Modules over Regular Algebras and Quantum Planes

by

Kaushal Ajitabh

M.Sc. (1989), Physics
Indian Institute of Technology, Kanpur, India

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Signature of author ______________________
Department of Mathematics

Certified by ______________________
Professor Michael Artin
Thesis Supervisor

Accepted by ______________________
Professor David Vogan
Chairman, Departmental Graduate Committee.

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Abstract

The first part of this thesis is a study of modules over elliptic algebras, especially, modules of Gelfand-Kirillov dimension 2. An elliptic algebra A is associated with a certain automorphism of a one-dimensional scheme E, generally an elliptic curve, and, every elliptic algebra defines a 'non-commutative projective plane' proj-A, sometimes called a quantum plane. Therefore, the study of modules translates into an interplay between the geometries of E and of quantum planes. First, we consider Cohen-Macaulay modules and their relation to geometry. The relation to geometry is studied by looking at 'the points of a given module M' and corresponding 'incidence relations' (a point p of E is said to be a point of M if there is a non-zero map from M to the point module N_p). Next, we study injective modules, the main objective being an explicit construction of a "residue complex" (a minimal injective resolution) for elliptic algebras, which we partially achieve. The second part of the thesis contains the construction of a residue complex (a minimal injective resolution) for regular algebras of dimension 2, which are twisted homogeneous coordinate rings of the projective line. Residue complexes for twisted coordinate rings have been previously constructed by geometric methods. Our method is algebraic, based on a unique factorization result for twisted coordinate rings, and (non-commutative) localizations of the algebra at orbits of points of the projective line.

Thesis Supervisor: Michael Artin
Title: Professor of Mathematics

CONTENTS

INTRODUCTION

This thesis has two parts: the first part (chapters 1 and 2) is a study of modules over elliptic algebras and geometry of quantum planes, and the second part (chapter 3) is a construction of "residue complex" (a minimal injective resolution) for regular algebras of dimension 2.

Elliptic algebras are the interesting cases of regular algebras of dimension 3 [ATV1]. It was shown in [AS] that a regular algebra of dimension 3 can have r generators and r relations (each of degree 5-r) where $r = 2$ or 3. To every regular algebra of dimension 3, there is an associated triple $(E, \sigma, \mathcal{L})$ [ATV1], where E is a scheme ($E \subset \mathbf{P}^2$ if $r = 3$, and $E \subset \mathbf{P}^1 \times \mathbf{P}^1$ if $r = 2$), σ is an automorphism of E, and $\mathcal{L}$ is an invertible sheaf on E whose global sections define a morphism $\pi : E \to \mathbf{P}^{r-1}$. The most interesting case of a regular algebra of dimension 3 is when $r = 3$, and E is a cubic divisor in the projective plane $\mathbf{P}^2$. This is the case we call the algebra an elliptic algebra. A Sklyanin algebra is a special example of an elliptic algebra where the cubic divisor E is an elliptic curve and σ is a translation on E. In an elliptic algebra A, there is a normalizing element g of degree 3, unique up to a scalar factor; and, there is an isomorphism of graded rings $B = A/gA \xrightarrow{\sim} \oplus_{n \geq 0} H^0(E, \mathcal{L}_n)$ where $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \mathcal{L}^{\sigma^{n-1}}$, and $\mathcal{L}^\sigma = \sigma^* \mathcal{L}$ [ATV1, Theorem 2].

Let A be an elliptic algebra. By a quantum plane [Ar], we mean the 'non-commutative projective plane' proj-A where proj-A is defined to be the quotient category gr-A/tor-A (gr-A is the category of finitely generated graded A-modules and tor-A is the subcategory of gr-A consisting of torsion modules). Thus, by definition, the algebro-geometric properties of quantum planes are described in terms of the modules over elliptic algebras.

The study of modules over elliptic algebras was started in [ATV1, ATV2], where the interest was primarily on modules of Gelfand-Kirillov (gk-) dimension 1. It was shown that the points of E parametrize certain A-modules called point modules. A point module N is a graded A-module with these properties: (i) $N_0 = k$, (ii) N_0 generates N, and (iii) the Hilbert function $h_N(n) = \dim_k(N_n) = 1$ for $n \geq 0$. We say that an A-module M is *Cohen-Macaulay* if $pd(M) = 3 - gk(M)$, (where $pd(M)$, resp. $gk(M)$ is the projective dimension, resp. Gelfand-Kirillov dimension of M), and say that it is *normal* if it is Cohen-Macaulay and its Hilbert function h_M satisfies the condition: $h_M(n) = 0$ for $n < 0$ and $h_M(0) \neq 0$. An A-module M is Cohen-

Macaulay if and only if $\mathrm{Ext}_A^q(M, A) = 0$ for $q \neq 3 - gk(M)$ [ATV2, §4]. The module $M^\vee = \mathrm{Ext}_A^{3-gk(M)}(M, A)$ is called the dual of M. The point modules are precisely the normal modules of gk-dimension 1 and multiplicity 1.

The first chapter of my thesis is a study of *Cohen-Macaulay* modules over elliptic algebras and their relation to the geometry of quantum planes. We first give (§I.2) a geometric interpretation of the duality $N \rightsquigarrow N^\vee$ between left and right (shifted) point modules, in terms of an auto-bijection ρ of the set of points of E. We later use this to describe the geometry of normal modules. Our main interest then is in normal modules of gk-dimension 2, simply called gk-2 normal modules. A gk-2 normal module has a minimal resolution of the form (I.3.2)

$$0 \to A(-j_1) \oplus \ldots \oplus A(-j_r) \to A(-i_1) \oplus \ldots \oplus A(-i_r) \to M \to 0$$

with $i_\nu, j_\nu \geq 0$. Since a minimal resolution is unique up to isomorphism, the sequence of indices $\tau = (i_1, \ldots, i_r; j_1, \ldots, j_r)$ is unique where we arrange $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$ in increasing order. We call this sequence τ the *type* of M. We show that the Hilbert function of a gk-2 normal module is convex and use this to conclude that there are only finitely many types of critical gk-2 normal modules in any given multiplicity. Next we consider the geometry of gk-2 normal modules. To study the relation to geometry, we look at the 'points of a given module M', i.e., the points p on the cubic divisor E, such that M has a non-zero map to corresponding point modules N_p. We prove that there is a natural way to describe the points of gk-2 normal modules (Theorem I.3.21): the points of a gk-2 normal module M are the zeroes of a certain global section s_M of an invertible sheaf $\mathcal{L}_\tau$ on E. It turns out that the invertible sheaf $\mathcal{L}_\tau$ depends only on the type τ of the module M, and the section s_M is independent, up to a scalar, of the choice of minimal resolution. The invertible sheaf $\mathcal{L}_\tau$ is given by the formula (I.3.19)

$$\mathcal{L}_\tau = \mathcal{L}^{p_\tau(\sigma)}.$$

Here $p_\tau(\sigma)$ is a polynomial in σ (related to the Hilbert series) and $\mathcal{L}^{p_\tau(\sigma)}$ is the action of p_τ on $\mathcal{L}$, where we consider Pic(E) as a module over the group ring $\mathbf{Z}<\sigma>$, the action of σ being the pull-back by the functor σ^*. We denote the divisor of zeroes of s_M by $\mathrm{div}(M)$ and call it the divisor of M. We describe the points of the dual, $M^\vee$, of a normal module M in terms of the points of M (Theorem I.3.32): we show that $\mathrm{div}(M^\vee(-\epsilon)) = \sigma^{\epsilon-1}\mathrm{div}(M)$, where ϵ is an integer such that $M^\vee(-\epsilon)$ is normal. The next results (Theorems I.3.36, 3.41) describe how the points of different gk-2 normal modules are related.

We study the gk-2 normal modules of multiplicity 2 in more detail (§I.4), especially, the *conic* modules and their geometry. A conic module is a module of the form $A/\phi A$, where ϕ is a quadratic: $0 \neq \phi \in A_2$. We give some results about the criticality and factorization of conics: we show, for example, that a quadratic can factorize into linear forms in at most four different ways, there being an example in a Sklyanin algebra where four distinct factorizations for a quadratic exist.

We introduce the notion of gk-1 equivalence between modules (§I.5): two finite A-modules are *gk-1 equivalent* if their images in the quotient category gr-$A/\mathrm{GK}_{\leq 1}$ are isomorphic, where $\mathrm{GK}_{\leq 1}$ is the localizing subcategory of gr-A consisting of modules with gk-dimension at most 1. The notion of gk-1 equivalence is useful because the category gr-$A/\mathrm{GK}_{\leq 1}$ is related to the category mod-Λ_0 (where $\Lambda = A[g^{-1}]$, g being a normalizing element of degree 3, unique up to a scalar). We conjecture that there is a type λ of gk-2 normal modules which is generic in the sense that every gk-2 critical module is gk-1 equivalent to a shift of a module of type λ; and, show how the maps to point modules can be used to prove this conjecture under a certain hypothesis. (The gk-1 equivalence is also used in the next chapter to classify indecomposable injective modules.)

The appendix A gives a classification of conics in the 'Weyl Plane', i.e., the quantum plane corresponding to the non-reduced case of the elliptic algebra where the associated cubic divisor E is the triple projective line $3\mathbf{P}^1$. We use some known results, and derive some others, about the module theory of the Weyl algebra to give a classification of conic modules up to gk-1 equivalence. They can roughly be classified as parabolic and hyperbolic. There is nothing novel about parabolic modules, but, hyperbolic modules *are "quantized"*: two critical hyperbolic modules $A/(yz+\alpha x^2)A$ and $A/(yz+\beta x^2)A$ are equivalent if and only if $\alpha - \beta \in \mathbf{Z}$. The last result is based on [Di] and [Mr1].

The second chapter is a study of injective modules over elliptic algebras. We say that an indecomposable injective module is of class $\mathcal{E}^j$ if it is an injective hull of a finite critical module of gk-dimension j. Our main objective behind studying injective modules was to construct a minimal injective resolution of A

$$0 \to A \to E^3 \to E^2 \to E^1 \to E^0 \to 0$$

where E^j would be a direct sum (with certain multiplicities) of the indecomposable injective modules of class $\mathcal{E}^j$. This would be an analogue of the residue complex for commutative projective plane, and could be called a "residue complex" for quantum plane. We show (Theorem II.1.12) that, up to an isomorphism, the indecomposable injective A-modules of class $\mathcal{E}^2$ are exactly the injective hulls, $E_A(M_\alpha)$, of gk-2 critical Cohen-Macaulay modules M_α, one from each gk-1 equivalence class α. Then we show (Theorem II.2.8) that the injective hull of the module $B = A/(g)$ is isomorphic to $Q/A_{(g)}$, where Q is the graded field of fractions of A and $A_{(g)}$ is localization of A with respect to the Ore subset consisting of homogeneous elements of $A - (g)$. On the other hand, the module Q/Λ (where $\Lambda = A[g^{-1}]$) is an injective module which decomposes as a direct sum of injective hulls of gk-2 critical modules *not* gk-1 equivalent to B. Thus we construct a resolution of A up to the E^2 term:

$$0 \to A \to Q \to Q/A_{(g)} \oplus Q/\Lambda \to \dots .$$

We do not know yet how the E^1 term and the "residue-map" $E^2 \to E^1$ look like. But we do describe the injective hull of a point module as a direct limit of essential extensions, and show that its gk-dimension is 1. In the final section, we obtain a Matlis-type duality (II.4.15) for certain non-commutative complete local rings; the motivation for this coming from the algebra $R = \hat{A}_{(g)}$, the g-adic completion of the local ring $A_{(g)}$. Thus for a finite graded right R-module M we have

$$\mathrm{Hom}_{R^\circ}(\mathrm{Hom}_R(M, E), E) \simeq M$$

where R° is the opposite ring of R and E is a bimodule (in our case, $Q/A_{(g)}$) which is an injective hull of $R/(\hat{g})$.

The final chapter of my thesis is a construction of a "residue complex" (a minimal injective resolution) for a regular algebra B of dimension 2. It is well-known that regular algebras of dimension 2 are twisted homogeneous coordinate rings of the projective line $\mathbf{P}^1_k$. To construct the resolution, we employ certain results which are valid for twisted homogeneous coordinate rings of general projective space $\mathbf{P}^n_k$. So, we start, in the first section, with a unique factorization result for the twisted coordinate rings $B(X, \sigma, \mathcal{L})$, where $X = \mathbf{P}^n_k$, σ is an automorphism of X, and $\mathcal{L} = \mathcal{O}_X(1)$. Recall that B is defined [AV] by $B = \oplus_{n\geq 0} H^0(X, \mathcal{L}_n)$ where $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \dots \otimes \mathcal{L}^{\sigma^{n-1}}$. The uniqueness of factorization is in terms of the orbits of the irreducible elements. The key fact we use is the equivariance $\sigma^*\mathcal{L} \simeq \mathcal{L}$. In the second section, we use the factorization to get a partial fraction decomposition for non-negative degree elements of the graded quotient field of the twisted coordinate ring $B(\mathbf{P}^1_k, \sigma, \mathcal{L})$. The partial fraction decomposition is used later to show the surjectivity of a "residue-map" in the complex. In the third section, we construct a "residue-complex" for a regular algebra B of dimension 2. Residue-complexes for general twisted coordinate rings have previously been constructed in [Ye1], by geometric methods. Our method is rather algebraic, using (non-commutative) localizations B_ω of the algebra B at *orbits* ω of points of $\mathbf{P}^1_k$. We define B_ω as the graded quotient ring of B with respect to the Ore subset $S_\omega = \{\phi \text{ homogeneous element of } B \mid \phi(p) \neq 0 \ \forall p \in \omega\}$. Let K be the $\mathbf{Z}$-graded quotient field of B, and $B' = \mathrm{Hom}_k(B, k)$. Then, there is an exact sequence

$$0 \to B \to K \to \oplus_\omega K/B_\omega \to B'(2) \to 0$$

which is a minimal injective resolution of B both as a left and right B-module. The injective B-module K/B_ω is a direct sum of injective hulls of point modules corresponding to the points in the orbit ω. We also consider the ("sheafified") version of this sequence in the category Proj$-B$, where we use a Grothendieck-Serre-type duality for the non-commutative projective scheme Proj$-B$. (Proj$-B$ = Gr$-B$/Tor$-B$, Gr-A is the category of all graded A-modules and Tor-A is the subcategory of Gr-B consisting of torsion modules). The appendix B describes and proves the Serre duality for regular algebras, using some known results about their dualizing complexes [Ye1].

I

COHEN-MACAULAY MODULES OVER ELLIPTIC ALGEBRAS

In this chapter we study modules over elliptic algebras, especially modules of Gelfand-Kirillov (gk-)dimension 2, and their relation to the geometry in the quantum plane.

1 Background and Preliminaries

Let k be a fixed field. Let A be an $\mathbf{N}$-graded connected k-algebra, finitely generated in degree 1. (A is *connected* if $A_0 = k$.) Such an algebra is called a *regular algebra* if it satisfies the following conditions:

1. A has finite global dimension d,
2. A has polynomial growth, i.e., $\dim_k(A_n) \leq an^\delta$ for some positive real numbers a and δ.
3. A is Gorenstein, i.e, if the left module $_Ak$ has a minimal graded resolution
$$0 \to P^d \to \cdots \to P^1 \to P^0 \to {_Ak} \to 0$$
of length d by projectives of finite type, then the transposed complex is a resolution of a right module isomorphic to $k_A(c)$, the shift of k_A to some degree c :
$$0 \leftarrow k_A(c) \leftarrow P^{d^*} \leftarrow \cdots \leftarrow P^{1^*} \leftarrow P^{0^*} \leftarrow 0.$$

The last condition can also be stated as follows: $\mathrm{Ext}^q_A({_Ak}, A) = 0$ if $q \neq d$ and $\mathrm{Ext}^d_A({_Ak}, A) \simeq k_A(c)$.

In this thesis, we will consider regular algebras of dimensions 2 and 3 only.

Regular algebras of dimension 3

A systematic study of regular algebras of (global) dimension 3 was started in [AS], where it was shown that there are two basic possibilities for a regular algebra of dimension 3: It has r generators and r relations, each of degree $s = 5 - r$, where $r = 2$ or 3. In [ATV1] these algebras were geometrically realized by associating to them certain triples $\mathcal{T} = (E, \sigma, \mathcal{L})$, where E is a scheme ($E \subset \mathbf{P}^2$ if $r = 3$, $E \subset \mathbf{P}^1 \times \mathbf{P}^1$ if $r = 2$) , σ is an automorphism of E, and $\mathcal{L}$ is an invertible sheaf on E whose global sections define a morphism $\pi : E \to \mathbf{P}^{r-1}$ (π is the inclusion of E in $\mathbf{P}^2$ if $r = 3$, or is the projection of E on the first factor $\mathbf{P}^1$ if $r = 2$). In all cases, $A_1 = H^0(E, \mathcal{L})$. There are four possibilities for the triples:

- *the elliptic case*:
  1. $r = 3$, E is a cubic divisor in $\mathbf{P}^2$, and $\mathcal{L} = \mathcal{O}_E(1)$
  2. $r = 2$, E is a divisor of bidegree (2,2) in $\mathbf{P}^1 \times \mathbf{P}^1$, and $\mathcal{L} = pr_1^* \mathcal{O}_{\mathbf{P}^1}(1)$
- *the linear case*:
  1. $r = 3$, $E = \mathbf{P}^2$, and $\mathcal{L} \approx \mathcal{O}_{\mathbf{P}^2}(1)$
  2. $r = 2$, $E = \mathbf{P}^1 \times \mathbf{P}^1$, and $\mathcal{L} \approx pr_1^* \mathcal{O}_{\mathbf{P}^1}(1)$

In case $r = 3$, the algebra A has a resolution of the form

$$(1.1) \qquad 0 \to A(-3) \xrightarrow{[xyz]} A(-2)^3 \xrightarrow{M} A(-1)^3 \xrightarrow{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} A \to {}_A k \to 0.$$

Here $A(\nu)$ denotes the shift of A by ν. The elements x, y, z and the entries of the 3×3 matrix M have degree 1. It follows from this resolution that, for these algebras (the case $r = 3$), the integer c in the condition (3) of the definition of regular algebras is 3, so that $\mathrm{Ext}_A^3({}_A k, A) \simeq k_A(3)$. These algebras have the same Hilbert function as that of a commutative polynomial ring in 3 variables: $\dim_k A_n = (n^2 + 3n + 2)/2$ for $n \geq 0$ and, therefore have Gelfand-Kirillov dimension 3. They are viewed as "twists" or "quantizations" of the polynomial ring $k[x, y, z]$ and are also called *quantum polynomial rings* in 3 variables [Ar]. We will mainly be interested in the elliptic case of a quantum polynomial ring, i.e., the case that our regular algebra A corresponds to a cubic divisor E in $\mathbf{P}^2$. In this case, the algebra A is simply called an *elliptic algebra.* Recall that an elliptic algebra is a (left and right) Noetherian domain [ATV1]. One of the most interesting cases is when the cubic divisor E is an elliptic curve, and σ is a translation on E. In this case, A is called a *Sklyanin algebra.* The relations defining a generic Sklyanin algebra A over an algebraically closed field can be put into the form

$$(1.2) \quad axy + byx + cz^2 = 0, \;\; ayz + bzy + cx^2 = 0, \;\; azx + bxz + cy^2 = 0.$$

The relations (1.2) define a regular algebra unless either $a^3 = b^3 = c^3$, or two of the three quantities (a, b, c) are zero [AS, ATV1].

In this chapter the symbol A will always denote an elliptic algebra unless otherwise stated. We will also assume the ground field k to be algebraically closed.

Given an invertible sheaf $\mathcal{M}$ on E, we denote by $\mathcal{M}^\sigma$ the pull-back $\sigma^*\mathcal{M}$. Thus, for an open subset U in E, $\mathcal{M}^\sigma(U)$ is canonically identified with $\mathcal{M}(\sigma U)$ [AV,§2]. Given a section $s \in H^0(E, \mathcal{M})$, and a point $p \in E$, we denote by s_p the canonical image of s in the localization $\mathcal{M}_p$ and by $s(p)$ its canonical image in $\mathcal{M}_p/m_p\mathcal{M}_p \simeq k(p)$, where m_p is the maximal ideal of the local ring $\mathcal{O}_p$. Given sections s_1 and s_2 of invertible sheaves $\mathcal{M}_1$ and $\mathcal{M}_2$ on E, we canonically identify $(s_1 \otimes s_2)(p)$ as $s_1(p) \otimes s_2(p)$ in $k(p)$. Given $s \in H^0(E, \mathcal{M})$, we denote by s^σ the canonical image of s under the natural k-linear isomorphism

$$H^0(E, \mathcal{M}) \xrightarrow{\sim} H^0(E, \mathcal{M}^\sigma).$$

Also, $s^\sigma(p) \in k(p)^\sigma$ is canonically identified with $s(\sigma p) \in k(\sigma p)$ under the natural isomorphism $k(p)^\sigma \xrightarrow{\sim} k(\sigma p)$. *For a section $s \in H^0(E, \mathcal{M})$ we will denote by* $\operatorname{div}(s)$ *the divisor of zeros of* s. We have $\operatorname{div}(s^\sigma) = \sigma^{-1}\operatorname{div}(s)$.

In an elliptic algebra A, there is a normalizing element g of degree 3, unique up to a scalar factor; and, there is an isomorphism of graded rings $B = A/gA \xrightarrow{\sim} \oplus_{n \geq 0} H^0(E, \mathcal{L}_n)$ where $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \mathcal{L}^{\sigma^{n-1}}$ [ATV1, Theorem 2]. The algebra B has Gelfand-Kirillov dimension 2, and is a domain if the cubic divisor E is reduced. We will often consider the homogeneous elements of B as global sections of the invertible sheaves $\mathcal{L}_n$. For a homogeneous element $\phi \in A$, we denote by $\bar{\phi}$ (or, by ϕ itself if no confusion arises) its image in B. Note that for homogeneous elements ϕ and ψ of B of degrees n and m respectively,

$$(\phi.\psi)(p) = \phi(p) \otimes \psi(\sigma^n p) \tag{1.3a}$$

and,

$$\operatorname{div}(\phi.\psi) = \operatorname{div}(\phi \otimes \psi^{\sigma^n}) = \operatorname{div}(\phi) + \sigma^{-n}\operatorname{div}(\psi). \tag{1.3b}$$

Non-commutative projective schemes, Quantum planes

We recall very briefly the notion of a non-commutative projective scheme associated to an N-graded non-commutative Noetherian algebra R over a field [Ar, AZ]. This definition is motivated by a fundamental theorem of Serre [Se]. Let $\mathrm{Gr}-R$ (resp. $\mathrm{gr}-R$) denote the category of all graded right R-modules (resp. graded Noetherian right R-modules) with morphisms being graded homomorphisms of degree 0, and let $\mathrm{Tor}-R$ (resp. $\mathrm{tor}-R$) denote the full subcategory of $\mathrm{Gr}-R$ (resp. $\mathrm{gr}-R$) consisting of $\mathbf{m}$-torsion modules, where $\mathbf{m}=\oplus_{n\geq 1}R_n$. (Recall that an element x of a graded module M is said to be $\mathbf{m}$-torsion if $x\mathbf{m}^n=0$ for some n. The set of torsion elements in a module form a graded submodule of M denoted by $t(M)$. A module M is called $\mathbf{m}$-torsion if $t(M)=M$. A Noetherian torsion module is necessarily finite k-dimensional.) The theorem of Serre states that if R is commutative and generated in degree 1, then the category $mod-\mathcal{O}_X$ of coherent sheaves on the (commutative) projective scheme $X=\mathbf{Proj}(R)$ is naturally equivalent to the quotient category $\mathrm{gr}{-}R/\mathrm{tor}{-}R$. In case R is non-commutative, the projective scheme $\mathrm{proj}-R$ is *defined* to be the quotient category $\mathrm{gr}{-}R/\mathrm{tor}{-}R$ along with two additional structures: one is the object $\mathcal{O}_R$ in $\mathrm{proj}-R$ which is the canonical image of the right R-module R_R in $\mathrm{proj}-R$, and the other is the shift functor $s_R:\mathcal{M}\rightsquigarrow\mathcal{M}(1)$ on $\mathrm{proj}-R$, induced by the shift functor s of $\mathrm{gr}-R$, $s(M)=M(1)$. So, by definition *"the projective scheme of R"* is the triplet $\mathrm{proj}-R=(\mathcal{C}_R,\mathcal{O}_R,s_R)$ where $\mathcal{C}_R=\mathrm{gr}{-}R/\mathrm{tor}{-}R$. Sometimes one also defines the *general projective scheme of* R, $\mathrm{Proj}{-}R$, as the corresponding triplet $\mathrm{Proj}{-}R=(\mathrm{Gr}{-}R/\mathrm{Tor}{-}R,\mathcal{O}_R,s_R)$. For a module M in $\mathrm{gr}-R$ we will generally denote its image in $\mathrm{proj}-R$ by the corresponding calligraphic letter $\mathcal{M}$. Note that

$$\mathrm{Hom}_{\mathrm{proj}\text{-}R}(\mathcal{M},\mathcal{N})=\lim_{n\to\infty}\mathrm{Hom}_{\mathrm{gr}\text{-}R}(M_{\geq n},N_{\geq n})$$

where $M_{\geq n}=\oplus_{k\geq n}M_k$, $N_{\geq n}=\oplus_{k\geq n}N_k$. Further, $\mathcal{M}\simeq\mathcal{N}$ in $\mathrm{proj}{-}R$ if and only if $M_{\geq n}\simeq N_{\geq n}$ in $\mathrm{gr}-R$ for large n.

When R is a quantum polynomial ring (in 3 variables), we call $\mathrm{proj}{-}R$ a non-commutative projective plane or a *quantum plane.* As remarked earlier, we will mainly be interested in the elliptic case of a quantum plane, i.e., the quantum plane defined by an elliptic algebra A, and, in the following, by a quantum plane we will mean this case only. Thus, the algebro-geometric properties of the quantum plane are, by definition, described in terms of A-modules. We will define below points, lines and, later, conics in the quantum plane in terms of certain modules called point, line or conic modules respectively. Given an A-module, one of the objectives would be to describe its points in the quantum plane.

Basic results about modules

All modules considered will be graded modules, unless otherwise specified. By the term A-module we will mean a graded left or right A-module. We denote the Gelfand-Kirillov (gk-) dimension of a module M by $gk(M)$. A module of gk-dimension m will simply be called a gk-m module. A *finite* A-module will mean a finitely generated graded A-module. A module M is said to be *locally finite* if each homogeneous component M_n is a finite dimensional vector space over k and is said to be *left bounded* if $M_n = 0$ for all $n << 0$. A finite module is clearly locally finite and left bounded. For a locally finite module M, we denote its Hilbert function by h_M: $h_M(n) = \dim_k(M_n)$. If no confusion arises then we also denote by h_M itself the Hilbert *series* of M: $h_M(t) = \sum \dim_k(M_n)t^n$. For a non-zero finite module M, the order of the pole of $h_M(t)$ at $t = 1$ equals $gk(M)$; and, the leading coefficient of the series expansion of $h_M(t)$ in powers of $(1-t)$ is a positive integer called the *multiplicity* of M, denoted $e(M)$ [ATV2, 2.21]. In fact, the Hilbert function of a finite module M is given by a polynomial $p(n)$ for large n: $h_M(n) = p(n)$ for $n >> 0$. Then the degree k of $p(n)$ is one less than $gk(M)$ and the leading coefficient of $p(n)$ is $e(M)/k!$. We denote by $M(\nu)$ the shift of M by degree ν: $M(\nu)_n = M_{\nu+n}$ and by $M_{\geq\nu}$ the *tail* (sub-)module defined by $(M_{\geq\nu})_n = M_n$ if $n \geq \nu$ and $(M_{\geq\nu})_n = 0$ if $n < \nu$.

Recall [C,§7] that every left bounded A-module M has a minimal projective resolution which is unique up to (non-unique) isomorphism, and if M is locally finite so are the projectives in the minimal resolution. A left bounded module P is projective in the category of graded modules if and only if it is free, i.e., isomorphic to a direct sum of shifts of the module A [ATV2, 2.6]. *We will use the notation* $A(i_1, \ldots, i_r) = \oplus_{k=1}^{r} A(i_k)$. Let M have a minimal resolution

$$0 \to P^n \to P^{n-1} \to \ldots P^1 \to P^0 \to M \to 0 \tag{1.4}$$

with, say, $P^k = A(-i_{1k}, \ldots, -i_{r_k k})$. (The negative sign is chosen to conform to later notations, 3.3.) The uniqueness of the minimal resolution implies that the sequence of indices $(i_{10}, \ldots, i_{r_0 0}\,; \ldots; i_{1n}, \ldots, i_{r_n n})$ is unique, where we take the indices $(i_{1k}, \ldots, i_{r_k k})$ in increasing order. We call this sequence the *type* of M. It follows again by the uniqueness of a minimal resolution that if the modules M_1 and M_2 are isomorphic, they are of the same type. We will generally denote a type by θ or τ.

We use the symbol $\mathrm{Hom}_A(M, N)$ to denote the graded group whose homogeneous component of degree ν consists of degree-preserving homomorphisms $M \to N(\nu)$. We use the notation $\mathrm{Hom}_{\mathrm{Gr}-A}(M,N)$ for the group of degree-zero homomorphisms in Gr$-A$. The categories Gr$-A$ and $A-$Gr have enough projectives and injectives [NV, Chapter A]. The derived functor of the graded $\mathrm{Hom}_A(M,N)$ is denoted by $\mathrm{Ext}^q_A(M,N)$. We will write $E^q(M)$ for $\mathrm{Ext}^q_A(M,A)$.

The key technique used to study modules is duality. If we denote by M a finite module or a finite complex of projectives, then M^D will denote its dual $\mathrm{RHom}_A(M,A)$. Thus, when M is represented by

$$0 \to P^k \to \cdots \to P^1 \to P^0 \to 0,$$

(for example, a projective resolution of M when M is a module) then M^D is represented by the transpose sequence

$$0 \leftarrow P^{k*} \leftarrow \cdots \leftarrow P^{1*} \leftarrow P^{0*} \leftarrow 0,$$

where $P^* = \mathrm{Hom}_A(P,A)$. The q-th cohomology of this complex is $\mathrm{Ext}^q_A(M,A)$. The biduality isomorphism $M \to M^{DD}$ expresses itself on Ext by a spectral sequence [ATV2, 2.33]

$$E_2^{p,q} = \mathrm{Ext}^p_A(\mathrm{Ext}^{-q}_A(M,A),A) \Rightarrow M.$$

It can be put into the standard first quadrant form, by reindexing it as

$$E_2^{p,q} = \mathrm{Ext}^p_A(\mathrm{Ext}^{d-q}_A(M,A),A) \Rightarrow M_{[d]}. \tag{1.5}$$

Here $M_{[d]}$ denotes the shift of the complex M by position d, where $d = 3$, the global dimension of the algebra A.

Recall that an A-module is said to be *critical* if it is not zero and every proper quotient has lower gk-dimension. A module M is called *pure* if the gk-dimension of all its non-zero submodules equals $gk(M)$. A critical module is pure. The gk-dimension provides a filtration of a finite A-module [ATV2, 2.29]:

Proposition 1.6 *Let M be a finite A-module of gk-dimension m.*

(i) There exists a finite filtration of M (called critical series of M) whose successive quotients are critical.

(ii) The sum M_ν of all submodules of M of gk-dimension $\leq \nu$ is a characteristic submodule of M, $gk(M_\nu) \leq \nu$, and there is a filtration

$$M_0 \subset M_1 \subset \ldots \subset M_{m-1} \subset M_m = M, \tag{1.7}$$

such that the quotient $M_\nu/M_{\nu-1}$ is pure ν-dimensional.

An important fact is that the filtration defined by the spectral sequence (1.5) is the same as the filtration defined by gk-dimension (1.7) [ATV2, §4].

We denote the projective dimension of a module M by $pd(M)$, and write $E^q(M)$ for $\mathrm{Ext}^q_A(M, A)$. An A-module M is called *Cohen-Macaulay* if $pd(M) = 3 - gk(M)$. This is equivalent to saying that $E^q(M) = 0$ for all $q \neq 3 - gk(M)$ [ATV2, §4]. An A-module is said to be *normal* if it is Cohen-Macaulay and satisfies the condition: $h_M(n) = 0$ for $n < 0$, $h_M(0) \neq 0$. For a Cohen-Macaulay module M there exists a unique integer m such that the shift $M(m)$ is normal. We call this shifted module *normal shift of M*. The *dual $M^\vee$* of an A-module M is defined to be the module $E^{3-gk(M)}(M)$. We collect the basic facts about the duality in the following proposition. We denote the augmentation ideal $A_{\geq 1}$ by $\mathbf{m}$.

Proposition 1.8. *Let M be a non-zero finite A-module of gk-dimension m. Then*

(i) $E^q(M) = 0$ if $q < 3 - m$.

(ii) $gk(M^\vee) = m$ and $e(M^\vee) = e(M)$.

(iii) If $m < 3$, then $M^\vee$ is Cohen-Macaulay.

(iv) There is a canonical map $\mu_M : M \to M^{\vee\vee}$ which is an isomorphism if and only if M is Cohen-Macaulay.

(v) $\ker(\mu_M)$ is the maximal submodule of M which has gk-dimension $< m$, and $gk(\mathrm{coker}\mu) \leq m - 2$. Thus, a Cohen-Macaulay module is pure.

(vi) $E^0(M) = M^* = 0$ if and only if $m < 3$.

(vii) The following conditions are equivalent:

(a)$E^3(M) = 0$,

(b) M is $\mathbf{m}$-torsion-free,

(c) $\mathrm{socle}(M) = \mathrm{Hom}_A(k, M) = 0$,

(d) $pd(M) < 3$.

The proofs are given in [ATV2, §2-4].

2 Duality among Point Modules

We recall the basic results about Cohen-Macaulay modules of gk-dimension 1 [ATV2, §6]. An A-module N of gk-dimension 1 is Cohen-Macaulay if and only if its socle is zero. The gk-1 normal modules of multiplicity 1 are precisely the point modules. A *point module* is an A-module with these properties: (i) $N_0 = k$, (ii) N_0 generates N, (iii) $\dim_k(N_n) = 1$ for all $n \geq 0$. It is known [ATV1, §3] that point modules are in bijective correspondence with points on the cubic divisor E. Given a point $p \in E$, we denote by N_p (resp. ${}_pN$) the corresponding right (resp. left) point module. In fact, the point module N_p (resp. ${}_pN$) is isomorphic to A/W_pA (resp. A/AW_p) where $W_p = \{l \in A_1 \mid l(p) = 0\}$ is the two-dimensional k-vector space of linear forms vanishing at p. The point modules are actually B-modules and one can write $N_p = B/W_pB$, ${}_pN = B/BW_p$. A point module is critical as an A- or a B-module [ATV2, 6.17]. As A is generated in degree 1, it follows that the only submodules of a point module N are the tails $N_{\geq k}$. The tails, being socle-free, would be again shifted point modules. In fact, the shift is related to the automorhism σ in the following way.

Proposition 2.1. *(i) $N_p(1)_{\geq 0}$ is isomorphic to $N_{\sigma p}$ as a right A-module. (ii) ${}_pN(1)_{\geq 0}$ is isomorphic to ${}_{\sigma^{-1}p}N$ as a left A-module.*

Proof. *(i)* Choose a linear form w such that $w(p) \neq 0$. Define a map $B \to N_p(1) = (B/W_pB)(1)$ by $b \mapsto wb$. As $(wb)(p) = w(p) \otimes b(\sigma p)$, we have $(wb)(p) = 0 \Leftrightarrow b(\sigma p) = 0$. Thus the map factors through an injective map $N_{\sigma p} \hookrightarrow N_p(1)$. Comparing dimensions one sees that $N_{\sigma p} \simeq N_p(1)_{\geq 0}$. It is similar to verify *(ii)*.□

Corollary 2.2. *(i)* $(N_p)_{\geq n} \simeq N_{\sigma^n p}(-n)$. *(ii)* ${}_pN_{\geq n} \simeq {}_{\sigma^{-n}p}N(-n)$.

Proof. By induction, it suffices to note that $(N_p)_{\geq 1} = (N_p(1))_{\geq 0}(-1) = N_{\sigma p}(-1)$. □

Let $\mathcal{N}_p$ denote the image, in proj$-A$, of the point module N_p. By *a point in the quantum plane* we mean an object of proj$-A$ which is isomorphic to $\mathcal{N}_p$ for some $p \in E$. Note that $\mathcal{N}_p \simeq \mathcal{N}_q$ is equivalent to $(N_p)_{\geq n} \simeq (N_q)_{\geq n}$ for some n, and hence to $p = q$ (Corollary 2.2). Thus points in the quantum plane are in bijective correspondence with point modules or with points on E, and sometimes we will identify a point in the quantum plane with a point on E.

It follows from Proposition (1.8 *iii, iv*) that $N \rightsquigarrow N^\vee = E^2(N)$ is a duality between left and right gk-1 Cohen-Macaulay modules of multiplicity 1. Here we describe this duality geometrically in terms of an auto-bijection ρ of the set of points of E.

By a linear form (in A) we mean a non-zero element of A_1 and, since $A_1 = B_1 = H^0(E, \mathcal{L})$, we regard it as a global section of $\mathcal{L}$. A linear form l

defines a line $\ell : l = 0$ in the projective plane $\mathbf{P}^2$. If no confusion arises, we will denote by l itself the line that $l \in A_1$ represents. We say that a linear form passes through a point, or two linear forms intersect at a point etc. if the corresponding lines have the same property. Recall that $\operatorname{div}(l)$ denotes the divisor of zeros of l, which has degree 3. This means the same thing as $(l.E)$, the intersection of the line (represented by) l and the cubic divisor E, counted with multiplicities.

Lemma 2.3. *Let w_1, w_2, l_1, l_2 be linear forms in A such that $l_1w_2 + l_2w_1 = 0$, then l_1, l_2 (resp. w_1, w_2) intersect at a point on E. Assume l_1, l_2 to be linearly independent, and let p (resp. $\tilde{p}$) be the point of intersection of l_1, l_2 (resp. w_1, w_2). Then*

$$\operatorname{div}(w_i) - \tilde{p} = \sigma(\operatorname{div}(l_i) - p) \qquad i = 1, 2. \tag{2.4}$$

Proof. Since $l_1 \otimes w_2{}^\sigma = -l_2 \otimes w_1{}^\sigma$, we get by (1.3)

$$\operatorname{div}(l_1) + \sigma^{-1}\operatorname{div}(w_2) = \operatorname{div}(l_2) + \sigma^{-1}\operatorname{div}(w_1). \tag{2.5}$$

If l_1, l_2 did not meet on E then $\operatorname{div}(l_1) = \sigma^{-1}\operatorname{div}(w_1), \operatorname{div}(l_2) = \sigma^{-1}\operatorname{div}(w_2)$. This would imply that $\sigma^*\mathcal{L} \simeq \mathcal{L}$, but we know that σ does not fix the class of $[\mathcal{L}]$ in $\operatorname{Pic}(E)$ [ATV1, Theorem 3]. So, l_1, l_2 intersect on E. Similarly it follows that w_1, w_2 intersect on E. If l_1, l_2 are linearly independent then so are w_1, w_2. Thus l_1, l_2 (resp. w_1, w_2) have a single point in common, say p (resp. $\tilde{p}$). It follows then from the relation (2.5) that $\operatorname{div}(l_i) - p = \sigma^{-1}(\operatorname{div}(w_i) - \tilde{p})$ for $i = 1, 2$, which is same as (2.4). □

Sometimes we write formula (2.4) as

$$(w_i.E) - \tilde{p} = \sigma((l_i.E) - p) \qquad i = 1, 2.$$

(2.6) Let p be a point on E, N_p the corresponding point module. Let $\{l_1, l_2\}$ be a basis of W_p. Then N_p has a minimal resolution of the form [ATV2,6.7]

$$0 \to A(-2) \xrightarrow{\begin{bmatrix} w_2 \\ w_1 \end{bmatrix}} A(-1) \oplus A(-1) \xrightarrow{[l_1, l_2]} A \to N_p \to 0. \tag{2.7}$$

with $l_1w_2 + l_2w_1 = 0$; $w_1, w_2 \in A_1$. In view of Lemma (2.3), it follows that w_1, w_2 intersect on E. We denote this point by $\rho(p)$. So far this depends upon the choice of a basis for W_p as well as the choice of w_1, w_2 in (2.7). We now show that for a given point p, $\rho(p)$ does not depend on the choice of basis for W_p nor on the choice of w_1, w_2 in (2.7). This is shown by the following intrinsic characterization of $\rho(p)$.

Proposition 2.8. *Let p be a point on E and $\rho(p)$ as defined in (2.6). Then*

$$(N_p)^\vee \simeq {}_{\rho(p)}N(2). \tag{2.9}$$

Proof. Since w_1, w_2 form a basis for the space $W_{\rho(p)}$ of linear forms vanishing at $\rho(p)$, we have ${}_{\rho(p)}N = A/A(w_1, w_2)$. Further, as $l_1w_2 + l_2w_1 = 0$, we have the following resolution of the left A-module ${}_{\rho(p)}N$

$$0 \to A(-2) \xrightarrow{[l_1, l_2]} A(-1) \oplus A(-1) \xrightarrow{\begin{bmatrix} w_2 \\ w_1 \end{bmatrix}} A \to {}_{\rho(p)}N \to 0. \tag{2.10}$$

On the other hand, applying the functor $\mathrm{Hom}_A(,A)$ to the resolution (2.7) of N_p gives the following complex of left A-modules

$$0 \to A \xrightarrow{[l_1, l_2]} A(1) \oplus A(1) \xrightarrow{\begin{bmatrix} w_2 \\ w_1 \end{bmatrix}} A(2) \to 0 \tag{2.11}$$

where the map $[l_1, l_2]$ is injective. We have used the fact that $\mathrm{Hom}(N_p, A) = 0$ (1.8 *vi*). Thus $(N_p)^\vee = \mathrm{Ext}^2(N_p, A) = A(2)/\mathrm{im}\begin{bmatrix} l_2 \\ l_1 \end{bmatrix}$ which by (2.10) is same as ${}_{\rho(p)}N(2)$. $\square$

Since two (left or right) point modules are isomorphic if and only if the corresponding points are equal, we see that we have a (well-defined) map

$$\rho : E \to E. \tag{2.12}$$

defined algebraically by (2.9). The map ρ is a bijection (Proposition 2.14 *i*). Thus the algebraic duality between (shifted) point modules has the following geometric interpretation in terms of the auto-bijection ρ of the points of E. For a point $p \in E$, $\rho(p)$ is the unique point on E with this property: for any line ℓ through p, the line ℓ' such that $\sigma(\ell.E - p) \subset \ell'.E$ passes through $\rho(p)$; in fact, $\ell'.E = \sigma(\ell.E - p) + \rho(p)$. For technical reason, to become clear later, we will call $\sigma^{-1}\rho p$ (rather than ρp) the point *dual* to p. We use the notations

$$p^\vee = \sigma^{-1}\rho(p), \qquad p^* = \sigma\rho^{-1}(p). \tag{2.13}$$

We collect the basic facts about the map ρ in the following proposition.

Proposition 2.14. *(i) The map ρ is a bijection.*

(ii) No point is self-dual, i.e., the map $\sigma^{-1}\rho$ has no fixed point.

(iii) ρ commutes with σ: $\rho\sigma(p) = \sigma\rho(p)$ for all $p \in E$.

(iv) In case of a Sklyanin algebra, $\rho = \sigma^{-2}$.

Proof. The statement (i) follows from the biduality isomorphism $N_p \simeq (N_p)^{\vee\vee}$ for point modules. More explicitly, one can show as in (2.8) that $({}_qN)^\vee \simeq (N_{\eta(q)})(2)$ for some point $\eta(q)$ on E. This gives another well-defined map $\eta : E \to E$. Now, the maps ρ and η are inverse to each other. Indeed, $({}_{\rho(p)}N)^\vee \simeq ((N_p)^\vee(-2))^\vee \simeq N_p(2)^{\vee\vee} \simeq N_p(2)$ so that $\eta\rho(p) = p$. Similarly, $\rho\eta(p) = p$.

To prove (ii), suppose $\sigma p = \rho p$ for some point p. Then, in the notations of (2.6), it follows from Lemma (2.3) that $\mathrm{div}(w_i) = \sigma(\mathrm{div}(l_i))$, implying that $\sigma^*\mathcal{L} \simeq \mathcal{L}$, which is a contradiction.

We prove (iii). From (2.1), we have a short exact sequence

$$0 \to N_{\sigma p}(-1) \to N_p \to k_A \to 0$$

which gives the following long exact sequence of cohomology

$$0 = E^2(k_A) \to E^2(N_p) \to E^2(N_{\sigma p}(-1)) \to E^3(k_A) \to 0.$$

Since $E^2(N_p) = {}_{\rho(p)}N(2)$, $E^2(N_{\sigma p}(-1)) = {}_{\rho\sigma(p)}N(3)$ and $E^3(k_A) = {}_Ak(3)$, we get an exact sequence

$$0 \to {}_{\rho(p)}N \to {}_{\rho\sigma(p)}N(1) \to {}_Ak(1) \to 0.$$

Thus we have an isomorphism ${}_{\rho(p)}N \simeq {}_{\rho\sigma(p)}N(1)_{\geq 0}$. Since, by (2.1), the latter is isomorphic to ${}_{\sigma^{-1}\rho\sigma(p)}N$, we get $\rho(p) = \sigma^{-1}\rho\sigma(p)$.

To prove (iv), choose p_0 as the origin on E and let σ be translation by the point q, in the group law of E, so that

$$p + q \sim \sigma p + p_0 \qquad \text{for all } p \in E \tag{2.15}$$

where $\sim$ means linear equivalence of divisors. Now let ℓ be a line through p, $\ell.E = p + p_1 + p_2$, say. Then (2.3) tells that $p + p_1 + p_2 \sim \rho p + \sigma p_1 + \sigma p_2$. Using (2.15) we get $\rho p + 2q \sim p + 2p_0$, hence $\rho p = \sigma^{-2} p$. □

3 Normal modules of GK-dimension 2

In this section we study Cohen-Macaulay modules of gk-dimension 2, and their "points" in the quantum plane. Recall the notation

$$A(i_1, \ldots, i_r) = \oplus_{k=1}^{r} A(i_k). \tag{3.1}$$

Proposition 3.2. *Let M be a Cohen-Macaulay A-module of gk-dimension 2 and multiplicity e. Then M has a minimal resolution of the form*

$$0 \to A(-j_1, \ldots, -j_r) \xrightarrow{f} A(-i_1, \ldots, -i_r) \to M \to 0 \tag{3.3}$$

where $r \leq \sum_{k=1}^{r}(j_k - i_k) = e$. Choosing $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_r)$ in increasing order, we have $i_k < j_k$ $(k = 1, \ldots, r)$. Further, if M is critical then $i_{k+1} < j_k$ $(k = 1, \ldots, r-1)$.

Proof. As $pd(M) = 1$, M has a minimal resolution of the form $0 \to P^1 \to P^0 \to M \to 0$, where P^0, P^1 are projective, hence free. A consideration of the Hilbert function shows that ranks of P^0 and P^1 are equal (say, r), and that $\sum_{k=1}^{r}(j_k - i_k) = e$.

Denote the map f in (3.3) by a matrix $[M]$. Choose $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_r)$ in increasing order. The entries m_{kl} of $[M]$ are homogeneous elements of A with $\deg(m_{kl}) = j_l - i_k$ if M is a right module, whereas $\deg(m_{kl}) = j_k - i_l$ if M is a left module. The minimality of the resolution (i.e., in this case, the minimality of the number of generators r) implies that there are no non-zero scalar entries in $[M]$, so the entries of $[M]$ are either zero or have positive degree. Now the assertion $i_k < j_k$ follows from the injectivity of the map f. For, if $j_k \leq i_k$ for some k, then $\mathrm{Im}(A(-j_1, \ldots, -j_k)) \subset (A(-i_1, \ldots, -i_{k-1}))$. (For $k = 1$ this means $\mathrm{Im}(A(-j_1)) = 0$.) Comparing dimensions, we see that this contradicts the injectivity of f.

To prove the final statement, suppose, on the contrary, that $j_k \leq i_{k+1}$ for some k. Then $\mathrm{Im}(A(-j_1, \ldots, -j_k)) \subset A(-i_1, \ldots, -i_k)$, and the quotient $Q = A(-i_1, \ldots, -i_k)/\mathrm{Im}(A(-j_1, \ldots, -j_k))$ will be a gk-2 module with multiplicity strictly less than e. This would imply that M has a proper gk-2 quotient, contradicting the hypothesis that M is critical. □

(3.4) For a gk-2 Cohen-Macaulay module M, the matrix $[M]$ described above depends on the choice of minimal resolution. However, as a minimal resolution is unique up to isomorphism, the matrices corresponding to different minimal resolutions of M will differ only by automorphisms of the free modules $\oplus A(-i_k)$ and $\oplus A(-j_k)$. Now note that the automorphisms of a free *right* A-module $\oplus_{k=1}^{r} A(-i_k)$ with $i_1 \leq \ldots \leq i_r$ are given by $r \times r$ matrices $[T]$, whose entries t_{kl} are zero if $i_l - i_k < 0$ and are homogeneous elements of A with $\deg(t_{kl}) = i_l - i_k$ otherwise. Such matrices are of the "upper *block*-triangular" form, i.e., there exists a partition $r = m_1 + \ldots + m_s$ such that $t_{kl} = 0$ if $k > m_1 + \ldots + m_\nu, l \leq m_1 + \ldots + m_\nu$ for all $\nu = 1, \ldots, s$; diagonal blocks being invertible square matrices with scalar entries. (Note. $s = 1$ is also possible.) Thus for a right gk-2 Cohen-Macaulay module M, if $[M_1]$ and $[M_2]$ are the matrices corresponding to two minimal resolutions, then $[M_2] = [S][M_1][T]$, for some automorphisms $[S]$ and $[T]$ of $\oplus A(-i_k)$ and $\oplus A(-j_k)$, respectively.

The *type* of a gk-2 Cohen-Macaulay module M is the unique set of indices $(i_1, \ldots, i_r; j_1, \ldots, j_r)$ appearing in its minimal resolution (3.3). If the modules M_1 and M_2 are isomorphic, then they are of the same type. To say that a gk-2 Cohen-Macaulay module is normal is same as saying that the index i_1 in (3.3) is 0. By a *gk-2 normal type* we mean the type of a gk-2 normal module. *We will usually denote a gk-2 normal type by τ.*

Proposition 3.5. *Let M_1, M_2 be gk-2 Cohen-Macaulay A-modules, and $\mathcal{M}_1, \mathcal{M}_2$ their images in (the quantum plane)* proj$-A$. *Then*

$$\mathrm{Hom}_{\mathrm{gr}\text{-}A}(M_1, M_2) \simeq \mathrm{Hom}_{\mathrm{proj}\text{-}A}(\mathcal{M}_1, \mathcal{M}_2).$$

Proof. Consider the exact sequence of cohomology by applying $\mathrm{Hom}_A(\ ,M_2)$ to the short exact sequence $0 \to (M_1)_{\geq n} \to M_1 \to T \to 0$ where T is a torsion module. To prove the proposition, it is sufficient to show that $\mathrm{Hom}_A(T, M) = 0$ and $\mathrm{Ext}^1_A(T, M) = 0$ for a torsion module T and a gk-2 Cohen-Macaulay module M. We can assume $T = k$. Then, the first is clear (1.8 *vii*). Now, using the resolution (3.3) and the fact $\mathrm{Ext}^i_A(k, A) = 0$ for $i \neq 3$, we get $\mathrm{Ext}^1_A(k, M) = 0$. □

We show that the Hilbert function of a gk-2 normal module M is convex, in the following sense: if we plot the function $h_M(n)$ for $n \in \mathbf{Z}$, and join the successive points by straight line segments, then the graph is convex (downwards). Further, in the notations of (3.2), $h_M(n) = en + c$ for $n \geq j_r - 2$, so that the graph is a straight line for $n \geq j_r - 2$. We prove the convexity as follows. For any finite A-module M, whose Hilbert function is $h_M(n)$, denote by $s_M(n)$ and $c_M(n)$, respectively, the *slope* and the *curvature* functions defined as:

(3.6) $\quad s_M(n) = h_M(n) - h_M(n-1) \qquad c_M(n) = s_M(n) - s_M(n-1).$

Note that, as we have defined, $c_M(n)$ is the difference of the upper and lower slopes at $(n-1)$, rather than at n.

Theorem 3.7. *Let M be a gk-2 normal module, as presented in (3.3). Then $c_M(n) = 0$ for $n < 0$, $n \geq j_r$, and $c_M(n) \geq 0$ for $0 \leq n < j_r$. If M is critical, then $c_M(n) \geq 1$ for $0 \leq n < j_r$.*

Proof. Starting with the Hilbert function of the free A-module $A(-i)$, we can show that

$$c_{A(-i)}(n) = \begin{cases} 0 & n < i, \\ 1 & n \geq i. \end{cases}$$

Since the Hilbert function is additive on exact sequences, so is the curvature function. Thus, using the resolution (3.3) we get

$$c_M(n) = \sum_{k=1}^{r} c_{A(-i_k)}(n) - \sum_{k=1}^{r} c_{A(-j_k)}(n) = \#\{i_k | i_k \leq n\} - \#\{j_k | j_k \leq n\}.$$

Let us denote the cardinalities of the two sets on the right hand side by $I(n)$ and $J(n)$ respectively. The inequalities $j_k > i_k$, for all k (3.2), imply that $I(n) \geq J(n)$, hence, $c_M(n) \geq 0$ for all n. For $n < 0$, $I(n) = J(n) = 0$, and, for $n > j_r$, $I(n) = J(n) = r$, thus completing the proof of the first part of the theorem. For a critical M, we have $j_k > i_{k+1}$ for $1 \leq k < r$, which implies that for $0 \leq n < j_r$, $I(n) \geq J(n) + 1$, thus $c_M(n) \geq 1$. □

Corollary 3.8. *There are only finitely many types of gk-2 critical normal modules of a given multiplicity e.*

Proof. The theorem implies that

$$e = \text{slope of the final line segment } = \sum_{n=0}^{j_r - 1} c_M(n) \geq j_r.$$

Thus we have bounds for all the parameters in (3.2): $0 < j_k \leq e, 0 \leq i_k < e$, $r \leq e$. This completes the proof. □

Example 3.9 [Case of multiplicity 1: Line modules].

For $e = 1$, the only critical normal type is $(0; 1)$ and the corresponding normal (right) module is $M_l = A/lA$ for some non-zero element l of A_1. The modules of this form are called *line modules*. Every element $0 \neq l \in A_1$ defines a line $\ell : \{l = 0\}$ in the ordinary projective plane $\mathbf{P}^2 = \mathbf{P}(A_1)$, and line modules are in canonical bijective correspondence with lines in the projective plane $\mathbf{P}^2$. A line module is critical [ATV2, 6.1] and its Hilbert function is $h_{M_l}(n) = n + 1$ for all $n \geq 0$. Let $\mathcal{M}_l$ denote the image of the line module M_l in proj$-A$. By *a line in the quantum plane* we mean an object of the quantum plane proj$-A$ which is isomorphic to $\mathcal{M}_l$, for some $0 \neq l \in A_1$. It follows from (3.5) that $\mathcal{M}_{l_1} \simeq \mathcal{M}_{l_2}$ if and only if $M_{l_1} \simeq M_{l_2}$. Thus, the lines in the quantum plane correspond bijectively to line modules and hence to lines in the ordinary projective plane $\mathbf{P}^2$.

Example 3.10 [Case of Multiplicity 2].

Let us classify the critical normal types in multiplicity 2. In the notation of (3.3), possible values of r are 1 and 2; the possible normal types are $(0;2)$ and $(0,0;1,1)$, and, the corresponding normal modules are presented by

$$0 \to A(-2) \to A \to M_1 \to 0,$$

$$0 \to A(-1) \oplus A(-1) \to A \oplus A \to M_2 \to 0.$$

We will study these modules in more detail in section 4.

Definition 3.11. *Let M be a right (resp. left) A-module and p a point of E. We say that p is a point of M if there is a non-zero A-homomorphism $M \to N_p$ (resp. $M \to {}_pN$).*

Since a point module N is a critical module of gk-dimension 1, it follows that every non-zero morphism $M \xrightarrow{\phi} N$ is surjective in high degrees (or, equivalently, the corresponding morphism $\mathcal{M} \to \mathcal{N}$ in the quantum plane proj$-A$ is surjective). In fact, $\mathrm{im}(\phi) = (N)_{\geq n}$ for some n. Note also that if p is a point of M then p is also a point of every submodule M' of M such that M/M' is finite-dimensional.

Remark 3.12 If p is a point of a right module M then $\sigma^n p$ is a point of $M(n)_{\geq 0}$ and $\sigma^{-n}p$ is a point of $M(-n)$ where $n \in \mathbf{N}$. Indeed, by induction over n it suffices to prove the statement for $n = 1$. Now a non-zero homomorphism $M \to N_p$ gives non-zero homomorphisms $M(1)_{\geq 0} \to N_p(1)_{\geq 0} \simeq N_{\sigma p}$ and $M(-1) \to N_p(-1) \hookrightarrow N_{\sigma^{-1}p}$ (2.1). In view of (2.1*ii*), replacing σ by σ^{-1} we get a similar result for a left module.

Remark 3.13 Let M be a right A-module and let M_B denote the B-module $M \otimes_A B = M/Mg$. Since point modules are B-modules, therefore p is a point of M if and only if there is a non-zero B-homomorphism $M_B \to N_p$. So the points of M can be described by considering just the B-module structure M_B.

We now show that the points of gk-2 normal modules of a given type τ can be described in terms of the divisor of zeros of certain sections of an invertible sheaf $\mathcal{L}_\tau$ on E. First we need a couple of lemmas about the zeroes of global sections. Recall that for a homogeneous element $\phi \in A$, $\bar{\phi}$ denotes its image in B, and is a global section of $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \ldots \otimes \mathcal{L}^{\sigma^{n-1}}$. Recall also that for a point $p \in E$, $W_p = \{l \in A_1 \mid l(p) = 0\}$.

Lemma 3.14. *Let ϕ be a homogeneous element of A. Then $\phi \in W_pA$ if and only if $\bar{\phi}(p) = 0$.*

Proof. Define $I_p = \{\phi \in A | \bar{\phi}(p) = 0\}$. Here, for an arbitrary $\phi \in A$, we say that $\bar{\phi}(p) = 0$ if $\bar{\phi}_i(p) = 0$ for all homogeneous components ϕ_i of ϕ. Then I_p is a right ideal of A containing W_pA. If $I_p \neq W_pA$ then A/I_p would be finite k-dimensional, since N_p is a critical gk-1 module. Thus $m^n \subset I_p$ for some $n > 0$. But this is a contradiction since for arbitrary large n there exist sections $\bar{\psi} \in B_n$ which do not vanish at p. □

(3.15) Recall that if $(E, \sigma, \mathcal{L})$ is a regular triple then so is $(E, \sigma^{-1}, \mathcal{L})$ [ATV1]. We denote by B° the twisted homogeneous coordinate ring of the triple $(E, \sigma^{-1}, \mathcal{L})$, i.e., $B^\circ = \oplus_{n \geq 0} H^0(E, \mathcal{L} \otimes \mathcal{L}^{\sigma^{-1}} \ldots \mathcal{L}^{\sigma^{-(n-1)}})$. There is a k-linear isomorphism called "transposition" in [Ye1, §6]:

$$\tau_n : B_n = H^0(E, \mathcal{L} \otimes \mathcal{L}^\sigma \ldots \otimes \mathcal{L}^{\sigma^n})^{(\sigma^*)^{-n+1}} \longrightarrow H^0(E, \mathcal{L} \otimes \mathcal{L}^{\sigma^{-1}} \ldots \mathcal{L}^{\sigma^{-(n-1)}}) = B^\circ_n.$$

Summing over all n we get a map $\tau : B \to B^\circ$, which is actually an anti-isomorphism of graded k-algebras [Ye1, 6.17]. (There should not be any confusion between the map τ and a type τ.) We will sometimes identify B° with the opposite algebra of B and a left B-module with a right B°-module. Since the points of an A-module M depend only upon the B-module M_B (Remark 3.13), we see that a statement about the points of a right A-module can be translated into a statement about the points of a left A-module by an appropriate replacement of σ by σ^{-1}. We will sometimes use the index $\imath$ in a formula to denote the parity of a module: $\imath$ will be 1 for a right module and -1 for a left module.

The left analogue of Lemma 3.14 is the following.

Lemma 3.16. *Let $\phi \in A_n$, then $\phi \in (AW_p)_n$ if and only if $\bar{\phi}(\sigma^{-(n-1)}p) = 0$.*

Proof. Clearly, $\phi \in (AW_p)_n \Leftrightarrow \bar{\phi} \in (BW_p)_n$. Since τ_1 is just identity, it follows that $\bar{\phi} \in (BW_p)_n$ if and only if $\tau(\bar{\phi}) \in (W_pB)_n$. By Lemma (3.14) this is equivalent to $\tau(\bar{\phi})(p) = 0$. Using the definition $\tau(\bar{\phi}) = (\bar{\phi})^{\sigma^{-(n-1)}}$, we see that this is same as $\bar{\phi}(\sigma^{-(n-1)}p) = 0$. □

Now to a given gk-2 normal type τ, we associate invertible sheaves $\mathcal{L}_\tau$ and ${}_\tau\mathcal{L}$ on E as follows.

Let $\mathbf{Z}[\sigma, \sigma^{-1}]$ be the Laurent polynomial ring, and $\mathbf{Z}[[\sigma, \sigma^{-1}]]$ the Laurent series ring. We will write $\frac{\sigma^n - 1}{\sigma - 1}$ to represent the element $\sum_{i=0}^{n-1} \sigma^i$. More generally, for $n, m > 0$ we put $\frac{\sigma^n - \sigma^m}{\sigma - 1} = (\frac{\sigma^n - 1}{\sigma - 1}) - (\frac{\sigma^m - 1}{\sigma - 1}) = \sum_{i=0}^{n-1} \sigma^i - \sum_{i=0}^{m-1} \sigma^j$. We define an association $M \rightsquigarrow p_M$ (M an A-module, $p_M \in \mathbf{Z}[[\sigma, \sigma^{-1}]]$) by

$$p_M(\sigma) = (1 - \sigma)^2 h_M(\sigma)$$

where $h_M(\sigma)$ is the Hilbert series of M. Note that for a gk-2 module M, $p_M \in \mathbf{Z}[\sigma, \sigma^{-1}]$ (the order of pole at 1 of the Hilbert series for a gk-2 module is 2). Alternatively, we could define this association by : (1) associating to the free A-module $A(-n)$ the element $\frac{\sigma^n}{1-\sigma}$ of $\mathbf{Z}[[\sigma, \sigma^{-1}]]$, and (2) requiring the association to be additive on exact sequences. Now note that p_M depends only on the *type* θ of M (1.4), so that we have, in fact, an association: $\theta \mapsto p_\theta$. For a gk-2 normal type $\tau = (i_1, \ldots, i_r; j_1, \ldots, j_r)$, we have

$$p_\tau(\sigma) = \frac{\sum_{k=1}^{r} (\sigma^{j_k} - \sigma^{i_k})}{\sigma - 1}. \tag{3.17}$$

We see that, for a gk-2 normal type τ, $p_\tau(\sigma)$ is actually a polynomial in σ.

Consider $\mathrm{Pic}(E)$ as a module over $\mathbf{Z}[\sigma, \sigma^{-1}]$, the action of σ being the pull-back by the functor σ^*. We denote the action of an element $\sum n_i \sigma^i$ of $\mathbf{Z}[\sigma, \sigma^{-1}]$ on an invertible sheaf $\mathcal{M}$ by: $(\mathcal{M})^{\sum n_i \sigma^i} = \otimes_i (\mathcal{M}^{\otimes n_i})^{\sigma^i} = \otimes_i (\mathcal{M}^{\sigma^i})^{\otimes n_i}$. In this notation,

$$\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \ldots \otimes \mathcal{L}^{\sigma^{n-1}} = \mathcal{L}^{\frac{\sigma^n - 1}{\sigma - 1}} \qquad (n > 0). \tag{3.18}$$

Now, for a gk-2 normal type τ, define

$$\mathcal{L}_\tau = \mathcal{L}^{p_\tau(\sigma)}. \tag{3.19}$$

Explicitly, we can write

$$\mathcal{L}_\tau = (\mathcal{L}_{j_1 - i_1})^{\sigma^{i_1}} \otimes (\mathcal{L}_{j_2 - i_2})^{\sigma^{i_2}} \ldots \otimes (\mathcal{L}_{j_r - i_r})^{\sigma^{i_r}}$$

where $\mathcal{L}_{j_k - i_k}$ are defined as in (3.18), which makes sense since $j_k - i_k > 0$.

We define ${}_\tau\mathcal{L}$ by replacing σ by σ^{-1} in the definition of $\mathcal{L}_\tau$; thus,

$${}_\tau\mathcal{L} = \mathcal{L}^{\frac{\sum_{k=1}^{r} (\sigma^{-j_k} - \sigma^{-i_k})}{\sigma^{-1} - 1}}. \tag{3.20}$$

More frequently, we will denote ${}_\tau\mathcal{L}$ by $\mathcal{L}_\tau^\circ$ and call it *the opposite of* $\mathcal{L}_\tau$.

Theorem 3.21. *Let τ be a gk-2 normal type and $\mathcal{L}_\tau$, $\mathcal{L}_\tau^\circ$ the associated invertible sheaves. To every gk-2 normal right (resp. left) A-module M of type τ there corresponds a global section s_M of $\mathcal{L}_\tau$ (resp. s_M° of $\mathcal{L}_\tau^\circ$) such that p is a point of M if and only if $s_M(p) = 0$ (resp. $s_M^\circ(p) = 0$).*

Proof. Let M be a right A-module presented as in (3.3), and let N_p be a point module. To give a non-zero A-homomorphism $M \to N_p = A/W_pA$ is equivalent to giving homogeneous elements ξ_k of A with $\deg(\xi_k) = i_k$ such that the entries of the row-vector $[\xi][M]$ are all in the ideal W_pA, but not all of ξ_k are in W_pA. This is equivalent (Lemma 3.14) to saying that the row vector $[\bar{\xi}_k(p)]$ is a non-trivial solution of the homogeneous equation

$$(3.22) \qquad [\bar{\xi}][\bar{M}](p) = 0$$

where a bar denotes the image in B. This equation is same as

$$(3.23) \qquad \sum_{k=1}^{r} (\bar{\xi}_k \otimes \bar{m}_{kl}^{\sigma^{i_k}})(p) = 0 \qquad l = 1, \ldots, r$$

where m_{kl} are the entries of $[M]$ and $\deg(m_{kl}) = j_l - i_k$. We now define a matrix X_M (which is different from $[M]$), the vanishing of whose determinant will give us a necessary and sufficient condition for the existence of a non-trivial solution to (3.23). Define

$$(3.24) \qquad (X_M)_{kl} = \bar{m}_{kl}^{\sigma^{i_k}} \in H^0(E, (\mathcal{L}_{j_l - i_k})^{\sigma^{i_k}}) \qquad l, k = 1, \ldots, r.$$

Recall that the matrix elements m_{kl} are either zero or have positive degree, so if $\bar{m}_{kl} \neq 0$ then $j_l - i_k > 0$ and writing $\mathcal{L}_{j_l - i_k}$ makes sense (3.18). Now set

$$(3.25) \qquad s_M = \det(X_M) = \sum_{\gamma \in S_r} \operatorname{sgn}(\gamma) (\bar{m}_{1,\gamma 1})^{\sigma^{i_1}} \otimes \ldots \otimes (\bar{m}_{r,\gamma r})^{\sigma^{i_r}}.$$

We claim that all non-zero terms within the summation sign in (3.25) are global sections of the same invertible sheaf $\mathcal{L}_\tau$. Indeed, the term in (3.25) corresponding to $\gamma \in S_r$ is non-zero only if $j_{\gamma k} - i_k > 0$ for all $k = 1, \ldots, r$. As $\bar{m}_{k,\gamma k}^{\sigma^{i_k}} \in H^0(E, (\mathcal{L}_{j_{\gamma k} - i_k})^{\sigma^{i_k}})$, to verify the claim, it is sufficient to check that

$$(\mathcal{L}_{j_{\gamma 1} - i_1})^{\sigma^{i_1}} \otimes \ldots \otimes (\mathcal{L}_{j_{\gamma r} - i_r})^{\sigma^{i_r}} = \mathcal{L}_\tau$$

if $j_{\gamma k} - i_k > 0$ for all k. This is clear because the left hand side is

$$\mathcal{L}^{\frac{\sum_{k=1}^{r} (\sigma^{j_{\gamma k}} - \sigma^{i_k})}{\sigma - 1}} = \mathcal{L}^{\frac{\sum_{k=1}^{r} (\sigma^{j_k} - \sigma^{i_k})}{\sigma - 1}} = \mathcal{L}_\tau.$$

We have shown that $s_M \in H^0(E, \mathcal{L}_\tau)$. Now, if there exists a non-zero morphism $[\xi] : M \to N_p$ then $[\bar{\xi}(p)]$ forms a non-trivial solution of (3.23). This implies that $s_M(p) = \det(X_M) = 0$. Conversely, if $s_M(p) = 0$ then we can solve (3.22) locally at p, i.e., we can find a non-trivial solution $[a_k(p)]$, $a_k(p) \in k(p)$, of the homogeneous system $\sum_{k=1}^r a_k(p) \otimes \bar{m}_{kl}^{\sigma^{i_k}}(p) = 0$ $(l = 1, \ldots, r)$. Since the maps from the space of global sections of the invertible sheaves $\mathcal{L}_{i_k}$ to $k(p)$ are all surjective, we see that there exists, in fact, a non-trivial global solution $[\bar{\xi}]$, $\bar{\xi}_k \in B_{i_k}$, of (3.22). This completes the proof in the case of right modules.

In view of the remarks in (3.15)), the result for left modules follows, abstractly, from that for right modules. Let us, however, explicitly write down the corresponding matrix, denoted by ${}_MX$ or X_M° for a left module M, vanishing of whose determinant gives a necessary and sufficient condition for the existence of a non-zero map from M to a point module. (We will need this in the proof of the next result.) We claim that X_M° is given by

$$(X_M^\circ)_{kl} = (\bar{m}_{kl})^{\sigma^{-(j_k-1)}} \in H^0(E, (\mathcal{L}_{j_k - i_l})^{\sigma^{-(j_k-1)}}) \tag{3.26}$$

and

$$s_M^\circ = \det(X_M^\circ). \tag{3.27}$$

To see that matrix X_M° is given by (3.26), let M be presented as in (3.3). To give a non-zero morphism $M \to {}_pN$ is equivalent to giving a column vector $[\xi]$ of homogeneous elements ξ_k in A with degree$(\xi_k) = i_k$ such that the entries of the column vector $[\bar{M}][\bar{\xi}]$ are in BW_p but not all $\bar{\xi_k}$ are in BW_p. Recall that $\deg(m_{kl}) = j_k - i_l$. As the degree of $([\bar{M}][\bar{\xi}])_k = \sum_{l=1}^r \bar{m}_{kl}\bar{\xi}_l$ is j_k $(k = 1, \ldots, r)$, we see, in view of Lemma (3.16), that $([\bar{M}][\bar{\xi}])_k \in BW_p$ is equivalent to $\sum_{l=1}^r (\bar{m}_{kl}\bar{\xi}_l)(\sigma^{-(j_k-1)}p) = 0$. This is same as

$$\sum_{l=1}^n \bar{m}_{kl}(\sigma^{-(j_k-1)}p) \otimes \bar{\xi}_l(\sigma^{-(i_l-1)}p) = 0. \tag{3.28}$$

Therefore the matrix vanishing of whose determinant gives a necessary and sufficient condition for the existence of a non-trivial solution of (3.28) is given by (3.26). □

Remark 3.29 As we have defined, the section s_M depends on the matrix $[M]$, i.e, on the choice of minimal resolution of M. We can show, however, that s_M *is independent, up to a scalar, of the choice of minimal resolution* *of* M. To see this, let $[M]$ and $[M'] = [U][M][V]$ be matrices representing the same module of a type $\tau = (i_1, \ldots, i_n; j_1, \ldots, j_n)$, where $[U]$ and $[V]$ have to be matrices as described in (3.4). Then, assuming the module is a right module, we get

$$(X_{M'})_{kl} = \sum_{r,s} (U_{kr} M_{rs} V_{sl})^{\sigma^{i_k}} = \sum_{r,s} (U_{kr})^{\sigma^{i_k}} \otimes (M_{rs})^{\sigma^{i_r}} \otimes (V_{sl})^{\sigma^{j_s}}.$$

Thus $X_{M'} = U' X_M V'$ where $U'_{kr} = (U_{kr})^{\sigma^{i_k}}$ and $V'_{sl} = (V_{sl})^{\sigma^{j_s}}$. Thus U', V' are again matrices of the form described in (3.4), whose diagonal blocks are invertible matrices with scalar entries. It follows that the determinants of U' and V' are scalars, and $s_{M'} = \det(X_{M'})$ is a scalar multiple of $s_M = \det(X_M)$. Thus $\operatorname{div}(s_M)$ is independent of resolution, and we will call it *the divisor of* M and denote it simply by $\operatorname{div}(M)$. Similarly, for a left module M we show that s_M° is independent of resolution, up to a scalar; and define $\operatorname{div}(M) = \operatorname{div}(s_M^\circ)$.

Remark 3.30 As k is algebraically closed, we note that *every gk-2 normal module does have a point.* It is not clear, however, whether a gk-2 normal module has a *surjective* map to some point module. In case where all the generators of M are in degree 0, it is trivial that every non-zero map to a point module is surjective. In particular this is true for all *principal* modules $A/\phi A$ where ϕ is a homogeneous element of A. Note that $\operatorname{div}(A/\phi A) = \operatorname{div}(\bar{\phi})$.

(3.31) Let M be a gk-2 Cohen-Macaulay A-module with minimal resolution (3.3). We will denote by ϵ_M (or, just ϵ if there is no confusion) the greatest index j_r in (3.2). It has the property that if M is normal then $M^\vee(-\epsilon)$ is also normal, as we will see below. Now we relate the divisors $\operatorname{div}(M)$ and $\operatorname{div}(M^\vee(-\epsilon))$.

Theorem 3.32. *Let M be a gk-2 normal A-module and ϵ as defined above (3.31). Then $M^\vee(-\epsilon)$ is normal, and*

$$\operatorname{div}(M^\vee(-\epsilon)) = \sigma^{\imath(\epsilon-1)}(\operatorname{div}(M))$$

where $\imath$ is 1 *for a right module and* -1 *for a left module.*

Proof. Let M be a right module presented as in (3.3). Taking the cohomology of (3.3) we get, with canonical identifications, the short exact sequence

$$0 \to \oplus_{k=1}^r A(i_k) \xrightarrow{[M]} \oplus_{k=1}^r A(j_k) \to M^\vee \to 0. \tag{3.33}$$

Here $[M]$ is the same matrix as in (3.3), but now the multiplication is on the right: $[v] \mapsto [v][M]$, $[v] \in \oplus_{k=1}^{r} A(i_k)$. First we observe that the normal shift of $M^\vee$ is precisely $M^\vee(-\epsilon)$, where $\epsilon = j_r$. Let us denote $M^\vee(-\epsilon)$ by M'. The type of M' is $\tau' = (i'_1, \ldots, i'_r; j'_1, \ldots, j'_r)$ where $i'_k = \epsilon - j_{r-k-1}$ and $j'_k = \epsilon - i_{r-k-1}$. The shift of (3.33) by $-\epsilon$ is a resolution of M', and the corresponding matrix $[M']$ is

$$(M')_{kl} = m'_{kl} = m_{r-k-1,r-l-1} = (M)_{r-k-1,r-l-1}.$$

Thus, in the notations of (3.26) and (3.24),

$$\begin{aligned}(X^\circ_{M'})_{kl} = (\bar{m}'_{kl})^{\sigma^{-(j'_k-1)}} &= (\bar{m}_{r-k-1,r-l-1})^{\sigma^{-(\epsilon-i_{r-k-1}-1)}} \\ &= ((X_M)_{r-k-1,r-l-1})^{\sigma^{-(\epsilon-1)}}\end{aligned}$$

It follows that $s^\circ_M = \det(X^\circ_{M'}) = (\det(X_M))^{\sigma^{-(\epsilon-1)}} = (s_M)^{\sigma^{-(\epsilon-1)}}$ and hence $\operatorname{div}(M') = \sigma^{\epsilon-1}\operatorname{div}(M)$.

Now, if M is a left A-module then the result follows from remarks in (3.15). □

We now obtain some results describing the geometry of gk-2 normal modules in terms of their points (Theorems 3.36, 3.41). For this, and further applications in the next chapter, we need to compute some Ext groups.

Lemma 3.34. *Let N_p, N_q be point modules, and $n \in \mathbf{Z}$. Then*

(i) $\operatorname{Hom}(N_q, N_p)_n = 0$ *for all p, q if $n < 0$; and, for $n \geq 0$,*

$$\operatorname{Hom}(N_q, N_p)_n = \begin{cases} 0 & q \neq \sigma^n p, \\ k & q = \sigma^n p. \end{cases}$$

(ii) $\operatorname{Ext}^1(N_q, N_p)_n = 0$ *for all p, q if $n < -1$;* $\operatorname{Ext}^1(N_q, N_p)_{-1} = k \oplus k$ *if $q = \sigma^{-1}p^*$, and k otherwise; and, for $n \geq 0$,*

$$\operatorname{Ext}^1(N_q, N_p)_n = \begin{cases} k & q = \sigma^n p, \\ k & q = \sigma^n p^*, \\ 0 & \textit{otherwise}. \end{cases}$$

(iii) $\operatorname{Ext}^2(N_q, N_p)_n = 0$ *for all p, q if $n < -2$;* $\operatorname{Ext}^2(N_q, N_p)_{-2} = k$ *for all p, q; and, for $n \geq -1$,*

$$\operatorname{Ext}^2(N_q, N_p)_n = \begin{cases} k & q = \sigma^n p^*, \\ 0 & \textit{otherwise}. \end{cases}$$

Recall that Hom and Ext are graded, and the subscript n denotes the degree of homogeneity.

Proof. (i) For a map $N_q \xrightarrow{\phi} N_p(n)$ $(\phi \in B_n)$ to be non-zero, we must have $\phi(p) \neq 0$ and $\phi W_q \subset W_p$. For this we must have $\sigma^n p = q$.

(ii) Let N_q have a resolution

$$0 \to A(-2) \xrightarrow{w=\begin{bmatrix} w_2 \\ w_1 \end{bmatrix}} A(-1) \oplus A(-1) \xrightarrow{l=[l_1,l_2]} A \to N_q \to 0.$$

Applying Hom(,N_p) gives the complex

$$0 \to \mathrm{Hom}(N_q, N_p) \to N_p \xrightarrow{l} N_p(1) \oplus N_p(1) \xrightarrow{w} N_p(2) \to 0$$

where the maps w and l are now right multiplications. Put $K = \ker(w), I = \mathrm{im}(l)$. We compute the dimensions of K_n, I_n.

The map l is trivially zero in degree $n < 0$. In degree $n \geq 0$, the map l is zero if and only if $l_1(\sigma^n p) = 0 = l_2(\sigma^n p)$, which is equivalent to $\sigma^n p = q$, because l_1, l_2 form a basis for W_q. Thus for $n < 0$ $\dim(I_n) = 0$; for $n \geq 0$, $\dim(I_n)$ is 0 if $q = \sigma^n p$ and 1 otherwise.

On the other hand, the map w is trivially zero in degree $n < -1$. In degree $n \geq -1$, the map w is zero if and only if $w_1(\sigma^{n+1}p) = 0 = w_2(\sigma^{n+1}p)$, which is equivalent to $\sigma^{n+1}p = \rho q$, because w_1, w_2 form a basis for $W_{\rho q}$ (2.3). Thus $\dim(K_n) = 0$ for $n < -1$; and, for $n \geq -1$, $\dim(K_n)$ is 2 if $q = \sigma^n p^*$ and 1 otherwise.

Knowing the dimensions of K_n and I_n we get formulas (ii) and (iii). □

Lemma 3.35. *Let M be a gk-2 normal A-module, and N a point module. Then*

$$\dim_k \mathrm{Ext}^1(M, N)_n = \dim_k \mathrm{Hom}(M, N)_n \qquad n \geq 0.$$

Proof. Let M be presented as in (3.3). Applying Hom(,N) we get the long exact sequence of cohomology

$$0 \to \mathrm{Hom}(M, N) \to \oplus_{k=1}^r N(j_k) \to \oplus_{k=1}^r N(i_k) \to \mathrm{Ext}^1(M, N) \to 0.$$

Since $i_k, j_k \geq 0$, and $\dim(N_n) = 1$ for $n \geq 0$, we have verified the claim. □

Theorem 3.36. *Let M be a gk-2 normal right A-module and p a point of M. The kernel of a non-zero homomorphism $M \xrightarrow{\phi} N_p$ is the shift $M'(-m)$ of a gk-2 normal module M' with $m = 0$ or 1 so we have a short exact sequence*

$$0 \to M'(-m) \to M \to (N_p)_{\geq n} \to 0 \tag{3.37}$$

for some n. Let S be the set of points of M, and put $S' = (S - \{p\}) \cup \{p^\vee\}$. Then the points $\sigma^m q\,(q \in S')$ are points of M', and, they exhaust the set of points of M except perhaps that $\sigma^m p$ could also be a point of M', where the last possibility definitely occurs if $\dim_k \mathrm{Hom}(M, N_p)_0 > 1$.

Needless to say, replacing σ by σ^{-1}, ρ by ρ^{-1} we get the corresponding result for a left module.

Proof. We know that $\mathrm{im}(\phi) = (N_p)_{\geq n} = N_{\sigma^n p}(-n)$ for some n. Let $K = \ker(\phi)$ so we have a short exact sequence: $0 \to K \to M \to N_{\sigma^n p}(-n) \to 0$. It is clear that $gk(K) = 2$. Considering the long exact sequence of cohomology of this short exact sequence, and using $E^q(M) = 0$ for $q \neq 1$, $E^q(N_{\sigma^n p}) = 0$ for $q \neq 2$, we get $E^q(K) = 0$ for $q \neq 1$, thus showing that K is a Cohen-Macaulay module. Let $M' = K(m)$ be the normal shift of K. As M is normal, $\dim_k(M_0) \geq 1$ and $\dim_k M_1 \geq 2$. Therefore, $\dim_k(K_0) \geq 0$ and $\dim_k K_1 \geq 1$. It follows that $0 \leq m \leq 1$.

In view of remarks in (3.12), note first that q is a point of K if and only if $\sigma^m q$ is a point of M'. So we show that points of S' are points of K. Now let N_q be an arbitrary point module. Applying $\mathrm{Hom}(\ ,N_q)$ to the above short exact sequence we get the exact sequence

(3.38)

$$\begin{array}{ccccccc}
0 & \to & \mathrm{Hom}(N_{\sigma^n p}(-n), N_q) & \to & \mathrm{Hom}(M, N_q) & \to & \mathrm{Hom}(K, N_q) \\
 & \to & \mathrm{Ext}^1(N_{\sigma^n p}(-n), N_q) & \to & \mathrm{Ext}^1(M, N_q) & \to & \mathrm{Ext}^1(K, N_q) \\
 & \to & \mathrm{Ext}^2(N_{\sigma^n p}(-n), N_q) & \to & 0 & &
\end{array}$$

We look at the sequence (3.38) for different values of q. First, put $q = p^\vee$, i.e., $q^* = p$. We see from (3.34 iii) that $\mathrm{Ext}^2(N_{\sigma^n p}(-n), N_{p^\vee})_0 \neq 0$. Thus $\mathrm{Ext}^1(K, N_{p^\vee})_0 \neq 0$, hence by (3.35), $\mathrm{Hom}(K, N_{p^\vee})_0 \neq 0$.

Suppose next that $q \neq p^\vee, p$. An application of (3.34) to the sequence (3.38) yields $\mathrm{Hom}(M, N_q)_0 \simeq \mathrm{Hom}(K, N_q)_0$. Thus $q \neq p, p^{\vee\vee}$ is a point of M if and only if it is a point of K.

Finally, putting $q = p$ in (3.38), and using $\mathrm{Hom}(N_{\sigma^n p}(-n), N_p)_0 \simeq k$, we see that if $\dim_k \mathrm{Hom}(M, N_p)_0 > 1$ then p is also a point of K, or $\sigma^m p$ a point of M'. This completes the proof. □

Theorem (3.36) generalizes the result known for the line modules [Ar, 2.10], as the following example shows. The "incidence relations" (3.37) are also important in classifying gk-2 normal modules up to gk-1 equivalence (see 5.10).

Example 3.39 ["Incidence relation" for line modules].

Let M_{l_1} be a line module, and p a point of M_{l_1}. Then Theorem (3.36) shows that the kernel of the surjective map $M_{l_1} \to N_p$ will be the shift, by -1 of a line module M_{w_1} (the only gk-2 normal modules of multiplicity 1 are line modules (3.9)). Thus the incidence relation (3.37) in this case is

$$0 \to M_{w_1}(-1) \to M_{l_1} \to N_p \to 0. \tag{3.40}$$

In fact, w_1 is the same as the one appearing in the equation $l_1 w_2 + l_2 w_1 = 0$ where l_2 is chosen such that $\{l_1, l_2\}$ is a basis for W_p. Indeed, the sequence (3.40) is same as the sequence

$$0 \to (A/w_1 A)(-1) \xrightarrow{l_2} A/l_1 A \to A/(l_1, l_2)A \to 0.$$

Also, we already know (2.3) that,

$$\operatorname{div}(M_{w_1}) = \sigma(\operatorname{div}(M_{l_1}) - p + p^{\vee}).$$

Theorem 3.41. *A gk-2 normal module M is not critical if and only if there exists an exact sequence*

$$0 \to M_1(-m_1) \to M \to M_2(m_2) \to T \to 0 \tag{3.42}$$

where M_1 and M_2 are gk-2 normal modules, M_2 is critical, T is m-torsion, and $m_1, m_2 \geq 0$. Further, if S, S_1, S_2 denote the set of points of the modules M, M_1, M_2 respectively then

$$S = \sigma^{-\imath m_1} S_1 \cup \sigma^{\imath m_2} S_2 \tag{3.43}$$

where $\imath$ is the parity of the module M.

Proof. The converse of the first statement being trivial, we only have to show the existence of (3.42) in case M is not critical. In that case, M has a gk-2 critical quotient P: $M \xrightarrow{\phi} P \to 0$. The canonical map $P \xrightarrow{\mu} P^{\vee\vee}$ is injective and $\operatorname{coker}(\mu) = T$ is torsion (1.8 v). Considering the cohomology we see that $E^j(P) = 0$ for $j \neq 1, 2$, $E^2(P) = E^3(T)$. Further, $P^{\vee\vee}$ is critical (by applying the next Lemma 3.45 to $P \to P^{\vee\vee}$).

Let $K = \ker(\phi)$. Now, M being pure (1.8v), K is a gk-2 module, and $E^0(K) = 0$ (1.8 vi). The cohomology of the sequence $0 \to K \to M \to P \to 0$, shows that $E^j(K) = 0$ for $j = 2, 3$. Thus K is Cohen-Macaulay. We have an exact sequence $0 \to K \to M \to P^{\vee\vee} \to T \to 0$. Now letting $M_1 = K(m_1)$ and $M_2 = P^{\vee\vee}(-m_2)$ be the normal shifts, we get (3.42). The fact that $m_1, m_2 \geq 0$ follows by comparing dimensions: Since M_1' injects into normal M, it must be a non-positive shift of a normal module M_1. Then M_2' must be a non-negative shift of a normal module.

To prove (3.43), we first claim that for a point module N_q,

$$\mathrm{Hom}_A(M_2, N_q)_0 \simeq \mathrm{Hom}_A(P(-m_2), N_q)_0. \tag{3.44}$$

To check this, we use the fact $\mathrm{Ext}^1_A(k, N_q) = k(1)$, which will be verified in (II.3.1). It follows from this that $\mathrm{Ext}^1_A(T, N_q)_0 = 0$ for a torsion module T with $T_n = 0$ for $n < 0$. Now consider the cohomology by applying $\mathrm{Hom}(,N_q)$ to the exact sequence $0 \to P(-m_2) \to M_2 \to T(-m_2) \to 0$. As $T(-m_2)$, being a quotient of the normal module M_2 has only positive degree elements, we have $\mathrm{Ext}^1(T(-m_2), N_q)_0 = 0$ and hence (3.44).

We prove $S \subset \sigma^{-\imath m_1} S_1 \cup \sigma^{\imath m_2} S_2$. The exact sequence $0 \to K \to M \to P \to 0$ shows that if p is a point of M then p is a point of K or a point of P. In the former case $\sigma^{\imath m_1} p$ will be a point of M_1 (Remark 3.12), and in the latter case $\sigma^{-\imath m_2}$ be a point of $P(-m_2)$ and therefore of M_2 (by 3.44).

Conversely, if q is a point of M_2 then $\sigma^{\imath m_2} q$ is a point of P, thus of M. So $\sigma^{\imath m_2} S_2 \subset S$. Finally we prove that $\sigma^{-\imath m_1} S_1 \subset S$. The cohomology of $0 \to M_1(-m_1) \to M \to P \to 0$ gives the exact sequence

$$M^\vee(-\epsilon) \to M_1^\vee(-\epsilon_1)(m_1 + \epsilon_1 - \epsilon) \to C \to 0$$

where $M^\vee(-\epsilon)$ and $M_1^\vee(-\epsilon_1)$ are normal and C is torsion. Now if p is a point of M_1 then $\sigma^{\imath(\epsilon_1 - 1)} p$ is a point of $M_1^\vee(-\epsilon_1)$ (Theorem 3.32). Then, by Remark 3.12, $\sigma^{\imath(\epsilon - m_1 - 1)} p$ is a point of $M_1^\vee(-\epsilon_1)(m_1 + \epsilon_1 - \epsilon)_{\geq 0}$, and hence also of $M^\vee(-\epsilon)_{\geq o} = M^\vee(-\epsilon)$. Applying Theorem (3.32) again, $\sigma^{-\imath m_1} p$ is a point of M. □

Lemma 3.45. *Let $\phi : M' \to M$ be a non-zero map from a critical module M' to a pure module M of equal gk-dimensions, say m. Then ϕ is necessarily injective, and, if the quotient has gk-dimension less than m then M is critical.*

Proof. If $\ker(\phi)$ were non-zero then by criticality of M', $M'/\ker(\phi)$ would have gk-dimension less than m. But this contradicts the purity of M. If the quotient has gk-dimension less than m, then $e(M') = e(M)$. To prove that M is critical, suppose, on the contrary, that P is a (necessarily, gk-m) submodule of M with $gk(M/P) = m$. We claim that $P \cap M' \neq 0$, and $M'/P \cap M'$ is a gk-m module. This would contradict the criticality of M', thus proving the result. To see that $P \cap M' \neq 0$, note that else $P \oplus M' \subset M$, which is impossible as $e(M') = e(M)$. To see that $gk(M'/P \cap M') = m$, note that we have an exact sequence $0 \to M'/P \cap M' \to M/P \to M/M' + P \to 0$ where $gk(M/M' + P) < m$. □

4 The Case of Multiplicity 2 : Conics in the Quantum Plane

(4.1) By a *quadratic* we mean a non-zero element of A_2. A right A-module of the form $A/\phi A$, where ϕ is a quadratic, will be called a *conic module.* More generally, a module of the form $M_\phi = A/\phi A$ where $0 \neq \phi \in A_d$ will be called a *principal module* of degree d. For a homogeneous element $\phi \in A_d$ we say that ϕ passes through p, or p is a point of ϕ, if p is a point of the corresponding principal module $A/\phi A$. This is equivalent to saying that there is a *surjective* A-homomorphism $A/\phi A \to N_p$, or, that $\bar{\phi}(p) = 0$ (Lemma 3.14). Here $\bar{\phi}$ is the image of ϕ in B. Assuming that $\bar{\phi} \neq 0$, we have $\operatorname{div}(A/\phi A) = \operatorname{div}(\bar{\phi})$ and $\deg(\operatorname{div}(\bar{\phi})) = 3.\deg(\phi)$. Let $\mathcal{M}_\phi$ denote the image of the conic module M_ϕ in proj$-A$. By a *conic in the quantum plane* we mean an object of the quantum plane proj$-A$ which is isomorphic to $\mathcal{M}_\phi$ for some quadratic ϕ. It follows from (3.5) that $\mathcal{M}_{\phi_1} \simeq \mathcal{M}_{\phi_2}$ if and only if $M_{\phi_1} \simeq M_{\phi_2}$. Thus conics in the quantum plane are in bijective correspondence with conic modules. We say that a conic (in the quantum plane) passes through a point p (in the quantum plane), or p is a point of the conic, if p is a point of the corresponding conic module. Recall that we have already identified points in the quantum plane with points of the cubic divisor E.

Proposition 4.2. *There exists a homogeneous element $\phi \in A_d$ ($\bar{\phi} \neq 0$) passing through any given $n\,(< 3d)$ points. There exists a homogeneous element $\phi \in A_d$, unique mod B_d (up to a unit in k), passing through given* $3d - 1$ distinct *points.*

Proof. Let $\{\bar{v}_i : i = 1, \ldots, 3d\}$ be a basis for B_d, so an arbitrary $\bar{\phi} \in B_d$ can be written as $\bar{\phi} = \sum_{i=1}^{3d} a_i \bar{v}_i$ for some $a_i \in k$. Then $\bar{\phi}(p) = 0$ is equivalent to the homogeneous equation $\sum_{i=1}^{3d} a_i \bar{v}_i(p) = 0$. We get therefore the first statement of the proposition. Now let ϕ and ψ be homogeneous elements of degree d passing through given $3d - 1$ distinct points. Both $\bar{\phi}$ and $\bar{\psi}$ are sections of the invertible sheaf $\mathcal{L}_d = \mathcal{L} \otimes \mathcal{L}^{\sigma} \ldots \otimes \mathcal{L}^{\sigma^{d-1}}$, therefore, $\operatorname{div}(\bar{\phi}) \sim \operatorname{div}(\bar{\psi})$. Further, since both divisors are effective of degree $3d$ and have $3d-1$ distinct points common between them, it follows that $\operatorname{div}(\bar{\phi}) = \operatorname{div}(\bar{\psi})$. As k is algebraically closed, $\bar{\phi} = \alpha\bar{\psi}$ $(\alpha \in k^*)$. □

In particular, since $A_2 = B_2$, we have

Corollary 4.3. *In the quantum plane, there exists a conic passing through any given $n < 6$ points and a unique conic passing through given 5 distinct points.*

(4.4) As described earlier (Example 3.10), there are just two types of gk-2 critical normal modules in multiplicity 2, namely,

$$0 \to A(-2) \xrightarrow{\phi} A \to A/\phi A = M_1 \to 0 \tag{4.5}$$

$$0 \to A(-1) \oplus A(-1) \xrightarrow{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} A \oplus A \to M_2 \to 0 \tag{4.6}$$

where ϕ is a quadratic and a, b, c, d are linear forms in A. Further, any other gk-2 normal module will be of the type $(0, k; 1, k+1)$ for some $k \geq 1$. Obviously, this is not critical.

The Hilbert functions of these modules are: $h_{M_1}(n) = 2n + 1, n \geq 0$; $h_{M_2}(n) = 2n + 2$, $n \geq 0$. Since both M_1, M_2 have generators in degree zero, they map *surjectively* to their point modules. The global sections s_{M_1} and s_{M_2} associated to these modules (3.21) are $s_{M_1} = \phi \in H^0(E, \mathcal{L} \otimes \mathcal{L}^{\sigma})$ and $s_{M_2} = a \otimes d - b \otimes c \in H^0(E, \mathcal{L} \otimes \mathcal{L})$.

(4.7) Let p be a point of a conic module M_ϕ. The kernel of the map $M_\phi \to N_p \to 0$ will be the shift, by -1, of a gk-2 normal module with multiplicity 2 (3.36). A consideration of the Hilbert function shows that this normal module is of the type M_2 (4.6). Thus there is an exact sequence

$$0 \to M_2(-1) \to M_\phi \to N_p \to 0. \tag{4.8}$$

We can describe the module M_2 in terms of M_ϕ as follows. Let $\{l_1, l_2\}$ be a basis of W_p. There exist linear forms b, d such that $l_1 b + l_2 d = 0$. Also, there exist linear forms a, c such that $\phi = l_1 a + l_2 c$. Then M_2 is given by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. To see this, we note that the kernel $M_2(-1) = \frac{(l_1, l_2)}{(\phi)}$, and claim that the resolution of $\frac{(l_1, l_2)}{(\phi)}$ is

$$(4.9) \qquad 0 \to A(-2) \oplus A(-2) \xrightarrow{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} A(-1) \oplus A(-1) \xrightarrow{[l_1\, l_2]} \frac{(l_1, l_2)}{(\phi)} \to 0.$$

Indeed, using the relations $l_1 b + l_2 d = 0$, $\phi = l_1 a + l_2 c$, we get the identity

$$(4.10) \qquad \begin{bmatrix} l_1 & l_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \phi & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \phi u$$

where $u, v \in A$. This implies two things: the composite map of (4.9) is zero, and the first map defined by the square matrix is injective. Now, comparing dimensions one sees that the sequence (4.9) is exact.

By Theorem (3.36), we see that $\rho(p)$ is a point of M_2, and, if $q \neq p$ is a point of ϕ then σq is a point of M_2. Thus, in the generic case, if $(p_1, p_2 \ldots, p_6 = p)$ are the six points of a conic ϕ then $(\sigma p_1, \sigma p_2, \ldots, \rho(p))$ would be the six points of M_2. Now suppose we map this M_2 to its "sixth point" $\rho(p)$. The kernel will be a conic module, say, $M_{\phi_1} = A/\phi_1 A$:

$$(4.11) \qquad 0 \to M_{\phi_1} \to M_2 \to N_{\rho(p)} \to 0.$$

The points of M_{ϕ_1} in the generic case would be $(\sigma p_1, \sigma p_2, \ldots, \sigma^{-1}\rho^2(p))$. One can indefinitely continue this process of mapping to point modules and get the "shifted" conics $M_{\phi_1}, M_{\phi_2}, \ldots$.

Conversely, we show that

Proposition 4.12. *Every critical module M_2 (of type (4.6)) is the kernel of a map from a conic module to some point module. Such a conic module is necessarily critical.*

Proof. Suppose first that module M_2 is given by a matrix $[M_2] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where b and d are independent linear forms intersecting on E. Then there exist linear forms l_1, l_2 (intersecting on E, say, at p) such that $l_1 b + l_2 d = 0$

(3.4). Now if we take the quadratic $\phi = l_1 a + l_2 c$, then following (4.7), it is clear that $M_2(-1)$ would be the kernel of the map $A/\phi A \to N_p \to 0$. Thus it suffices to show that every critical module M_2 of type (5.6) is isomorphic to one of the same type in which the linear forms in the second column are linearly independent, intersecting on E. Recall that every $X \in \mathbf{GL}(2,k)$ gives an automorphism of $A \oplus A$ (or, of $A(-1) \oplus A(-1)$). So replacing $[M_2]$ by $X^{-1}[M_2]Y$ for $X, Y \in \mathbf{GL}(2,k)$ gives an isomorphic module.

Now let M_2 be a critical module of type (4.6) given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It follows from criticality of M_2 that the pairs (a,b), (c,d), (a,c), and (b,d) are linearly independent. We further claim that a,b,c,d all can not meet at the same point p in $\mathbf{P}^2$. For, if they did, $c+\gamma a$ and $d+\gamma b$ will be linearly dependent for some $\gamma \in k$, implying that the module corresponding to the matrix $\begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}[M_2]$ is not critical, and hence also the module M_2 is not critical. Now consider the locus (in $\mathbf{P}^2$) of the point of intersection of the lines $\alpha a + b, \alpha c + d$ as α varies over k. As a,b,c,d do not meet at the same point, this locus would intersect the cubic E for some value of α. Then the module corresponding to the matrix $\begin{bmatrix} a & \alpha a + b \\ c & \alpha c + d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ is the required module, to which M_2 is isomorphic. The last statement follows from Lemma 3.45.□

We now give some results about the criticality of a conic module and factorization of a quadratic. In the classical (commutative) case, one knows that a conic module $A/\phi A$ is not critical, if and only if ϕ factorizes into two linear forms, if and only if three points of the conic lie on one line. This, in turn, is equivalent to the existence of an exact sequence of the form $0 \to (A/l_2A)(-1) \to A/\phi A \to A/l_1 A \to 0$. In the quantum case, we have the following:

Proposition 4.13. *A conic-module $M_\phi = A/\phi A$ is not critical if and only if there exists an exact sequence*

$$(4.14) \qquad 0 \to M_{l_1}(-m-1) \to M_\phi \to M_{l_2}(m) \to T \to 0$$

where M_{l_1} and M_{l_2} are line modules, T is $\mathfrak{m}$-torsion, and $m \geq 0$. Further, if S, S_1, S_2 are the set of points of M_ϕ, M_{l_1}, M_{l_2} respectively then $S = \sigma^{-m-1}S_1 \cup \sigma^m S_2$.

Proof. This is an immediate consequence of Proposition (3.41), for, the only gk-2 normal modules of multiplicity 1 are line modules. The fact that the integers m_1, m_2 of (3.42)) are related by $m_1 = m_2 + 1$ follows from a consideration of the Hilbert functions. The Hilbert function of $A/\phi A$ is $2n+1\,(n \geq 0)$ and that of a line module is $n+1\,(n \geq 0)$. Therefore, for large n, we must have $((n-m_1)+1)+((n+m_2)+1) = 2n+1$, showing that $m_1 = m_2 + 1$. □

Remark 4.15. It turns out that m in (4.14) can take *all* non-negative integral values. See (A.12) in the appendix A for an example. Proposition (4.13) has following geometric meaning in the generic case where the conic ϕ has six distinct points: If M_ϕ is not critical then there exist lines ℓ_1, ℓ_2 and an $m \in \mathbf{N}$ such that three points of M_ϕ lie on $\sigma^{-m-1}\ell_1$ and the other three points on $\sigma^m \ell_2$.

In the following, we will say that factorizations $\phi = l_1 l_2$ and $\phi = l_1' l_2'$ of a quadratic ϕ into linear forms are distinct if l_1 is linearly independent of l_1' and l_2 is linearly independent of l_2'. As A is a domain, this is equivalent to linear independence of *one* of the pairs. Also, we will denote the line represented by a linear form l by the same symbol l. We use the notations $l.E = \operatorname{div}(l)$, $\phi.E = \operatorname{div}(\phi)$, and $(l.E)^\vee = \sigma^{-1}\rho(l.E)$.

Remark 4.16. Let ϕ be a quadratic in A, and $D = (\phi.E)$ its divisor. The following are equivalent: (i) ϕ factorizes into linear forms, (ii) D contains the divisor $l.E$ of a linear form l, (iii) D contains the divisor $\operatorname{div}(l^\sigma) = \sigma^{-1}(l.E)$ for some linear form l. We need only check that (ii) (resp. (iii)) implies (i). Indeed, the invertible sheaf $\mathcal{O}(D) \simeq \mathcal{L} \otimes \mathcal{L}^\sigma$. Put $L = l.E$ Thus we see that if (ii) (resp. (iii)) holds, then $\mathcal{O}(D-L) \simeq \mathcal{L}^\sigma$ (resp. $\mathcal{O}(D-\sigma^{-1}L) \simeq \mathcal{L}$). Hence there exists a divisor $L' = l'.E$ of some linear form l', such that $D = L + \sigma^{-1}L'$ (resp. $D = \sigma^{-1}L + L'$), so that ϕ equals ll' (resp. $l'l$), up to a scalar multiple.

Proposition (4.17). *A quadratic ϕ can factorize into two linear forms in at most four distinct ways. If a quadratic ϕ factorizes in four distinct ways, then*

$$\phi.E = l.E + (l.E)^\vee \tag{4.18}$$

for some linear form l.

Proof. For a conic ϕ that does not factorize into linear forms the statement is trivial. So let $\phi = l_1 w_2$ be a fixed factorization of ϕ (this unusual notation is chosen to conform to the notations in (2.1), which we use below). Put $Z = w_2.E \cap \rho(l_1.E)$.

Now suppose $\phi = l_2 w_1$ is a factorization distinct from $l_1 w_2$. As $l_1 w_2 - l_2 w_1 = 0$, we see from Lemma (2.1) that l_1, l_2 intersect on E, say at p and then w_1, w_2 intersect at $\rho(p)$, and the linear forms l_2, w_1 are determined, up to a scalar, by

$$(4.19) \quad (l_2.E) = \sigma^{-1}(w_2.E - \rho(p)) + p \qquad (w_1.E) = \sigma(l_1.E - p) + \rho(p).$$

As $\rho(p) \in Z$, we have thus defined a map from the set {factorizations of ϕ distinct from $l_1 w_2$ } to Z. In view of (4.19), this map is injective. Thus the number of distinct factorizations of ϕ is at most $1 + \#\{Z\} \leq 4$. This also shows that for four distinct factorizations to exist, Z must have three distinct points and $\rho(l_1.E) = (w_2.E)$. Thus, $\phi.E = l_1.E + \sigma^{-1}(w_2.E) = l_1.E + \sigma^{-1}\rho(l_1.E) = l_1.E + (l_1.E)^{\vee}$. □

As the condition for multiple factorization is a closed condition in $\mathbf{P}(A_2)$, the projective space of quadratics, we note that there exists an open dense subset in $\mathbf{P}(A_2)$ in which unique factorization holds. In other words, unique factorization for quadratics holds generically.

Example 4.20 [Quadratics factorizing in four different ways].

We give examples where a quadratic in an elliptic algebra factorizes in four different ways. Suppose l is a linear form with three distinct points p_1, p_2, p_3 such that $\rho p_1, \rho p_2, \rho p_3$ are also on a line, say corresponding to a linear form w. Put $\phi = lw$. Then

$$(4.21) \quad \begin{aligned} \phi.E &= (p_1 + p_2 + p_3) \ + \sigma^{-1}(\rho p_1 + \rho p_2 + \rho p_3) \\ &= (p_1^{\vee} + p_2^{\vee} + p_3) \ + \sigma^{-1}(\sigma p_1 + \sigma p_2 + \rho p_3) \\ &= (p_1^{\vee} + p_2 + p_3^{\vee}) \ + \sigma^{-1}(\sigma p_1 + \rho p_2 + \sigma p_3) \\ &= (p_1 + p_2^{\vee} + p_3^{\vee}) \ + \sigma^{-1}(\rho p_1 + \sigma p_2 + \sigma p_3). \end{aligned}$$

Here all the parentheses are the divisors of some linear form. Thus ϕ factorizes in four distinct ways. For a concrete example, where this occurs, consider the Sklyanin algebra with the parameter values $a = 1, b = -1, c = -1$ (1.2). This is an elliptic algebra with relations $xy - yx = z^2$, $yz - zy = x^2$, $zx - xz = y^2$. Now consider the quadratic $\phi = zx$. Then one checks that

$$zx = -\frac{1}{2\alpha^2}[x + \alpha y - \alpha^2 z][x - \alpha y - \alpha^2 z]$$

where α is a cube root of unity.

5 GK-1 Equivalence

We introduce the notion of gk-1 equivalence. In the next chapter (§II.1), this will be used to classify injective indecomposable A-modules. Here, we show how the maps to point modules ("incidence relations" (3.37)) can be used to classify finite gk-2 modules up to gk-1 equivalence.

(5.1) Consider the full subcategory $\mathrm{GK}_{\leq 1}$ of $\mathrm{gr}-A$ whose objects are modules of gk-dimension at most one. This is a thick abelian subcategory; for, in any exact sequence

$$0 \to N' \to N \to N'' \to 0$$

in $\mathrm{gr}-A$, N belongs to $\mathrm{GK}_{\leq 1}$ if and only if N' and N'' belong to $\mathrm{GK}_{\leq 1}$. Let $\Sigma = \Sigma_{GK_{\leq 1}}$ denote the class of morphisms s of $\mathrm{gr}-A$ such that $\ker(s)$ and $\mathrm{coker}(s)$ belong to $\mathrm{GK}_{\leq 1}$. Then by a general result [PP, Theorem 4.7.8], the quotient category $\Sigma^{-1}(\mathrm{gr}-A) = \mathrm{gr}-A/\mathrm{GK}_{\leq 1}$ is defined.

Definition 5.2. *Let M_1 and M_2 be finite right A-modules. M_1 is said to be gk-1 equivalent to M_2, denoted $M_1 \sim M_2$, if the canonical projection of M_1 in the quotient category $\mathrm{gr}-A/\mathrm{GK}_{\leq 1}$ is isomorphic to that of M_2.*

This is obviously an equivalence relation on the objects of gr–A. Similarly one defines gk-1 equivalence on $A-\mathrm{gr}$. Needless to say, if there is a map $M_1 \xrightarrow{\phi} M_2$ such that $gk(\ker\phi) \leq 1$ and $gk(\mathrm{coker}\phi) \leq 1$, then M_1 is gk-1 equivalent to M_2.

(5.3) Assume that no positive integral power of σ fixes the class of $[\mathcal{L}]$ in $\mathrm{Pic}(E)$, i.e., the index s_0 of the algebra A [ATV2, §7] is infinite. This hypothesis has the consequences that a gk-1 module is critical if and only if it is a shifted point module; and, thus every gk-1 module is annihilated by some power of g [ATV2,§7]. Then the notion of gk-1 equivalence is related to g-torsion. To make it more precise, let tor–g be the full subcategory of gr–A consisting of the g-torsion modules. (An A-module M is said to be g-torsion if every element of M is annihilated by some power of g, which is equivalent to saying that $Mg^n = 0$ for some n, in case M is finite.) Now $\mathrm{tor}-g$ is a thick abelian localizing subcategory, so the quotient category $\mathrm{gr}-A/\mathrm{tor}-g$ is defined. The quotient category $\mathrm{gr}-A/\mathrm{tor}-g$ can also be described as follows. Let $\Lambda = A[g^{-1}]$ be the $\mathbf{Z}$-graded quotient ring obtained by adjoining the inverse of the normalizing element g, and let $T : \mathrm{gr}\text{–}A \to \mathrm{gr}\text{–}\Lambda;\ M \rightsquigarrow M[g^{-1}]$ be the canonical (exact) localization functor. Then, by a general result [PP, Theorem 4.7.11], tor$-g$ is the "kernel"of the functor T, and the category gr$-\Lambda$ is equivalent to gr$-A/$tor$-g$.

Now since every A-module of gk-dimension ≤ 1 is g-torsion, therefore, by the universal property of the quotient category, the canonical projection functor P : gr$-A \to$ gr$-A/$tor$-g$ factors through gr$-A/\mathrm{GK}_{\leq 1}$ and we have the following commutative diagram

(5.4)
$$\begin{tikzcd} \mathrm{gr}-A \arrow[r] \arrow[d, "A\otimes\Lambda"'] \arrow[dr, "P"] & \mathrm{gr}-A/\mathrm{GK}_{\leq 1} \arrow[d, dotted] \\ \mathrm{gr}-\Lambda \arrow[r, "\sim"] & \mathrm{gr}-A/\mathrm{tor}-g \end{tikzcd}$$

Our main purpose now is to show that if the following Hypothesis (5.5) is true, then the type $(0,\ldots,0;1,\ldots,1)$ of gk-2 normal modules is generic in the sense that every gk-2 finite critical module is gk-1 equivalent to a shift of a module of such type.

Hypothesis 5.5. *Every gk-2 critical normal module M has a submodule $M'(-m)$ such that M' is normal of the same type as M and maps surjectively to at least one point module.*

This remains unproved, but seems to be valid. A partial result which supports this hypothesis is the following.

Proposition 5.6. *Every gk-2 critical normal module M has a normal submodule M' mapping surjectively to a point module.*

Proof. Every gk-2 normal module has a non-zero map to some point module. Let $M \to N_p$ a non-zero map. If this map is not surjective then the kernel M_1 will be a gk-2 critical *normal* module (Theorem 3.36). Thus if the proposition were false, we will obtain an infinite chain of gk-2 critical normal modules

(5.7)
$$\ldots \subset M_{i+1} \subset M_i \ldots \subset M_1 \subset M$$

such that the quotients M_i/M_{i+1} are shifted point modules of the form $N(-m)$ with $m > 0$. Now a consideration of the Hilbert function shows that (5.7) would contradict the convexity (Theorem 3.7) of the Hilbert function of some M_i. □

Definition 5.8. *A module is said to be of λ-type if it is a gk-2 normal module of the type* $(\underbrace{0,\ldots,0}_{r \text{ times}};\underbrace{1,\ldots,1}_{r \text{ times}})$, *i.e., it has a minimal resolution of the form*

$$0 \to A^r(-1) \to A^r \to L \to 0$$

for some r.

Lemma 5.9. *Let M be a gk-2 normal module with multiplicity e, having a minimal resolution of the form (3.3). The following are equivalent*

(i) M is a module of λ-type,

(ii) the Hilbert function of M is $h_M(n) = \dim_k(M_n) = en + e, \quad n \geq 0$,

(iii) r, the minimal number of generators of M $= e$.

Proof. We show $(i) \Rightarrow (ii)$. Let r be the (minimal) number of generators of M. Then $h_M(n) = \dim_k(M_n) = r(\dim_k(A(-1)_n) - \dim_k(A_n)) = rn + r(n \geq 0)$. Thus $e = r$ and we get (ii). Now, suppose (ii) holds and let M be presented as in (3.3). Then $e = \dim_k(M_0) = \#\{i_k | i_k = 0\} \leq r$. As $r \leq e$ (3.2), we get (iii). Finally, $(iii) \Rightarrow (i)$ follows from the conditions $\sum_{k=1}^{k=r=e}(j_k - i_k) = e$ and $0 \leq i_k < j_k$. □

Theorem 5.10. [Assume that Hypothesis (5.5) is true, then] *Every gk-2 critical module is gk-1 equivalent to a shift of a module of λ-type.*

Proof. We first claim that for any gk-2 critical normal module M whose Hilbert function is $(m_0, m_1, m_2, \ldots)$, there exists a normal module P with Hilbert function $(m_1 - m_0, m_2 - m_0, \ldots)$ such that $M(\mu) \sim P$ for some $\mu \geq 0$.

To prove the claim, put $M = K^0$. By hypothesis (5.5), we can find a normal module M^0 with the same Hilbert function as K^0 such that $M^0 \subset K^0(\nu_0)$ for some $\nu_0 \geq 0$, and there is a surjective map $M^0 \to N_p$ to some point module N_p. The kernel K^1 of this map has Hilbert function $(m_0 - 1, m_1 - 1, , m_2 - 1, \ldots)$. If $m_0 > 1$, then K^1 is normal, and by hypothesis (5.5) again, there exists a normal module M^1 with same Hilbert function as K^1, such that $M^1 \subset K^1(\nu_1)$ for some $\nu_1 \geq 0$ and M^1 has a surjective map to some point module. The kernel K^2 of this map has Hilbert function $(m_0-2, m_1-2, m_2-2, \ldots)$. It is clear now that repeating the same process m_0 times, we can find normal modules M^i, K^i $(i = 0, \ldots, m_0-1)$, and a shifted normal module K^{m_0}, such that $M^i \subset K^i(\nu_i)$ for some $\nu_i \geq 0$, $K^{i+1} \subset M^i$

(in particular, $M^i \sim K^i(\nu_i)$, $K^{i+1} \sim M^i$) and the Hilbert function of K^{m_0} is $(0, m_1 - m_0, m_2 - m_0, \ldots,)$. Taking $P = K^{m_0}(1)$ our claim is proved.

Now let M be any gk-2 critical module. In the canonical map $M \xrightarrow{\mu} M^{\vee\vee}$, $gk(\mathrm{coker}\mu) \leq 0$ (1.8 v). Thus, M is gk-1 equivalent to its Cohen-Macaulay bidual. So we can assume M to be Cohen-Macaulay, and indeed normal. Let the Hilbert function of M be $h_M(n) = m_n$. Then $m_{n+1} \geq m_n$ for all n, and m_n is an arithmetic progression for $n \geq s$ for some s; $m_n = en + c$; $(n \geq s, c \in \mathbf{Z})$.

Now apply the claim first to M to find a normal module P^0 such that $P^0 \sim M(\mu_0)$ $(\mu_0 \geq 0)$ and the Hilbert function of P^0 is $(m_1 - m_0, m_2 - m_0, m_3 - m_0, \ldots)$; then apply the claim to P^0 to find a normal module P^1 such that $P^1 \sim P^0(\mu_1)$ $(\mu_1 \geq 0)$ and the Hilbert function of P^1 is $(m_2 - m_1, m_3 - m_1, \ldots)$, and so on. We see that after s number of applications of the claim, we will get a normal module P^s such that the Hilbert function of P^s is $(m_{s+1} - m_s, m_{s+2} - m_s, \ldots)$ and $P^s \sim M(\mu)$ where $\mu = \mu_0 + \mu_1 + \ldots + \mu_s$. Since $m_n = en + c$ for $n \geq s$, we see that the Hilbert function of P^s is $(e, 2e, 3e, \ldots)$ so that P^s is a module of λ-type (Lemma 5.9), and theorem is proved. $\square$

Remark 5.11 By results in section 4, we know that the theorem is true in multiplicity 2: every gk-2 critical module of multiplicity 2 is gk-1 equivalent to a module of the λ-type (0,0;1,1). In multiplicity 2, however, it is also true that every gk-2 critical module is gk-1 equivalent to a conic module (4.12).

Appendix A Conics in the "Weyl Plane"

By the Weyl plane we mean that special case of the quantum plane, in which the cubic divisor E associated with the regular algebra A is $3\mathbf{P}^1$, the triple projective line. In this case the relations in A are simply [ATV2, §8]

(A.1) $$xy - yx = 0 \quad xz - zx = 0 \quad yz - zy = x^2$$

In this section A will always denote this regular algebra only. In this case $g = x^3$, up to a scalar, and the algebra $\Lambda_0 = A[g^{-1}]_0$ is the Weyl algebra W (hence the terminology); $W = k\langle p, q\rangle/(pq - qp = 1)$, where $p = yx^{-1}, q = zx^{-1}$ [ATV2,§8]. Here we study the conics in the Weyl plane. Using the automorphisms of the algebra, we will classify the conics up to gk-1 equivalence.

Proposition A.2. *The group of (graded) automorphisms of A is the direct product of k^* and the semidirect product* $\mathbf{SL}(2,k) \ltimes k^2$*:*

$$\mathrm{Aut}(A) = k^* \times (\mathbf{SL}(2,k) \ltimes k^2).$$

Explicitly, every automorphism θ of A is given by a linear transformation (also denoted by θ) of A_1

$$\begin{bmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{A.3}$$

such that $\Delta_1 = b_2c_3 - b_3c_2 = {a_1}^2$.

Proof. First it is clear that any (graded) automorphism of A is given by a linear transformation $\theta \in \mathbf{GL}(3,k)$ of A_1:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Imposing the commutativity conditions (6.1) we see that $\Delta_1 = b_2c_3 - b_3c_2 = a_1^2, \Delta_2 = b_1c_3 - b_3c_1 = 0, \Delta_3 = b_1c_2 - b_2c_1 = 0$. The conditions $\Delta_2 = 0 = \Delta_3$ force $b_1 = 0 = c_1$, giving (A.3). Now, any θ of the form (A.3) is $\begin{bmatrix} a & 0 \\ T & X \end{bmatrix}$ for some $T \in k^2, X \in \mathbf{GL}(2,k), a \in k^*$ with $\det(X) = a^2$. We see that it factorizes as $\begin{bmatrix} a & 0 \\ T & X \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ a^{-1}T & a^{-1}X \end{bmatrix}$, where the matrix on the right hand side is an element of $\mathbf{SL}(2,k) \ltimes k^2$. Uniqueness of such a factorization being clear, the proof is complete. □

We now define the canonical forms for a quadratic in A. *Throughout this section, we will write $\phi \sim \psi$ for two quadratics ϕ and ψ if they are equal up to a unit in k.* A quadratic ϕ in A is said to be *parabolic* (resp. *hyperbolic*) if there exists an automorphism θ of A such that $\theta(\phi) \sim y^2 + \alpha zx$, $\alpha \in k^*$ (resp. $\theta(\phi) \sim yz + \alpha x^2$, $\alpha \in k$). A conic ϕ is said to be *degenarate* if there exists an automorphism θ of A such that $\theta(\phi) \sim xl$, or, $\theta(\phi) \sim y^2 + \alpha x^2$, where l is a linear form and $\alpha \in k$. A conic module M_ϕ is said to be parabolic, hyperbolic or degenerate if the quadratic ϕ is of the corresponding form.

Proposition A.4. *A quadratic in A is exactly one of the three forms: parabolic, hyperbolic, or degenerate.*

Proof. Let $\phi = ay^2 + bz^2 + cx^2 + 2gyx + 2fzx + 2hyz$ be an arbitrary quadratic. Suppose first that $a = 0 = b$. If further $h = 0$ then $\phi = xl$ where l is the linear form $cx+2gy+2fz$. If $h \neq 0$ then $\phi = 2h[(y+\frac{f}{h}x)(z+\frac{g}{h}x)+(\frac{ch-2fg}{2h^2})x^2]$. So the automorphism $x \mapsto x, y \mapsto y + \frac{f}{h}x, z \mapsto z + \frac{z}{h}x$ transforms ϕ into the hyperbolic form.

Suppose now that $a \neq 0$ or $b \neq 0$. For the sake of definiteness, suppose $a \neq 0$. Then the automorphism $x \mapsto x, y \mapsto y + \frac{h}{a}z + \frac{g}{a}x, z \mapsto z$ transforms ϕ into the form $\phi = y^2 + b'z^2 + 2f'zx + c'x^2$. In this form if $b' = 0 = f'$ then ϕ is degenerate. If $b' = 0, f' \neq 0$ then the automorphism $x \mapsto x, y \mapsto y, z \mapsto z + c'x/2f'$ transforms ϕ into the parabolic form. If $b' \neq 0$ then the automorphism $x \mapsto \delta x, y \mapsto y + \gamma(z + \frac{f'}{b'}x), z \mapsto y - \gamma(z + \frac{f'}{b'}x)$ (where $\gamma^2 = -b', \delta^2 = -2\gamma$) transforms ϕ into the hyperbolic form.

We have shown that every quadratic has at least one of the three forms. To show that the three forms are mutually exclusive, it suffices to show that there exists no automorphism of A which transfoms one canonical form into a different one. Having an explicit description of automorphisms of A (A.3), this is straightforward to verify. □

(A.5) Consider the localization functor: $\mathrm{Gr}-A \to \mathrm{Mod}-\Lambda_0 : M \rightsquigarrow M[g^{-1}]_0$. Recall that, in the case of Weyl plane, we have $g = x^3$, and $\Lambda_0 = W = k<p,q>/(pq-qp = 1)$, where $p = yx^{-1}, q = zx^{-1}$. Thus the localization functor in this case is just $M \rightsquigarrow M[x^{-1}]_0$. For a homogeneous element ϕ of A of degree n, define its *dehomogenization* $\bar{\phi}$ of ϕ as $\bar{\phi} = x^{-n}\phi$. Then, for a principal module $M_\phi = A/\phi A$ localization just means $M_\phi \rightsquigarrow M_{\bar{\phi}}$ where $M_{\bar{\phi}} = W/\bar{\phi}W$. By a *quadratic in W* we mean dehomogenization $\bar{\phi}$ of some quadratic ϕ in A, i.e., an element of the form $ap^2 + bq^2 + c + fq + gp + hpq$ where $a, b, c, f, g, h \in k$. Quadratics in W correspond bijectively to quadratics in A. We say that a quadratic $\bar{\phi}$ in W is parabolic (resp. hyperbolic, resp. degenerate) if the corresponding quadratic ϕ in A is of the same form. It follows from Proposition (A.4) that every quadratic in W is exactly one of these three forms. The W-module $W/\bar{\phi}W$ will be called parabolic, hyperbolic or degenerate if $\bar{\phi}$ is of the corresponding form. The W-modules of the form W/lW, $l = \alpha p + \beta q + \gamma$, will be simply called line modules over W.

(A.6) Let $\mathrm{Aut}(W)$ denote the group of automorphisms of the Weyl algebra. It contains the semidirect product $\mathbf{SL}(2,k) \ltimes k^2$; because, every linear transformation $p' = a_2 + b_2p + c_2q$, $q' = a_3 + b_3p + c_3q$ with $b_2c_3 - b_3c_2 = 1$ gives an automorphism of W. We therefore have a group-homomorphism: $\mathrm{Aut}(A) \to \mathrm{Aut}(W)$, $\theta \mapsto \bar{\theta}$ given by the projection of $\mathrm{Aut}(A)$ on the factor $(\mathbf{SL}(2,k) \ltimes k^2)$; i.e., $\begin{bmatrix} a & 0 \\ T & X \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ a^{-1}T & a^{-1}X \end{bmatrix}$. We will denote the image of this homomorphism by $\mathrm{Aut}_A(W)$. Note that there are automorphisms of the Weyl algebra, not in $\mathrm{Aut}_A(W)$; see, for example, [Di]. However, as $\overline{\theta(\phi)} = \bar{\theta}(\bar{\phi})$ (ϕ a homogeneous element of A), it follows that we can transform the quadratics in W to canonical forms just by using an automorphism in $\mathrm{Aut}_A(W)$ only. Equivalently speaking, a quadratic $\bar{\phi}$ in W is parabolic (resp. hyperbolic, resp. degenerate) if there exists an automorphism $\bar{\theta} \in \mathrm{Aut}_A(W)$ such that $\bar{\theta}(\bar{\phi}) \sim p^2 + \alpha q$, $\alpha \in k^*$ (resp. $\bar{\theta}(\bar{\phi}) \sim pq + \alpha$, $\alpha \in k$; resp. $\bar{\theta}(\bar{\phi}) \sim \alpha p + \beta q + \gamma$ or $p^2 + \alpha$, $\alpha, \beta, \gamma \in k$). This would allow us to lift the results about quadratic modules over W to the ones about conic modules over A.

(A.7) Let θ be an automorphism of A. Two principal modules M_{ϕ_1} and M_{ϕ_2} are isomorphic (resp. gk-1 equivalent) if and only if $M_{\theta(\phi_1)}$ and $M_{\theta(\phi_2)}$ are. M_ϕ is critical if and only if $M_{\theta(\phi)}$ is. Let $\bar{\theta} \in \mathrm{Aut}_A(W)$ be an automorphism of W induced by the automorphism θ of A. Then $M_{\bar{\phi}_1} \simeq M_{\bar{\phi}_2}$ if and only if $M_{\bar{\theta}(\bar{\phi}_1)} \simeq M_{\bar{\theta}(\bar{\phi}_2)}$ and $M_{\bar{\phi}}$ is simple if and only if $M_{\bar{\theta}(\bar{\phi})}$ is. Also, in view of (5.4), A-modules M_{ϕ_1} and M_{ϕ_2} are gk-1 equivalent if and only if $M_{\bar{\phi}_1} \simeq M_{\bar{\phi}_2}$ as W-modules.

The main results about the quadratic W-modules are given by the following theorem.

Theorem A.8. *(i) A parabolic or a line W-module is simple. Let M_ϕ be a hyperbolic W-module and $pq + \alpha$ a canonical form of ϕ. Then M_ϕ is simple if and only if $\alpha \in k - \mathbf{Z}$.*

(ii) For a quadratic ϕ in W, $W/\phi W \simeq W/(p^2 + \alpha q)W$, ($\alpha \in k^*$) if and only if $\phi \sim (p^2 + \alpha q)$.

(iii) For a quadratic ϕ in W, $W/\phi W \simeq W/(pq + \alpha)W$ ($\alpha \in k - \mathbf{Z}$) if and only if $\phi \sim pq + \alpha + n$ for some $n \in \mathbf{Z}$. Further, $W/\phi W \simeq W/pqW$ if and only if $\phi \sim pq + n$ for some $n \in \mathbf{N}$, and $W/\phi W \simeq W/qpW$ if and only if $\phi \sim qp - n$ for some $n \in \mathbf{N}$.

We will prove this at the end of the section. Using the Theorem and remarks in (A.7), we get the following corollary.

Corollary A.9. *No parabolic W-module is isomorphic to a hyperbolic W-module. Two parabolic W-modules M_ϕ and $M_{\phi'}$ are isomorphic if and only if $\phi \sim \phi'$. Two hyperbolic W-modules M_ϕ and $M_{\phi'}$ are isomorphic if and only if there exist canonical forms $pq + \alpha$ of ϕ and $pq + \beta$ of ϕ' such that (i) $\alpha, \beta \in k - \mathbf{Z}, \alpha - \beta \in \mathbf{Z}$; or, (ii) $\alpha, \beta \in \mathbf{Z}_{\geq 0}$; or, (iii) $\alpha, \beta \in \mathbf{Z}_{<0}$.*

Remark (A.10) For every $n \in \mathbf{N}$ there exists a commutative diagram in Mod–W with exact rows

$$\text{(A.11)} \qquad \begin{array}{ccccccccc} 0 & \to & \frac{W}{qW} & \xrightarrow{p^{n+1}} & \frac{W}{(pq+n)W} & \longrightarrow & \frac{W}{(pq+n,p^{n+1})W} & \to & 0 \\ & & \big\downarrow \wr\, c_n & & \big\downarrow q^n & & \big\downarrow \wr\, q^n & & \\ 0 & \to & \frac{W}{qW} & \xrightarrow{p} & \frac{W}{pqW} & \longrightarrow & \frac{W}{pW} & \to & 0 \end{array}$$

where $c_n = (-1)^n n!$. The commutativity of the left square follows from $q^n p^{n+1} = (-1)^n n!\, p \bmod pqW$. The exactness of the first row follows from the equality $(pq+n)W \cap p^{n+1}W = p^{n+1}qW$. In particular, $W/(pq+n)W \overset{q^n}{\xrightarrow{\sim}} W/pqW$ is an isomorphism (as mentioned in Theorem (A.8 *iii*)).

There is a corresponding commutative diagram in gr–A with exact rows, which on localization gives the previous diagram:

$$\begin{array}{ccccccccc} 0 & \to & \frac{A}{zA}(-n-1) & \xrightarrow{y^{n+1}} & \frac{A}{(yz+nx^2)A} & \longrightarrow & \frac{A}{(yz+nx^2,y^{n+1})A} & \to & 0 \\ & & \big\downarrow c_n x^{2n} & & \big\downarrow z^n & & \big\downarrow z^n & & \\ 0 & \to & \frac{A}{zA}(n-1) & \xrightarrow{y} & \frac{A}{yzA}(n) & \longrightarrow & \frac{A}{yA}(n) & \to & 0 \end{array}$$

It follows that the hyperbolic A-modules A/yzA and $A/(yz+nx^2)A$ are gk-1 equivalent. Also, it follows that there is an exact sequence of A-modules

$$\text{(A.12)} \qquad 0 \to \frac{A}{zA}(-n-1) \xrightarrow{y^{n+1}} \frac{A}{(yz+nx^2)A} \xrightarrow{z^n} \frac{A}{yA}(n) \to T \to 0$$

where T is a torsion module. This shows that, for all $n \in \mathbf{N}$, the conics defined by $yz + nx^2$ are not critical. This verifies the claim made in (4.15).

Similarly, we have an isomorphism $W/(qp-n)W \overset{p^n}{\xrightarrow{\sim}} W/qpW$ and a gk-1 equivalence $A/(zy - nx^2)A \xrightarrow{y^n} A/zyA$, for all $n \geq 0$. However, $W/pqW \not\simeq W/qpW$ (Theorem A.8 *iii*).

The above results about the quadratic W-modules translate into those about the conic A-modules as follows.

Theorem A.13. *(i) A parablic A-module is critical. Let M_ϕ be a hyperbolic A-module, and $yz + \alpha x^2$ a canonical form of ϕ. Then M_ϕ is critical if and only if $\alpha \notin \mathbf{Z}$.*

(ii) No parabolic A-module is gk-1 equivalent to a hyperbolic A-module. Two parabolic A-modules M_ϕ and $M_{\phi'}$ are gk-1 equivalent if and only if $\phi \sim \phi'$. Two hyperbolic A-modules M_ϕ and $M_{\phi'}$ are gk-1 equivalent if and

only if there exist canonical forms $yz + \alpha x^2$ of ϕ and $yz + \beta x^2$ of ϕ' such that (a) $\alpha, \beta \in k - \mathbf{Z}, \alpha - \beta \in \mathbf{Z}$; or, (b) $\alpha, \beta \in \mathbf{Z}_{\geq 0}$; or, (c) $\alpha, \beta \in \mathbf{Z}_{<0}$.

Proof. (i) In view of (A.7), Theorem (A.8) and Remarks in (A.10) it suffices to show that for a parabolic or hyperbolic ϕ, criticality of M_ϕ follows from the simplicity of $M_{\bar{\phi}}$. Suppose M_ϕ is not critical. Then by the (4.13), there exists an exact sequence

$$0 \to M_{l_1}(-m-1) \xrightarrow{\psi} M_\phi \to M_{l_2}(m) \to T \to 0 \tag{A.14}$$

where l_1, l_2 are linear forms. Localization would give a non-zero submodule of $M_{\bar{\phi}}$, contradicting the simplicity of $M_{\bar{\phi}}$; unless, $l_1 \sim x$ in which case localization of M_{l_1} is zero. In case $l_1 \sim x$, we claim that ϕ factorizes as $\phi \sim xl$ for some linear form l. This would mean that ϕ is degenerate, a contradiction. Now, to verify the claim, note that we have $\psi x = \phi\psi'$ for some ψ'. As $A/(x)$ is a domain, we see that $\phi \in (x)$ or $\psi' \in (x)$. But the latter can not hold, else, ψ will be a multiple of ϕ, implying that the *map* multiplication by ψ in (A.14) is zero. Now (ii) follows from (A.7), Corollary (A.9) and Remark (A.10). □

The rest of this section is devoted to the proof of Theorem (A.8).

Since for a quadratic $\bar{\phi}$ in W and an automorphism $\bar{\theta} \in \mathrm{Aut}_A(W)$, the module $M_{\bar{\phi}}$ is simple if and only if $M_{\bar{\theta}(\bar{\phi})}$ is simple, therefore, in proving (i) of (A.8), we are reduced to the canonical forms. In that case the results are known [MR1, 4.4], [Di] and [Ba, 2.1,2.2].

Here we prove (ii) and (iii) using some of the methods of [MR1]. We denote $W/\phi W$ by W_ϕ etc. (Hereon, we will drop the bar from a quadratic in W.) Let ϕ, ψ be quadratics in W. Applying Hom($,W_\psi$) to the exact sequence $0 \to \phi W \to W \to W_\phi \to 0$ we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & H(W_\phi, W_\psi) & \to & H(W, W_\psi) & \to & H(\phi W, W_\psi) & \to & E^1(W_\phi, W_\psi) & \to 0 \\
& \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\
0 \to & \ker\phi & \to & W_\psi & \xrightarrow{\phi} & W_\psi & \to & \mathrm{coker}\phi & \to 0
\end{array}$$

where H and E denote Hom and Ext respectively, and the map ϕ is right multiplication by $\phi : a + \psi W \mapsto a\phi + \psi W$. In particular, if W_ϕ and W_ψ are *simple* modules then

$$W_\phi \simeq W_\psi \Leftrightarrow \mathrm{Hom}(W_\phi, W_\psi) \neq 0 \Leftrightarrow \ker\phi \neq 0. \tag{A.15}$$

The question whether $\ker(\phi) \neq 0$ will now be reduced to the question of existence of polynomial (or, Laurent polynomial, in hyperbolic case) solutions to a certain system of differential equations. Recall that the Weyl algebra $W = k\langle p, q\rangle/(pq-1)$ is the ring of formal differential operators on the polynomial ring $k[q]$, with $p = d/dq$. An element of W can be written uniquely as a polynomial in p with coefficients on the right from $k[q]$. Now we treat the parabolic and hyperbolic cases separately.

Parabolic case: Proof of (A.8 ii). Let $\psi = p^2 + cq$ be a canonical parabolic. Below we will write $\phi = p^2 + r$ where $r = cq$, an element of $k[q]$ of degree 1. $W_\psi = W/\psi W$ is a free right $k[q]$-module of rank 2 having basis elements $1, p$, so an element of W_ψ can be written uniquely as $\alpha + p\beta$ ($\alpha, \beta \in k[q]$). Now we compute the action of the multiplication map by ϕ on this module, where ϕ is an arbitrary quadratic. Write $\phi = ap^2 + pt + s$ where t and s are polynomials in q of degrees at most 1 and at most 2, respectively. Put $\lambda = s - ar$. Using the relation $fp = -f' + pf$ for $f \in k[q]$, we get, by direct computation,

$$(\alpha + p\beta)\phi = \xi + p\eta \bmod \psi \qquad \alpha, \beta \in k[q] \tag{A.16}$$

where

$$\xi = a\alpha'' - t\alpha' + \lambda\alpha + 2ar\beta' - rt\beta + ar'\beta \tag{A.17a}$$

$$\eta = -2a\alpha' + t\alpha + a\beta'' - t\beta' + \lambda\beta \tag{A.17b}$$

and a prime denotes derivative with respect to q. Thus $\alpha + p\beta \in \ker\phi$ if and only if there exists a polynomial solution (α, β) to the system of differential equations: $\xi = 0, \eta = 0$. We rewrite these equations as

$$a\alpha'' - t\alpha' + \lambda\alpha = rt\beta - 2ar\beta' - ar'\beta \tag{A.18a}$$

$$-2a\alpha' + t\alpha = t\beta' - a\beta'' - \lambda\beta \tag{A.18b}$$

Proposition A.19. *If deg(r) = 1, and $(t, \lambda) \neq (0,0)$ then there is no nontrivial polynomial solution $(\alpha, \beta) \neq (0,0)$ to the system (A.18).*

Proof. We prove this by a degree comparison. First we show that $\alpha = 0 \Leftrightarrow \beta = 0$. Suppose $\alpha = 0$. Then the equations become

$$rt\beta = 2ar\beta' + ar'\beta \qquad t\beta' - \lambda\beta = a\beta''.$$

Let $\deg\beta = m \geq 0$ then comparing the degrees of both sides of the first equation gives

$$\deg\beta + \deg r + \deg t = \deg(\text{LHS}) = \deg(\text{RHS}) \leq \deg\beta + \deg r - 1.$$

Hence $\deg t \leq -1$ which shows that $t = 0$. Then the second equation forces $\lambda = 0$. Thus $\alpha = 0 \Rightarrow \beta = 0$. By a similar degree-comparison, we see that $\beta = 0 \Rightarrow \alpha = 0$.

Therefore, for a non-trivial solution, we must have $\alpha \neq 0, \beta \neq 0$. Let $\deg\alpha = n, \deg\beta = m$ $(n, m \geq 0)$. Again, we compare the degrees of the two sides of (A.18). We consider three cases. First, suppose $\deg\lambda \geq 1$, $t \neq 0$. Then (A.18a) and (A.18b) give respectively

$$n + \deg\lambda = m + \deg r + \deg t \qquad n + \deg t = m + \deg\lambda.$$

This implies $2n = 2m + \deg r = 2m + 1$, an impossibility. One similarly shows that we get absurdities in the cases $(\deg\lambda = 0, \lambda \neq 0, t \neq 0)$ and $(\lambda = 0, t \neq 0)$. This completes the proof because we are now left with the option $(t, \lambda) = (0, 0)$. □

We have shown that $\ker(\phi) = 0$ unless $(t, \lambda) = (0, 0)$. Now the condition $(t, \lambda) = (0, 0)$ is equivalent to $\phi = a\psi$. This proves (A.8 *ii*).

Hyperbolic case: Proof of (A.8 *iii*). Let $\psi = pq + \omega$, $(\omega \in k)$ be a canonical hyperbolic. Suppose first that $\omega \notin \mathbf{Z}$. Then, $W/\psi W$ is isomorphic to the right W-module $k[q, q^{-1}]$ where, on $f \in k[q, q^{-1}]$, the action of q is just the multiplication by q, and the action of p is $fp = -(f' + \omega q^{-1} f)$ [Di]. Hereon, we identify $W/\psi W$ with $k[q, q^{-1}]$.

We show that for an arbitrary quadratic ϕ, $W_\phi \simeq W_\psi$ implies $\phi \sim \psi + m$ for some $m \in \mathbf{Z}$. As W_ψ is simple (A.8*i*), we must have, in view of (A.15), $\ker\phi \neq 0$ where ϕ is multiplication by ϕ on W_ψ. Write $\phi = ap^2 + pt + s$ where t and s are polynomials in q of degrees at most 1 and at most 2 respectively. The action of multiplication on an element $f \in W_\psi = k[q, q^{-1}]$ is

$$f\phi = af'' + (2a\omega q^{-1} - t)f' + (a\omega^2 q^{-2} - a\omega q^{-2} - \omega q^{-1} t + s)f \mod(pq + \omega).$$

It follows that $f \in \ker\phi$ if and only if f satisfies the differential equation

$$\text{(A.20)} \qquad a(f'' + 2\omega q^{-1} f' + \omega^2 q^{-2} f - \omega q^{-2} f) - t(f' + \omega q^{-1} f) + sf = 0$$

(where we have collected together the terms "equidimensional" in q.) Now let $f = \sum_{k=n}^{m} c_k q^k$; $c_n \neq 0, c_m \neq 0$. Considering the coefficients of q^{m+2} and

q^{m+1} in (A.20) we see that s must reduce to a constant. Writing $s = c$, and $t = hq + g$ $(c, h, g \in k)$, the equation becomes

$$a(f'' + 2\omega q^{-1} f' + \omega^2 q^{-2} f - \omega q^{-2} f) - g(f' + \omega q^{-1} f) + (cf - hqf' - h\omega f) = 0.$$

Equating the coefficient of q^m to zero we get $c - mh - \omega h = 0$, i.e., $c = h(m + \omega)$. On the other hand equating the coefficient of q^{n-2} to zero we get $a(\omega + n)(\omega + n - 1) = 0$. Since we assumed $\omega \notin \mathbf{Z}$, $a = 0$. Now, equating the coefficient of q^{n-1} to zero we get $-g(n + \omega) = 0$. Again, since $\omega \notin \mathbf{Z}$, we have $g = 0$. Thus ϕ reduces to $\phi = h(pq + m + \omega), m \in \mathbf{Z}$, as required.

Conversely, we show that the hyperbolic modules corresponding to $pq + \omega$ and $pq + \omega + m$ $(\omega \notin \mathbf{Z}$, $m \in \mathbf{Z})$ are isomorphic. Clearly, it suffices to show this for $m = 1$, in which case we have an isomorphism $W/(pq + \omega + 1) \overset{q}{\xrightarrow{\sim}} W/(pq + \omega)$.

The remaining part (case $\omega \in \mathbf{Z}$) of the Theorem (A.8 *iii*) is taken care of by the following proposition.

Proposition A.21. *Let ϕ be a quadratic in W. Then (i) $W/\phi W \simeq W/pqW$ if and only if $\phi \sim pq + n, n \in \mathbf{N}$. (ii) $W/\phi W \simeq W/qpW$ if and only if $\phi \sim qp - n, n \in \mathbf{N}$.*

Proof. We prove (i), the proof of (ii) being similar. It is shown in (A.10) that $W/(pq + n) \overset{q^n}{\xrightarrow{\sim}} W/pq$ for $n \in \mathbf{N}$. Here we show the converse. Write again $\phi = ap^2 + pt + s$, where t (resp. s) is a polynomial in q of degree at most 1 (resp. at most 2). We first claim that $a = 0$ and $\deg s = 0$.

Suppose $a \neq 0$, then $W/\phi W$ is a free right $k[q]$-module of rank 2. Let $W/pqW \xrightarrow{\sim} W/\phi W$ be an isomorphism. It must be left multiplication by some element ψ of $W/\phi W$, say, $\psi = \alpha + p\beta$ where $\alpha, \beta \in k[q]$. So we have $(\alpha + p\beta)pq = 0 \bmod \phi$. Since $W/\phi W$ is q-torsion-free, this means $(\alpha + p\beta)p = 0$ and the map actually vanishes on the larger ideal $pW \supset pqW$, showing that the map can not be an isomorphism. So we must have $a = 0$. Now, $W/\phi W \simeq W/pqW$ also implies that $\mathrm{Hom}(W/\phi W, W/pW) \neq 0$, i.e., $\ker(\phi : W/pW \to W/pW) \neq 0$. As $W/pW \simeq k[q]$, we must have $0 \neq \alpha \in k[q]$ such that $\alpha\phi = 0 \bmod p$, i.e., $\alpha(pt + s) = 0 \bmod p$. This means

$$t\alpha' = s\alpha. \qquad \text{(A.22)}$$

By comparison of degrees, it follows that $\deg s = 0$.

Now write $t = hq + g, s = c$ $(h, g, c \in k)$ so that $\phi = p(hq + g) + c$. We must have $h \neq 0$, else ϕ reduces to a linear form $gp + c$ and $W/(gp + c)W$

is not isomorphic to W/pqW [MR1]. So taking $h = 1$, (A.22) becomes $(q+g)\alpha' = c\alpha$. Let $\deg\alpha = m \geq 0$. Consideration of the coefficient of q^m gives $m = c$, and the solution: $\alpha = \alpha_0(q+g)^m$. To finish the proof we show that $g = 0$. For this, consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \frac{W}{(q+g)W} \arrow[r, "p^{m+1}"] & \frac{W}{(p(q+g)+m)W} \arrow[r] \arrow[d, "\wr\, (q+g)^m"] \arrow[dr, "\alpha"] & \frac{W}{pW} \arrow[r] & 0 \\
0 \arrow[r] & \frac{W}{qW} \arrow[r, "p"] & \frac{W}{pqW} \arrow[r] & \frac{W}{pW} \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact (A.10) and the triangle is commutative. Since $(q+g)^m p^{m+1} = 0 \bmod pW$, but $(q+g)^m p^{m+1} \neq 0 \bmod pqW$ we see that there is a non-zero map $W/(q+g)W \longrightarrow W/qW$, which shows that $g = 0$. (Here, we have used the fact that $W/(q+\mu)W \simeq W/(q+\nu)W$ for $\mu, \nu \in k$ if and only if $\mu = \nu$ [MR1].) $\square$

II

INJECTIVE MODULES OVER ELLIPTIC ALGEBRAS

In this chapter we study injective modules over elliptic algebras. Our main objective behind studying injective modules over elliptic algebra was to construct a minimal injective resolution (*"residue-complex"*) for elliptic algebras. We have been successful in constructing the resolution only up to a certain term (§2). In the first section, we give basic results about indecomposable injective modules and their gk-dimension. In the second section, we study injective modules of gk-dimension 2 and construct the injective resolution up to the term corresponding to indecomposable injective modules of gk-dimension 2. In the third section, we consider the injective hullls of point modules (gk-1 critical modules) and show that their gk-dimension is 1. In the final section, we prove a Matlis-type duality for certain non-commutative complete local rings. The proof follows [Ma] with some 'non-commutative' modifications.

In sections 1-3 of this chapter, we denote by A an elliptic algebra, and by B the algebra A/gA, g being a normalizing element of degree 3, unique up to a scalar. (When necessary, we will use the symbol R to denote an arbitrary $\mathbb{N}$*-graded connected Noetherian k-algebra.) We assume that the cubic divisor E associated to A is reduced, so that the algebra B is a domain. We also assume that no positive integral power of σ fixes the class of $[\mathcal{L}]$ in $Pic(E)$, i.e., the index s_0 of the algebra A [ATV2, §7] is infinite. This hypothesis has the consequence that a gk-1 module is critical if and only if it is a shifted point module; and, thus every gk-1 module is annihilated by some power of g [ATV2,§7].*

1 Indecomposable Injective and Critical Modules

In this section, we state the basic results about indecomposable injective modules (which are injective hulls of critical modules, Lemma 1.2i).

Let R be an $\mathbb{N}$-graded connected Noetherian k-algebra. We will denote by R° the opposite algebra of R and by R^{e} the algebra $R \otimes_k R^{\circ}$. Thus an

(R,R)-bimodule is a right R^e-module. There is a diagram of restriction functors

$$
\begin{tikzcd}
\mathrm{Gr}-R^e \arrow[r, "\mathrm{Res}_{R^\circ}"] \arrow[d, "\mathrm{Res}_R"'] & \mathrm{Gr}-R^\circ \arrow[d, "\mathrm{Res}_k"] \\
\mathrm{Gr}-R \arrow[r, "\mathrm{Res}_k"'] & \mathrm{Gr}-k
\end{tikzcd}
\tag{1.1}
$$

in which all the functors are exact, and the functors Res_R and Res_{R° map injectives to injectives, and projectives to projectives [Ye1, 2.1]. By an R-module we mean a right or left graded R-module. For an R-module M, $E_R(M)$ denotes its graded injective hull. *For bimodules M and E we say that E is an injective hull of M in* $\mathrm{Gr}-R$ *and in* $R-\mathrm{Gr}$ *if* $\mathrm{Res}_R(E)$ *is an injective hull of* $\mathrm{Res}_R(M)$ *in* $\mathrm{Gr}-R$ *and* $Res_{R^\circ}(E)$ *is an injective hull of* $Res_{R^\circ}(M)$ *in* $\mathrm{Gr}-R^\circ$. Hereon, the restriction functors will be implicit, for the sake of legibility.

Lemma 1.2. *(i) E is an indecomposable injective R-module if and only if $E = E_R(M)$ for some finite critical R-module M.*

(ii) A non-zero homomorphism between two critical R-modules of same gk-dimension is essential injective. In particular, for an elliptic algebra A, if there exists a non-zero homomorphism between two gk-2 critical A-modules then they are gk-1 equivalent.

Proof. (i) If E is an indecomposable injective R-module, then E is an injective hull of every one of its non-zero submodule. Since E contains a finite R-module and hence a finite critical R-module M, $E = E_R(M)$. Conversely, if M is a finite critical module, then M is uniform. For, every submodule of M has the same gk-dimension and the same multiplicity as M has, and therefore M can not contain the direct sum of two non-zero submodules. M being uniform, $E_R(M)$ is indecomposable.

(ii) follows directly from Lemma (I.3.45). □

We say that an R-module F is a *maximal gk-m essential extension of an R-module M* if it is an essential extension of M, has gk-dimension m, and there is no module of gk-dimension m essentially and properly containing F. The following lemma will be later used in relating the gk-dimension of injective hull of a critical module M to that of M. Recall that for an arbitrary module, not necessarily finitely generated, the gk-dimension is defined as the supremum of the gk-dimensions of its finitely generated submodules.

Lemma 1.3. *Let M be a finite R-module of gk-dimension m, F a maximal gk-m essential extension of M, and $E = E_R(M)$. The following are equivalent*

(i) $gk(E) = m$,

(ii) $E \simeq F$,

(iii) For every injective map $0 \to M \xrightarrow{\phi} X$, *we have a map* $X \xrightarrow{\phi'} F$, *extending* $M \to F$, *i.e., the diagram below commutes.*

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr] & X \arrow[d, dotted] \\ & F \end{tikzcd}$$

(iv) Every extension $0 \to M \to X \to P \to 0$ *by a finite module P, drops to an extension of gk-dimension m i.e., for every extension* $0 \to M \to X \to P \to 0$, *there exists a commutative diagram*

$$\begin{array}{ccccccc} 0 \to & M & \to & X & \to & P & \to 0 \\ & \| & & \downarrow & & \downarrow & \\ 0 \to & M & \to & X' & \to & P' & \to 0 \end{array} \tag{1.4}$$

such that $gk(X') = m$.

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear. $(iii) \Rightarrow (iv)$ is also clear, because given an extension $0 \to M \to X \to P \to 0$ there exists a map $X \to F$ extending the inclusion map $M \hookrightarrow F$. Now take X', in (1.4), to be the image of $X \to F$. To show $(iv) \Rightarrow (i)$, let E' be a finite submodule of E. Then there exists a commutative diagram

$$\begin{array}{ccccccc} 0 \to & M & \to & E' & \to & P & \to 0 \\ & \| & & \phi\downarrow & & \downarrow & \\ 0 \to & M & \to & Y & \to & P' & \to 0 \end{array}$$

with $gk(Y) = m$. Now, $0 = \ker(\phi) \cap M \subset E'$, and E' being an essential extension of M, $\ker(\phi) = 0$. Thus ϕ is injective and $gk(E') \leq gk(Y) = m$. □

Remark. (1.5). Let F be a maximal gk-m essential extension of an R-module M. Then $\mathrm{Ext}^1(P, F) = 0$ for all R-modules P with $gk(P) \leq m$. Indeed, let $0 \to F \to X \to P \to 0$ be an extension. Then $gk(X) = m$. Now X contains a submodule Y such that $F \oplus Y$ is an essential submodule of

X and $F = (F \oplus Y)/Y$ is an essential submodule of X/Y ([GW, 3.22]). As $gk(X/Y) = m$, we have $F \xrightarrow{\sim} X/Y$, $Y = P$, and $X = F \oplus P$.

(1.6) Let M_1 and M_2 be two critical R-modules of gk-dimensions m_1 and m_2 respectively. If $E_A(M_1) \simeq E_A(M_2)$ then, by a general theorem [GW, 4ZC], there exist (essential) submodules $P_1 \subset M_1$ and $P_2 \subset M_2$ such that $P_1 \simeq P_2$. Since M_1 and M_2 are pure modules, $gk(P_1) = m_1$ and $\mathrm{gk}(P_2) = m_2$. It follows that $m_1 = m_2$. This shows that we can classify indecomposable injective R-modules into classes $\mathcal{E}^j$ such that an indecomposable injective module E is of class $\mathcal{E}^j$ if and only if $E = E_R(M)$ for some finite critical R-module M of gk-dimension j. It turns out (1.14, 3.10) that in the case of elliptic algebras, the gk-dimension of an injective indecomposable module of class $\mathcal{E}^j$ is also j.

(1.7) Now let A be an elliptic algebra. The indecomposable injective modules will be of classes $\mathcal{E}^j$ with $j = 0, 1, 2, 3$. Our main motive behind studying injective modules over elliptic algebra was to construct an injective resolution

$$0 \to A \to E^3 \to E^2 \to E^1 \to E^0 \to 0 \tag{1.8}$$

of A where we expect E^j would be a direct sum, with certain multiplicities, of indecomposable injective modules of class $\mathcal{E}^j$ (injective hulls of finite critical A-modules of gk-dimension j). This would be an analogue of the *"residue-complex"* in the commutative case, and could be called a residue-complex for quantum planes. In the next section, we construct the complex up to the E^2 term. We have not been able to construct the full complex.

The only critical modules in gk-dimension 0 are the shifts of the simple module k. It is standard that the (A, A)-bimodule $A' = \mathrm{Hom}_k(A, k)$ is an injective hull of k in both Gr$-A$ and $A-$Gr. It is clear that the gk-dimension of A' is 0. The critical modules of gk-dimension 1 are precisely the shifts of the point modules. We return to injective hulls of these modules in section 3, where we show that the gk-dimension of an injective hull of a point module is 1. Here we show that, the indecomposable injective modules of class $\mathcal{E}^2$ have gk-dimension 2, and can be classified, up to isomorphism, in terms of the gk-1 equivalence classes. This classification is used in the next section to construct the "residue-complex" up to the E^2 term.

Proposition 1.9. *Let M be a gk-2 finite critical A-module. Then the following conditions are equivalent*

(i) M is gk-1 equivalent to B,
(ii) M is g-torsion, i.e., $M[g^{-1}] = 0$,
(iiii) M has a g-torsion element.
Under these conditions, $E_A(M) \simeq E_A(B)$.

Proof. $(i) \Rightarrow (ii)$ is clear because if $M \sim B$ then, by (I.5.4)), $M[g^{-1}] = 0$; and, $(ii) \Rightarrow (iii)$ is trivial. We now show $(iii) \Rightarrow (i)$ and $E_A(M) \simeq E_A(B)$. We prove this for a right A-module. Of course, a similar proof works for left modules. Let $x \in M$ be such that $xg = 0$. Since g is normalizing, $xAg = 0$, and thus the (essential) sub-A-module xA of M is a B-module. Being a submodule of a critical module, xA is a critical A-module, as well as a critical B-module, of gk-dimension 2. Now $xA \sim M$ (Lemma 1.2 ii). To complete the proof, it suffices to show that for any gk-2 finite critical B-module P, $B \sim P$ (as A-modules) and $E_A(P) \simeq E_A(B)$. Since $P = \mathrm{Hom}_B(B, P) \neq 0$, there exists a non-zero B- and hence a non-zero A-homomorphism $\phi : B \to P(n)$ for some $n \geq 0$. By Lemma (1.2ii), $P(n) \sim B$, or, $P \sim B(-n)$. Since multiplication by a homogeneous element of degree n, $B(-n) \to B$ gives gk-1 equivalence of $B(-n)$ with B, it follows that P is gk-1 equivalent to B. Since all monomorphisms considered above are essential, $E_A(P) \simeq E_A(B)$. □

Corollary 1.10. *Let M be a finite critical A-module,* not *gk-1 equivalent to B, and $E = E_A(M)$. Then E has a natural Λ-module structure, compatible with its A-module structure. Further, $E \simeq E_\Lambda(M[g^{-1}])$ as Λ-modules.*

Proof. To prove the first statement, it suffices to show that E is g-torsion-free and g-divisible [GW, 9.12]. Being an injective A-module, E is automatically A-divisible (in the graded sense) and, in particular, g-divisible. To show that E is g-torsion-free, suppose, on the contrary, that $0 \neq x \in E$ such that $xg = 0$. There exists an $a \in A$ such that $0 \neq xa \in M$. But then $xag = 0$, because g is normalizing. This is a contradiction since M, being not gk-1 equivalent to B, is g-torsion-free by the Proposition 1.9. The last statement is a standard fact [GW, 9.16b]: E being g-torsion-free injective A-module is also injective as Λ-module, and, since the injection $M \hookrightarrow E$ factors through $M[g^{-1}]$, we have $E \simeq E_\Lambda(M[g^{-1}])$. □

Proposition 1.11. *Let M_α, M_β be two gk-2 finite critical A-modules. Then $E_A(M_\alpha) \simeq E_A(M_\beta)$ (as A-modules) if and only if M_α is gk-1 equivalent to M_β.*

Proof. In case, $M_\alpha \sim M_\beta \sim B$, we get, by Proposition (1.9), $E_A(M_\alpha) \simeq$

$E_A(M_\beta) \simeq E_A(B)$. Suppose now that $M_\alpha \sim M_\beta \not\sim B$. By Corollary (1.10), $E_A(M_\alpha) \simeq E_\Lambda(M_\alpha[g^{-1}]); E_A(M_\beta) \simeq E_\Lambda(M_\beta[g^{-1}])$ as Λ-modules. But also, $M_\alpha \sim M_\beta$ implies that $M_\alpha[g^{-1}] \simeq M_\beta[g^{-1}]$ as Λ-modules (I.5.4). Thus $E_A(M_\alpha) \simeq E_A(M_\beta)$ as Λ-modules and hence as A-modules.

Conversely, if $E_A(M_\alpha) \simeq E_A(M_\beta)$ then there exist essential submodules $M'_\alpha \subset M_\alpha$, $M'_\beta \subset M_\beta$ such that $M'_\alpha \simeq M'_\beta$ as A-modules. But as $M_\alpha \sim M'_\alpha$ and $M_\beta \sim M'_\beta$, (Lemma 1.2 *ii*), we have $M_\alpha \sim M_\beta$. □

Theorem 1.12. *The indecomposable injective A-modules of class $\mathcal{E}^2$ are exactly the injective hulls, $E_A(M_\alpha)$, of gk-2 critical Cohen-Macaulay modules M_α, one from each gk-1 equivalence class.*

Proof. In view of (1.2 *i*) and Proposition (1.11), we only have to show that within a gk-1 equivalence class, we can restrict to a Cohen-Macaulay module. For this it suffices to check that for a gk-2 finite critical module M, its Cohen-Macaulay bidual $M^{\vee\vee}$ is critical and $E_A(M) \simeq E_A(M^{\vee\vee})$. But this is immediate from Lemma (I.3.45). □

(1.13) The description of indecomposable injective modules of class $\mathcal{E}^3$ is trivial. First, note that A is a critical A-module [ATV2, 2.30 v]. Up to an isomorphism, $E_A(A)$ is the only object in class $\mathcal{E}^3$. More precisely, if M is a gk-3 critical A-module then $E_A(M) \simeq E_A(A)$. Indeed, since $M = \mathrm{Hom}_A(A, M) \neq 0$, we see by Lemma (1.2 *ii*) that there is an essential monomorphism $0 \to A \to M(n)$, and $E_A(A) \simeq E_A(M(n))$, or, $E_A(A(-n)) \simeq E_A(M)$ (for some $n \geq 0$). But as a multiplication by a homogeneous element of degree n gives an essential monomorphism $A(-n) \hookrightarrow A$, we have $E_A(A) \simeq E_A(M)$.

Proposition 1.14. *Let M be a gk-2 critical A-module, and E an injective hull of M. Then $gk(E) = 2$.*

Proof. Let F be the maximal gk-2 essential extension of F. To prove the proposition, we verify that $\mathrm{Ext}^1(P, F) = 0$ for all finite modules. As a finite module has a finite filtration whose successive quotients are critical (I.1.6*i*), it suffices to verify this for critical modules P. In view of Remark (1.5), we can assume that P is gk-3 critical. Then there is exact sequence $0 \to A(-n) \to P \to P' \to 0$ (1.13) where P' is gk-2, thus $\mathrm{Ext}^1(P', F) = 0$ (1.5). Considering the cohomology of this sequence, we see that $\mathrm{Ext}^1(P, F) = 0$. □

2 Injective Modules of GK-dimension 2

In this section we construct an injective resolution ("residue complex" (1.8)) of A up to the E^2 term, i.e., the term corresponding to the indecomposable injective modules of class $\mathcal{E}^2$. Our construction of the E^2 term is as a direct sum of two injective factors: one corresponds to an injective hull of the critical A-module $B = A/(g)$ and the other decomposes as a direct sum (with certain multiplicities) of injective hulls of gk-2 critical modules not gk-1 equivalent to B.

Let Q be the $\mathbf{Z}$-graded quotient field of A. This (A, A)-bimodule is an injective hull of A in $\mathrm{Gr}-A$ and in $A-\mathrm{Gr}$ and corresponds to the E^3 term in (1.8).

To construct the E^2 term, we start with the injective hull of the gk-2 critical A-module B. It turns out that the bimodule $Q/A_{(g)}$ is an injective hull of B in $\mathrm{Gr}-A$ and $A-\mathrm{Gr}$ (Proposition 2.8). Here $A_{(g)}$ is the localization of A with respect to the Ore subset

$$(2.1) \qquad S = \{s, \text{homogeneous element of } A | s \notin (g)\}.$$

That such localizations exist would follow from the fact that the ideal (g) satisfies the Artin-Rees property.

Proposition 2.2. *The following conditions on a (two-sided) ideal J of a right Noetherian ring R are equivalent*

(i) If I is a right ideal of R then $I \cap J^n \subset IJ$ for some n,

(ii) If M is a finitely generated right R-module and N is an essential submodule of M with $NJ = 0$ then $MJ^n = 0$ for some n,

(iii) If M is a finitely generated right R-module and N a submodule of M with $NJ = 0$, then $N \cap MJ^n = 0$ for some n.

This is standard [MR2, Theorem 4.2.2].

A (two-sided) ideal J of a right Noetherian ring R is said to have the right *Artin-Rees property* if it satisfies the equivalent conditions of Proposition (2.2). One can similarly define the left Artin-Rees property.

Proposition 2.3. *Let R be a right Noetherian ring and J an ideal generated by normalizing elements. Then J has the right Artin-Rees property.*

This is also standard [MR2, Proposition 4.2.6].

In particular, we see that in an elliptic algebra A, the ideal $J = (g)$

has the Artin-Rees property. This has the following consequence for the injective hull of B over A.

Proposition 2.4. *Let $E = E_A(B)$ be an injective hull of B in* $\mathrm{Gr}-A$. *Then E is g-torsion: $E = \cup_{n=1}^{\infty} E_n$ where $E_n = \{x \in E | xg^n = 0\}$. Further, $E_n = E_{A/g^n}(B)$.*

Of course, a similar statement holds for an injective hull of B in $A-\mathrm{Gr}$.

Proof. Given $x \in E$ we show that $xg^n = 0$ for some n. Let $M = xA$, $N = M \cap B$. Now E being a uniform module, $0 \neq N$ is an essential submodule of M. Clearly, $Ng = 0$. Now apply Proposition (1.2) to conclude that $xAg^n = 0$. The last statement follows from a standard result [GW, 4E]: if I is a two-sided ideal in a ring A, M a right A/I-module, $E = E_A(M)$, then $\mathrm{ann}_E(I) = E_{A/I}(M)$. Here we take $M = B$ and $I = (g)^n$. □

We now turn to localization of A. Let S be defined as in (2.1). Since (g) is a completely prime ideal (i.e., $B = A/(g)$ is a domain), we see that S is a multiplicative subset. In standard notations,

$$S = \mathcal{C}_A(g) \overset{\text{def}}{=} \{s \text{ homogeneous element of } A | s \text{ is regular in } A/(g)\}.$$

Lemma 2.5. $\cap_{n=1}^{\infty}(g)^n = 0$. *Consequently, any homogeneous element $a \in A$ can be written as $a = g^n s$ with a unique $n \in \mathrm{N}$ and a unique $s \in S$.*

Proof. The first assertion follows simply by noting that $(g)^n \subset A_{\geq 3n}$ and $\cap_n A_{3n} = 0$. For a given homogeneous element $a \in A$, choose the largest integer $n \in \mathrm{N}$ such that $a \in (g)^n$. Then $a = g^n s$ for some $s \in S$. Uniqueness of this representation is evident, as A is a domain. □

We now show that S is a (left and right) denominator subset of A. We first prove that the image of S in $A/(g)^n$ is an Ore subset, and then lift this to A, using the Artin-Rees property of $J = (g)$. We denote the image of S in A/g^n by S itself, if there is no confusion.

Proposition 2.6. *S is a (left and right) Ore subset in A.*

Proof. We check the right Ore condition. We first show, by induction over n, that for all n, the homomorphic image of S in $A/(g)^n$ satisfies the right Ore condition. The image of S in $B = A/(g)$ is $\mathcal{C}_B(0) = \{b \in B | b \neq 0\}$. Since B is a Noetherian domain, this is an Ore subset. Thus the statement is true for $n = 1$. To complete the induction, one has to show that for arbitrary homogeneous elements $a \in A, s \in S$, if there exist homogeneous elements $a_n \in A, s_n \in S$ such that $as_n - sa_n \in (g)^n$ then there exist homogeneous elements $a_{n+1} \in A, s_{n+1} \in S$ such that $as_{n+1} - sa_{n+1} \in (g)^{n+1}$. Let $as_n - sa_n = g^n r$, $r \in A$. Let s^* be a homogeneous element such that $sg^n = g^n s^*$. By (2.5), $s^* \in S$. Now there exist homogeneous elements $r' \in A, s' \in S$ such that $rs' - s^* r' \in (g)$. Then $as_n s' - s(a_n s' + g^n r') = g^n rs' - sg^n r' = g^n(rs' - s^* r') \in (g)^{n+1}$. Thus we can take $s_{n+1} = s_n s' \in S$ and $a_{n+1} = a_n s' + g^n r'$, to complete the induction.

We now prove that S satisfies the right Ore condition in A. Let $a \in A$, $s \in S$ be homogeneous elements. Since $J = (g)$ satisfies the right Artin-Rees property, therefore, for some n,

$$(aA + sA) \cap (g)^n \subset (aA + sA).(g) = (ag) + (sg).$$

The right Ore condition for S in $A/(g)^n$ gives homogeneous elements $a' \in A$ and $s' \in S$ such that $as' - sa' \in (g)^n$. Thus, $as' - sa' \in (ag) + (sg)$. Therefore, one has homogeneous elements $b, c \in (g)$ such that $as' - sa' = ab + sc$. Then $a(s' - b) = s(a' + c)$, and $s' - b \in S$. This completes the proof. □

(2.7) Note that $\mathrm{ass}_A(S) := \{a \in A | as = 0 \text{ for some } s \in S\} = 0$. Thus, S is a denominator subset. We denote by $A_{(g)}$ the **Z**-graded localization of A with respect to S. The canonical map $A \to A_{(g)}$ is injective and $A_{(g)}$ is a domain with the quotient field Q. Further, $(g)A_{(g)}$ is a two-sided principal ideal which we will denote simply by (g), if there is no possibility of confusion. It defines an exhaustive filtration on $A_{(g)}$, which we simply call the g-adic filtration. We will consider the g-adic completion $\hat{A}_{(g)}$ and a Matlis-type duality for $\hat{A}_{(g)}$-modules in the last section.

The g-adic filtration is separated:

$$\cap_{n=1}^{\infty} (g)^n.A_{(g)} = 0.$$

Hence, every non-zero element of $A_{(g)}$ can be written as $g^n(uv^{-1})$ for a unique $n \in \mathbf{N}$ where $u, v \in A - (g)$. Every non-zero element of Q can be written as $g^n(uv^{-1})$ for a unique $n \in \mathbf{Z}$ such that $u, v \in A - (g)$. This defines a discrete valuation on Q. The ring $A_{(g)}$ is the valuation ring of Q for this valuation, $(g)A_{(g)}$ is the (unique) maximal ideal of $A_{(g)}$, and $A_{(g)}/(g)A_{(g)}$, the residue field of the valuation, is isomorphic to K_B, the **Z**-graded field of fractions of B. It follows that the global dimension of $A_{(g)}$ is 1.

Proposition 2.8. *The (A, A)-bimodule $Q/A_{(g)}$ is an injective hull of B in Gr-A and in A-Gr.*

Proof. As the global dimension of $A_{(g)}$ is 1, the exact sequence

$$0 \to A_{(g)} \to Q \to Q/A_{(g)} \to 0 \tag{2.9}$$

shows that $Q/A_{(g)}$ is an injective module in $\mathrm{Gr}-A_{(g)}$ and in $A_{(g)}-\mathrm{Gr}$, and therefore, an injective module in Gr–A and in A–Gr. Now fix a homogeneous element $t \in S$ of degree 3. Then the right (resp. left) A-homomorphism $A \to Q$ mapping the generator 1 to tg^{-1} (resp. $g^{-1}t$ defines an essential monomorphism $B = A/g \hookrightarrow Q/A_{(g)}$ in $\mathrm{Gr}-A$ (resp. $A-\mathrm{Gr}$).□

We now look at the contribution to the E^2 term from other gk-2 critical modules, i.e., ones not gk-1 equivalent to B. In the notation of (1.12), we write the indecomposable injective modules of class $\mathcal{E}^2$ as injective hulls $E_A(M_\alpha)$ where M_α's are critical gk-2 Cohen-Macaulay modules, one from each gk-1 equivalence class α. *We denote the gk-1 equivalence class of B by β.* We now show that there is an (A, A)-bimodule which decomposes as direct sums (with certain multiplicities) of the injective hulls $E_A(M_\alpha)$ with $\alpha \neq \beta$. Recall that $\Lambda = A[g^{-1}]$ is the $\mathbf{Z}$-graded localization of A with respect to the Ore subset $\{g^n | n \geq 0\}$.

Theorem 2.10. *The (A, A)-bimodule Q/Λ is a g-torsion-free injective module in Gr-A and in A-Gr. Further, Q/Λ decomposes as*

$$Q/\Lambda = \oplus_{\alpha \neq \beta} \mu_\alpha E_A(M_\alpha) \tag{2.11}$$

in $\mathrm{Gr}-A$ *(resp.* $A-\mathrm{Gr}$*), where the direct sum runs over gk-1 equivalence classes different from β (that of B), and $\mu_\alpha \geq 1$.*

Proof. As the global dimension of Λ is 1 [ATV2, §7], the exact sequence

$$0 \to \Lambda \to Q \to Q/\Lambda \to 0$$

shows that Q/Λ is an injective module in Gr–Λ and in Λ–Gr, and therefore an injective module in Gr–A and in A–Gr. It is clear that Q/Λ is g-torsion-free.

To complete the proof, it is sufficient to show that there is a non-zero A-homomorphism from every gk-2 critical Cohen-Macaulay module $M_\alpha, (\alpha \neq \beta)$ to Q/Λ. Indeed, if $\phi : M_\alpha \to Q/\Lambda$ is a non-zero homomorphism then ϕ is necessarily injective; else, one would have a gk-1 submodule $M_\alpha/\mathrm{ker}\phi$ of Q/Λ which would give g-torsion to Q/Λ. Finally, the decomposition would be direct because $E_A(M_\alpha)$ and $E_A(M_{\alpha'})$ can not intersect within Q/Λ for $\alpha \neq \alpha'$.

Let us therefore exhibit a non-zero A-homomorphism from such M to Q/Λ. We show this in Gr$-A$, similar argument can be used in $A-$Gr. Let M have the minimal resolution as in (Chapter I, 3.3)

$$0 \to \oplus_{k=1}^{r} A(-j_k) \xrightarrow{[M]} \oplus_{k=1}^{r} A(-i_k) \to M \to 0.$$

To give a non-zero morphism from M to Q/Λ is equivalent to giving a row-vector $[q] \in Q^r - \Lambda^r$ such that $[q][M] \in \Lambda^r$. Now M having gk-dimension < 3, $M \otimes_A Q = 0$ ([ATV2, Proposition 2.30 v]). Thus the matrix $[M]$ is invertible in $\mathrm{End}_Q(Q^r)$. In particular, there exists a row-vector $[q] \in Q^r$ such that

$$(2.12) \qquad [q][M] = [1, 1, \ldots, 1] \quad \in \Lambda^r.$$

We claim that this $[q]$ has the required property, i.e., $[q] \notin \Lambda^r$. Suppose, on the contrary, that $[q] \in \Lambda^r$. Write $q_i = (g^{m_i})^{-1} a_i$, $(m_i \geq 0)$, $a_i \notin (g)$. Take $m = \max\{m_i | i = 1, \ldots, r\} = m_{i_0}$, say. Multiplying the equation (2.12) on the left by g^m we get

$$(2.13) \qquad [b][M] = [g^m, g^m, \ldots, g^m]$$

where $[b] \in A^r$, and $b_{i_0} = a_{i_0} \notin (g)$. This, however, means that there is a non-zero A-homomorphism from M to $B = A/g$. But M and B being critical, this would imply that M is gk-1 equivalent to B (Proposition 1.9), contrary to hypothesis. □

(2.14) Now the sequence of **Z**-graded (A, A)-bimodules

$$0 \to A \to Q \to Q/\Lambda \oplus Q/A_{(g)} \to ..$$

is exact. Indeed, $\Lambda \cap A_{(g)} = A$ inside Q. It is also clear that the map $Q/A \to Q/\Lambda \oplus Q/A_{(g)}$ is essential. This, therefore, gives a minimal injective resolution up to the E^2 term.

3 Injective Hulls of Point Modules.

In this section, we study the indecomposable injective modules of class $\mathcal{E}^1$, i.e, injective hulls of point modules (gk-1 finite critical modules). Using the computations of certain Ext groups, we describe the possible essential extensions of a point module. It turns out, as in the case of "commutative projective plane", that an injective hull of a point module has gk-dimension 1; in other words, there are no essential gk-2 extensions of a point module.

Proposition 3.1. *Let N_p be a point module. Then* $\mathrm{Ext}^1(k, N_p) = k(1)$,

$$\mathrm{Ext}^2(k, N_p) = k(2) \oplus k(2), \quad \mathrm{Ext}^3(k, N_p) = k(3).$$

Proof. Applying Hom($,N_p$) to the resolution of k_A (which would be transpose of the sequence (Chapter I, 1.1)), we get the complex

(3.2)

$$0 \to N_p \xrightarrow{[xyz]} N_p(1) \oplus N_p(1) \oplus N_p(1) \xrightarrow{M} N_p(2) \oplus N_p(2) \oplus N_p(2) \xrightarrow{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} N_p(3) \to 0.$$

Here M is the transpose of the matrix M in (I,1.1). Denote the three maps in (3.2) by ϕ_1, ϕ_2, ϕ_3 respectively. First note that the matrix M has rank 2 at every point p [ATV1, §1]. Thus the dimension of the k-space $(\ker(\phi_2)_n)$ is 1 for $n \geq -1$ and 0 otherwise; that of $(\mathrm{im}(\phi_2)_n)$ is 2 for $n \geq -1$ and 0 otherwise. The maps ϕ_1 and ϕ_3 are non-zero in degrees $n \geq 0$, and $n \geq -2$ respectively. Thus the dimension of $(\mathrm{im}(\phi_1)_n)$ is 1 for $n \geq 0$ and 0 otherwise; that of $(\ker(\phi_3)_n)$ is 2 for $n \geq -2$ and 0 otherwise. Knowing the dimensions of the kernels and images of the maps, we now get the proposition. □

In fact, the non-split extension of N_p by $k(1)$, corresponding to $\mathrm{Ext}^1(k, N_p) = k(1)$, is given by

$$(3.3) \qquad 0 \to N_p \to N_{\sigma^{-1}p}(1) \to k(1) \to 0$$

as we already know (Chapter I, 2.1). An essential extension of a point module by a finite-dimensional (gk-0) module is realized by successive extensions of the form (3.3). We have a direct system $N_p \hookrightarrow N_{\sigma^{-1}p}(1) \hookrightarrow N_{\sigma^{-2}p}(2) \hookrightarrow \cdots$. We denote this direct system by $\mathbf{N}_p$, and the direct limit by N_p^∞:

$$(3.4) \qquad N_p^\infty = \varinjlim \mathbf{N}_p = \varinjlim N_{\sigma^{-n}p}(n).$$

Note that we have $N_p^\infty(k) \simeq N_{\sigma^k p}^\infty$ for $k \in \mathbf{Z}$.

Lemma 3.5. *Let (M_i) be a direct system of A-modules, X a finite A-module. There is a natural isomorphism*

$$(3.6) \qquad \varinjlim \mathrm{Ext}^i(X, M_i) \xrightarrow{\sim} \mathrm{Ext}^i(X, \varinjlim M_i) \qquad (i \geq 0).$$

Proof. Indeed, the category of finite A-modules has enough projectives, and both sides in (3.6) are δ-functors on that category, coeffaceable for $i > 0$. As the natural homomorphism

$$\varinjlim \operatorname{Hom}(X, M_i) \to \operatorname{Hom}(X, \varinjlim M_i)$$

is an isomorphism for a finite (-ly presented) module X, the lemma is proved. □

In view of (3.1), we conclude from (3.5) that $\operatorname{Ext}^i(k, N_p^\infty) = 0$ for all $i \geq 0$. In fact,

$$\text{(3.7)} \qquad \operatorname{Ext}^i(T, N_p^\infty) = 0 \qquad (i \geq 0)$$

for all torsion modules T, because T is a direct limit of finite torsion modules T_α, and $\operatorname{Ext}^i(\lim_\to T_\alpha, N_p^\infty) = \lim_\leftarrow \operatorname{Ext}^i(T_\alpha, N_p^\infty) = 0$.

Thus, the module N_p^∞ is the maximal gk-0 essential extension of N_p. Now let $0 \to N_p^\infty \to X \to P \to 0$ be an essential extension of N_p^∞ by a pure (i.e., torsion-free) gk-1 module P. As P has a filtration whose successive quotients are shifted point modules, we see that the essential extension of N_p^∞ by P is realized by successive extensions of N_p^∞ by shifted point modules. The following result shows what are such possible essential extensions of N_p^∞. Recall that $p^* = \rho^{-1}\sigma(p)$ (Chapter I, 2.13).

Lemma 3.8. $\operatorname{Ext}^i(N_q^\infty, N_p^\infty) \simeq \operatorname{Ext}^i(N_q, N_p^\infty)$ *for* $i \geq 0$. *Hence*

$$\operatorname{Hom}(N_q^\infty, N_p^\infty)_0 = \begin{cases} k & q = p, \\ 0 & q \neq p. \end{cases}$$

$$\operatorname{Ext}^1(N_q^\infty, N_p^\infty)_0 = \begin{cases} k & q = p \ \text{or}\ q = p^*, \\ 0 & \text{otherwise.} \end{cases}$$

$$\operatorname{Ext}^2(N_q^\infty, N_p^\infty)_0 = \begin{cases} k & q = p^*, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Applying $\operatorname{Hom}(\ , N_p^\infty)$ to the sequence $0 \to N_q \to N_q^\infty \to T \to 0$, and using (3.7), we verify the first claim. Now, $\operatorname{Ext}^i(N_q^\infty, N_p^\infty)_0 \simeq \operatorname{Ext}^i(N_q, N_p^\infty)_0 = \lim_{\longrightarrow} \operatorname{Ext}^i(N_q, N_{\sigma^{-n}p}(n))_0$ (3.6). Thus, the formulas follow from the computations (Chapter I, 3.34). □

This lemma gives us the possible gk-1 essential extensions of N_p^∞. Let

$$\text{(3.9)} \qquad 0 \to N_p^\infty \to T_p \to N_p^\infty \to 0 \qquad 0 \to N_p^\infty \to F_p \to N_{p^*}^\infty \to 0$$

be the "first" essential extensions corresponding to the non-vanishing Ext^1 in (3.8). One can check that $\mathrm{Ext}^1(N_q^\infty, T_p) = 0$ unless $q = p$ or p^*, and $\mathrm{Ext}^1(N_q^\infty, F_p) = 0$ unless q is one of p, p^*, p^{**}. (These claims follow by applying Hom $(N_q^\infty, \ )$ to the sequences (3.9), and using (3.7) and (3.8) repeatedly.) We see that there is a two-parameter family of essential gk-1 extensions of N_p^∞, starting with (3.9). Geometrically, the injective hull of a point looks like a 'thickening' of the point along the tangential direction, and a 'thickening' along the $*$-orbit of the point.

Let E denote the maximal gk-1 essential extension of N_p. Our main purpose now is to show that there are no gk-2 essential extensions of N_p (or of E), so that E is actually an injective hull of N_p. In other words, the gk-dimension of an injective hull of a point module is 1. This is similar to the case in the commutative projective plane, where the result is quickly implied by the Artin-Rees property of the maximal ideal corresponding to the given point. In the quantum case, we use the Artin-Rees property of the (two-sided) ideal (g) (2.3); but, this is not sufficient to conclude the result and we need some more computations as the following proof shows.

Proposition 3.10. *The gk-dimension of an injective hull of a point module is 1.*

Proof. We verify the equivalent condition (iv) of Lemma 1.3. Let N_p be a point module, and let E denote a maximal gk-1 essential extension of N_p. Let $0 \to N_p \xrightarrow{j} X \to P \to 0$ be an extension of N_p by a finite module P. Since the ideal (g) satisfies the Artin-Rees property (2.3), there exists an integer $n \geq 1$ such that $Xg^n \cap N_p = 0$. Thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & N_p & \to & X & \to & P & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & N_p & \to & X/Xg^n & \to & P' & \to & 0 \end{array}$$

where $P'g^n = 0$ so that P', being a A/g^n-module, has gk-dimension 2. We are reduced to verifying the condition (iv) of Lemma 1.3 for a gk-2 g-torsion module P, where we can further assume P to be critical. Being a gk-2 critical g-torsion module, P is gk-1 equivalent to $B = A/gA$ (Proposition 1.9). In fact, we have an exact sequence (see the proof of 1.9)

$$0 \to B(-n) \to P \to S \to 0,$$

for some $n \geq 0$, and some gk-1 module S. Now let $0 \to N_p \to X \to P \to 0$ be a given extension. To verify $1.3iv$ we show equivalently (see $1.3iii$) that there is a map $X \to E$, extending the map $N_p \to E$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & N_p & \to & Y & \to & B(-n) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & N_p & \to & X & \to & P & \to & 0 \end{array}$$

where the square on the right is cartesian (a pull-back of $X \to P$, $B(-n) \to P$). As $X/Y \simeq S$ has gk-dimension 1, and $\mathrm{Ext}^1(S,E) = 0$ (1.5), it suffices to show that there is a map $Y \to E$, extending the map $N_p \to E$. This is shown by the following Lemma. □

Lemma 3.11 *Let $0 \to N_p \to X \to B(-n) \to 0$ $(n \geq 0)$ be a non-split extension. There exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & N_p & \to & X & \to & B(-n) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & N_p & \to & Y & \to & N_{\sigma^n p^*}(-n) & \to & 0 \end{array}$$

and therefore a map $X \to E$, extending the map $N_p \to E$.

Proof. Consider the surjective map $B \to N_{\sigma^n p^*} \to 0$, and let K be the kernel. The cohomology of the exact sequence $0 \to K \to B \to N_{\sigma^n p^*} \to 0$ gives

$$(3.12) \qquad \begin{array}{cccccc} 0 & \to & \mathrm{Hom}(B, N_p(n))_0 & \to & \mathrm{Hom}(K, N_p(n))_0 & \to \\ & & \mathrm{Ext}^1(N_{\sigma^n p^*}, N_p(n))_0 & \to & \mathrm{Ext}^1(B, N_p(n))_0 & \to \end{array}$$

We used the fact that $\mathrm{Hom}(N_{\sigma^n p^*}, N_p(n)) = 0$ (I,3.34). As the dimensions of the first, third and fourth terms in (3.12) are all 1 (I.3.34, I.3.35), to complete the proof of the lemma it is suffices to show that the dimension of the second term in (3.12) is also 1. We do this by an explicit computation.

In the rest of this proof, E will denote the cubic divisor associated with A (and not the injective hull !). Let $\{l_1, l_2\}$ be a basis for the space of linear forms vanishing at $\sigma^n p^*$, where we choose l_1 such that $l_1.E = \sigma^n p + \sigma^n p^* + q$, for some $q \in E$. Let w_1, w_2 be linear forms such that $l_1 w_2 + l_2 w_1 = 0$ (I.2.6), so that w_1, w_2 pass through $\rho(\sigma^n p^*) = \sigma^{n+1} p$, and $w_1.E = 2\sigma^{n+1} p + \sigma q$ (I.2.3). Put $g = l_1 \phi_2 + l_2 \phi_1$ for $\phi_1, \phi_2 \in A_2$. Then we have exact sequences

$$0 \to K = (l_1, l_2)/(g) \to A/(g) \to A/(l_1, l_2) \to 0,$$

$$0 \to A(-2) \oplus A(-3) \xrightarrow{\begin{bmatrix} w_1 & \phi_1 \\ w_2 & \phi_2 \end{bmatrix}} A(-1) \oplus A(-1) \xrightarrow{[l_2 l_1]} (l_1, l_2)/(g) \to 0.$$

To give a non-zero map $K \to N_p(n)$ is equivalent to giving a row vector $[\xi_1, \xi_2]$, such that $\xi_1, \xi_2 \in A_{n+1}$, $[\bar{\xi_1}(p), \bar{\xi_2}(p)] \neq [0,0]$, and

$$(1)\ (\bar{\xi_1}\bar{w_1} + \bar{\xi_2}w_2)(p) = 0, \quad (2)\ (\bar{\xi_1}\bar{\phi_1} + \bar{\xi_2}\bar{\phi_2})(p) = 0$$

where a bar denotes the image in B. As w_1, w_2 intersect at $\sigma^{n+1}p$, equation (1) is trivially satisfied. Now we claim that $\bar{\phi_1}(\sigma^{n+1}p) = 0, \bar{\phi_2}(\sigma^{n+1}p) \neq 0$, so that (2) implies $\bar{\xi_2}(p) = 0$, and hence the dimension of $\mathrm{Hom}(K, N_p(n))_0$ will be 1.

To check the claim, note that $g = l_1\phi_2 + l_2\phi_1$ means $\bar{l_1}\bar{\phi_2} = -\bar{l_2}\bar{\phi_1}$ so we have

$$l_1.E + \sigma^{-1}(\phi_2.E) = l_2.E + \sigma^{-1}(\phi_1.E) \tag{3.13}$$

where $l_1.E = \sigma^n p + \sigma^n p^* + q$. As l_2 is independent of l_1, $\sigma^n p, q \notin l_2.E$, so we have $\sigma^n p + q \subset \sigma^{-1}(\phi_1.E)$, or $\sigma^{n+1}p + \sigma q \subset \phi_1.E$. Thus $\bar{\phi_1}(\sigma^{n+1}p) = 0$. Now if $\bar{\phi_2}(\sigma^{n+1}p) = 0$, then the left hand side of (3.13) would contain $2\sigma^n p$. But as $\sigma^n p \notin l_2.E$, we would get $2\sigma^n p + \sigma q \subset \phi_1.E$. But as $2\sigma^n p + \sigma q = w_1.E$, this means that ϕ_1 factorizes (I,4.16) as $\phi_1 = w_1 l$ for some linear form l, implying that g itself factorizes: $g = l_1\phi_2 + l_2\phi_1 = l_1\phi_2 + l_2 w_1 l = l_1(\phi_2 - w_2 l)$. This is a contradiction because we assumed that the cubic divisor E was reduced, i.e., B was a domain. □

4 Matlis Duality for some Non-commutative Rings

In this section, we prove a Matlis-type duality for the ring $\hat{A}_{(g)}$, the g-adic completion of the local ring $A_{(g)}$. It turns out that the duality holds under more general condition: we do not require A to be as special as a regular algebra. The proof closely follows that in the commutative case [Ma], with some non-trivial modifications. The key ingredient is Proposition 4.11, which also seems to follow from some more general results of [LM]. Here we use a direct method based on [Ma].

Hypothesis and Notation (4.1) We assume that A is a non-commutative N-graded connected Noetherin domain over a field k. Let $J = (g)$ be a completely prime ideal of A generated by a homogeneous normalizing element g. Thus, $B = A/(g)$ is a graded Noetherian domain. We denote by Q (resp.

K) the $\mathbf{Z}$-graded quotient field of A (resp. B). Regard K as an (A, A)-bimodule by restricting its (B, B)-bimodule structure. Assume that there is an (A, A)-bimodule E, containing K as a sub-(A, A)-bimodule, such that E is an injective hull of B in Gr-A and in A-Gr.

(4.2) By Proposition (2.3), J has left and right Artin-Rees property. Denote by S the multiplicative subset consisting of the homogeneous elements of $A - (g)$. It is easily seen that (2.4) - (2.6) hold in this case (we did not use any hypothesis on A, other than that mentioned in (2.1) above, while proving (2.4) - (2.6)). Thus, A is localizable at the prime ideal J. We denote by $A_{(g)}$ the $\mathbf{Z}$-graded quotient ring $AS^{-1} = S^{-1}A$. Then $A_{(g)}$ is a local ring with the maximal ideal $(g)A_{(g)}$. In fact, the discussion in (2.7) shows that $A_{(g)}$ is a discrete valuation ring of Q with residue field K and $(g)A_{(g)}$ is a principal prime ideal which we will simply denote by $J = (g)$, if no confusion occurs. This defines an exhaustive filtration (simply called g-adic filtration) on $A_{(g)}$. We denote by $\hat{A}_{(g)}$ the g-adic completion of $A_{(g)}$: $\hat{A}_{(g)} = \lim_{\leftarrow} \frac{A_{(g)}}{(g)^n A_{(g)}}$.

The g-adic filtration is separated:

$$\cap_{n=1}^{\infty} (g)^n A_{(g)} = 0.$$

Therefore [B1, II,§3.9], the canonical map $A_{(g)} \to \hat{A}_{(g)}$ is injective (with a dense image). We denote by $\hat{J}$ the g-adic completion of J. Then $\hat{J} = (g)\hat{A}_{(g)}$; and it is a prime principal ideal in $\hat{A}_{(g)}$. We denote the image of g in $\hat{A}_{(g)}$ by $\hat{g}$. There is a canonical isomorphism of rings [B2, III,§2.12, Proposition 15]

$$A_{(g)}/J \simeq \hat{A}_{(g)}/\hat{J} \tag{4.3}$$

Both these rings are isomorphic to K, the quotient field of B.

(4.4) It can be proved as in (2.4) that

$$E = \cup_{n=1}^{\infty} E_n \qquad = \cup_{n=1}^{\infty} E'_n \tag{4.5}$$

where

$$E_n = \{x \in E \mid xJ^n = 0\}, \qquad E'_n = \{x \in E \mid J^n x = 0\}. \tag{4.6}$$

As g is a normalizing element, E_n and E'_n are sub-bimodules of E, and $E_n = E_{\mathrm{Gr}-(A/J^n)}(B)$; $E'_n = E_{(A/J^n)-\mathrm{Gr}}(B)$. It is clear that $K \subset E_1$, $K \subset$

E'_1. Since $K = E_B(B)$ in Gr$-B$ and $B-$Gr, it follows that $E_1 = K = E'_1$. We now show, by induction over n, that $E_1 = E'_1$ implies that $E_n = E'_n$ for all n. Indeed, suppose $x \in E_n$ is a homogeneous element. Then xJ^{n-1} is a right submodule of E such that $xJ^{n-1} \subset E_1 = E'_1$. Thus $JxJ^{n-1} = 0$, showing that $Jx \subset E_{n-1}$. By induction hypothesis, $E_{n-1} = E'_{n-1}$, hence $0 = J^{n-1}Jx = J^n x$, thus completing the induction. Also note that E being a uniform module, E_n are essential submodules in Gr-A and in A-Gr.

Lemma 4.7. *E is S-torsion-free and S-divisible as left and right A-module. In other words, the maps $x \mapsto xs$ and $x \mapsto sx$, where $x \in E$ and $s \in S$, are bijective.*

Proof. E being an injective module in Gr-A and in A-Gr is certainly A-divisible, hence S-divisible. Now we use induction over n to show that E_n is S-torsion-free. If $x \in E_1$ such that $xs = 0$, then $x = 0$, because $E_1 = K$ is S-torsion-free. Now suppose $x \in E_n$, $x \notin E_{n-1}$ such that $xs = 0$ for some $s \in S$. Then $0 \neq xg^{n-1} \in E_1$, and, $0 = xsg^{n-1} = xg^{n-1}s'$ for some $s' \in S$, showing (by induction hypothesis) that $xg^{n-1} = 0$, a contradiction. □

This lemma shows that E has a natural structure of $(A_{(g)}, A_{(g)})$-bimodule, and as such it is an injective hull of $A_{(g)}/(g)A_{(g)}$ in Gr$-A_{(g)}$ and in $A_{(g)}-$Gr [GW, 9.16]. Also, E_n are sub-$(A_{(g)}, A_{(g)})$-bimodules of E, and $K \simeq A_{(g)}/(g)A_{(g)}$ as $(A_{(g)}, A_{(g)})$-bimodules.

In the following, we will write R for the ring $\hat{A}_{(g)}$.

Proposition 4.8. *E is, in a natural way, an injective (R, R)-bimodule, and as such it is an injective hull of $R/\hat{J}$ in* Gr$-R$ *and in* $R-$Gr. *Let $\hat{E}_n = \{x \in \hat{E} | x\hat{J}^n = 0\}$ (resp. $\hat{E}'_n = \{x \in \hat{E} | \hat{J}^n x = 0\}$). Then $\hat{E}_n = E_n$ (resp. $\hat{E}'_n = E'_n$).*

Proof. Let $a \in R$, then there exists a Cauchy sequence $\{a_n\}$ in $A_(g)$ such that $a_n \to a$. Let $x \in E$, then $x \in E_k$ for some $k \geq 0$. There exists an integer N such that $a_n - a_m \in J^k$ for some $n > m \geq N$. Now if we define $xa = xa_N$ (here xa_N is defined from the module-structure of E over $A_{(g)}$), then $xa_n = xa_m$ for $n > m \geq N$. It is easily seen that this definition makes E a right R-module. One similarly defines a left R-module structure on E. It is clear that the two structures are compatible. It is also clear that with this definition, E_n are sub-(R, R)-bimodules of E, and $K \simeq R/\hat{J}$ as (R, R)-bimodules.

Let $\hat{E}$ be an injective hull of E in Gr$-R$. As $0 \to E_1 = K \to E$ is an essential monomorphism in Gr$-A$, and hence in Gr$-R$, $\hat{E}$ is also an injective hull of $K = R/\hat{J}$ in Gr$-R$. We show that $\hat{E} = E$. First note that $\hat{E}$

is indecomposable as an R-module. Since E is injective in $\mathrm{Gr}-A_{(g)}$, $\hat{E}$ would split as $\hat{E} = E \oplus C$ in $\mathrm{Gr}-A_{(g)}$. We now show that C is actually an R-module, hence by indecomposability of $\hat{E}$ as an R-module, $C = 0$ and $\hat{E} = E$. Now, one sees as in (2.4) that $\hat{E} = \cup_{n=1}^{\infty} \hat{E}_n$ where $\hat{E}_n = \{x \in \hat{E} | x\hat{J}^n = 0\}$. $\hat{E}_n$ are sub-bimodules of $\hat{E}$. Suppose $x \in C$. Then $x \in \hat{E}_k$ for some k. If $a \in R$, then there exists $a' \in A_{(g)}$ such that $a = a' \pmod{\hat{J}^k}$. Thus $xa = xa' \in C$, showing that C is an R-module, as claimed.

Similarly, one shows that E is an injective hull of $R/\hat{J}$ in $R-\mathrm{Gr}$. Now, it is easy to check that $\hat{E}_n = \{x \in E | x\hat{J}^n = 0\} = \{x \in E | xg^n = 0\} = E_n$.□

Lemma 4.9. *Let $x \in E_n, x \notin E_{n-1}$, then $l.ann(x) = r.ann(x) = \hat{J}^n$. Thus, $l.annE_n = r.annE_n = \hat{J}^n$, and E is a faithful module in* $\mathrm{Gr}-R$ *and in* $R-\mathrm{Gr}$.

Proof. By Theorem (2.8), $\hat{J}^n \subset \mathrm{l.ann}(x)$, $\hat{J}^n \subset \mathrm{r.ann}(x)$. Now suppose $a \in R$ such that $xa = 0$. Let $a = \hat{g}^k.b$ where $\hat{g}$ is the image of g in R, and $b \notin \hat{J}$. Then E being torsion-free for $R - \hat{J}$, it follows that $x\hat{g}^k = 0$. Thus, by hypothesis on x, $k \geq n$, and $a \in \hat{J}^n$. We have shown that $\mathrm{r.ann}(x) = \hat{J}^n$. Similarly one gets the left version. Now, $\mathrm{r.ann}(E_n) = \cap_{k=1}^{n} \hat{J}^k = \hat{J}^n$. Finally, $\mathrm{r.ann}(E) = \cap_{x \in E} \mathrm{r.ann}(x) = \cap_{n=1}^{\infty} \hat{J}^n = 0$. □

Lemma 4.10. *Let R be a graded ring, M a right R-module, C an injective right R-module. Let $y \in C$, $x \in M$ be homogeneous elements. Then $r.ann(x) \subset r.ann(y)$ if and only if there exists an $f \in \mathrm{Hom}_R(M, C)$ such that $y = f(x)$.*

This is [M, Proposition 2.8]. A similar result holds for the left-modules.

In the following, we use the notations:

$$\mathrm{Hom}_R(M, N) = \oplus_n \mathrm{Hom}_{\mathrm{Gr}-R}(M, N(n)),$$

$$\mathrm{Hom}_{R^\circ}(M, N) = \oplus_n \mathrm{Hom}_{R-\mathrm{Gr}}(M, N(n)).$$

We put

$$H = \mathrm{Hom}_R(E, E) \qquad H' = \mathrm{Hom}_{R^\circ}(E, E).$$

H and H' are graded rings. Since E is faithful as a right or left R-module (Lemma 4.9), we can identify R as a subring of H (resp. H') consisting of left (resp. right) multiplications by elements of R. We now show that

Proposition 4.11. *Every right (resp. left) R-homomorphism of E into itself can be realized by left (resp. right) multiplication by exactly one element of R.*

Proof. Define $H_i = \{h \in H | h(E_i) = 0\}$ and $H'_i = \{h \in H' | h(E_i) = 0\}$. Then $\cap H_i = 0 = \cap H'_i$. Since $f(E_i) \subset E_i$ for all f in H or in H', it follows that H_i (resp. H'_i) is a two-sided graded ideal of H (resp. H'). There is a linear topology on H (resp. H') defined by the filtration $\{H_i\}$ (resp. $\{H'_i$). Since $\hat{J}^n \subset H_n$ (resp. $\hat{J}^n \subset H'_n$), the topology induced on R by this topology on H (resp. H') is the same as the g-adic topology on R.

We will show by induction over i that if $f \in H$ (resp. $g \in H'$) then there exist elements $p_i \in R$ (resp. $q_i \in R$) such that $f = p_i \bmod H_i$ (resp. $g = q_i \bmod H'_i$). Here p_i (resp. q_i) means left (resp. right) multiplication. Now, this is evident for $i = 1$, for then $E_1 = E'_1 = R/\hat{J}$. Suppose that we have gotten p_i and q_i with the above property for $i = 1, \ldots, k$. If $(f - p_k)(x) = 0$ for all $x \in E_{k+1}$ then we simply take $p_{k+1} = p_k$. Else, choose $x_0 \in E_{k+1}, x_0 \notin E_k$ so that $(f - p_k)(x_0) \neq 0$. Note that $\text{l.ann}(x_0) = \hat{J}^{k+1}$ (Lemma 4.9). Now $x_0 g \in E_k$, and therefore $0 = (f - p_k)(x_0 g)$, showing that $(f - p_k)x_0 \in E_1$. Put $\psi = f - p_k \in H_k$, thus $\psi(x_0) \in E_1$.

Now E being a uniform (left) R-module, there exist $r_1, r_2 \in R$ such that $r_1 x_0 = r_2(f - p_k)x_0 \neq 0$. As $(f - p_k)x_0 \in E_1$, $r_2 \notin \hat{J}$, so r_2 is invertible, and $r = r_2^{-1} r_1$ satisfies

$$r x_0 = (f - p_k)x_0 = \psi x_0$$

Now let x be an arbitrary element of E_{k+1}. As $\text{l.ann}(x) \supset \hat{J}^{k+1} = \text{l.ann}(x_0)$, therefore, there exists $f_0 \in H'$ such that $f_0(x_0) = x$ (Lemma 4.10). By induction hypothesis, $f_0 = s \bmod H'_k$ for some $s \in R$, say, $f_0 = s + h_0$ for some $h_0 \in H'_k$. Since $g x_0 \in E_k$, $g h_0(x_0) = h_0(g x_0) = 0$. Thus $h_0(x_0) \in E_1$ and $\psi(h_0(x_0)) = 0$. This implies that

$$\psi(x) = \psi(f_0(x_0)) = \psi(x_0 s + h_0(x_0)) = \psi(x_0 s) = \psi(x_0)s = r x_0 s.$$

On the other hand, since $h_0 \in H'_k$ and $\psi(x_0) \in E_1$, therefore, $0 = h_0(\psi(x_0)) = h_0(r x_0) = r h_0(x_0)$. Thus

$$r x = r(f_0(x_0)) = r x_0 s + r h_0(x_0) = r x_0 s.$$

We have shown that $\psi(x) = r x$ for all $x \in E_{k+1}$. Thus, $f = p_k + r \bmod H_{k+1}$, and the induction is complete.

We have associated with $f \in H$ a sequence $\{p_k\}$ of elements of R such that $f = p_k \bmod H_k$. If $n \leq m$ then $p_n - p_m = (p_n - f) + (f - p_m) \in H_n + H_m = H_n$. Hence $(p_n - p_m)E_n = 0$. By Lemma (4.9), $p_n - p_m \in \hat{J}^n$. Thus $\{p_k\}$ is a Cauchy sequence in R, and there exists $a \in R$ such that $p_k \to a$. As $\hat{J}^n \subset H_n$, $p_k \to a$ in H. But $p_k \to f$ in H, therefore $f = a \in R$. One similarly completes the proof for the left-version. $\square$

Let R^n (resp. E^n) denote the direct sum of n copies of the bimodule R (resp. E). Let S be a right submodule of R^n, B a left submodule of E^n. We define

$$S' = \{(x_i) \in E^n | \sum x_i r_i = 0, \forall (r_i) \in S\} \tag{4.12}$$

$$B' = \{(r_i) \in R^n | \sum x_i r_i = 0, \forall (x_i) \in B\}. \tag{4.13}$$

Note that S' is a left submodule of E^n and B' is a right submodule of R^n.

Theorem 4.14. *Let S be a right submodule of R^n, B a left submodule of E^n. Then*

(i) $S' \simeq \mathrm{Hom}_R(R^n/S, E)$ as left R-modules, $E^n/S' \simeq \mathrm{Hom}_R(S, E)$ as left R-modules, and $S'' = S$ as right R-modules.

(ii) $B' \simeq \mathrm{Hom}_{R^\circ}(E^n/B, E)$ as right R-modules, $R^n/B' \simeq \mathrm{Hom}_{R^\circ}(B, E)$ as right R-modules, and $B'' = B$ as left R-modules.

Corollary 4.15 [Matlis Duality]. *Let M be a finite right R-module, N a left submodule or factor module of E^n, then*

$$\mathrm{Hom}_{R^\circ}(\mathrm{Hom}_R(M, E), E) \simeq M \tag{4.16}$$

$$\mathrm{Hom}_R(\mathrm{Hom}_{R^\circ}(N, E), E) \simeq N. \tag{4.17}$$

Proof. We prove the theorem, the corollary being immediate from the theorem.

(i): It is clear that $S' \simeq \mathrm{Hom}_R(R^n/S, E)$ as left R-modules. Now from the exact sequence

$$0 \to S \to R^n \to R^n/S \to 0$$

we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}_R(R^n/S, E) & \to & \mathrm{Hom}_R(R^n, E) & \to & \mathrm{Hom}_R(S, E) & \to & 0 \\ & & \downarrow \wr & & \downarrow \wr & & & & \\ 0 & \to & S' & \to & E^n & \to & E^n/S' & \to & 0 \end{array}$$

The top row is exact because E is injective. The vertical maps are isomorphisms. Thus, $\mathrm{Hom}_R(S, E) \simeq E^n/S'$.

Now it is clear that $S \subset S'' \subset R^n$. Suppose that $(s_i) \in S'', (s_i) \notin S$. Let (u_i) be the generators for R^n/S as right R-module. Then $0 \neq \sum u_i s_i = y \in$

R^n/S. As $\hat{(J)}$ is the maximal ideal of R, we have r.ann$(y) \subset \hat{J}$. There exists, by Lemma (4.10), a map $h \in \mathrm{Hom}_R(R^n/S, E)$ such that $h(y) \neq 0$. But as $(h(u_i)) \in S'$, and $(s_i) \in S''$, we get $h(y) = h(\sum u_i s_i) = \sum h(u_i)s_i = 0$, which is a contradiction. Hence, $S = S''$.

(ii) The map $\Phi : B' \to \mathrm{Hom}_{R^o}(E^n/B, E)$ defined as $\Phi(r_i)[(\bar{y}_i)] = \sum y_i r_i$, $(r_i) \in B'$,$(y_i) \in E^n$ is a well-defined right R-linear map. This is injective; for, if $\sum y_i r_i = 0$ for all $(y_i) \in E^n$ then $Er_i = 0$ for all i, and therefore by Lemma (4.9) $r_i = 0$. Now, let $f \in \mathrm{Hom}_{R^o}(E^n/B, E)$ be given. Denote by E_i the ith component of E^n. The composite left R-linear map $E \xrightarrow{\sim} E_i \hookrightarrow E^n \to E^n/B \xrightarrow{f} E$ must be a right multiplication by some element $r_i \in R$ (Proposition 4.11). Thus, if $(y_i) \in E^n$, then $f[(\bar{y}_i)] = \sum y_i r_i$. Now for $(x_i) \in B, 0 = \sum x_i r_i = f[(\bar{x}_i)] = 0$, thus $(r_i) \in B'$. As $\Phi(r_i) = f$, this shows that Φ is surjective, and $\mathrm{Hom}_{R^o}(E^n/b, E) \simeq B'$.

From the exact sequence

$$0 \to B \to E^n \to E^n/B \to 0$$

one derives the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & B' & \to & R^n & \to & R^n/B' & \to & 0 \\
 & & \downarrow\wr & & \downarrow\wr & & & & \\
0 & \to & \mathrm{Hom}_{R^o}(E^n/B, E) & \to & \mathrm{Hom}_{R^o}(E^n, E) & \to & \mathrm{Hom}_{R^o}(B, E) & \to & 0
\end{array}$$

The bottom row is exact because E is injective. The vertical maps are isomorphisms. It follows that $R^n/B' \simeq \mathrm{Hom}_{R^o}(B, E)$.

Again it is clear that $B \subset B'' \subset E^n$. Suppose that $(x_i) \in B''$ but $(x_i) \notin B$. Since l.ann$((\bar{x}_i)) \subset \hat{J}$, there exists a map $f \in \mathrm{Hom}_{R^o}(E^n/B, E)$ such that $f((\bar{x}_i)) \neq 0$ (Lemma 4.10). But $f = \Phi(r_i)$ for some $(r_i) \in B'$, thus, $f((\bar{x}_i)) = \Phi(r_i)(\bar{x}_i) = \sum x_i r_i = 0$, since $(x_i) \in B''$. This contradiction completes the proof. □

Since σ fixes the class $[\mathcal{L}]$ of $\mathcal{L} = \mathcal{O}_X(1)$ in $\mathrm{Pic}(X)$, there is an equivariance $e : \sigma^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$, which gives a k-linear isomorphism

$$(1.3) \qquad e\sigma^* : H^0(X,\mathcal{L}) \xrightarrow{\sigma^*} H^0(X,\mathcal{L}^\sigma) \xrightarrow{e} H^0(X,\mathcal{L}).$$

Note that, by (1.2) we have

$$(1.4) \qquad \mathrm{div}(e\sigma^*(\phi)) = \sigma^{-1}\mathrm{div}(\phi),\ \phi \in H^0(X,\mathcal{L}).$$

More generally, for every $m \in \mathbf{Z}$, we have an equivariance $e^{\sigma^m} = (\sigma^*)^m(e) : \mathcal{L}^{\sigma^{m+1}} \xrightarrow{\sim} \mathcal{L}^{\sigma^m}$, induced by the pull-back of the morphism e by the functor $(\sigma^*)^m$. Following the notation of [Ye1], we put $e^m_{m+1} := e^{\sigma^m}$. For arbitrary $n, m \in \mathbf{Z}$ let $e^n_{n+m} : \mathcal{L}^{\sigma^{n+m}} \xrightarrow{\sim} \mathcal{L}^{\sigma^n}$ be the equivariance such that the recursive relation $e^n_{n+m+1} = e^n_{n+m} \circ e^{n+m}_{n+m+1}$ is satisfied. In this notation, $e = e^0_1$.

(1.5) In this section only, we denote by A the *commutative* coordinate ring of the projective space, $A = \oplus_{n\geq 0} H^0(X,\mathcal{L}^{\otimes n})$ ($X = \mathbf{P}^n_k$ and $\mathcal{L} = \mathcal{O}_X(1)$). Thus A is a polynomial ring in $n+1$ variables with the usual grading. For a (graded) automorphism τ of A, we define a twisted graded k-algebra A^τ as follows. We put $(A^\tau)_n = A_n$ as underlying k-vector spaces, and define the multiplication in A^τ by

$$(1.6) \qquad \phi.\psi = \phi \otimes \tau^n\psi \qquad \phi \in A^\tau_n, \psi \in A^\tau_m$$

where $\otimes$ will denote the multiplication in A. In particular, if σ is an automorphism of X then the isomorphism $e\sigma^*$ (1.3) on $A_1 = H^0(X,\mathcal{L})$ induces a graded automorphism of the algebra A which we denote by τ_σ. This defines the twisted ring A^{τ_σ}. Note that in this notation

$$(1.7) \qquad \mathrm{div}((\tau_\sigma)^n\phi) = \sigma^{-n}\mathrm{div}(\phi), \qquad \phi \in A^{\tau_\sigma}.$$

Of course, A^{τ_σ} depends on the choice of equivariance.

(1.8) Let $B = B(X,\sigma,\mathcal{L})$ be the *twisted homogeneous coordinate ring* of $X = \mathbf{P}^n_k$, obtained by twisting the invertible sheaf $\mathcal{L}$ by the automorphism σ [AV]. Thus, by definition, $B = \oplus_{n\geq 0} B_n$, where $B_n = H^0(X,\mathcal{L}_n)$; $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \ldots \otimes \mathcal{L}^{\sigma^{n-1}}$. For $\phi \in B_n, \psi \in B_m$ the multiplication is given by $\phi\psi = \phi \otimes \psi^{\sigma^n}$. We show now that, using equivariance, we can identify the twisted coordinate ring $B(X,\sigma,\mathcal{L})$ with the twist A^{τ_σ}.

(1.9) Choose an equivariance $e : \mathcal{L}^\sigma \xrightarrow{\sim} \mathcal{L}$, which gives isomorphisms $e^0_n : \mathcal{L}^{\sigma^n} \xrightarrow{\sim} \mathcal{L}$. There is a k-linear isomorphism, sometimes called an "untwisting"

III

RESIDUE COMPLEX FOR REGULAR ALGEBRAS OF DIMENSION 2

Our final objective in this chapter is to construct a "residue-complex" (a minimal injective resolution) for a regular algebra B of dimension 2. It is well-known that regular algebras of dimension 2 are twisted homogeneous coordinate rings of the projective line $\mathbf{P}^1_k$ (3.1). Residue-complexes for general twisted coordinate rings have previously been constructed in [Ye1], by geometric methods. Here the method is rather algebraic, using a unique factorization result in twisted coordinate rings of the projective spaces, and (non-commutative) localizations of the algebra at *orbits* of points. We also look at the complex in the category Proj$-B$, where we use a Grothendieck-Serre-type duality for regular algebras.

1 A Unique Factorization in Twisted Coordinate Rings

In this section, we prove a unique factorization theorem for twisted homogeneous coordinate rings of the projective spaces $\mathbf{P}^n_k$.

(1.1) *Througout this section, k will be an algebraically closed field, X the projective scheme $\mathbf{P}^n_k$, and $\mathcal{L}$ the invertible sheaf $\mathcal{O}_X(1)$.* For an invertible sheaf $\mathcal{M}$ and an automorphism σ of X, we denote by $\mathcal{M}^\sigma$ the pull-back invertible sheaf $\sigma^*\mathcal{M}$. There is a natural k-linear isomorphism

$$H^0(X,\mathcal{M}) \xrightarrow{\sigma^*} H^0(X,\mathcal{M}^\sigma). \tag{1.1}$$

For a global section $\phi \in H^0(X,\mathcal{M})$ we denote by ϕ^σ its canonical image under the above isomorphism. We denote by $\mathrm{div}(\phi)$ the divisor of zeros of ϕ. We have the relation

$$\mathrm{div}(\phi^\sigma) = \sigma^{-1}\mathrm{div}(\phi). \tag{1.2}$$

map [Ye2, §3]:
(1.10)

$$u_n : B_n = H^0(X, \mathcal{L} \ldots \otimes \mathcal{L}^{\sigma^{n-1}}) \xrightarrow{\mathrm{id}\otimes e_1^0 \ldots \otimes e_{n-1}^0} H^0(X, \mathcal{L}^{\otimes n}) = A_n = (A^{\tau_\sigma})_n.$$

Summing over all n , we get a k-linear map $u : B \to A^{\tau_\sigma}$. Note that u_1 is just identity. One can verify [Ye2, 3.3.2] that, for $\phi \in B_n, \psi \in B_m$

$$\begin{aligned} u(\phi\psi) &= u(\phi) \otimes (\tau_\sigma)^n u(\psi), \text{ in } A \\ &= u(\phi).u(\psi), \text{ in } A^{\tau_\sigma}. \end{aligned} \tag{1.11}$$

Thus $u : B \xrightarrow{\sim} A^{\tau_\sigma}$ is an isomorphism of graded k-algebras.

Hereon, we will write $B = A^{\tau_\sigma}$. Since $B_n = A_n^{\tau_\sigma} = A_n$, a " homogeneous element" will mean a homogeneous element of A or B. We will denote the multiplication in $B = A^{\tau_\sigma}$ simply by $\phi\psi$ and that in A by $\otimes$. Further, we introduce the notation

$$\sigma(\phi) := (\tau_\sigma)^{-1}(\phi) \qquad \phi \text{ homogeneous.} \tag{1.12}$$

This simplifies the formulas (1.6, 1.7) as

$$\phi\psi = \phi \otimes \sigma^{-n}\psi, \quad \operatorname{div}(\sigma^i\phi) = \sigma^i \operatorname{div}(\phi), \tag{1.13}$$

where ϕ, ψ are homogeneous elements, with $\deg(\phi) = n$. Thus

$$\operatorname{div}(\phi\psi) = \operatorname{div}(\phi) + \sigma^{-n}\operatorname{div}(\psi). \tag{1.14}$$

Lemma 1.15. *Let $\phi \in B_n$ and $\psi \in B_m$, then*
(i) $\phi\psi = (\sigma^{-n}\psi)(\sigma^m\phi)$,
(ii) $\sigma^i(\phi\psi) = (\sigma^i\phi)(\sigma^i\psi)$ for all $i \in \mathbf{Z}$.

Proof. Indeed, $\phi\psi = \phi \otimes \sigma^{-n}\psi = \sigma^{-n}\psi \otimes \sigma^{-m}\sigma^m\phi = (\sigma^{-n}\psi)(\sigma^m\phi)$. And, $\sigma^i(\phi\psi) = \sigma^i(\phi \otimes \sigma^{-n}\psi) = \sigma^i\phi \otimes \sigma^{-n}\sigma^i\psi = (\sigma^i\phi)(\sigma^i\psi)$. □

Definition 1.16. *A homogeneous element π of positive degree is said to be irreducible in A (resp. in B) if it can not be written as a product of two (homogeneous) elements of (strictly lower) positive degrees; i.e., $\pi = \phi_1 \otimes \phi_2$ (resp. $\pi = \phi_1\phi_2$) implies* $\deg\phi_1 = 0$ *or* $\deg\phi_2 = 0$. *A homogeneous element of positive degree is said to be reducible if it is not irreducible. Two homogeneous elements ϕ_1 and ϕ_2 are said to be associates if $\phi_1 = u\phi_2$ where u is a unit in k.*

We write $\phi_1 \sim \phi_2$ if ϕ_1 and ϕ_2 are associates.

Proposition 1.17. *Let ϕ be a homogeneous element. Then the following are equivalent: (i) ϕ is irreducible in B, (ii) ϕ is irreducible in A, (iii)* $\operatorname{div}(\phi)$ *is a prime divisor in X.*

Proof. Equivalence of (ii) and (iii) is well-known. We prove $(i) \Leftrightarrow (ii)$. Clearly, if $\phi = \phi_1\phi_2$ is reducible in B, then $\phi = \phi_1 \otimes \sigma^{-n_1}\phi_2$ is reducible in A, where $n_1 = \deg\phi_1$. Conversely, if $\phi = \phi_1 \otimes \phi_2$ is reducible in A, then $\phi = \phi_1 \otimes \sigma^{-n_1}\sigma^{n_1}(\phi_2) = \phi_1(\sigma^{n_1}\phi_2)$ is reducible in B, where $n_1 = \deg\phi_1$. □

Corollary 1.18. *If π is an irreducible element of B then $\sigma^m\pi$ is irreducible for all $m \in \mathbf{Z}$.*

In view of Proposition 1.17, irreducible elements in B can be identified with irreducible elements in A. Thus we can call a homogeneous element irreducible without referring whether in A or in B. However, Lemma 1.15 shows that unique factorization theorem for A does not carry over to B; for, letting π be an irreducible element of degree r, we have $\pi\pi = (\sigma^{-r}\pi)(\sigma^r\pi)$ in B, and $\sigma^{-r}\pi, \sigma^r\pi$ are in general not equal to π. Our main purpose now is to show that we still have a "uniqueness" statement about factorization in B, in terms of the orbits.

(1.19) We generally use π to denote an irreducible element. Let π be an irreducible element of degree r. We denote the product $\pi(\sigma^r\pi)\ldots(\sigma^{(n-1)r}\pi)$ in B by $[\pi]^n$, which has degree nr. Note that $[\pi]^n = \pi^{\otimes n}$ where $\pi^{\otimes n}$ denotes the n times product of π in A. We have the relations $\operatorname{div}([\pi]^n) = n\operatorname{div}(\pi)$, and $\sigma^i([\pi]^n) = [\sigma^i\pi]^n$ (1.15*ii*).

(1.20) The cyclic group $<\sigma>$ acts on the set of prime divisors of X. By an *orbit* ω we mean an orbit of this group action. Sometimes we will use the (informal) notation $\pi \in \omega$ (π an irreducible element, ω an orbit) to mean $\operatorname{div}(\pi) \in \omega$, and the notation ω_π to mean the orbit ω such that $\pi \in \omega$. (The map $\phi \mapsto \operatorname{div}(\phi)$ is a bijection between the set of irreducible elements, up to associates, and the set of prime divisors of X.) Thus we say that two irreducible elements are in the same orbit if their divisors are in the same orbit. For an orbit ω and $n \in \mathbf{N}$ we define a "valuation" map

$$v_\omega^n : B_n \to \mathbf{Z}, \quad v_\omega^n(\phi) = \sum_{Y \in \omega} \operatorname{mult}_Y(\operatorname{div}(\phi)). \tag{1.21}$$

It is clear that for $\phi \in B_n, \psi \in B_m$ one has

$$v_\omega^{m+n} = v_\omega^n(\phi) + v_\omega^m(\psi).$$

We will drop the superscript on v if there is no confusion. For a homogeneous element ϕ we define the *orbit-support* of its divisor $\mathrm{div}(\phi)$ as the set $\{\omega | v_\omega(\phi) \neq 0\}$. This will be denoted by $\mathrm{supp}_{\mathrm{orb}}(\phi)$. A homogeneous element whose orbit-support consists of a single orbit will be called *uni-orbital.* A uni-orbital element whose orbit-support is $\{\omega\}$ will generally be denoted by a symbol of the form Π_ω.

(1.22) As B is noetherian, any homogeneous element of B can be written as a product of irreducible elements. A uni-orbital element Π_ω will be a product of irreducible elements of the same orbit so it can be written in the form

$$\begin{aligned} \Pi_\omega =& [\sigma^{i_1}\pi]^{n_1}[\sigma^{i_2}\pi]^{n_2}\dots[\sigma^{i_k}\pi]^{n_k} \\ =& (\sigma^{m_1}\pi)^{\otimes n_1} \otimes (\sigma^{m_2}\pi)^{\otimes n_2} \dots \otimes (\sigma^{m_k}\pi)^{\otimes n_k} \end{aligned} \tag{1.23}$$

for some irreducible element π, and $m_\nu = i_\nu - (\sum_{j=1}^{\nu-1} n_j)r$, r being the degree of π. Further, $\sigma^{m_\nu}\pi$ in (1.23) can be chosen to be *distinct* irreducible elements in the orbit ω_π. In case σ is of infinite order, Π_ω can be written uniquely as in (1.23) with i_ν in strictly increasing order.

Theorem 1.24[Unique Factorization for Twisted Rings]. *Let ϕ be a homogeneous element of B. Then ϕ can be written in the form*

$$\phi = \Pi_{\omega_1}\Pi_{\omega_2}\dots\Pi_{\omega_n}. \tag{1.25}$$

where Π_{ω_i} are uni-orbital elements, and $\omega_i \neq \omega_j$ for $i \neq j$. Further, if

$$\phi = \Pi_{\omega'_1}\Pi_{\omega'_2}\dots\Pi_{\omega'_m}$$

is another factorization of this form then $m = n$, there is a permutation $\tau \in S_n$ such that $\omega'_i = \omega_{\tau(i)}$ $(1 \leq i \leq n)$, and

$$\Pi_{\omega'_i} \sim \sigma^{s_i}(\Pi_{\omega_{\tau(i)}})$$

for some $s_i \in \mathbf{Z}$.

Proof. Choose a factorization of ϕ into irreducible elements. Using Lemma (1.15 i) repeatedly, one can collect all the irreducible factors of the same orbit and thus write ϕ as a product of uni-orbital elements as in (1.25).

We now show the uniqueness part. Since $\mathrm{supp}_{\mathrm{orb}}(\phi) = \{\omega_1, \omega_2, \dots, \omega_n\} = \{\omega'_1, \omega'_2, \dots, \omega'_m\}$ and $\omega_i \neq \omega_j, \omega'_i \neq \omega'_j$ for $i \neq j$, we have $n = m$ and a permutation $\tau \in S_n$ such that $\omega'_i = \omega_{\tau(i)}$. Now let us just write Π_j for Π_{ω_j} and

Π'_j for $\Pi_{\omega'_j}$. Let r_i (resp. r'_i) be the degree of an irreducible element in the orbit ω_i (resp. ω'_i), and let $\deg(\Pi_i) = k_i r_i$, $\deg(\Pi'_i) = k'_i r'_i$. We have

$$\begin{aligned}\operatorname{div}(\phi) &= \operatorname{div}(\Pi_1) + \sigma^{-k_1 r_1}\operatorname{div}(\Pi_2) + \ldots + \sigma^{-(k_1 r_1 + k_2 r_2 + \ldots k_{n-1} r_{n-1})}\operatorname{div}(\Pi_n) \\ &= \operatorname{div}(\Pi'_1) + \sigma^{-k'_1 r_1'}\operatorname{div}(\Pi'_2) + \ldots + \sigma^{-(k'_1 r_1' + \ldots k'_{n-1} r'_{n-1})}\operatorname{div}(\Pi'_n)\end{aligned}$$

Since the divisors of Π_i (resp. Π'_i) are supported on a single orbit ω_i (resp. ω'_i), it follows that

$$\sigma^{-\sum_{j=1}^{i-1} k'_j r'_j}\operatorname{div}(\Pi'_i) = \sigma^{-\sum_{j=1}^{\tau(i)-1} k_j r_j}\operatorname{div}(\Pi_{\tau(i)})$$

thus proving that $\Pi'_i \sim \sigma^{s_i}\Pi_{\tau(i)}$, where $s_i = -\sum_{j=1}^{\tau(i)-1} k_j r_j + \sum_{j=1}^{i-1} k'_j r'_j$. $\square$

2 Partial Fraction Decomposition

In this section we obtain a partial fraction decomposition for non-negative degree elements of the graded quotient field of the twisted coordinate ring $B = B(X, \sigma, \mathcal{L})$ where $X = \mathbf{P}^1_k$. This is done by using an isomorphism between the degree zero components of the graded quotient fields of the twisted and commutative coordinate rings (Proposition 2.1). This isomorphism itself holds for the general case $B = B(\mathbf{P}^n_k, \sigma, \mathcal{L})$, so, we keep the discussion general until we get to the proof of the decomposition theorem. As in §1, we denote by A the commutative coordinate ring, and identify B with the algebra A^{τ_σ} (1.9). The homogeneous elements of A and B will be identified, as before. Let K_A and K_B denote the $\mathbf{Z}$-graded quotient fields of A and B respectively.

Proposition 2.1. *The map*

$$(2.2) \qquad v : (K_B)_0 \to (K_A)_0 \qquad fg^{-1} \mapsto f \otimes g^{-1}$$

is an isomorphism of fields.

Proof. First we check that this is well-defined. Suppose $f_1 g_1{}^{-1} = f_2 g_2{}^{-1}$ in $(K_B)_0$ and let n be the degree of f_1, m that of f_2. Then by definition of equality in the Ore quotient ring, there exist homogeneous elements h_1 and h_2 such that $f_1 h_1 = f_2 h_2$ and $g_1 h_1 = g_2 h_2 \neq 0$. Then, by (1.13),

$f_1 \otimes \sigma^{-n}h_1 = f_2 \otimes \sigma^{-m}h_2$ and $g_1 \otimes \sigma^{-n}h_1 = g_2 \otimes \sigma^{-m}h_2$. But this then shows that $f_1 \otimes g_1^{-1} = f_2 \otimes g_2^{-1}$ in $(K_A)_0$.

Using the algebra of fractions in an Ore quotient ring, one can verify that the map defined above is a homomorphism of fields. It is clearly bijective. □

Finally, we define the divisor of a homogeneous element of K_B. This will be used in the next section.

Definition 2.3. *Let fg^{-1} be a homogeneous element of K_B of degree n. The divisor of fg^{-1}, denoted by* $\mathrm{div}(fg^{-1})$ *is* $\mathrm{div}(f) - \sigma^{-n}\mathrm{div}(g)$.

This is well-defined. For, let $f_1g_1^{-1} = f_2g_2^{-1}$ in K_B be homogeneous of degree n. Then there exist homogeneous elements h_1, h_2 in B such that $f_1h_1 = f_2h_2$ and $g_1h_1 = g_2h_2 \neq 0$. Let m_1 be the degree of g_1 and m_2 that of g_2. Then we have

$$\begin{aligned}\mathrm{div}(f_1) + \sigma^{-(m_1+n)}\mathrm{div}(h_1) &= \mathrm{div}(f_2) + \sigma^{-(m_2+n)}\mathrm{div}(h_2) \\ \mathrm{div}(g_1) + \sigma^{-m_1}\mathrm{div}(h_1) &= \mathrm{div}(g_2) + \sigma^{-m_2}\mathrm{div}(h_2)\end{aligned}.$$

It follows that $\mathrm{div}(f_1) - \sigma^{-n}\mathrm{div}(g_1) = \mathrm{div}(f_2) - \sigma^{-n}\mathrm{div}(g_2)$, thus verifying the claim.

It is straight-forward to verify the following Lemma.

Lemma 2.4. *(i) Let $f_1g_1^{-1}$ and $f_2g_2^{-1}$ be homogeneous elements of K_B of degrees n and m respectively. Then*

$$\mathrm{div}((f_1g_1^{-1})(f_2g_2^{-1})) = \mathrm{div}(f_1g_1^{-1}) + \sigma^{-n}\mathrm{div}(f_2g_2^{-1}).$$

(ii) Let fg^{-1} be a homogeneous element of degree n then $\mathrm{div}((fg^{-1})^{-1}) = -\sigma^n\mathrm{div}(fg^{-1})$.

(iii) Let $f_1g_1^{-1}$ and $f_2g_2^{-1}$ be homogeneous elements of the same degree n. If $\mathrm{div}(f_1g_1^{-1}) = \mathrm{div}(f_2g_2^{-1})$ *then $f_1g_1^{-1} = f_2g_2^{-1}$ (up to a unit in k.)*

Now we specialize to $B = B(\mathbf{P}^1_k, \sigma, \mathcal{L})$, and, of course, identify it with A^{τ_σ} where A is the commutative polynomial ring in two variables. Since k is algebraically closed, the only irreducible elements are the linear forms $0 \neq l \in B_1 = A_1$ and the only prime divisors are the points $p \in X = \mathbf{P}^1_k$. Thus, in view of (1.22), a uni-orbital element Π_ω can be written as

$$\begin{aligned}\Pi_\omega &= [\sigma^{i_1}l]^{n_1}[\sigma^{i_2}l]^{n_2}\dots[\sigma^{i_k}l]^{n_k} \\ &= (\sigma^{m_1}l)^{\otimes n_1} \otimes (\sigma^{m_2}l)^{\otimes n_2}\dots\otimes(\sigma^{m_k}l)^{\otimes n_k}\end{aligned} \tag{2.5}$$

for a linear form in the orbit ω, such that $m_\nu = i_\nu - \sum_{j=1}^{\nu-1} n_j$, and $\sigma^{m_\nu} l$ are *distinct* linear forms. We will call (2.5) a canonical form of Π_ω if $\sigma^{m_\nu} l$ are distinct. Recall (1.19) that

$$[l]^n = l(\sigma l)(\sigma^2 l)\ldots(\sigma^{n-1} l) = \underbrace{l \otimes l \otimes \ldots \otimes l}_{n \text{ times}} = l^n, \tag{2.6}$$

where *we will simply write l^n for the n times product $l \otimes l \ldots \otimes l$ in A.* In the following, we will sometimes put $\frac{f}{g}$ for fg^{-1} (in K_B), just for convenience in writing.

Theorem 2.7 [Partial Fraction Decomposition]. *(i) Let fg^{-1} be a homogeneous element of K_B of degree $r \geq 0$. Then fg^{-1} can be decomposed in K_B as*

$$fg^{-1} = h + \sum_{\text{orbits}\,\omega} f_\omega {\Pi_\omega}^{-1} \tag{2.8}$$

where h and f_ω are homogeneous elements (of B), Π_ω are uni-orbital elements (1.20) and $f_\omega = 0$ for all except finitely many ω.

(ii) Further, if $\Pi_\omega = [\sigma^{i_1} l]^{n_1} \ldots . [\sigma^{i_k} l]^{n_k}$ is in a canonical form (2.5) then, in K_B,

$$\frac{f}{\Pi_\omega} = b_o + \frac{b_1}{[\sigma^{m_1} l]^{n_1}} + \frac{b_2}{[\sigma^{m_2} l]^{n_2}} + \ldots + \frac{b_k}{[\sigma^{m_k} l]^{n_k}} \tag{2.9}$$

where $m_\nu = i_\nu - \sum_{j=1}^{\nu-1} n_j$, and b_i are some homogeneous elements (of B).

(iii) Finally, a fraction of degree zero of the form $\frac{h}{[l]^n}$ can be decomposed in K_B as

$$\frac{h}{[l]^n} = c_0 + c_1 \frac{w}{l} + c_2 \frac{[w]^2}{[l]^2} + \ldots + c_n \frac{[w]^n}{[l]^n} \tag{2.10}$$

for some $c_i \in k$ and a linear form w not in the orbit of l.

Proof. We prove (i). Let w be a linear form corresponding to a point which is not in the orbits of the zeroes of f and g. Consider the degree zero element $fg^{-1}([w]^r)^{-1}$ in K_B. Its image in $(K_A)_0$ via the isomorphism $v(2.1)$ has a decomposition, in K_A as

$$v(fg^{-1}([w]^r)^{-1}) = h \otimes (w^r)^{-1} + \sum_{l \in A_1, l \neq w} a_l \otimes (l^{\nu(l)})^{-1}$$

where $\nu(l)$ is a positive integer, h and a_l are homogeneous elements and $a_l = 0$ for all but finitely many l. Apply v^{-1} to get

$$fg^{-1}([w]^r)^{-1} = (h)([w]^r)^{-1} + \sum_{l \in B_1, l \neq w} (a_l)([l]^{\nu(l)})^{-1}.$$

Now multiply on the right by $[w]^r$ to get

$$fg^{-1} = h + \sum_{l \in B_1, l \neq w} (a_l)([l]^{\nu(l)})^{-1}[w]^r = h + \sum_{l \in B_1, l \neq w} a_l(\sigma^{\nu(l)}[w]^r)(\sigma^r[l]^{\nu(l)})^{-1}.$$

Now the sum on the right hand side can be written as $\sum_\omega \left(\sum_{l \in \omega} b_l(\sigma^r[l]^{\nu(l)})^{-1} \right)$, (where $b_l = a_l(\sigma^{\nu(l)}[w]^r)$) and the inner summmation could be performed in K_B to get $f_\omega {\Pi_\omega}^{-1}$ as required.

(ii): Let the degree of $f{\Pi_\omega}^{-1}$ be r. Again let w be a linear form corresponding to a point which is not in the orbits of the zeroes of f and Π_ω. Consider the degree zero element $f([w]^r\Pi_\omega)^{-1} = f{\Pi_\omega}^{-1}([w]^r)^{-1}$ in K_B. Its image in $(K_A)_0$ is

$$\begin{aligned} v(f([w]^r\Pi_\omega)^{-1}) &= f \otimes ([w]^r\Pi_\omega)^{-1} = f \otimes (w^r \otimes \sigma^{-r}\Pi_\omega)^{-1} \\ &= f \otimes (w^r)^{-1} \otimes \{(\sigma^{-r+m_1}l)^{n_1} \otimes \ldots \otimes (\sigma^{-r+m_k}l)^{n_k}\}^{-1} \end{aligned}$$

where $m_\nu = i_\nu - \sum_{j=1}^{\nu-1} n_j$, and $\sigma^{m_\nu}l$ are distinct linear forms. This has a decomposition, in $(K_A)_0$, as

$$a_0 \otimes (w^r)^{-1} + a_1 \otimes ((\sigma^{-r+m_1}l)^{n_1})^{-1} + \ldots a_k \otimes ((\sigma^{-r+m_k}l)^{n_k})^{-1}.$$

Now applying the inverse v^{-1} and multiplying on the right by $[w]^r$ we get (2.9) with $b_0 = a_0, b_j = a_j(\sigma^{n_j}[w]^r), j = 1, \ldots k$.

(iii): Choose w not in the orbit of l. Now, $v(h([l]^n)^{-1}) = h \otimes (l^n)^{-1}$, and, by classical result, the latter decomposes in $(K_A)_0$, as

$$c_0 + c_1(w \otimes l^{-1}) + c_2(w^2 \otimes (l^2)^{-1} \ldots + c_n(w^n \otimes (l^n)^{-1}.$$

Applying the inverse v^{-1} we get (2.10). □

3 Residue Complex

It is well-known that regular algebras of dimension 2 are exactly the twisted coordinate rings of the projective line $\mathbf{P}^1_k$. A sketch of the proof follows.

Proposition 3.1. *Regular algebras of global dimension 2 are exactly the twisted coordinate rings of the projective line.*

Proof. Let R be a regular algebra of dimension 2, and let

$$0 \to P^{r_2} \xrightarrow{M} R^{r_1}(-1) \xrightarrow{x} R \to {}_R k \to 0 \tag{3.2}$$

be a minimal resolution of the left module ${}_R k$, where x denotes the row vector of generators for the algebra, and P^{r_2} denotes a free module of rank r_2. Note that $r_j = \dim_k \mathrm{Tor}_j^R(k_R, {}_R k)$. Using the Gorenstein condition, we have a resolution of the right module $k_R(c)$ for some c,

$$0 \leftarrow k_R(c) \overset{M^t}{\leftarrow} P^{r_2} \leftarrow R^{r_1}(1) \leftarrow R \leftarrow 0. \tag{3.3}$$

Now (3.3) shows two things: first, since the Tor-dimensions are symmetric with respect to left and right, we get $r_2 = 1$. Put $r_1 = r$. Second, as M^t also generates the radical, therefore $M^t = \tau x$ for some automorphism $\tau \in \mathbf{GL}(r,k)$. Thus degrees of the entries of M are all 1, and the resolution (3.2) actually looks

$$0 \to R(-2) \xrightarrow{(\tau x)^t} R^r(-1) \xrightarrow{x} R \to {}_R k \to 0.$$

A Hilbert-series computation shows that for polynomial growth, r must be 2. Thus we see that R is the algebra $k< x_0, x_1> /(x_0(\tau x_1) - x_1(\tau x_0))$, for some $\tau \in \mathbf{PGL}(2,k)$. Now let $A^{\tau^{-1}}$ (see 1.5) be the algebra, which is the twist of the commutative polynomial ring $A = k[x_0, x_1]$ by the automorphism τ^{-1} defined by the matrix τ^{-1}. As $x_0(\tau x_1) - x_1(\tau x_0) = 0$ also in $A^{\tau^{-1}}$ (1.15i), we see that $A^{\tau^{-1}}$ is a quotient of R. Since the Hilbert functions of all these algebras $R, A, A^{\tau^{-1}}$ are the same, we get $R = A^{\tau^{-1}}$. The latter being identified with the twisted coordinate ring $B = B(\mathbf{P}^1_k, \tau, \mathcal{L})$ (see 1.9), the proof is complete. □

Note that $\mathrm{Ext}^2_B(k, B) = k(2)$.

Let $B = B(\mathbf{P}^1_k, \sigma, \mathcal{L})$ be a regular algebra of (global) dimension 2, so we identify $B = A^{\sigma^{-1}}$. Note that the gk-dimension of B is also 2. By a B-module we will mean a right or left B-module. For a B-module M, we denote by $E_B(M)$ the graded injective hull of M. An indecomposable injective B-module is said to be of class $\mathcal{E}^i$ if it is an injective hull of a critical B-module of gk-dimension i. Here we construct an injective resolution for B,

$$0 \to B \to E^2 \to E^1 \to E^0 \to 0 \tag{3.4}$$

where E^i is a (B,B)-bimodule which decomposes in $\mathrm{Gr}-B$ (resp. $B-\mathrm{Gr}$) as a direct sum of indecomposable injective right (resp. left) B-modules of class $\mathcal{E}^i$. This would be an analogue of the commutative "residue-complex" (for A), and could be called a 'residue-complex for the quantum projective line' (intuitively, we think of a quantum projective line as the twist of the commutative projective line by the automorphism σ, and B as the coordinate ring of the quantum projective line).

The only critical B-modules in gk-dimension 0 are shifts of the simple module k. The (B,B)-bimodule $B' = \mathrm{Hom}_k(B,k)$ is an injective hull of k in $\mathrm{Gr}-B$ and in $B-\mathrm{Gr}$. Note that B' is completely $\mathbf{m}$-torsion, where $\mathbf{m}$ is the augmentation ideal $B_{\geq 1}$. Also note that $\mathrm{Hom}_B(k,B') \simeq k$. The B-module B is critical [ATV2, 2.30 v] of gk-dimension 2. Let $K = K_B$ be the $\mathbf{Z}$-graded quotient field of B. Then K is a (B,B)-bimodule and it is an injective hull of B in $\mathrm{Gr}-B$ and in $B-\mathrm{Gr}$. Up to isomorphism, this is the only indecomposable injective module of class $\mathcal{E}^2$.

We now consider the injective hulls of critical B- modules of gk-dimension 1. The only critical (right) B-modules of gk-dimension 1 are the (right) point-modules N_p, i.e., modules of the form B/l_pB where $l_p \in B_1$ is a linear form vanishing at $p \in X = \mathbf{P}^1_k$. Thus, the point modules are in bijective correspondence with the points of X. Since N_p is just a right B-module, one can not expect the injective hull $E_B(N_p)$ to carry a natural bimodule structure. However, if we take the direct sum N_ω of the point modules over an orbit ω then, as we see below, N_ω is a (B,B)-bimodule and there is a (B,B)-bimodule which is an injective hull of N_ω in $\mathrm{Gr}-B$ and in $B-\mathrm{Gr}$. So, define N_ω as

$$N_\omega = \oplus_{p\in\omega} N_p. \tag{3.5}$$

Then N_ω has a left B-module structure, compatible with the usual right module structure. This is defined as follows. Let $a = (\bar{a_p}) \in N_\omega$, where $\bar{a_p}$ denotes the component of a in N_p, $a_p \in B$. For $\phi \in B_n$, we define $\phi a \in N_\omega$ such that its component in N_p is

$$(\phi a)_p = \overline{\phi . a_{\sigma^n p}} \tag{3.6}$$

where the multiplication under the bar sign is in B. This is well-defined, since if $(a_p)(p) = 0$ for all $p \in \omega$, then

$$(\phi a_{\sigma^n p})(p) = \phi(p) \otimes (a_{\sigma^n p}(\sigma^n p)) = 0.$$

Since the right B-module structure on N_ω is given by $(a\phi)_p = \overline{a_p \phi}$, we see that the two module structures are compatible and N_ω is a (B,B)-bimodule.

We will see (3.11) that the bimodule K/B_ω is an injective hull of N_ω in Gr–B and in B–Gr, where B_ω is the "localization of B at the orbit ω" to be defined now.

For an orbit ω, define the set

$$(3.7) \qquad S_\omega = \{\phi \text{ homogeneous element of } B | \phi(p) \neq 0, \forall p \in \omega\}$$

We will write ω_p for the orbit of p and S_p for S_{ω_p}.

Proposition 3.8. *S_ω is a left and right Ore denominator subset of B.*

Proof. It is clear that S_ω is a multiplicative subset. Now S_ω satisfies the right Ore condition; for, if $\phi \in B, \psi \in S_\omega$ with $\deg(\phi) = r, \deg(\psi) = s$, then by (1.15),

$$\psi.\sigma^s\phi = \phi.\sigma^r\psi$$

and $\sigma^r\psi \in S_\omega$. Finally, we need to check

$$\operatorname{ass}(S_\omega) \stackrel{\text{def}}{=} \{b \in B | bs = 0 \text{ for some } s \in S_\omega\} = 0.$$

As B is a domain, this is trivial. One similarly verifies the left version of the result. □

Definition 3.9. *The* **Z***-graded localization of B at an orbit ω is $B_\omega = BS_\omega^{-1}$.*

Note that since S_ω is both a left and a right denominator subset of B, B_ω as defined above is also the left quotient of B with respect to S_ω. Since $\operatorname{ass}(S_\omega) = 0$, the canonical map $B \to B_\omega$ is injective and the **Z**-graded quotient field of B_ω is same as K_B.

In the following Lemma, $\operatorname{div}(fg^{-1})$ is as defined in (2.3).

Lemma 3.10. *Let ω be an orbit and let fg^{-1} be a homogeneous element of K_B. Then $fg^{-1} \in B_\omega$ if and only if $\operatorname{mult}_p(\operatorname{div}(fg^{-1})) \geq 0$ for all $p \in \omega$.*

Proof. Suppose $fg^{-1} \in B_\omega$, then $fg^{-1} = f_1g_1^{-1}$ in K_B for some homogeneous elements f_1, g_1 such that $g_1 \in S_\omega$. As $\operatorname{mult}_p(\operatorname{div}(g_1)) = 0$ for all $p \in \omega$, $\operatorname{mult}_p(\operatorname{div}(fg^{-1})) \geq 0$ for all $p \in \omega$. Conversely, suppose that $\operatorname{mult}_p(\operatorname{div}(fg^{-1})) \geq 0$ for all $p \in \omega$, and write $\operatorname{div}(fg^{-1}) = D_1 - D_2$ where D_1 and D_2 are effective divisors without a common point. Thus $\operatorname{mult}_p(D_2) = 0$ for all $p \in \omega$. Let f_1, g_1 be homogeneous elements in B such that $D_1 = \operatorname{div}(f_1)$ and $D_2 = \operatorname{div}(g_1)$. It follows that $\operatorname{mult}_p(\operatorname{div}(g_1)) = 0$ for all $p \in \omega$. Thus $g_1 \in S_o$ and $fg = f_1g_1^{-1}$ up to a unit in k (2.4 *iii*). □

Theorem 3.11. *The (B,B)-bimodule K/B_ω is an injective hull of N_ω in* $\mathrm{Gr}-B$ *and in* $B-\mathrm{Gr}$.

Proof. First we note that the global dimension of B_ω is 1. To see this, just recall that B has global dimension 2 and the only graded-simple module k_B of B of projective dimension 2 is annihilated by the localization functor, $k_B \otimes_B B_\omega = 0$.

Now, the exact sequence

$$0 \to B_\omega \to K \to K/B_\omega \to 0$$

shows that K/B_ω is an injective B_ω-module in $\mathrm{Gr}-B_\omega$ and in $B_\omega-\mathrm{Gr}$, therefore, an injective module in $\mathrm{Gr}-B$ and in $B-\mathrm{Gr}$.

We now show that there is an essential monomorphism $N_\omega \hookrightarrow K/B_\omega$ in $\mathrm{Gr}-B$. In the following, for a point $p \in X$, we will denote by p itself a (chosen) linear form vanishing at p. Fix a linear form r not in the orbit ω. Now, for an arbitrary $q \in \omega$, define a right B-linear homomorphism of degree zero $N_q \to K/B_\omega$ by mapping the generator of N_q to rq^{-1} mod B_ω. This is a well-defined injective map, since, for $b \in B$ one has $rq^{-1}b \in B_\omega$ if and only if $b \in qB$ (Lemma 3.10). Summing over all $q \in \omega$, we get a B-homomorphism $N_\omega \to K/B_\omega$, mapping $a \in N_\omega$ to $\sum_{q\in\omega} \frac{r}{q}.a_q$, where a_q is the component of a in N_q. We claim that this is an essential monomorphism. First, we check the injectivity. Let p be a fixed point in the orbit ω, so that every $q \in \omega$ is $\sigma^m p$ for some $m \in \mathbf{Z}$. Now, showing the injectivity is the same as showing that if for some homogeneous elements $a_k \in B$ $(k = 1, \ldots, n)$ and some indices j_k $(k = 1, \ldots, n)$, with $\sigma^{j_k}p \neq \sigma^{j_l}p$ for $k \neq l$, one has

$$\frac{r}{\sigma^{j_1}p}a_1 + \frac{r}{\sigma^{j_2}p}a_2 + \ldots + \frac{r}{\sigma^{j_n}p}a_n \quad \in B_\omega \tag{3.12}$$

then $a_k(\sigma^{j_k}p) = 0$ for all k.

By the definition of summation in the quotient ring, the sum in (3.12) is

$$f = rs^{-1}(s_1a_1 + s_2a_2 + \ldots + s_na_n)$$

where s and s_k are given by

$$s = \sigma^{j_1-(n-1)}p.\sigma^{j_2-(n-2)}p \ldots \sigma^{j_n}p \qquad s = s_k.\sigma^{j_k}p.$$

Thus we have

$$\mathrm{div}(s) = \sigma^{-(n-1)}(\sum_{k=1}^{n} \sigma^{j_k}p) \qquad \mathrm{div}(s_k) = \sum_{l\neq k} \sigma^{j_l-(n-1)}p \tag{3.13}$$

Now, using Lemma (2.4 i),

$$\operatorname{div}(f) = r - (\sum_{k=1}^{n} \sigma^{j_k} p) + \sigma^{(n-1)} \operatorname{div}(\sum_{k=1}^{n} s_k a_k).$$

Since $f \in B_\omega$, $\operatorname{mult}_q(\operatorname{div}(f)) \geq 0$, $\forall q \in \omega$. It follows that for all l, $\sigma^{j_l} p \in \sigma^{(n-1)} \operatorname{div}(\sum_{k=1}^{n} s_k a_k)$, so

$$(\sum_{k=1}^{n} s_k a_k)(\sigma^{j_l - (n-1)} p) = 0 \qquad \forall l.$$

In conjunction with (3.13), we get $(s_l a_l)(\sigma^{j_l-(n-1)} p) = 0$ for all l. Noting that $\deg(s_l) = n - 1$, we have $a_l(\sigma^{j_l} p) = 0$, as required.

We now show that the map is essential. This means to show that for every non-zero homogeneous element $fg^{-1} \in K/B_\omega$, there exists an element $b \in B$, such that $fg^{-1}b \in N_\omega$. For this it suffices to show that, for every non-zero homogeneous element $fg^{-1} \in K - B_\omega$, there exist non-zero homogeneous elements $a, b \in B$ such that $fg^{-1}.b = rq^{-1}.a$ for some $q \in \omega$. (Here, r is the fixed linear form used in the definition of the map $N_\omega \to K/B_\omega$.) Write $\operatorname{div}(fg^{-1}) = D_1 - D_2$ where D_1 and D_2 are effective divisors without a common point. Since $fg^{-1} \notin B_\omega$, there exists $q \in \omega$ such that $\operatorname{mult}_q(D_2) \geq 1$. Then $(D_2 - q + r)$ is an effective divisor. Let ab^{-1} be a homogeneous element in K such that $\operatorname{div}(ab^{-1}) = D_1 - (D_2 - q + r)$. It follows that $\operatorname{div}(fg^{-1}) = \operatorname{div}(rq^{-1}ab^{-1})$. Thus $fg^{-1} = rq^{-1}ab^{-1}$, and $fg^{-1}b = rq^{-1}a$, up to a unit in k (2.4 iii).

As for the left B-module structure on N_ω, we define a left B-linear map $N_o \to K/B_\omega$ as follows. Choose a fixed linear form r not in the orbit ω. Let $a = (\bar{a}_p)$ be a homogeneous element of N_ω of degree n. Define the map by $a \mapsto \sum_{p\in\omega} a_{\sigma^{-n}p}.\frac{r}{p} \bmod B_\omega$. One can check as above that this map is well-defined, left B-linear, and essential injective. □

Theorem 3.14. *The sequence*

$$(3.15) \qquad 0 \to B \to K \to \oplus_\omega \frac{K}{B_\omega} \to B'(2) \to 0,$$

where the first map is the canonical injection and the second is the direct sum of canonical projections, is a minimal injective resolution of B in Gr-B and B-Gr.

Proof. First, all the modules in the above sequence are injective both in Gr-B and B-Gr. The exactness of the sequence at the first three terms is clear; for, $\cap_\omega B_\omega = B$ (inside K).

Next, the map $K \to \oplus_\omega K/B_\omega$ is surjective in all non-negative degrees. This follows from the partial fraction decomposition in $(K)_n$ for $n \geq 0$ (Theorem 2.7). Indeed, let $(a_\omega b_\omega^{-1})$ be a homogeneous element of non-negative degree in $\oplus_\omega K/B_\omega$, where $a_\omega b_\omega^{-1}$ is its component in K/B_ω. Then by (2.7 i), there exist homogeneous elements $f_\omega {g_\omega}^{-1}$ of K such that g_ω is uni-orbital, and $a_\omega b_\omega^{-1} = f_\omega g_\omega^{-1}$ in K/B_ω. Then the element $fg^{-1} = \sum_\omega f_\omega g_\omega^{-1}$ in K maps to $(a_\omega b_\omega^{-1})$ in $\oplus_\omega K/B_\omega$. Let us denote this map by π for the moment. Because of surjectivity of the map π in all non-negative degrees, coker(π) is $\mathbf{m}$-torsion. Since the global dimension of B is 2, it follows that coker(π) is an injective module (both in Gr$-B$ and in $B-$Gr). The only such $\mathbf{m}$-torsion injective module is $\oplus_i B'(l_i)$ for certain shifts l_i. Thus we have the exact sequence

$$(3.16) \qquad 0 \to B \to K \to \oplus_\omega K/B_\omega \xrightarrow{\pi} \mathrm{coker}(\pi) = \oplus_i B'(l_i) \to 0.$$

Now if we compute $\mathrm{Ext}^2_B(k, B)$ using the injective resolution (3.16) of B, we get $\mathrm{Ext}^2_B(k, B) = \oplus_i k(l_i)$. Since we already know that $\mathrm{Ext}^2_B(k, B) = k(2)$, it follows that coker$(\pi) = B'(2)$, and this completes the proof.

The minimality of the resolution is clear, as the maps $B \to K$, and $K/B \to \oplus_\omega K/B_\omega$ are essential injections. □

(3.17) The map $\oplus_\omega K/B_\omega \to B'(2)$ in (3.15) can be described as follows, by considering the ("sheafified") version of the exact sequence (3.15) in the category Proj$-B$. Since $B'(2)$ is $\mathbf{m}$-torsion, its image in Proj$-B$ vanishes. Thus, we get an exact sequence in Proj$-B$:

$$(3.18) \qquad 0 \to \mathcal{B} \to \mathcal{K} \to \oplus_\omega \frac{\mathcal{K}}{\mathcal{B}_\omega} \to 0.$$

[This should be compared with the commutative case

$$0 \to \mathcal{O} \to \mathcal{K} \to \mathcal{K}/\mathcal{O} \to 0$$

where $\mathcal{O}$ is the structure sheaf on $\mathbf{P}^1_k$, $\mathcal{K}/\mathcal{O} = \oplus i_*(\frac{K}{\mathcal{O}_p})$, $i : \{p\} \hookrightarrow X$ being the inclusion.]

Taking cohomology in Proj$-B$ we get the exact sequence (3.15) back:

(3.19)

$$
\begin{array}{ccccccccc}
0 & \to & \underline{H}^0(\mathcal{B}) & \to & \underline{H}^0(\mathcal{K}) & \to & \underline{H}^0(\oplus_\omega \frac{\mathcal{K}}{\mathcal{B}_\omega}) & \to & \underline{H}^1(\mathcal{B}) & \to & 0 \\
 & & \| & & \| & & \| & & \downarrow \wr & & \\
0 & \to & B & \to & K & \to & \oplus_\omega \frac{K}{B_\omega} & \to & B'(2) & \to & 0
\end{array}
$$

where the isomorphism $\underline{H}^1(\mathcal{B}) \xrightarrow{\sim} B'(2)$ comes from the Serre duality (B.15).

Appendix B Serre Duality for Regular Algebras

A non-commutative analogue of dualizing complexes was introduced for graded algebras over a field in [Ye1]. Reinterpreting some of the results in [Ye1], we deduce Grothendieck-Serre-type duality for the non-commutative projective scheme proj$-A$ where A is a Noetherian regular algebra. This result was obtained jointly with J.J. Zhang.

(B.1) Let us review the notations first. For generalities and basic results about the non-commutative projective schemes see [AZ]. Let A be an $\mathbf{N}$-graded algebra over a field k. Recall that Gr$-A$ (resp. gr$-A$) is the category of all graded right A-modules (resp. Noetherian graded right A-modules) with morphisms being graded homomorphisms of degree zero, and Tor$-A$ (resp. tor$-A$) is the full subcategory of Gr$-A$ (resp. gr$-A$) consisting of all $\mathbf{m}$-torsion modules (resp. Noetherian $\mathbf{m}$-torsion modules). Then the quotient category Gr$-A/$Tor$-A$ (resp. gr$-A/$tor$-A$) exists and is denoted by Proj$-A$. The global section functor H^0 is defined by $H^0(\mathcal{M}) = \mathrm{Hom}_{\text{proj-}A}(\mathcal{A}, \mathcal{M})$, and the derived functors of H^0 are denoted by H^i. We also define $\underline{H}^i(\mathcal{M}) = \oplus_n H^i(\mathcal{M}(n))$. As usual, $\mathrm{Hom}_A(M, N)$ denotes the graded group of homomorphisms and $\mathrm{Hom}_{\text{Gr-}A}(M, N)$ its degree-zero component; similarly for Ext. The *cohomological dimension of* proj$-A$ is defined to be $\max\{n |\ H^n(\mathcal{M}) \neq 0$ for some object $\mathcal{M}$ in proj$-A\}$.

Definition B.2. *Let* proj$-A$ *be the non-commutative projective scheme of a graded algebra A. Suppose that* proj$-A$ *has finite cohomological dimension p. A* dualizing sheaf *for* proj$-A$ *is an object ω_A in* proj$-A$, *together with a* trace *isomorphism $t : H^p(\omega_A) \to k$, such that for all objects $\mathcal{M}$ of proj(A), the natural pairing*

(B.3) $$\mathrm{Hom}_{\text{proj-}A}(\mathcal{M}, \omega_A) \times H^p(\mathcal{M}) \to H^p(\omega_A) \to k$$

gives an isomorphism

$$\theta_0 : \mathrm{Hom}_{\mathbf{proj}\text{-}A}(\mathcal{M}, \omega_A) \xrightarrow{\sim} (H^p(\mathcal{M}))^*$$

where $*$ denotes the dual space.

Clearly, if a dualizing sheaf for proj $-A$ exists, then it is unique up to a unique isomorphism, i.e., if (ω_A, t) and (ω'_A, t') are two dualizing sheaves for proj $-A$ then there is a unique isomorphism $\phi : \omega_A \xrightarrow{\sim} \omega'_A$ such that $t = t' \circ H^p(\phi)$. This follows from the fact that (ω_A, t) represents the functor $\mathcal{M} \rightsquigarrow (H^p(\mathcal{M}))^*$.

Denote by A° the algebra opposite to A, and by A^e the algebra $A \otimes_k A^\circ$. Let A' be the A^e-module $\mathrm{Hom}_k(A, k)$. For an A-module M and an A°-module N, there are natural isomorphisms

(B.4) $\quad \mathrm{Hom}_A(M, A') \xrightarrow{\sim} \mathrm{Hom}_k(M, k) \qquad \mathrm{Hom}_{A^\circ}(N, A') \xrightarrow{\sim} \mathrm{Hom}_k(N, k)$

Recall [AZ] that for an object M of gr $-A$, whose projection in proj $-A$ is denoted by $\mathcal{M}$, we have

(B.5) $$\underline{H}^i(\mathcal{M}) = \lim_{n\to\infty} \mathrm{Ext}_A^{i+1}(A/A_{\geq n}, M) \qquad (i > 0).$$

(B.6) Let A be a Noetherian regular algebra of finite global dimension. Thus if the global dimension of A is d then $\mathrm{Ext}^{\cdot}_A(k, A) \xrightarrow{\sim} k(l)[-d]$ for some integer l. Here (l) denotes the degree shift while $[d]$ denotes the shift of the complex. Therefore, there is an isomorphism

(B.8) $$t : H^{d-1}(\mathcal{A}(-l)) \xrightarrow{\sim} k.$$

Further, For any object M in gr $-A$, one has a natural isomorphism

(B.9) $$\mathrm{Hom}_{\mathbf{gr}-A}(M, A) \simeq \mathrm{Hom}_{\mathbf{proj}\text{-}A}(\mathcal{M}, \mathcal{A}).$$

To see this, we just check that $\mathrm{Hom}_A(M_{\geq n}, A) \xrightarrow{\sim} \mathrm{Hom}_A(M, A)$. Now this follows by applying $\mathrm{Hom}_A(,A)$ to the sequence $0 \to M_{\geq n} \to M \to T \to 0$ (where T is a torsion module), and noting that $\mathrm{Ext}^i_A(T, A) = 0$ for $i = 0, 1$.

Theorem B.10 [Serre duality for regular algebras]. *Let A be a Noetherian regular algebra of dimension $d \geq 2$. Then the cohomological dimension* *of* proj$-A$ *is* $p = d-1$, *and the pair* $(\mathcal{A}(-l), t)$ *is a dualizing sheaf for* proj$-A$. *Thus*

(i) for every $\mathcal{M}$ *in* proj$-A$, *the natural pairing*

$$\text{(B.11)} \qquad \mathrm{Hom}_{\text{proj-}A}(\mathcal{M}, \mathcal{A}(-l)) \times H^{d-1}(\mathcal{M}) \to H^{d-1}(\mathcal{A}(-l)) \xrightarrow{t} k$$

is a perfect pairing of finite-dimensional vector spaces over k.

(ii) for each $i \geq 0$, *there is a natural functorial isomorphism*

$$\text{(B.12)} \qquad \mathrm{Ext}^i_{\text{proj-}A}(\mathcal{M}, \mathcal{A}(-l)) \xrightarrow{\sim} H^{d-1-i}(\mathcal{M})^*$$

which for $i = 0$ *is the one induced by the pairing* (B.11).

Proof. By [Ye1, Proposition 4.17, with $R = A(-l)[d]$], we have a natural isomorphism

$$\text{(B.13)} \qquad \lim_{n\to\infty} \mathrm{Ext}^i_A(A/A_{\geq n}, M) \xrightarrow{\sim} \mathrm{Hom}_{A^\circ}(\mathrm{Ext}^{d-i}_A(M, A(-l)), A')$$

and, therefore, by (B.4):

$$\text{(B.14)} \qquad \lim_{n\to\infty} \mathrm{Ext}^i_A(A/A_{\geq n}, M) \xrightarrow{\sim} \mathrm{Ext}^{d-i}_A(M, A(-l))^*$$

The left hand side is just $\underline{H}^{i-1}(\mathcal{M})$. Now if $i > d$, then $\mathrm{Ext}^{d-i}_A(M, A(-l)) = 0$, and, by (B.14), we have $\underline{H}^{i-1}(\mathcal{M}) = 0$. Thus, the cohomological dimension of proj-A is at most $d-1$. But since $\underline{H}^{d-1}(\mathcal{A}) \simeq k(l)$ (B.7), the cohomological dimension is $d-1$.

Now setting $i = d$ in (B.14) we see that

$$\mathrm{Hom}_{\text{gr-}A}(M, A(-l)) \simeq H^{d-1}(\mathcal{M})^*.$$

As $\mathrm{Hom}_{\text{gr-}A}(M, A(-l)) \simeq \mathrm{Hom}_{\text{proj-}A}(\mathcal{M}, \mathcal{A}(-l))$, the pair $(\mathcal{A}(-l), t)$ represents the functor $\mathcal{M} \rightsquigarrow H^{d-1}(\mathcal{M})^*$. This proves (i).

The proof of (ii) is a standard cohomological argument. There is a natural transformation from $\mathrm{Ext}^i_{\text{proj-}A}(., \mathcal{A}(-l))$ to $H^{d-i-1}(.)^*$. Both sides are contravariant δ-functors indexed by $i \geq 0$. For $i = 0$, there is an isomorphism by (B.14). Thus to prove that the two are isomorphic, it is sufficient to show that both the functors are coeffaceable for $i > 0$. Recall that a functor F is coeffaceable if for each object $\mathcal{M}$, there is an epimorphism $f : \mathcal{N} \to \mathcal{M}$ such that $F(f) = 0$. Now given an object $\mathcal{M}$ of proj$-A$, $\mathcal{M}$ is a quotient of $\mathcal{P} = \oplus_j \mathcal{A}(-r_j)$ for some $r_j << 0$. Then

$\mathrm{Ext}^i_{\text{proj-}A}(\mathcal{P}, \mathcal{A}(-l)) = \oplus_j H^i(\mathcal{A}(-l + r_j)) = 0$ for $i > 0$, by (B.14). Also, $H^{d-i-1}(\mathcal{P})^* = \oplus_j H^{d-i-1}(\mathcal{A}(-r_j))^* = 0$ for $i > 0$, again by (B.14). Thus the two functors are coeffaceable for $i > 0$, hence isomorphic. □

Corollary B.15. $\underline{H}^{d-1}(\mathcal{A}) \simeq A'(l)$.

Proof. We use the duality (B.11) for $\mathcal{M} = \mathcal{A}(-n)$ for $n \in \mathbf{Z}$, to conclude that $H^{d-1}(\mathcal{A}(-n)) = \mathrm{Hom}_k(A_n, k(l))$, and thus $\underline{H}^{d-1}(\mathcal{A}) \simeq A'(l)$. □

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