January 1981

LIDS-P-1027
(Revised)

OSP No. 87049

EQUIVALENCE RELATIONS IN QUEUING MODELS OF MANUFACTURING NETWORKS

by

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This research was carried out in the M.I.T. Laboratory for Information and Decision Systems with support extended by National Science Foundation Grant DAR78-17826.

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ABSTRACT

Fundamental equivalence properties, which include transfer line reversibility, are established for queuing models of manufacturing networks. The basic tool used for arriving at these properties is the analysis of hole (or empty space) motion in the networks. Specifically, it is shown that networks can be grouped into equivalence classes, where members of the same class can have different layouts. The relationship among the performance measures of members of the same class is exhibited.

A simple queuing network model is used to introduce and prove those results. However, the same properties can be shown to hold for more realistic models.

I. Introduction

In this paper we present some fundamental equivalence properties for queuing models of manufacturing networks. The basic tool used for arriving at these properties is the analysis of hole (or empty space) motion in the network. Specifically we show that networks can be grouped into equivalance classes, where members of the same class can have different layouts. We also exhibit the relationship among the performance measures of members of the same class. These results are of interest to designers of manufacturing systems as well as other systems that can be modelled as networks of queues.

The model which we use as a vehicle for illustrating the equivalance result is rather simple. It is a variation on one that has been used extensively by queueing theorists to model networks of queues (e.g. Jackson (1963)). New features have been added to model finite storage capacities and assembly and disassembly operations. The equivalence results, however, are not model-dependent. In Ammar (1980) some of the ideas presented here are explored in the context of another model.

The idea of focusing on hole motion in a network is inspired by a similar type of tool used to study semiconductor devices. Newell (1979) uses the motion of holes in analyzing the approximate behavior of queues in tandem.

We show in this paper that transfer line reversibility is a consequence of the results we obtain. That is, reversal of the order of operations in a transfer line leaves the production rate unchanged. This was conjectured by Hillier and Boling (1977) and proven for a rather general model of transfer lines by Muth (1979) and Dattatreya (1978). Another proof of transfer line reversibility, similar to the one by Muth, is by Yamazaki and Sakagegawa (1975). However, we also show that, although production rate is unaffected by line reversal, another performance measure, the mean in-process inventory, does change.

2. THE MODEL

In this section we discuss a model of manufacturing networks. It is important to emphasize that this model has been chosen for its simplicity rather than its accurate depiction of the manufacturing process. This allows us to explain the equivalence results to follow in the simplest possible way, without getting entangled in modelling details. For a presentation of some of these equivalence ideas in the context of a more realistic model the reader is referred to Ammar (1980). In that model, each machine's processing time is fixed, and machines fail at random times and require random lengths of time to repair. The results presented here, however, are valid in both models.

2.1 Definitions and Network Operation

Each machine i in a manufacturing network can be fed by, i.e., receives parts from, a set, U(i), of buffers called the upstream buffers. Machine i in turn feeds a set of downstream buffers, D(i). Note that the number of buffers in U(i) is not necessarily equal to those in D(i). (See Figure 2.1).

Each buffer j feeds exactly one machine, its down-stream machine, d_j, and is fed by exactly one machine, its upstream machine, u_j. (See Figure 2.2). All buffers are assumed to have finite capacities: buffer j can hold no more that N_j parts.

A machine takes one part from each of its upstream buffers. These parts are assembled into a single item. The machine then disassembles that item, depositing one part into each of its downstream buffers. We assume that the time to perform the whole assembly-disassembly operation is an exponentially distributed random variable with rate μ_i.

We call the time between input of parts to be assembled and output of disassembled product a cyle. For mathematical convenience we assume that parts that are being operated on reside in their respective upstream buffers. That is , the machine has no capacity to hold parts.

The transformation into disassembled product is assumed to take place instantaneously at the time of output. Thus machines act as transfer mechanisms between buffers.

A machine is said to be blocked if any of its downstream buffers is full. We also say that a machine is starved when any of its upstream buffers is empty. A machine which is neither starved nor blocked, upon the completion of a cyle, starts a new cycle immediately. A starved machine cannot start a new cycle until each of its upstream buffers has at least one part. A blocked machine, which is not starved, does not start its cycle until none of its downstream buffers is full.

There are two special cases of machines. An input machine is one that does the first processing on some of the raw material entering the system. It is assumed that buffers upstream of input machines contain an unlimited supply of the required raw material, so that an input machine is never starved. Output machines are those from which some of the final product emerges. We assume that the buffers downstream of output machines have infinite capacity. Hence output machines are never blocked.

2.2 System State and Markov Process Formulation

We define the state of a system at time t as

$$s(t) = (n_1(t),\ldots,n_{k_B}(t)) ,$$

where k_B is the number of buffers in the system and $n_i(t)$ is the number of parts in buffer i at time t. Each $n_i(t)$ satisfies

$$0 \leq n_i(t) \leq N_i . \tag{2.1}$$

Also we define the indicator variable

$$I_j(t) = \begin{cases} 0 & \text{if machine } j \text{ is starved } \underline{\text{or}} \text{ blocked at time } t \\ 1 & \text{otherwise} \end{cases} \tag{2.2}$$

and

$$I(t) = (I_1(t),\ldots, I_{k_M}(t)) \tag{2.3}$$

where k_M is the number of machines.

According to our exponential processing time assumption, if machine i is not starved or blocked at time t, it completes a cycle at time t+δt with probability

$$\mu_i \delta t + O(\delta t) . \tag{2.4}$$

Thus the number of parts in each buffer in D(i) increases by one at time t + δt with probability given by (2.4). That is,

$$\begin{aligned} \text{Prob}[n_j(t+\delta t) - n_j(t) = 1 \,|\, I_i(t) = 1] \\ = \mu_i \delta t + O(\delta t) \end{aligned} \tag{2.5}$$

for all $j \in D(i)$.

Similarly the number of parts in each of the buffers in U(i) decreases by one at time $t + \delta t$ with the same probability (2.4) or

$$\text{Prob}\ [n_\ell(t + \delta t) - n_\ell(t) = -1 | I_i(t) = 1]$$

$$= \mu_i \delta t + 0(\delta t)$$

for all $\ell \in U(i)$.

However, if machine is is blocked or starved at time t it completes a cycle at time $t + \delta t$ with probability 0 since a blocked or starved machine cannot work.

Thus

$$\text{Prob}[n_j(t + \delta t) - n_j(t) = 0 | I_i(t) = 0]$$
$$= \text{Prob}[n_\ell(t + \delta t) - n_\ell(t) = 0 | I_i(t) = 0] = 1 \tag{2.7}$$

for all $j \in D(i)$ and $\ell \in U(i)$.

In addition, no buffer gains or loses more than one part. We summarize this behavior in Table 2.1. For the meaning of the subscripts see Figure 2.2.

$n_i(t + \delta t) - n_i(t)$	$\text{Prob}[\bullet | I_{d_i}(t), I_{u_i}(t)]$
1	$\mu_{u_i} I_{u_i}(t)\delta t$
-1	$\mu_{d_i} I_{d_i}(t)\delta t$
0	$1 - \mu_{u_i} I_{u_i}(t)\delta t - \mu_{d_i} I_{d_i}(t)\delta t$
$m, \lvert m \rvert \geq 2$	0

Table 2.1 Buffer Level Transition Probabilities

Using Table 2.1 and the memoryless property of the exponential distribution we can obtain for each pair of states, s_1 and s_2, the following quantities

$$T(s_2, s_1, \delta t) = \text{Prob}[s(t + \delta t) = s_2 | s(t) = s_1] \tag{2.8}$$

The quantities (2.8) can be used to construct a continuous time discrete state Markov process with M states of the form (2.1), where

$$M = \prod_{i=1}^{k_B} (N_i + 1) \tag{2.9}$$

2.3 Steady State Probabilities and Performance Measures

This Markov process may be multiple-chained. Thus we cannot define steady state probabilities for those systems. However given a starting state we can define steady state probabilities as follows:

$$P_{s_0}(s_1) = \lim_{t \to \infty} \text{Prob}[s(t) = s_1 | s(0) = s_0] \ . \tag{2.10}$$

We use probabilities (2.10) to calculate performance measures of a manufacturing network, in particular, production rate and mean in-process inventory.

Let i be an output machine. We define

$$R_i(s_0) = \text{limiting production rate from machine } i \text{ given that } s(0) = s_0. \tag{2.11}$$

For all buffers j in the network we also define

$$\bar{n}_j(s_0) = \text{limiting mean in-process inventory at buffer } j \text{ given that } s(0) = s_0. \tag{2.12}$$

The production rate or machine i is the rate at which it produces parts when not starved. Thus

$$R_i(s_0) = \mu_i \text{ Prob[machine i not starved} | s(0) = s_0]$$

$$= \mu_i \sum_{\substack{n_j=1 \\ j\in U(i)}}^{N_j} \sum_{\substack{n_\ell=0 \\ \ell\notin U(i)}}^{N_\ell} P_{s_0}(n_1,n_2,\ldots,n_{k_B}) \quad . \tag{2.13}$$

The mean in-process inventory at buffer j is calculated using the definition of the expected value of a random number as follows

$$\bar{n}_i(s_0) = \sum_{\text{all states}} n_i P_{s_0}(n_1,n_2,\ldots,n_{k_B}) \quad . \tag{2.14}$$

2.4 Example

Consider the example of the three-machine two-buffer assembly system shown in Figure 2.3. The five parameters of the system are

$N_i \qquad i = 1,2$

and

$\mu_j \qquad j = 1,2,3 \; .$

For this system we can draw the state transition rate diagram as in Figure 2.4. Here

$$s(t) = (n_1(t), n_2(t)) \; .$$

For this example, the Markov process describing the system is single chained, and hence the steady state probabilities are independent of the starting state. Hence

$$P_{s_0}(s_1) = P(s_1) \quad \text{for all } s_1,\ s_0 . \tag{2.15}$$

The performance measures of the assembly system can be calculated as follows

$$R_2(s_0) = R_2 = \mu_2 \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} P(n_1,n_2) \tag{2.16}$$

and

$$\overline{n}_i(s_0) = \overline{n}_i = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} n_i P(n_1,n_2) \tag{2.17}$$

for i = 1,2.

3. DUALITY

In this section we define a duality concept in the context of the manufacturing networks described earlier. We then state and prove a duality theorem that relates the dynamics of the dual systems. This duality is extended to a more general equivalence idea in the next section.

3.1 Part-Hole Duality

We define a hole in a manufacturing network as an empty space. (See Newell (1979) Sevast'yanov (1962), Gordon and Newell (1967).) Thus a buffer of capacity N that contains n parts has N-n holes.

In our manufacturing network model holes or empty spaces move in the opposite direction of parts. (see Figure 3.1). At the start of a cycle, a machine takes one part from each of its upstream buffers, which increases the number of empty spaces or holes in them by one. Also at the end of a cycle, when a machine deposits the disassembled product into its downstream buffers it is decreasing the number of holes in each of these buffers by one. Thus every end-of-cycle is an event of part production, while every beginning-of-cycle is a hole production event. Since every end-of-cycle must have a corresponding beginning-of-cycle, every part production event has a corresponding hole production event.

Note that a full buffer has no holes, and an empty buffer is full of holes. Hence a machine starved of parts is blocked by holes and also a machine blocked by parts is starved of holes. Also note that where we assume an infinite supply of parts, this is equivalent to having infinite room for holes. Similarly infinite room for parts is equivalent to an infinite supply of holes. Table 3.1 summarizes the part-hole duality ideas introduced here.

Parts	Holes
n	N-n
∞ room	∞ supply
∞ supply	∞ room
starvation	blockage
blockage	starvation

Table 3.1 Part-Hole Duality

3.2 Dual Systems

A manufacturing network (M) is i-dual to another network (M') if part motion in buffer i of M' corresponds to hole motion in buffer i of M, and if otherwise the networks M and M' are identical. Note that this condition requires buffer i to have the same capacity in both networks.

To explain this definition consider the primal network M in Figure 3.2. Buffer i has upstream machine u_i and downstream machine d_i. Holes in buffer i move from machine d_i to machine u_i. Thus the dual system M' has machine u_i as the downstream machine of buffer i while d_i is its upstream machine (See Figure 3.3). (Note that in system M' the labels u_i and d_i are no longer consistent with earlier notation).

We establish the convention of superscripting all quantities pertinent to the dual system with a prime. For example, $n_i(t)$ and

$n'_i(t)$ denote the number of parts in buffer i at time t in M and M' respectively.

3.3 Duality Theorem

Theorem 3.1

For the two i-dual systems M and M':

$$T(s(t + \delta t), s(t), \delta t) = T'(s'(t + \delta t), s'(t), \delta t) \tag{3.1}$$

whenever

$$n'_j(t) = n_j(t) \text{ and } n'_j(t + \delta t) = n_j(t + \delta t) \quad \text{for all } j \neq i \tag{3.2}$$

and

$$n'_i(t) = N_i - n_i(t) \text{ and } n'_i(t+\delta t) = N_i - n_i(t+\delta t). \tag{3.3}$$

Conditions (3.2) and (3.3) are called the duality conditions.

This theorem states that if the states of systems M and M' satisfy the duality conditions at times t and t+δt, then the probability of transition between such states in system M is the same as that for system M'.

Before proceeding to prove Theorem 3.1 we note the following facts:

1. When the duality conditions hold
   $$n_i(t+\delta t) - n_i(t) = -(n'_i(t+\delta t) - n'_i(t)) . \tag{3.4}$$
   This can be shown from condition (3.3).

2. When the duality conditions hold
   $$I(t) = I'(t), \tag{3.5}$$
   This can be shown as follows:

a. For all machines j, $j \neq u_i$ and $j \neq d_i$, $I_j(t) = I'_j(t)$. This follows from (3.2).

b. If $I_{u_i}(t) = 0$, this implies that machine u_i is blocked or starved in the primal system M.

<u>Case 1</u> If machine u_i is starved in the primal then it is starved in the dual, hence $I'_{u_i}(t) = 0$.

<u>Case 2</u> If machine u_i is blocked in the primal not because buffer i is full, then it is blocked in the dual as well and $I'_{u_i}(t) = 0$.

<u>Case 3</u> If u_i is blocked in the primal because buffer i is full, then buffer i is empty in the dual and machine u_i is starved in the dual system. Thus $I'_{u_i}(t) = 0$.

c. If $I_{u_i}(t) = 1$, then machine u_i is neither starved nor blocked in the dual system and $I'_{u_i}(t) = 1$. Thus $I_{u_i}(t) = I'_{u_i}(t)$.

d. Similarly we can show that $I_{d_i}(t) = I'_{d_i}(t)$.

Therefore when the duality conditions holds,

$I(t) = I'(t)$.

We now let

$$q_i(t) = (n(t),\ldots,n_{i-1}(t),\ n_{i+1}(t),\ldots,n_{k_B}(t)) \quad . \tag{3.6}$$

That is $q_i(t)$ is a description of the state of the system <u>except</u> for the state of buffer i at time t. Note that $q_i(t)$ and $n_i(t)$ provide a complete description of the state at time t. Also let

$$q_i(t+\delta t) - q_i(t) = (n_1(t+\delta t) - n_1(t), \ldots, n_{k_B}(t+\delta t) - n_{k_B}(t)) . \quad (3.7)$$

Proof of Theorem 3.1

We have

$$T(s(t+\delta t), s(t), \delta t) = \text{Prob}[s(t+\delta t) | s(t)] \quad (3.8)$$

$$= \text{Prob}[q_i(t+\delta), n_i(t+\delta t) | q_i(t), n_i(t)] \quad (3.9)$$

In (3.9) we condition the state at time $t+\delta t$ on the state at time t. Consider now conditioning the difference in buffer levels between $t+\delta t$ and t on the value of the buffer level at time t. Hence

$$T(s(t+\delta t), s(t), \delta t) = \text{Prob}[q_i(t+\delta t) - q_i(t), n_i(t+\delta t) - n_i(t) | q_i(t), n_i(t)] \quad (3.10)$$

The only useful information in the value of the condition $q_i(t)$, $n_i(t)$ is whether machines are starved or blocked and thus

$$T(s(t+\delta t), s(t), \delta t) = \text{Prob}[q_i(t+\delta) - q_i(t), n_i(t+\delta t) - n_i(t) | I(t)] . \quad (3.11)$$

When the duality conditions hold we have

$$q_i(t) = q_i'(t) \text{ and } q_i(t+\delta t) = q_i'(t+\delta t) \quad (3.12)$$

and

$$q_i(t+\delta t) - q_i(t) = q_i'(t+\delta t) - q_i'(t) \quad (3.13)$$

using (3.13), (3.4) and (3.5) we can write

$$T(s(t+\delta t),s(t),\delta t)$$

$$= \text{Prob}[q_i'(t+\delta t)-q_i'(t),n_i'(t)-n_i'(t+\delta t)\,|\,I'(t)] \tag{3.14}$$

Using the same argument with which we constructed (3.11) from (3.9) we can say

$$\text{Prob}[q_i'(t+\delta t)-q_i'(t),n_i'(t)-n_i'(t+\delta t)\,|\,I'(t)]$$

$$= \text{Prob}[q_i'(t+\delta t),n_i'(t+\delta t)\,|\,q_i'(t),n_i'(t)]$$

$$= T'(s'(t+\delta t),s'(t),\delta t) \quad . \tag{3.15}$$

Hence

$$T(s(t+\delta t),s(t),\delta t) = T'(s'(t+\delta t),s'(t),\delta t) \tag{3.16}$$

We have thus proven Theorem 3.1.

3.4 Steady-State Behavior of the Dual Systems

Theorem 3.1 can now be used to relate the steady state probabilities of the two systems.

The relationship is simply stated as follows

$$P_{s(0)}(s) = P_{s'(0)}(s'), \tag{3.17}$$

where

$$s = (n_1,\ldots,n_{k_B})$$

and

$$s' = (n_1',\ldots,n_{k_B}') \quad ,$$

whenever in the i-dual systems M and M'

$$n_j(0) = n'_j(0), \text{ for } j \neq i,$$

$$n_i(0) = N_i - n'_i(0),$$

$$n_j = n'_j, \text{ for } j \neq i$$

and

$$n_i = N_i - n'_i .$$

This result can be seen easily from the definition

$$P_{s(0)}(s) = \lim_{t\to\infty} \text{Prob}[s(t) = s|s(0)] . \tag{3.19}$$

Using Theorem 3.1 it is clear that

$$\text{Prob}[s(t) = s|s(0)] = \text{Prob}[s'(t) = s'|s'(0)] \tag{3.20}$$

and (3.17) follows.

We can use (3.17) to relate the performance measures of the two i-dual systems. This is best illustrated in the duality example that follows.

3.5 Duality Example

Let system M be the three-machine two-buffer assembly system in Figure 2.3, and let M' be the three-machine two-buffer transfer line in Figure 3.4. In both systems machines with the same label have the same mean processing times and buffers with the same labels have the same capacities. Thus system M' is the 2-dual of system M.

The state transition rate diagram for the three-machine transfer line is in Figure 3.5. Note that this transition diagram can be obtained from the one for the three-machine assembly system (Figure 2.4) by a relabelling of the state space according to the duality conditions. That is,

$$n_1'(t) = n_1(t)$$

and

$$n_2'(t) = N_2 - n_2(t) \ .$$

As a consequence of the duality theorem we can obtain the following steady state result: (Recall that since the Markov process is single-chained we omit the subscript of the initial state).

$$P(n_1,n_2) = P'(n_1',n_2') \tag{3.21}$$

whenever

$$n_1' = n_1 \tag{3.22}$$

and

$$n_2' = N_2 - n_2 \ . \tag{3.23}$$

For system M we have

$$R_2 = \mu_2 \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} P(n_1,n_2) \tag{3.24}$$

Using (3.21), (3.22), and (3.23) we can rewrite (3.24) as

$$R_2 = \mu_2 \sum_{n_1'=1}^{N_1} \sum_{n_2' \neq 0}^{N_2-1} P'(n_1',n_2') \tag{3.25}$$

We recognize the right hand side of (3.25) as the rate at which machine 2 in the transfer line (M') moves parts from buffer 1 to buffer 2. Since no parts are being rejected or destroyed this is equal to the rate at which machine 3 moves parts out of the system. (For a proof of conservation of flow in a related model, see Ammar (1980)). Hence

$$R_2 = R'_3 \tag{3.26}$$

i.e. the production rate of system M is the same as that for system M'.

We also have

$$\bar{n}_1 = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} n_1 P(n_1, n_2) \tag{3.27}$$

This can be rewritten as

$$\bar{n}_1 = \sum_{n'_1=0}^{N_1} \sum_{n'_2=0}^{N_2} n'_1 P'(n'_1, n'_2) \ . \tag{3.28}$$

Thus

$$\bar{n}_1 = \bar{n}'_1 \tag{3.29}$$

or the mean in-process inventory in buffer 1 of the assembly system (M) is the same as that for buffer 1 of the transfer line (M'). Similarly we can show that

$$\bar{n}_2 = N_2 - \bar{n}'_2 \ . \tag{3.30}$$

4. EQUIVALENCE

Let system M' be the i-dual of system M, and M'' be the j-dual of system M'. If i=j then system M and M'' are identical. If i≠j then we have three distinct systems M,M', and M''. We can say that by Theorem 3.1

$$T(s(t+\delta t),s(t),\delta t) = T'(s'(t+\delta t),s'(t),\delta t) \quad (4.1)$$

whenever

$$n'_\ell(t+\delta t) = n_\ell(t+\delta t), \quad (4.2)$$

$$n'_\ell(t) = n_\ell(t) \qquad \text{for } \ell\neq i, \quad (4.3)$$

$$n'_i(t+\delta t) = N_i-n_i(t+\delta t), \quad (4.4)$$

and

$$n'_i(t) = N_i-n_i(t). \quad (4.5)$$

Also

$$T'(s'(t+\delta t),s'(t),\delta t) = T''(s''(t+\delta t),\ s''(t),\ \delta t)) \quad (4.6)$$

whenever

$$n''_\ell(t+\delta t) = n'_\ell(t+\delta t) \quad (4.7)$$

$$n''_\ell(t) = n'_\ell(t) \text{ for } \ell\neq j\ , \quad (4.8)$$

$$n''_j(t+\delta t) = N_j - n'_j(t+\delta t), \quad (4.9)$$

and

$$n''_j(t) = N_j - n'_j(t)\ . \quad (4.10)$$

By combining conditions (4.2) through (4.5) with conditions (4.7) through (4.10) we have

$$n_\ell''(t+ t) = n_\ell(t+\delta t), \tag{4.11}$$

$$n_\ell''(t) = n_\ell(t), \quad \ell\neq i, \quad \ell\neq j , \tag{4.12}$$

$$n_i''(t+\delta t) = N_i - n_i(t+\delta t), \tag{4.13}$$

$$n_i''(t) = N_i - n_i(t), \tag{4.14}$$

$$n_j''(t+\delta t) = N_j - n_j(t+\delta t), \tag{4.15}$$

and

$$n_j''(t) = N_j - n_j(t). \tag{4.16}$$

Furthermore conditions (4.11) through (4.16) imply that

$$T(s(t+\delta t),s(t),\delta t)) = T''(s''(t+\delta t),s''(t),\delta t) \tag{4.17}$$

It is clear that we can construct another system M''' which is k-dual to M'', where $k\neq i$, and $k\neq j$. A similar result to (4.17) can now be established. We say that such systems generated using one or more duality steps are equivalent.

4.1 Equivalence Classes

A k_B-buffer and k_M-machine equivalence class contains all systems with k_B buffers and k_M machines that are equivalent to each other. That is, any two members of an equivalence class can be derived from each other through one or more duality steps.

Let us denote i-duality as follows

$$M \overset{i}{\leftrightarrow} M' .$$

That is system M' is the i-dual of system M.

One method of generating all members of an equivalence class from a representative member is through Figure 4.1. Note that not all duality relations are shown in the figure.

Since each buffer can have parts flowing in either direction, the number of members of a $k_B - k_M$ equivalence class is 2^{k_B}.

Also note that there are in general several $k_B - k_M$ equivalence classes. For example, the two three-buffer, four-machine systems in Figure 4.2 cannot be derived from each other through any series of duality steps. Thus they belong to different equivalence classes.

All members of the same equivalence class are related by Theorem 3.1, and their performance measures are also related by the steady state consequences of Theorem 3.1. These comments are best illustrated by the examples in the next section.

4.2 Equivalence Example

Consider the two-buffer three-machine equivalence class illustrated in Figure 4.3. This class has four members (2^2): a three-machine assembly system (A), a forward (F) and a reversed (R) three-machine transfer line, and a three-machine disassembly system (D). Recall that all buffers and machines of members of the same equivalence class that have the same label are identical.

In the following discussions we use the superscripts A, F, R, or D to indicate the system which a quantity describes.

In sections 2 and 3 we have encountered systems A and F. To summarize that discussion we have shown that

$$R_3^F = R_2^A \tag{4.18}$$

and

$$\bar{n}_1^F = \bar{n}_1^A \tag{4.19}$$

and

$$\bar{n}_2^F = N_2 - \bar{n}_2^A \quad . \tag{4.20}$$

By similar argument we can show that for systems A and R the following holds

$$R_1^R = R_2^A \quad , \tag{4.21}$$

$$\bar{n}_1^R = N_1 - \bar{n}_1^A \quad , \tag{4.22}$$

and

$$\bar{n}_2^R = \bar{n}_2^A \quad . \tag{4.23}$$

Also we can relate systems R and D as follows

$$R_1^R = R_3^D = R_1^D \quad , \tag{4.24}$$

$$\bar{n}_1^D = \bar{n}_1^R \quad , \tag{4.25}$$

and

$$\bar{n}_2^D = N_2 - \bar{n}_2^R \quad . \tag{4.26}$$

We can now use these results to relate systems that are not the duals of each other but are equivalent.

For example we can say that for systems F and R:

$$R_3^F = R_1^R \tag{4.27}$$

$$\bar{n}_1^F = N_1 - \bar{n}_1^R \tag{4.28}$$

and

$$\bar{n}_2^F = N_2 - \bar{n}_2^R \tag{4.29}$$

These relations follow immediately from (4.18) through (4.23).

4.3 A Note on Transfer Line Reversibility

In the example of the previous section systems F and R were a three-machine transfer line and its reverse. We have shown that they have the same production rates but different mean-in-process inventories.

Consider now a k-machine transfer line with k-1 buffers. If the order of the machines and buffers in the transfer line is reversed we obtain the reversed transfer line (RTL). We call the original line the forward transfer line (FTL) for emphasis.

System RTL can be obtained from FTL by considering a series of k-1 dual systems as follows:

$$\text{FTL} \overset{1}{\leftrightarrow} \bullet \overset{2}{\leftrightarrow} \bullet \text{----} \overset{k-2}{\leftrightarrow} \bullet \overset{k-1}{\leftrightarrow} \text{RTL}$$

It is clear that FTL and RTL are equivalent systems. Thus we can use Theorem 3.1 to show that

$$T^{FTL}(s(t+\delta t),\ s(t),\delta t) = T^{RTL}(s'(t+\delta t);s'(t),\delta t) \tag{4.30}$$

whenever

$$n_i'(t+\delta t) = N_i - n_i(t+\delta t) \quad \text{and} \tag{4.31}$$

$$n_i'(t) = N_i - n_i(t)\ . \tag{4.32}$$

for all i. Therefore

$$P^{FTL}(n_1,\ldots,n_K) = P^{RTL}(n_1',\ldots,n_K') \tag{4.33}$$

whenever

$$n_i' = N_i - n_i \qquad \text{for all i.} \tag{4.34}$$

By using (4.33) and the performance measures formulas we can show that

$$R^{FTL} = R^{RTL} \tag{4.35}$$

and

$$\bar{n}_i^{FTL} = N_i - \bar{n}_i^{RTL}, \quad \text{for all i.} \tag{4.36}$$

Hence, in general, if a transfer line is reversed its production rate remains unchanged. This result has been shown by Muth (1979) for a rather general transfer line model. However the effect of line reversal on mean in-process inventory does not appear anywhere in the literature. Here we have shown that the mean in-process inventory can be increased or decreased through line reversal.

5. Conclusions and Future Research

In this paper we have demonstrated duality and equivalence ideas in the context of a simple manufacturing network model. Some of these ideas are explored in Ammar (1980) for a more realistic (and complex) model of assembly systems.

The concept of i-duality was explained in Section 3 and it was shown that state transition probabilities are related in i-dual systems. Steady-state probabilities and performance measures of the dual systems were also shown to be related.

The idea of the equivalence of two networks was explored in Section 4. An example of establishing an equivalence class as shown. Finally we demonstrated how equivalence can be used to determine how transfer line reversal can affect performance.

Future research in this area will have to deal with extending these equivalence ideas to more general manufacturing network models such as the one in Muth (1979).

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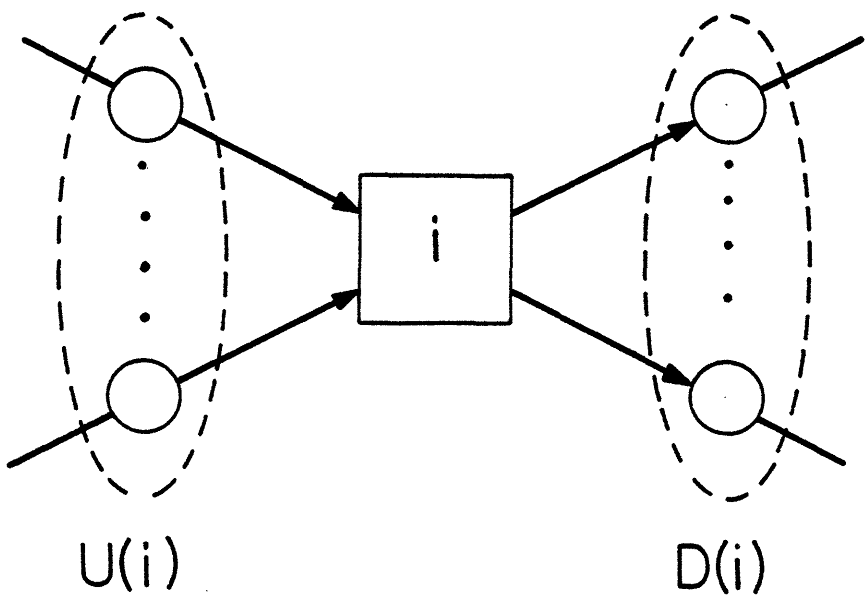


Figure 2.1 Upstream and Downstream Buffers

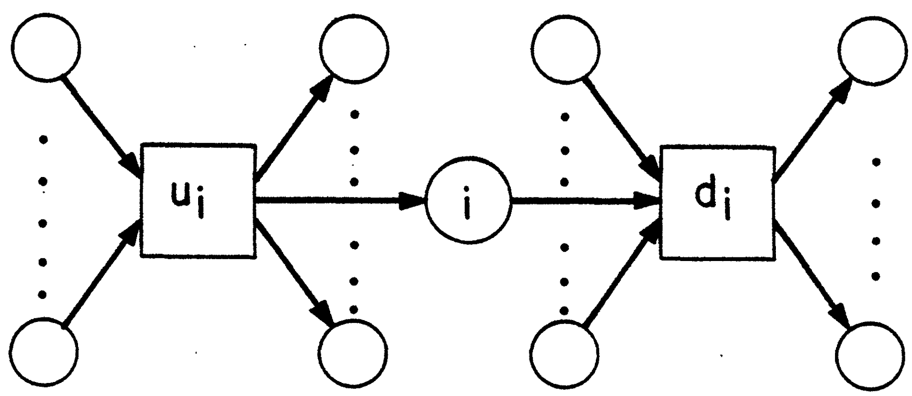


Figure 2.2 Upstream and Downstream Machines

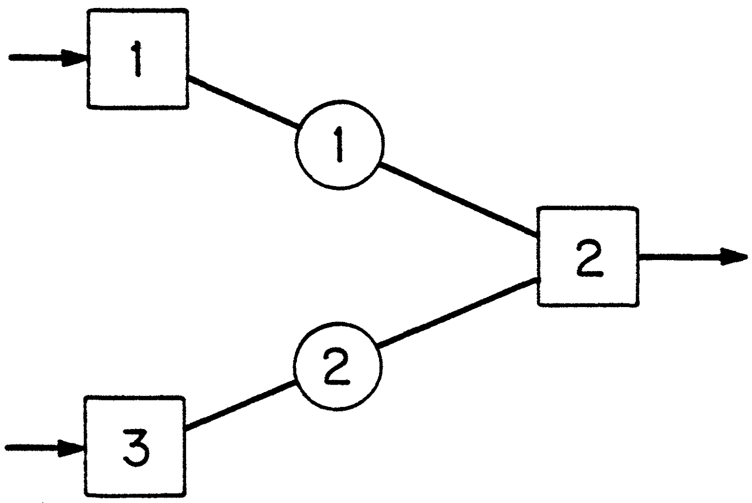


Figure 2.3 Three-Machine Two-Buffer Assembly System

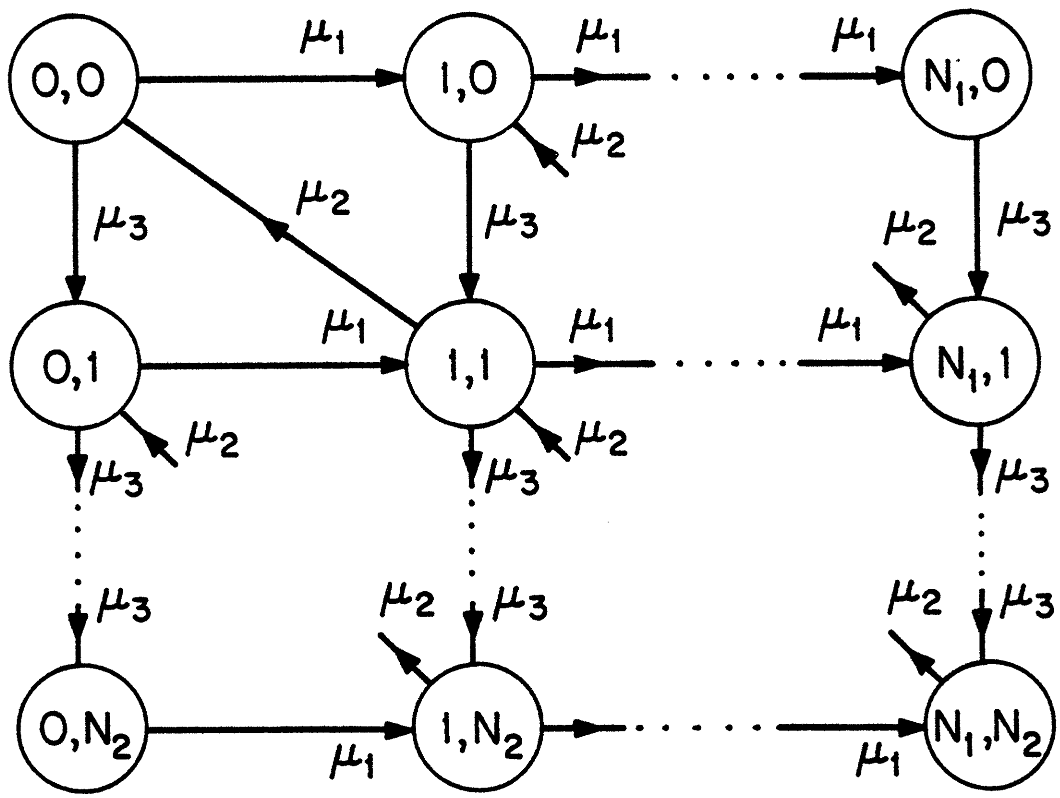
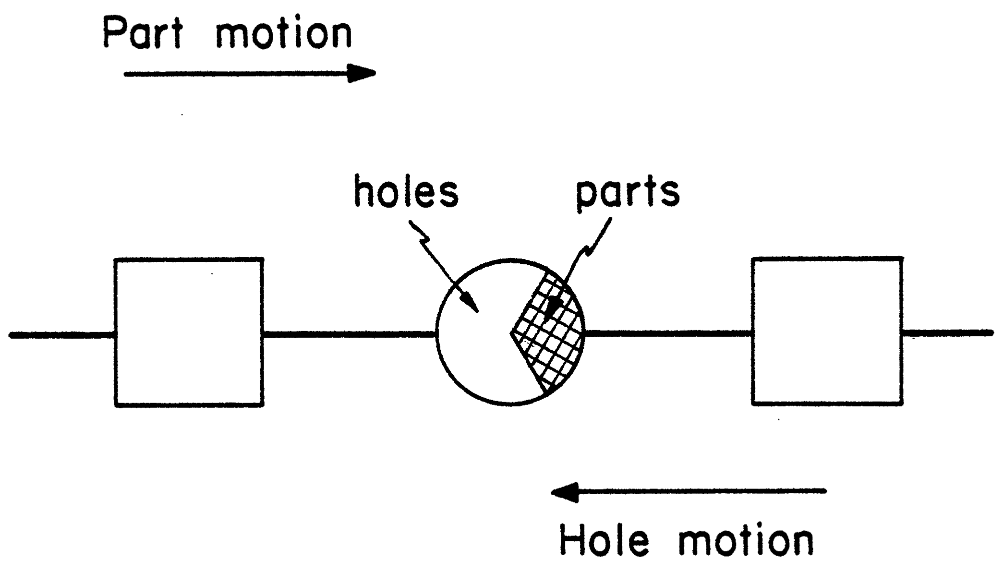


Figure 2.4 State Transition Rate Diagram For System of Figure 2.3



Figuree 3.1 Parts and Holes

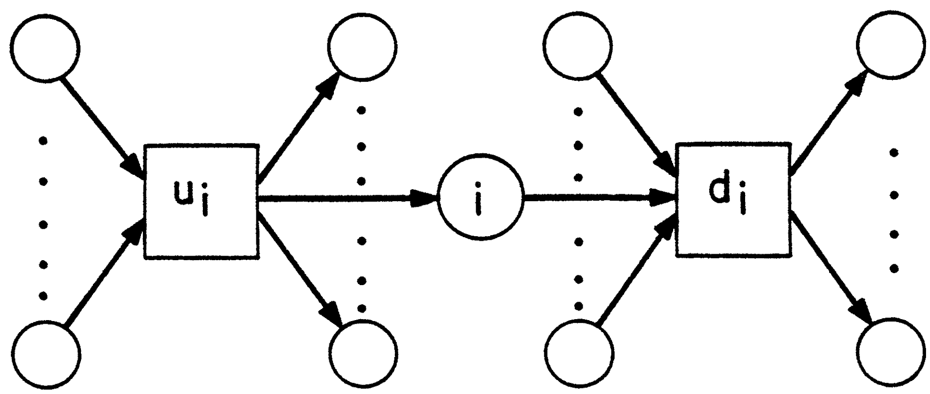


Figure 3.2 Primal System M

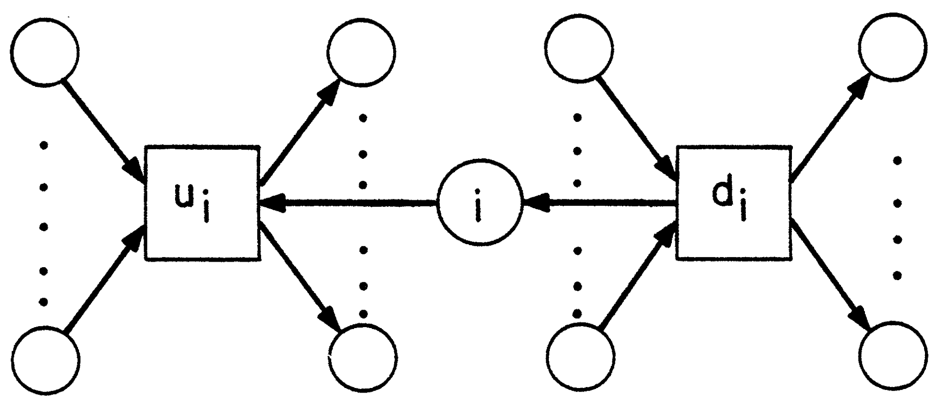


Figure 3.3 Dual System M'

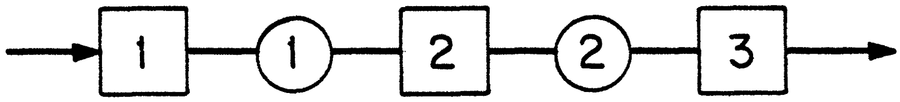


Figure 3.4 Three-Machine Transfer Line

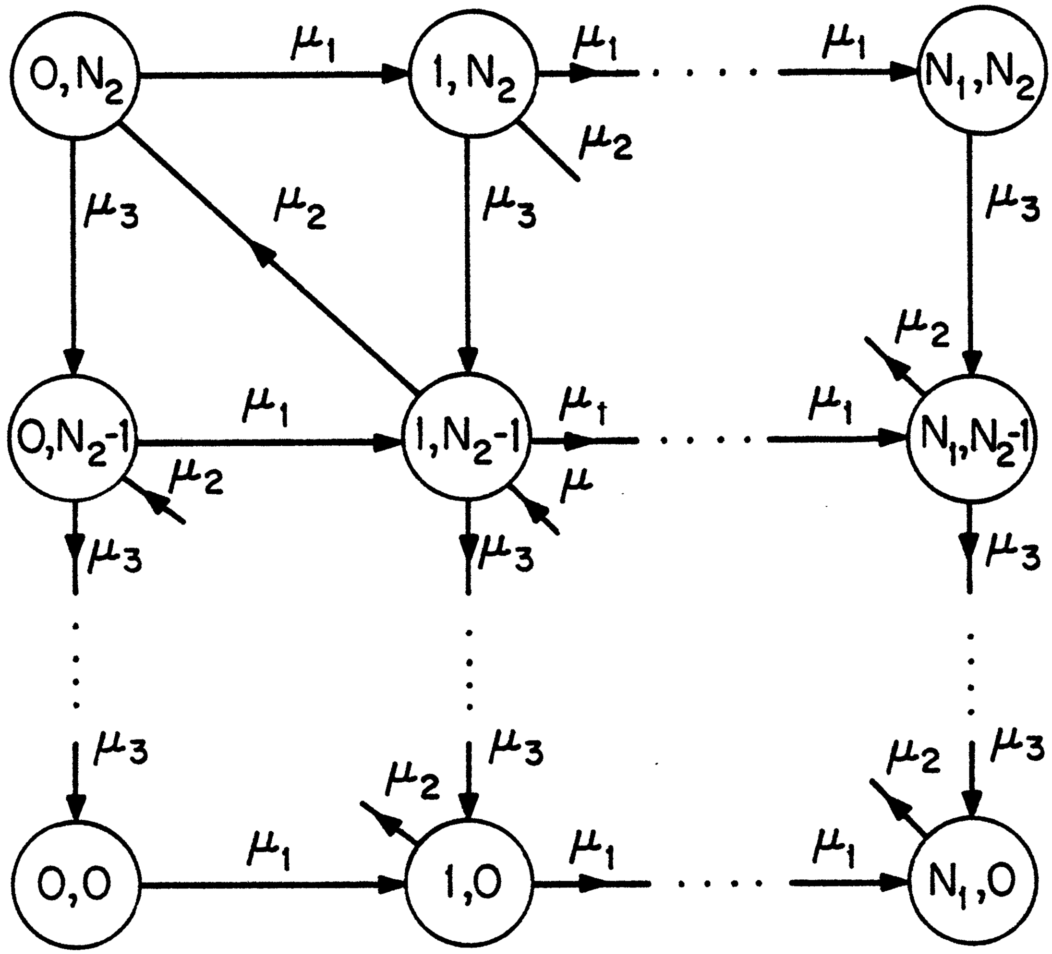


Figure 3.5 State Transition Rate Diagram for the Three-Machine Transfer Line

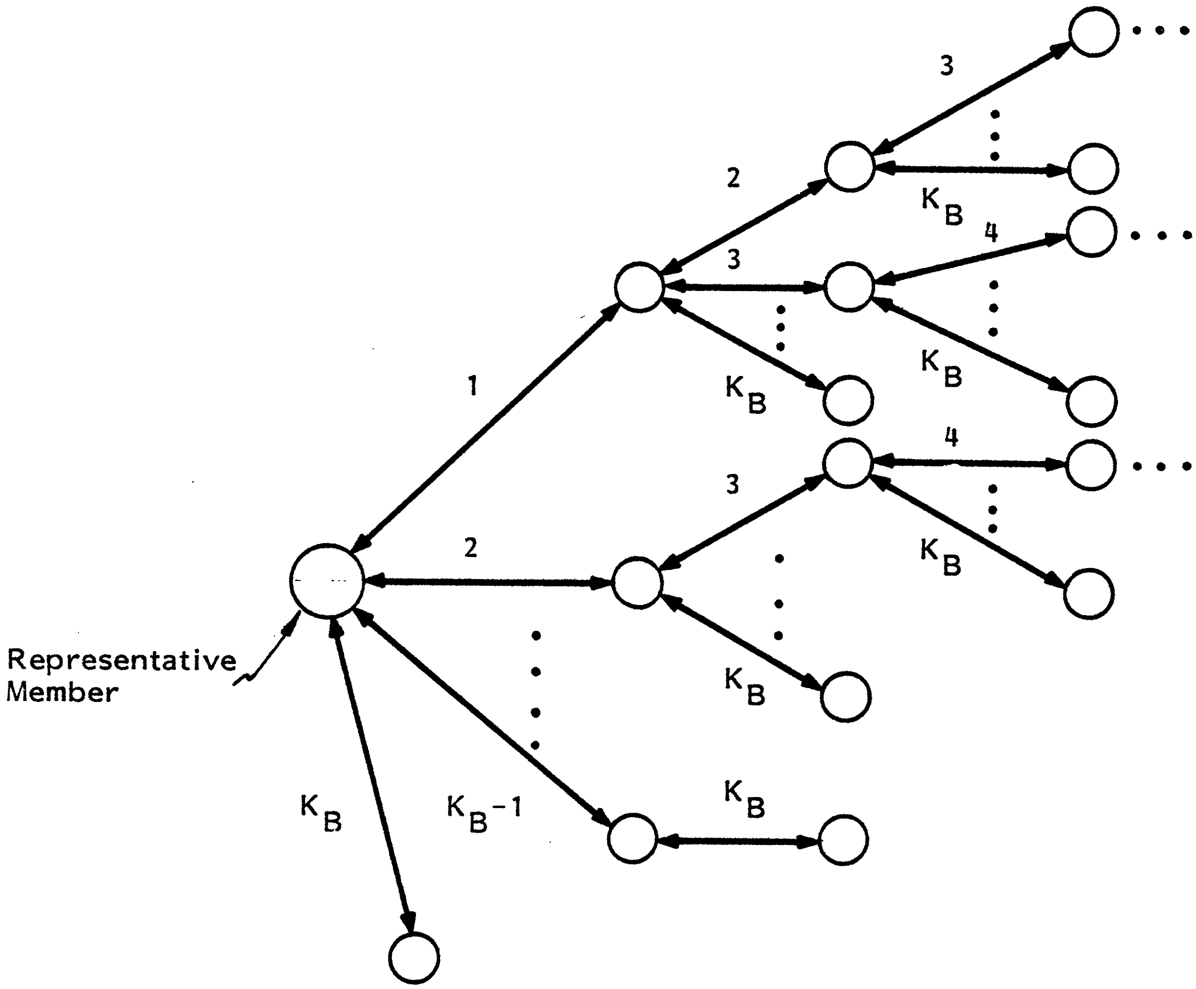
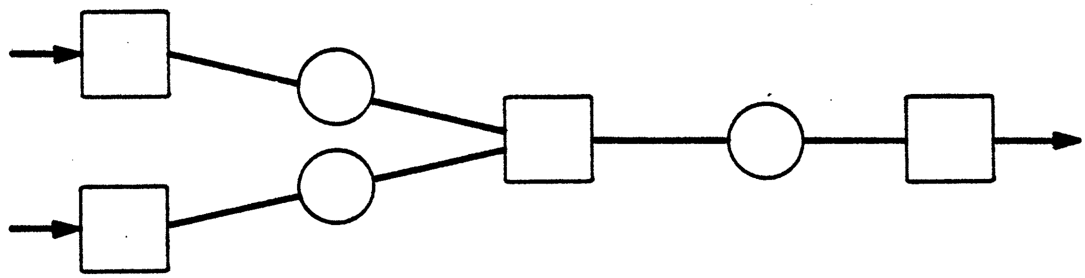
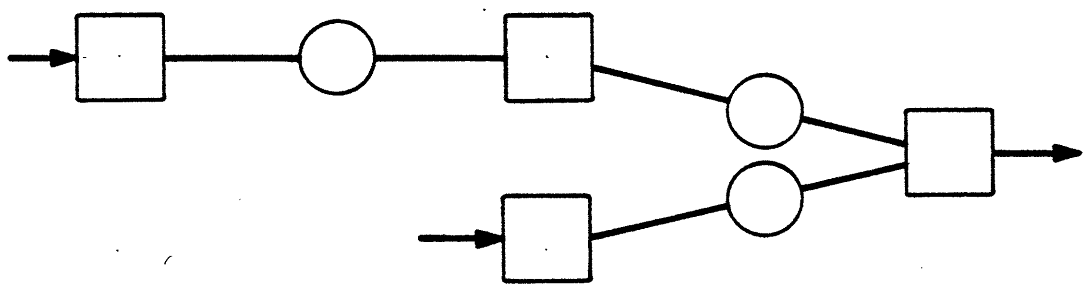


Figure 4.1 Generating All Memebers of an Equivalence Class



(a)



(b)

Figure 4.2 Two Four-Machine, Three-Buffer Systems

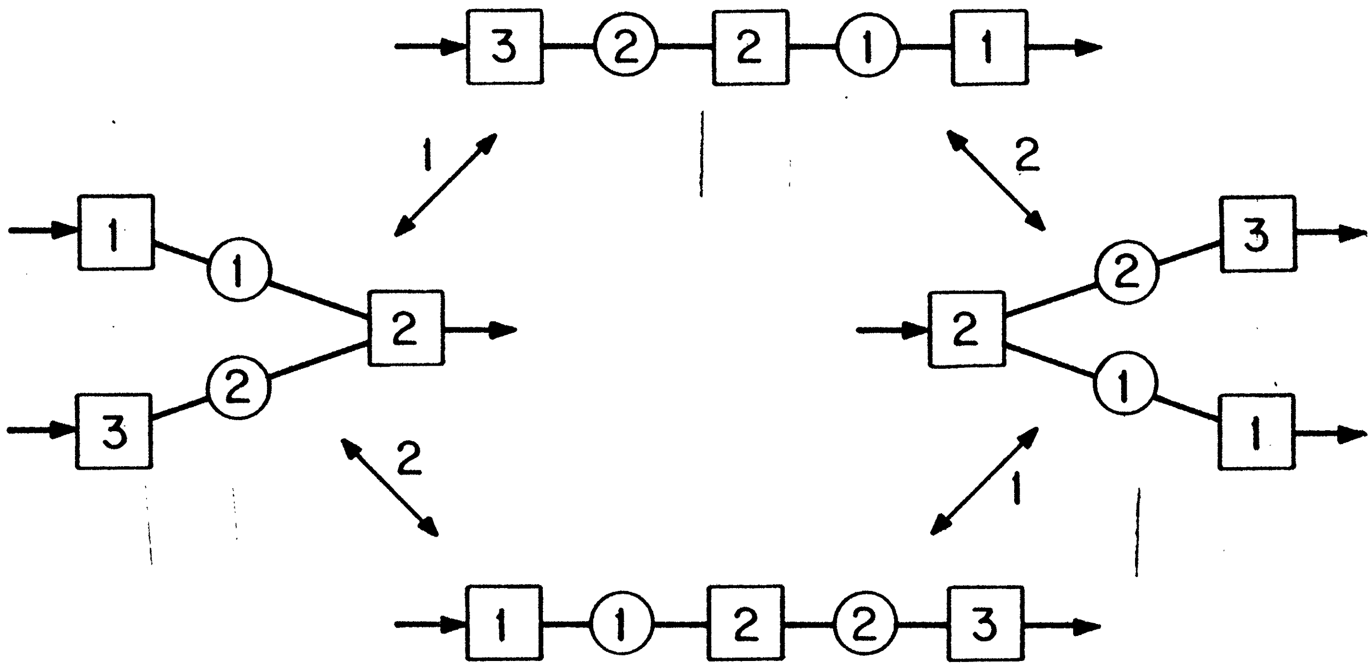


Figure 4.3 A Two-Buffer, Three-Machine Equivalence Class