

STUDY OF THE APPLICATIONS OF THE NONLINEAR
SCHRODINGER EQUATION

by

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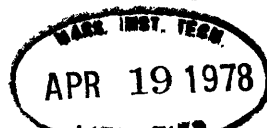
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Submitted January 20, 1978, to the Department of Electrical Engineering and Computer Science in partial fulfillment of the requirements for the degree of Bachelor of Science.

Abstract

Three dimensional lower hybrid plasma waves are examined, where the third dimension is treated as a perturbation of the two dimensional problem. The derivation of the lower hybrid wave from the general Harris equation is shown. The dispersion of the wave is balanced by introducing a nonlinearity, the ponderomotive force. The resulting nonlinear equation is reduced by the multiple scales technique to the three dimensional nonlinear Schrodinger equation. To carry out this reduction, a plane wave solution is assumed. The three dimensional nonlinear Schrodinger equation is solved by perturbing the two dimensional Schrodinger soliton solutions. This results in a set of coupled second order differential equations. A method of numerically

integrating these equations is discussed and used to obtain a wavenumber vs coupling coefficient (i.e. growth rate) graph. The meaning and the accuracy of this data is discussed. Using other techniques, predictions are made of what the data should be.

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Notation

References occurring in the text are placed in brackets, thus: [Acl]. Usual notation is not included below. The numbers in parentheses refer to the equations in the text. It should be noted that in Section 2.4 there are some operators having the same symbol as used elsewhere. These operators are not included below. Also the subscript s stands for species unless otherwise stated.

ω_p	plasma frequency (2.1.1)
f	distribution function (2.1.1)
Ω	cyclotron frequency (2.1.1)
V_{TH}	thermal velocity (2.1.4)
$k_{\perp}, k_{\parallel}, a, b, c$	coefficients in dispersion relation (2.1.31)
F	pondermotive force (2.2.21)
χ	plasma dielectric tensor (2.3.5)
ψ	angle between k_x and k_z (2.3.11, B.2)
ϵ	small parameter (2.3.19)
ϵ^i	small ordering parameter (2.4.8a-10a)
$k_{\perp 0}, k_{\parallel 0}, a_0, b_0, c_0, A_0, B_0$	dispersion relation coefficients (2.3.24-25)
$\phi(x, y, z)$	plane wave solution (2.4.1)
P, Q, R	derivative expressions (2.4.8a-10a)

$\partial/\partial x_i, \partial/\partial y_i, \partial/\partial z_i$	multiple time scales notation (2.4.8b-10b)
c_1	constant from nonlinear terms in dispersion relation (2.4.11)
v_g	group velocity (2.4, B.9.10)
T_e, T_i, T_s	temperature (2.3.23)
E	Schrodinger energy variable (5.1.3)
η, τ, ξ	rescaling factors (2.4.21-22)
c_2	constant of integration 93.2.6)
$\hat{g}_1(\), \hat{k}(\), \hat{R}(\), \kappa_n$	Gelfand-Levitan parameters (3.2.19)
Γ	coupling coefficient
H	integration step size (4.1.1)
R	slowly varying function (5.1, only)
ψ', α, β	boundary condition variables (C.7, 9, 10)
T	dummy time variable (A.1.13)
θ	angle in Figure A.1
θ'	angle in Figure A.2

Chapter One

Introduction

Today's research in thermonuclear fusion could be broadly divided into two major areas, the heating and the confinement of a plasma. When people first began to consider fusion, they thought that electric currents would be sufficient to heat the plasma to fusion temperature. However, as the temperature increases, the resistivity decreases. Thus, at a temperature less than that needed for fusion, heating by currents becomes impractical. Many techniques have been suggested for heating the plasma and many have been tried. One method is the excitation of low frequency plasma waves with electromagnetic radiation. If these low frequency waves can be excited, it is possible that ion heating could result. This method has the advantage that ions are heated directly. The ions are what we want to fuse, so it is they that must be heated. But difficulties arise with the stability of the waves. If the excited wave damps out or breaks up, what good is it?

Recently, the lower hybrid wave has received much attention. In this thesis, we will discuss the lower hybrid dispersion relation in the reduced form of the nonlinear Schrodinger equation. This will be considered in three dimensions. The applicability of the results obtained here to the fusion problem is not known.

The question of lower hybrid stability in three dimensions is interesting in its own right as a nonlinear wave phenomena, and this is how we approach the problem.

We begin by using the three-dimensional Harris dispersion relation for longitudinal waves. The plasma we are considering has no d.c. electric fields and has a magnetic field along the \hat{z} axis. The plasma is being excited at $x = 0$ by a waveguide. The major assumption being made is that there is little or no variation in the \hat{y} direction. This allows us to consider the \hat{y} direction as a perturbation of the two dimensional problem. After showing the derivation of the lower hybrid dispersion relation, the question of wave stability arises. Is the lower hybrid wave stable to perturbations in the \hat{y} direction? Clearly if the wave has only dispersive terms, it will break up or damp out. Therefore, to balance the dispersion, we include in the model some nonlinearity. This nonlinearity is introduced in the form of the ponderomotive force. This force causes a modulation in the number density of the plasma and thus enters the dispersion relation. After including the nonlinearity, we resort to the multiple time scales method to reduce the dispersion relation to the three dimensional nonlinear Schrodinger equation. Once again the \hat{y} direction is considered as a perturbation and the nonlinearity is ordered the same as the dispersion term $(\partial^2 \phi / \partial y^2)$. This ordering balances the dispersion with a nonlinearity.

What has been described above is included in Chapter Two. Chapter Three deals with trying to solve the three dimensional equation. However, since there are no known solutions, we must resort to perturbation theory. Once again the \hat{y} direction is considered as a perturbation of the two dimensional equation. Doing this, we must solve the two dimensional equation, but the general solutions are not used. Rather, a class of solutions known as solitons are considered. They are stable in two dimensions. These solutions are used with perturbation theory to yield a coupled set of equations. These equations are not easily solved; and in Chapter Four, we describe a computer method of solution which is then used to obtain the data given in Section 4.2. To verify the data, Chapter Five is devoted to making some predictions about what the solutions should look like.

Chapter Two

Lower Hybrid Dispersion Relation

2.1 Expansion of the Harris Equation

To obtain a plasma wave dispersion relation, one can use either the fluid model or the more general kinetic theory approach. Here it is advantageous to use the kinetic theory approach because we wish to include thermal effects which are better handled in kinetic theory. In this chapter, we wish to show the derivation of the lower hybrid dispersion relation from the longitudinal Harris equation. To balance instabilities due to thermal effects, we introduce the nonlinearity due to the ponderomotive force. This ponderomotive force is discussed in Section 2.2. In Section 2.3, we use the expression for this force obtained in Section 2.2 to modulate the number density of the lower hybrid wave. Finally in Section 2.4, we describe the reduction by multiple time scales of the dispersion relation to the nonlinear cubic Schrodinger equation.

To begin, we assume a plasma of many species in a d.c. magnetic field, B , which points along the \hat{z} axis. There is no d.c. electric field. We excite the plasma at $x = 0$ with a waveguide and we also assume there is little variation in the \hat{y} direction. For each species, s , we have a plasma frequency

($\omega_{ps}^2 = 4\pi n_{os} q_s^2 / m_s$) and a cyclotron frequency ($\Omega_s = q_s B / m_s c$), where m_s is the mass, q_s the charge, and n_{os} is the number density of the species s . We also assume that each species is described by an equilibrium (denoted by a o subscript) distribution function in velocity space (f_{os}). This distribution is a function of velocity both parallel ($v_{\parallel s}$) and perpendicular to the \hat{z} axis ($v_{\perp s}$). These subscripts \parallel and \perp are used with other variables as well to show orientation with respect to the \hat{z} axis. With the above assumptions, the longitudinal Harris equation derived in Appendix A is,

$$k^2 = \sum_s \omega_{ps}^2 \int_{-\infty}^{\infty} dv_{\parallel s} \int_0^{\infty} 2\pi v_{\perp s} dv_{\perp s} \frac{\sum_N J_N^2(k_{\perp} v_{\perp s} / \Omega_s)}{(k_{\parallel} v_{\parallel s} - \omega + \Omega_s N)} \left[k_{\parallel} \frac{\partial f_{os}}{\partial v_{\parallel s}} + \frac{N \Omega_s}{v_{\perp s}} \frac{\partial f_{os}}{\partial v_{\perp s}} \right], \quad (1)$$

where k is the wavenumber (which can be directed \parallel or \perp to the \hat{z} axis) and ω is the angular frequency. The term $\sum_N J_N^2$ is a summation of the Bessel function squared over all orders.

To obtain the lower hybrid dispersion relation, we must expand the Harris equation with the following assumption,

$$\Omega_i \ll \omega \ll \Omega_e. \quad (2)$$

Here i stands for ions and e for electrons. Also it is important to note that since we are dealing with three dimensions,

$$k_{\perp}^2 = k_x^2 + k_y^2. \quad (3)$$

In the expansion, we need to decide which terms to keep. As stated earlier thermal effects cause instabilities (such as dispersion) in the plasma and these we want to consider. Since the thermal terms are of higher order we keep the k_x^4 , k_z^4 , and $k_x^2 k_z^2$ terms. However, because we are considering the third dimension (the \hat{y} direction) as a perturbation, we will disregard thermal effects and only retain first order terms in k_y^2 . Another consideration is the nature of the distribution function f_0 . First it should be noted that this is a velocity distribution in three dimensions. A good assumption for plasmas is to assume a Maxwellian distribution. We have

$$f_0 = \frac{1}{\sqrt{\pi^3} v_{\text{THS}}^3} \exp \left[- \frac{v_{\parallel s}^2}{v_{\text{THS}}^2} - \frac{v_{\perp s}^2}{v_{\text{THS}}^2} \right], \quad (4)$$

where v_{THS} is the thermal velocity of species s .

To actually carry out this expansion, we must simultaneously Taylor expand in two variables. This is algebraically

complicated. Therefore, to illustrate the procedure we will break it up into several phases, by first setting $k_x^2 = 0$ and then $k_z^2 = 0$, but by so doing we will lose the expression for the cross terms $(k_x^2 k_z^2)$. We will just quote the results for these terms [Ka2]. To begin we assume

$$k_{\parallel} = 0, \quad k_{\perp}^2 = k_x^2.$$

As a result, we have eliminated the first term of the Harris equation which we can handle later. We also do not consider the third dimension. However since $k_{\perp}^2 = k_x^2 + k_y^2$, we can just add the results from using k_x^2 to the results from k_y^2 to get k_{\perp}^2 . Therefore we obtain

$$k_x^2 = \sum_s \omega_{ps}^2 \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{\infty} 2\pi v_{\perp} dv_{\perp} \frac{\sum N J_N^2(k_{\perp} v_{\perp} / \Omega_s)}{\Omega_s N - \omega} \frac{N \Omega_s}{v_{\perp}} \frac{\partial f_{os}}{\partial v_{\perp}}. \quad (5)$$

The derivative of f_o with respect to v_{\perp} , is

$$\frac{\partial f_{os}}{\partial v_{\perp}} = - \frac{2v_{\perp}}{v_{THS}^2} f_{os}. \quad (6)$$

At this point we can substitute equation (6) into equation (5).

$$k_x^2 = - \sum_s \omega_{ps}^2 \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{\infty} 4\pi v_{\perp} dv_{\perp} \frac{\sum_N J_N^2(k_{\perp} v_{\perp} / \Omega_s)}{\Omega_s N - \omega} \cdot \frac{2\Omega_s}{v_{THS}^2} \frac{1}{\sqrt{\pi}^3 v_{THS}^3} \exp \left[- \frac{v_{\parallel}^2}{v_{THS}^2} - \frac{v_{\perp}^2}{v_{THS}^2} \right]. \quad (7)$$

To simplify the above, we need to use the following equation, [Nol],

$$\int_0^{\infty} \frac{dy}{\sqrt{\pi}} e^{-y^2} J_N^2(Ay) = e^{-\lambda} I_N(\lambda) \quad (8)$$

where $\lambda = A^2/2$, where we identify $\lambda_s = k_{\perp}^2 v_{THS}^2 / 2\Omega_s^2$ and $y^2 = v_{\perp}^2 / v_{THS}^2$. To do this it is noted that the argument of J_N is both multiplied and divided by v_{THS} . This gives $J_N(k_{\perp} v_{THS} y / \Omega_s)$. Also note that I_N is the modified Bessel function. Using relation (8) with equation (7) we obtain

$$k_x^2 = \omega_{ps}^2 \sum_N \frac{N\Omega_s}{\omega - N\Omega_s} \frac{2}{v_{THS}^2} e^{-\lambda_s} I_N(\lambda_s),$$

$$\lambda_s = \frac{k_{\perp}^2 v_{THS}^2}{2\Omega_s^2}. \quad (9)$$

Letting $N = \pm 1, \pm 2$ to eliminate higher order terms, and using

the expansions of $e^{-\lambda_s}$ and $I_N(\lambda_s)$ to second order, we can reduce the above equation to a series of terms,

$$k_x^2 = \omega_{ps}^2 \left(\frac{k_\perp^2}{(\omega - \Omega_s)\Omega_s} - \frac{k_\perp^2}{(\omega + \Omega_s)\Omega_s} \right) + \omega_{ps}^2 \left(\frac{v_{THS}^2}{4} \frac{\Omega_s}{\omega - 2\Omega_s} \frac{k_\perp^4}{\Omega_s^4} - \frac{v_{THS}^2}{4} \frac{\Omega_s}{\omega + 2\Omega_s} \frac{k_\perp^4}{\Omega_s^4} \right). \quad (10)$$

After using some algebra we can combine some of the terms to get

$$k_x^2 = \frac{\omega_{ps}^2 k_x^2}{\omega^2 - \Omega_s^2} + \frac{\omega_{ps}^2 v_{THS}^2 k_x^4}{\Omega_s^2 (\omega^2 - 4\Omega_s^2)}. \quad (11)$$

So far everything we have done has been general. The above applies to all longitudinal waves. We have reduced the Harris dispersion relation to a series of terms to order k_x^4 . Now we can use the assumption made at the beginning of the section (i.e. equation (2)) to make the equation specifically for the lower hybrid wave.

To make the dispersion equation specific, we must consider each ordering of equation (2) and each species separately. First just using $\omega \ll \Omega_e$ and considering electrons in the first term of the RHS of equation (11), we obtain

$$- \frac{\omega_{pe}^2 k_x^2}{\Omega_e^2} . \quad (12)$$

Now considering the same assumptions for the second term of equation (11) we get,

$$- \frac{\omega_{pe}^2 v_{THE}^2 k_x^4}{4\Omega_e^4} . \quad (13)$$

Last using $\omega \gg \Omega_i$ and ions we get for the first term,

$$\frac{\omega_{pi}^2 k_x^2}{\omega^2} . \quad (14)$$

For the second term, we obtain

$$\frac{\omega_{pi}^2 v_{THi}^2 k_x^4}{\omega^4} . \quad (15)$$

Combining equations (12)-(15), we have the lower hybrid dispersion relation for the k_x direction,

$$k_x^2 = \left(- \frac{\omega_{pe}^2}{\Omega_e^2} + \frac{\omega_{pi}^2}{\omega^2} \right) k_x^2 + \left(- \frac{\omega_{pe}^2 v_{THE}^2}{4\Omega_e^4} + \frac{\omega_{pi}^2 v_{THi}^2}{\omega^4} \right) k_x^4 . \quad (16)$$

Since we are finished with the k_x part of the dispersion

equation, we can proceed to solve for the k_z term. We let

$$k_{\perp} = 0. \quad (17)$$

But it should be noted the argument of the Bessel function contains k_{\perp} . To handle this we need to find the limit of $\sum_N J_N(\Omega)$ as Ω goes to zero.

$$\sum_N J_N(\Omega) \rightarrow N^2 \quad \text{as } \Omega \rightarrow 0. \quad (18)$$

Making these substitutions into equation (1) and noting that

$$\int_{-\infty}^{\infty} \frac{\partial f_{OS}}{\partial v_{\parallel}} dv_{\parallel} = \int_{-\infty}^{\infty} \partial f_{OS}, \quad (19)$$

we obtain

$$k_{\parallel}^2 = \sum_s \omega_{ps}^2 \int_0^{\infty} 2\pi v_{\perp} dv_{\perp} \int_{-\infty}^{\infty} k_{\parallel} \cdot \frac{\partial f_{OS}}{k_{\parallel} v_{\parallel} - \omega + \Omega_s N}. \quad (20)$$

To simplify the above, we have to integrate by parts. Following this approach we let,

$$u = \frac{k_{\parallel}}{k_{\parallel} v_{\parallel} - \omega + \Omega_s N}, \quad dv = \partial f_{OS}, \quad (21)$$

and therefore we obtain,

$$du = - \frac{k_{\parallel}^2 dv_{\parallel}}{(k_{\parallel} v_{\parallel} - \omega + \Omega_S N)^2}, \quad v = f. \quad (22)$$

The integration by parts finally yields,

$$f_{OS} \left(\frac{k_{\parallel}}{k_{\parallel} v_{\parallel} - \omega + \Omega_S N} \right) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{\infty} 2\pi v_{\perp} dv_{\perp} f \cdot \frac{k_{\parallel}^2}{(k_{\parallel} v_{\parallel} - \omega + \Omega_S N)^2}. \quad (23)$$

Last, we integrate over f which just gives 1 times everything else, and our dispersion relation becomes,

$$k_{\parallel}^2 = \omega_{PS}^2 \frac{k_{\parallel}^2}{(k_{\parallel} v_{\parallel} - \omega + \Omega_S N)^2}. \quad (24)$$

We have reached the point where we need to apply the specific assumptions that will reduce this general equation to the lower hybrid dispersion relation. First note that

$$\Omega_S N \ll k_{\parallel} v_{\parallel}. \quad (25)$$

With this relation and $\omega \ll \Omega_e$, we obtain the following from equation (24),

$$k_{\parallel}^2 \omega_{pe}^2 \left[\frac{1}{\omega^2} - \frac{k_{\parallel}^2 v_{THe}^2}{\omega^4} \right]. \quad (26)$$

Now considering ions and $\omega \gg \Omega_i$, we obtain

$$k_{\parallel}^2 \omega_{pi}^2 \left[\frac{1}{\omega^2} - \frac{k_{\parallel}^2 v_{THi}^2}{\omega^4} \right]. \quad (27)$$

Combining the above two equations, we have the complete lower hybrid dispersion relation for the k_{\parallel} direction,

$$k_{\parallel}^2 = k_{\parallel}^2 \left[\omega_{pe}^2 \left(\frac{1}{\omega^2} - \frac{k_{\parallel}^2 v_{THe}^2}{\omega^4} \right) + \omega_{pi}^2 \left(\frac{1}{\omega^2} - \frac{k_{\parallel}^2 v_{THi}^2}{\omega^4} \right) \right]. \quad (28)$$

What we have done so far is found the dispersion relation for the k_x and $k_z (= k_{\parallel})$ directions, but there are still the cross terms (i.e. $k_{\parallel}^2 k_{\perp}^2$). Using Taylor series expansions in k_{\perp} and k_z and also using equation (2) which is the lower hybrid assumptions, the following for the cross terms can be obtained [Ka2],

$$k_{\parallel}^2 k_{\perp}^2 = k_{\parallel}^4 k_{\perp}^4 \left[\frac{1}{3} \frac{\omega_{pe}^2}{\omega^2} \frac{v_{THE}^2}{\Omega_s^2} + 2 \frac{\omega_{pi}^2}{\omega^2} \frac{v_{THi}^2}{\omega^2} \right]. \quad (29)$$

Now we have found all parts of the lower hybrid dispersion equation, we just need to combine them together. We should realize that because we are assuming $\exp[i\omega t - ik \cdot \Omega]$, that the k 's represent derivatives. However, to properly place the terms with respect to the derivatives requires that we keep account of their position from the beginning. This is not easily done and we will state the result of doing so [Ka2],

$$\begin{aligned} & \frac{\partial}{\partial x} k_{\perp} \frac{\partial}{\partial x} \phi + \frac{\partial}{\partial z} k_{\parallel} \frac{\partial}{\partial z} \phi + a \frac{\partial^4}{\partial x^4} \phi + b \frac{\partial^4}{\partial x^2 \partial z^2} \phi + c \frac{\partial^4}{\partial z^4} \phi \\ & = 0. \end{aligned} \quad (30)$$

The ϕ is the potential which was in each part of the lower hybrid equation, but we avoided writing it on both sides of the equations. The other factors are as follows,

$$\begin{aligned} k_{\perp} &= 1 + \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2}, & k_{\parallel} &= 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2}, \\ a &= \frac{1}{4} \frac{\omega_{pe}^2}{\Omega_e^2} \frac{v_{THE}^2}{\Omega_e^2} + \frac{\omega_{pi}^2}{\omega^2} \frac{v_{THi}^2}{\omega^2}, & b &= -\frac{1}{3} \frac{\omega_{pe}^2}{\omega^2} \frac{v_{THE}^2}{\Omega_e^2} + 2 \frac{\omega_{pi}^2}{\omega^2} \frac{v_{THi}^2}{\omega^2}, \end{aligned} \quad (31)$$

$$c = \frac{\omega_{pe}^2}{\omega^2} \frac{v_{The}^2}{\omega^2} + \frac{\omega_{pi}^2}{\omega^2} \frac{v_{THi}^2}{\omega^2} . \quad (32)$$

Last, we look at the k_y dependence of the lower hybrid dispersion equation. This follows exactly the k_x derivation except higher order terms are eliminated and we get

$$k_y^2 \phi . \quad (33)$$

This gives us a $\partial^2/\partial y^2 \phi$ term in equation (30). The complete three dimensional equation is as follows,

$$\begin{aligned} \frac{\partial}{\partial x} k_{\perp} \frac{\partial}{\partial x} \phi + \frac{\partial^2}{\partial y^2} \phi + \frac{\partial}{\partial z} k_{\parallel} \frac{\partial}{\partial z} \phi + a \frac{\partial^4}{\partial x^4} \phi + b \frac{\partial^4}{\partial x^2 \partial z^2} \phi \\ + c \frac{\partial^4}{\partial z^4} \phi = 0 . \end{aligned} \quad (34)$$

2.2 The Ponderomotive Force

In the derivation of the lower hybrid dispersion relation, we have just considered linear effects, some of which were higher order thermal effects. As stated earlier these thermal effects result in the dispersion of a linear wave. We want to balance this dispersion with the ponderomotive force. This force is nonlinear and is a result of a particle interacting with oscillating E and B fields. Since we are only considering oscillating E and B fields, we can set,

$$E_0 = 0 \quad \text{and} \quad B_0 = 0. \quad (1)$$

Using the equation of motion for an electron,

$$m \frac{dv}{dt} = -e[E + v \times B], \quad (2)$$

we can let each variable be perturbed around an equilibrium point. In other words,

$$v = v_0 + v_1 + v_2 + \dots, \quad E = E_0 + E_1 + E_2 + \dots, \\ B = B_0 + B_1 + B_2 + \dots \quad (3)$$

Each term is a smaller perturbation around the equilibrium.

Now separating according to order, we obtain

$$m \frac{dv_0}{dt} = - e [E_0 + v_0 \times B_0] = 0, \quad (4)$$

$$m \frac{dv_1}{dt} = - e [E_1 + v_0 \times B_1 + v_1 \times B_0] = -e E_1, \quad (5)$$

$$m \frac{dv_2}{dt} = - e [E_2 + v_1 \times B_1]. \quad (6)$$

We wish to solve the above set of equations for $n \frac{dv_2}{dt}$ which is the ponderomotive force. To do this we need to find expressions for E_2 and B_1 in terms of v_1 . Using the concept of Fourier transforms, we know that all derivatives with respect to time just involve multiplying by ω . We are assuming that $E = E_1 \cos \omega t$. From Maxwell's equations we have

$$\nabla \times E_1 = - \frac{dB_1}{dt}, \quad (7)$$

which is the following using Fourier transforms,

$$\nabla \times E_1 = - \omega B_1. \quad (8)$$

Therefore,

$$B_1 = \frac{\nabla \times E_1}{-\omega} . \quad (9)$$

Now we have to find an expression for E_1 in terms of v_1 . Equation (5) gives us such an expression,

$$m\omega v_1 = -eE_1 . \quad (10)$$

Solving for E_1 ,

$$E_1 = -\frac{m\omega}{e} v_1 . \quad (11)$$

This can be substituted into equation (9) to give us an expression for B_1 totally in terms of v_1 ,

$$B_1 = \frac{m}{e} \nabla \times v_1 . \quad (12)$$

To find an expression for E_2 , we must Taylor expand E_1 about an equilibrium point r_0 . Doing this we have,

$$E_1 = E_1(r_0) + (r_1 \cdot \nabla)E_1 + \dots . \quad (13)$$

E_2 is equivalent to the second term in the above expression,

$$E_2 = (r_1 \cdot \nabla)E_1. \quad (14)$$

To solve for r_1 , we use

$$\frac{dr_1}{dt} = v_1. \quad (15)$$

This yields,

$$r_1 = \frac{v_1}{\omega}. \quad (16)$$

Using equation (16) and equation (11) in equation (14), we obtain an expression for E_2 totally in terms of v_1 ,

$$E_2 = \frac{m}{e} (v_1 \cdot \nabla)v_1. \quad (17)$$

Substituting the expressions for E_2 and B_1 into equation (6), we now have the ponderomotive force,

$$m \frac{dv_2}{dt} = -m[(v_1 \cdot \nabla)v_1 + v_1 \times (\nabla \times v_1)]. \quad (18)$$

The above expression is the ponderomotive force for a general wave [Ch1], but the lower hybrid wave is longitudinal. Therefore, since for longitudinal waves,

$$\nabla \times \mathbf{E}_1 = 0 \quad (19)$$

which from equation (11) implies that

$$\nabla \times \mathbf{v}_1 = 0, \quad (20)$$

we can reduce our expression to the following

$$\mathbf{F} = - m(\mathbf{v} \cdot \nabla)\mathbf{v}. \quad (21)$$

Remember that when we derived the ponderomotive force, we only considered one particle of one species. To generalize to many particles and many species, we write,

$$\mathbf{F}_s = - m_s N(\mathbf{v}_s \cdot \nabla)\mathbf{v}_s, \quad (22)$$

where N is the number of particles and s stands for species.

2.3 Modulation by the Ponderomotive Force

We begin with the equation derived in the last section,

$$F_s = - m_s N(\mathbf{v}_s \cdot \nabla) \mathbf{v}_s. \quad (1)$$

This is the ponderomotive force that will result in a modulation of the number density of the plasma. To use equation (1), we first have to find an expression for the velocity. This is possible through the plasma dielectric tensor. The expression for the velocity can be substituted into the momentum equation to find the number density modulation. The new number density is substituted into the lower hybrid dispersion relation to yield a nonlinear equation.

First to find v , we use

$$J = n_s f_s v_s, \quad (2)$$

where this expression is equivalent to the following,

$$J = - \frac{i\omega \chi E}{4\pi}. \quad (3)$$

χ is the plasma dielectric tensor and ω is the angular

frequency. Equating equations (2) and (3), we can obtain an expression for v_s in terms of χ ,

$$v_s = - \frac{i\omega\chi E}{4\pi n_s q_s} . \quad (4)$$

To solve for v_s , we need to find χ which for a plasma is of the form, [Krl],

$$\chi = \begin{bmatrix} \chi_{xx} & \chi_{xy} & 0 \\ \chi_{yx} & \chi_{yy} & 0 \\ 0 & 0 & \chi_{zz} \end{bmatrix} . \quad (5)$$

Up to this point, everything regarding the ponderomotive force has been completely general. Here we are going to specialize the derivation of the density modulation to longitudinal waves by letting

$$E = - \nabla\phi . \quad (6)$$

By substituting equations (3) and (6) into the expression for v_s , we obtain

$$\mathbf{v}_S = \frac{i\omega}{4\pi n_S q_S} \begin{bmatrix} \chi_{xx} & \chi_{xy} & 0 \\ \chi_{yx} & \chi_{yy} & 0 \\ 0 & 0 & \chi_{zz} \end{bmatrix} \cdot \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \phi \quad (7)$$

Multiplying out equation (7), we can get the equation into vector form.

$$\mathbf{v}_S = \frac{i\omega}{4\pi n_S q_S} \left[\left(\chi_{xx} \frac{\partial}{\partial x} \phi + \chi_{xy} \frac{\partial}{\partial y} \phi \right) \hat{i} + \left(\chi_{yx} \frac{\partial}{\partial x} \phi + \chi_{yy} \frac{\partial}{\partial y} \phi \right) \hat{j} + \left(\chi_{zz} \frac{\partial}{\partial z} \phi \right) \hat{k} \right] . \quad (8)$$

To calculate the ponderomotive force, we need to find $(\mathbf{v}_S \cdot \nabla)$,

$$(\mathbf{v}_S \cdot \nabla) = \frac{i\omega}{4\pi n_S q_S} \left(\chi_{xx} \frac{\partial}{\partial x} \phi \frac{\partial}{\partial x} + \chi_{xy} \frac{\partial}{\partial y} \phi \frac{\partial}{\partial x} + \chi_{yx} \frac{\partial}{\partial x} \phi \frac{\partial}{\partial y} + \chi_{yy} \frac{\partial}{\partial y} \phi \frac{\partial}{\partial y} + \chi_{zz} \frac{\partial}{\partial z} \phi \frac{\partial}{\partial z} \right) . \quad (9)$$

This is a rather cumbersome equation and this has to be multiplied by \mathbf{v}_S again. One should note that there are x , y , and z components. These are not all needed if one should happen to be larger than the others. At zero frequency, the electrons can be considered infinitely magnetized; therefore,

they travel along the field lines which is the z direction. The z component is definitely the largest, so

$$\begin{aligned}
 F_z = & - \frac{\omega^2}{4\pi \omega_{pe}^2} \left[\left(\chi_{xx} \frac{\partial}{\partial x} \phi \frac{\partial}{\partial x} \right) \chi_{zz} \frac{\partial}{\partial z} \phi + \left(\chi_{xy} \frac{\partial}{\partial y} \phi \frac{\partial}{\partial x} \right) \chi_{zz} \frac{\partial}{\partial z} \phi \right. \\
 & + \left(\chi_{yx} \frac{\partial}{\partial x} \phi \frac{\partial}{\partial y} \right) \chi_{zz} \frac{\partial}{\partial z} \phi + \left(\chi_{zz} \frac{\partial}{\partial z} \phi \frac{\partial}{\partial z} \right) \chi_{zz} \frac{\partial}{\partial z} \phi \\
 & \left. + \left(\chi_{yy} \frac{\partial}{\partial y} \phi \frac{\partial}{\partial y} \right) \chi_{zz} \frac{\partial}{\partial z} \phi \right]. \quad (10)
 \end{aligned}$$

With the equation for the modulating force fully derived, we need to find the expressions for the components of the dielectric tensor. To make these expressions applicable to the lower hybrid wave, we assume that ω is approximately ω_{pi} . From this we get

$$\begin{aligned}
 \chi_{xx}^e = \chi_{yy}^e = \frac{\omega_{pe}^2}{\Omega_e^2}, \quad \chi_{xy}^e = -\chi_{yx}^e = -i \frac{\omega_{pe}^2}{\omega \Omega_e}, \\
 \chi_{zz}^e = -\frac{\omega_{pe}^2}{\omega^2} \quad (11)
 \end{aligned}$$

$$\chi_{xx}^i = \chi_{yy}^i = -\frac{\omega_{pi}^2}{\omega^2}, \quad \chi_{xy}^i = \chi_{yx}^i = -i \frac{\Omega_i \omega_{pi}^2}{\omega^3}$$

$$\chi_{zz}^i = - \frac{\omega_{pi}^2}{\omega^2} . \quad (12)$$

Looking at the above relations, one can see that χ_{xy} and χ_{yx} are imaginary. Also, we must again realize that the y dimension is a perturbation on the two dimensional equation. Therefore, we want to neglect this term. If we did not we would be perturbing the ponderomotive force which is itself a perturbation. We do not want a perturbation of a perturbation. With all of this in mind, we can substitute equations (11) and (12) into equation (10) to get

$$F_{ze} = - \frac{1}{4\pi} \frac{\partial}{\partial z} \left[- \frac{\omega_{pe}^2}{\Omega_e^2} \left| \frac{\partial}{\partial x} \phi \right|^2 + \frac{\omega_{pe}^2}{\omega^2} \left| \frac{\partial}{\partial z} \phi \right|^2 \right] \quad (13)$$

$$F_{zi} = - \frac{1}{4\pi} \frac{\partial}{\partial z} \left[\frac{\omega_{pi}^2}{\omega^2} \left| \frac{\partial}{\partial x} \phi \right|^2 + \frac{\omega_{pi}^2}{\omega^2} \left| \frac{\partial}{\partial z} \phi \right|^2 \right] . \quad (14)$$

With this expression for the ponderomotive force, we can find the density modulation by using the \hat{z} component of the momentum equation. It must be remembered that we have derived the ponderomotive force with the assumption that there were no dc E and B fields. Therefore, we get the following for the momentum equation,

$$F_{sz} = - \nabla p_z = - kT \nabla n_s = - kT_s \frac{\partial}{\partial z} n_s, \quad (15)$$

where k is Boltzmann's constant and T is the temperature. Summing over electrons and ions and substituting equations (13) and (14) for F_z in equation (15), for $\partial n_s / \partial z$ we get,

$$\frac{\partial n_s}{\partial z} = \frac{1}{kT_{e+i}} \frac{1}{4\pi} \frac{\partial}{\partial z} \left[\frac{\omega_{pe}^2}{\Omega_e^2} \left| \frac{\partial \phi}{\partial x} \right|^2 - \frac{\omega_{pe}^2}{\omega^2} \left| \frac{\partial \phi}{\partial z} \right|^2 - \frac{\omega_{pi}^2}{\omega^2} \left| \frac{\partial \phi}{\partial x} \right|^2 - \frac{\omega_{pi}^2}{\omega^2} \left| \frac{\partial \phi}{\partial z} \right|^2 \right]. \quad (16)$$

We can simplify equation (16) by using the cold lower hybrid dispersion relation,

$$1 - \frac{\omega_{pi}^2}{\omega^2} - \cos^2 \psi \frac{\omega_{pe}^2}{\omega^2} + \sin^2 \psi \frac{\omega_{pe}^2}{\Omega_e^2} = 0,$$

where

$$\psi = \tan^{-1} \frac{k_x}{k_z}. \quad (17)$$

Taking the terms in the brackets of equation (16), grouping

them together, and remembering that $\partial/\partial y = 0$, so that

$$\left| \frac{\partial}{\partial x} \right|^2 + \left| \frac{\partial}{\partial z} \right|^2 = |\nabla|^2,$$

we get

$$-\frac{\omega_{pi}^2}{\omega^2} |\nabla\phi|^2 - \frac{\omega_{pe}^2}{\omega^2} \left| \frac{\partial\phi}{\partial z} \right|^2 + \frac{\omega_{pe}^2}{\Omega_e^2} \left| \frac{\partial\phi}{\partial x} \right|^2. \quad (18)$$

Assuming ψ to be small in equation (17) and using ϵ to denote a small parameter, we obtain the following,

$$1 = \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{\omega^2} - \epsilon^2 \frac{\omega_{pe}^2}{\Omega_e^2}. \quad (19)$$

Multiplying this equation through by $|\nabla\phi|^2$,

$$|\nabla\phi|^2 = \frac{\omega_{pi}^2}{\omega^2} |\nabla\phi|^2 + \frac{\omega_{pe}^2}{\omega^2} |\nabla\phi|^2 - \epsilon^2 \frac{\omega_{pe}^2}{\Omega_e^2} |\nabla\phi|^2. \quad (20)$$

In deriving the expression for the ponderomotive force, we assumed that the \hat{z} component was much larger than the \hat{x} or \hat{y} component. Therefore, we can say that $|\partial\phi/\partial z|^2 \sim |\nabla\phi|^2$ and $|\partial\phi/\partial x|^2 \sim 0$ or $|\partial\phi/\partial x|^2 = \epsilon^2 |\nabla\phi|^2$. Using

these assumptions, we reduce equation (18) to

$$-\frac{\omega^2 p_i}{\omega^2} |\nabla\phi|^2 - \frac{\omega^2 p_e}{\omega^2} |\nabla\phi|^2 + \frac{\omega^2 p_e}{\Omega_e^2} \epsilon^2 |\nabla\phi|^2. \quad (21)$$

From equation (20) we realize that this is no more than $-|\nabla\phi|^2$. Therefore, equation (16) is simply

$$\frac{\partial n_s}{\partial z} = - \frac{1}{kT_{e+i}} \frac{1}{4\pi} \frac{\partial}{\partial z} |\nabla\phi|^2. \quad (22)$$

Note that T_{e+i} is $T_e + T_i$ and will be expressed simply as T_s from now on.

With this simplified form for the differential equation in n_s , we can integrate to obtain

$$n_s = n_o \left[1 - \frac{1}{4\pi} |\nabla\phi|^2 / (n_o T_s) \right]. \quad (23)$$

This is the equation for the modulation of the number density [Jol]. Plugging it into the lower hybrid dispersion relation (equation 2.1.32) and only allowing modulation of the lower order terms, we obtain

$$\frac{k_{\perp o}}{k_{\parallel o}} \frac{\partial^2 \phi}{\partial x^2} - \frac{k_{\parallel o}}{k_{\parallel o}} \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial y^2} + a_o \frac{\partial^4 \phi}{\partial x^4} + b_o \frac{\partial^4 \phi}{\partial x^2 \partial z^2} + c_o \frac{\partial^4 \phi}{\partial z^4}$$

$$+ \frac{1}{4\pi} A_0 \frac{\partial}{\partial x} \left| \frac{|\nabla\phi|^2}{n_0 T_s} \frac{\partial\phi}{\partial x} \right| + \frac{1}{4\pi} B_0 \frac{\partial}{\partial z} \left| \frac{|\nabla\phi|^2}{n_0 T_s} \frac{\partial\phi}{\partial z} \right| = 0 \quad (24)$$

where $k_{\perp 0}$, $k_{\parallel 0}$, a_0 , b_0 , and c_0 are the equilibrium values of the corresponding coefficients described in equations (2.1.31) and (2.1.32), except n_0 replaces n . The other coefficients are defined as follows,

$$A_0 = - \frac{\omega_{pe}^2}{\Omega_e^2} + \frac{\omega_{pi}^2}{\omega^2}, \quad B_0 = \frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pi}^2}{\omega^2}. \quad (25)$$

2.4 Multiple Time Scales Perturbation

To completely solve the equation for the lower hybrid dispersion relation (equation 2.3.24) would be difficult. Therefore we will resort to perturbation theory. The idea is to assume a solution that is a perturbation of the solution to the linear equation. In the limit of only considering linear effects, this assumed solution should reduce to the linear solution. Morales and Lee, [Mol], used this idea and assumed a solution with an explicit x dependence (besides the linear solution). With perturbation theory, they obtained the modified Korteweg de Vries equation. Yet their analysis assumes a real solution which means energy propagates away from and toward the source. Obviously, power should only propagate away from the source, since this is how the plasma is being excited. More is said concerning this in Appendix B.

To fulfill the above criterion, assume a plane wave solution,

$$\phi(x, y, z) = \tilde{\phi}(x', y', z') e^{(ik_z z - ik_x x)}, \quad (1)$$

where $x' = x$, $y' = y$, and $z' = z - v_g x$. $\tilde{\phi}(z - v_g x)$ is a solution to the nondispersive lower hybrid equation. The method of multiple time scales involves ordering the terms of an assumed

solution according to the importance of their spatial (or temporal) variations, (see [Nal] for a description of this method). Equation (1) contains a non-dispersive solution, $\hat{\phi}(z - v_g x)$, with some added dependencies, x' and y' (these model the nonlinearity), that modulate an exponential. With this information, we proceed to ordering.

What exactly are terms we want to order? Because the three dimensional dispersion relation (2.3.24) contains $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, these are what we want to order. Solving for the above mentioned derivatives by taking first $\partial\phi/\partial x$ where ϕ is as given in equation (1),

$$\frac{\partial}{\partial x} \rightarrow -ik_x - v_g \frac{\partial}{\partial z'} + \frac{\partial}{\partial x'}. \quad (2)$$

The same is done for $\partial\phi/\partial y$ and $\partial\phi/\partial z$ to yield,

$$\partial/\partial y \rightarrow \partial/\partial y' \quad (3)$$

and

$$\partial/\partial z \rightarrow ik_z + \partial/\partial z'. \quad (4)$$

Each term representing the derivatives has to be ordered. The terms resulting from the exponential terms are of first order; those resulting from the nondispersive solution are second order; and finally the x' , y' dependencies are of highest

order. Consequently,

$$|ik_x| \gg \left| v_g \frac{\partial}{\partial z'} \right| \gg \left| \frac{\partial}{\partial x'} \right| \quad (5)$$

$$|ik_z| \gg \left| \frac{\partial}{\partial z'} \right| . \quad (6)$$

Since we have considered there to be no variations in the \hat{y} direction, the \hat{y} direction can be treated as a perturbation of the two dimensional equation. Therefore,

$$\left| \frac{\partial}{\partial y'} \right| \approx \left| v_g \frac{\partial}{\partial z'} \right| . \quad (7)$$

This ordering information can be conveniently expressed using the small parameter ϵ ,

$$\frac{\partial}{\partial x} = P = -ik_x - \epsilon v_g \frac{\partial}{\partial z'} + \epsilon^2 \frac{\partial}{\partial z'} \quad (8a)$$

$$\frac{\partial}{\partial z} = Q = ik_z + \epsilon \frac{\partial}{\partial z'} \quad (9a)$$

$$\frac{\partial}{\partial y} = R = \epsilon \frac{\partial}{\partial y'} . \quad (10a)$$

Using the multiple time scales notation, we define the following

$$\frac{\partial}{\partial x_0} = -ik_x, \quad \frac{\partial}{\partial x_1} = -v_g \frac{\partial}{\partial z'}, \quad \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x'} \quad (8b)$$

$$\frac{\partial}{\partial z_0} = ik_z, \quad \frac{\partial}{\partial z_1} = \frac{\partial}{\partial z'}, \quad \frac{\partial}{\partial z_2} = 0 \quad (9b)$$

$$\frac{\partial}{\partial y_0} = 0, \quad \frac{\partial}{\partial y_1} = \frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial y_2} = 0. \quad (10b)$$

Before continuing, we want to simplify the nonlinear terms of the dispersion relation (2.3.24) into a more manageable form. Since these terms are perturbations, the derivatives of ϕ in equation (1) just bring down a constant (ik_z or $-ik_x$) from the exponential. Therefore,

$$\frac{1}{4\pi} A_0 \frac{\partial}{\partial x} \left[\frac{|\nabla\phi|^2}{n_0 T_S} \frac{\partial}{\partial x} \phi \right] + \frac{1}{4\pi} B_0 \frac{\partial}{\partial z} \left[\frac{|\nabla\phi|^2}{n_0 T_S} \frac{\partial}{\partial z} \phi \right] = c_1 |\phi|^2 \phi e^{(ik_z z - ik_x x)} \quad (11)$$

where c_1 is a constant. Using this, we can change equation

(2.3.24) into

$$E\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi + \text{nonlinear terms} = 0_1. \quad (12)$$

The E operator above can be further specified by substituting the definitions in equations (8a - 10a) for the derivatives in (2.3.24),

$$E(P, Q, R) = R^2 + \underline{k}_\perp P^2 - |\underline{k}_\parallel| Q^2 + aP^4 + bP^2Q^2 + cQ^4. \quad (13)$$

As a brief summary before progressing into some lengthy algebra, we will repeat the main steps of the perturbation theory. First we assumed a plane wave solution to the dispersion relation. Then we ordered the derivatives (with respect to x , y , and z) according to importance of spatial variation. Finally, we simplified the nonlinear term. Now we must find expressions for each order of the small parameter ϵ . To obtain successively better solutions to the equation in question (2.3.24), solve each equation of order ϵ . But it is important to remember that all higher order solutions must be consistent with lower order solutions. The result of solving the lower order equation will be used to solve the higher order equation. To continue, the P ,

Q, and R terms in equation (13) are to be expanded and orders of ϵ equated.

$$\epsilon^0 \rightarrow \underline{k}_\perp (-k_x^2) - |\underline{k}_\parallel| (-k_z^2) + ak_x^4 + bk_x^2 k_z^2 + ck_z^4 = 0 \quad (14a)$$

or we can generalize the above to,

$$\epsilon^0 \rightarrow E\left(\frac{\partial}{\partial x_0}\right) + E\left(\frac{\partial}{\partial y_0}\right) + E\left(\frac{\partial}{\partial z_0}\right) = 0 \quad (14b)$$

where $\partial/\partial x_i$ is the i -th order term of the expansion of the derivative as defined in equations (8b)-(10b), we evaluate the operator E (equation 13) with the zero order terms of the derivative expansion, (i.e. $P = -ik_x$, $Q = ik_z$, and $R = 0$). Continuing,

$$\epsilon^1 \rightarrow E_P \frac{\partial}{\partial x_1} + E_Q \frac{\partial}{\partial z_1} + E_R \frac{\partial}{\partial y_1} = 0 \quad (15)$$

where E_j refers to the derivative of E with respect to the variable j . For each order of ϵ expansion the zero order expressions are used to evaluate each term. Finally,

$$\epsilon^2 \rightarrow E_P \frac{\partial}{\partial x_2} + E_Q \frac{\partial}{\partial z_2} + E_R \frac{\partial}{\partial y_2} + \frac{1}{2} \left(E_{PP} - 2E_{PQ} \frac{E_P}{E_Q} + E_{QQ} \frac{E_P^2}{E_Q^2} \right) \frac{\partial^2}{\partial x_1^2}$$

$$\begin{aligned}
& + \frac{1}{2} \left[E_{RR} - 2E_{PR} \frac{E_R}{E_P} + E_{PP} \frac{E_R^2}{E_P^2} \right] \frac{\partial^2}{\partial Y_1^2} \\
& + \frac{1}{2} \left[E_{QQ} - 2E_{QR} \frac{E_R}{E_Q} + E_{RR} \frac{E_R^2}{E_Q^2} \right] \frac{\partial^2}{\partial Z_1^2}, \tag{16}
\end{aligned}$$

Let us analyze these equations to extract some physical sense. The zero order term (equation 14a) is just the lower hybrid dispersion relation without the nonlinear terms and without the \hat{y} dependence. This is what we hoped. Our perturbation theory reduces to the original problem. The first order terms give an expression for the group velocity, v_g . The group velocity is an unknown in the second order equation. Therefore, by using the expression for v_g obtained from the first order terms, v_g can be eliminated from the second order equation. Before solving for v_g and substituting it into equation (16), we can eliminate some of the terms in equation (16). First notice that since all terms are evaluated by the zero order expressions only the second derivative with respect to R is nonzero. Since $\partial/\partial x'$ terms are of highest order, any terms with a $\partial/\partial x'$, $(\partial/\partial x_2)$, multiplied by one of the thermal term coefficients (a , b , c , which are higher order dependencies of the linear dispersion equation) can be neglected as higher order. Therefore from equation (16) we get,

$$\frac{\partial^2 \phi}{\partial Y'^2} + \underline{k}_\perp \left(-2ik_x \frac{\partial \phi}{\partial x'} \right) + [(\underline{k}_\parallel - 6a k_x^2 - bk_z^2) v_g^2 - 4b k_x k_z v_g + (-|\underline{k}_\parallel| - bk_x^2 - 6c k_z^2)] \left(\frac{\partial \phi}{\partial z'} \right)^2 = 0 \quad (17)$$

where v_g is solved from equation (16). Solving equation (15) for \underline{k}_\perp , yields

$$\underline{k}_\perp = \frac{|\underline{k}_\parallel| + (ak_x^4 + bk_x^2 k_z^2 + ck_z^4)}{k_x^2} . \quad (18)$$

This is substituted into equation (17). The first two terms of (17) pose no problems, but the last term is extremely complex. We need to simplify this somehow. We have already said that the thermal terms are of high order, so expand this term in equation (17) by incorporating this information. By balancing degrees of smallness, we can obtain the answer. Since this is difficult, we will just state the results, [Ka2],

$$- \frac{3}{k_z^2} (ak_x^4 + bk_x^2 k_z^2 + ck_z^4) \frac{\partial^2 \phi}{\partial z'^2} . \quad (19)$$

Since we have found all of coefficients of the different

terms, adding equations (17), (19), and (11) gives the following,

$$\begin{aligned} \frac{\partial^2}{\partial y'^2} \phi - 2ik_x k_{\perp} \frac{\partial}{\partial x'} \phi - \frac{3}{k_z^2} (ak_x^4 + bk_x^2 k_z^2 + ck_z^4) \frac{\partial^2}{\partial z'^2} \phi \\ + c_1 |\phi|^2 \phi = 0. \end{aligned} \quad (20)$$

Dividing equation (21) by $2/c_1$ and rescaling the parameters by

$$\eta = -\frac{c_1}{2} y', \quad \tau = -\frac{c_1}{k_x k_{\perp}} x', \quad (21)$$

$$\xi = -\frac{2k_z^2 c_1}{3} \frac{1}{(ak_x^4 + bk_x^2 k_z^2 + ck_z^4)}, \quad (22)$$

yields the following,

$$-\phi_{\eta\eta} + i\phi_{\tau} + \phi_{\xi\xi} + 2|\phi|^2 \phi = 0. \quad (23)$$

Therefore, the three dimensional nonlinear lower hybrid dispersion relation reduces by a multiple time scales approach to the

three dimensional nonlinear Schrodinger equation. Note that in the equation there are two second derivatives. These are of opposite signs so non-Laplacian. This makes for interesting behavior.

Chapter Three

The Nonlinear Schrodinger Equation

3.1 The Two Dimensional Equation

In the previous chapter, the three dimensional nonlinear Schrodinger equation was obtained from a multiple time scales reduction of the lower hybrid dispersion relation. In this chapter, we would like to find a way of solving this equation. Unfortunately, there are no known solutions. As a first step in looking at the three dimensional solutions, we will investigate the behavior of the known two dimensional solutions in three dimensions. Therefore, we resort to perturbation theory. This is reasonable because throughout the derivation of the dispersion relation the third dimension has been considered as a perturbation. We need to find the solutions of the two dimensional equation and perturb it to obtain a solution for the three dimensional problem. We will just consider one specific group of two dimensional solutions. These are solitons.

Solitons are pulse shaped nonlinear waves that asymptotically keep their shape and velocity after colliding with each other. When two solitons collide, they emerge only shifted by a phase factor. In other words, solitons do not destroy each other upon collision. Solitons are found in some systems where

nonlinearity balances dispersion. Asymptotically, a solution to such a system will consist of N (where N is an integer) solitons and a small background radiation. This occurs no matter what the initial conditions are.

3.2 The Soliton Solutions

In the previous section, we discussed the properties of solitons. Now we want to find the soliton solution for the two dimensional nonlinear Schrodinger equation so that it can be used to find a solution to the three dimensional equation. Beginning with the Schrodinger equation,

$$i\phi_t + \phi_{xx} + 2|\phi|^2\phi = 0. \quad (1)$$

Assume a solution of the form,

$$\phi = A(x, t) e^{i\phi(x, t)}. \quad (2)$$

Substituting equation (2) into equation (1) and separating the real and imaginary parts, respectively the equations are

$$-A\phi_t + A_{xx} - A(\phi_x)^2 + 2A^3 = 0, \quad (3)$$

$$A_t + 2A_x\phi_x + A\phi_{xx} = 0. \quad (4)$$

Using the traveling wave assumption ($\xi = x - ut$) in equation (4) the equation becomes

$$\frac{A_{\xi}}{A} = - \frac{\phi_{\xi\xi}}{2\phi_{\xi} - u} . \quad (5)$$

Integrating, we obtain

$$A = c_2(u - 2\phi_{\xi})^{-1/2} \quad (6)$$

where c_2 is a constant of integration. Substituting this result into equation (3) results in a differential equation,

$$A_{\xi\xi} + \frac{u^2 A}{4} - \frac{1}{4c_2^2 A} + 2A^3 = 0. \quad (7)$$

To integrate this equation multiply both sides by A_{ξ} and letting

$$f(\phi) = \frac{1}{4c_2^2 A^3} - \frac{u^2 A}{4} - 2A^3, \quad (8)$$

gives the expression

$$\frac{1}{2} (A_{\xi})^2 = \int f(\phi) dA. \quad (9)$$

Let $g(A) = \int f(\phi) dA$ and the expression for A_{ξ} is

$$A_{\xi} = \sqrt{g(A)}. \quad (10)$$

Rearranging equation (10), yields

$$\xi - \xi_0 = \int \frac{dA}{\sqrt{g(A)}}. \quad (11)$$

If the constants of integration are properly chosen, the soliton is

$$\xi - \xi_0 = \operatorname{sech}(A).$$

The technique used above only gives one soliton solution, yet there are N such solutions. To obtain the N soliton expression, we can use the inverse scattering method. The general method will be outlined to give an idea of the procedure involved. We begin with some general nonlinear equation,

$$\phi_t = N[\phi(x, t)] \quad (12)$$

where N is some general nonlinear operator. If there is operator L such that

$$L\psi = \lambda\psi \quad (13)$$

and an operator B such that

$$\psi_t = B\psi \quad (14)$$

that satisfies the following relationship

$$L_t = BL - LB \quad (15)$$

then we can use L and B to find the N soliton solutions. Basically we start with some initial conditions at $t = 0$ and find the scattering data (reflection and transmission coefficients) for $t = 0$. Equation (14) gives the time evolution of this scattering data for $x = \infty$. Using equation (13) we can find $\phi(x, t)$. To do this inversion, we must use the Gelfand-Levitan equation,

$$\begin{aligned} \hat{g}_1(x, y, t) + k(x + y, t) + \int_x^\infty k(y + y', t) \hat{g}_1(x, y', t) dy' \\ = 0 \end{aligned} \quad (16)$$

where

$$k(x + y, t) = \hat{R}(x + y) + \sum_{n=1}^N m_n e^{-\kappa_n(x + y)}, \quad (17)$$

$$\hat{R}(x + y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k) e^{ik(x + y)} dk \quad (\text{The Fourier Transform}),$$
(18)

$$\phi(x) = -2 \frac{d}{dx} \hat{g}_1(y, x).$$
(19)

The κ_n 's are the eigenvalues of the scattering problem defined by equation (13). This procedure will yield the N soliton solutions for the differential equation specified in equation (12).

3.3 The Derivation of the Coupled Equations

In the previous section, we discussed the soliton solution of the two dimensional nonlinear Schrodinger equation. We did not obtain a general solution to the two dimensional problem. However, we can use the specific solution (i.e. the soliton solution) with a perturbation to help us understand the properties of three dimensional equation. Following the perturbation technique used by Schmidt, [Scl], we will obtain a set of coupled second order differential equations. Beginning with the three dimensional nonlinear Schrodinger equation (equation 2.4.23),

$$i\phi_{\tau} + \phi_{\xi\xi} - \phi_{\eta\eta} + 2|\phi|^2\phi = 0. \quad (1)$$

We assume a solution of the form,

$$\phi = \phi_0(\xi, \tau) + \phi_1(\xi, \tau) \sum_N (A_N \sin Nk_{\eta}\eta + B_N \cos Nk_{\eta}\eta), \quad (2)$$

where $\phi_0(\xi, \tau)$ is the solution to the two dimensional equation,

$$i\phi_{0\tau} + \phi_{0\xi\xi} + 2|\phi_0|^2\phi_0 = 0. \quad (3)$$

The second term on the right hand side of equation (2) is the perturbation where we have assumed a general waveform described

by a Fourier series with k_n as the excitation wavenumber, and A_N and B_N as the Fourier coefficients. Thus, we have a general perturbation which is assumed to be small compared to $\phi_0(\xi, \tau)$.

Recalling from the previous section, the two dimensional soliton solution is

$$\phi_0(\xi, \tau) = A \operatorname{sech}(As) \exp \left[i \frac{1}{2} u_e (\xi - u_c \tau) \right] \quad (4)$$

with $s = \xi - u_e \tau$ and $A = (1/2)(u_e^2 - 2u_e u_c)^{1/2}$. Now, we substitute equation (2) into equation (1) and we note the last term on the left hand side is nonlinear,

$$2|\phi_0 + \phi_1 \Sigma[\]|^2 (\phi_0 + \phi_1 \Sigma[\]), \quad (5)$$

where $[\] = A_N \sin Nk_n \eta + B_N \cos Nk_n \eta$. Using

$$|\phi|^2 = \phi \phi^*,$$

we expand equation (5) and linearize by eliminating all terms in high powers of ϕ_1 . To get an equation describing the perturbation (ϕ_1), we disregard the terms only dependent on ϕ_0 . This equation for the perturbation is

$$i\phi_{1\tau} + \phi_{1\xi\xi} + \sum_N N^2 k_n^2 \phi_1 + 4|\phi_0|^2 \phi_1 + 2\phi_0^2 \phi_1^* = 0. \quad (6)$$

The $\sum_N N^2 k_n^2 \phi_1$ term results from taking the second derivative of ϕ_1 with respect to η .

Doing a general stability analysis, we assume a solution to the perturbation equation (equation 6) of the form,

$$\phi_1 = [f(AS) + ig(AS)] \exp\left[i \frac{1}{2} u_e (\xi - u_c \tau) + \gamma \tau\right], \quad (7)$$

where f , g , and γ are real and A and S are defined as before. Substituting equation (7) into equation (6) and using a change of variables, $z = AS$ and $t = \tau$, the following equation in f and g results,

$$\begin{aligned} A^2 f_{zz} + 6A^2 \operatorname{sech}^2(z) f + \sum_N N^2 k_n^2 f - A^2 f + i\gamma f + iA^2 g_{zz} \\ + 2iA^2 \operatorname{sech}^2(z) g + i \sum_N N^2 k_n^2 g - iA^2 g - \gamma g = 0. \end{aligned} \quad (8)$$

Separating the real and imaginary parts of equation (8) and redefining

$$k^2 = \frac{k^2 \eta}{A^2} \quad \text{and} \quad \Gamma = \frac{\gamma}{A^2}, \quad (9)$$

results in the following coupled set of equations,

$$f_{zz} + 6 \operatorname{sech}^2(z)f - (1 - N^2k^2)f = \Gamma g \quad (10)$$

$$g_{zz} + 2 \operatorname{sech}^2(z)g - (1 - N^2k^2)g = -\Gamma f. \quad (11)$$

Chapter Four
The Computer Solution

4.1 The method of solution

This section is devoted to explaining the numerical integration method used to obtain the data given in Section 4.2. Finally Section 4.3 is devoted to discussing the meaning of the data and the accuracy of the program.

Since the $\text{sech}^2(z)$ functions are assumed to be slowly varying, it can be approximated by a series of line segments. We solve for each line segment and match these solutions. One method which achieves this is the predictor--corrector integration scheme, [Acl]. We will describe this technique for a first order differential equation. This is sufficient since one can easily generalize to higher orders by expressing the higher order differential equation as a system of first order equations. We have two functions; one is the derivative; the other is the solution we are seeking. What we do is fit a polynomial to the last three points of the derivative. These points are evenly spaced by some spacing, H , which is the step size. A parabola is used to give a much better approximation than a line. We extrapolate this parabola and integrate using the following,

$$y_1 - y_0 = \int_{t_0}^{t_1} \frac{dy}{dt} dt = \int_{t_0}^{t_1} dy. \quad (1)$$

For a parabola equation (1) becomes,

$$y_{1p} = y_{-3} + \frac{4H}{3} (2y'_{-2} - y'_{-1} + 2y'_0). \quad (2)$$

The p stands for predicted. This gives us a guess at the next value of our solution. We substitute this predicted value back into the differential equation to obtain y'_{1p} . Now repeat the process and use y'_{1p} for fitting the derivative points. This time there is no need to extrapolate. We are looking for y_{1c} (c stands for corrected) and we know the value of the derivative at that point. So we use

$$y_{1c} = y_{-1} + \frac{H}{3} (y'_{-1} + 4y'_0 + v'_{1p}). \quad (3)$$

As the final step, substitute y_{1c} back into the differential equation to obtain y'_{1c} . Then proceed using the above method to get the next value and so on.

The problem with the predictor-corrector method is that we assume we have three points to fit the parabola to. What we need is some other method to give us these initial points.

The technique used was the Bulirsch and Stoer integration, [Acl]. Here we use a line extrapolation process to obtain the first three points. To get the first point use

$$y_1 = y_0 + H y_0' . \quad (4)$$

Note that y_0 and y_0' are the conditions to be specified. Equation (4) is just a description of finding a slope, as is shown below. We have

$$y_1 - y_0 = H y_0' = H \frac{dy_0}{dt} , \quad (5)$$

and since $H = dt$, we really have

$$dy_0 = y_1 - y_0 \quad (6)$$

which is the slope. For the next two steps we use one point as the starting point and use the slope at the second point to get a value for the third point that is distance H away. The equation used is

$$y_N = y_{N-2} + 2H y_{N-1}' . \quad (7)$$

To find the values $y'_N - 1$, we merely use the answer from the integration before and substitute it into the differential equation. Therefore, we have a method for integration a differential equation.

The coupled set of equations involve boundary conditions determined by the starting values. How can we use the method outlined above to solve this boundary value problem? What we do is guess the starting conditions and perform the integration to see if we obtain a valid answer. For example, if at $x = 0$ we have the function $y = 1$ and at $x = 1$ we have $y = 2$ then we must pick an initial slope y' . With this estimate we integrate until we get to $x = 1$. Here we check to see if $y = 1$. If it does we have a solution; if not then we must guess another value for y' . If there is only one boundary conditions, then we need to have some physical idea of what should happen at infinity. With our problem we know that none of the solutions grow at infinity; they must decay to zero or oscillate.

The problem has an added difficulty. Because the $\text{sech}^2(z)$ potential wells extend from minus to plus infinity, the boundary conditions must be satisfied at infinity. What is infinity? A value must be picked, but we must specify the values for the functions f and g and the slopes of f and g such that to the left of minus infinity the solution decays to zero or oscillates. In general these conditions are found by assuming

zero for the value of the $\text{sech}^2(z)$ function and solving for B^2 and setting $e^{-\alpha z}$ (where α is real part of B) to zero, thus preventing the function from blowing up at minus infinity. Then we can solve for the boundary conditions. See Appendix C.

Until now there has been no discussion of how to handle the coupling between the two equations. The coupled part of the equations is treated as any other part. An example will explain. Solving for f_0 , g_0 , f_0' , and g_0' (the subscript shows the order of the value and the ' means derivative), we integrate to find the next value of f (i.e. f_1) using f_0 , f_0' and g_0 times the coupling coefficient. With these we integrate using the predictor-corrector method discussed above, thus finding f_1 . We find g_1 by using g_0 , g_0' , and f_0 . Now the process continues in the same manner. To find f_2 we substitute the previously obtained values into equations (7) and use g_1 times the coupling coefficient for that part of the equation. After we find the first three points, we use equations (2) and (3) as described above to find the rest of the function values.

Upon finishing the integration, we must decide if the value chosen for the coupling coefficient is correct or not. The solution should either begin at zero and go to zero at infinity, or should oscillate to zero at both plus and minus infinity, or should be a completely oscillatory function. If the solution

obtained from the integration does not fit one of the above, then we must refine the value of the coupling coefficient. Using this technique, we can obtain a graph of Γ vs. k^2 . This is shown in the next section.

4.2 The Data

Table 1 below gives the numbers obtained by using the method discussed in the previous section.

<u>k</u>	<u>k²</u>	<u>Γ</u>
0	0	0
.1	.01	.1
.2	.04	.2
.3	.09	.35
.35	.1225	.41
.4	.16	.37
.45	.2025	.3
.5	.25	.27
.6	.36	.23
.7	.49	.2
.8	.64	.16
.9	.81	.11
.95	.9025	.05
1.0	1.0	0
1.1	1.21	0
>1.1	>1.21	0

Table 1

Figure 1 shows the f function for $k^2 = 0$ and $\Gamma = 0$ and Figure 2 shows the g function under the same conditions for k^2 and Γ . Figure 3 is a plot of the values given in Table 1.

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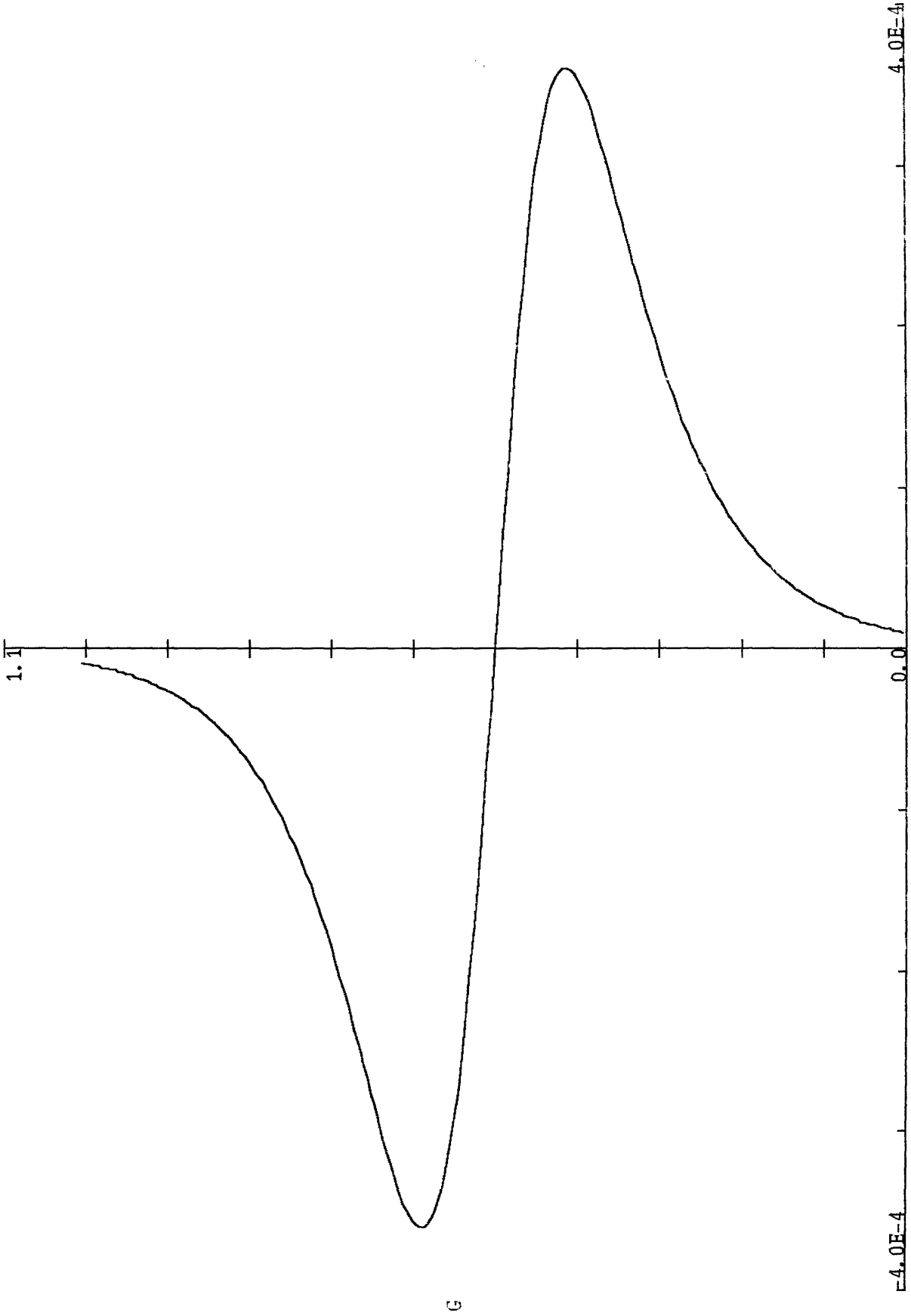
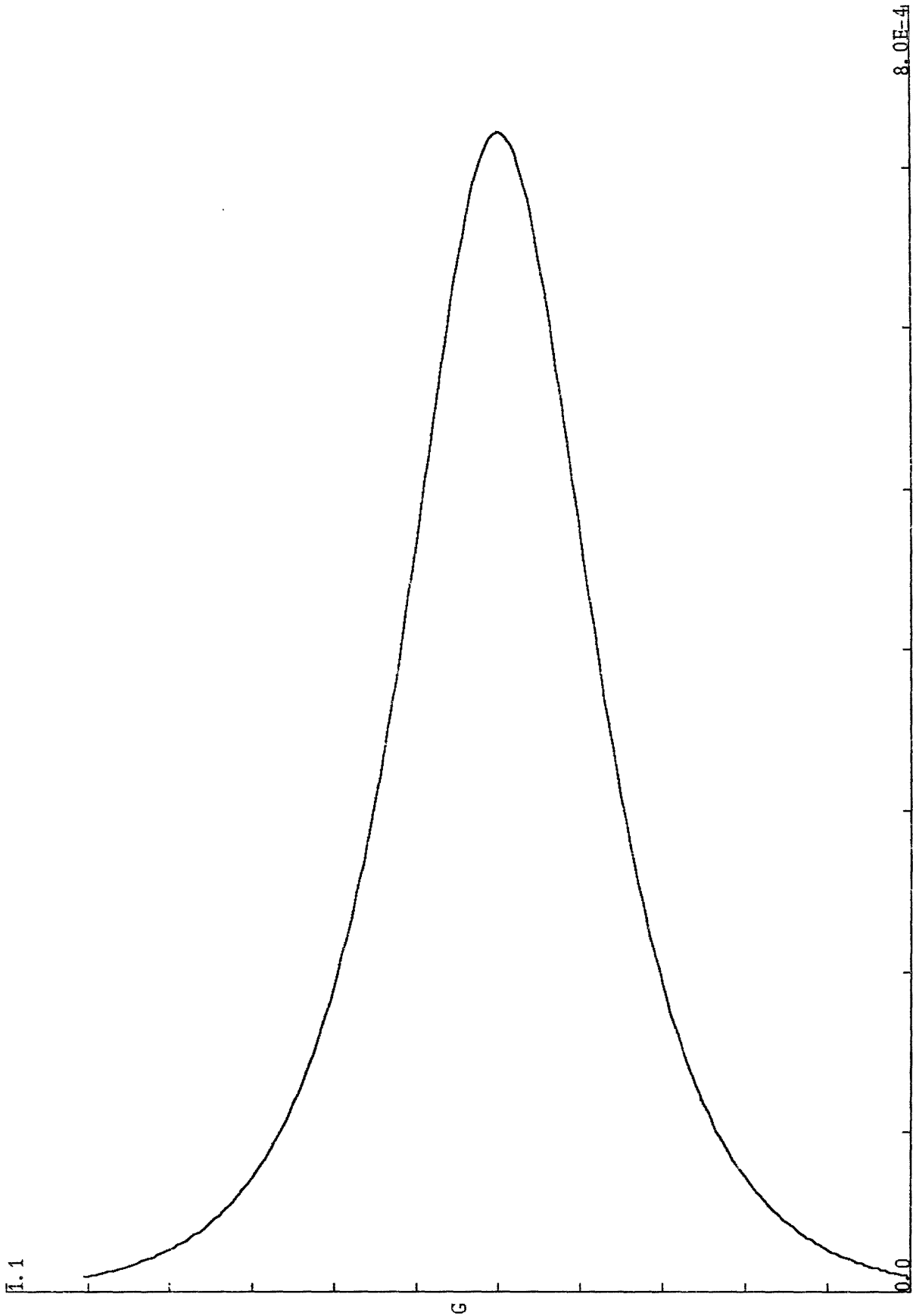


FIGURE 1 : f function K² = 0 $\Gamma = 0$

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$X_{min} = 0.0$ $X_{max} = 8.0E-4$ $Y_{min} = 0.0$ $Y_{max} = 1.1$

FIGURE 2: g function $K_2 = 0$ $\Gamma = 0$

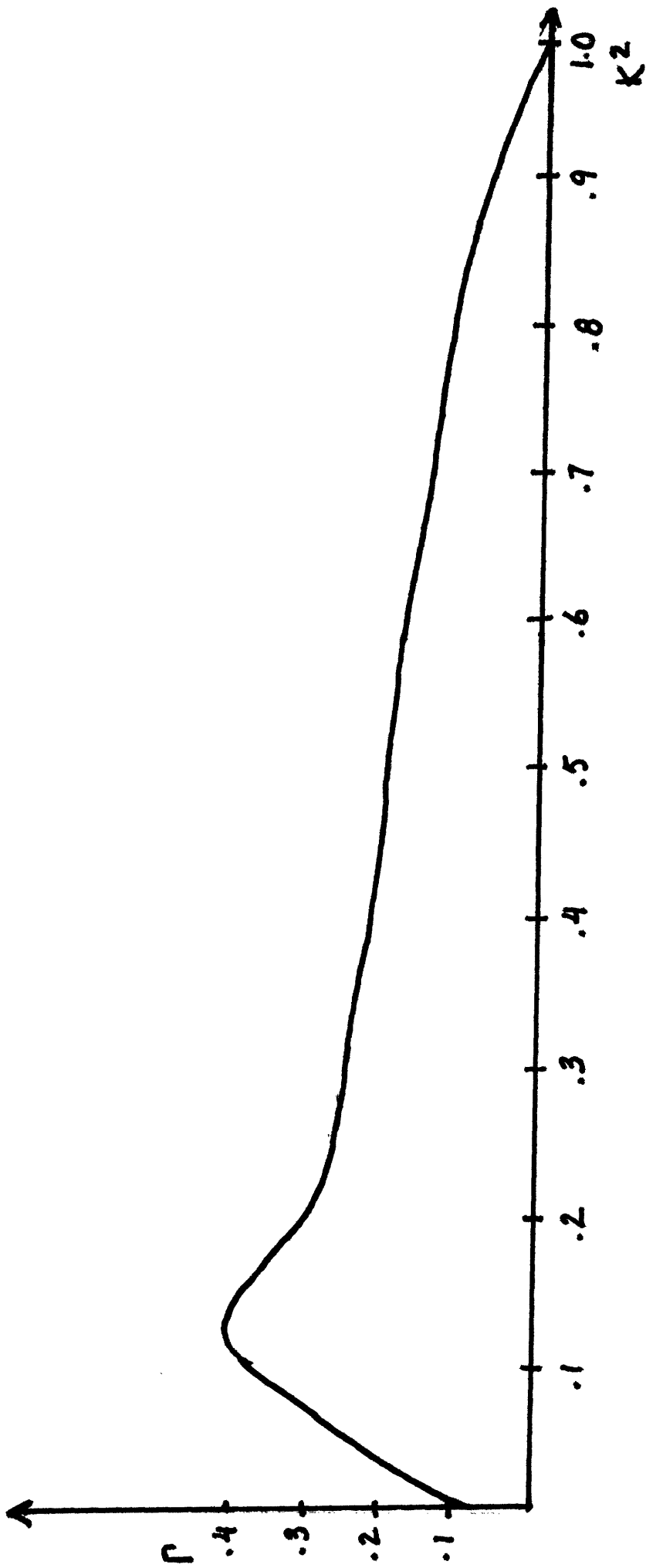


TABLE 1

ρ vs. K^2

4.3 The Meaning of the Results

The important result from the numerical integration is that there is a continuous spectrum for $k^2 > 1$. For this region Γ equals zero. So it seems that there could be stable waves for these corresponding wavenumbers. For the discrete region ($0 < k^2 < 1$), we note that there is only one growth rate. Therefore, if the excitation pulse is composed of only one spatial frequency, it will grow through space and other nonlinear effects would have to be considered. If the excitation pulse is composed of many spatial frequencies, it is hard to say what will occur. The different frequencies could interact to produce some sort of stable pulse. This is a question that needs to be considered in further research.

At this point it is appropriate to say something about the compute program, (see Appendix D). The program was written on MACSYMA, a system developed by the MATHLAB at MIT. As far as the uncoupled second order differential equation is concerned, the program worked quite well. The integration scheme was checked against known solutions of Schrodinger well problems and gave reliable results. Unfortunately, some problems developed when a fourth order equation was tried. The specification of the boundary conditions at minus infinity had to be very accurate to guarantee that the solution decayed at plus

infinity. To obtain accurate values of Γ was hard due to some instabilities inherent in the technique used. These problems did not manifest themselves in the second order equation. It seems that coupling the two second order equations together intensified the instability making extremely accurate results difficult to obtain. However, the results obtained are reasonable. This is discussed in the next chapter.

Chapter Five

Predictions for Solutions of the Coupled Equations

5.1 The Schrodinger Well Analogy

In the previous chapter, we obtained values for the coupling coefficient versus the wavenumbers (k) by numerically integrating the coupled set of equations. But how can we be sure the values are correct? This chapter is devoted to predicting what the results should be by using some approximation techniques. Also we will be able to gain a better physical understanding of the problem and the results. In Section 5.1, we compare the problem to that of a Schrodinger well which gives a physical meaning to the $\Gamma = 0$ solutions. Section 5.2 deals with finding the bounds on the coupling coefficient. The last section discusses the behavior of the solution at infinity and gives a general shape for the Γ vs k^2 curve.

To relate our problem to the Schrodinger well problem, let $N = 1$ and the coupling coefficient in the coupled equations equal zero; we then have

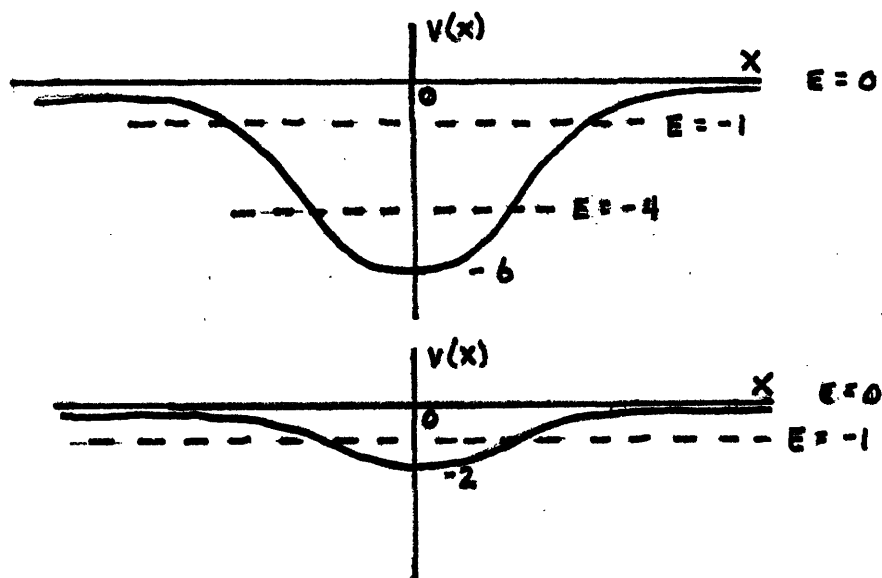
$$f_{zz} + [6 \operatorname{sech}^2(z) - (1 - k^2)]f = 0 \quad (1)$$

$$g_{zz} + [2 \operatorname{sech}^2(z) - (1 - k^2)]g = 0. \quad (2)$$

Comparing equations (1) and (2) with the general form of the time independent Schrodinger equation,

$$\phi_{xx} - (v - E)\phi = 0, \quad (3)$$

we realize that we have two potential well problems just by redefining some terms. Let the potential $v = -6 \operatorname{sech}^2(z)$ or $-2 \operatorname{sech}^2(z)$ and the energy (or eigenvalues) $E = -(1 - k^2)$. Using Landau and Lifshitz, [Lal], the discrete eigenvalues for $v = -6 \operatorname{sech}^2(z)$ are $E = -4$ and $E = -1$. For $v = -2 \operatorname{sech}^2(z)$, $E = -1$ is the only eigenvalue. Also for $E > 0$ the spectrum is continuous. Pictorially,



To find the analytic expressions for the eigenfunctions at

$E = -1$ and $E = -4$ for the above pictured potential wells, we refer back to Chapter 3. In deriving the coupled set of equations, we assumed the solution of the equation describing the perturbation to have a real and an imaginary part. This resulted in the two wells. We can treat each well separately to obtain an expression for the eigenfunctions. First considering the real function (f) , we let $k = 0$ to obtain $E = -1$. From equation (3.3.6) the equation for the perturbation is

$$i\phi_{1\tau} + \phi_{1\xi\xi} + 6|\phi_0|^2\phi_1 = 0 \quad (4)$$

for ϕ_1 real. ϕ_0 is the soliton solution of the two dimensional equation,

$$i\phi_{0\tau} + \phi_{0\xi\xi} + 2|\phi_0|^2\phi_0 = 0. \quad (5)$$

If we take the derivative of equation (5) with respect to ϵ ,

$$i(\phi_{0\xi})_{\xi} + (\phi_{0\xi})_{\xi\xi} + 6|\phi_0|^2\phi_{0\xi} = 0, \quad (6)$$

we realize that $\phi_{0\xi}$ satisfies the perturbation equation (4).

Therefore, the derivative of the soliton solution for the two dimensional equation is the $k^2 = 0$ solution of the real part of the coupled equations. The shape of this curve is shown in Figure 1.

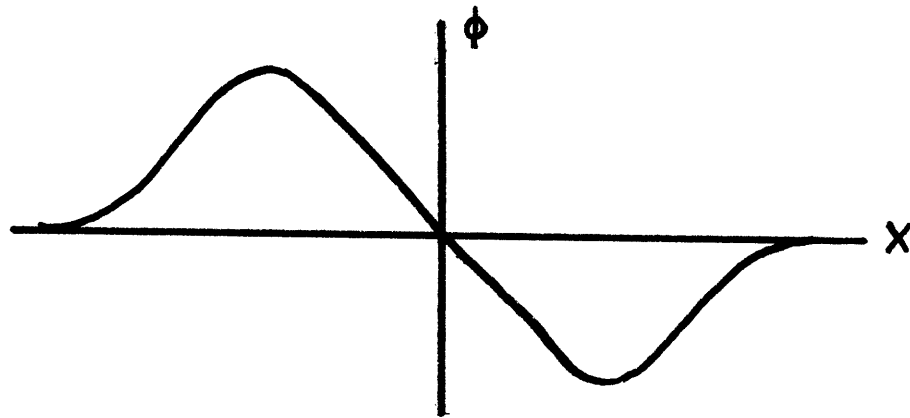
The same procedure can be used to solve for the eigenfunction of the imaginary equation (g). Again let $k^2 = 0$, thus $E = -1$. For an imaginary function the perturbation equation is

$$i\phi_{1\tau} + \phi_{1\xi\xi} + 2|\phi_0|^2\phi_1 = 0. \quad (7)$$

Comparing equation (7) with equation (5), we realize that ϕ_0 , the two dimensional soliton solution, is a solution to the perturbation equation. This function is shown in Figure 2. From the properties of self-adjoint equations, the lower eigenfunction for the $6 \operatorname{sech}^2(z)$ potential at $E = -4$ has no zero crossings and is a pulse shape but not necessarily $\operatorname{sech}(z)$.

Comparing the above analysis with the results obtained by the numerical integration in Figures 4.1 and 4.2, we notice that the graphs are the same. For $k^2 = 0$ and $\Gamma = 0$ the coupled equations reduce to two uncoupled Schrodinger well problems where the value for E (which equals $-(1 - k^2)$) corresponds to an allowable eigenvalue for both potential wells. In other words, for both wells at $k^2 = 0$ there are valid solutions without the need of a coupling coefficient. As we

increase E , we no longer have valid solutions for $\Gamma = 0$. This is completely analogous to obtaining solutions for a Schrodinger well.



FOR $U_0 = 6$ $E = 1$

FIGURE ONE: THE DERIVATIVE OF THE SOLITON SOLUTION.

FOR $U_0 = 2$ $E = 1$

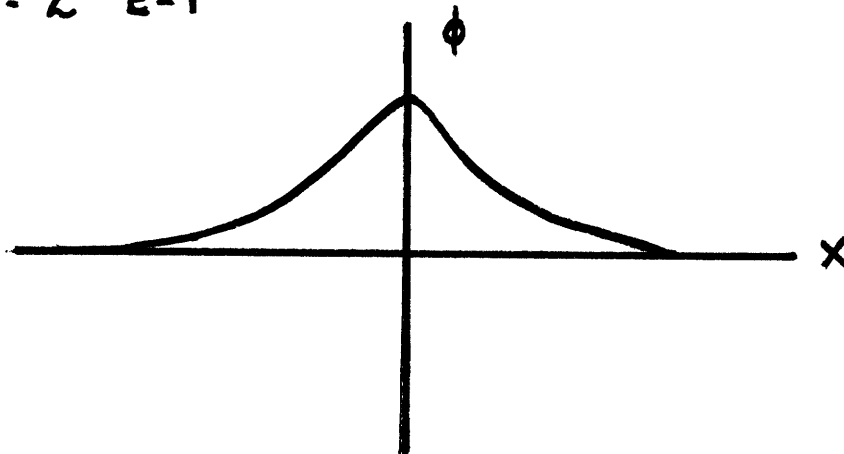


FIGURE TWO: THE SOLITON SOLUTION.

5.2 The Bounds on the Coupling Coefficient

Do the values for the coupling coefficient obtained in Chapter Four make sense? Assuming the coupling coefficient to be real, we can obtain its upper bound. Recalling the set of equations (equations 3.3.10 and 3.3.11 with $N = 1$)

$$f_{zz} + 6 \operatorname{sech}^2(z)f - (1 - k^2)f = \Gamma g \quad (1)$$

$$g_{zz} + 2 \operatorname{sech}^2(z)g - (1 - k^2)g = -\Gamma f. \quad (2)$$

Assuming a solution,

$$f \sim e^{Bz}, \quad g \sim e^{Bz}, \quad (3)$$

and substituting into equations (1) and (2), yields

$$f(B^2 + 6 \operatorname{sech}^2(z) - (1 - k^2)) = \Gamma g \quad (4)$$

$$g(B^2 + 2 \operatorname{sech}^2(z) - (1 - k^2)) = -\Gamma f. \quad (5)$$

Let $E = -(1 - k^2)$ and let $B^2 = 0$ for turning point solutions.

Taking the determinant, the set of equations becomes

$$(6 \operatorname{sech}^2(z) + E)(2 \operatorname{sech}^2(z) + E) = -\Gamma^2. \quad (6)$$

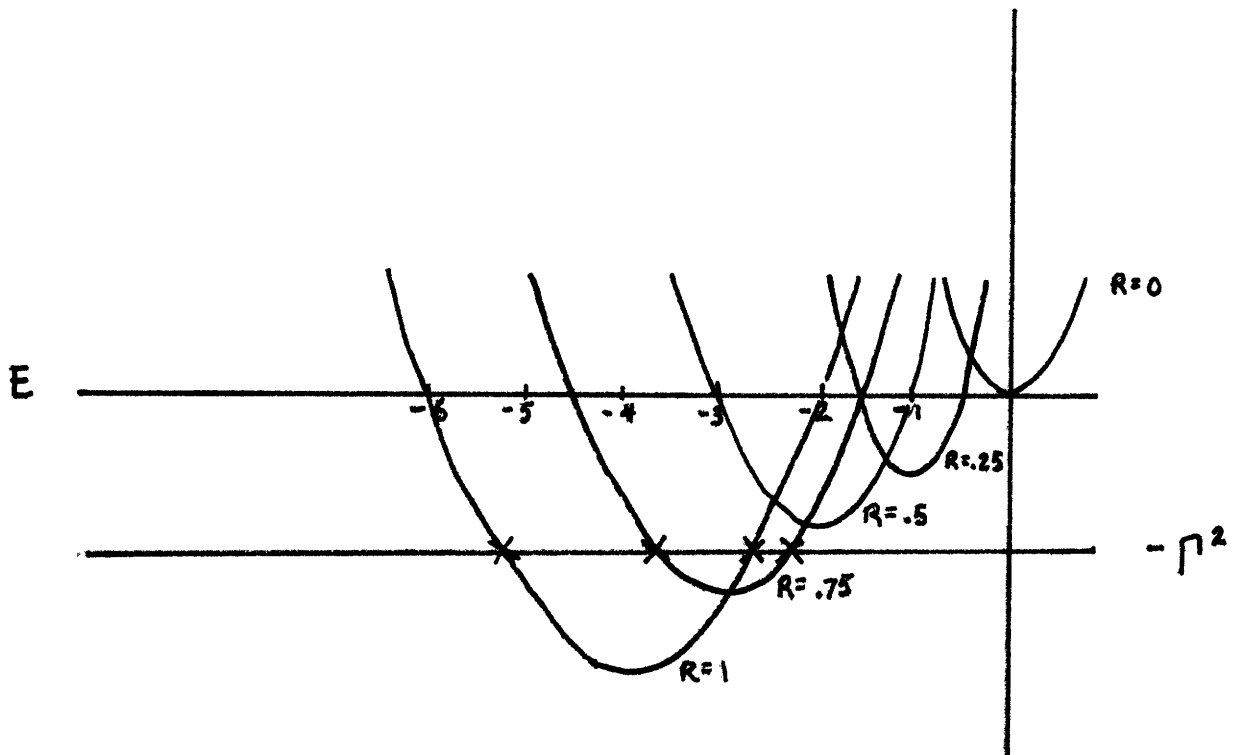
We can assume $\operatorname{sech}(z)$ varies slowly such that we can define R as a slowly varying function with values between zero and one. Substituting in R ,

$$(6R + E)(2R + E) = -\Gamma^2. \quad (7)$$

The value of E (the eigenvalue) and where E intersects the $\operatorname{sech}(z)$ well (the value of R) determines what Γ is required to yield a valid solution. The solution must oscillate or decay to zero at both infinities. Therefore, we can vary R to observe the effect it has on the values for the coupling coefficient.

We can use graphical methods to study equation (7). Allowing R to vary between zero and one in steps of .25, we can plot each side of equation (3). The left hand side yields parabolas as shown in the following figure; the x 's mark the places where $B^2 = 0$. Note that $-\Gamma^2 \leq 0$ to assure that Γ is real as we assumed earlier. As is shown, E can vary between one and minus six,

$$-6 \leq E \leq 0. \quad (8)$$



In Section 5.1, we found that at $E = -1$ there exist two valid solutions with $\Gamma = 0$. Since this value of E is the smallest E for which this is true, we will begin by letting $E = -1$ in equation (7).

The R giving the maximum coupling coefficient is found from equation (7) to be

$$R = 1/3. \tag{9}$$

Using this value for R , the E giving the maximum coupling is,

$$E = -4/3. \tag{10}$$

With $R = 1/3$ and $E = -4/3$, equation (7) gives a value for Γ ,

$$\Gamma = 2/3. \tag{11}$$

This is the maximum coupling coefficient. As one realizes from Chapter Four, the coupling coefficient is always smaller than two-thirds. The data agrees with the above analysis.

5.3 Behavior at Infinity

By assuming that we are observing the solution to the coupled equations far from the potential wells, we can obtain an understanding of what type of spectrum (i.e. continuous or discrete) results from a given value of k^2 . Beginning with the coupled equations (equations 3.3.10 and 3.3.11) and assuming $N = 1$, we have

$$f_{zz} + 6 \operatorname{sech}^2(z)f - (1 - k^2)f = \Gamma g \quad (1)$$

$$g_{zz} + 2 \operatorname{sech}^2(z)g - (1 - k^2)g = -\Gamma f. \quad (2)$$

As we let $z \rightarrow \pm\infty$, we realize that $\operatorname{sech}^2(z) \rightarrow 0$. Therefore, we can write the following,

$$f_{zz} - (1 - k^2)f = \Gamma g \quad (3)$$

$$g_{zz} - (1 - k^2)g = -\Gamma f. \quad (4)$$

Assuming the usual e^{Bz} dependence for f and g , we get

$$f[B^2 - (1 - k^2)] = \Gamma g \quad (5)$$

$$g[B^2 - (1 - k^2)] = -\Gamma f. \quad (6)$$

Eliminating f and g and solving for B^2 yields,

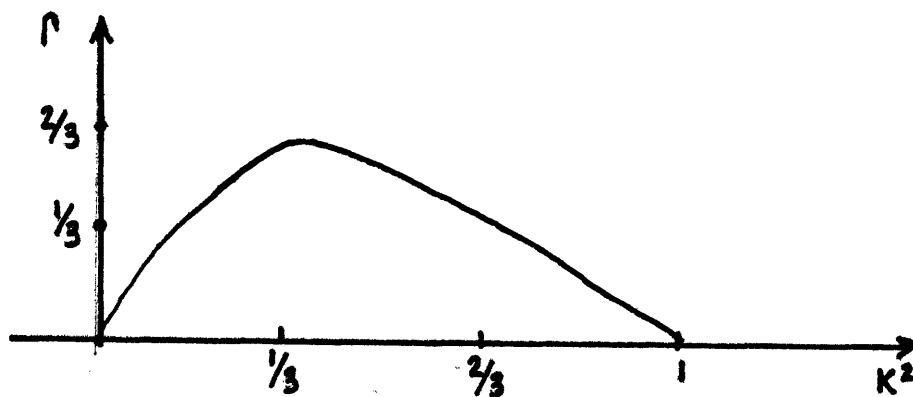
$$B^2 = 1 - k^2 + i\Gamma. \quad (7)$$

We have obtained an equation describing the behavior of the solutions to the coupled set of equations far from the effects of the potential wells. Moving in from infinity, these potentials will greatly influence the solution. However, from the definition of the problem, we can not have exponentially growing functions at infinity. The solutions must decay or oscillator to be valid. We can immediately see what happens at infinity for a given k^2 by looking at equation (7). If k^2 is greater than one, there is a negative real part which results in an oscillatory function. Γ is zero since if it is not it will contribute some exponential part which is not wanted. The function is already valid. Therefore, we can conclude that for $k^2 > 1$ the spectrum is continuous and any value of $k^2 > 1$ is permissible; also $\Gamma = 0$. The solutions are completely oscillatory.

For the range of k^2 between zero and one, the real part of equation (7) is positive; hence we have an exponential part to our solution which must be balanced by a suitable coupling coefficient. We can interpret these results in a more physical

physical manner by considering the Schrodinger well problem. Letting $E = -(1 - k^2)$, we note that for $k^2 = 1$, $E = 0$. This is the transition between the discrete and the continuous spectrum. This is just the conclusion we obtained above. What occurs at $k^2 = 0$? In this case $E = -1$. But we remember from Section 5.1 that for both wells (depth six and two) $E = -1$ was a valid eigenvalue. There is no need for a coupling coefficient. We have covered all the k^2 from $k^2 = 0$ to infinity by going from $E = -1$ to $E = \infty$. With $\Gamma \neq 0$, we can get valid solutions for the energies between $E = -1$ and $E = 0$. Although these are not valid solutions with $\Gamma = 0$.

From the three sections in this chapter, we can construct a graph of Γ vs k^2 . The solutions at $k^2 = 0$ are known; the maximum Γ is $2/3$ at $k^2 = 1/3$; the spectrum is continuous for $k^2 > 0$. Therefore, the plot of Γ vs k^2 is



The above conclusions agree with the results of the numerical integration in Chapter Four. The graph above looks very similar to that in the previous chapter. Overall the numerical integration scheme has given answers that are predictable by the techniques used above.

Chapter Six
Conclusions and Suggestions

What we have done in this thesis is show the development of a model for the lower hybrid plasma wave in three dimensions. Most of this had been previously done by others, but we have shown the complete details of each step to give the reader a much better understanding of the physics involved and where the equations come from. We have used some important and powerful perturbation techniques to give us a reasonable equation that we can solve. We have found that in three dimensions there is a range of k 's in the \hat{y} direction resulting in instability.

The question arises as to what kind of instability this is. Do the wave pulses (solitons) grow to some saturation level or do they break up? Are there strange nonlinear effects that cause the development of stable waveforms? It would be good if a nonlinear equation (for example the nonlinear three dimensional Schrodinger equation) could be directly integrated. Recently some questions have arisen concerning the validity of the plane wave assumption used with the multiple time scales reduction of the dispersion relation. These are questions for further research.

APPENDIX A

The Harris Dispersion Relation

A.1 Perturbation of the Vlasov Equation

The derivation of the Harris dispersion relation is a complicated procedure involving some expansions rarely used. We hope to simplify this derivation by pointing out the principles used in the derivation and showing as much of the algebra as is considered reasonable. We begin with the Vlasov equation developed from macroscopic kinetic theory. We assume a velocity distribution function ($f_0(v)$) that is perturbed by a high order distribution function ($f_1(x, v, t)$). The Vlasov equation is

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + \frac{q}{m} (E + v \times \bar{B}) \frac{\partial f}{\partial v}, \quad (\text{in one dimension}). \quad (1)$$

Now we linearize about an equilibrium point by assuming zero order quantities perturbed by first order quantities. This not only applies to the velocity distribution function, but to the electric (E) and magnetic (B) fields as well.

Therefore we have,

$$f = f_0(v) + f_1(v, x, t), \quad E = E_0 + E_1, \quad B = B_0 + B_1. \quad (2)$$

(Note, we assume a magnetized plasma with no dc electric fields.) Substituting equation (2) into the Vlasov equation (equation 1) and separating with respect to orders of f , we get

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} + \frac{q}{m} \frac{\mathbf{v} \times \mathbf{B}_0}{c} \cdot \frac{\partial f_1}{\partial \mathbf{v}} = - \frac{q}{m} \left(\mathbf{E}_1 + \frac{\mathbf{v} \times \mathbf{B}_1}{c} \right) \cdot \frac{\partial f_0}{\partial \mathbf{v}} . \quad (3)$$

To solve this equation, we will use the method of unperturbed orbits. What we do is assume the perturbation is small, small enough such that single particle motion is unaffected by the perturbation in fields. We establish boundary conditions on a set of relations that will simplify the left hand side (L.H.S.) of equation (3). These relations have to be something that will make the L.H.S. of equation (3) into a full derivative. Therefore,

$$\frac{d\mathbf{x}'}{dt} = \mathbf{v}'(t'), \quad \frac{d\mathbf{v}'}{dt} = \frac{q}{m} \frac{\mathbf{v}' \times \mathbf{B}_0}{c} . \quad (4)$$

Equation (3) becomes

$$\frac{df_1}{dt} = - \frac{q}{m} \left(\mathbf{E}_1 + \frac{\mathbf{v}' \times \mathbf{B}_0}{c} \right) \cdot \frac{\partial f_0}{\partial \mathbf{v}} \quad (5)$$

with the use of the following boundary conditions,

$$x'(t' = t) = x, \quad v'(t' = t) = v. \quad (6)$$

Solving equation (5) for the perturbed velocity distribution (f_1), we get,

$$f_1(x, v, t) = -\frac{q}{m} \int_{-\infty}^t dt' \left[E_1(x', t') + \frac{v' \times B_1}{c} \right] \frac{\partial f_0}{\partial v}. \quad (7)$$

We wish to solve equation (7) by first assuming longitudinal waves (the lower hybrid wave is longitudinal) and then evaluating $\partial f_0 / \partial v$ by using the unperturbed orbit equations. In assuming longitudinal waves we have,

$$v \times B_1 = 0 \quad (8)$$

as well as

$$E = -\nabla\phi = -ik\phi. \quad (9)$$

We have taken the Laplace-Fourier transform of $\phi(x, t)$ so we have

$$\phi(k, \omega) e^{i(k \cdot x - \omega t)}. \quad (10)$$

Substituting equation (10) into our equation for $f_1(x, v, t)$ (equation 7) we obtain the following

$$f_1(x, v, t) = \frac{q}{m} \int_{-\infty}^t dt' e^{i(k \cdot x' - \omega t')} ik\phi(k, \omega) \cdot \frac{\partial f_0}{\partial v}. \quad (11)$$

To evaluate the dot products in equation (11) we can use Fig. 1,

$$k \cdot \frac{\partial f_0}{\partial v} = k_{\perp} \frac{v_x'}{v_{\perp}} \frac{\partial f_0}{\partial v_{\perp}} + k_{\parallel} \frac{\partial f_0}{\partial v_{\parallel}}. \quad (12)$$

Now we need expressions for v_x' and x' of equation (11). This is where the equations of unperturbed orbits enter. These equations are derived from the force equation assuming $E = 0$ and B is finite and uniform, [Ch1].

$$v_x' = v_{\perp} \cos(\theta - \Omega T), \quad v_y' = v_{\perp} \sin(\theta - \Omega T),$$

$$v_z = v_{\parallel}. \quad (13)$$

$$x' = x - \frac{v_{\perp}}{\Omega} [\sin(\theta - \Omega T) - \sin \theta],$$

$$y' = y + \frac{v_{\perp}}{\Omega} [\cos(\theta - \Omega T) - \cos \theta]. \quad (14)$$

$$z' = v_{\parallel} T + z, \quad (15)$$

where θ is an angle dependent on initial conditions as shown in Fig. 1 and Ω is the plasma cyclotron frequency (qB/mc), and T is the variable for time. Substituting equations 12-15 into equation (11) and noting the change of limits we have the full equation for the perturbation of the velocity distribution assuming longitudinal waves

$$\begin{aligned}
 f_1 = & \frac{iq}{m} \phi(k, \omega) e^{i(k \cdot x - \omega t)} \int_{-t}^0 dT \left[k_{\perp} \cos(\theta - \Omega T) \frac{\partial f_0}{\partial v_{\perp}} \right. \\
 & \left. + k_{\parallel} \frac{\partial f_0}{\partial v} \right] x \exp i \left[- \frac{k_{\perp} v_{\perp}}{\Omega} (\sin(\theta - \Omega T) - \sin \theta) \right. \\
 & \left. + k_{\parallel} v_{\parallel} T - \omega T \right].
 \end{aligned} \tag{16}$$

A.2 The Expansion of the First Order Velocity Distribution Function

In the previous section (A.1), we developed the expression for the first order distribution function (f_1). Now we want to expand this using the assumption of asymptotic behavior ($t \rightarrow \infty$) and also continue to only consider longitudinal waves. Before continuing, there are important identities which will be used throughout the expansion of the equation, [No1],

$$\exp[\pm iR \sin \theta] = \sum_{N=-\infty}^{\infty} J_N(R) \exp[\pm iN\theta] \quad (1)$$

$$\exp[\pm iR \cos \theta] = \sum_{N=-\infty}^{\infty} J_N(R) \exp[\mp iN\theta] \quad (2)$$

$$R \cos \theta \exp[\pm iR \sin \theta] = \sum J_N(R) N \exp[\pm iN\theta] \quad (3)$$

$$R \sin \theta \exp[\pm iR \cos \theta] = - \sum J_N(R) N \exp[\mp iN\theta] \quad (4)$$

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \exp[iN\theta] = \delta[N], \quad (5)$$

where J_N is the N -th order Bessel function and $\delta[N]$ is the Dirac delta function. Also it should be noted that

$$\int d^3v \dots = \iint 2\pi v_{\perp} dv_{\perp} dv_{\parallel} \int_0^{2\pi} \frac{d\theta}{2\pi} \dots \quad (6)$$

From the properties of a general distribution function, we know

$$n = \int f \, dv^3 \quad (7)$$

where n is the number density. From Poisson's equation with the electrostatic assumption,

$$k^2 \phi = 4\pi\rho = 4\pi \sum_s e_s \int d^3v f_{1s}, \quad (8)$$

where s signifies the species. Substituting equation (A.1.16) into equation (8) using the relation in equation (6), we obtain our dispersion relation,

$$k^2 = 4\pi \sum_s e_s \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2\pi v_{\perp} \, dv_{\perp} \, dv_{\parallel} \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{iq}{m} e^{i(k \cdot x - \omega t)}$$

$$\int_{-\infty}^0 dT \left[k_{\perp} \cos(\theta - \Omega_s T) \cdot \frac{\partial f_0}{\partial v_{\perp}} + k_{\parallel} \frac{\partial f_0}{\partial v_{\parallel}} \right]$$

$$\cdot \exp i \left[- \frac{k_{\perp} v_{\perp}}{\Omega_s} (\sin(\theta - \Omega_s T) - \sin \theta) \right.$$

$$\left. + k_{\parallel} v_{\parallel} T - \omega T \right]. \quad (9)$$

Taking the first term of equation (9),

$$\begin{aligned}
& \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{iq}{m} e^{i(k \cdot x - \omega t)} \int_{-\infty}^0 dT k_{\perp} \cos(\theta - \Omega_s T) \frac{\partial f_0}{\partial v_{\perp}} \\
& \exp i \left[- \frac{k_{\perp} v_{\perp}}{\Omega_s} \cdot (\sin(\theta - \Omega_s T) - \sin \theta) \right. \\
& \left. + k_{\parallel} v_{\parallel} T - \omega T \right], \tag{10}
\end{aligned}$$

and using the identity of equation (1), we obtain the following

$$\begin{aligned}
& \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{iq}{m} e^{i(k \cdot x - \omega t)} \int_{-\infty}^0 dT k_{\perp} \cos(\theta - \Omega_s T) \frac{\partial f_0}{\partial v_{\perp}} \sum_N J_N \left(\frac{k_{\perp} v_{\perp}}{\Omega_s} \right) \\
& \cdot e^{iN\theta} \exp i \left[\frac{k_{\perp} v_{\perp}}{\Omega_s} \sin(\theta - \Omega_s T) + (k_{\parallel} v_{\parallel} - \omega) T \right]. \tag{11}
\end{aligned}$$

Using identity (3) this can be further reduced to

$$\begin{aligned}
& \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{iq}{m} e^{i(k \cdot x - \omega t)} \int_{-\infty}^0 dT \frac{\partial f_0}{\partial v_{\perp}} \frac{\sum_N J_N^2(k_{\perp} v_{\perp} / \Omega_s) N \cdot k_{\perp}}{k_{\parallel} v_{\parallel} / \Omega_s} \\
& \exp i[\Omega_s T N + T \cdot (k_{\parallel} v_{\parallel} - \omega)]. \tag{12}
\end{aligned}$$

Integrating equation (12) we obtain a compact expression, we have

$$\sum_N \frac{k_{\perp} J_N^2(k_{\perp} v_{\perp} / \Omega_s) N}{i \frac{k_{\perp} v_{\perp}}{\Omega_s} (N \Omega_s + k_{\parallel} v_{\parallel} - \omega)} \frac{\partial f_0}{\partial v_{\perp}} . \quad (13)$$

Taking the second term of equation (9),

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \frac{i q}{m} e^{i(k \cdot x - \omega t)} \int_{-\infty}^0 dT k_{\parallel} \frac{\partial f_0}{\partial v_{\parallel}} \exp i \left[- \frac{k_{\perp} v_{\perp}}{\Omega_s} (\sin(\theta - \Omega_s T) - \sin \theta) + k_{\parallel} v_{\parallel} T - \omega T \right] \quad (14)$$

and using identity (1) twice we get,

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \frac{i q}{m} e^{i(k \cdot x - \omega t)} \int_{-\infty}^0 dT k_{\parallel} \sum_N J_N^2 \exp i[\Omega_s T + T(k_{\parallel} v_{\parallel} - \omega)] \cdot \frac{\partial f_0}{\partial v_{\parallel}} . \quad (15)$$

Integrating we obtain

$$\sum_N \frac{J_N^2(k_{\perp} v_{\perp} / \Omega_s) k}{i(k_{\parallel} v_{\parallel} - \omega + \Omega_s N)} \frac{\partial f_0}{\partial v_{\parallel}} . \quad (16)$$

Combining equations (13) and (16) and substituting these results

into equation (9), we have

$$k^2 = \sum_s \omega_{ps}^2 \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{\infty} 2\pi v_{\perp} dv_{\perp} \frac{\sum N J_N^2(k_{\perp} v_{\perp} / \Omega_s)}{(k_{\parallel} v_{\parallel} - \omega + \Omega_s N)} \left[k_{\parallel} \frac{\partial f_{os}}{\partial v_{\parallel}} + \frac{N \Omega_s}{v_{\perp}} \frac{\partial f_{os}}{\partial v_{\perp}} \right], \quad (17)$$

where

$$\omega_{ps}^2 = \frac{4\pi n_s q^2}{m_s}$$

which is the plasma frequency for the associated species. This is the Harris dispersion relation for k_{\perp} in the \hat{x} direction as is shown in Fig. 1.

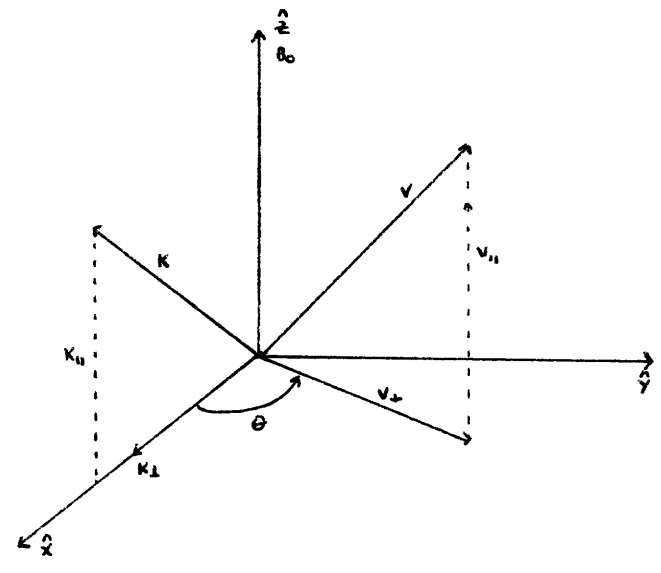


FIGURE 1

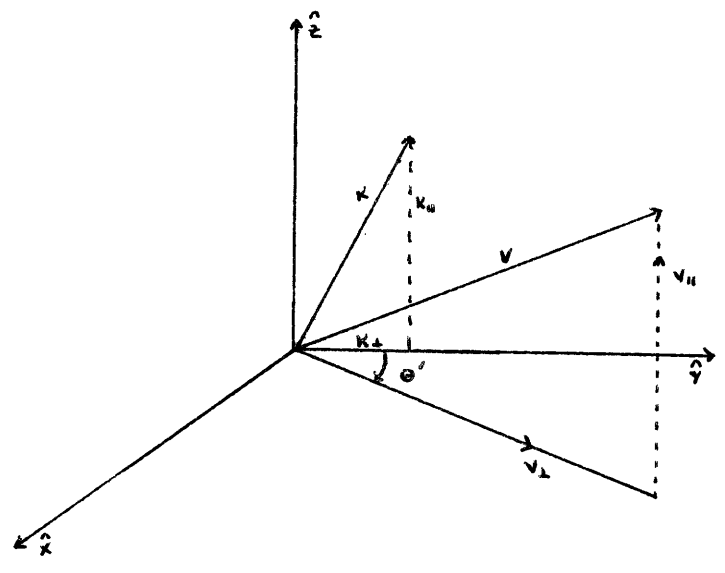


FIGURE 2

A.3 The Extension to Three Dimensions

Since in the last section we obtained the Harris dispersion relation for k_{\perp} in the \hat{x} direction, we just need to make the necessary changes to find the equation for k_{\perp} in the \hat{y} direction. This will extend the Harris equation to three dimensions (k_{\parallel} in \hat{z} direction, $k_{\perp x}$, $k_{\perp y}$). It seems reasonable to assume that the Harris relation will be the same for $k_{\perp y}$ as for $k_{\perp x}$. This can be shown to be true. In the following, we make the necessary changes to prove the above. The first change occurs in taking the dot product in equation (A.1.12); now we have

$$\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = k_{\perp} \frac{v_y'}{v_{\perp}} \frac{\partial f_0}{\partial v_{\perp}} + k_{\parallel} \frac{\partial f_0}{\partial v_{\parallel}} \quad (1)$$

which can be clearly seen in Fig. 2. Also we must let

$$\phi = \phi(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{y}' - \omega t')} \quad (2)$$

Using the expressions for \mathbf{y}' and \mathbf{v}_y' that result from the unperturbed orbit derivation (shown in equations (13) and (14), we obtain a new expression for the first order velocity distribution function,

$$\begin{aligned}
f_{\perp} = & \frac{iq}{m} \phi(k, \omega) e^{i(k \cdot y - \omega t)} \int_{-t}^0 dT \left[k_{\perp} \sin(\phi - \Omega_s T) \frac{\partial f_0}{\partial v_{\perp}} \right. \\
& + k_{\parallel} \left. \frac{\partial f_0}{\partial v_{\parallel}} \right] \cdot \exp i \left[\frac{k_{\perp} v_{\perp}}{\Omega_s} (\cos(\theta - \Omega T) - \cos \theta) \right. \\
& \left. + k_{\parallel} v_{\parallel} T - \omega T \right] . \tag{3}
\end{aligned}$$

Looking at the second term of equation (3), we see that the only difference in this case and the $k_{\perp x}$ case is in the exponent of equation (3). From identity (A.2.2) we note the only effect is in the sign of the exponent; but since we use this identity twice, this sign difference cancels. We obtain an answer similar to the expression obtained for $k_{\perp y}$,

$$\sum_N \frac{J_N^2(k_{\perp} v_{\perp} / \Omega_s) k_{\parallel}}{i(k_{\parallel} v_{\parallel} - \omega + \Omega_s N)} \frac{\partial f_0}{\partial v_{\parallel}} . \tag{4}$$

For the first term in equation (3), we obtain a minus sign in front of the summation because we have used identity (A.2.4) instead of (A.2.3). But returning to Fig. 2, we notice that the angle θ' is in the $-\hat{\theta}$ direction. This results in a negative sign in front of the $\sin(\theta - \Omega T)$ term, thereby, cancelling the other negative sign. Consequently, just as we proposed, the Harris dispersion relation is the same for $k_{\perp x}$

and $k_{\perp y}$. Therefore, the full three dimensional equation is shown in equation (A.2.17), just remember k_{\perp}^2 is $k_{\perp x}^2 + k_{\perp y}^2$.

Appendix B

Power Flow and Spatial Causality

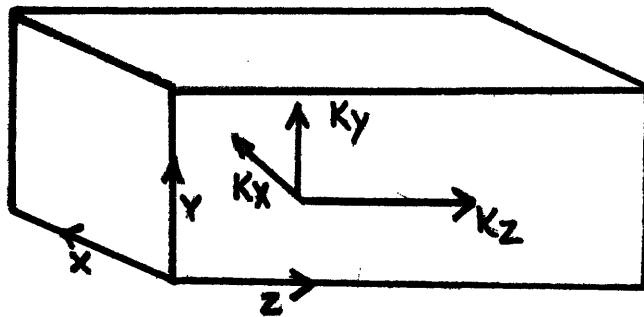
In assuming a solution to the lower hybrid dispersion relation (equation (2.3.24)), we considered a plane wave. This assures us that power flows in the direction of the moving wavefront. If in a non-dispersive media,

$$\phi = \phi_0 \exp[ik_x x + ik_z z], \quad (1)$$

then the wave is moving in the positive \hat{x} and positive \hat{z} directions. The power flow is in the same directions. Attaching a waveguide to the plasma chamber, the waveguide will act as a source exciting the plasma. But we cannot be too hasty and assume that positive k_x will cause propagation in the positive \hat{x} direction. We must analyze this more carefully using the group velocity. Another consideration is the excitation pulse. What prevents it from causing some response in the waveguide instead of just in the plasma?

Considering the problem of power flow, we want to find the signs of the wavenumbers causing power flow into the plasma. In a dispersive medium, the dispersion relation must be used to find the group velocity. It is the group velocity that gives

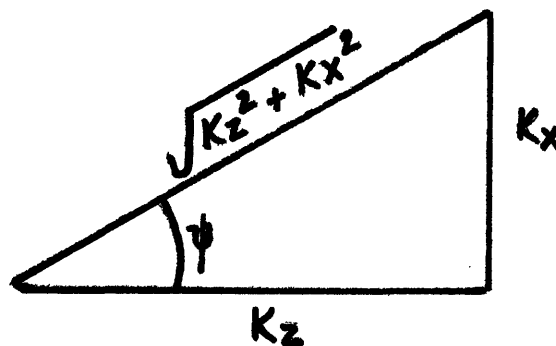
the direction of power flow. Since the \hat{y} direction is a perturbation its effects will be disregarded. Note $k_x > k_z$ for the same order mode (see the picture below).



Using the linear lower hybrid dispersion relation,

$$\omega^2 = \omega_{pi}^2 + \omega_{pe}^2 \cos^2 \psi, \quad (2)$$

(where ω_{pe} , i is the plasma frequency) we can solve for v_g . Considering the \hat{x} and \hat{z} directions, $\cos \psi$ is defined by the diagram below,



Therefore,

$$\cos \psi = \frac{k_z}{\sqrt{k_z^2 + k_x^2}} . \quad (3)$$

Substituting equation (2) into equation (1) yields

$$\omega^2 = \omega_{pi}^2 + \omega_{pe}^2 \frac{k_z^2}{k_z^2 + k_x^2} . \quad (4)$$

Simplifying the above expression by using

$$\omega_{pi}^2 = \frac{m_e}{m_i} \omega_{pe}^2 \quad (5)$$

$$k_z^2 = \frac{m_e}{m_i} k_x^2 \quad (6)$$

where m_e is the electron mass and m_i is the ion mass, we obtain

$$\omega^2 = 2\omega_{pe}^2 \frac{k_z^2}{k_x^2} . \quad (7)$$

The group velocity is defined as

$$v_{gx} = \frac{\partial \omega}{\partial k_x} \quad \text{and} \quad v_{gz} = \frac{\partial \omega}{\partial k_z}, \quad (8)$$

From the above expression and equation (6), the group velocities evaluated in the \hat{x} and \hat{z} directions are

$$v_{gx} = -\sqrt{2} \omega_{pe} \frac{k_z}{k_x^2} \quad (9)$$

and

$$v_{gz} = \sqrt{2} \omega_{pe} \frac{1}{k_x}. \quad (10)$$

The criterion that power flows in the positive \hat{x} and positive \hat{z} directions requires v_{gx} and v_{gz} to be positive. This can be accomplished by allowing k_z to be negative and k_x positive. These are the necessary signs for the wavenumbers.

Now consider the problem of having the excitation pulse cause some response in the waveguide instead of in the plasma. Remember that we have already specified that the excitation carrying the power should move into the plasma. Therefore, there can be no response for x and z less than zero assuming the excitation occurs at $z = 0$ and $x = 0$. This is a problem of spatial causality. It is completely analogous to causality in time. How can we guarantee that the excitation pulse results

in a causal response? If an excitation results in a causal response, its real and imaginary parts are related by the Hilbert transform. We must be sure that the excitation pulse meets this criterion to obtain spatial causality, [Op1].

Appendix C
The Boundary Conditions

The boundary conditions are found for z equal to minus infinity. Therefore, the $\text{sech}^2(z)$ functions are equal to zero. Using this in the coupled set of equations from Chapter Three yields

$$f_{zz} - (1 - k^2)f = \Gamma g \quad (1)$$

$$g_{zz} - (1 - k^2)g = -\Gamma f. \quad (2)$$

Assume a solution of the form e^{Bz} giving the following,

$$[B^2 - (1 - k^2)]f = \Gamma g \quad (3)$$

$$[B^2 - (1 - k^2)]g = -\Gamma f. \quad (4)$$

Eliminating g and f ,

$$B^2 - (1 - k^2) = \pm i\Gamma. \quad (5)$$

Using complex notation,

$$B^2 = \sqrt{(1 - k^2)^2 + \Gamma^2} e^{\pm i\psi'} \quad (6)$$

where

$$\psi' = \tan^{-1} \left(\frac{\Gamma}{1 - k^2} \right). \quad (7)$$

Therefore,

$$B = (1 - k^2 + \Gamma^2)^{1/4} e^{\pm i\psi'/2}. \quad (8)$$

Taking the real and imaginary parts,

$$\alpha = (1 - k^2 + \Gamma^2)^{1/4} \cos \left(\frac{1}{2} \psi' \right) \quad (9)$$

$$\beta = (1 - k^2 + \Gamma^2)^{1/4} \sin \left(\frac{1}{2} \psi' \right) \quad (10)$$

we then have

$$f = e^{\pm \alpha z} [\cos(\beta z) + \sin(\beta z)], \quad (11)$$

and

$$g = e^{\pm \alpha z} [\cos(\beta z) + \sin(\beta z)]. \quad (12)$$

These two equations will give the boundary values for f and g if we eliminate the $e^{-\alpha z}$ solution. This is done to prevent the solution from blowing up at minus infinity. Remember the solution must oscillate or decay to zero at plus and minus infinity. To find the starting slopes, we just take the first derivative of equation (11) and equation (12),

$$f' = g' = [\alpha(\sin(\beta z) + \cos(\beta z) + \beta(\cos(\beta z) - \sin(\beta z)))] e^{\alpha z}. \quad (13)$$

Appendix D
The Computer Program

The following page is a copy of the program used to perform the numerical integration of the coupled set of equations. This program is written in the MACSYMA language and is called by using,

$$\text{lam 5}(p, Q, i, h, z1, u, s, t, g1). \quad (1)$$

The variables correspond to the following:

p	initial value of f
Q	initial slope of f
i	initial value of g
h	initial slope of g
z1	the wavenumber k
u	the spatial variable used to specify infinity
s	the integration step size
t	the number of integration points
g1	the coupling coefficient.

```

lam5(p,g,i,h,z1,u,s,t,s1):=block([l,m,j,n,e,f,a,b,c,d,a2,b2,x,y,w,z,x2,y2],
array(l,float,t), array(m,float,t), array(j,float,t), array(w,float,t),
array(n,float,t), array(z,float,t), array(e,float,t), array(f,float,t),
array(g,float,t),
l[0]:p, m[0]:a, j[0]:i, n[0]:h,
for k:0 step 1 thru t do
  (s[k]:.1*s*(k+1),
  u:u+s,
  if k<4 then
    (e[k]:a, f[k]:-(z1**2-1+6*sech(u)**2)*p+s1*j[k], w[k]:h,
    z[k]:-(z1**2-1+2*sech(u)**2)*i-s1*l[k],
    if k=0 then (l[k+1]:p+s*e[k], m[k+1]:a+s*f[k],
    j[k+1]:i+s*w[k], n[k+1]:h+s*z[k]),
    if k=1 then (l[k+1]:l[0]+2*s*e[k], m[k+1]:m[0]+2*s*f[k],
    j[k+1]:j[0]+2*s*w[k], n[k+1]:n[0]+2*s*z[k]),
    if k>1 then (l[k+1]:p+2*s*e[k], m[k+1]:a+2*s*f[k],
    j[k+1]:i+2*s*w[k], n[k+1]:h+2*s*z[k]),
    p:l[k+1], a:m[k+1], i:j[k+1], h:n[k+1])
  else
    (a:l[k-4]+4*s/3*(2*e[k-3]-e[k-2]+2*e[k-1]),
    b:m[k-4]+4*s/3*(2*f[k-3]-f[k-2]+2*f[k-1]),
    a2:j[k-4]+4*s/3*(2*w[k-3]-w[k-2]+2*w[k-1]),
    b2:n[k-4]+4*s/3*(2*z[k-3]-z[k-2]+2*z[k-1]),
    x:b, y:-(z1**2-1+6*sech(u)**2)*a+s1*j[k],
    x2:b2, y2:-(z1**2-1+2*sech(u)**2)*a2-s1*l[k],
    l[k]:l[k-2]+s/3*(e[k-2]+4*e[k-1]+x),
    m[k]:m[k-2]+s/3*(f[k-2]+4*f[k-1]+y),
    j[k]:j[k-2]+s/3*(w[k-2]+4*w[k-1]+x2),
    n[k]:n[k-2]+s/3*(z[k-2]+4*z[k-1]+y2),
    e[k]:m[k], f[k]:-(z1**2-1+6*sech(u)**2)*l[k]+s1*j[k],
    w[k]:n[k], z[k]:-(z1**2-1+2*sech(u)**2)*j[k]-s1*l[k]),
    print(l[k],m[k],j[k],n[k])),
graph2(1,s,"l","s"), graph2(j,s,"j","s"));

```

*

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