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ALMOST STABILIZABILITY SUBSPACES AND HIGH GAIN FEEDBACK

by

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ABSTRACT

The class of "almost stabilizability subspaces" is introduced as the state-space analog of the class of stable but not necessarily proper transfer functions. Almost stabilizability subspaces can be considered as candidate closed-loop eigenspaces associated with infinitely fast and stable modes. We derive the basic properties of these subspaces, and show that they can be approximated by regular stabilizability subspaces. The relation with high gain feedback is elaborated upon in a number of applications.

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1. Introduction

The concept of "almost-invariance" was introduced by J.C. Willems in a series of recent papers [1-3] as a geometric means of studying high gain feedback and more generally, asymptotic phenomena in linear systems. In a sense, almost invariant subspaces provide a state-space parallel to the frequency-domain use of nonproper transfer functions. Another such parallel was made quite explicit by Hautus [4], who linked the class of "stabilizability subspaces" (which had already appeared, without being named as such, in the work of Wonham and his colleagues in the seventies [5]) to the set of stable proper transfer functions, which plays a prominent role in recent research like [6-8].

Of course, an important role is also played by the class of stable but not necessarily proper transfer functions. In this paper, we shall identify the corresponding state-space concept, which we shall term "almost stabilizability subspace". These subspaces can be thought of as candidate closed-loop eigenspaces associated with infinitely fast and stable modes.

The formal definition will be given in Section 2, along with a number of basic properties. In Section 3, the key result is proven that every almost stabilizability subspace can be obtained as the limit of a sequence of stabilizability subspaces. Applications are given in Section 4. A few examples will be discussed of known results that can be re-interpreted in terms of almost stabilizability subspaces. Most of the section, however, is devoted to new results on the problem of stabilization by high gain feedback.

Throughout this paper, we shall work with a fixed finite-dimensional time-invariant linear system, given by

$$x'(t) = Ax(t) + Bu(t) \qquad (x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}) \tag{1.1}$$

(augmented by an observation equation in subsection 4D). The state space $\mathcal{X}$, the input space $\mathcal{U}$, the system mapping $A:\mathcal{X} \to \mathcal{X}$, and the input mapping $B:\mathcal{U} \to \mathcal{X}$ are all taken over the real field $\mathbb{R}$, but the obvious complexifications will be used where needed without change of notation. The complex number field is denoted by $\mathbb{C}$. The nullspace and the range of a linear mapping M will be written as ker M and im M, respectively.

2. Definition and Basic Properties

Recall the following definition from [4]:

Definition 2.1: A subspace $\mathcal{S}$ of $\mathcal{X}$ is a stabilizability subspace if there exists an $F: \mathcal{X} \to \mathcal{U}$ such that $(A+BF)\mathcal{S} \subset \mathcal{S}$ and the restriction of A+BF to $\mathcal{S}$ is stable.

We have the following characterizations of this concept.

Proposition 2.2 [4]. $\mathcal{S}$ is a stabilizability subspace if and only if for every $x \in \mathcal{S}$ there exist stable strictly proper rational functions $\xi(s)$ and $\omega(s)$ such that $\xi(s) \in \mathcal{S}$ (for all s) and

$$x = (sI-A)\xi(s) + B\omega(s). \tag{2.1}$$

Proposition 2.3 [9]. $\mathcal{S}$ is a stabilizability subspace if and only if

$$(sI-A)\mathcal{S} + \text{im } B = \mathcal{S} + \text{im } B \tag{2.2}$$

for all s in the right half-plane.

Almost (A,B)-invariant subspaces and almost controllability subspaces were introduced in [1] (see also [10] for a pure algebraic treatment, based on a discrete-time interpretation). Recall the basic result [1] that a subspace $\mathcal{V}_a$ is almost (A,B)-invariant if and only if it can be written as the sum of an (A,B)-invariant subspace [5] and an almost controllability subspace. This is one motivation for the following definition.

Definition 2.4. A subspace $\mathcal{S}_a$ of $\mathcal{X}$ is an almost stabilizability subspace if it can be written in the form $\mathcal{S}_a = \mathcal{S} + \mathcal{R}_a$, where $\mathcal{S}$ is a stabilizability subspace, and $\mathcal{R}_a$ is an almost controllability subspace.

Further motivation is provided by the next result.

Proposition 2.5. The following are equivalent:

(i) $\mathcal{S}$ is an almost stabilizability subspace.

(ii) For every $x \in \mathcal{S}$ there exist stable rational functions $\xi(s)$ and $\omega(s)$ such that $\xi(s) \in \mathcal{S}$ (for all s) and

$$x = (sI-A)\xi(s) + B\omega(s). \tag{2.3}$$

(iii) The inclusion

$$\mathcal{S} \subset (sI-A)\mathcal{S} + \text{im } B \tag{2.4}$$

holds at all points s in the right half-plane.

The proof follows closely the lines of [10], and will therefore be omitted. Additional support for Def. 2.4 comes from the following proposition.

Proposition 2.6. A subspace $\mathcal{S}$ is a stabilizability subspace if and only if it is both an (A,B)-invariant subspace and an almost stabilizability subspace.

Proof. Combine Prop. 2.3 and Prop. 2.5 (iii) with the observation that a subspace $\mathcal{S}$ is (A,B)-invariant if and only if

$$(sI-A)\mathcal{S} \subset \mathcal{S} + \text{im } B \tag{2.5}$$

for some $s \in \mathbb{C}$ (cf. [5], p. 88), which is easily seen to be equivalent to the statement that (2.5) holds for all $s \in \mathbb{C}$.

The following direct-sum decomposition is an immediate consequence of the general decomposition given in [2].

Proposition 2.7. Every almost stabilizability subspace can be written in the form $S_a = R + \tilde{R}_a + \tilde{S}$, where R is a controllability subspace, $\tilde{R}_a$ is a sliding subspace (i.e., an almost controllability subspace that does not contain any nonzero controllability subspace, see [2]), and $\tilde{S}$ is a coasting subspace (i.e., an (A,B)-invariant subspace that does not contain any nonzero controllability subspace, see again [2]) such that the restriction of A+BF to $\tilde{S}$ is stable for any F such that $(A+BF)\tilde{S} \subset \tilde{S}$.

It is easily checked that the sum of two almost stabilizability subspaces is again an almost stabilizability subspace. Hence, there is a unique largest almost stabilizability subspace in any given subspace K, and we shall denote it by $S_a^*(K)$. The following characterization closely parallels the one given by Hautus [4] for the largest stabilizability subspace in K, which we shall denote by $S^*(K)$. The proof follows the lines of [4] and [10] and will be omitted.

Proposition 2.8. $S_a^*(K)$ equals the set of all $x \in K$ for which there exist stable rational functions $\xi(s)$ and $\omega(s)$ such that $\xi(s) \in K$ and

$$x = (sI-A)\xi(s) + B\omega(s) \qquad (2.6)$$

Removing the restriction "$x \in K$", we obtain the following subspace, which will turn out to be more useful.

Definition 2.9. $S_b^*(K)$ equals the set of all $x \in X$ for which there exist stable rational functions $\xi(s)$ and $\omega(s)$ such that $\xi(s) \in K$ and (2.6) holds.

This subspace can be interpreted as the set of all vectors that can serve as initial values for stable and possibly impulsive trajectories that stay in K for all time ([2], [10]). Geometrically, $S_b^*(K)$ can be characterized as follows. We write $\mathcal{R}_a^*(\mathcal{K})$ for the largest almost controllability subspace in $\mathcal{K}$, and define $\mathcal{R}_b^*(\mathcal{K}) = A\mathcal{R}_a^*(\mathcal{K}) + \text{im } B$ as in [2].

Proposition 2.10. $\mathcal{S}_b^*(\mathcal{K}) = \mathcal{S}^*(\mathcal{K}) + \mathcal{R}_b^*(\mathcal{K})$.

The proof can be given without difficulty, using the methods of [2] and/or those of [10]. Actually, [2] uses the above formula as the definition of the subspace $\mathcal{S}_b^*(\mathcal{K})$ (see Thm. 18; note the printing error).

A convenient way of finding out whether a given subspace is an almost stabilizability subspace is given by the following rank test (cf. [10]).

Proposition 2.11. Let $\mathcal{S}_a$ be a given subspace. Write $\dim \mathcal{S}_a = k$, $\dim (\mathcal{S}_a + A\mathcal{S}_a + \text{im } B) = r$. Choose a basis for $\mathcal{X}$ such that the first k basis vectors span $\mathcal{S}_a$ and the first r basis vectors span $\mathcal{S}_a + A\mathcal{S}_a + \text{im } B$. Let the matrices of A and B with respect to this basis and a given basis in $\mathcal{U}$ be

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix} \tag{2.7}$$

Then $\mathcal{S}_a$ is an almost stabilizability subspace if and only if

$$\text{rank} \begin{pmatrix} sI-A_{11} & B_1 \\ -A_{21} & B_2 \end{pmatrix} = r \tag{2.8}$$

for all s in the right half-plane.

An alternative route leads via the definition and the algorithms given in [2] and [5].

3. Asymptotic Properties

We shall use the following (quite common) notion of convergence for sequences of subspaces.

Definition 3.1. A sequence $\{V^n\}$ of subspaces of fixed dimension k is said to converge to a k-dimensional subspace V if the following is true. Let $\{x_1,\ldots,x_k\}$ be a basis for V. For every $\varepsilon>0$, one can find an N such that for all $n \geq N$ there exists a basis $\{x_1^n,\ldots,x_k^n\}$ of V^n which satisfies $||x_i^n-x_i|| < \varepsilon$ for all $i = 1,\ldots,k$.

The main result of this section is:

Theorem 3.2. For every almost stabilizability subspace S_a, there exists a sequence of stabilizability subspaces $\{S^n\}$ which converges to it.

The proof will be given through a series of lemmas. The first one of these can be proved by standard means.

Lemma 3.3. Suppose that $V = V_1 + \ldots + V_r$, and suppose also that we have sequences $\{V_j^n\}$ converging to V_j, for each $j = 1,\ldots,r$. Then the subspaces $V_1^n,\ldots,V_r^n$ are independent for all sufficiently large n. Moreover, if we define $V^n = V_1^n + \ldots + V_r^n$ for these n, then the sequence $\{V^n\}$ converges to V.

Lemma 3.4. Every almost controllability subspace R_a can be written as a direct sum $R_a = \hat{R}_a^1 + \ldots + \hat{R}_a^r$, where for each $j = 1,\ldots,r$ there exist a mapping $F: X \rightarrow U$, a vector $b \in \operatorname{im} B$, and an integer $k \geq 0$, such that

$$\hat{R}_a^j = \operatorname{span}\{b, (A+BF)b,\ldots, (A+BF)^k b\} . \qquad (3.1)$$

Proof. This is immediate from the geometric characterization of almost controllability subspaces given in [1].

Lemma 3.5. Every subspace $\hat{R}_a$ of the form (3.1) can be obtained as the limit of a sequence defined by

$$S^n = \tilde{S}^n + \text{span}\{b,(A+BF)b,\ldots,(A+BF)^{k-1}b\} \tag{3.2}$$

where $\tilde{S}^n$ is a (one-dimensional) stabilizability subspace.

Proof. For all sufficiently large n, the mapping $I + \frac{1}{n}(A+BF)$ will be invertible, and so we can define $\tilde{S}^n$ by

$$\tilde{S}^n = \text{span}\{(I+\frac{1}{n}(A+BF))^{-1}b\} \; . \tag{3.3}$$

With $\hat{R}_a$ given by (3.1) and S^n by (3.2) and (3.3), it is easily verified that

$$S^n = (I + \frac{1}{n}(A+BF))^{-1}\,\hat{R}_a \tag{3.4}$$

which shows that the sequence $\{S^n\}$ converges to $\hat{R}_a$. It remains to show that $\tilde{S}^n$ is a stabilizability subspace. To see this, note that

$$\begin{aligned}(sI-A)(I + \tfrac{1}{n}(A+BF))^{-1}b &= \\ &= (sI-(A+BF))(I + \tfrac{1}{n}(A+BF))^{-1}b + BF(I + \tfrac{1}{n}(A+BF))^{-1}b = \\ &= (s+n)(I + \tfrac{1}{n}(A+BF))^{-1}b + BF(I + \tfrac{1}{n}(A+BF))^{-1}b - nb \; .\end{aligned} \tag{3.5}$$

As a consequence, we have

$$(sI-A)\tilde{S}^n + \text{im } B = \tilde{S}^n + \text{im } B \tag{3.6}$$

for all s in the right half-plane (in fact, for all $s \neq -n$). By Prop. 2.3, this gives the desired result.

Proof (of Thm. 3.1). It follows from Prop. 2.7 and Lemma 3.4 that S_a can be written as the direct sum of a stabilizability subspace and a number of subspaces of the form (3.1). Using Lemma 3.3, we can now complete the proof by a repeated application of Lemma 3.5.

If we think of almost controllability subspaces as invariant subspaces for the infinite modes of a closed-loop system (called into existence by infinite-gain feedback), then the theorem can be interpreted as saying that, in this context, "infinity" can always be read as "minus infinity".

It is essential for the proof that the stable part of the complex plane contains points of arbitrarily large modulus. In the discrete-time situation, one takes the interior of the unit circle as the stable part of the complex plane, and it would be possible to modify the definitions of "stabilizability subspace" and "almost stabilizability subspace" accordingly. But it is then no longer true, as can be seen from simple examples, that every almost stabilizability subspace can be obtained as the limit of a sequence of stabilizability subspaces.

4. Applications

A. "Almost" Disturbance Decoupling

The following set-up is considered in [2]. Let

$$\begin{cases} x'(t) = Ax(t) + Bu(t) + Gq(t) \\ z(t) = Hx(t) \end{cases} \tag{4.1}$$

constitute a linear system, in which x(t) and u(t) have the usual meanings, q(t) is considered as a disturbance input, and z(t) represents a "cost" function which we want to keep close to zero. After state feedback u(t) = Fx(t), the system (4.1) becomes

$$\begin{cases} x'(t) = (A+BF)x(t) + Gq(t) \\ z(t) = Hx(t) \end{cases} \tag{4.2}$$

and the influence of q(t) on the cost function z(t) is given by

$$\tilde{z}(t) = \int_0^t He^{(A+BF)(t-s)}\, Gq(s)\,ds \tag{4.3}$$

As a measure of the degree to which the feedback F succeeds in attenuating the disturbance, we may use the ratio of the L_p-norm of $\tilde{z}(\cdot)$ to the L_p-norm of $q(\cdot)$, where p satisfies $1 \leq p \leq \infty$:

$$\alpha_p(F) = ||\tilde{z}||_{L_p(0,\infty)} \Big/ ||q||_{L_p(0,\infty)} . \tag{4.4}$$

We can now state the following result, which identifies the situations in which it is possible to eliminate the influence of the disturbance 'almost' completely, while the stability of the closed-loop system is assured.

Theorem 4.1. For the system (4.1), the following two statements are equivalent:

(i) For every $\varepsilon > 0$, there exists a state feedback mapping F_ε such that $A+BF_\varepsilon$ is stable and $\alpha_p(F_\varepsilon) < \varepsilon$.

(ii) The pair (A,B) is stabilizable (i.e., there are no unstable uncontrollable eigenvalues), and

$$\text{im } G \subset \mathcal{S}_b^*(\ker H). \tag{4.5}$$

The proof of this statement can be given along the lines of [2], and it will be omitted here. The corresponding result in [2] does not require the stability of the closed-loop system, so that there is no stabilizability requirement on the pair (A,B) and (4.5) is replaced by the condition that im G should be contained in $\mathcal{V}_b^*(\ker H)$.

B. Singular Optimal Control

Consider the linear system

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ z(t) = Hx(t) \end{cases} \tag{4.6}$$

with associated cost functional

$$J_\varepsilon(x_0) = \min_u \int_0^\infty (||z(t)||^2 + \varepsilon^2||u(t)||^2)\,dt\ . \tag{4.7}$$

It is assumed that the pair (A,B) is stabilizable, the pair (H,A) is detectable, B is full column rank, and H is full row rank. Under these conditions, the following result has been given by Francis [11].

Theorem 4.2. $J_\varepsilon(x_0)$ tends to 0 as $\varepsilon \downarrow 0$ if and only if

$$x_0 \in S_b^*(\ker H) \tag{4.8}$$

where the stability set from which the subspace $S_b^*(\ker H)$ is defined has to be taken equal to the closed left half-plane.

A similar result is given in [2]. Of course, Francis did not formulate his theorem in terms of almost invariant subspaces, but it can be seen from the algorithms he uses that the subspace constructed in [11] is exactly $S_b^*(\ker H)$, in the above interpretation. The fact that the closed left half-plane is important here (rather than the open LHP) can be made plausible by a continuity argument.

C. Solvability of a Rational Matrix Equation

Let $R_1(s)$ and $R_2(s)$ be strictly proper rational transfer matrices, of sizes $p\times m$ and $p\times r$, respectively. Under various circumstances, it is important to know whether the equation

$$R_1(s)X(s) = R_2(s) \tag{4.9}$$

has a solution in the set of stable rational transfer matrices. Suppose that we have a realization for the transfer matrix $[R_1(s)\ \ R_2(s)]$:

$$[R_1(s)\quad R_2(s)] = H(sI-A)^{-1}[B\quad G] \tag{4.10}$$

Then we should be able to state the solvability conditions for (4.9) in terms of the matrices H,A,B, and G. Indeed, the following result was essentially proved by Benqtsson [12]:

Theorem 4.3. Suppose that the realization given by (4.10) is observable. Then the equation (4.9) has a stable rational solution if and only if

$$\text{im } G \subset \mathcal{S}_b^*(\ker H). \tag{4.11}$$

This condition is quite closely related to the condition of Thm. 4.2, and, in fact, the proof of Thm. 4.2 in [11] is based on Bengtsson's result. Obviously, there is also a close connection between Thm. 4.3 and Thm. 4.1. Connections of this sort are discussed in [3] and [4].

D. High Gain Feedback

We shall now concentrate on an application of a different type, involving the notion of "almost stabilizability subspace" itself rather than a $\mathcal{S}_b^*$-space. Our concern will be with dynamic output feedback rather than state feedback, so we consider the controlled and observed linear system

$$\begin{cases} x'(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \quad . \end{cases} \tag{4.12}$$

In addition to our earlier notational conventions, we denote the output space by $\mathcal{Y}$, and we set dim $\mathcal{Y} = p$. We shall need the following concept. A subspace $\mathcal{T}$ of $\mathcal{X}$ will be called a minimum-phase input subspace if there exists a mapping $T: \mathcal{Y} \to \mathcal{X}$ such that:

$$\mathcal{T} = \text{im } T \tag{4.13}$$

$$\det CT \neq 0 \tag{4.14}$$

$$\det(sI-A)\det C(sI-A)^{-1}T \neq 0 \qquad \text{for } s \in \mathbb{C}_+ . \tag{4.15}$$

Of course, (4.15) says that the transfer function $C(sI-A)^{-1}T$ should have no unstable zeros ([13], p.41), and this motivates the terminology.

Note, however, that we also require CT to be invertible. A characterization in terms of the subspace $\mathcal{T}$ itself can be given as follows.

Lemma 4.4. A subspace $\mathcal{T}$ is a minimum-phase input subspace if and only if $\mathcal{T} \oplus \ker C = \mathcal{X}$ and the restriction of PA to ker C is stable, where P is the projection onto ker C along $\mathcal{T}$.

Proof. Suppose that $\mathcal{T}$ is a minimum-phase input subspace, and let T be a mapping satisfying (4.13-15). It follows from (4.14) that $\mathcal{T} \oplus \ker C = \mathcal{X}$. We see that $T(CT)^{-1}$ also satisfies (4.13-15), so we may as well assume that CT = I. Then the projection P onto ker C along $\mathcal{T}$ is simply given by

$$P = I - TC. \tag{4.16}$$

The subspace ker C is, of course, invariant for PA, and the factor mapping induced by PA on the quotient space $\mathcal{X}/(\ker C)$ is clearly 0. Therefore, we have

$$\det(sI-PA) = s^p \det(sI-PA|_{\ker C}) . \tag{4.17}$$

Now, consider the following manipulations, in which we use the determinant equality $\det(I+MN) = \det(I+NM)$, the rule $A(sI-A)^{-1} = s(sI-A)^{-1} - I$, and the fact that CT = I.

$$\begin{aligned} \det(sI-PA) &= \det(sI-(I-TC)A) = \\ &= \det(sI-A)\det(I+TCA(sI-A)^{-1}) = \\ &= \det(sI-A)\det(I+CA(sI-A)^{-1}T) = \\ &= \det(sI-A)\det(I-CT + sC(sI-A)^{-1}T) = \\ &= s^p \det(sI-A)\det C(sI-A)^{-1}T . \end{aligned} \tag{4.18}$$

Comparing this with (4.17), we see that (4.15) implies that $PA|_{\ker C}$ is stable.

For the converse, suppose that we have a subspace $\mathcal{T}$ that satisfies the condition of the lemma. Then it is easily verified that there exists a (unique) mapping T: $\mathcal{Y} \to \mathcal{X}$ such that TC = I-P, and that this mapping satisfies (4.13) and (4.14). (In fact, CT=I.) Moreover, it is immediate from (4.17) and (4.18) that (4.15) holds.

The following result was proved in [14] (Lemma 2.12), see also [15] (Thm. 4.4 and Lemma 5.1).

Theorem 4.5. Suppose that, for the system (4.12), we have a stabilizability subspace $\mathcal{V}$ that contains a minimum-phase input subspace. Let the dimension of $\mathcal{V}$ be k. Then the system (4.12) can always be stabilized by dynamic output feedback of the form

$$\begin{cases} w'(t) = A_c w(t) + G_c y(t), & w(t) \in \mathcal{W} \\ u(t) = F_c w(t) + Ky(t) \end{cases} \tag{4.19}$$

where the order of the feedback dynamics (i.e., dim $\mathcal{W}$) is equal to k-p. In particular, if k=p, then the system (4.12) can be stabilized by direct output feedback alone. We now want to prove the same theorem, but with "stabilizability subspace" replaced by "almost stabilizability subspace". The idea is that there are stabilizability subspaces arbitrarily close to a given almost stabilizability subspace (Thm. 3.2) and, on the other hand, any subspace that is close enough to a minimum-phase input subspace will itself be a minimum-phase input subspace. Let us first formally establish the latter fact.

<u>Lemma 4.6</u> Let $\mathcal{T}$ be a minimum-phase input subspace, and let $\{\mathcal{T}_n\}$ be a sequence of subspaces converging to $\mathcal{T}$. Then $\mathcal{T}_n$ is a minimum-phase input subspace for all sufficiently large n.

<u>Proof.</u> Let T be a mapping satisfying (4.13-15); as in the proof of Lemma 4.4, we may assume that CT = I. By the definition of convergence, there exists a basis $\{x_1,\dots,x_p\}$ for $\mathcal{T}$ and a corresponding basis $\{x_1^n,\dots,x_p^n\}$ for each $\mathcal{T}_n$, such that $\{x_i^n\}$ converges to x_i for each i=1,...,p. Define $y_i = Cx_i$ (i=1,...,p). Then $\{y_1,\dots,y_p\}$ is a basis for $\mathcal{Y}$, and $x_i = Ty_i$ (i=1,...,p). Define $T_n: \mathcal{Y} \to \mathcal{X}$ for each n by $T_n y_i = x_i^n (i = 1,\dots,p)$. Obviously, we have $T_n \to T$. Hence, we also have $CT_n \to CT = I$ which implies that CT_n will be invertible for all sufficiently large n. In other words, $\mathcal{T}_n \oplus \ker C = \mathcal{X}$ for these values of n. The projection along $\mathcal{T}_n$ onto ker C is given by

$$P_n = I - T_n(CT_n)^{-1}C. \tag{4.20}$$

By the continuity of matrix inversion and multiplication, it follows that $P_n \to I-T(CT)^{-1}C = I-TC = P$, the projection onto ker C along $\mathcal{T}$. This implies, in particular, that the sequence of mappings $\{P_nA|_{\ker C}\}$ converges to $PA|_{\ker C}$. By the continuity property of the eigenvalues ([16], p. 191), it follows that $P_nA|_{\ker C}$ is stable for all sufficiently large n. An appeal to Lemma 4.4 now completes the proof.

We are now in a position to prove the main result of this section.

<u>Theorem 4.7</u>. Suppose that, for the system (4.12), we have an almost stabilizability subspace $\mathcal{V}$ that contains a minimum-phase input subspace

$\mathcal{T}$. Let the dimension of $\mathcal{V}$ be k. Then the system (4.12) can always be stabilized by dynamic output feedback of the form (4.19), where the order of the feedback dynamics is equal to k-p.

Proof. Let $\{\mathcal{V}_n\}$ be a sequence of stabilizability subspaces converging to $\mathcal{V}$. Then there exists a sequence $\{\mathcal{T}_n\}$, with $\mathcal{T}_n \subset \mathcal{V}_n$ for each n, such that $\{\mathcal{T}_n\}$ converges to $\mathcal{T}$. According to Lemma 4.6, $\mathcal{T}_n$ will be a minimum-phase input subspace for all sufficiently large n. Take such an n, and apply Thm. 4.5 to the corresponding $\mathcal{V}_n$ and $\mathcal{T}_n$.

We immediately have the following corollary.

Corollary 4.8. Suppose that the system (4.12) is square and minimum-phase, and also suppose that the matrix CB is invertible. Then the system (4.12) can be stabilized by direct output feedback alone, i.e., there exists K such that A+BKC is stable.

Proof. It suffices to note that im B is an almost controllability subspace and that the assumptions of the corollary imply that im B is also a minimum-phase input subspace. The result then follows from an application of Thm. 4.7.

In order to obtain a gain matrix K as in the corollary, what one has to do is to form a sequence of stabilizability subspaces that will approximate the almost stabilizability subspace im B. In another terminology, this comes down to selecting a pattern along which p closed-loop poles will go off to infinity. For the purposes of illustration, let us suppose that we want to create a closed-loop eigenvalue with multiplicity p at the point $-\varepsilon^{-1}$, for $\varepsilon>0$. The corresponding stabilizability subspace is

$$\mathcal{V}_\varepsilon = (I+\varepsilon A)^{-1}(\mathrm{im}\ B). \tag{4.21}$$

The mapping T_ε which satisfies $CT_\varepsilon = I$ and $\mathrm{im}\ T_\varepsilon = \mathcal{V}_\varepsilon$ is, of course, given by

$$T_\varepsilon = (I+\varepsilon A)^{-1}B[C(I+\varepsilon A)^{-1}B]^{-1},$$

and it is easy to verify that $F_\varepsilon \in \underline{F}(\mathcal{V}_\varepsilon)$ if and only if

$$F_\varepsilon(I+\varepsilon A)^{-1}B = -\varepsilon^{-1}I. \tag{4.23}$$

As indicated in [15] (see the proof of Lemma 5.1), the corresponding gain matrix K_ε is then given by

$$K_\varepsilon = F_\varepsilon T_\varepsilon = -\frac{1}{\varepsilon}[C(I+\varepsilon A)^{-1}B]^{-1}\ . \tag{4.24}$$

From the theory developed above, it follows that the closed-loop mapping $A+BK_\varepsilon C$ will have an eigenvalue of multiplicity p at the point $-\varepsilon^{-1}$, while the other n-p eigenvalues will approach the zeros of $\det(sI-A)\det C(sI-A)^{-1}B$ as ε goes to zero. Of course, this can also be verified by direct computation.

The hypotheses of Cor. 4.8 are well-known to provide excellent circumstances for a study of high-gain feedback. In root-locus terms, they mean that there are only first-order asymptotic root loci, and that the finite termination points are all stable. Recently, these hypotheses have turned up in a study of robust controller design via LQG techniques [17] and an investigation into controller design for largely unknown systems [18].

For single-input-single-output systems, it is quite easy to prove Cor. 4.8 by a root-locus argument. Indeed, in this case there is only one pole that goes off to infinity as the gain is increased, and so stability can be guaranteed by selecting the right sign of the gain. The modern multivariable root-locus techniques (see, e.g., [19]) allow this argument to be extended to the multi-input-multi-output case. However, it seems that the following are advantages of the 'geometric' approach that has been developed here. First, we have essentially broght down the amount of asymptotic analysis involved in obtaining the result to a mere application of some standard continuity arguments from matrix theory. Second, our result is general in the sense that it is not tied up with any specific asymptotic pole pattern. Third, the result appears here as a corollary of a more general theorem (Thm. 4.5). The last point is perhaps the most important: it suggests that the methods presented here might also be helpful in studying more difficult situations (higher-order asymptotics, non-minimum-phase systems). Continued research will have to show if this is indeed true.

5. Conclusions

The purpose of this paper has been to introduce the concept of "almost stabilizability subspaces". We gave a number of equivalent characterizations of this class of subspaces, and linked it to the class of stable but not necessarily proper transfer functions. We established the important fact that almost stabilizability subspaces can be viewed as limits of regular stabilizability subspaces. Several applications were discussed, and special emphasis has been placed on the role that almost stabilizability subspaces can play in the study of high gain feedback. The results that we obtain suggest that we might have a way here to develop a general theory, which escapes the one-parameter framework that is so often characteristic both for root-locus and for LQG techniques. However, our results are only preliminary, and much work in this direction remains to be done.

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