HUMAN VISUAL PERCEPTION OF STRUCTURE

by

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ABSTRACT

Visual perception is viewed as the interaction between external
forces, generated by the pattern of stimulation at the retinal mosaic
and internal forces, which tend to organize the visual input in the
form of increasingly complex units.

Some psychophysical experiments, using computer-generated patterns
as stimuli, were conducted in order to find out the basic laws that the
internal organizing forces follow. Based on the results of these
experiments, a theoretical model was constructed. In it, the process
of perceptual organization is considered to occur hierarchically through
a sequence of similarly structured stages of processing.

The application of this theoretical framework to the study of
perceptual phenomena is exemplified by two particular cases, which are
discussed in detail: The perception of patterns of organized orientation
in textured fields (flow patterns), and the perception of symmetry.
Finally, a plausible scheme for the implementation of this theory in the
form of a computational symbolic model is suggested.

THESIS SUPERVISORS: Whitman Richards
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FOREWORD

This thesis is largely based on the ideas and intuitions of Dr. Manuel V. Cerrillo, from whom I had the fortune of receiving instruction and inspiration during 1973-74 and to whom I am deeply indebted. The central idea of it: That of applying the constructs of Group Theory to the study of perceptual phenomena, must be credited to him, as well as the conjecture that has in a large part guided my research: That if a pattern "fascinates" an observer it is because it is the representation of some underlying processing structure of the brain.

I also want to express my deep gratefulness to Profs. Whitman Richards and Henry Zimmermann, from both of whom I have received, not only advise and valuable suggestions, but also friendship and understanding.

Also my thanks to Charlie Lynn, who assisted me in the implementation of the software system, and to Cynthia, who typed the manuscript.

Two monks were arguing about a flag.

One said: "The flag is moving."

The other said: "Wind is moving."

The Sixth Patriarch happened to be passing by.

He told them:

"Not the flag, not the wind;

 mind is moving."

From "The Gateless Gate"

(A Collection of Zen Stories)

INTRODUCTION

0.0 General Considerations

The motivation for this work is to answer the question "How is
it that we see?", or, in other words, "Which are the processes
underlying visual experience?" The complete answer to these questions
represents a monumental task, and at present several disciplines, such
as Experimental Psychology, Artificial Intelligence, Neurophysiology,
Pattern Recognition, Brain Theory, etc., converge toward its solution.

There is, however, a basic issue that is only very seldomly discussed:
What do we mean by "visual experience"? Very often the problem of vision
is oversimplified, and to "see" a scene is identified with the task of
computing a verbal description of it. This problem is difficult enough,
but it is important to recognize that there is much more in visual
perception than assigning verbal labels to "objects".

If we pay attention to what we actually see, instead of thinking
about it, we find that our visual experience is richer than any verbal
description. This fact becomes evident as we observe a pictoric
representation of a landscape - i.e., a non-verbal description - and
compare the visual image with a literary description of the same scene
or, further, with a list of the "objects" it contains. Moreover, we all
recognize that certain visual patterns have qualities such as simplicity,
elegance, unity, beauty, harmony, etc., but all these words denote
experiences which are almost impossible to be described verbally, and
which cannot be explained by a theory which considers visual perception

merely as the assignment of verbal labels.

A further consequence of the "labelling" models is the belief that perception is based on the "detection" of certain features or structures that are "out there". This concept can be very useful in the construction of "pattern recognition" devices, but as a model of human perception it can be very misleading because, as I will show in Chapter 1, it is contradicted by many experimental facts.

A more realistic framework is to consider that the main function performed by the processes that underlie visual experience is to organize the input information in such a way as to construct what we call "reality", i.e., our world of surfaces, textures, structures, objects, etc. From this viewpoint, perception is considered to be the result of the interaction of two systems of "forces":[+] External forces originated by the pattern of stimulation at the retinal mosaic, ("proximal stimulus") and internal organizing forces resulting from the topological structure of the neural network.[*]

At this point it is impossible to determine experimentally what "subjective experience" is (and maybe it will never be possible).

_____

[+]Throughout this work, I will use the term "force" in a metaphorical sense. However, as I will show in Chapter 1, in many cases the behavior of the physical and the psychological forces is analogous.

[*]This idea was originally formulated by the "Gestalt" psychologists See, for example Koffka (1935) and Kohler (1947).

However, it seems reasonable to suppose that it arises as a correlate
of physiological processes occurring in the brain.[*] If this is the
case, it follows that there must be an isomorphism between the structure
of subjective experience and that of the underlying brain processes.
If we adopt this point of view, then we may say that the structures we
perceive in the world are constructed, rather than detected, by the
brain. The subjective perception of structure is the correlate of
internal organizing processes acting upon the visual input.

This does not mean, of course, that all the organizations we see
are purely imaginary. Our survival demonstrates that the structures
constructed by the brain are generally well correlated with the objective
structures occurring in the physical world. For example, the internal
construction of a region of the visual field as a segregated unit is
generally correlated with the existence of a corresponding unit in the
physical environment (i.e., an "object"). A corollary of this theory
is that our perception along a "continuum" between Chaos and Order, must
be restricted to a small region within it by our perceptual apparatus,
in the same way as our perception of the electromagnetic waves is
restricted to a small region of the spectrum. This conjecture is supported
by the fact that when we observe an unstructured (chaotic) pattern, such
as dynamic "noise", the brain tends to impose structure upon it, thus
limiting our potential perception of "disorder", and on the other hand,
a pattern with too high a degree of organization tends to fragment and

---

[*] See Globus (1973)

be perceived as a collection of lower order "substructures". This
will be discussed in more detail in Chapters 1 (#1.0.1) and 3 (#3.3).

0.1 Research Strategy: Ambiguous Patterns as Stimuli:

In order to study scientifically the organizational processes
of the brain, our first task is to find an experimental method by means
of which we can observe them in action.

If, as we have conjectured, visual experience results from the
interaction between the driving force towards organization, produced
by the internal processes, and the constraints upon their action imposed
by the proximal pattern of stimulation, then, whenever these external
constraints are weak enough, the action of the internal forces will
become apparent.

Several methods have been used by various investigators to generate
these "weak" stimuli, such as the use of short time exposures, low
intensity and after-images (see Koffka, 1935). However, one disadvantage
of these methods is that the forces operating in these special conditions
are not necessarily the same as those which operate in our everyday
perception.

A different approach is to generate a pattern which is "ambiguous"
in the sense that the external forces generated by it produce equal
stresses towards several possible organizations. In this case, the
action of the internal forces is manifested in the "choice" of the
configuration that we actually see. Note that in this case, a pattern
is ambiguous only with respect to the external forces (it is "externally"

ambiguous), and not with respect to the perceived configuration, which
may be unique. For example, consider Fig (1). If we analyze it
carefully, we will note that each point is surrounded by many neighbours,
and therefore, many different clustering schemes are possible at a local
level. However, when we observe it as a whole, a "spiral flow" pattern
appears unambiguously, and the dots are clustered along spiral
trajectories (see chapter 2). As a second example, consider Fig (2).
In this case also, from all the clustering schemes that would be plausible
based on dot proximity only, the system selects unambiguously to cluster
together sets of points that form 2 or 4-fold symmetric configurations.

On the other hand, perceptual ambiguity results when the internal
driving force towards the formation of several (externally) feasible
organizations is equal, and it is experienced subjectively as an alter-
nation between them, as in the familiar case of the Necker cube.

It is also interesting to note that a pattern is ambiguous only with
respect to a set of organizational processes. Thus, in our examples, the
Necker cube is ambiguous with respect to the three-dimensional organization
of plane figures, while the dot patterns are ambiguous with respect to the
processes that group together sets of "places" in the plane. This
represents an additional advantage of their use as stimuli, since it allows
us to study each process separately. Throughout this thesis I will use
as stimuli mainly ambiguous patterns produced by sets of equally sized
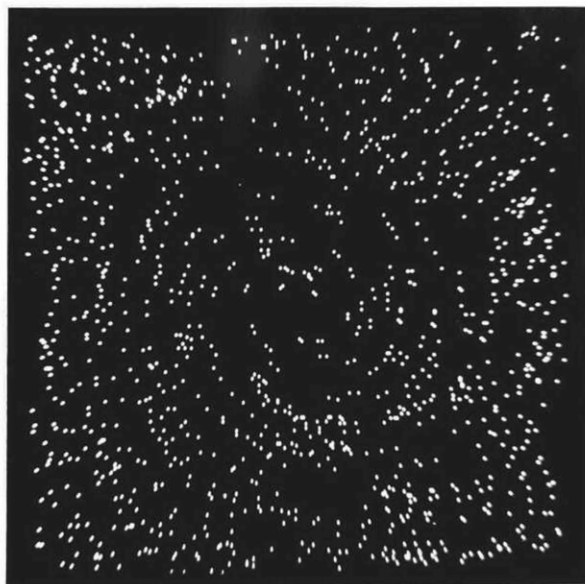dots located in the plane or in the 3-space.

Fig 1. A Flow Pattern generated by a rotation and an expansion in the
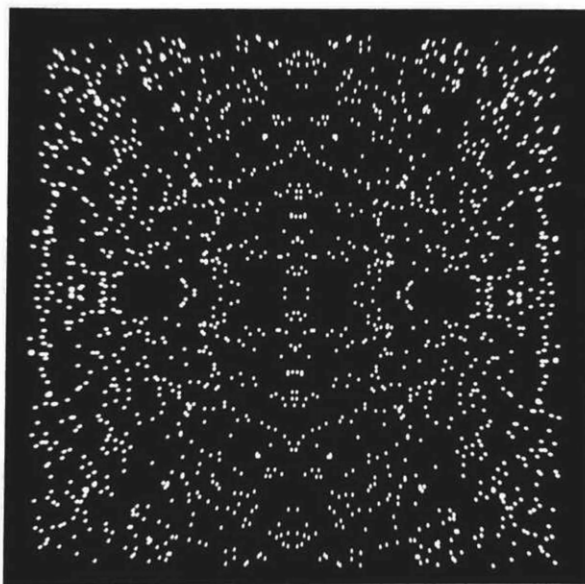   x direction: CFLW (RRDM (400), θ = .104, xexp = 1.05, yexp = 1).*

Fig 2. A 4-fold Symmetry: CSYM(RRDM(400), 4)

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*This string of symbols represents the "program" used to generate the
figure. (See Appendix 1)

0.2   Objectives of This Work.

The objectives of this work are to establish the basic laws that
the internal organizational forces follow, and based on these laws, to
formulate a mathematical model of the processes underlying perceptual
organization up to the level of the figure/ground articulation of the
field.

To do this, the first task will be to perform psychophysical
experiments in order to demonstrate experimentally the functional
reality of the internal organizing forces, and to find out the principles
on which their action is based.

In Chapter 1 I will discuss the results of these experiments and
formulate the main strategies of perceptual organization, namely:

i)   The tendency of the system to form structures that are
      representable as mathematical groups ("dense solutions").

ii)   The hierarchical process of formation of increasingly complex
      segregated "units" (internally organized in the form of group
      structures).

These principles will be presented formally as a mathematical model
of the processes underlying visual organization, in which the whole
system is viewed as a sequence of similarly structured stages of
processing. The functions performed by each stage are to take the set
of units formed by the preceding stage and to divide it into subsets
with the maximum possible degree of internal organization (clusters),
which will then serve as input units for the next stage of the process.

In chapters 2 and 3 I will present two examples of the application
of this theoretical and experimental framework to the study of the
visual perception of organized patterns: Chapter 2 is an analysis
of texture perception with special emphasis on the organization of the
direction of the microelements in a textured field, and chapter 3 is
an analysis of symmetry perception. Most of the patterns used as stimuli
were generated using a specially developed software system implemented
in the PDP-11 computer of the image processing facility of the Cognitive
Information Processing Group at the Research Laboratory of Electronics.
A technical description of the generating procedures is presented in
Appendix 1. A stereoscope is provided (see inside back cover) so that
the reader can observe the 3-dimensional patterns included in this work.

1. A MODEL OF THE PROCESSES UNDERLYING VISUAL ORGANIZATION

Our main thesis is that the basic process of visual perception
is not "feature detection", but rather organizational processes whose
result is the formation of perceptual units. These processes articulate
the whole field in the form of specific figure/ground configurations
and depend on the interaction between external forces, generated by the
pattern of stimulation at the retinal mosaic, and internal organizing
forces that result from the topology of the neural network. The
first task, then, is to prove experimentally the functional reality of
these processes. This will be done by means of several experiments.

1.0 Functional Reality of Organizing Processes:

1.0.1 Organization of dynamic noise.

If we switch a conventional TV set to an empty channel and adjust
the contrast and brightness controls until we see a dark field with
white dots in a sort of Brownian motion, and then fixate our gaze on an
arbitrary point on the screen, the movement will become organized during
a few seconds in the form of closed rings, spirals, rotating flower-like
patterns, etc. (Mackay, 1959) It is clear that we are constructing these
patterns and not "detecting" them.

An even stronger effect occurs if we break the temporal synchro-
nization of our two eyes, delaying the signal that comes through one eye
with respect to the other (see Ross, 1976). This can be done using the
so-called "Pulfrich effect": If the intensity of the signal coming through

one eye is dimmed (e.g., by placing a neutral filter over the eye),
the processing time in that eye will be increased, and so, when
observing an object in motion, a relative disparity will result,
and its value will depend upon the direction and speed of the
movement. (Lit (1949 , Levick (1972)) If we observe the TV screen
under these conditions, we would expect to perceive dots moving at
different speeds and depth planes, some to the right and in front of
the screen, and others to the left and behind it, as well as
uncorrelated dots moving upwards and downwards. However, what we
actually see is a transparent cylinder rotating around the plane of
the screen. (To my knowledge, this effect was first observed by
L. Kaufman at Sperry Rand in 1965.)

1.0.2 Random dot interference patterns.

If we superimpose upon a set of random dots a second version of
itself, but slightly rotated, then a new phenomenon occurs: (see
Fig (3)). The whole field becomes organized according to a single
unifying principle; we no longer perceive a set of unrelated dots,
but a global rhythm that expresses itself throughout the whole figure
(Glass, 1971).

If we now slide slowly one random dot pattern with respect to
the other, we see that the whole organization moves in a direction
perpendicular to that of the inducing sliding.[*] Now we ask: What

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[*]It is easily shown geometrically that the composition of a rotation
and a translation is equivalent to another rotation about a different
center. See, for example, Budden (1972)
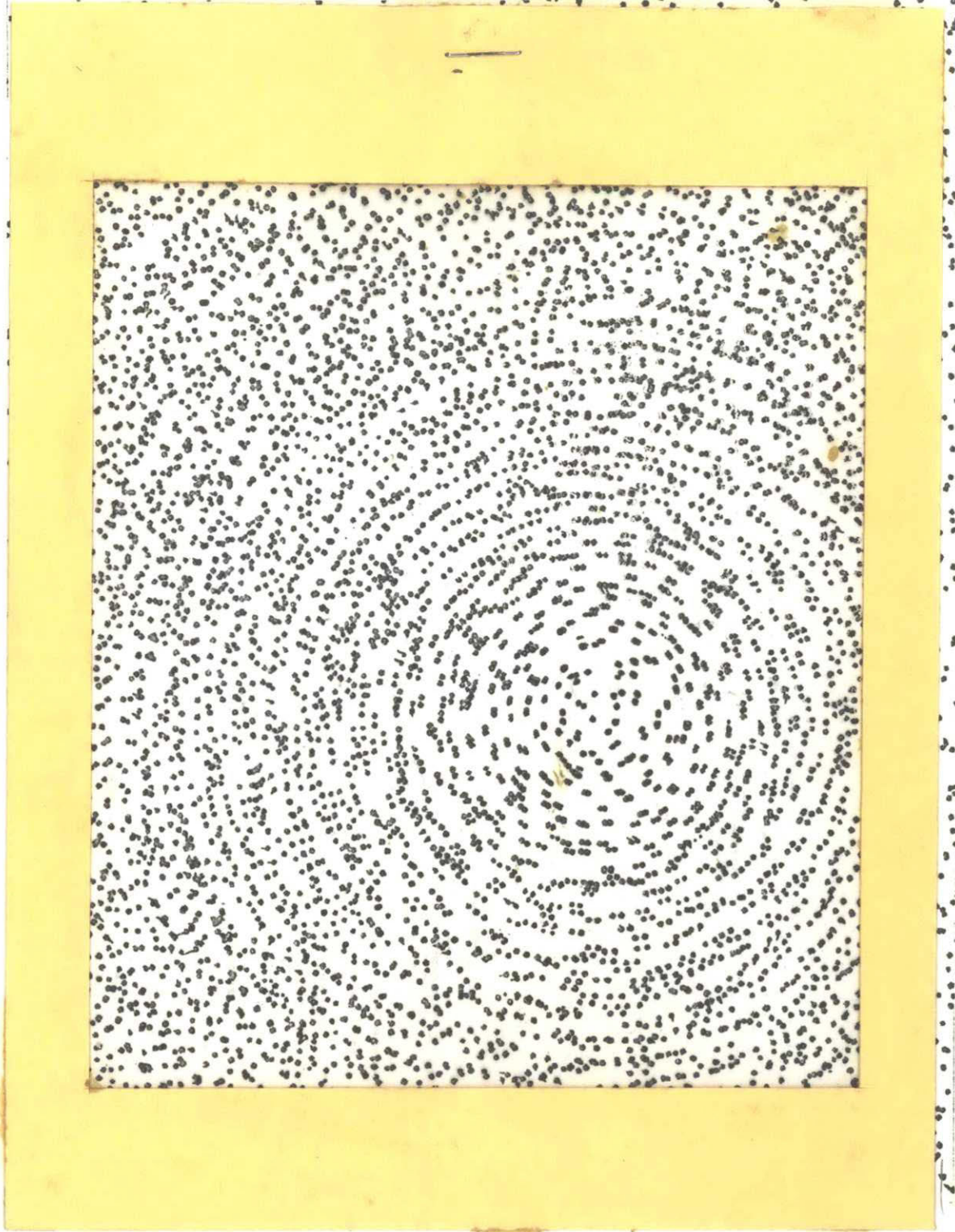
Fig 3   As we slide the transparency along the vertical direction,
        we perceive a circular flow pattern moving along the
   horizontal direction.

is it that moves? Consider Figs (4a) and (4b) produced at two discrete positions of the sliding pattern. If we analyze their detailed micro-configurations, we find them completely different, but nevertheless they appear very similar when observed globally. Their similarity lies in their overall organization, and this, in turn, results from a perceptual process of construction (see chapter 2 for a more detailed discussion). We must conclude, then, that these organizing processes have a functional reality, since they can induce sensations of movement where nothing is "really" moving and at the same time, this sensation of movement masks the "real" movement (the sliding movement of the set of dots)!

1.0.3 Interference patterns produced by regular systems of points.

If instead of random dots we take a regular system of points[*] (see Fig (5)), we can observe the same principles acting in a more complex way: As we rotate slowly the transparency with respect to the stationary pattern, we observe how a new organization is formed which appears to contract and expand as we rotate the pattern to the right and left.

This new organization is also perceived as a unified whole, as the representation of a single organizing principle, and, as in the case of the random pattern, the perception of expansion and contraction is induced by the internal organizing processes.

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[*]The original idea is due to Cerrillo (1975)
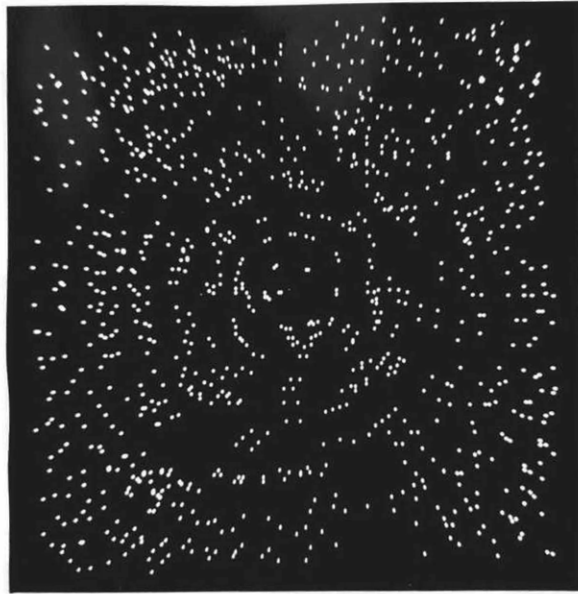
(a)
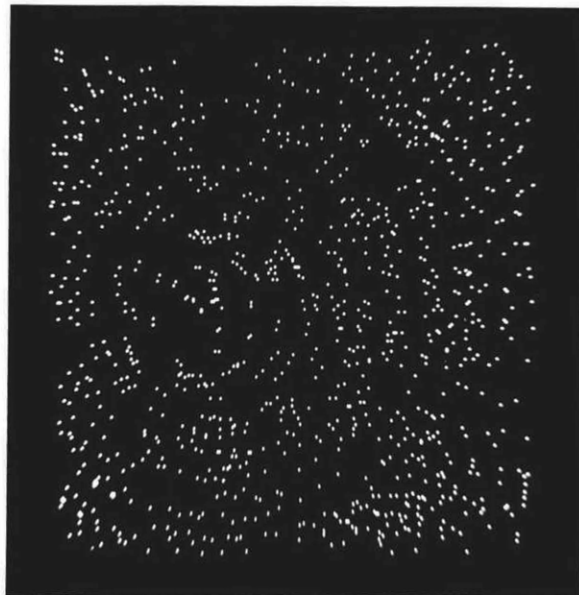
(b)

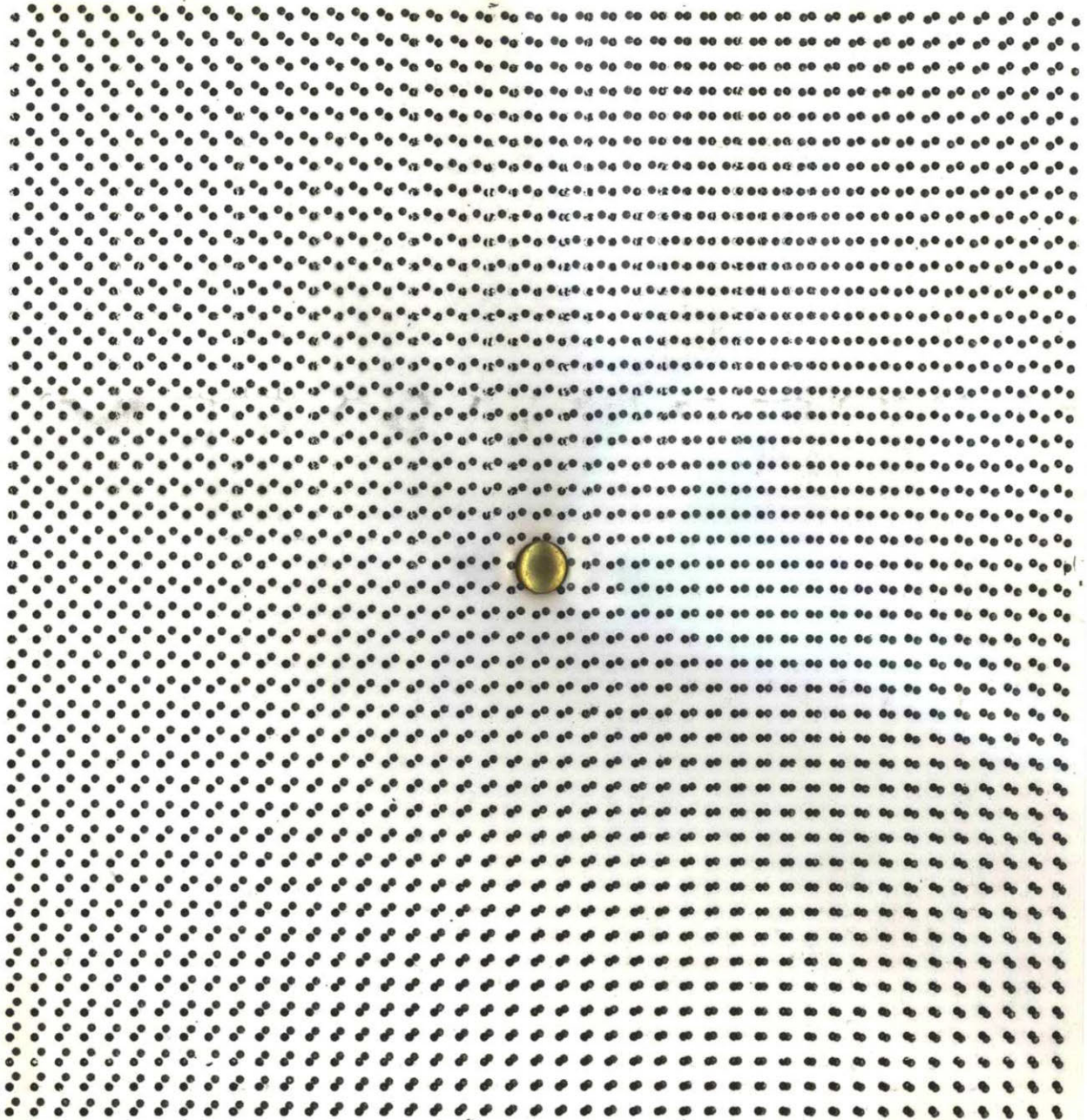Fig 4.   Two discrete positions of the moving circular flow.

Fig 5   As we rotate slowly the transparency, we perceive a pattern
         expanding and contracting.

1.0.4 Monocular Organization and Fusion Time of Stereo Pairs:

The functional reality of the organizing forces can also be demonstrated by the following experiment: Figures (6a) and (6b) are two stereoscopic pairs, i.e., they will form a three-dimensional percept when viewed through a stereoscope.* Although the disparity is the same in both cases, the fusion time (the total time elapsed from the moment we look through the stereoscope to the moment when a stable 3-dimensional image is perceived) is much greater for Fig (6b) than for Fig (6a), as may be easily verified by the reader. This is so, because the monocular organization (the small "ellipse") in the former case has to be broken before the 3-dimensional percept can be formed. The monocular organization may thus be viewed as a resistance which hinders fusion.

1.1 A Perceptually Ambiguous Figure Allows Us to Observe the Internal
   Organizing Forces in Action:

1.1.1 Formation of alternative configurations:

Let us now consider Fig.(7);** When we look at it for some time, we observe how different configurations appear and disappear alternatively. The specific configurations that are formed depend on the subject, the viewing distance and angle, the viewing time, etc. Some of the most commonly observed configurations are the following (see Fig (8)).
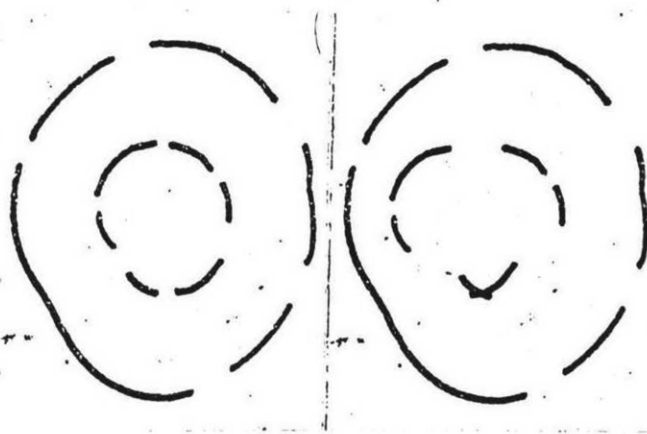
a) Systems of circles of several sizes

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*The reader interested in stereopsis is referred to the excellent work by Julesz (1971).

**See the Appendix for a description of the generating procedure.
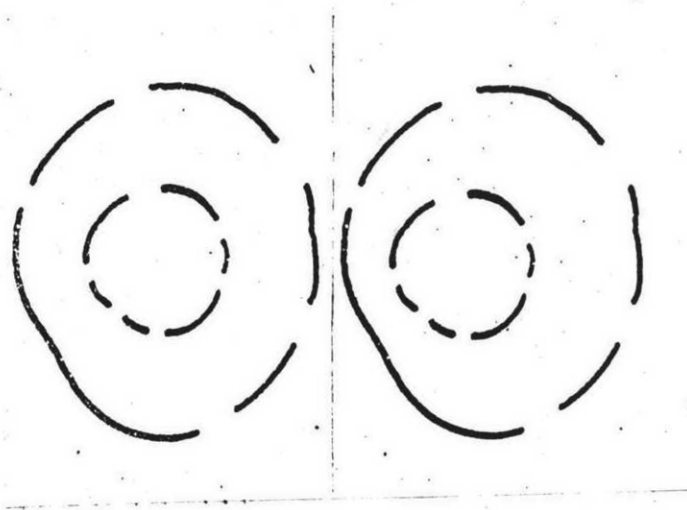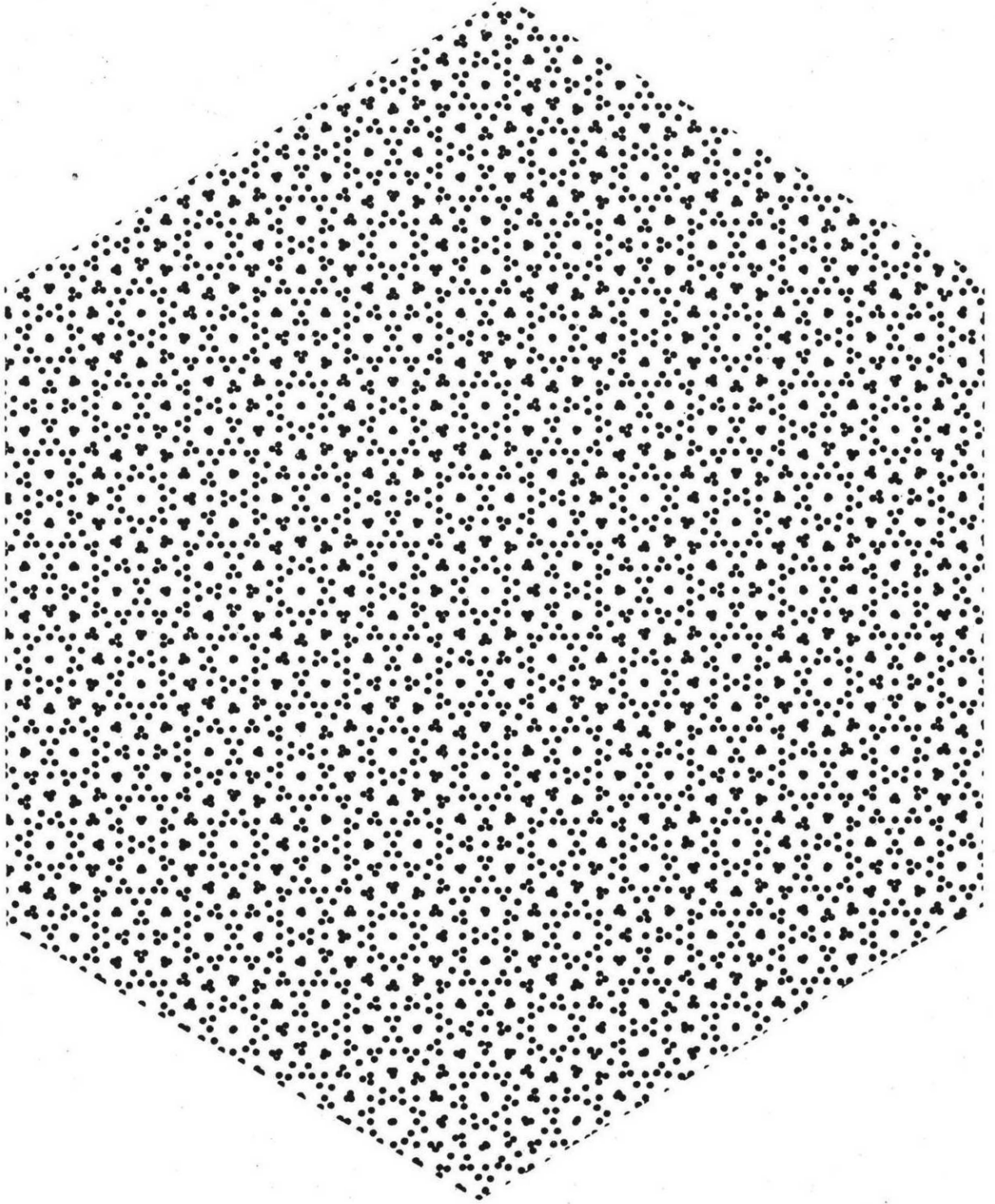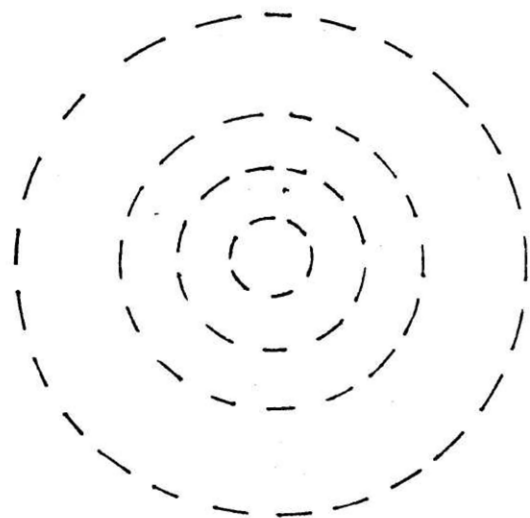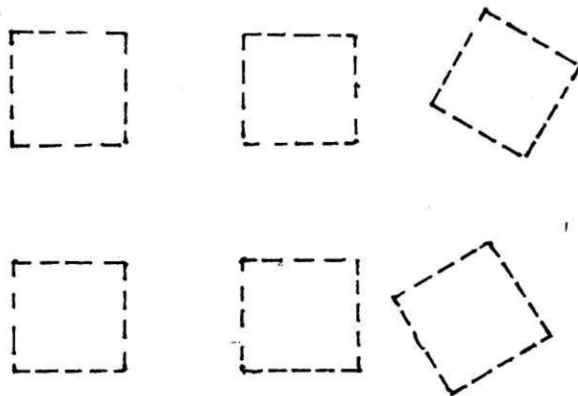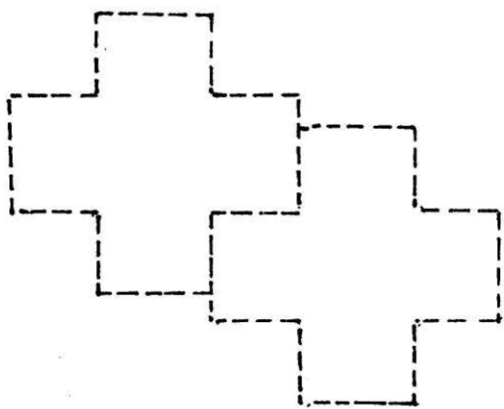
(a)

(b)

Fig 6. Monocular Organization and Fusion Time

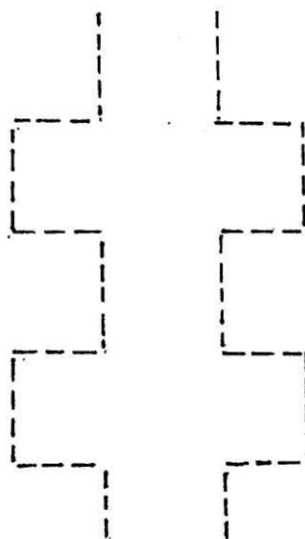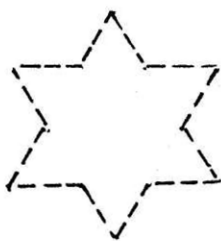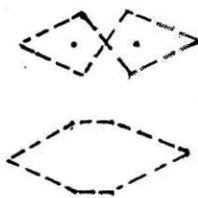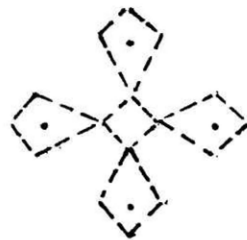Fig. 7 Two Congenial Groups: CROT(RTES(240,10,10,90),3)

(a)

(b)

(c)

(d)

(e)   (f)   (g)

Fig 8.   Some Configurations Perceived in Fig (7).

b) Systems of squares at different orientations (0°, 30°, and 60°), although those at 30° and 60° are only seldom seen.

c) Systems of "crosses", also at different orientations

d) Repetitive Patterns

e) Stars

f) "Lozenges"

g) 4-fold-symmetric figures, etc.

1.1.2 Common characteristics of the configurations perceived in Fig (7).

In spite of the large variety of figures formed, some general principles can be established:

a) It is very difficult to observe two (or more) different configurations at the same time. When one figure is formed, the dots that don't contribute to it are disregarded as "noise".

b) All the figures that are spontaneously formed have the following characteristics:

   b1) Closure: All the configurations are closed figures.[*]

   b2) Regularity: All are regular figures: They have at least 4-fold symmetry.

   b3) Unity: One tends to perceive a unified field, instead of isolated figures, i.e., once we see, say, a square, we tend to see the whole figure as a regular system of squares. This tendency is also evident in the effect of the surroundings on the perceived configurations: a square mask (fig (9)) will make the perception of squares easier.

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[*] Except for the repetitive patterns of Fig (8d) in which case the tendency to unify the field dominates over the tendency to closure
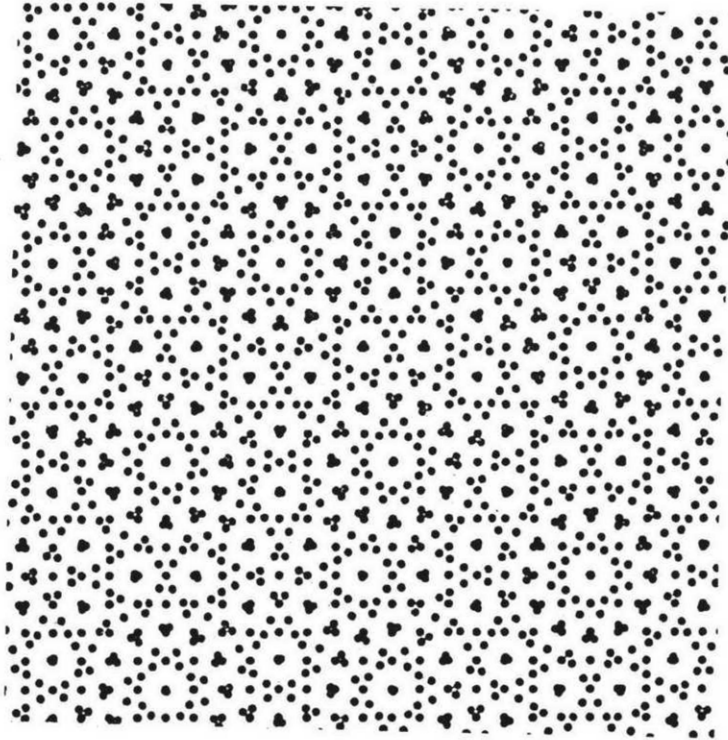
Fig 9.   The Square Configurations are enhanced when a Square Mask is
         put over Fig (7).

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b4) The figures with vertical axes of symmetry dominate over those

with tilted axes. This can be verified with the following

experiment: Observe the figure until you see the vertical

system of squares (or crosses). Now, rotate the whole figure

60°, so that a different pair of vertices of the hexagon is

aligned with the vertical. You will lose the perception of the

original system of squares (which now should appear tilted

60°), and see instead a new vertical system.

1.1.3 Conclusions from the experiment.

It is clear that the processes underlying the perception of
the different configurations are processes of construction, of
organization, and not of "detection" or "recognition". What exists in
reality" (in the retinal mosaic) is just a set of isolated dots[*], and
from this raw material the brain constructs different configurations.
These are not arbitrary: From the enormous number of possible alternatives,
the brain selects only those that are congruent with its basic laws.
When we see these patterns, we are not detecting forms that exist "out
there", we are observing our internal organizing forces in action, and
from this observation, we can infer the basic laws that these forces follow.

1.2 The Basic Laws of Internal Organization

The pattern of the dynamic behavior of the visual system is the

_____

[*]Even this is a consequence of organization: the proximal stimulus
in only an array of receptors of which some receive more stimulation
than others. The perception of a "dot" is in itself the result of a
grouping process (c.f: Koffka (1935))

same as that of any natural phenomenon: Let us assume a hypothetical

initial state of equilibrium with no activity in the system: A

completely homogeneous field. As soon as we introduce an inhomogeneity

in the system, a driving force is generated that will tend to move the

configuration of the system towards a new state of equilibrium.

A model of the system, therefore, should contain a general

description of the conditions that characterize the state of equilibrium

(i.e., the direction of the driving force), as well as a description

of the modus operandi of the system, i.e., the way in which these

conditions tend to be reached.

A model of this sort should be able to explain how the miscroscopic

processes that constitute the modus operandi give rise to macroscopic

observable phenomena; in other words, how the local processes define

the global state of the system.

1.2.1 The driving force: the formation of group structures.

Our hypothesis is that the direction of the driving force of the

system is such that the final results of perceptual organization will

tend to form configurations that represent a mathematical group.[*]

A configuration is a representation of a group if the variation

of some attribute (s) of its elements is generated by a group of

transformations. Consider for example fig (10). The "elements" of the

pattern are the clusters marked in Fig (10-B), and the position and

orientation attributes of them vary accordingly with the transformations

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[*]See, for example, Budden (1972), p.   for the formal definition of
a mathematical group.
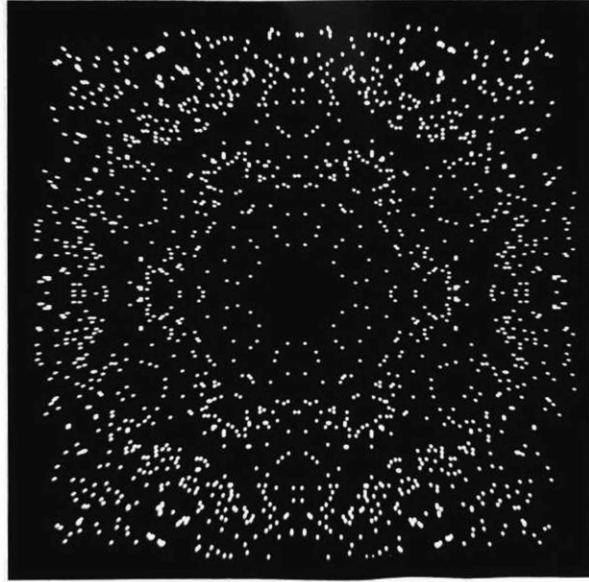
(a): CSYM (RRDM(250), 3)
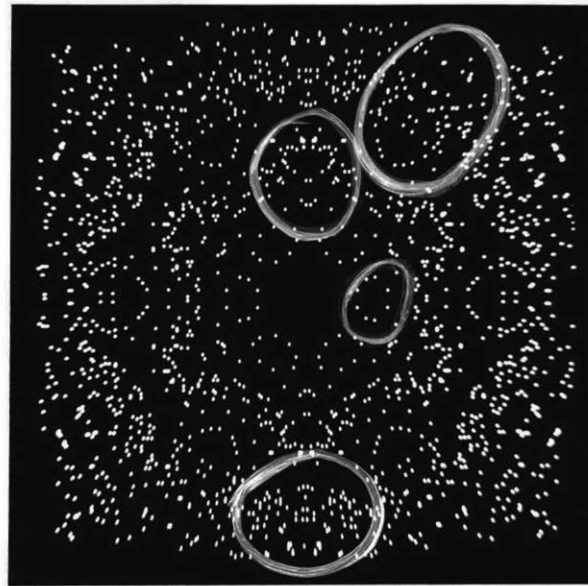
(b)

Fig 10.   The perception of a Kaleidoscope (a) is based on the variation
          of the position and orientation of the clusters marked in (b)

of the dihedral group D_3.* In fact, any pattern that is perceived as
"symmetric" can be considered as the representation of a group.

This principle does not mean, of course, that all the configurations
that we perceive are representations of groups. We must not forget that
the external forces impose constraints upon the organizing processes, and
our actual perceptions are always the result of the interaction of these
two systems of forces, in the same way the actual shape of a drop of
oil in water will not be circular if we agitate the water. It means
however, that whenever the external forces are weak enough as to give to
the system degrees of freedom, it will tend to form group structures.

Let us return to our experiments with ambiguous patterns and reexamine
them in this light.

1.2.2 Group theoretical analysis of our experimental results.

a) Consider the group formed by all the spatial rotations about
the Z axis.** It is a continuous group generated by the transformation
matrix:

$$Z_\varphi = \begin{bmatrix} \cos\varphi - & \sin\varphi \\ \sin\varphi - & \cos\varphi \end{bmatrix}$$

As the real parameter φ runs through the values in the interval $[0, 2\pi]$.
The patterns formed as we fixate on a point of a dynamic random field
(#1.0.1) are clearly representations of this group.

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*See Budden (1972) ch. 13.

**See Smirnov (1961), Chapter 9.

b) Similarly, the rotating cylinder formed when we delayed the signal coming through one eye (#1.0.1) is a representation of the group of spatial rotations about the x axis.

c) Consider once more the group of rotations defined by the transformation matrix Z_φ . We can define a two-dimensional coordinate system in it[*] by allowing the parameter to vary as a continuous function of two real variables (u,v) subject to the condition that to the point (0,0) corresponds a value of $\varphi = 0$ (i.e., the identity transformation). If we consider the transformation of this group to act upon the orientation of the line segments formed by joining pairs of corresponding neighboring points in Fig (4a)[**] (see 1.0.2), we may consider this figure to be a representation of this two-dimensional topological group (see Appendix 2, #A.2.2), the coordinates (x,y) of the middle points of each line segment being a linear transformation of the parameters (u,v). There exists the difficulty that the variation of the orientation of the line segments in our pattern is discrete, while the corresponding group is continuous. Nevertheless, we still can consider it as being a "sampled" representation of the continuous group, the "samples" being taken at random values of (x,y).

d) The patterns formed by the superposition of two regular systems of dots (#1.0.3) are representations of the group generated by two equal orthogonal translations in the plane. The dynamic expanding/contracting pattern formed as we rotate the transparency may be considered the

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[*]See Pontryagin (1966) pp. 96, 283 for the technical details.

[**]Since this figure is generated by superimposing to a set of random dots a second pattern formed by applying to the first a transformation T (in this case a rotation), two points (x_1,y_1) and (x_2,y_2) are "corresponding points" if $(x_1,y_1) = T[x_2,y_2]$ or $(x_2,y_2) = T[(x_1,y_1)]$.

representation of a one-parameter continuous group of orthogonal
translations.

It is important to note that in this case, the "element" that
undergoes the transformations of the group (the "fundamental region"),
in this case the system of "concentric squares", is in itself the
representation of a group (the dihedral group D_2). This means that
once a group structure is formed, it constitutes a segregated unit
which may be used as a building block for constructing higher order
organizations.

e) This phenomenon can be observed also in Fig (7). The figures
that are formed (#1.1.1) are all representations of discrete groups of
transformations consisting of rotations and reflections (dihedral groups);
however, once one of these units is formed, the brain tries to organize
the whole field as a group of translations, using the formed figure as
the fundamental region. Since this is not possible, the structure is
unstable, and the brain tries an alternate organization using a different
unit as building block.

f) A closed contour is a representation of a one-parameter*
continuous group, the transformation of the group being a rotation
acting on the slope of the contour. Thus the brain groups the line
segments of the internal "ellipse" of Fig (6) (#1.0.4) to form this
stable monocular organization, which is strong enough as to retard fusion.

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*A contour may be represented as a complex valued function on one
real parameter.

1.2.3 Groups and the "Law of Pragnanz".

If we examine which are the perceptual characteristics of a group structure, we find that they are:

a) Regularity ("good shape").

b) Closure.

c) Unity and Simplicity (in the sense that there is a single organizing principle for the whole pattern.

d) Completeness (at a global level).

e) Continuity ("good continuation" - at a local level).

Therefore, we recognize in our "driving force towards group structures" a more general and formal reformulation of the Gestalt "Law of Pragnanz" (See Koffka (1935)).

1.3 The modus operandi of the organizing processes.

Up to this point we have established the direction in which the system tends to move when inhomogeneities are introduced, and we have found a tendency to form group structures. We will now examine possible ways in which these structures are constructed.

The current doctrine of neurophysiology is the hypothesis that perceptual processes are organized as successive layers of cells with increasing size and complexity of their receptive fields, thus implying a local-to-global processing hierarchy. However, it is important to recognize the highly parallel nature of neurophysiological systems,

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*See, for example, Lettvin, et al (1959) and Hubel and Wiesel (1962)

and the incredible richness and complexity of the interconnections of
their networks, which implies the existance of feed-forward and feed-
back between successive "stages" of a process, as well as lateral connec-
tions between different subsystems.

These facts make it practically impossible at this point to construct
detailed faithful models of a perceptual system; however, it is possible
to deduce from the analysis of the results of psychophysical experiments
certain general principles that these perceptual processes must follow:

1.3.1 Principle of unit formation

The basic law that the perceptual organizing processes follow
is the tendency to form segregated units of increasing size and complexity
in a hierarchical way. This means that the most stable organization
of a large field is not that in which elementary "atoms" or micro-
elements are held together by a single global force, but rather, a
tree-like structure in which the microelements are held together in
small clusters or "molecules", which in turn, are grouped together
into larger units, and so on until the whole field is organized.

We observe two forces in action as the cause of this behavior:
Cohesive forces which tend to form clusters, and "surface tension"
forces, which segregate these clusters as independent, complete units.
As the analogy with the physical process from which I am borrowing this
term suggests (the formation of soap bubbles, for example), both forces
arise from a single underlying principle: the attraction forces between
units. These forces, in turn, follow two important laws: the

"principle of homogeneity" and the "principle of relative proximity".

1.3.2 Principle of homogeneity:

This principle, first observed by the Gestalt Psychologists, can be stated as follows: "Homogeneity in a field generates cohesive forces and inhomogeneities generate segregating forces." This means that if a region in the retinal mosaic receives uniform stimulation, it will form a unit, while discontinuities in stimulation (edges, lines) will segregate the unit form the rest of the field. For example, in a dot pattern such as those we have considered, each dot will form a segregated unit. Similarly, a closed contour will generate forces that will segregate the area in its interior from the rest of the field. It is possible, however, for several of these units to interact and form larger units within which the stimulation is no longer uniform. This leads us to our next principle.

1.3.3 Principle of relative proximity:

This principle states that the attraction force between two units varies directly with their size, "strength", and with the degree of similarity between them, and inversely with the distance between their centers of gravity.

This principle is slightly recursive in the sense that the "strength" of a unit depends on the local attraction forces between the sub-units that form it, as well as on the degree of its internal organization. In this sense, the maximum internal cohesion or "strength"

is obtained by complete homogeneity within the unit, or by a close
packing of the subunits in the form of a group configuration (see
#1.2.1). However, the strength of any particular unit is, at this
point, impossible to predict theoretically, and more experimental
research in this direction is needed.[*]

A second difficulty arises from the notion of "similarity".
One possible way around it would be to consider the values of a set
of attributes of each unit, together with their spatial coordinates,
as the coordinates of a point in an n-dimensional Euclidean space.
The attraction force between units, then, will be considered to vary
inversely with an "effective inter-unit distance" which is simply the
Euclidean distance between the two points so defined. The trouble lies
not only in the selection and quantification of a set of "relevant
attributes," but also in the fact that the "weight" or scale factor for
a given attribute may be a variable which depends on more global
considerations (see #1.3.4 below).

In summary, we find that at this point a more precise and quantita-
tive formulation of this principle is not possible, but it will give us
useful qualitative indications in the general case. For simple cases,
it reduces to the Gestalt principle of proximity, and some attempts have
been made for building quantitative models of its functioning (see
O'Callaghan (1974 a and b)). This principle, however, does not tell
us anything about the way in which global organizations are built from

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[*] A plausible method for determining the perceived strength of an
organization is by determining psychophysically the total amounts of
noise that, when added to the unit, will destroy it.

local operations. The following principle points in this direction:

1.3.4 Formation of local groups ("dense solutions").

This principle can be stated as follows: The system tends to structure the field in such a way as to maximize the formation of clusters of "proximal"[*] elements, so that within each cluster, either:

a) The position of "similar" elements can be expressed as the result of applying to a subset of them a discrete group of transformations, or

b) The variation of some attribute of the elements can be expressed as a smooth and continuous function of their spatial position.

This principle is a consequence of the general driving force of the system (see #1.2.1) operating at a local level. While this is obvious in the former case, in the latter it requires some clarification. Let us examine both cases in more detail:

Case (a): The formal expression of this condition is presented in Appendix 2 (#A.2.1). Another form of stating it is to say that a set of elements will form a cluster if the spatial configuration of the set is invariant under the transformations of a discrete group G which belongs to a set of groups $\mathcal{G}$. Once this cluster is formed, it will be perceived as "symmetric". Three points of this definition need clarification:

1. The set $\mathcal{G}$ of groups that will produce a symmetry percept must be determined by means of psychophysical experimentation. From the

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[*] i.e., elements which are at a distance such that the attraction forces between them is above a certain threshold. See #1.3.3.

experiments that I have done, I know that at least all the dihedral

groups D_n and the cyclic groups of order greater than 3 should be

included in it.

2. We have required that all the elements that will form the cluster

must be "similar". By this I mean that the values of a subset of the

attributes of each element must remain approximately constant for

all the elements of the set; In other words, something must remain

invariant under the transformations of the group that takes one

element of the set into another, if the group is to be constructed at all.

3. Since in this case the cohesive forces operate between "corresponding"

elements (i.e., sets of elements such that the transformations of the

group map one into the other), the condition of "proximity" between

the elements must be interpreted in the sense that the maximum distance

between corresponding elements must be smaller than the "critical distance"

at which the attraction forces between them fall below a given threshold.

This means that the maximum distance from an element to the axis

(or center) of symmetry of the configuration must not exceed a "critical

value" - which depends on the size and internal organization of the

element - if it is to belong to the cluster. All these points will be

discussed in more detail in chapter 3.

Case (b): Let us now examine the connection between the driving force

towards the formation of groups and condition (b):

In Appendix 2, #A.2.2 are presented the conditions that a set of

elements must satisfy to be a "sampled" representation of a local group.

These conditions are fulfilled only if the variation of the attribute

within the cluster can be expressed as a continuous and smooth function

of the spatial coordinates of the elements. The interior of such a

cluster (i.e., excluding the elements in the boundary) will be a

representation of a local group, for all the transformations we will be

considering (rotations, translations, etc.).

This grouping process can be implemented as a local operator which

will cluster together two elements X_1 and X_2 whenever the condition:

$$\text{if} \quad d(X_1, X_2) < E_1$$
$$\text{then} \quad |a_1 - a_2| < E_2 \quad \text{is satisfied.}$$

(where a_1 and a_2 are the values of the considered attribute for the

elements X_1, X_2; E_1 and E_2 are arbitrarily small numbers and the function

d(   ) computes the Euclidean distance between the two elements.)

However, if the variation of the attribute is closed within the

cluster (see appendix 2), it will be the representation of a complete

continuous group. It is found that such clusters (see Fig (11)) are

more stable than others which, though fulfilling condition (b) are only

local groups (Fig (12)). This will be discussed in more detail in

Chapter 2.

It is important to note once more[*] that the actual variation of

the attribute is discrete in the cases we will be considering, and so,

if the pattern is to be an adequate "sampled" perceptual representation of
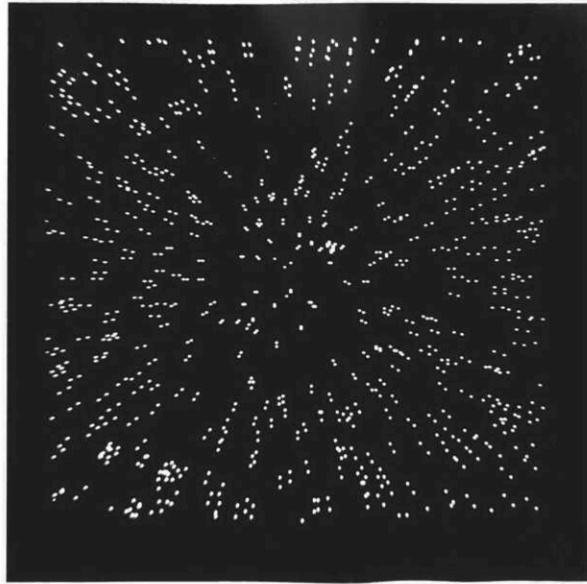
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[*]See #1.2.2-c

Fig 11.   Flow pattern generated by an expansion:
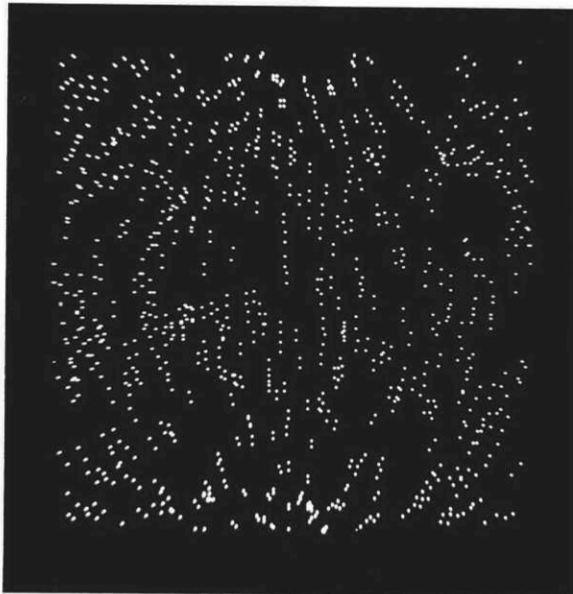   CFLW (RRDM (400), exp=1.05)

Fig 12.   Flow pattern generated by a bilinear transformation:
          OFCT (RRDM (400), x0=0, y0=.035)

a continuous group, the "sampling density" - i.e., the number of elements
per unit area, must be high enough; otherwise, the cluster will not be
formed.

Another important consideration is related with the question of
similarity: The perceptual similarity upon which these grouping
processes are based is constructed, rather than detected: it results
from a search over the set of attributes of the elements until one of
them is found to form a "dense solution".

In summary, the consequence of this principle (either condition
(a) or (b)), is that the whole field (i.e., the set of all the elements
in the field) is partitioned into a set of clusters which are internally
organized as (local or global, discrete or continuous) groups. Once a
cluster is formed, it is no longer considered perceptually as a set of
isolated elements, but as a continuous, "dense", segregated entity.
For this reason, we will call a group-structured cluster a "dense
solution". A good example of this is found in the stereograms constructed
by Julesz (1971): Even when in some of them the microelements are
relatively sparse, we never perceive them as sets of dots floating at the
same depth, but always as smooth, "dense" surfaces with speckles on them.

Moreover, once a cluster is constructed, it becomes a new "unit"
with a new set of attributes, and therefore, may be considered as an
"element" for a new clustering process.

In this sense, each application of this principle is a "stage"
of a global organizing process, which results in the formation of a
"system of clusters". The dimension of the system is equal to the

number of clusters it contains, and its "order" is the number of
stages necessary to build it from the simplest (homogeneous) microelements.

1.3.5 Solution of local ambiguities; Rival and congenial organizations

Very often a system of clusters is not uniquely determined by the
rules formulated above. In this case, the ambiguity is solved by
selecting the system whose clusters will form a dense solution at the
next level. In other words, if a certain stage produces as its output
a set of alternative systems of clusters, the process will select the
system - i.e., the set of clusters, which can itself be partitioned into
a system of higher order clusters: Local ambiguity is solved with basis
on more global considerations.

It is possible, however, that two mutually exclusive systems of
clusters form different dense solutions at the next level. In this
case, the two resulting organizations are "rival", and they both
cannot be perceived at the same time. If only one of them has global
closure (i.e., is the representation of a global group), it will
dominate over the other. In other cases, both will alternate in time
(see for example Fig 7), or they will mutually cancel each other and
no organization will be perceived. Examples of each of these cases will
be presented in the next chapter.

On the other hand, when two (or more) organizations can be
perceived at the same time they are said to be congenial. This can
happen in the following ways:

a) Both organizations occur at successive levels: One of them
is a system of clusters which act as units to be clustered at the
next stage, and produce the other organization. As an example, see
Fig (13). In it, two "spiral flow" clusters are formed at some
stage of the process and at the next stage, they are clustered
together to form a representation of the discrete group D_1: a
bilateral symmetry.

b) The two organizations are based on the variation of two independent
attributes, such as size and color, orientation and depth, etc., so that
the formation of clusters based on the variation of one attribute, does
not interfere with the formation of those based on the variation of the
other. As an example, see Fig (14): When these figures are viewed
through a stereoscope, a hemisphere appears in depth, but the "spiral
"flow" is not destroyed.

Note that in both types of congenial organizations, each organization
retains its individuality, and is often enhanced, but at the same time,
a new percept is formed: "symmetric spirals" or "sphirical spiral flow".
The whole is clearly different from the sum of its parts.

1.3.5 General validity of the "principle of search for dense solutions"

The principle of "search for dense solutions" finds application
in a wide variety of situations. The reader is invited to reexamine
the examples presented thus far in this light. I will present some
more examples to show the general validity of this principle in the
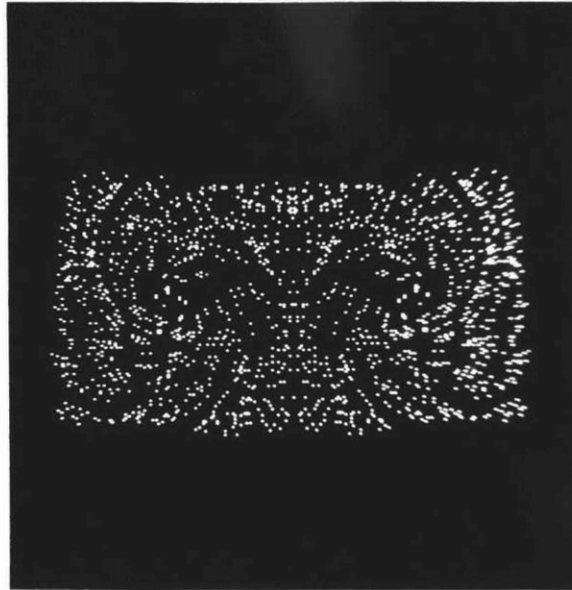perceptual organization processes:

Fig 13.   Symmetric Spiral flow:
CSYM (SHF2(RRDM(400), exp=1.05, θ=5°), 160), 2)

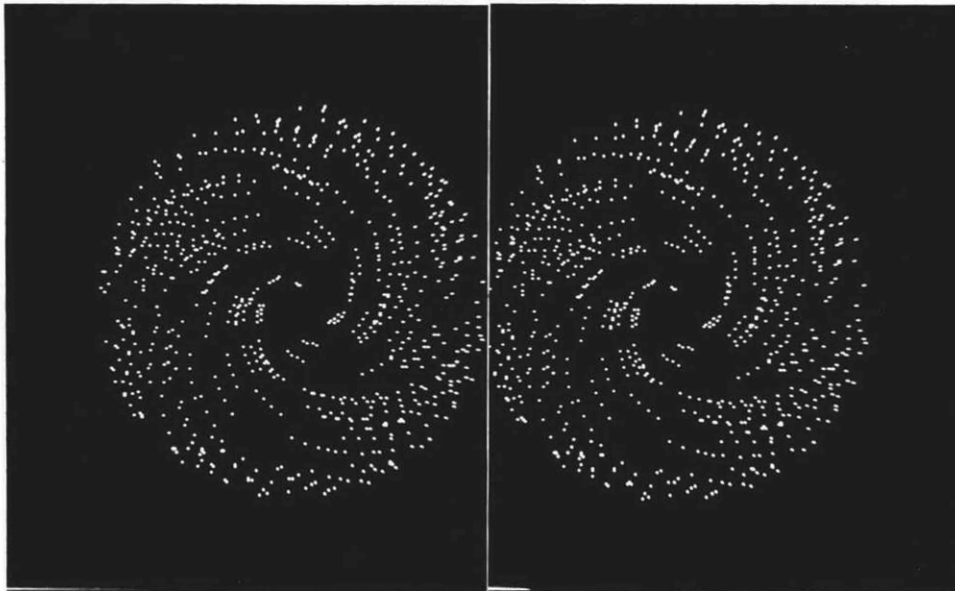Fig 14. Spherical spiral flow:
right: CFLW (RRDM (400), exp=1.05, θ=5°);
left: SHFS (CFLW (RRDM(400), exp=1.05, θ=5°), MAXSH=8)

a) Good Continuation:

Consider the random pattern of Fig (15). After observing it
for a while, we note that sets of points tend to cluster together in
the form of straight or smoothly curved lines. According to our model
the mechanism for the formation of such trajectories is the following:
At a first stage, pairs of proximal dots will group together to form
small "line clusters", which will have, therefore, an <u>orientation</u>
attribute.[*] These units, in turn, could be grouped together in many
ways. However, we recognize that a smooth trajectory is a representation
of a continuous local group: The transformation of the group is a
rotation, and it acts upon the slope of the trajectory, i.e., upon the
orientation of the line clusters involved. Since a contour can be
represented as a complex valued function of a real parameter t, we can
make the magnitude of the rotations depend also on t, and in this way,
define a one-dimensional coordinate system on the local group, and satisfy
the conditions required for cluster formation (see #A.2.2). It is
interesting to note that in this particular case, the application of
this principle reduces to the Gestalt law of "Good Continuation".

b) Movement Perception:

Some experiments on movement perception reported by Johansson
(1976) make clear that when a set of points are moving at the same
time, the visual system tends to build a dense solution, grouping
together all the points that have a common movement component. The

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[*]Such line clusters may be considered representations of the group D_1.
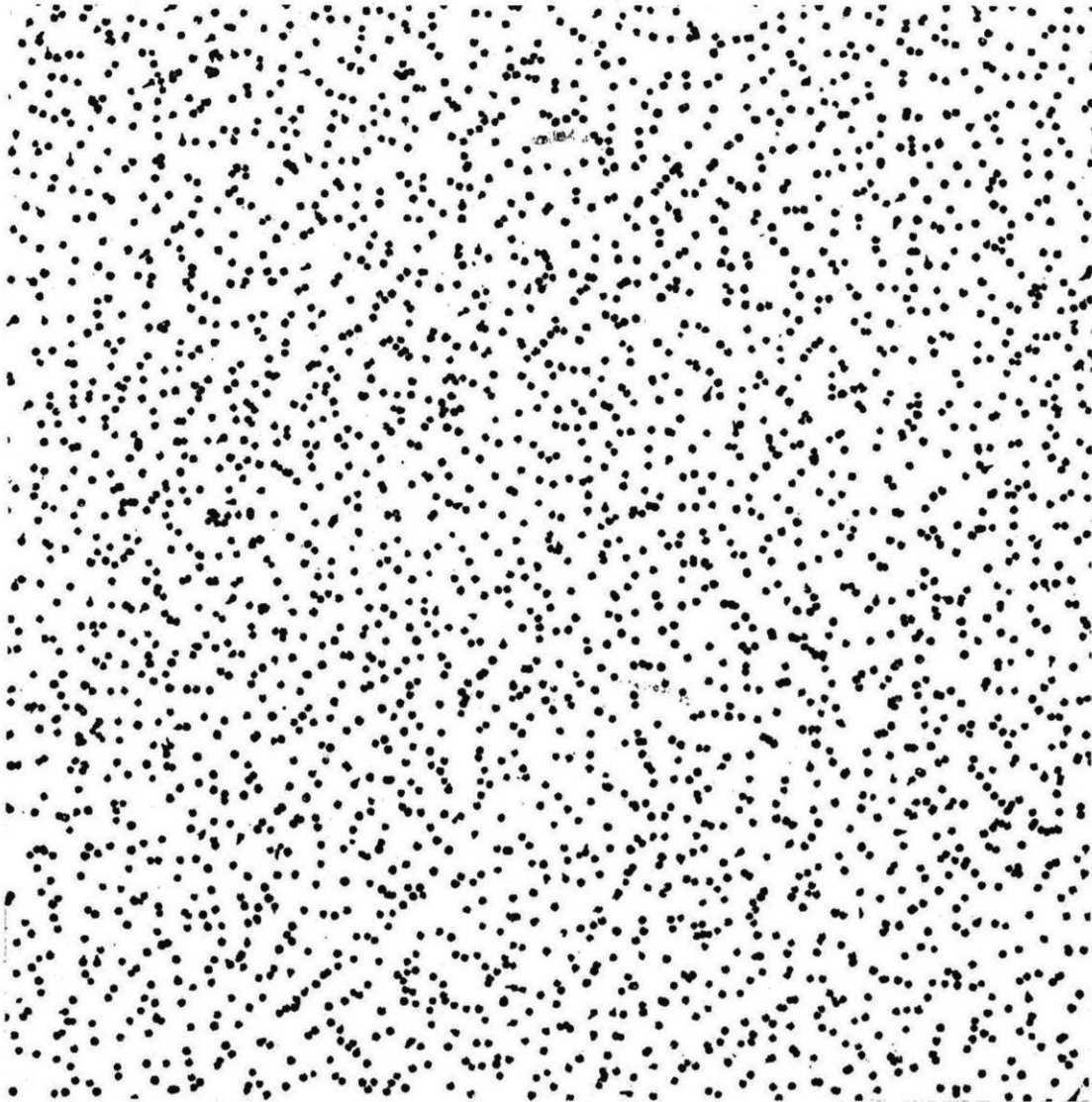
Fig 15. Random dot pattern

result of this process is that we tend to perceive rigid bodies
in motion, rather than separate particles moving in a complex way. Thus,
if a point describes a cycloid with its motion, and at the same time a
second point moves horizontally, if the motion components of both particles
in the horizontal direction coincide, the subjects will not perceive the
cycloid but rather a rigid wheel moving (see Fig (16)).

c) Stereopsis:

The mechanism underlying the perception of depth from a random dot
stereogram (Julesz, 1971), involves pairing together the corresponding
points in the right and left components of the pair, and computing the
relative disparity. However, such a stereogram is always ambiguous,
because since all the dots involved are similar, a great number of pairing
strategies (false targets) is possible. The visual system, however,
selects always the dense solution that corresponds to a smooth surface: [*]
a two dimensional continuous local group whose transformation acts on the
depth attribute of the elements.

1.4 Conclusion: A Model of the Processes Underlying Perceptual
     Organization.

Let us sum up what we have done up to this point:

First, I presented several experiments which showed the functional
existence of internal organizing forces acting upon the pattern of
stimulation in the retinal mosaic. From the analysis of the results of
the experiments, the basic laws that rule the dynamic behavior of these

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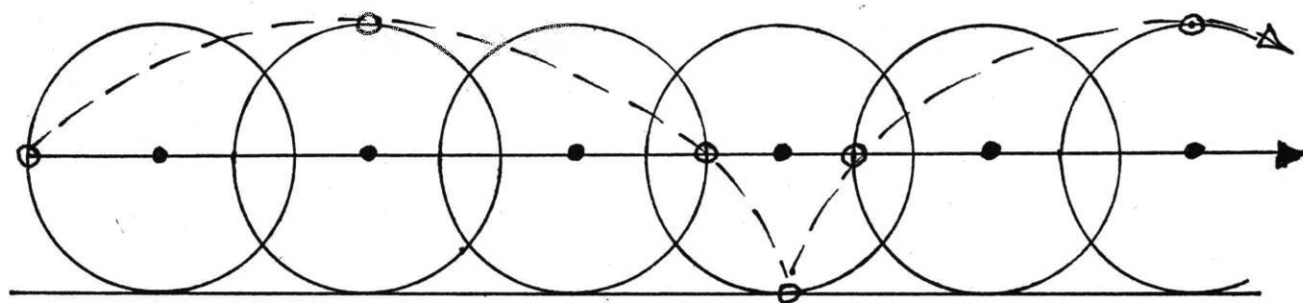[*]This process is called "Global Stereopsis" by Julesz.

Fig 16. (From Johansson (1971))

processes were deduced. We distinguished between the driving force of the system – the tendency to form structures that represent mathematical groups, – and the principles that the local behavior follows, and from which global phenomena result: The principles of Unit Formation, Homogeneity, Relative Proximity, and the Search for Dense Solutions. All these principles can be put together in the following way:

The process of organization of the visual input can be represented by a set of equivalent processes that occur sequentially. Each of them can be represented as a system whose input is a set of n systems of clusters (see #1.3.4 and #1.3.5) S_1, S_2, ...S_n , and whose output is a new set of m systems of clusters Q_1, Q_2,.....Q_m . The output set is formed by selecting from the input set the system S_k whose clusters form one or more dense solutions. The effect of this process is such that:

a) The order (see #1.3.4) of the output systems Q_1, ...Q_m is equal to the order of S_k plus 1 (increasing complexity of the organization).

b) m is less than n, which implies that the new configuration is less ambiguous than the old one.

c) The dimension of each output system Q_1, ...Q_m is smaller than the dimension of S_k (The new organization is more global than the old one).

In other words, each stage of the overall process increases the "globality" of the organization, and the complexity and size of the input units for the next stage, decreasing at the same time the ambiguity of the configuration (i.e., the number of mutually exclusive cluster systems that can coexist at a given stage.)

The general structure of the process occurring at each stage
follows directly from the application of the principles we have
discussed. They imply a search over the attribute space[*] of the
clusters of each input cluster system (S_i) until a dense solution is
found. This, in turn, implies a mechanism for the verification of the
variation of the attribute within each new cluster, and a special
procedure for the detection of boundary points and the construction of
cluster boundaries (see for example, O'Callaghan (1974b)). The relative
degree of "globality" of the process is defined by the principle of
relative proximity (#1.3.3).

This general scheme is valid for every stage of the overall
process. Note, however, that the first stage always implies the applica-
tion of the principle of homogeneity, that is to say, the first step
of the organizational process is to construct units out of regions of
uniform stimulation.

The last stage should produce as its output a global, non-ambiguous
organization of the whole field, either as a single group structure, or
as a set of segregated-internally organized units (objects). When this
is not possible (i.e., when the ambiguities cannot be solved), it will
produce a set of different organizations that will alternate in time.

At this point, the articulation of the field in the form of a
specific figure/ground configuration takes place: This process may
be influenced by high level phenomena such as intention, learning, etc.,

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[*]This search must be done by specialized mechanisms, each one of them
looking only at its particular attribute (color, disparity, orientation,
etc.).

and is complex enough as to deserve a special study in its own right, but we recognize that it is largely determined by autonomous laws that are congruent with our basic principles of organization (See Koffka (1935)).

It is from this raw material – the field organized in terms of segregated units or "objects" – that the processes that lead to recognition, verbal labeling, frame activation (see Minsky (1974)), etc. depart, although in certain cases they may influence the control of lower processing stages.

Rather than a finished product, the model I have presented in this chapter should be considered as a general framework within which future sensible research can take place. For this reason, more attention has been given to the potential generality of its application than to specific computational details. It is clear that the perceptual organization along any particular dimension (attribute) presents its own problems. In the next chapter I will discuss in some detail how this theory can be applied to the study of a particular form of organization: The structured variation of the orientation of the elements in some textured fields.

2, PATTERNS OF ORGANIZED ORIENTATION IN TEXTURED FIELDS

2.0 What "Texture" is and Why it is Important.

The word "texture" is commonly used with a variety of meanings: Thus,
we speak of the texture of a literary work or of a musical composition,
as well as of the texture of an object or of a specific material. The
term is very difficult to define, partly because it denotes a mode of
perception, rather than a physical property: We perceive a pattern as tex-
ture when we do not analyze its precise microconfiguration, but rather, we
represent the pattern internally in some compressed form.

Several attempts have been made in order to define these compressed
descriptions, for instance, by means of overall statistical measures
(Muerle (1970), Julesz (1975), Hawkins (1970), Rosenfeld (1970)) or by
the magnitudes of the coefficients of the Fourier transform of the pattern
(Anderson (1969), Huang (1963), Jayaramamurthy (1973)). However, an
analysis of the artistic representations of texture (such as it is found in
the works of Rembrandt and Van Gogh[*]) shows how a pattern with a very
different "compressed description" (according to a "statistical" or
"frequency composition" criterion) may be a perceptually more faithful
representation of the real texture than a photograph of it. Since what is
conserved in these kinds of representations are only certain structural
properties of the pattern (see below), these facts lead us to conjecture
that a texture is represented internally as a set of congenial organiza-
tions of the attributes of the "texture elements".

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[*] See, for example, "Rembrandt, The Complete Etchings"' Boom, K. Ed.,
Abrams (1970) and "Van Gogh: Paintings, Drawings and Prints"
Phaidon (1974).

Therefore, we will consider a texture as a particular form of structure, capable of forming segregated units within a complex visual field. As such, its perception (or rather, its perceptual construction) should follow the general principles outlined in the last chapter.

The importance of its study may be better understood in the context of the general question: How is it that we perceive "objects" as segregated units? According to our model, an object is either a connected region of the visual field within which several attributes form a set of congenial structures, or (if the object is partially occluded by another) a set of similarly structured disjoint regions so that in a new hypothetical region formed by interpolation between them, the same attributes form a higher-order global structure.*

This set of congenial structures is formed by looking for "dense solutions" (see #1.3.4) along several attributes - such as color, depth, or "texture" - within the object, and the more attributes intervene, the stronger will be the perceived internal structure of the object, the more it will "stand out" from its surroundings.

On the other hand, the fact that we can easily segregate and recognize objects from black and white photographs, from which the color and depth "cues" are absent, implies that uniform texture within a region is sufficient to segregate it as a unit. Conversely, the art of camouflage (in animals or military vehicles) is a clear demonstration of how an otherwise familiar object can be given a texture similar to that of its surroundings in order to mask its perceptual segregation.

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*Of course, in the perception of real objects, frame activation (Minsky (1974)) and recognition also play an important role, specially in the "interpolation" between disjoint regions.

Let us return now to the question of the processes underlying

the perceptual construction of texture. According to our general model,

they should be structured as a sequence of clustering processes of

increasing complexity and globality. The first stage of the process

corresponds to the application of the principle of homogeneity, which

results in the segregation of the smallest discernible uniform units of

the field – which we shall call microelements – from the uniform back-

ground. These microelements may belong to a single "family", i.e.,

they may all be similar with respect to a set of attributes, or they

may form several "families".[*] In successive stages, several congenial

structures will result from the search for dense solutions along

several dimensions of the attribute space for each family of microelements.

Not all the attributes have the same importance from the viewpoint

of human visual perception. Let us return to the analysis of the artistic

representation of textures. It reveals that one of the most important

structures is that which results from the organization of the orientation

of the major axis of a family of microelements. We will call such a

structure a "Flow pattern", and its "canonical representation" may be a

pattern formed by straight line segments of uniform width and intensity

placed at random locations on a uniform background (see Fig (17)).

Formally, a flow pattern may be considered a representation of a

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[*]A "family" is the set of all the microelements of a pattern that have
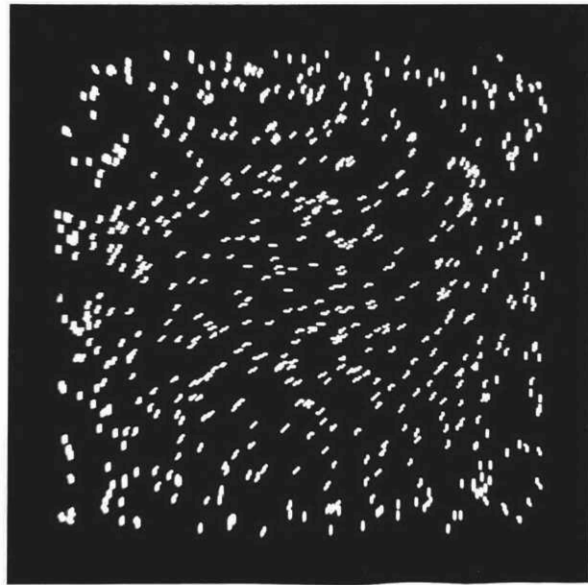approximately the same values for a given subset of their attributes.

Fig 17. Canonical Representation of a Flow Pattern:
NNB (RRDM(400)), $\theta=\frac{\pi}{260}\sqrt{x^2 + y^2}$, d=1,2,3,4)

continuous group of transformations - the transformation of the
group being a rotation acting on the slope of the line segments, with
a two-dimensional coordinate system defined in it (see #1.2.2-c,
#1.3.4-b and #A.2.2). As it is clear from the analysis of artistic
representations that do not use color, such as etchings or drawings,
this kind of structure alone is capable of producing intense cohesive
forces within a region, thus segregating it as a unit from its
surroundings, and in many cases (with the help of contextual infor-
mation) it completely characterizes the texture of an object. We will
now perform some experiments to find out some more about the processes
underlying its perceptual construction.

For doing this, I will follow the same strategy that has guided us
thus far:
I will construct ambiguous stimuli, so that we can observe in the
resulting percept the action of the internal organizing forces.

A pattern with these characteristics may be generated by
superimposing upon a set of randomly located similar dots a transformed
version of itself (see Glass (1971)). Figures (4a), (11), and (12) are
formed using as the overall transformation a uniform rotation, an
expansion and a bilinear transformation, respectively.

However, it will be more useful for our discussion to define
a more general generating algorithm:

2.0.1 Let R be a set of similar dots located at random positions on
a uniform background.

For each point $(X_i\ Y_i)^* \in R$, generate a "corresponding point" $(X_j,\ Y_j)$ using the functions:

$$X_j = X_i + d \cos \theta$$
$$Y_j = Y_i + d \sin \theta$$

d and θ are computed using two independent functions

$$d = F_d\ (X_i,\ Y_i)$$
$$\theta = F_\theta\ (X_i,\ Y_i)$$

The algorithm used by Glass may be considered as a particular case; for example, a uniform expansion will be generated by the functions:

$$d = K\sqrt{X_i^2 + Y_i^2}$$
$$\theta = \text{atan}\ \frac{Y_i}{X_i}$$

where K is the expansion factor.

The general experimental scheme will be the following: We will use several pairs of generating functions (F_d, F_θ), and in each case we will ask whether we perceive a global integrated pattern (as in Figs (4a), (11) and (12)) or not (see Fig 18).

The results of these experiments, and the conclusions that can be drawn from them now follow:

2.1 Orientation and Length of the Microelements.

As a working hypothesis, let us suppose that pairs of neighbouring dots are clustered together in the second stage of the overall

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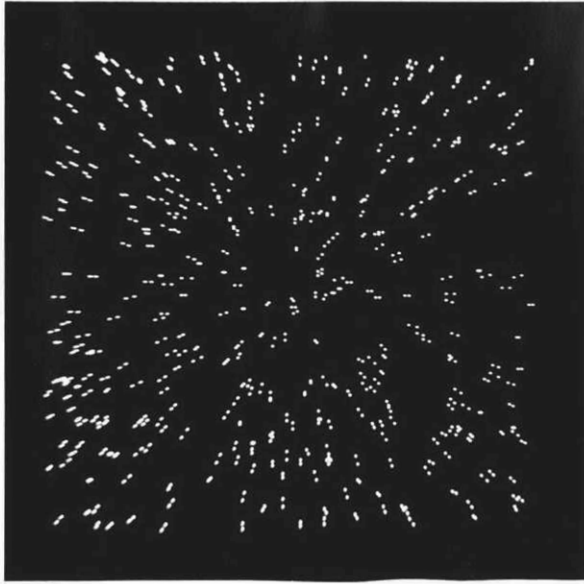*Selecting as origin of coordinates the center of the pattern.

Fig 18. Random Distance and Organized Angle:
NNB (RRDM(360), θ=atan(y/x), d=1+3·rand)
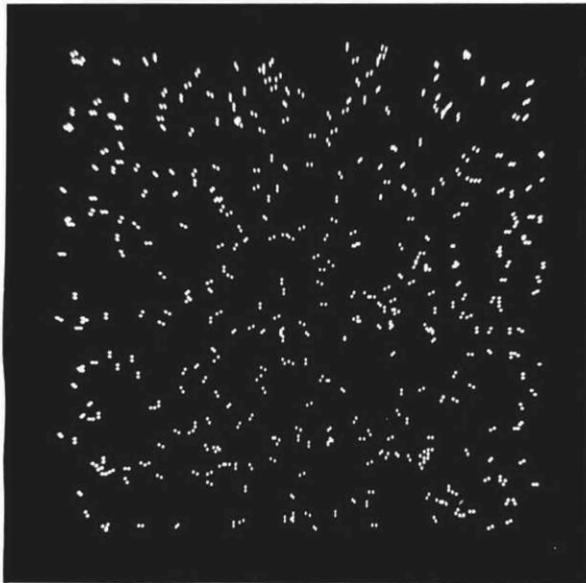

Fig 19. Constant Distance and Random Angle:
NNB (RRDM(360), θ=2π·rand, d=2)

process* to form a set of alternative systems of LINE** clusters
(see #1.3.5). If this is the case, subsequent clustering must take
place along the two attributes of such LINEs, namely length_ (distance
between the dots of the pair), and orientation or slope.

To study these processes, I generated patterns in which each of
these attributes is varied independently. The results of these
experiments indicate the following:

2.1.1 Length

The distance "d" between the dots of each pair must be within
certain limits (dmin $\leq$ d $\leq$ dmax) if the pattern is to be perceived.
The lower limit corresponds to the distance at which the two dots fuse
into one, while dmax is fixed by the properties of the dots (size,
contrast, etc. See #1.3.3). However, there is no restriction in the
form of F_d, and it can even be a random function, as in the case of
Fig (18), where an expansion pattern is strongly perceived. On the
other hand, the organized variation of distance alone does not produce
any global structure percept, as is evident in Figures (19) and (20).

2.1.2 Orientation

As the definition of "flow pattern" implies, it will be perceived

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*The dots themselves, as separate units were formed in the first stage.

**In a practical implementation of the model in the form of a symbolic
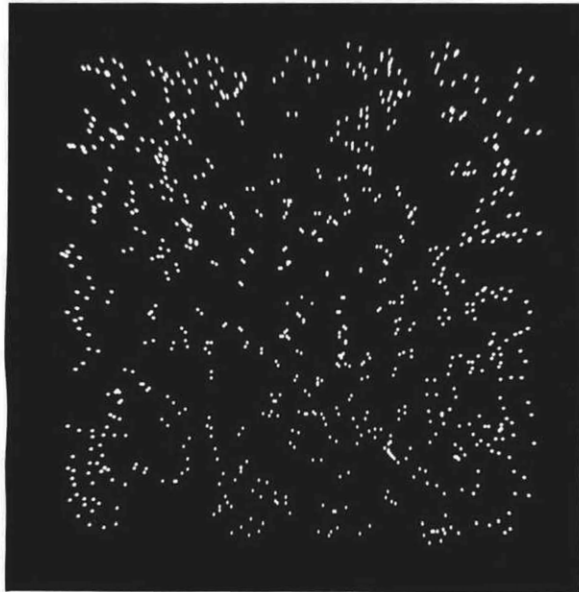process (see #4.1), the symbol "LINE" would be the "name" of such clusters.

Fig 20: Linear Variation of Distance and Random Angle:
NNB (RRDM(360)), $\theta = 2\pi$ rand, $d = .05\sqrt{x^2 + y^2}$

only if the variation of the orientation of the LINEs represents a

local continuous group. In #1.3.4-b we showed that this condition

is equivalent to the requirement that the absolute value of the

difference between the orientations of two neighbouring LINEs is smaller

than a certain threshold E_2. This number represents the maximum

rate of spatial variation of the orientation that can be integrated

perceptually, and its actual value must be determined experimentally.

Fig (21) shows a pattern in which the actual variation is too fast, and

thus, the visual system cannot integrate it as a global pattern.

2.2 Noise Organization:

If we add to a flow pattern (formed by a set of correlated pairs

of dots randomly placed) a new set of random dots, (uncorrelated)

the perceptual organization will not be destroyed, even if we add as

many uncorrelated dots as correlated pairs. This is the case of Fig (22).

This shows how a flow pattern is a "dense" solution, in the sense that,

even if the LINE clusters are located at discrete positions, the whole

field is organized in a continuous form, and this organization is

communicated to the noise.

2.3 Trajectories:

When we observe a flow pattern such as Fig (23) we perceive a

number of "trajectories" or clusters formed apparently by the casual

alignment of pairs of correlated dots. We might think that the

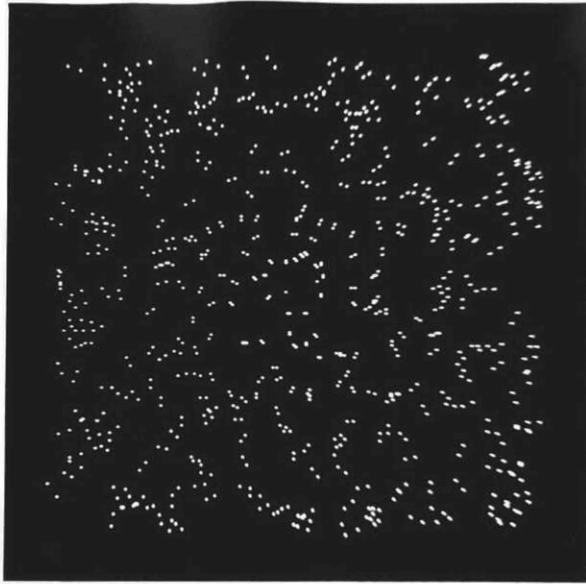perception of a flow pattern is based on the perception of these

-62-

Fig 21. The angle varies too fast:
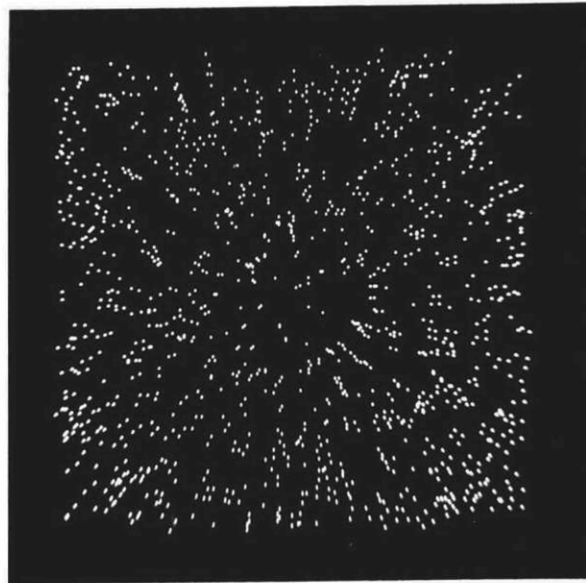NNB(RRDM(400), $\theta = \frac{8\pi}{260}\sqrt{x^2+y^2}$, $d=.05\sqrt{x^2+y^2}$)

Fig 22. A Flow Pattern can organize noise:
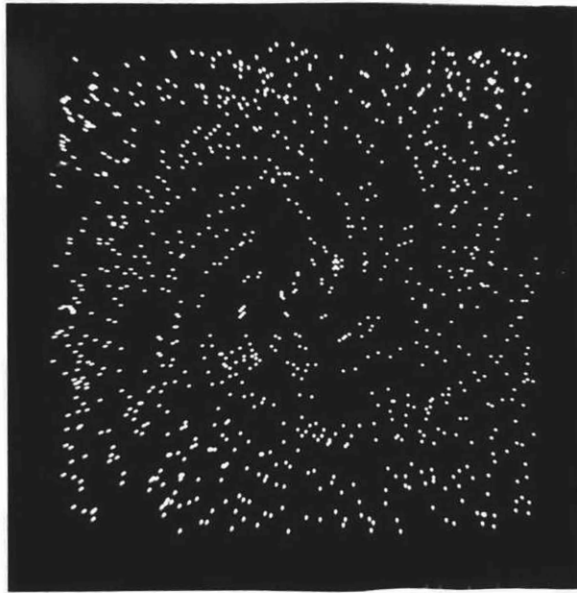CFLW (RRDM,(360), Kexp=1.05) $\oplus$ RRDM$_2$(360)

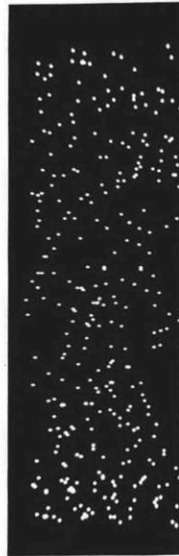Fig 23.   Spiral Flow
CFLW(RRDM(400), θ=5°,kexp=1.05)

Fig 24.

trajectories. However, this is not so: In fact, the trajectories are induced by the flow pattern, that is to say, we perceive the trajectories as a consequence of our perception of the whole structure. This can be demonstrated by the following experiments:

a) Observe Fig (24): we cannot detect any system of organized trajectories. However, Fig (24) is in the right-most strip of Fig (23), and when we see it in the context of the whole pattern, the induced trajectories become evident.

b) Fig (25) was generated in such a way as to prevent the accidental alignment of pairs of dots (see Appendix 1, A.1.1.2);); therefore no radial trajectories are present. Nevertheless, the expansion pattern is as clear as in Fig (11) in which the trajectories do appear. However, if we add noise to Fig (25) we obtain the pattern represented by Fig (26) in which not only the expansion pattern is apparent, but some radial trajectories now become visible.

2.4 Inter-dot Spacings:

An analogous argument can be made if we reverse the figure/ground configuration of these patterns: If we observe Fig (23) focusing our attention, not in the dots or in the lines connecting them, but in the shapes of the white "blobs" that appear between them, i.e., in the inter-dot spacings, we see that, although they are irregular, the majority of them have an elongated shape, and the direction of their major axis follows that of the overall spiral pattern. The following experiment shows, however, that the shape of these blobs is induced by the flow pattern and not viceversa:
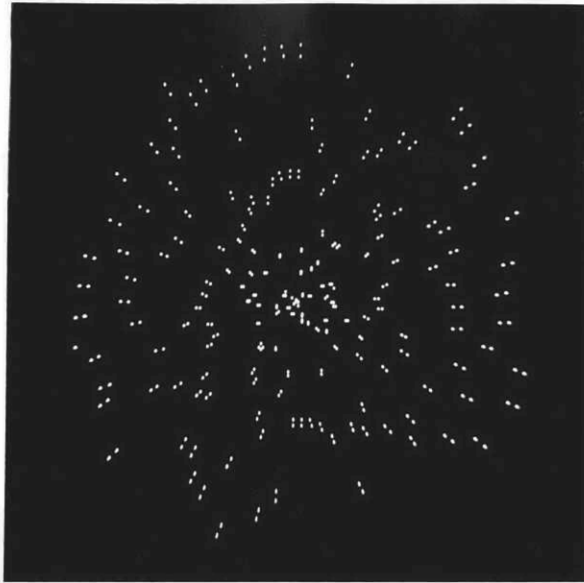
Fig 25. An Expansion Pattern is perceived even if
no radial trajectories are present:
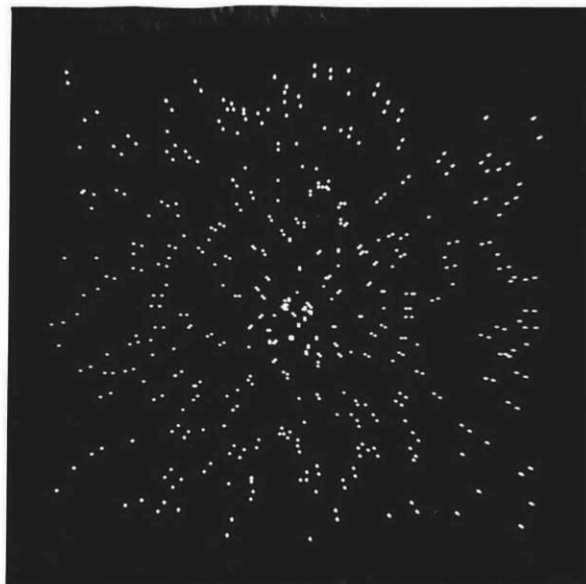CFLW(SPGEN(4,4(360 points)), kexp=1.05)

Fig 26. When noise is added to the pattern of
Fig 25, radial trajectories appear

Consider Fig (27-a). Here, the shapes of the inter-dot spacings do not follow any general rule. However, if we add to each dot of the configuration a second dot along a specific direction, a set of elongated blobs with parallel major axis is formed (Fig (27-b)). The direction of the formed blobs changes as we change the direction along which the correlated dots are added to the same initial configuration, as can be observed in Fig (27-c). On the other hand, if the correlated dots are not ordered along any specific direction, the orientation of the major axis of the blobs will not be organized either.

2.5 Dense Solutions:

If the perception of flow patterns is not based on the formation of accidental trajectories nor on the shapes of the inter-dot spacings, we must conclude that our hypothesis is true, namely that the flow percept is based on the formation of small LINE molecules defined by pairs of neighbouring points, and on the construction of a global relation between the orientation of such lines. However, a sizeable problem remains: If we analyze carefully a pattern such as Fig (23), we find that in the neighborhood of any given point there are several points, and so, several possible line molecules with different orientations that might be formed; thus the global organization should be perceptually ambiguous (which it is not). It can be easily verified that a simplistic criterion such as the selection of the nearest neighbor will fail, especially as we move away from the center of the pattern.
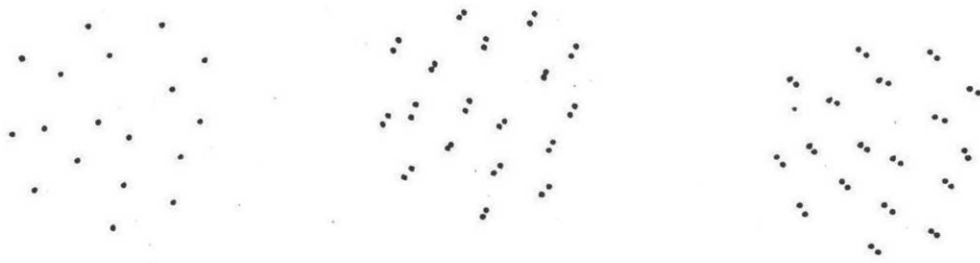
(a)   (b)   (c)

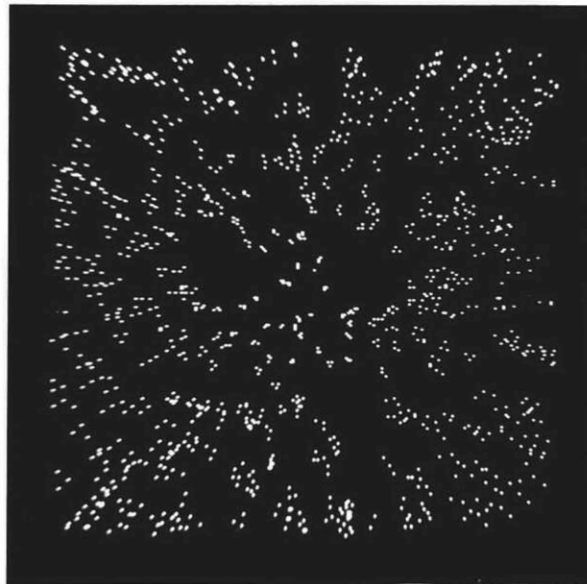Fig 27.   A Flow Pattern induces the shape
          of the inter-dot spacings

Fig 28.   The system looks for dense solutions:
          CFLW (RRDM(360), Kexp=1.05) ⊕ NNB(RRDM(360)
          d=.05$\sqrt{x^2 + y^2}$, θ=2πrand)

Moreover, we can construct a pattern deliberately ambiguous in the following way: For each point (X_i, Y_i) in the original set we generate a corresponding point at a distance d_i and angle θ_i, where θ_i is completed as a function $F_\theta (X_i, Y_i)$ corresponding to a given global organization (e.g. an expansion pattern). Then, we generate a second corresponding point at the <u>same</u> distance d_i, but with a random angle θ_i' .

In other words, for each point (X_i, Y_i) we generate two corresponding points: (X_i', Y_i') at :

$$X_i' = X_i + d_i \cos F_\theta (X_i, Y_i)$$

$$Y_i' = Y_i + d_i \sin F_\theta (X_i, Y_i),$$

and (X_i'', Y_i'') at:

$$X_i'' = X_i + d_i \cos (2\pi \cdot \text{rand})$$

$$Y_i'' = Y_i + d_i \sin (2\pi \cdot \text{rand}),$$

where rand is a pseudo-random number uniformly distributed between 0 and 1. The pattern of Fig (28) is formed in this way. We see that, in spite of all our efforts, the brain is not confused, and we still perceive a clear expansion pattern.[*]

It is clear, then, that the ambiguities at a local level are solved by means of the application of a more global criterion, namely a search

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[*] The perception of this pattern represents a remarkable feat of the system: it is comparable to separating a signal from the noise in which it is immersed, when we do not know a priori what the signal is, and when the ratio of "signal dots" to "noise dots" is one.

for dense solutions (see #1.3.4)

2.6   Flow Patterns in the 3-space.

Another experiment which supports the hypothesis that the perception of a flow pattern is based on the formation of LINE clusters at an earlier stage is the following:

It is possible to generate a stereo pair in which both the original and the transformed sets of dots that form the flow pattern are organized independently in depth. In this case it is found that the global structure is perceived only if both corresponding dots of each pair are at approximately the same depth. This was proved by constructing a pattern in which it was possible to vary continuously the relative depth of the original and transformed sets of dots (see Appendix 1, #A.1.1-d) Figures (29a) and (29b) show the patterns that are formed at the extreme situations, i.e., when the corresponding dots lie in the same and in two very different depth planes. When viewed through a stereoscope, it is clear that Fig (29a) forms an expansion pattern, while in Fig (29b) no organization is apparent.

2.7   The Perception of Flow Patterns in the Context of Our General Model.

The results of the experiments discussed above, are consistent with our model for the perceptual construction of organizations. According to it, the perception of a flow pattern is a sequential process consisting of the following stages:
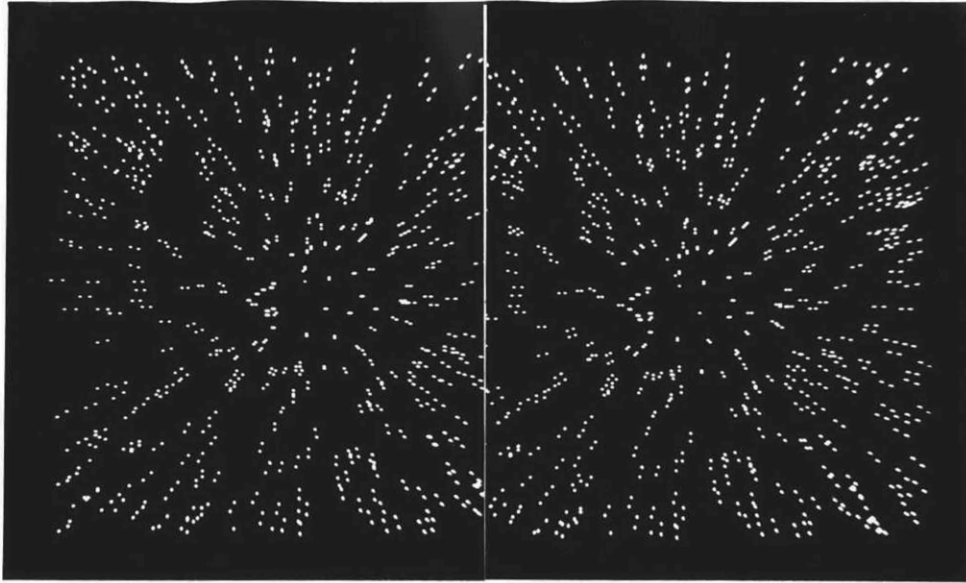
Fig (29a) A 3-D Expansion Pattern
           right: CFLW(RRDM(400), Kexp=1.05)
           left: SHFR1(CFLW(RRDM(400), Kexp=1.05),
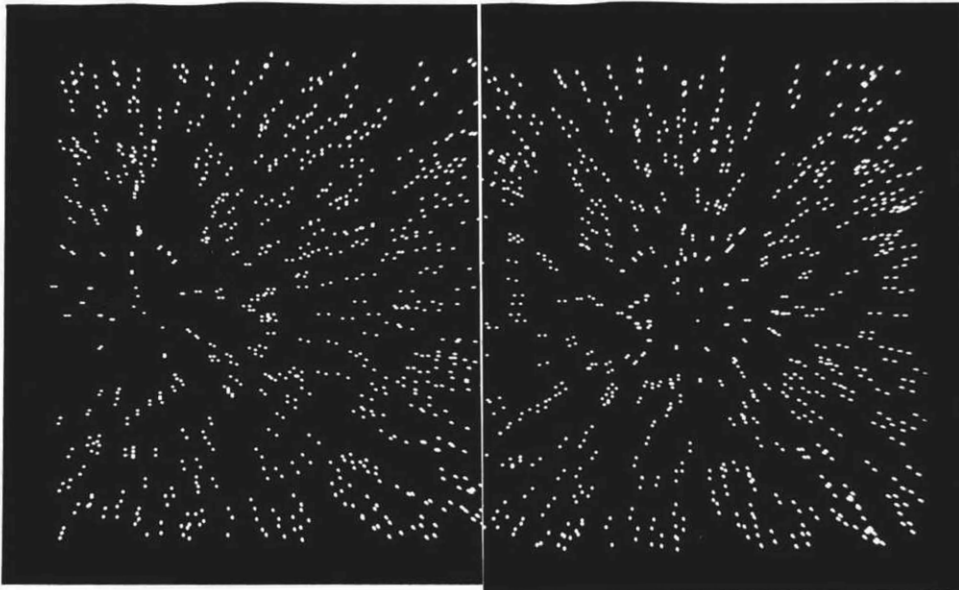   SHMAX=4, DSH=0)

Fig (29b) When the 2 points of each pair are in different
           depth planes, no global pattern is perceived.
           right: CFLW(RRDM(400), Kexp=1.05)
           left: SHFR1(CFLW(RRDM(400), Kexp=1.05),
                   SHMAX=4, DSH=4)

1: Application of the principle of homogeneity. The dots are formed

as segregated units.

2: A set of mutually exclusive systems of LINE clusters is formed,

each cluster consisting of a pair of proximal dots.

3: From the set formed in (2), the system of LINEs that form a

dense solution is selected. The output of the process is a

single unambiguous organization.

Of course, in a practical implementation of this model, it is not
necessary to form all the possible systems of LINEs, prior to the
selection of the dense solution. In practice, these two processes
could be merged into one. which would form the final configuration in
one pass, solving the local ambiguities by looking at the slopes of
previously formed clusters in a relatively small neighborhood, and
clustering together the pair of dots that form the LINE such that the
sum of the absolute value of the differences between its direction and
that of its neighbors is minimized.

2.8 Rival Organizations:

What will happen, however, if several mutually exclusive dense
solutions exist?

The generation of a second corresponding point in the neighbor-
hood of each point (X_i, Y_i) in the original set, at the same distance as
the first one, but using a different function $F_\theta^{(2)}(X_i, Y_i)$, gives rise
to three rival organizations: For each point (X_i, Y_i) we generate:

$$(X_1, Y_1) \text{ at } X_1 = X_i + d_i \cos (F_\theta^{(1)}(X_i, Y_i)$$
$$Y_1 = Y_i + d_i \sin (F_\theta^{(1)}(X_i, Y_i)$$

$$\text{and } (X_2, Y_2) \text{ at } X_2 = X_i + d_i \cos (F_\theta^{(2)}(X_i, Y_i))$$
$$Y_2 = Y_i + d_i \sin (F_\theta^{(2)}(X_i, Y_i))$$

and so, we are implicitly generating a third relation between (X_1, Y_1) and (X_2, Y_2). Figures (30a, b and c) were generated in this way. As we can observe, in general the effect is to destroy or weaken all the organizations involved, but in some cases, one organization dominates over the others; the rules for this dominance depend upon the relative strength of each of the alternative organizations which in turn depends on such factors as their global symmetry, closure, the relative rate of spatial variation of the slope, etc., in a complex way. Thus, a pattern with a highly symmetric, closed, global organization, such as an expansion pattern, will dominate over structures that are not representations of complete, continous groups (Fig 30b). When all the organizations involved have equal strength, the most likely result is mutual cancellation (Figs 30a, and 30c).

2.9 Conclusion:

In this chapter, we have shown experimentally the fact that the orientation of the major axis of elongated microelements can form global structures if its spatial variation follows the transformation of a (global or local) continuous group. We have shown how this global organization can induce the perception of trajectories and define the
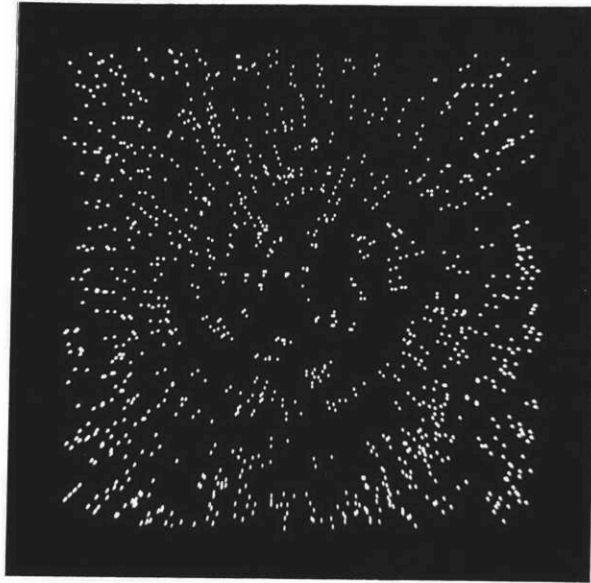
Fig (30a)   Rival Structures: Expansion and Rotation:
RRDM(400) ⊕ CFLW(RRDM(400), θ =5; Kexp=1)⊕
CFLW(RRDM(400), θ=0, Kexp=1.05)
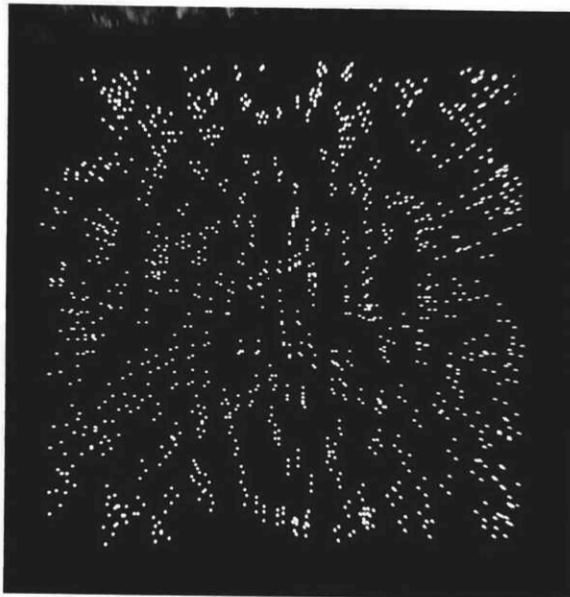

Fig (30b)   Expansion and Bilinear Transformation:
RRDM(400) ⊕ CFLW(RRDM(400), θ =0, Kexp=1.05)
⊕OFCT (RRDM(400), X0=0, Y0=.035)

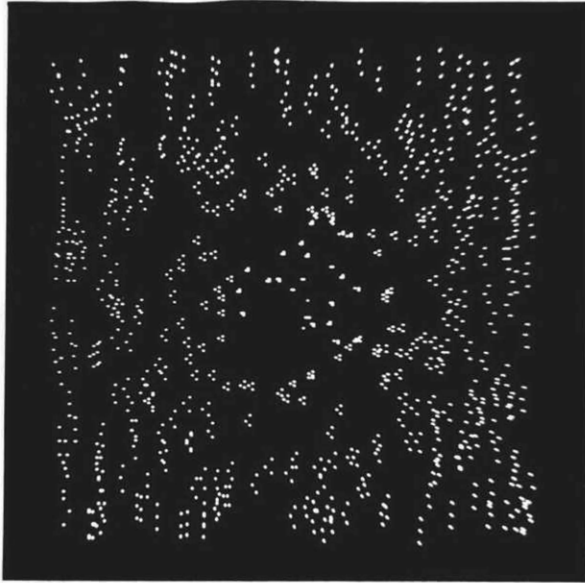Fig (30c) Two Rival Structures of Equal Strength
RRDM(400) + NNB(RRDM(400), $\theta=(\pi/260)\sqrt{x^2+y^2}$,
d=.05 x^2+y^2
+ NNB(RRDM(400), $\theta=(-\pi/260)\sqrt{x^2+y^2}$,
d=.05$\sqrt{x^2+y^2}$)

shapes of the inter-element spacings in the pattern, and how the underlying internal forces can organize noise and solve local ambiguities. Finally, we have shown how these experimental findings are consistent with our general theoretical framework, and can be explained by our general model for the perceptual construction of organizations.

It is interesting to note that the same model can be applied directly to the perception of "texture gradients", i.e., structures formed by the variation of the size or density attributes of a family of microelements, which constitute very important cues for the perception of depth and relative size, and are also preserved in the artistic representation of textures. It would be desirable to perform in this connection, some psychophysical experiments in the same style as those described in this chapter.

In the next chapter, we will revisit the flow patterns in the context of the perception of symmetry, and will show how they can function as units (internally structured clusters) with their particular form of organization (expansion, circular, spiral, etc.) predicated of them as an attribute, and thus serve as building blocks for the construction of higher order organizations.

3. PERCEPTION OF SYMMETRY:

3.0 Introduction:

In the last chapter, we discussed the construction of structures based on the smooth and continuous variation of one attribute, and showed how they can be adequately represented by a dense enough set of "samples" located at random positions. We conjectured that any texture can be represented by a small number of congenial structures of this kind.

However, the internal driving force towards the formation of group structures (see #1.21) is not confined to this case; in fact, the most significant and fascinating structures are those that arise from the representation of discrete groups (see Figs (31) and (32)). Such representations ("symmetry stimuli") can be built by applying to an arbitrary pattern ("theme" or fundamental region) a discrete group of transformations. An excellent collection of examples of symmetric patterns occurring in art and nature can be found in Weyl (1952). In contrast with the continuous case, these structures are characterized by the fact that the attributes which undergo the transformations (usually the position and orientation of the elements) take only a definite set of values. From this fact it follows that we can assign to each element of the pattern a specific set of "corresponding elements".

One consequence of this fact is that these organizations are more "global" than those we have discussed so far, in the sense that in order to construct them perceptually, one needs to consider simultaneously
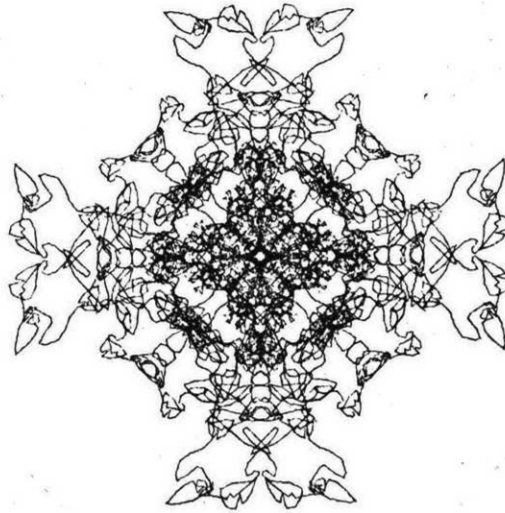
Fig 31. The Transformations of a Dihedral
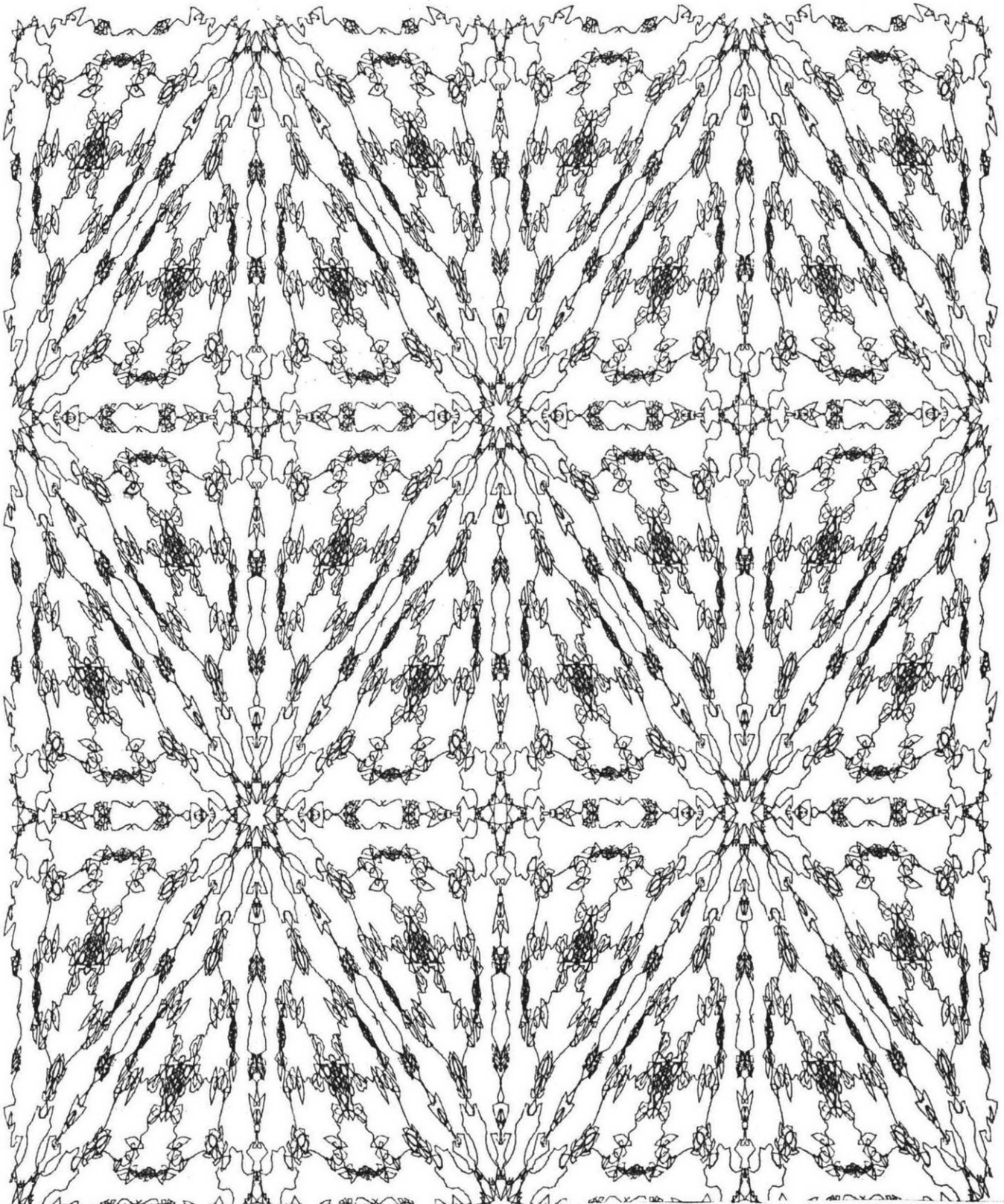Group Acting on a Random Walk

Fig 32. A Group Formed by Translations and Barycentric
Transformations acting on a Random Walk

a set of elements which might be distributed over a relatively

large region.

In spite of these differences, our general theoretical framework

can still be applied, and a plausible process for the perception of

symmetry can be postulated using the constructs of our model.

3.1   The Internal Tendency Towards Symmetry:

Following our experimental strategy, we will generate stimuli

that induce weak external forces, and learn from the resulting percept

the direction that the internal organizing forces follow:

a)   As a first case, let us recall the analysis of Fig (7) (#1.1):

The percepts resulting from the observation of this pattern show clearly

how under ambiguous conditions, the system always chooses to construct

symmetric configurations over all possible patterns.

b)   As a second example, consider Figure (33). When we observe it, we

certainly get a strong impression of bilateral symmetry; however, a closer

look will reveal that it is absolutely non-symmetric in a strict sense.

A similar phenomenon occurs in Fig (34): Fig (34a) is a

representation of the dihedral group D_6. In it, each corresponding point

is mapped into the next by a rotation of 60°; Fig (34b) appears to have

the same configuration in spite of the fact that in this case, the angles

of rotation have a deviation of 10% from the "ideal" case.

In this case, as well as in Fig (33), the internal forces produce

a "dislocation" of the actual pattern in order to form a regular

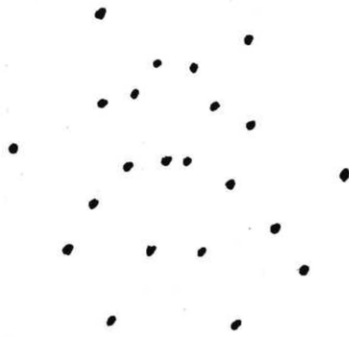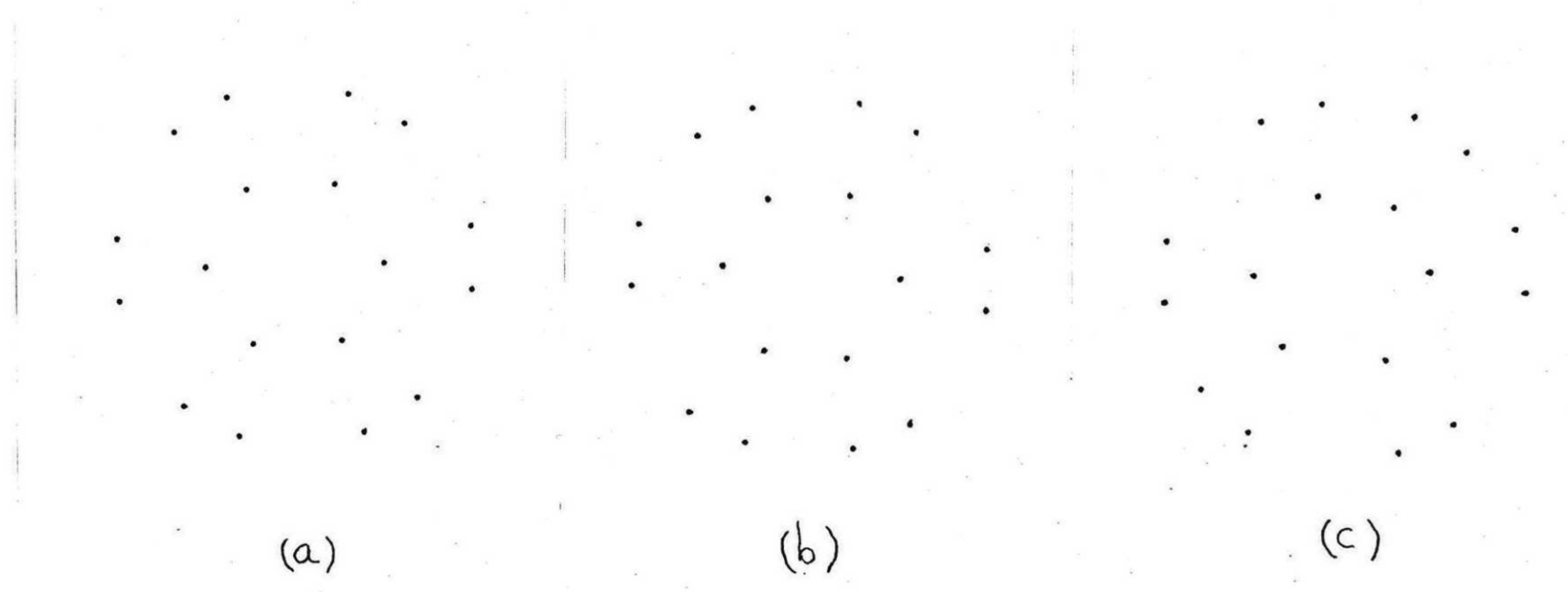(i.e., group-structured) configuration. For larger deviations (Fig 34c)

Fig 33.

(a)   (b)   (c)

Fig 34. 12-Fold central symmetry: a) Ideal ease; b) 10% perturbation in the transformations;
         c) 20% perturbation.

the pattern is generally no longer perceived as regular. However,
the internal effort towards compensation of the deviations is
experienced as a tension or disequilibrium: The pattern is
aesthetically displeasing.

3.2 Basic Principles of Symmetric Organization

The basic principles that the perceptual construction of
symmetries follows must be deduced from psychophysical experimentation.
Some of them are the following:

3.2.1 Predominance of vertical symmetric axis:

The internal processes responsible for the construction of
bilateral symmetries show an asymmetry in their operation: The
construction of structures with vertical symmetry axes dominates over
all other possible orientations. This fact can be observed in situations
in which the operating forces are so weak that we are near the threshold
between perceiving and not-perceiving the symmetry. These conditions
can be obtained in several ways:

a) The organizing forces are weak:

An interesting set of experiments performed by Paraskevoupulos (1968)
with small children showed that the internal organizing forces that
construct symmetric organizations evolve with chronological age. So,
very small children (of age less than 5) do not seem to distinguish
between a random and a symmetric pattern, while older ones (age 8) do.

At intermediate ages, however, when the internal forces are weak, bilateral symmetry about a vertical axis is perceived, while about a horizontal axis, it is not.

b) The external forces are ambiguous:

Recall once more the analysis of Fig (7) (#1.1.2-b4). It is clear that the vertically symmetric configurations dominate, even though the external forces (in the sense of physical proximity between dots) favor equally the configurations with axes at 30° and 60°.

c) The external forces are weak:

A strongly distorted symmetry can be constructed by using a group of barycentric transformations instead of rotations and reflexions (see Smith (1971)). In this case, the symmetry percept is just "above threshold", and results only if the symmetry axis is precisely vertical, disappearing even with a small tilt of the axis. This may be appreciated by observing Fig (35): The symmetric percept results only if we view it along one of the diagonals of the rectangle defined by the figure, tilting it at the same time about the Z axis.

3.2.2 Similarity of the correspondency elements

In chapter 1 (#1.3.4) we formulated the principle of the search for dense solutions, which is the basic process underlying symmetry perception. For the construction of these solutions we required from each set of corresponding points to be "similar" in the sense that some subset of their attributes should remain invariant under the transformations of the group. However, not all the attributes have the same
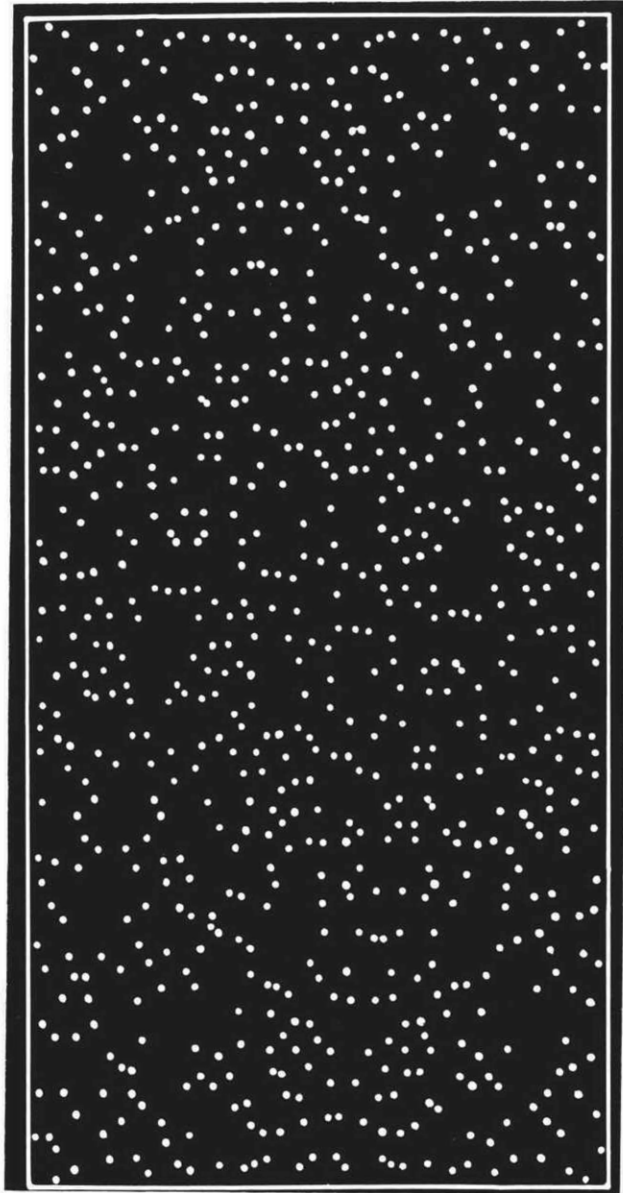
Fig 35.   A Group of Barycentric Transformations.
          This figure was generated originally by Dr. M.
          Cerrillo (See Smith (1970) for details on the
          generating procedure)

importance in this kind of perceptual similarity.

Thus, Fig (36a) still gives us a strong sense of symmetry even though the size of the corresponding elements is practically the only invariant; on the other hand, if size is not maintained, the symmetry percept is lost (Fig 36b).

It is interesting to note that the similarity between a set of elements is greatly enhanced if they are arranged in a group structure. This suggests that the perception of similarity is a consequence of the internal organization of the field, since it is always the invariant conserved under the transformations of a group.

3.2.3 Similarity of organizations of the fundamental region

If all the corresponding fields (the fundamental region and its transformations) of a pattern are strongly structured in a similar way, a symmetric configuration will be perceived, in spite of the asymmetry of the microstructure. Thus, Fig (37) appears bilaterally symmetric, and this perception is caused not only by the similarity of the flow structure at both sides of the field (spiral), but by the symmetric operation implicit in the change of the sense of rotation of the spiral flows.

The fact that the brain spontaneously tends to form the symmetric structures based on some high-order attribute of the organizations can also be verified by the following experiment:

If we superimpose upon a dynamic random pattern (such as the screen of a TV switched to an empty channel) a transparency of a pattern formed
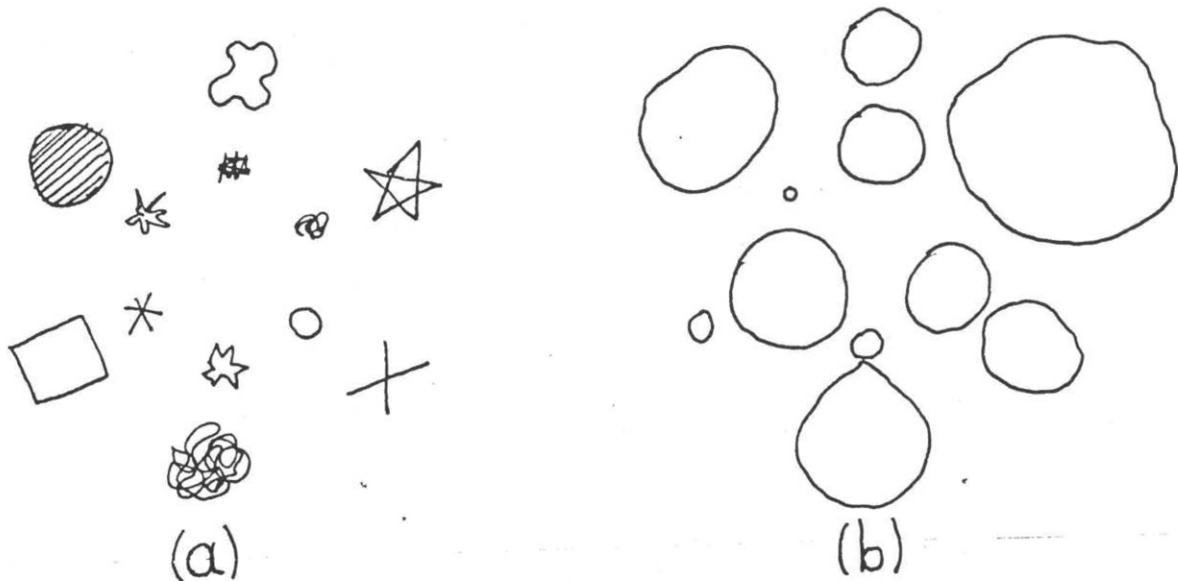
Fig 36.   In order to perceive a symmetry, the corresponding
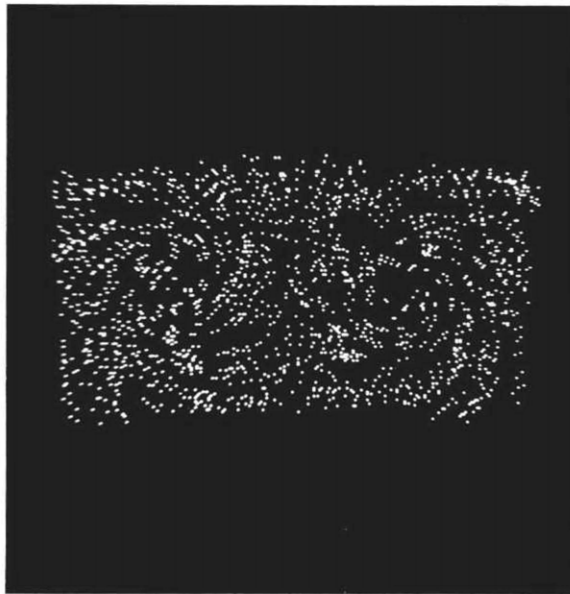elements must be similar in size.

Fig 37.   Similarity of organization of the corresponding fields
results in a symmetry percept, in spite of the asymmetry
of the micro-configuration

by equispaced radial lines (Fig 38), a complementary circular flow will

be generated internally, and perceived as an organized motion of the

noise (see McKay (1961)). The sense of rotation of this induced flow

will vary from subject to subject, and very often will alternate in time.

However, if we put two identical transparencies of radial lines, side by

side on the screen, the sense of rotation of the induced patterns will

always be such that one is the mirror image of the other, i.e., a group

structure based on the sense of rotation will be constructed.

This means that it should be possible to generate higher order "flow

symmetries" using only local rules for generating the complement of a

random pattern, the only condition being that equivalent points

(according to the symmetry) undergo similar transformations. For example

a 4-fold symmetric flow pattern will be generated by a function F_θ

(see #2.0.1) that fulfills the following condition:

$$F_\theta \ (X, \ Y) = F_\theta \ (-X, \ -Y) = -F_\theta \ (\ X,-Y) = -F_\theta \ (-X, \ Y) \text{ for all X, Y}$$

Figure (39) was generated in this way.

In order to construct higher order central symmetries, it is

convenient to tile the plane with equal triangles, and then use

barycentric coordinates to define equivalent points (see Smith (1970)).

For example, using the configuration of Fig (40a), the pattern of Fig.

(40b) was constructed using as F_θ a function of the barycentric coordinates

of each point (X, Y), with respect to the triangle where it lies.

Fig (41a and b) represents "strict" 4 and 6-fold symmetric patterns,

respectively, so that a comparison with Figs (39) and (40b) can be made.
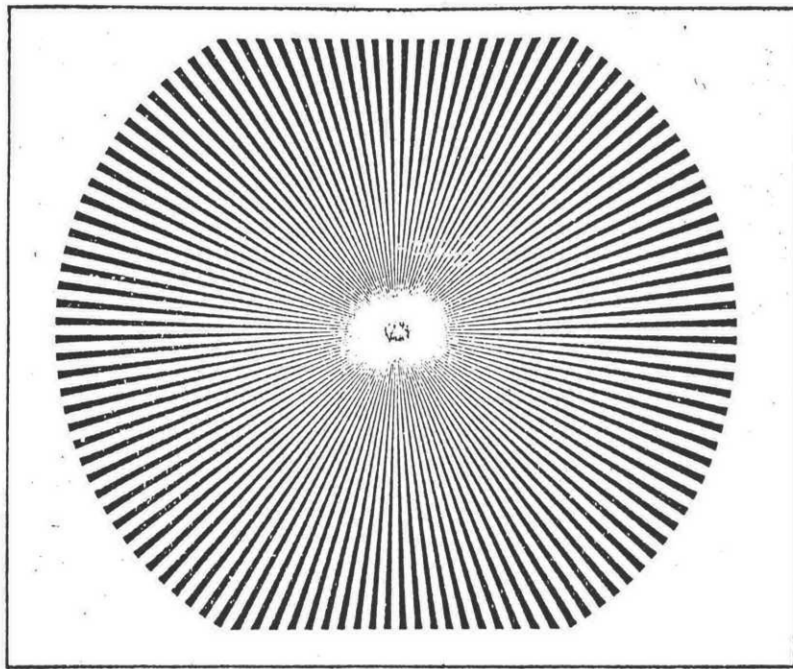
Fig 38. Equispaced Radial Lines
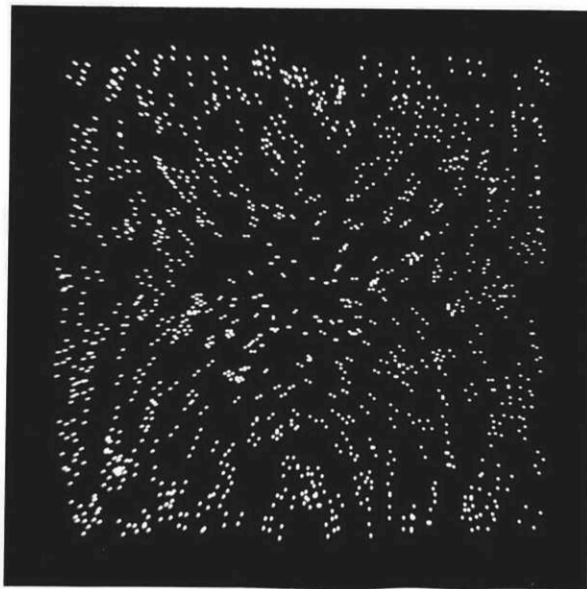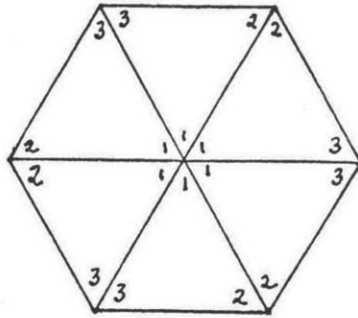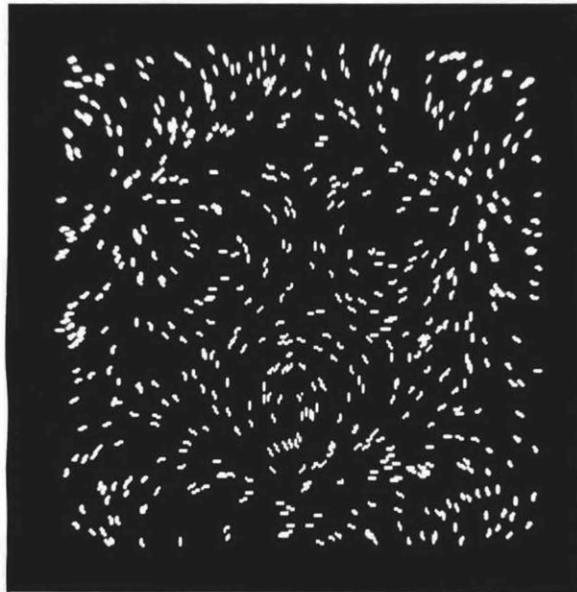

Fig 39. 4-Fold Symmetric Flow:
NNB (RRDM(400)), $\theta = \text{sign} \left(-\frac{\pi}{260}\sqrt{x^2 + y^2}, x \cdot y\right)$,
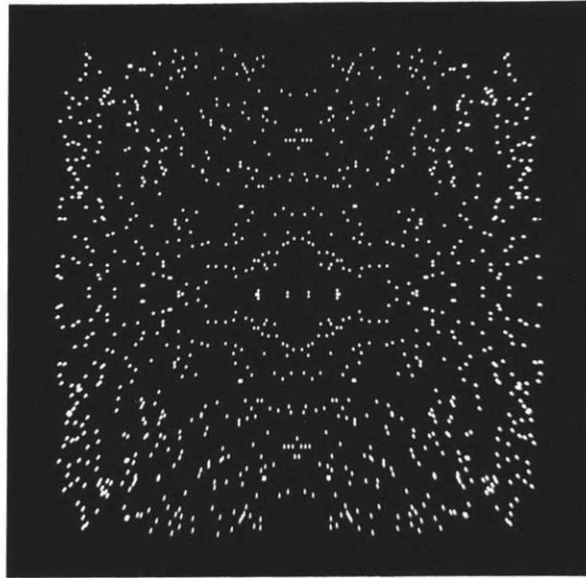$d = .05\sqrt{x^2 + y^2}$

(a)

b) NNB(RRDM(550), $\theta=8(1-B1) + 2.094$, d=1,2,3).

Fig 40.   The 6-fold Symmetric Flow (b) is generated
          using the grid (a)

a) CSYM(RRDM(400), 2)

b) CSYM(RRDM(400), 3)

Fig 41. 4-Fold and 6-Fold Kaleidoscopes

It is clear that these patterns will approach the "perfect symmetry" percept as the microconfiguration is less visible, and the flow organization stronger. This occurs as the diameter of the dots becomes smaller, and as the number of dots increases.

. Let us examine all these experimental facts in the context of our general model.

3.3 A model for the perception of symmetry:

The perception of symmetry can be explained in terms of our general model, (see #1.4) provided that an adequate interpretation of the principles of search for dense solutions and relative proximity (see #1.3.4a and #A.2.1) is made. This means that the symmetric structuring of a field occurs hierarchically at consecutive stages of a sequential process, corresponding to each successive level of organization (complexity of the clusters) an increasing effective distance from the center or the axis of symmetry. If the symmetric organization at one of these levels is not possible, the field is nevertheless structured, although the resulting organization is weaker than in the case of complete organization at all levels.[*] Thus in Fig (42) a structure is formed near the axis at the level of organization corresponding to the formation of dots (second stage in the overall process), but since there are no strong organizations in the left and right halves of the field, the symmetry percept is lost as we move

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[*] A similar idea is found in Winston (1970)
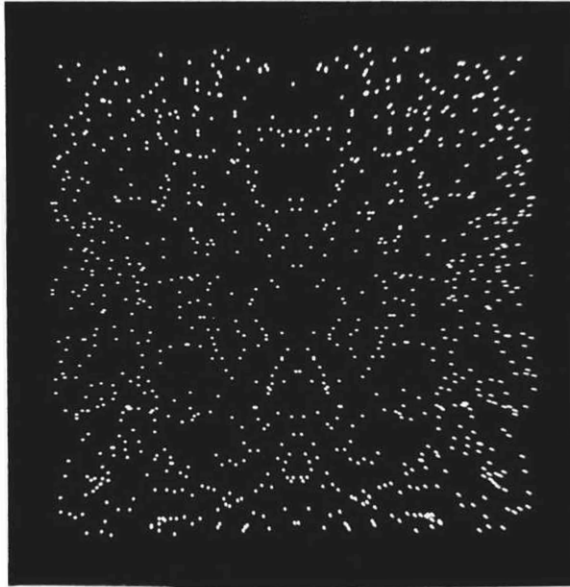
Fig 42. Bilateral Symmetry:
        CSYM(RRDM(400), 1)

away from the axis. On the other hand, in Figs (37, 39, and

40b), the symmetric structure is formed away from the axis at a

level of organization corresponding to a flow pattern (see Chapter

2) but the organization near the axis is missing, since the micro-

structure is asymmetric. Finally, Fig (13) is an example of a complete

organization of the field. A further consequence of this processing

structure is that there is a limit in the complexity of the organizations

we can perceive, which results from the principle of relative proximity;

a given structure (for example, a high-order central symmetry), can

"hold" its elements only up to a certain maximum distance from the center

(which depends on the size and internal organization of the elements),

and at larger distances, the structure will "break" into lower-order,

locally more stable substructures. This will happen at very short

distances from the center if the alternative locally-stable configurations

are also relatively structured. Such is the case of Figure 7, in

which the global central symmetry corresponding to the group D_{12}

(i.e., a 24 fold symmetry) which this pattern has (see #A.1 15b) is

impossible to perceive, since it breaks into locally stable 4 or

8 fold symmetric substructures.

3.3.1 An algorithm for the formation of bilaterally symmetric clusters:

According to our model, the construction of high order central symmetries (v,gr, Fig (10a)) is based on the position and orientation of bilaterally symmetric clusters which must be formed at a previous stage. The mechanism for the formation of these clusters is, thus, fundamental for the functioning of the system; it is very difficult to tell which is the actual process used by the brain, but it is possible to design a practical algorithm for a simulation of this process based on the following property: If we define the major axis of a cluster as the maximum distance that one can find between any pair of elements belonging to the cluster, then, if the cluster is bilaterally symmetric, its axis of symmetry either coincides with or is perpendicular to the major axis; if the cluster has two (equal) major axes, then the axis of symmetry either coincides with one of them, or with one of the bi-sectors of the two. This property allows a great reduction on the number of possible positions and inclinations through which a computer program would have to search in order to test if a given cluster is bilaterally symmetric.[*]

3.3.2 Congenial structures:

Figures (13, 37, 39 and 40b) should be considered congenial structures (#1.3.5) in the sense that a flow pattern is used as a segregated unit to build a higher order structure based on its attributes. However, other types of interaction are also possible.

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[*] see Fig (5), #1.0.3

For example, Fig (43) was constructed by adding noise (i.e. random
dots) to a square lattice of points until the structure is no longer
perceivable. If we superimpose it upon another version of itself
rotated 95° (Figure (44)), the flow patterns (systems of concentric
squares) formed by interference[*] appear, though they are difficult
to perceive. On the other hand, if we superimpose Fig (43) with its
mirror reflection, a bilateral symmetry is formed although it is also
very weak (Fig (45)). However, if we allow these two structures (the
bilateral symmetry and the flow pattern) to interact, by rotating the
mirror reflection slightly about a point located in the axis, a mutual
enhancement of the 2 congenial structures occurs as we can see in Fig 46.
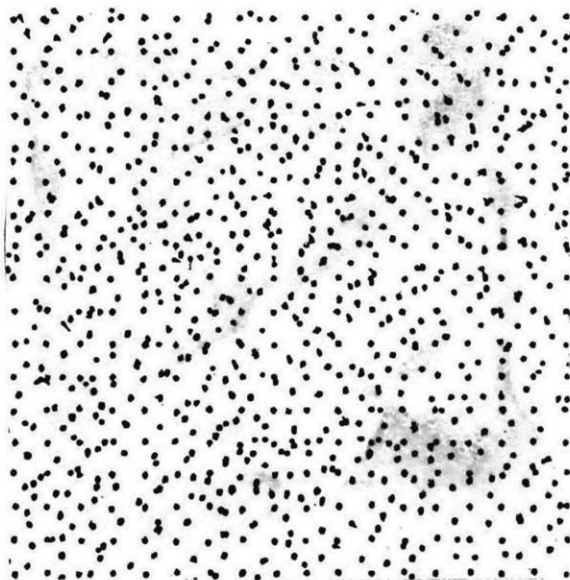
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[*]See Fig (5), #1.0.3

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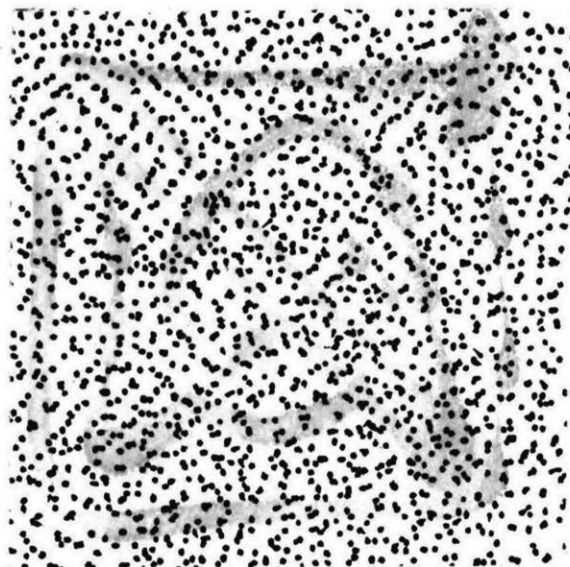Fig 43. Noise Added to a Square Lattice of Points

Fig 44. Fig (43) Superimposed upon another Version of
Itself Rotated 95°
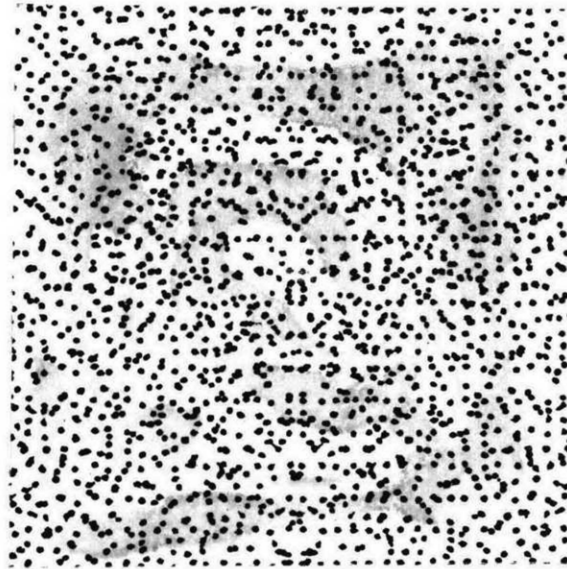
Fig 45.   Fig(43) Superimposed upon its Mirror Reflection
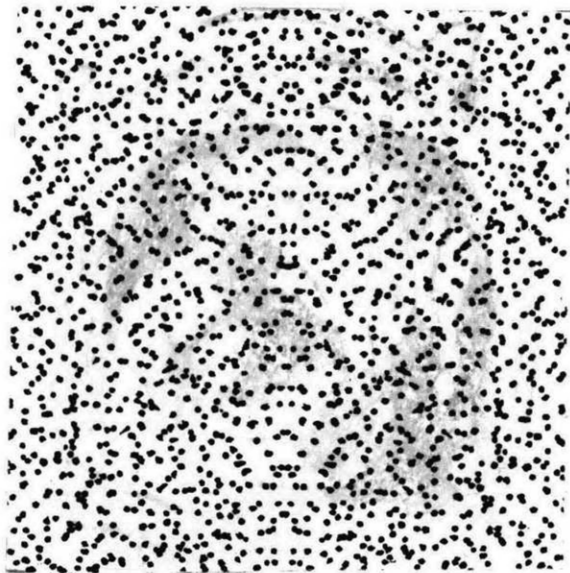
Fig 46.   Congenial Flow and Symmetry

4. CONCLUSIONS AND SUGGESTED AREAS FOR FURTHER RESEARCH

4.0 Conclusions:

Throughout this thesis I have presented experimental evidence to support the claim that "reality" - i.e., our subjective perceptual experience, is indeed our personal construction: it results from the organization of the visual input (the pattern of stimulation at the receptors). The dynamic behavior of these organizing processes appears to be directed by a tendency (driving force) towards the formation of Group structures (#1.1.1, #1.3.4). The way in which these processes act was presented as a mathematical model (#1.4) in which visual organization was viewed as a sequence of similarly structured processing stages, each resulting in an increased "globality" and complexity of the organization, and, at the same time, a decreased ambiguity of the configuration. Each stage of processing represents the application of the overall driving force of the system at a local level (i.e., the formation of Local Groups or dense solutions*), subject to the constraints that result from the principle of Relative Proximity (#1.3.3), which imposes both a lower limit to the number of elements per unit area that must be present if a local structure is to be formed, and an upper limit to its spatial extension.

Some examples were given of the way in which, by the introduction of the concept of congenial structures (#1.3.5) this general model can be

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* The principle of homogeneity (#1.3.2) may be considered as a particular case of this tendency.

applied to a wide variety of situations (#1.3.6). In particular,
chpaters 2 and 3 represent instances of the application of this
general theoretical and experimental framework to the analysis of
two concrete perceptual phenomena. In chapter 2, we studied in detail
the construction of one of the most important structures that
characterize a textured field: The patterns of organization of the
orientation of the microelements. In chapter 3, we studied the perception
of different types of symmetries, and postulated a model for the structure
of the processes on which it is based.

4.1 A Plausible Implementation of the Model

The ideas presented here are still far from being an accurate
model of the processes underlying perceptual organization, but I think
they point in the right direction. More precise, quantitative
formulations of its use should be done together with more experimen-
tation.* A plausible criterion to guide this experimentation and evaluate
the results, is the implementation of the theory is the form of a
computer model which, given a stimulus such as the ones we have used here
as its input, should produce at its output the same organization that the
human observer perceives.

It is clear that this implementation presents specific problems
for each particular application (perception of symmetry, texture, etc.)
However, a plausible general scheme is the use of symbolic processes,

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*For example, in #1.3.3 (the formulation of the principle of relative
proximity), I pointed out the need of determining experimentally the
value of the parameters and the form of the function to compute the
"attraction force" between two units.

the essence of which is to give a "name" to a set of data which is to
be considered a segregated unit. This naming permits the assignment
of a set of properties or attributes to the unit, as well as the future
reference to it by subsequent processes. In this sense, each stage of
the global operation of the system may be implemented as a symbolic
process which looks for dense solutions in the attribute space of a set
of input "primitives", and assigns a new name, and a new set of attributes
to each subset of these primitives that forms a "cluster" (#1.3.4). The
specific set of relevant attributes varies for each grouping process and
for each system of clusters, but in general, the position of the center
of gravity, the orientation of the major axis, the intensity, type of
symmetry present, and some global measure which characterizes its shape
should be included. An example of this style of implementation is found
in Marr (1975).

4.2 Possible Extensions of the Group Theoretical Approach to the
Study of Higher Level Processes

The patterns that I have used as stimuli in this work represent
the interaction of two opposite principles: randomness and order
(in the form of group structures). Aside from their usefulness as
stimuli for psychophysical experimentation, such patterns are
characterized by the great fascination they exert on the observer, and
by the fact that they are "meaningful" and significant. They
"make sense". In fact, group structures have always fascinated mankind,

as an analysis of the artistic representations of any culture will

reveal.[*] What is even more interesting is the fact that they have always

and everywhere been used as religious and magic figures, as symbols of

the most profound and incomprehensible mysteries of the human mind. Thus

we find representations of groups in the pyramids of Egypt, and in

the Gothic Cathedrals in Europe; in the structure of the 64 hexagrams of

the ancient Book of Changes (I Ching) regarded as the basic source of

Chinese philosophical thought, and in the Tibetan Mandalas on figures for

meditation; in the archetypical astrological figure and in the religious

sculptures from ancient Mexico.

If, following Jung's interpretation[**] we regard religious and magical

imagery as a symbolic representation of man's and woman's inner reality,

we must deduce from these facts that group structures should appear in

the organization of higher mental processes.

Also supporting this conjecture is the work of Piaget (1971, 1974)

on the development of intelligence. After long and careful observation

and experimentation with children, Piaget concluded that the mental

processes underlying intelligent behavior (not only logical and

mathematical capabilities, but also the ability to recognize the conser-

vation of weight, volume, length, etc. for example) are structured in

a group-like fashion.

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[*] See Weyl (1952), and also Budden (1972), who presents examples of
group structures present in musical compositions

[**] See Jung (1958)

I do not mean to say that a straightforward extension of the model presented in this thesis will explain these higher mental processes; I believe, however, that there is enough evidence as to justify the attempt of extending the use of Group theoretical constructs and of ad hoc ambiguous psychophysical stimuli to the study of these more complex cases.

APPENDIX 1

A.1 Description of the Main Generating Procedures for the Patterns used as Stimuli in this Thesis:

I developed an interactive software system for generating random dot patterns, organizing them in different ways, and displaying them, either on a CRT, or on an electrostatic plotter.

The system is structured in the form of a driving program and a collection of subroutines. Each dot is represented internally by a pair of numbers which correspond to its Cartesian coordinates with respect to the center of the pattern. The space allocated to them is structured in the form of 4 buffers of variable length which is specified by means of a set of pointers to the addresses of the initial locations of each buffer. This structure permits an efficient use of the available memory, and at the same time gives great flexibility to the system, since the set of dots stored in each buffer may be manipulated (i.e., transformed or displayed) independently.

The functions performed by the driving program are:

a) To update the values of the pointers each time a new set of points is generated or transformed.

b) To call the next routine specified by the user:

The calling sequence for each routine has no arguments, and so, the addition of new routines to the system is very easy (the points and pointers are all in common storage, and each routine asks the user to input the values it needs whenever it is called).

The routines that I have implemented thus far, and the functions they perform are as follows. In these descriptions, the symbol "rand" will always denote an internally generated-uniformly destributed pseudorandom number whose value is between 0 and 1; the variables whose names appear in capital letters are parameters whose value is specified by the user at execution time, and finally, if the routine applies a transformation, (x, y) will denote the coordinates of the original points, and (x', y'), the transformed coordinates.

The user can always specify the buffer from which the original points are to be taken, and the one in which the transformed points are to be stored.

A.1.1   RRDM

This routine generates a set of N "random" points within a square of size SIZ, i.e., it generates two independent sets of random numbers with values between -SIZ and +SIZ. The probability distribution of these numbers may be either uniform or triangular. The actual shape of the the distribution depends on the slope SL (see Fig (47)). This means that the points will be either uniformly distributed over the square, concentrated near the center, or concentrated near the corners.

A.1.2   CSYM

This routine applies to a set of points all the transformations of the dihedral group of order 2N (D_N), i.e., it generates a kaleidoscope with N axes of symmetry.
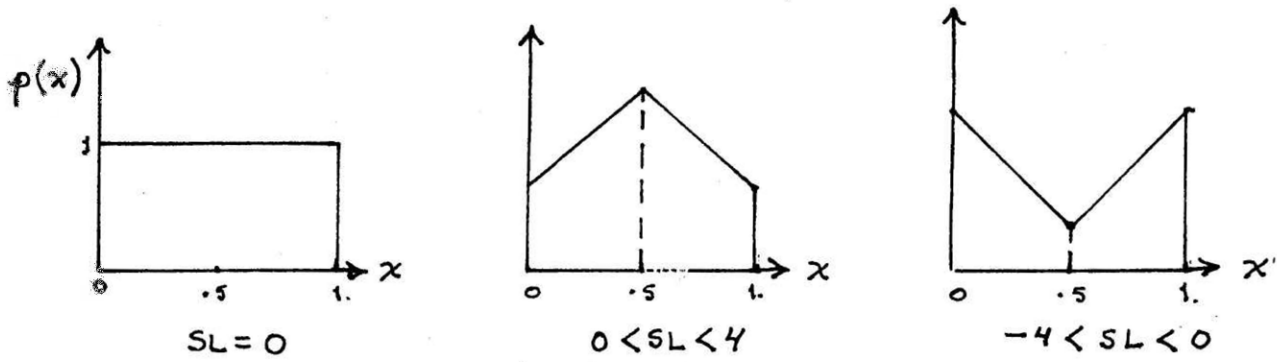
Fig 47.   Probability Distributions Used by the Routine RRDM
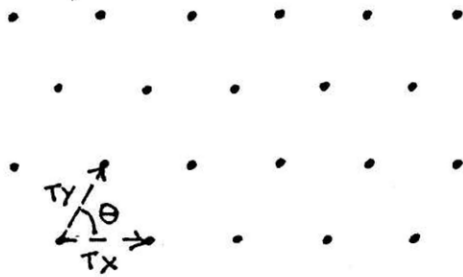

Fig 48.   Regular System of Points Generated by RTES
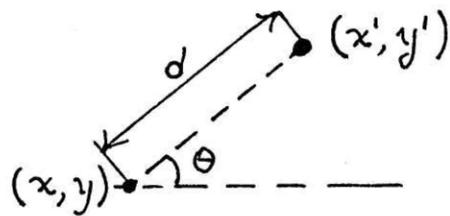
Fig 49.   Corresponding Point Generated by NNB

As it is well known, the transformations of this group are reflexions and rotations, and can be represented by the linear transformations:

$$x' = x(\cos\alpha\cos\varphi + \sin\alpha\sin\varphi) + y(-\cos\alpha\sin\varphi + \sin\alpha\cos\varphi)$$

$$y' = y(\sin\alpha\cos\varphi - \cos\alpha\sin\varphi) + y(-\sin\alpha\sin\varphi - \cos\alpha\cos\varphi)$$

where $\alpha = \dfrac{\pi}{N}$, and $\varphi = K\alpha$, $K = 1, 3, \ldots, 2N-1$,

and:

$$x' = x\cos\varphi - y\sin\varphi$$

$$y' = x\sin\varphi + y\cos\varphi, \quad \varphi = K\alpha, \quad K = 2, 4, \ldots, 2N$$

This routine, as well as CROT and CFLW (described below) has the possibility of adding a controlled amount of "noise", PN to each transformation. This means that for each point (x_i, y_i) belonging to the original set, a corresponding point $T_k(x_i, y_i)$, for $T_k \in D_N$, will be generated with a probability (1-PN), and with a probability PN, another random point, whose spatial variation is in the same range as that of (x_i, y_i) (see #A.1.1), will be generated instead.

A.1.3 CROT:

This routine applies to a set of points all the transformations of the cyclic group of order N (C_N), i.e., the group of rotations represented by the transformations:

$$x' = x\cos\frac{2\pi K}{N} - y\sin\frac{2\pi K}{N}$$

$$y' = x\sin\frac{2\pi K}{N} + y\cos\frac{2\pi K}{N}, \quad K = 1, 2, \ldots N$$

Adding an amount PN of noise to each transformation (see #A.1.2, above).

A.1.4 CFLW

This routine applies to a set of points, a transformation consisting

of a rotation, an expansion, or a combination of both, about an

arbitrary center. Formally, this transformation is represented as:

$x' = FX (x-XO) \cos (THETA) - FY (y-YO) \sin (THETA) + XO$

$y' = FX (x-XO) \sin (THETA) + FY (y-YO) \cos (THETA) + YO$

where FX is the expansion factor in the X direction,

and FY is the expansion factor in the Y direction.

THETA is the angle of rotation, and (XO, YO) are the coordinates

of the center of rotation and/or of expansion.

Note that if FX = FY = 1, and THETA = 0, the transformation reduces

to the identity transformation. The routine also allows the addition of

an amount PN of "noise" as the transformation is applied (see #A.1.2).

A.1.5 RTES

This routine generates a regular system of points within a square of

size SIZ, using as the generating operations two translations of magnitudes

TX and TY, with an angle THETA between them (see Fig (48)).

A.1.6 OFCT

This routine applies to a set of points the bilinear transformation:

$$z' = \frac{-(z - c \cdot ZO)}{-(z) \cdot \overline{ZO} - c} \cdot C; \qquad x' = real (z')$$

$$y' = imag (z')$$

where the complex variable $z = x + iy$;

$$\underline{z0} = \bar{X}0 + i \cdot Y0 \; ;$$
$$\overline{z0} = X0 - i \cdot Y0$$

(X0 and Y0 are real parameters supplied by the user), and

$\mathbf{C}$ = a real constant

A.1.7 BARC

This routine applies to a set of points a barycentric transformation (see Smith (1970)), which corresponds to mapping the points referred to an original triangle with vertices at $(0,0)$, $(1,0)$ and $(1,1)$ into a transformed triangle with vertices at (X1, Y1), (X2, Y2) and (X3, Y3)

Formally, the transformation is:

$$x' = x (X2-X1) + y (X3-X2) + X1$$

$$y' = x (Y2-Y1) + y (Y3-Y2) + Y1$$

The routines SHF2, SHFR, SHFS, and SHFR1 whose description follows are used to generate stereo pairs which, when viewed through a stereoscope will appear organized in different depth planes. They operate on the principle that a point whose coordinates are (X, Y) on the left component of the pair, and (X + SHF, Y) on the right component, will appear at a height:

$$h = K \cdot SHF$$

over the "reference plane" (plane of zero disparity) when viewed stereoscopically (see Julesz (1971)).

The constant K depends on the interocular distance of the subject and on the optical characteristics of the stereoscope, and its value must be determined experimentally.

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A.1.8   SHF2

This routine applies to a set of points a constant shift of
SHF units* to the right, i.e., it applies the transformation:

$x' = x + SHF$

$y' = y$

Its effect is that all the points of the set will appear on a
plane at a certain height over the reference plane.

A.1.9   SHFR

This routine applies to each point of a set a random shift whose
absolute value is less than SHMAX, i.e., it applies the transformation:

$x' = x + SHMAX (2 \cdot rand - 1)$

$y' = y$

The function of this routine, when applied to a set of random
points is to produce a stereo pattern of points distributed
randomly in space (i.e., with random x, y and z coordinates). The
visual effect of such a pattern is similar to that of a stereo
photograph of a snowfall.

A.1.1.0   SHFR1

This routine operates on two sets of points, i.e., on points
stored on 2 consecutive buffers; it applies a random shift of

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*
The units in which SHF is specified are picture elements in an
array of 256 x 256 pels.

absolute value less than SHMAX to each point belonging to the first set, and the same shift multiplied by a constant factor to each corresponding point of the second set. Formally, we can express the transformation by the algorithm:

1:   sh = SHMAX (2 · rand - 1);

2:   For j = 1, n   do

$$(x'_{j1} = x_{j1} + sh ;$$

$$y'_{j1} = y_{j1}$$

$$x'_{j2} = x_{j2} + sh \cdot DSH$$

$$y'_{j2} = y_{j2})$$

where (x_{j1}, y_{j1}) and (x_{j2}, y_{j2}) are the corresponding j^{th} points of the sets 1 and 2. The purpose of this routine is to generate an expansion along the Z direction; DSH acts as a scaled expansion factor.

A.1.1.1   SHFS

This routine applies to each point from a set a shift proportional to the distance from the point to the origin of coordinates (the center of the pattern). Formally, the transformation is:

$$x' = x + K\sqrt{x^2 + y^2}$$

$$y' = y$$

The purpose of this routine is to generate a stereo pair in which the points are organized in depth in the form of a hemisphere.

A.1.1.2 SPGEN:

This routine generates a set of "random" points in such a way
as to prevent the formation of "radial" trajectories, i.e., given an
point (x, y) belonging to the original pattern, the generating procedure
guarantees that no other point will be generated near the first along
the straight line passing through (0,0) and (x, y). This is achieved
by generating the points using the following algorithm:

    1: t: = 0;

    2: n: = 2 · RMAX/DR;

    3: r: = RMAX·(2·rand − 1);

    4: call polar (r,t);

    5: For j=1, n do

        (r: = (r + DR) modulo RMAX;

         call polar (r,t));

    6: t: = t + DT;

    7: if (t < 360°) go to 3;

    8: end

where DR, RMAX and DT are parameters supplied by the user.
Their meaning is the following:

    RMAX = radius of the overall pattern

    DR = minimum distance at which two points are allowed to lie
         along a line passing through the center of the pattern.

    DT = angle between two successive radial lines.

Polar is a routine which transforms from polar (r, t) to rectangular (x, y) coordinates and stores the generated point in the corresponding buffer.

A.1.1.3 NNB

This routine generates, for each point in a given set, a corresponding point at a distance d and at an angle t (see Fig (49)). i.e., it applies the transformation:

$$x' = x' + d \cdot \cos t$$

$$y' = y' = d \cdot \sin t$$

d and t are computed using two funcitons F_d and F_θ supplied by the user (at compilation time). The main functions I have used are:

a) For the distance:

1) d = K (constant distance)

2) $d = K\sqrt{x^2 + y^2}$ (linear variation with the distance from the origin of coordinates)

3) $d = K_1 + K_2 \cdot \text{rand}$ (controlled random variation)

b) For the angle:

1) $t = 2 \cdot \text{rand}$ (random variation)

2) $t = K\sqrt{x^2 + y^2}$ (linear variation with the distance from the origin)

3) $t = K_1(1-B1) + K2$ (linear variation with the value of a barycentric coordinate B1 (see #3.2.3)

4) $t = \text{sign} (K\sqrt{x^2 + y^2}, X)$ (where the function sign (a, b) assigns the sign of b to the absolute value of a)

5) $t = \text{sign} (K\sqrt{X^2 + Y^2}, X \cdot Y)$

With this routine it is possible to generate any arbitrarily defined
flow pattern.

A.1.1.4 DISP and DISP1

These are sets of high-level and assembly language routines used
to display a set of points (i.e., the contents of any buffer) either on
a CRT or on an electrostatic plotter.

A.1.1.5 Generation of Specific Patterns

To generate and display a specific organized pattern, it is
necessary to call several of these routines in sequence. In this sense,
this system may be thought of as a high-level language designed for
organizing noise, each call to a routine (together with the specification
of the necessary input parameters) corresponding to a "command" of the
language. After this routine is executed, the control returns to the
driving program, and the system is ready to accept the next command.

In order to clarify these points, I will now present the "programs"
(i.e., the sequences of commands or routine calls) necessary to generate
some organizations. Recall that whenever the control is transferred to
a given subroutine, it asks the user for the specific parameters it
needs for its operation. In the following examples, these parameters
will appear as numbers within parentheses.

a) To generate a kaleidoscope, such as Fig (10a), only three commands
are needed:

1: RRDM (180, 200, 1, 0); generates a set of 180 random points

within a square 200 units wide on buffer 1, with uniform

distribution (slope = 0)

2: CSYM (1, 6, 1); Applies to the contents of buffer 1 the

transformations of the group D_6, and adds the transformed

points to buffer 1.

3. DISP1 (1); Displays the contents of buffer 1 on the electro-

static plotter.

b) A pattern such as that of Figure 7 is generated by the following

program:

1: RTES (240, 10, 10, 90); Generates a regular system of points

within a square 240 units wide using two equal

translations of 10 units with an angle of 90° between

them, i.e., it generates a square lattice of points.

2: CROT (1, 3, 1); Applies to the points that are in buffer 1

the transformations of the group C_3 (i.e., 3 rotations

about the center of the pattern of 0°, 120°, and 240°),

and leaves the result in buffer 1.

3: DISP (1); Displays the contents of buffer 1 on the CRT.

It is interesting to note that the intriguing perceptual

properties of Figure 7 obtain only if the center of rotation (i.e.,

the center of the pattern) coincides with a point of the square lattice,

as in the preceding example. Since the points of the lattice are generated

starting from the left lower corner, this condition is equivalent to

the requirement that the size of the square divided by 2 be an integer

multiple of the translation size (in this case, 10 x 12 = 240/2);

If the center of rotation coincides not with a point, but with the

center of a square of the lattice (for example, using the command:

RTES (250, 10, 10, 90)) a pattern such as that of Fig (50) results (note

now the overall central symmetry is much more visible in this case), and

if it doesn't coincide neither with a point nor with the center of a

square, (as with the command RTES (245, 10, 10, 90)) the resulting

pattern appears almost random (Fig (51)).

c) To generate a pattern of "symmetric spirals", such as that of Fig

(13), the following sequence of commands is needed:

    1: RRDM (300, 120, 1, 0); Generates a set of 300 random points

          within a square of 120 x 120 units with uniform distribution

          and stores them on buffer 1.

    2: CFLW (1, 1.05, 1.05, 5, 1); Applies to the points stored in

          buffer 1 a uniform expansion in the X and Y directions with

          an expansion factor of 1.05, and then rotates the expanded

          pattern 5°; it adds the transformed pattern to buffer 1.

    3: SHF2 (1, 60, 1); Shifts the set of dots stored in buffer 1

          60 units to the right, and leaves the result in buffer 1

          deleting the previous contents.

    4: CSYM (1, 1, 2); Applies to the points stored in buffer 1 the

          transformations of the group D_1 (i.e., a mirror reflexion

          about a vertical axis) and leaves the reflected pattern in

          buffer 2.

    5: DISP1 (1, 2); Displays the contents of buffers 1 and 2 on the

          electrostatic plotter.
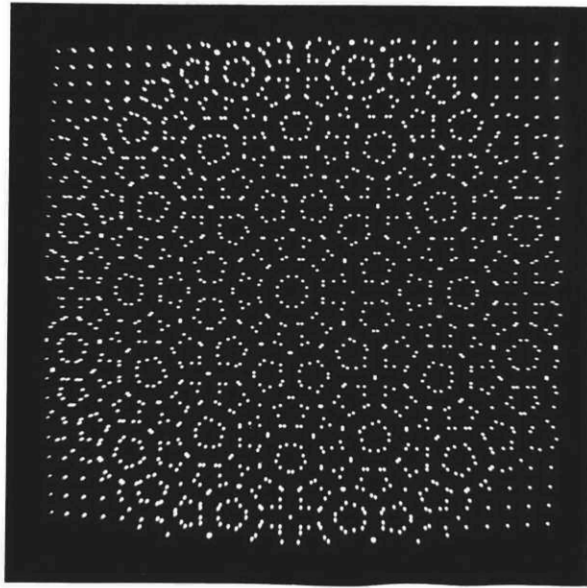
Fig 50. The Overall 24-fold Central Symmetry is Perceived
         if the center of Rotation coincides with the center of
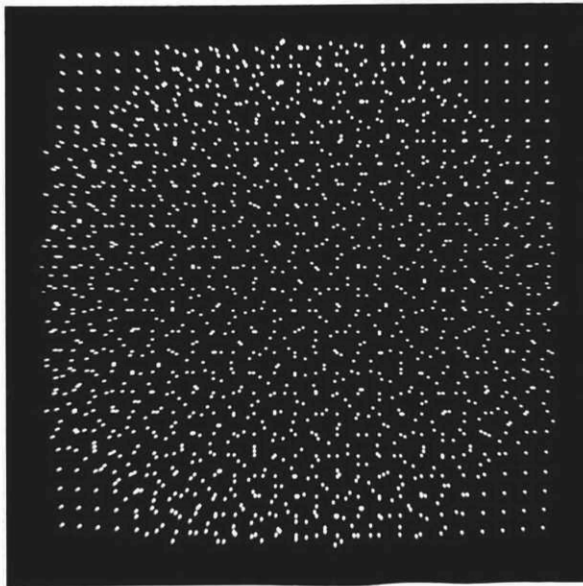   a Square: CROT (RTES(250,10,10,90), 3)

Fig 51. An arbitrary Center of Rotation produces an
         almost random pattern: CROT (RTES(245,10,10,90),3)

d) To generate a "three dimensional expansion", i.e., a flow pattern
in which the pairs of corresponding points appear at random heights, but
the two points of each pair appear at the same depth, such as the stereo
pattern of Fig (29a), the following program is used:

1: RRDM (400, 250, 1, 0); Generates a set of 400 uniformly
distributed random points within a square of 250 x 250
units, and stores them on buffer 1.

2: CFLW (1, 1.05, 1.05, 0, 2); Applies to the contents of buffer
1 a uniform expansion in the X and Y direction with an
expansion factor of 1.05; stores the expanded pattern in
buffer 2.

3: DISP1 (1, 2); Displays on the electrostatic plotter the contents
of buffers 1 and 2; the pattern so generated is the left
component of the pair.

4: SHFR1 (1, 2, 4, 1); Applies to each point stored on buffer 1 a
random shift of absolute value less than 4, and to the
corresponding point on buffer 2, the same shift (multiplied
by 1).

5: DISP1 (1, 2); Displays on the electrostatic plotter the contents
of buffers 1 and 2. This is the right component of the
stereo pair. If buffer 1 and buffer 2 are displayed
independently, and a transparency is made of the pattern
contained in buffer 2, by sliding slowly this transparency
over the patterns of buffer 1, while viewing the stereo
pair through a stereoscope, one sees 2 patterns of dots
randomly located in depth, one moving with respect to the

other along the Z axis. Only when the relative depth of both sets

of dots "match", i.e., when both corresponding dots of each pair appear

at the same depth, a three dimensional expansion pattern appears

(c.f. #2.6).

I hope that these examples will give the reader a fair idea of the

way in which most of the patterns that appear in this thesis were

generated. The patterns of Figures (31) and (32) were constructed using

similar algorithms, with the difference that the original and transformed

sets of points were connected by smooth curves, (using a modified spline-

fitting technique) and the resulting patterns were displayed using

an incremental pen plotter. The listings of the programs, as well as

a DEC Tape containing the source code are available at the RLE head-

quarters.

APPENDIX 2: MATHEMATICAL CONSIDERATIONS RELATED TO THE PRINCIPLE

OF FORMATION OF LOCAL GROUPS (#1.3.4)

A.2.1 Discrete Groups:

A set X of similar and proximal elements will form a cluster if:

a) It can be partitioned into m disjoint sets:

$$Y_1, \ Y_2, \ \ldots \ Y_m$$

each with p elements, i.e.,

$$Y_1 = \left\{ x_{11}, \ \ldots \ x_{1p} \right\}$$
$$\vdots$$
$$Y_m = \left\{ x_{m1}, \ \ldots \ x_{mp} \right\}$$

$$x_{ij} \in X \quad \text{for } i = 1, \ldots m; \quad j = 1, \ldots p,$$

and

$$Y_1 \cup Y_2 \ \ldots \cup Y_m = X$$

$$Y_i \cap Y_j = \emptyset \text{ for } i \neq j$$

b) There exists a discrete group $G \in \mathcal{G}$ such that for every transformation $g \in G$,

$$g \left[x_{ij} \right] = x_{ej}, \quad j=1, \ p$$

Where

$$x_{ij} \in Y_i$$

$$x_{ej} \in Y_e$$

and $Y_i \neq Y_e$ if g is not the identity transformation.

A.2.2 Continuous Groups:

Consider a set $X = \left\{ x_i \right\}$ of elements, each with a set of attributes; let us represent by a_i the value of the attribute of the element x_i whose variation we are considering, and by $N(x_i)$ the set of elements $x_j \in X$ which are in the neighborhood of x_i.

The set X will be a "sampled" representation of a continuous local group[*] if, when assigning the identity transformation to an arbitrary element $x_0 \in X$, the following conditions are fulfilled:

1: For every three transformations f_1, f_2, f_3, such that the elements x_1, x_2, x_3, x_{12}, x_{23}, and x_{123}, corresponding to $a_1 = f_1(a_0)$; $a_2 = f_2(a_0)$; $a_3 = f_3(a)$; $a_{12} = f_1 \, f_2(a_0)$ $a_{23} = f_2 \, f_3(a_0)$; and $a_{123} = (f_1 \, f_2) \, f_3(a_0)$ all belong to X, the relation

$$f_1 \, (f_2 \, f_3) = (f_1 \, f_2) \, f_3 \quad \text{holds.}$$

(the "product" $f_1 \, f_2$ denotes composition of transformations)

2) There exist some pairs of transformations f_1, f_2 such that:

   a) if to $a_1 = f_1 \, (a_0)$ and to $a_2 = f_2 \, (a_0)$ correspond the elements

   $x_1 \in X$ and $x_2 \in X$, then to:

   $a_{12} = f_1 \, f_2 \, (a_0)$ corresponds $x_{12} \in X$

   b) For every pair of elements

   $x_{1n} \in N \, (x_1)$ and $x_{2n} \in N \, (x_2)$

   corresponding to the transformations:

   $a_{1n} = f_{1n} \, (a_0)$

   $a_{2n} = f_{2n} \, (a_0)$

there exists an element $x_{12n} \in N(x_{12})$

---

[*]See #1.2.2-c and #1.3.4. Also see Pontryagin (1966), p. 137

corresponding to:

$$a_{12n} = f_{\imath n} \, f_{2n} \, (a_o)$$

3)   If for a transformation f_1 corresponding to $x_{\imath} \in X$ exists a left

inverse $f_{-\imath}$ such that

to $a_{-\imath} = f_{-\imath} \, (a_o)$ corresponds $x_{-\imath} \in X$,

and $a_o = f_{-\imath} \, f_{\imath} \, (a_o)$,

then, for every element $x_{\imath n} \in N(x_{\imath})$ corresponding to the

transformation

$$a_{\imath n} = f_{\imath n} \, (a_o)$$

there exists an element $x_{\imath n} \in N(x_{-\imath})$ corresponding to the

transformation

$$a_{-\imath n} = f_{-\imath n} \, (a_o), \text{ such that}$$

$$a_o = f_{-\imath n} \, f_{\imath n} \, (a_o)$$

4)   If to $a_{\imath} = f_{\imath} (a_o)$ corresponds $x_{\imath} \in X$

and to $a_{\imath n} = f_{\imath n} (a_o)$ corresponds $x_{\imath n} \in N (x_{\imath})$

then

$$|a_{\imath} - a_{\imath n}| < \epsilon \qquad \text{(an arbitrarily small number)}$$

Since the position of any element x_i is defined by the real

vector $\bar{x}_i$, this last condition means that there is a coordinate system

defined in the local group which is a linear transformation of the

coordinate system in which the elements are defined[*], i.e., the

parameters of the transformations of the group can be represented by

a continuous single-valued function $g(\bar{x}_i)$. The dimension of $\bar{x}$

---

[*] see Pontryagin (1966), p. 283.

is the dimension of the group.

   If, in addition to the above conditions, for <u>every</u> pair of elements $x_1 \in X$ and $x_2 \in X$ corresponding to the transformations:

$$a_1 = f_1(a_o) \text{ and } a_2 = f_2(a_o)$$

the elements $x_{-1} \in X$, $x_{-2} \in X$, $x_{12} \in X$ corresponding to:

$$a_{-1} = f_{-1}(a_o) ; \qquad a_o = f_{-1} \, f_1(a_o)$$

$$a_{-2} = f_{-2}(a_o) ; \qquad a_o = f_{-2} \, f_2(a_o)$$

$$a_{12} = f_1 f_2(a_o)$$

also exist, the set X represents a (global) continuous group.

(the variation of the attribute is closed within X).

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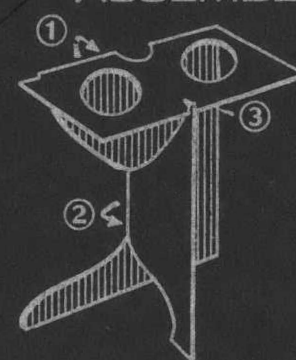
Toward light source

KEEP

this line parallel to
the bottom of stereo picture.

Fold

HOW TO USE

1. Place viewer with open end toward light source. Both stereo pictures must be illuminated equally well, with no shadow on them from the viewer.

2. Make sure the bottom edge of the stereo picture is parallel to the line between the lenses. (Some defects of vision may necessitate a slight twist off the parallel.)

3. If you wear glasses, use them while viewing.

4. In some cases, focusing may be improved by raising or lowering the viewer.

5. Give your eyes fair amount of time to fuse the stereo pictures; only 8% of people cannot see stereo.

STEREO VIEWER
taylor-merchant
corporation
New York 36, N.Y.

Fold

HOW TO ASSEMBLE

1. Fold lens panel.

2. Fold support panel.

3. Lock support to lens panel.