$$

$$(199) \quad 0 \leq g_i^T - \alpha_i^T(y_i)$$

$$(200) \quad 0 \leq \lambda_i^T(y_i)$$

$$(201) \quad 0 = \lambda_i^T(y_i) (g_i^T - \alpha_i^T(y_i))$$

$$(202) \quad b_i \frac{dg_i^T}{d\tau} = \left(\frac{E\{\lambda_i^T(y_i)\}}{b_i} - \frac{1}{N} \sum_{j=1}^N \frac{E\{\lambda_j^T(y_j)\}}{b_j} \right).$$

The adjustment of the quotas can be interpreted as shifting resources toward those teammates whose expected increase in team utility per unit resource is above the average. An interesting point to note is that

$$\begin{aligned} (203) \quad \frac{d}{d\tau} (b_1 g_1^T + \dots + b_N g_N^T) &= \sum_i b_i \frac{dg_i^T}{d\tau} \\ &= \sum_i \frac{E\{\lambda_i^T(y_i)\}}{b_i} - \frac{1}{N} \sum_i \sum_j \frac{E\{\lambda_j^T(y_j)\}}{b_j} \\ &= 0. \end{aligned}$$

If the original allocation of the resources was feasible, $b'g^0 = c$ then the quotas will remain feasible. The decision functions are continuous adjusted to optimize given the current quota system. The procedure is also convergent to the optimal quotas and decision rules in the case of discrete, finite range of random variables

Theorem: The solution algorithm converges to the optimal decision functions $\hat{\alpha}(\gamma)$ and optimal quotas \hat{g} when the random variables

have discrete, finite ranges.

Proof: Define the Lagrangian for the original problem by

$$(204) L = \sum_{e_0=1}^{r_0} \sum_{e_1=1}^{y_1} \dots \sum_{e_N=1}^{y_N} U(a_{1e_1}, \dots, a_{Ne_N}, \theta_{e_0}) \phi(y_{1e_1}, \dots, y_{Ne_N}, \theta_{e_0}) + \\ \sum_{i=1}^N \sum_{e_i=1}^{y_i} \lambda_{ie_i} (g_i - a_{ie_i}) \phi(y_{ie_i}) + \delta (c - b'g)$$

Define a Lyapunov function as the difference between $L(\hat{a}, \hat{g}, \hat{\lambda}, \hat{\delta})$ and $L(\alpha^\tau, g^\tau, \lambda^\tau, \delta^\tau)$ and notice that $L[\alpha^\tau, g^\tau, \lambda^\tau, \delta^\tau]$ is a non-decreasing function of τ :

$$(205) \frac{dL}{d\tau} = \sum_i (E \{ U_{a_i} | y_{ie_i} \} - \lambda_{ie_i}^\tau) \phi_i(y_{ie_i}) \frac{d a_{ie_i}^\tau}{d\tau} + \\ \sum_i \sum_{e_i} (g_i^\tau - a_{ie_i}^\tau) \phi_i(y_{ie_i}) \frac{d \lambda_{ie_i}^\tau}{d\tau} + \sum_i \left(\sum_{e_i} \lambda_{ie_i}^\tau \phi_i(y_{ie_i}) - b_i \right) \frac{d g_i^\tau}{d\tau} \\ + (c - b'g^\tau) \frac{d \delta^\tau}{d\tau}$$

But $c - b'g^\tau = 0$ at all stages τ

$$(206) E \{ U_{a_i} | y_{ie_i} \} - \lambda_{ie_i}^\tau = 0 \text{ at all stages } \tau$$

$$(207) (g_i^\tau - a_{ie_i}^\tau) \lambda_{ie_i}^\tau = 0 \text{ at all stages } \tau$$

Hence

$$(208) \frac{dL}{d\tau} = \sum_i \left(\sum_{e_i} \lambda_{ie_i}^\tau \phi_i(y_{ie_i}) - b_i \right) \frac{d g_i^\tau}{d\tau} \\ = \sum_i \left(\frac{E \{ \lambda_i^\tau \}}{b_i} \right)^2 - \frac{1}{N} \left(\sum_i \frac{E \{ \lambda_i^\tau \}}{b_i} \right)^2 \geq 0 \text{ for all } \tau$$

Hence the Lyapunov is non increasing. Q.E.D.

Without attempting to enter the ideological dialogue as to whether prices or quantities should be the guiding force in an economic organization or system, it should be pointed out that in practice of quantity targets are much more prevalent than shadow prices in planning procedures. The planners typically allocate

resources tentatively then seek information as to what direction the allocation should be shifted to improve the organizations payoff.

f. Appendix: Convergence of Gradient Methods in Abstract Spaces⁽⁷⁾

The team problem in its general form requires the selection of functions to maximize a functional, and therefore a gradient solution procedure generally defines a continuum of differential equations. The analysis of the convergence of a gradient algorithm must be based not on the theory of finite dimensional differential equations but on the theory of differential equations in abstract spaces. This appendix does not pretend to euclidiate concrete results; rather it shall state some conjectures on this topic.

Because the team problem is a variational problem, gradient convergence will be discussed in terms of related problems of optimal control and the calculus of variations. Suppose the following optimal control problem is to be solved

$$(209) \quad \max_u \int_0^T I(x, u, t) dt$$

$$\text{s.t.} \quad \dot{x} = f(x, u, t)$$

$$x(0), x(T) \quad \text{given.}$$

A gradient solution procedure for this problem is defined by

$$(210) \quad \frac{\partial u(t; \tau)}{\partial \tau} = H_u(x(t; \tau), u(t; \tau), P(t; \tau)) \quad 0 < t < T$$

where H_u is the partial derivative with respect to U of the Hamiltonian $H = I(x, u, t) + P'f(x, u, t)$. The multiplier function P and state function x will optimally satisfy

$$(211) \quad \dot{x} = \frac{\partial H}{\partial P} = f(x, u, t)$$

$$(212) \quad \dot{p} = - \frac{\partial H}{\partial x} = - I_x - P f_x$$

The team problem is formulated more like the calculus of variations' problem

$$(213) \quad \max_x \int_0^T I(x, \dot{x}, t) dt \quad x(0), x(T) \text{ Given.}$$

How would the gradient solution of this problem relate to the above formulation for optimal control? Noticing that in this case $\dot{x} = f(x, u, t) = u$, we can see that

$$(214) \quad \frac{\partial u}{\partial \tau} = I_{\dot{x}}(x, \dot{x}, t) + P(t)$$

Integrating $\dot{p}(t) = -I_x - P f_u = -I_x$ backwards we have

$$(215) \quad \frac{\partial \dot{x}(t; \tau)}{\partial \tau} = I_{\dot{x}} + \int_t^T I_x dt \\ = \int_t^T (I_x - \frac{d}{dt} I_{\dot{x}}) dt$$

or

$$(216) \quad \frac{\partial x(t; \tau)}{\partial \tau} = I_x(x(t; \tau), \dot{x}(t; \tau), t) - \frac{d}{dt} I_{\dot{x}}(x(t; \tau), \dot{x}(t; \tau), t).$$

Thus the gradient solution of the calculus of variations defines a differential equation on some space of functions $x(t)$. What problems do we have if, as in the above case, this space is infinite dimensional (such as a general Banach space over the real numbers)? Unfortunately many of the strong results of finite dimensional system of differential equations do not carry over to Banach spaces of infinite-dimensions. For example, Peano's theorem in n-dimensional Euclidean space, "If $\dot{x} = f(x, t)$ then continuity of f in the neighborhood of (x_0, t_0) implies the existence of a local solution," cannot be generalized to the infinite-dimensional case.⁽⁸⁾ The underlying difficulty in this example is that the infinite-dimensional closed unit ball is not necessarily compact.

This last fact has disturbing implications because the stability theorem underlying Lyapunov's Second Method requires the minimization of the Lyapunov function over a closed ball around the singularity.⁽⁹⁾ If the ball is compact then a minimum must exist by the Weierstrass theorem. However, Lakshmikantham has shown that "many problems concerning the behavior or solutions of ordinary differential system can be made to depend on scalar differential equations".⁽¹⁰⁾ The following extension of Lyapunov's method to infinite dimensional Banach spaces is based on the dependence between the infinite dimensional problem and a scalar problem.

Let $u \in B$, where B is Banach space over the field of real numbers and $t \in J = [t_0, \infty)$. A Cauchy differential equation is

$$(217) \quad \frac{du}{dt} = f(t, u) \quad t \geq t_0$$

$$u(t_0) = u_0$$

where $f: J \times B \rightarrow B$.

Assumption 1: $V \in C[R_+ \times B, R_+]$, a functional, and for $(t, x_1), (t, x_2) \in R_+ \times B$

$$(218) \quad |V(t, x_1) - V(t, x_2)| \leq L(t) \|x_1 - x_2\|$$

where $L(t) \geq 0$ and is continuous.

Assumption 2: There exists a function $g \in C[R_+ \times R_+, R_+]$ such that for each $(t, x) \in R_+ \times B$

$$(219) \quad g(t, V(t, x)) \geq \limsup_{h \rightarrow 0_+} \frac{1}{h} (V(t+h, x+h f(t, x)) - V(t, x)).$$

Assumption 3: For each $(t_0, r_0) \in R_+ \times R_+$, the maximal solution $r(t, t_0, r_0)$

of the scalar initial value problem: $\dot{r} = g(t, r)$ and $r(t_0) = r_0$, exists in the future.

Assumption 4: $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$, $V(t, 0) \equiv 0 \quad t \in R_+$.

Assumption 5: There exists a function $b: R_+ \rightarrow R_+$ such that $b'(r) > 0$

and $b(\|x\|) < V(t, x) \quad (t, x) \in R_t \times B$

Theorem: If $r=0$ is asymptotically stable for $\dot{r} = g(t, r)$, $r(t_0) = r_0$
then $u=0$ is asymptotically stable for $\dot{u} = f(t, u)$, $u(t_0) = u_0$. (11)

The finite-dimensional analogue of the calculus of variations problem is to select (x_1, \dots, x_T) to

$$(220) \max J = \sum_{t=0}^{T-1} I(x_t, x_{t+1}, t), \quad x_0, x_T \text{ given.}$$

The gradient solution algorithm for this problem at stage τ is defined by

$$(221) \dot{x}_+(\tau) = I_1(x_+(\tau), x_{t+1}(\tau), t) + I_2(x_{t-1}(\tau), x_t(\tau), t-1).$$

Theorem: If $I(x_t, x_{t+1}, t)$ is strictly concave in x_t and x_{t+1}
for all t and there exists a unique optimum $(\hat{x}_1, \dots, \hat{x}_{T-1})$ then
the gradient procedure for any initial vector $(x_1(0), \dots, x_{T-1}(0))$
converges to the optimum.

Proof: Either $D = \frac{1}{2} \|x(\tau) - \hat{x}\|^2$ or $D = J(\hat{x}) - J(x(\tau))$ is easily shown to
be a Lyapunov functions.

The following conjecture is merely stated. Its truth seems likely
in view of the above two theorems, but has not be verified by the
author.

Conjecture: If $I(x, \dot{x}, t)$ is strictly concave in x, \dot{x} for all t and
twice differentiable and if there exists a unique extremal
to the calculus of variations problem then the gradient
solution for any initial function $x(t; 0)$ converges to the
extremal satisfying Euler's equation.

Getting back to team theory, it is conjectured that strict
concavity of $u(A, \theta)$ in A for all θ is sufficient for convergence
of gradient solutions of the team's problem to the decision rules

that satisfy person-by-person optimality conditions. Exploration of this proposition is currently being carried out by the author.

FOOTNOTES

Section I

- (1) Marschak and Radner (1972), page 60.
- (2) To be more precise, the constraint must be satisfied almost everywhere, i.e., except on a set of measure zero.
- (3) See J.S. Jordan (1973), and M.J. Beckmann (1958), for the theory and application of quotas in a team problems.
- (4) J.S. Jordan (1973).
- (5) T. Groves and R. Radner (1972).

Section II

- (1) O. Lange (1936).
- (2) K. Arrow and L. Hurwicz (1960).
- (3) G. Dantzig and P. Wolfe (1960).
- (4) E. Malinvaud (1967).
- (5) M. Weitzman (1970).
- (6) G. M. Heal (1969).

Section III

- (1a) Arrow and Hurwicz (1960).
- (1b) See the general formulation in the previous subsection on "Decision Making with Identical Information".

- (2) This a system of first order linear differential equations with constant coefficients as can be seen by the following calculations:

$$\frac{\partial W}{\partial q_{1j}} = [\mu_1 P_1 \theta_1 (\pi_{j1}^1 + \pi_{j2}^1) + \mu_1 P_2 \theta_2 (\pi_{j1}^2 + \pi_{j2}^2)] - c_{11} \phi_1(y_{1j}) a_{1j}(\tau) - c_{12} (\pi_{j1}^1 P_1 + \pi_{j1}^2 P_2) a_{21}(\tau) - c_{12} (\pi_{j2}^1 P_1 + \pi_{j2}^2 P_2) a_{22}(\tau)$$

$$\frac{\partial W}{\partial q_{2m}} = [\mu_2 P_1 \theta_1 (\pi_{1m}^1 + \pi_{2m}^1) + \mu_2 P_2 \theta_2 (\pi_{1m}^2 + \pi_{2m}^2)] - c_{22} \phi_2(y_{2m}) a_{21m}(\tau) - c_{12} (\pi_{1m}^1 P_1 + \pi_{1m}^2 P_2) a_{11}(\tau) - c_{12} (\pi_{2m}^1 P_1 + \pi_{2m}^2 P_2) a_{12}(\tau)$$

where $\phi_1(y_{1j}) = P_1 (\pi_{j1}^1 + \pi_{j2}^1) + P_2 (\pi_{j1}^2 + \pi_{j2}^2)$
and $\phi_2(y_{2m}) = P_1 (\pi_{1m}^1 + \pi_{2m}^1) + P_2 (\pi_{1m}^2 + \pi_{2m}^2)$.

- (3) "Vertical decomposition" is the name M. Weitzman gave to a solution algorithm for a particular linear programming problem he used as an exercise in his course on Central Planning at M.I.T. in 1975. I would like to thank Professor Weitzman for his help on the general formulation of the problem.
- (4) See A. Whinston (1966) or J.E. Hass (1972).
- (5) Arrow and Hurwicz (1960), pp.50-51.
- (6) The adjustment of quotas in this manner is identical to the gradient procedure in G.M. Heal (1969).
- (7) I would particularly like to thank H. Varian for discussions on this points. In addition, K. Arrow, Y.C. Ho and R. Solow all provided comments on the topic.
- (8) See Ladas and Lakshmikantham (1972), pp.128-132 for counter-examples.
- (9) Krieder, et. al. (1968), page 412.
- (10) Laokshmikantham (1964), page 392.
- (11) Ladas and Laokshmikantham (1972), pp.167-170.

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BIOGRAPHICAL NOTE

James D. Hess was born February 9, 1949, in Ann Arbor, Michigan. From Satellite High School in Satellite Beach, Florida, he entered Princeton University in 1967. In 1971 he received, simultaneously, a B.S.E. in Electrical Engineering and an A.B. in Economics. He is married, with a three year old son, Clint.