A cohomological interpretation of the scalar product on the elliptic class functions

by

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Abstract

Let G be a connected reductive p-adic group with compact center and let C be the set of regular elliptic conjugacy classes. There is a unique measure dc on C such that for any $f \in \mathcal{H}(G) = \mathbf{C}_c^\infty(G)$ with support in the set G^e of regular elliptic elements G, we have $\int_G f(g)dg = \int_C \check{f}(c)dc$ where $\check{f}$ is the function on C given by $\check{f}(x) = \int_G f(gxg^{-1})dg$ for $x \in G^e$. Kazhdan conjectured that for representations π and τ of G of finite length, $\int_C \Theta_\tau(c)\Theta_\pi(c^{-1})dc$ is equal to the Euler-Poincaré chracteristics of π and τ, where Θ_π (and Θ_τ) is the character of π (and τ). On G^e, Θ_π is a finite linear combination of characters Θ_{π_i} of irreducible tempered representations π_i. Each π_i is a direct summand of the representation induced from an irreducible square-integrable representation σ_i of a Levi subgroup. In this paper we prove the conjecture when all σ_i are cuspidal.

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1 Introduction

Let F be a nonarchimedean local field of characteristic zero and let G be a connected reductive F-group with compact center. If π is a representation of G of finite length, then it is known that the distributional character Θ_π is a locally integrable function on G which is locally constant on the set of regular elements [10]. Let C be the set of regular elliptic conjugacy classes and let $\mathcal{H}$ be the Hecke algebra of G. There is a unique measure dc on C such that for any $f \in \mathcal{H}$ with support in the set G^e of regular elliptic elements of G, $\int_G f(g)dg = \int_C \check{f}(c)dc$ where $\check{f}$ is the function on C given by $\check{f}(x) = \int_G f(gxg^{-1})dg$ for $x \in G^e$. Let τ be another representation of G of finite length. Kazhdan conjectured that

$$\int_C \Theta_\tau(c)\Theta_\pi(c^{-1})dc = \sum_{j=0}^{\infty}(-1)^j \dim\ \mathrm{Ext}^j_G(\pi,\tau) \tag{1}$$

This is a generalization of the fact that irreducible characters of a finite (or compact) group are orthonormal. Really, if G is compact, then $G^e = G$ and every G-module is projective since all representations of G are completely reducible. So the above equation is a direct result of the orthonormality of irreducible characters.

The lefthand side of the above equation is the scalar product which Kazhdan has defined in [11]. He showed that

$$<\tau,\pi>= \int_C \Theta_\tau(c)\Theta_\pi(c^{-1})dg$$

defines a non-degenerate scalar product on $\bar{R}(G) = R(G)/R_I(G)$, the Grothendieck group of G-modules of finite length modulo induced representations. (See §2.1 below for more precise definition.) Let $A(G)$ be the set of $f \in \mathcal{H}$ whose orbital integral over any non-elliptic regular conjugacy class vanishes. He proved this by showing that there exists an isomorphism $\phi : \bar{R}(G) \xrightarrow{\sim} \bar{A}(G) = A(G)/[\mathcal{H},\mathcal{H}]$ which respects the action of the Bernstein center. Then $< \tau, \pi >$ is just the natural trace pairing between τ and $\phi(\pi)$. The map $\phi : \bar{R}(G) \rightarrow \bar{A}(G)$ is an explicit realization of the well-known but rather vague philosophy that there is a duality between representations and conjugacy classes. We prove Equation 1 by establishing a cohomological interpretation of this map $\phi : \bar{R}(G) \rightarrow \bar{A}(G)$ as follows. *If π is a G-module of finite length, then $\phi(\pi)$ is the "rank" of its dual $\pi^\vee$.* Here the "rank" means the rank of $\pi^\vee$ as a module over $\mathcal{H}$ [4]. A precise definition of the rank of a G-module and the implication of Equation 1 from this statement are given in §2.2.

From the Langlands classification theorem for p-adic groups, it follows that $\bar{R}(G)$ is spanned by irreducible tempered representations. And it is not difficult to show that the righthand side of Equation 1, which is the Euler-Poincaré characteristics, defines a bilinear form on $\bar{R}(G) \times \bar{R}(G)$. So it is enough to prove Equation 1 when τ and π are irreducible tempered representations. Any irreducible tempered representation π of G is a direct summand of $i_{GM}(\sigma_\pi)$, the representation induced from an irreducible square-integrable representation σ_π of a standard Levi subgroup M. We will prove the conjecture when σ_π is cuspidal.

2 The scalar product

2.1 The scalar product

In this subsection we briefly review a part of [11]. Let G be a reductive p-adic group with compact center as before and let $\mathcal{H}$ be the Hecke algebra of G (the space of locally constant functions on G with compact support). Then the category of smooth representations of G is equivalent to the category $\mathcal{H}(G)^\wedge$ of non-degenerate $\mathcal{H}$-modules. Let $\mathcal{C}$ be the center of this category [4]. The center of an abelian category is the ring of endomorphisms of the identity functor. For example, the center of the category of modules over a ring with identity is just the center of the ring. In our case $\mathcal{C}$ is the ring of endomorphisms of $\mathcal{H}$ which commute with both left and right actions of G on $\mathcal{H}$. This is the set of invariant distributions z on G such that $z * \mathcal{H} \subset \mathcal{H}$.

Let $R_Z(G)$ be the Grothendieck group of representations of G of finite length and let $R(G) = R_Z(G) \otimes \mathbf{C}$. The natural bilinear form $\mathrm{Irr}\ (G) \times \mathcal{H} \to \mathbf{C}$ given by $< \pi, h >=$ trace $\pi(h^*)$ (where Irr(G) is the set of equivalence classes of irreducible representations of G and $h^*(g) = h(g^{-1})$) induces a bilinear form $<,>: R(G) \times \mathcal{H} \to \mathbf{C}$ which respects the action of $\mathcal{C}$ (i.e. $< z \cdot \pi, h >=< \pi, z \cdot h >$, for $z \in \mathcal{C}, \pi \in R(G), h \in \mathcal{H}$). The kernel of this pairing in $\mathcal{H}$ is the set of $f \in \mathcal{H}$ whose orbital integral over any regular conjugacy class in G vanishes [11, Theorem 0], and is equal to $[\mathcal{H}, \mathcal{H}]$. Let $A(G)$ be the set of $f \in \mathcal{H}$ such that the orbital integral of f over any non-elliptic regular conjugacy class in G vanishes, and let $R_I(G)$ be the subspace of $R(G)$ generated by representations which are (parabolically) induced from representations of finite length of proper Levi subgroups. Then $A(G) = \{f \in \mathcal{H} | < \Pi, f >= 0, \forall \Pi \in R_I(G)\}$ and $R_I(G) = \{\Pi \in R(G) | < \Pi, f >= 0, \forall f \in A(G)\}$ [11, Theorem A]. Since $R_I(G)$ is an $\mathcal{C}$ -submodule of $R(G)$, $A(G)$ is also a $\mathcal{C}$ -submodule of $\mathcal{H}$. Let $\bar{R}(G) = R(G)/R_I(G), \bar{A} = A(G)/[\mathcal{H}, \mathcal{H}]$, we get a non-degenerate bilinear form $<,>: \bar{R}(G) \times \bar{A}(G) \to \mathbf{C}$ which respects the actions of $\mathcal{C}$.

There exists an isomorphism of $\mathcal{C}$ -modules $\phi : \bar{R}(G) \to \bar{A}(G)$ such that for any irreducible representation π of G, $\int_G \phi(\pi)(gxg^{-1})dg = \Theta_\pi(x), \forall x \in G^e$ [11, Theorem E]. (Caution: An element $\bar{f}$ of $\bar{A}(G)$ is not a function, so the value of $\bar{f}$ at a point of G doesn't make sense. But still its orbital integrals over elliptic regular conjugacy classes are well-defined.) Hence $< \pi_1, \pi_2 > \stackrel{\mathrm{def}}{=} < \pi_1, \phi(\pi_2) >$ defines a non-degenerate scalar product on $\bar{R}(G)$ such that $< z \cdot \pi_1, \pi_2 >=< \pi_1, z \cdot \pi_2 >$, for $z \in \mathcal{C}$.

From the Weyl integration formula we can see easily that

$$< \pi_1, \pi_2 >= \int_C \Theta_{\pi_1}(c)\Theta_{\pi_2}(c^{-1})dc$$

2.2 Ranks and Euler-Poincaré characteristics of G -modules

Let A be a ring with an identity element and let P be a finitely generated projective (left) A-module. We have a canonical isomorphism $\mathrm{End}_A(P) \cong P^* \otimes_A P$. If $u = \sum_i x_i^* \otimes x_i \in \mathrm{End}_A(P)$, $\sum x_i^*(x_i)$ is a well-defined element of $A/[A, A]$, called the *trace* of u and denoted by $r_P(u) = r(P, u)$ [3]. The *rank* of P, denoted by $r_P = r(P)$, is defined to be $r(P, \mathrm{id}_P)$. The trace map $r_P : \mathrm{End}_A(P) \to A/[A, A]$ satisfies 1)additivity; $r_{P\oplus Q}(u \oplus v) = r_P(u) + r_Q(v)$, 2)linearity; $r_P(u+v) = r_P(u)+r_P(v)$, 3)commutativity; $r_P(vu) = r_Q(uv)$ if $u : P \to Q$ and $v : Q \to P$, and 4) is universal with respect to above properties in obvious sense. Let M be an A-module of type (FP) (i.e. it has a finite resolution by finitely generated projective modules). Let $0 \to P_n \to \cdots \to P_0 \to M \to 0$ be such a resolution. If $u \in \mathrm{End}_A(M)$, then it extends to an endomorphism (u_i) of the complex. We define $r_M(u) = r(M, u)$ to be $\sum_{i=0}^n (-1)^i r(P_i, u_i)$. $r(M, u)$ is well-defined and satisfies the analogous properties as r_P [3, Section 2]. (In this case, additivity means $r(M, u) = r(M', u') + r(M'', u'')$ for an exact sequence $0 \to (M', u') \to (M, u) \to (M'', u'') \to 0$ of modules of type (FP) with endomorphisms.) Suppose A is an algebra over a field k and let N be an A-module which is a finite dimensional k-vector space. For $a \in A$, the multiplication by a defines a k-linear endomorphism a_N of N. $a \mapsto \mathrm{Trace}(a_N)$ induces a k-linear map $\chi_N : A/[A, A] \to k$. If P is a finitely generated projective A-module and $u \in \mathrm{End}_A(P)$, then $\mathrm{Hom}_A(P, N)$ is a finite dimensional k-vector space and $\chi_N(r(P, u)) = \mathrm{trace}(\mathrm{Hom}_A(P, N); u^*)$ where u^* is the map induced by u [3, Proposition 4.2].

This shows that if M is an A-module of type (FP), then all $\mathrm{Ext}^i_A(M,N)$ are finite dimensional and

$$\chi_N(r(M,u)) = \sum_{i=0}^{\infty}(-1)^i \mathrm{trace}(\mathrm{Ext}^i_A(M,N); u^*)$$

In particular, $\chi_N(r(M)) = \sum_{i=0}^{\infty}(-1)^i \dim_k \mathrm{Ext}^i_A(M,N)$.

If M is an A-module of type (FP), then any direct summand M_0 of M is also of type (FP) and $r(M_o) = r(M, p_{M_0})$, where $p_{M_0} : M \to M$ is the projection onto M_0.

We fix a minimal parabolic subgroup P_0 of G, a maximal split torus A_0 in P_0 and a good maximal compact subgroup K_0 of G once for all. Let K be a congruence subgroup of K_0. Then $\mathcal{H}(G)^\wedge_K$, the category of G-modules which are generated by K-fixed vectors is a direct summand of $\mathcal{H}(G)^\wedge$ and is equivalent to the category of modules over $\mathcal{H}_K(G)$, the Hecke algebra of G with respect to K [4, Section 2].

Let π be a representation of G of finite length. Then π has a finite resolution by finitely generated projective G-modules [14, Proposition 37]. Let $0 \to P_n \to \cdots \to P_0 \to \pi \to 0$ be such a resolution. Choose a small congruence subgroup K such that π and P_i are generated by K-fixed vectors. Then $0 \to P_n^K \to \cdots \to P_0^K \to \pi^K \to 0$ is a resolution of the $\mathcal{H}_K(G)$-module π^K by finitely generated projecitive $\mathcal{H}_K(G)$-modules. We define the *rank* of π, denoted by $r_\pi = r(\pi)$, to be the rank of π^K as an $\mathcal{H}_K(G)$-module. The natural map $\mathcal{H}_K/[\mathcal{H}_K, \mathcal{H}_K] \to \mathcal{H}/[\mathcal{H}, \mathcal{H}]$ is injective [12, Theorem B]. We will consider $r(\pi)$ as an element of $\mathcal{H}/[\mathcal{H}, \mathcal{H}]$.

Proposition 1 *r_π does not depend on the choice of K.*

Proof. Shown in the remarks following Proposition 38 in [14].

Proposition 2 *Let π be a G-module of finite length. Then 1) $r_\pi \in \bar{A}(G) = A(G)/[\mathcal{H}, \mathcal{H}]$ and 2) $r_\pi = 0$ if π is a representation induced from a proper Levi subgroup of G.*

Proof. 1) Write $[\tau, h]$ for $\mathrm{tr}(\tau(h))$ where τ is an admissible representation of G and $h \in \mathcal{H}$. By [11, Theorem A], it's enough to show $[i_{GM}(\sigma), r_\pi] = 0$ for all proper standard Levi subgroup M of G and $\sigma \in \mathrm{Irr}\ M$, where i_{GM} is the usual unitary induction functor [9]. Fix (M, σ) and let $\Psi(M)$ be the group of unramified characters of M. $\Psi(M)$ has a natural structure of complex algebraic torus and for $\forall h \in \mathcal{H}, \psi \mapsto [\tau_\psi, h]$ defines a regular function on $\Psi(M)$, where $\tau_\psi = i_{GM}(\psi\sigma)$. On the other hand, from the properties of ranks of G-modules stated above $[\tau_\psi, r_\pi] = \sum_{i=0}^{\infty}(-1)^i \dim \mathrm{Ext}^i_G(\pi, \tau_\psi) \in \mathbf{Z}$. So $\psi \mapsto [\tau_\psi, r_\pi]$ is a constant function on $\Psi(M)$. We will show that $[\tau_\psi, r_\pi] = 0$ for some $\psi \in \Psi(M)$ and this will prove the first part of the proposition.

Each $z \in \mathcal{C}$ acts on τ_ψ by a scalar, say, $z(\tau_\psi)$ [4, Proposition 2.11]. Let h_z be the function on $\Psi(M)$ given by $h_z(\psi) = z(\tau_\psi)$. Then $z \mapsto h_z$ defines a ring homomorphism from $\mathcal{C}$ to the ring of regular functions on $\Psi(M)$. The image of this map, which is described in [4, Section 2], contains a non-constant function. In particular, there exist $z \in \mathcal{C}$ and $\psi \in \Psi(M)$ such that $z(\pi) \neq z(\tau_\psi)$. Now from a general fact, the actions of z on $\mathrm{Ext}^i_G(\pi, \tau_\psi)$ induced from the actions of z on the first and the second variables are the same and equal to multiplications by $z(\pi)$ and $z(\tau_\psi)$, respectively. This is a contradiction unless $\mathrm{Ext}^i_G(\pi, \tau_\psi) = 0$. Hence $\mathrm{tr}(\tau_\psi(r_\pi)) = 0$.

2) Now suppose $\pi = i_{GN}(\rho)$ where N is a proper Levi subgroup of G and $\rho \in \mathrm{Irr}N$. To prove $r_\pi = 0$, it's enough to show $[\tau, r_\pi] = 0$ for any irreducible representaion τ of G [11, Theorem 0]. Let $\pi_\psi = i_{GN}(\psi\rho)$ for $\psi \in \Psi(N)$. We claim that $\psi \mapsto [\tau, r(\pi_\psi)]$ is a regular function on $\Psi(N)$. Once this is proven, the same argument as above shows $[\tau, r(\pi_\psi)] = 0$.

In [5], Bernstein showed that i_{GN} is left adjoint to an exact functor $\bar{r}_{NG}$, which is the *restriction along the opposite parabolic subgroup* (see Proposition 4 and the remark before it). From this fact, we can see easily that $\mathrm{Ext}^j_G(i_{GN}(\rho), \tau) = \mathrm{Ext}^j_N(\rho, \bar{r}_{NG}(\tau))$ for all j. So $[\tau, r(\pi_\psi)] = \sum_j(-1)^j \mathrm{Ext}^j_G(i_{GN}(\psi\rho), \tau) = \sum_j(-1)^j \mathrm{Ext}^j_N(\psi\rho, \bar{r}_{NG}(\tau)) = [\bar{r}_{NG}(\tau), r(\psi\rho)]_N$, where $r(\psi\rho)$ is the rank of N-module $\psi\rho$ and $[,]_N$ is the trace pairing on N. We claim that $r(\psi\rho) = \psi^{-1}r(\rho)$. It's enough to show that if ρ is a projective N-module then $\psi\rho$ is also projective and $r(\psi\rho) = \psi^{-1}r(\rho)$. Clearly, $\mathrm{Hom}_N(\psi\rho, \tau) = \mathrm{Hom}_N(\rho, \psi^{-1}\tau)$. So $\psi\rho$ is projective. $r(\psi\rho)$ is uniquely determined by the property $[\tau, r(\psi\rho)] = \dim \mathrm{Hom}_N(\psi\rho, \tau)$ for all irreducible N-modules. $\dim \mathrm{Hom}(\psi\rho, \tau) = \dim \mathrm{Hom}(\rho, \psi^{-1}\tau) = [\psi^{-1}\tau, r(\rho)] = [\tau, \psi^{-1}r(\rho)]$. So $r(\psi\rho) = \psi^{-1}r(\rho)$ and $\psi \mapsto [\bar{r}_{NG}(\tau), \psi^{-1}r(\rho)]_N$ is clearly a regular function on $\Psi(N)$. Q.E.D.

For $\pi, \tau \in \mathrm{Irr}G$, put $\chi_G(\tau, \pi) = \sum_j (-1)^j \dim \mathrm{Ext}^j_G(\pi, \tau)$ the Euler-Poincaré characteristics of π and τ. It follows from the long exact sequence for Ext that $\chi_G(\cdot, \cdot)$ can be extended to a bilinear form on $R(G) \times R(G)$.

Corollary 1 $\chi_G(\tau, \pi) = 0$ *if either* τ *or* π *is induced from a representation of a proper Levi subgroup.*

Hence, $\chi_G(\cdot, \cdot)$ defines a bilinear form on $\bar{R}(G) \times \bar{R}(G)$. It follows from the Langlands classification theorem for p-adic groups that $\bar{R}(G)$ is spanned by classes of tempered representations [11, Proposition 1.1]. So to prove $<, > = \chi_G(,)$ on $\bar{R}(G)$, it's enough to show $< \tau, \pi > = \chi_G(\tau, \pi)$ when τ and π are irreducible tempered representations. An irreducible tempered representation is a direct summand of $i_{GM}(\sigma)$ where σ is an irreducible square integrable representation of M. Let $\bar{R}(G)'$ be the subspace of $\bar{R}(G)$ generated by irreducible tempered representations that are summands of $i_{GM}(\sigma)$ where M and σ run over all standard Levi subgroups and all irreducible unitary cuspidal representations of them, respectively. In this paper we will prove the following theorem.

Theorem 1 *Suppose* G *is connected, then* $< \tau, \pi > = \chi_G(\tau, \pi)$ *for* $\tau \in \bar{R}(G)$ *and* $\pi \in \bar{R}(G)'$.

On the other hand, from properties of ranks of G-modules, we can see the following

Lemma 1 *Let $\phi : \bar{R}(G) \to \bar{A}(G)$ be the isomorphism stated in Section 2.1 and let π be a representation of finite length. If $\phi(\pi) = r_\pi^*$ then $< \tau, \pi >= \chi_G(\tau, \pi)$ for all $\tau \in \bar{R}(G)$.*

In the following we will show $\phi(\pi) = r_\pi^*$ if $\pi \in \bar{R}(G)'$.

Remark. If π is cuspidal, then this is aleady known. Let v be in the space of π such that $(v, v) = 1$ and let $h(x) = d_\pi^{-1}(\pi(x^{-1})v, v) \in \mathcal{H}$, where d_π is the formal degree of π. Then it is not difficult to show that 1) $< \pi', h >= \delta_{\pi,\pi'}$ for any irreducible tempered representation π' of G and 2) $\check{h}(x) = \theta_\pi(x^{-1})$ for all $x \in G^e$ [11, Proposition 5.3]. Since π is a projective G-module (see the remark following Proposition 3 below), 1) implies that $h \in A(G)$ and that its image $\bar{h}$ in $\bar{A}(G)$ is the rank of π.

3 The local trace formula

In this section we review a part of [2]. From now on we assume that G is connected.

3.1 R-groups

Recall that we have fixed a minimal parabolic subgroup P_0 of G and a maximal split torus A of G in P_0. Define standard Levi subgroups of G in the usual way. Let $\mathcal{L}$ be the set of standard Levi subgroups and let $\Pi_2(M)$ be the set of equivalence classes of irreducible square integrable representations of M. Let $M \in \mathcal{L}$ and $\sigma \in \Pi_2(M)$. Consider $W_\sigma = \{w \in W^M | \sigma^w \cong \sigma\}$ where $W^M = N_G(A_M)/M$ is the Weyl group of G with respect to A_M, the split component of the center of M. For each $w \in W_\sigma$ fix an isomorphism $\alpha_w : \sigma^w \xrightarrow{\sim} \sigma$. Then the normalized operator $I(w, \sigma)$: $i_{GM}(\sigma) \to i_{GM}(\sigma)$ is given by $I(w, \sigma) = \alpha_w \lambda(w) R_{w^{-1}Pw|P}(\sigma)$ where $R_{w^{-1}Pw|P}(\sigma) : i_{GP}(\sigma) \to i_{Gw^{-1}Pw}(\sigma)$ is the normalized intertwining operator given in [1] and λ is the left translation.

Lemma 2 *There exist normalizing factors such that the normalized intertwining operators* $I(w,\sigma)$ *satisfy the following conditions.*

1. *Let* L *be a standard Levi subgroup containing* M *and let* $w \in W_\sigma \cap W_L^M$ *where* $W_L^M = N_L(A_M)/M$ *is the Weyl group of* L *with respect to* A_M*. Then* $I(w,\sigma) = i_{GL}(I_L(w,\sigma))$ *where* $I_L(w,\sigma)$ *is the normalized intertwining operator of* $i_{LM}(\sigma)$*.*

2. $I(w,\sigma^\vee) = (I(w,\sigma)^{-1})^\vee$

Proof. This follows directly from [1, Theorem 2.1]

Let $W_\sigma^o = \{w \in W_\sigma | I(w,\sigma)$ is a scalar.$\}$ then W_σ^o is a normal subgroup of W_σ and $R_\sigma = W_\sigma/W_\sigma^o$ can be identified with a subgroup of W_σ so that W_σ is the semi-direct product of W_σ^o by R_σ. R_σ is called the R-group of σ. $w \mapsto I(w,\sigma)$ a projective representation of R_σ. And we can always find a finite central extension $1 \to Z_\sigma \to \tilde{R}_\sigma \to R_\sigma \to 1$, a fuction $\xi_\sigma : \tilde{R}_\sigma \to \mathbf{C}^*$ and a character χ_σ of Z_σ such that 1) $\xi_\sigma(zw) = \chi_\sigma(z)\xi_\sigma(w)$, for $z \in Z_\sigma, w \in \tilde{R}_\sigma$ and 2) $w \mapsto \tilde{I}(w,\sigma) = \xi_\sigma(w)^{-1}I(w,\sigma)$ is a representation of $\tilde{R}_\sigma$ [2, Section 2]. There is a bijection $\rho \leftrightarrow \pi_\rho$ between the set $\Pi(\tilde{R}_\sigma, \chi_\sigma)$ of the irreducible representations ρ of $\tilde{R}_\sigma$ such that $\rho|_{Z_\sigma} = \chi_\sigma$ and inequivalent irreducible components of $i_{GM}(\sigma)$. More precisely, the natural representation $\mathcal{R}$ of $\tilde{R}_\sigma \times G$ on the space of $i_{GM}(\sigma)$ decomposes as

$$\mathcal{R} = \bigoplus_{\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} (\rho^\vee \otimes \pi_\rho)$$

and each π_ρ is irreducible and distinct [*loc.cit.*]. This shows in particular, for $w \in \tilde{R}_\sigma, f \in \mathcal{H}$ we have

$$\mathrm{tr}(\tilde{I}(w,\sigma)i_{GM}(\sigma)(f)) = \sum_{\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \mathrm{tr}(\rho^\vee(w))\mathrm{tr}(\pi_\rho(f)) \tag{2}$$

3.2 Trace Paley-Wiener theorem

Consider the set of triplets $\tau = (M, \sigma, w)$ with $M \in \mathcal{L}, \sigma \in \Pi_2(M), w \in \tilde{R}_\sigma$. The Weyl group $W = W_G$ of G with respect to A acts on this set naturally. For each $\tau = (M, \sigma, w)$, we define a distribution $\Theta(\tau)$ on G by $\Theta(\tau, f) = \mathrm{tr}(\tilde{I}(w, \sigma) i_{GM}(\sigma)(f))$. Clearly, these distributions are invariant under the action of W and satisfy $\Theta(z\tau, f) = \chi_\sigma(z)^{-1}\Theta(\tau, f), z \in Z_\sigma$ where $z\tau = (M, \sigma, zw)$. Let $T'(G)$ be the set of triplets $\tau = (M, \sigma, w)$ such that $\chi_\sigma \equiv 1$ on $\{z \in Z_\sigma | zw$ and w are conjugated in $\tilde{R}_\sigma\}$. These triplets are said to be *essential.* (If τ isn't essential, then $\Theta(\tau) \equiv 0$ by the above argument.) Let $\mathbf{a} = \mathrm{Hom}(X(A), \mathbf{R})$ be the real Lie algebra of A and let $\tilde{R}_{\sigma,reg}$ be the set of $t \in \tilde{R}_\sigma$ such that the subspace $\mathbf{a}^w$ of $\mathbf{a}$ fixed under the action of w is (0). Define $T(G)$ to be the set of W-orbits in $T'(G)$ and let $T_{ell}(G) = \{\tau = (M, \sigma, w) \in T(G) | w \in \tilde{R}_{\sigma,reg}\}$

$T'(G)$ has a natural structure of analytic manifold, whish is isomorphic to a disjoint union of compact tori [2, Section 3]. $T(G)$ is then a quotient space of $T'(G)$ and $T_{ell}(G)$ is the union of connected components of the minimal dimension, which is zero in our case. Let ϕ be a fuction on $T'(G)$ such that 1) ϕ is supported on only finitely many components, 2) $\phi(\tau^s) = \phi(\tau), s \in W$, i.e. ϕ is a function on $T(G)$, 3) $\phi(z\tau) = \chi_\tau(z)^{-1}\phi(\tau), z \in Z_\sigma$, and 4) on each connected component of $T'(G)$, ϕ is a Paley-Wiener function. Then the trace Paley-Wiener theorem [7] says there exists $f \in \mathcal{H}$ such that $\phi(\tau) = \Theta(\tau, f)$ for all $\tau \in T'(G)$. Note that such an f is unique modulo $[\mathcal{H}, \mathcal{H}]$ and if ϕ is supported on $T_{ell}(G)$, then $f \in A(G)$ [11, Theorem 0 and A].

Remark. The trace Paley-Wiener theorem in [7] is not stated in this form and I could not find a proof of the above statement which appears in [2, Section 3]. But the proof is quite simple if ϕ is supported on $T_{ell}(G)$ and this is the only case we need. Let $R_t(G) \subset R(G)$ be the subspace spannedd by classes of tempered representations of G and let $R_{t,I}(G)$ be the subspace generated by $i_{GL}(\sigma), L \in$

$\mathcal{L}, L \neq G, \sigma \in \Pi_{temp}(L)$. By inverting the formula 2 as $\phi(\pi_\rho) = |\tilde{R}_\sigma|^{-1} \sum_{r \in \tilde{R}_\sigma} \mathrm{tr}(\rho(r))\phi(\tau_r)$ where $\tau_r = (M, \sigma, r)$, we get a linear form $\phi : R_t(G) \to \mathbf{C}$. If ϕ is supported on $T_{ell}(G)$ then $\phi \equiv 0$ on $R_{t,I}(G)$. (See the arguments in the proof of Lemma 3 below.) So ϕ defines a linear form on $\bar{R}_t(G) = R_t(G)/R_{t,I}(G)$. Since the natural map $R_t(G)/R_{t,I}(G) \to \bar{R}(G)$ is an isomorphism [11, Proposition 1.1], the linear form $\phi : R_t(G) \to \mathbf{C}$ can be extended uniquely to a linear form $\phi : R(G) \to \mathbf{C}$ such that $\phi \equiv 0$ on $R_I(G)$. Now the trace Paley-Wiener theorem in [7] says that ϕ comes from an element f of $\mathcal{H}$.

3.3 The local trace formula

We define a measure on $T_{ell}(G)$ by

$$\int_{T_{ell}(G)} \theta(\tau) d\tau = \sum_{\tau=(M,\sigma,w)\in T_{ell}(G)} |\tilde{R}_{\sigma,w}|^{-1}\theta(\tau)$$

where $\tilde{R}_{\sigma,w}$ is the centralizer of w in $\tilde{R}_\sigma$.

For $\tau = (M, \sigma, w) \in T(G)$, let $\tau^\vee = (M, \sigma^\vee, w)$. Let θ be a function on $T(G)$ supported on $T_{ell}(G)$ such that the function $\tau \to \theta(\tau^\vee)$ satisfies the conditions 1) - 4) in the previous subsection (the fourth condition is empty in this case). To θ, we can associate a distribution Θ on G by $\Theta(f) = \int_{T_{ell}(G)} \theta(\tau)\Theta(\tau, f)d\tau$. Θ is a finite linear combination of tempered chracters, hence is a locally integrable fuction on G. Let $\mathbf{a}_M$ be the real Lie algebra of M and let θ' be the function on $T(G)$ given by $\theta'(\tau) = |d(w)|\theta(\tau^\vee)$ for $\tau = (M, \sigma, w)$, where $d(w) = \det(1 - w)|_{\mathbf{a}_M}$. Then θ' also satisfies the conditions 1) - 4) above, hence there exists $f \in \mathcal{H}$ such that $\Theta(\tau, f) = \theta'(\tau)$ for all

$\tau \in T(G)$. Now it was shown in the proof of Theorem 6.1 in [2] that

$$\int_G f(gxg^{-1})dg = \Theta(x), \text{ for all } x \in G^e$$

4 Reduction of the theorem

We fix $M \in \mathcal{L}$ and $\sigma \in \Pi_2(M)$ such that $\tilde{R}_{\sigma,reg} \neq \emptyset$. We also fix an irreducible representation $\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)$ of $\tilde{R}_\sigma$. To ρ, there corresponds an irreducible component π_ρ of $i_{GM}(\sigma)$ as before.

4.1 Rank and character of π_ρ

Consider the decompositon $\mathcal{R} = \bigoplus_{\lambda \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \lambda^\vee \otimes \pi_\lambda$ of the representation of $\tilde{R}_\sigma \times G$ on the space of $i_{GM}(\sigma)$. The projection map onto the $\rho^\vee$-component is $p_\rho = \sum_{w \in \tilde{R}_\sigma} |\tilde{R}_\sigma|^{-1} \dim(\rho) \mathrm{tr}(\rho(w)) \tilde{I}(w, \sigma)$. Since the multiplicity of π_ρ in $i_{GM}(\sigma)$ is $\dim(\rho)$, the rank of π_ρ is

$$\begin{aligned} r(\pi_\rho) &= \dim(\rho)^{-1} r(i_{GM}(\sigma), p_\rho) \\ &= \sum_{w \in \tilde{R}_\sigma} |\tilde{R}_\sigma|^{-1} \mathrm{tr}(\rho(w)) r(i_{GM}(\sigma), \tilde{I}(w, \sigma)) \end{aligned}$$

Lemma 3 *If $w \notin \tilde{R}_{\sigma,reg}$, then $r(i_{GM}(\sigma), \tilde{I}(w, \sigma)) = 0$.*

Proof. We need more facts about R-groups. Let $\mathbf{a} = \mathrm{Hom}(X(A), \mathbf{R})$ be the real Lie algebra of A. For each $M \in \mathcal{L}$, there is a natural embedding $\mathbf{a}_M = \mathrm{Hom}(X(A_M), \mathbf{R}) \hookrightarrow \mathbf{a}$ [2, Section 1]. For any $w \in \tilde{R}_\sigma$, the fixed subspace $\mathbf{a}_M^w$ of $\mathbf{a}_M$ under w is of the form $\mathbf{a}_L$ for some Levi subgroup $L \in \mathcal{L}$ containing M [2, Section 2]. Let $R_\sigma^L = W_L^M \cap R_\sigma$, where $W_L^M = N_L(A_M)/M$ is the Weyl group of L with respect to A_M. Then R_σ^L is the R-group of σ relative to L [*loc.cit.*]. Put $\tilde{R}_\sigma^L$ be the inverse image of R_σ^L in $\tilde{R}_\sigma$. Then as in the case of G, we have the decompositon $\mathcal{R}_L = \bigoplus_{\nu \in \Pi(\tilde{R}_\sigma^L, \chi_\sigma)} \nu^\vee \times \pi_\nu$

of the representation of $\tilde{R}^L_\sigma \times L$ on the space of $i_{LM}(\sigma)$. Since the normalization factors are chosen such that for $t \in R^L_\sigma \subset R_\sigma, I(t,\sigma) = i_{GL}(I_L(t,\sigma)) : i_{GL}(i_{LM}(\sigma)) \to i_{GL}(i_{LM}(\sigma))$, we see that $\mathcal{R} = i_{GL}(\mathcal{R}_L)$, in other words we have the decompositon $\mathcal{R}|_{\tilde{R}^L_\sigma \times G} = \bigoplus_{\nu \in \Pi(\tilde{R}^L_\sigma, \chi_\sigma)} \nu^\vee \otimes i_{GL}(\pi_\nu)$. On the other hand, it's easy to prove the following

Lemma 4 *Let $\alpha : S \to GL(V)$ be a finite dimensional representation of a finite group S, and let E be a module of type (FP) over a* $\mathbf{C}$*-algebra. Then $r(V \otimes_{\mathbf{C}} E, \alpha(s)) = tr(\alpha(s))r(E)$.*

If $w \notin \tilde{R}_{\sigma,reg}$, then $\mathbf{a}^w_M = \mathbf{a}_L \neq (0)$. So $L \neq G, w \in \tilde{R}^L_\sigma$ and we have $r(i_{GM}(\sigma), \tilde{I}(w,\sigma)) = \sum_{\nu \in \Pi(\tilde{R}^L_\sigma, \chi_\sigma)} \mathrm{tr}(\nu^\vee(w)) r(i_{GL}(\pi_\nu)) = 0$ since the rank of an induced representation is zero by the Proposition 2. Q.E.D.

We have seen that

$$\begin{aligned} r(\pi_\rho) &= \sum_{w \in \tilde{R}_{\sigma,reg}} |\tilde{R}_\sigma|^{-1} \mathrm{tr}(\rho(w)) r(i_{GM}(\sigma), \tilde{I}(w,\sigma)) \\ &= \sum_{w \in \{\tilde{R}_{\sigma,reg}\}} |\tilde{R}_{\sigma,w}|^{-1} \mathrm{tr}(\rho(w)) r(i_{GM}(\sigma), \tilde{I}(w,\sigma)) \\ &= \sum_{w \in \{\tilde{R}_{\sigma,reg}\}} h_w \end{aligned}$$

where $\{\tilde{R}_{\sigma,reg}\}$ denotes the set of conjugacy classes in $\tilde{R}_\sigma$ and we have put

$h_w = |\tilde{R}_{\sigma,w}|^{-1} \mathrm{tr}(\rho(w^{-1})) r(i_{GM}(\sigma), \tilde{I}(w^{-1},\sigma))$ for each $w \in \{\tilde{R}_{\sigma,reg}\}$.

For $\tau = (N, \delta, t) \in T(G)$, we have

$$\begin{aligned} \Theta(\tau, h_w) &= \mathrm{tr}(\tilde{I}(t,\delta) i_{GN}(\delta)(h_w)) \\ &= \sum_{\lambda \in \Pi(\tilde{R}_\delta, \chi_\delta)} \mathrm{tr}(\lambda^\vee(t)) \mathrm{tr}(\pi_\lambda(h_w)) \\ &= \sum_{\lambda \in \Pi(\tilde{R}_\delta, \chi_\delta)} \mathrm{tr}(\lambda^\vee(t)) |\tilde{R}_{\sigma,w}|^{-1} \mathrm{tr}(\rho(w^{-1})) \sum_j (-1)^j \mathrm{tr}\left(\mathrm{Ext}^j_G(i_{GM}(\sigma), \pi_\lambda); \tilde{I}(w^{-1},\sigma)^*\right) \end{aligned}$$

On the other hand, it's easy to see

$$\begin{aligned} & \textstyle\sum_{\lambda\in\Pi(\tilde{R}_\delta,\chi_\delta)} \mathrm{tr}(\lambda^\vee(t)) \sum_j (-1)^j \mathrm{tr}\left(\mathrm{Ext}^j_G(i_{GM}(\sigma),\pi_\lambda); \tilde{I}(w,\sigma)^*\right) \\ = \quad & \textstyle\sum_j (-1)^j \mathrm{tr}\left(\mathrm{Ext}^j_G(i_{GM}(\sigma), i_{GN}(\delta)); \tilde{I}(w,\sigma)^* \tilde{I}(t,\delta)_*\right)\end{aligned}$$

where $\tilde{I}^*$ (resp. $\tilde{I}_*$) is the action on Ext induced by the action on the first (resp. the second) variable. We have proven the following

Lemma 5 *The rank $r(\pi_\rho)$ of π_ρ is equal to $\sum_{w\in\{\tilde{R}_{\sigma,reg}\}} h_w$ and $h_w \in \mathcal{H}/[\mathcal{H},\mathcal{H}]$ is determined uniquely by the following property. For any $\tau = (N,\delta,t) \in T(G)$,*

$$\Theta(\tau, h_w) = |\tilde{R}_{\sigma,w}|^{-1}\ tr(\rho(w^{-1})) \sum_j (-1)^j tr\left(Ext^j_G(i_{GM}(\sigma), i_{GN}(\delta)); \tilde{I}(w^{-1},\sigma)^* \tilde{I}(t,\delta)_*\right)$$

By inverting the equation 2 we get

$$\begin{aligned}\Theta_{\pi_\rho}(f) &= |\tilde{R}_\sigma|^{-1} \sum_{w\in\tilde{R}_\sigma} \mathrm{tr}(\rho(w))\mathrm{tr}(\tilde{I}(w,\sigma) i_{GM}(\sigma)(f)) \\ &= |\tilde{R}_\sigma|^{-1} \sum_{w\in\tilde{R}_\sigma} \mathrm{tr}(\rho(w))\Theta(\tau_w, f)\end{aligned}$$

for all $f \in \mathcal{H}$, where $\tau_w = (M,\sigma,w)$. If $w \notin \tilde{R}_{\sigma,reg}$ then $\Theta(\tau_w)$ is a linear combination of induced characters as shown in the proof of Lemma 3, hence $\Theta(\tau_w)|_{G^e} \equiv 0$. So

$$\begin{aligned}\Theta_{\pi_\rho}|_{G^e} &= |\tilde{R}_\sigma|^{-1} \sum_{w\in\tilde{R}_{\sigma,reg}} \mathrm{tr}(\rho(w))\Theta(\tau_w) \\ &= \sum_{w\in\{\tilde{R}_{\sigma,reg}\}} |\tilde{R}_{\sigma,w}|^{-1}\mathrm{tr}(\rho(w))\Theta(\tau_w) \\ &= \int_{T_{ell}(G)} \theta(\tau)\Theta(\tau)d\tau\end{aligned}$$

where θ is the function on $T(G)$ defined by

$$\theta(N,\delta,t)=\begin{cases}\mathrm{tr}(\rho(t)) & \text{if } (N,\delta)=(M,\sigma) \text{ and } t\in\tilde{R}_{\delta,reg}\\ 0 & \text{otherwise}\end{cases}$$

In other words, on G^e, Θ_{π_ρ} is the distribution Θ corresponding to the function θ as in the previous section. Let θ' be the fuction on $T(G)$ associated to θ as before and let $f_{\pi_\rho}\in\mathcal{H}$ be such that $\Theta(\tau,f_{\pi_\rho})=\theta'(\tau),\forall\tau\in T(G)$. Then we have seen that $\int_G f_{\pi_\rho}(gxg^{-1})dg=\Theta_{\pi_\rho}(x),\forall x\in G^e$, so $\int_G f^*_{\pi_\rho}(gxg^{-1})dg=\Theta_{\pi_\rho}(x^{-1})$. We expect the image of $f^*_{\pi_\rho}$ in $\bar{A}(G)$ to be the rank of π_ρ.

4.2 Reduction of the theorem

Lemma 6 *Let $\tau=(N,\delta,t)\in T(G)$ and $h\in\mathcal{H}$. Then $\Theta(\tau,h^*)=\Theta(\tilde{\tau},h)$, where $\tilde{\tau}=(N,\delta^\vee,t^{-1})$*

Proof. We can choose normalizing factors, Z_δ and χ_δ in such a way that $\tilde{R}_{\delta^\vee}=\tilde{R}_\delta,\chi_{\delta^\vee}=\chi_\delta^{-1}$ and the representation $\mathcal{R}^\vee$ of $\tilde{R}_{\delta^\vee}\times G$ on $i_{GM}(\sigma^\vee)$ is the contragradient of the representation $\mathcal{R}$ of $\tilde{R}_\delta\times G$ on $i_{GM}(\sigma)$ [2, Section 3]. In other words, we have the decomposition $\mathcal{R}^\vee=\bigoplus_{\nu\in\Pi(\tilde{R}_\delta,\chi_\delta)}\nu\otimes\pi_\nu^\vee$. So

$$\begin{aligned}\Theta(\tau,h^*) &= \sum_{\nu\in\Pi(\tilde{R}_\delta,\chi_\delta)}\mathrm{tr}(\nu^\vee(t))\mathrm{tr}(\pi_\nu(h^*))\\ &= \sum_\nu \mathrm{tr}(\nu(t^{-1})\mathrm{tr}(\pi_\nu^\vee(h))\\ &= \Theta(\tilde{\tau},h)\end{aligned}$$

We have seen that $f^*_{\pi_\rho}$ is the function such that for $\tau=(N,\delta,t)\in T(G)$

$$\Theta(\tau,f^*_{\pi_\rho})=\begin{cases}|d(t)|\mathrm{tr}(\rho(t^{-1})) & \text{if } (N,\delta)=(M,\sigma) \text{ and } t\in\tilde{R}_{\delta,reg}\\ 0 & \text{otherwise}\end{cases}$$

And such an f_{π_ρ} is unique modulo $[\mathcal{H},\mathcal{H}]$. For each $w \in \{\tilde{R}_{\sigma,reg}\}$, let O_w be the Z_σ-orbit of w in $\{\tilde{R}_\sigma\}$ and let $f_w \in \mathcal{H}$ be such that

$$\Theta(\tau, f_w) = \begin{cases} |d(t)|\mathrm{tr}(\rho(t^{-1})) & \text{if } (N,\delta,t) \in \text{the } Z_\sigma\text{-orbit of } (M,\sigma,w) \text{ in } T(G) \\ 0 & \text{otherwise} \end{cases}$$

We know that $f_w \in A(G)$ and is unique modulo $[\mathcal{H},\mathcal{H}]$. Clearly, $f^*_{\pi_\rho} = \sum_{w\in\{\tilde{R}_{\sigma,reg}\}/Z_\sigma} f_w$ and $r(\pi_\rho) = \sum_{w\in\{\tilde{R}_{\sigma,reg}\}} h_w = \sum_{w\in\{\tilde{R}_{\sigma,reg}\}/Z_\sigma} |O_w| h_w$ since $h_{zw} = h_w$. We want to show that for each $w \in \{\tilde{R}_{\sigma,reg}\}/Z_\sigma$, the image of f_w in $\bar{A}(G)$ is $|O_w|h_w$. This will prove Theorem 1 for $\pi = \pi_\rho$. This is equivalent to say that $\Theta(\tau, f_w) = |O_w|\Theta(\tau, h_w)$ for all $\tau \in T(G)$. We have proved the following

Lemma 7 *The following statement implies that $\phi(\pi_\rho) = r(\pi_\rho)^*$ where $\phi : \bar{R}(G) \to \bar{A}(G)$ is the map stated in §2.1. For $\forall \tau = (N,\delta,t) \in T(G)$,*

$$tr(\rho(w^{-1})) \sum_j (-1)^j tr\left(Ext^j_G(i_{GM}(\sigma), i_{GN}(\delta)); \tilde{I}(w^{-1},\sigma)^* \tilde{I}(t,\delta)_*\right) \tag{3}$$

$$= \begin{cases} |\tilde{R}_{\sigma,w}||O_w|^{-1}|d(t)|tr(\rho(t^{-1})) & \text{if } (N,\delta) = (M,\sigma) \text{ and } t \in O_w \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

From now on, we will assume σ *is cuspidal* unless stated otherwise. First we have the following

Lemma 8 *Let (M,σ) be as before and let $N \in \mathcal{L}$ and $\delta \in \Pi_2(N)$. If (M,σ) and (N,δ) are not conjugated (by an element of W_0), Then*

$$Ext^j_G(i_{GM}(\sigma), i_{GN}(\delta)) = 0, \text{ for all } j \geq 0$$

Proof. There exists a standard Levi subgroup L contained in N and an irreducible cuspidal representation π of L such that δ is a subquotient of $i_{NL}(\pi)$. (L,π) is unique up to conjugation and is called *the infinitesimal character* of δ [7, Section 2.1]. We claim that (L,π) and (M,σ) are not conjugated. Suppose they are conjugated. Then replacing N by suitable conjugate of it, we may assume that N contains M properly and δ is a subquotient of $i_{NM}(\sigma)$. But the Plancherel formula says that $i_{NM}(\sigma)$ contains no square integrable irreducible component. ($i_{NM}(\sigma)$ belongs to the continuous spectrum.) The claim is proven. Any element of the center $\mathcal{C}$ acts by scalar on $i_{GM}(\sigma)$ and $i_{GN}(\delta)$ [4, Proposition 2.11]. And since the infinitesimal character (L,π) of $i_{GN}(\delta)$ is not conjugated to that of $i_{GM}(\sigma)$ (which is just (M,σ)), there exists $z \in \mathcal{C}$ such that $z|_{i_{GM}(\sigma)} \neq z|_{i_{GN}(\delta)}$. Now by the same argument as in the proof of Proposition 2, $\mathrm{Ext}^j_G(i_{GM}(\sigma), i_{GN}(\delta)) = 0, \forall j \geq 0$. Q.E.D.

We have seen that to veryfy the equation 4, we only need to check the case $(N,\delta) = (M,\sigma)$. Note if we replace t by $zt, z \in Z_\sigma$ then the both sides of the equation 4 are changed by $\chi_\delta(z)^{-1}$. Hence if the equation 4 is true for $\tau = (N,\delta,t)$ then it's true for $z\tau = (N,\delta,zt)$ also. We have proven that the following proposition implies $\phi(\pi_\rho) = r^*_{\pi_\rho}$ and hence Theorem 1 for the case $\pi = \pi_\rho$.

Proposition 3 *Let $M \in \mathcal{L}$ and let σ be an irreducible unitary cuspidal representation of M. Then*

$$\sum_j (-1)^j tr\left(Ext^j_G(i_{GM}(\sigma), i_{GM}(\sigma)); \tilde{I}(w^{-1},\sigma)^* \tilde{I}(w,\sigma)_*\right) = |d(w)||\tilde{R}_{\sigma,w}||O_w|^{-1}$$

and

$$\sum_j (-1)^j tr\left(Ext^j_G(i_{GM}(\sigma), i_{GM}(\sigma)); \tilde{I}(w^{-1},\sigma)^* \tilde{I}(t,\sigma)_*\right) = 0$$

if t is not cojugate to $zw, z \in Z_\sigma$.

Since the center of G is compact, each irreducible cuspidal representation of G splits the category of G-modules [8, Theorem 2.44]. This means that such a representation is a projective G-module. So if $M = G$ in the above, then we have $R_\sigma = (e)$ and $\mathrm{Ext}^j_G(\sigma, \sigma)$ is 0 if $j > 0$ and $\mathbf{C}$ if $j = 0$. So the proposition is obviously true in this case. From now on we will assume $M \neq G$.

5 Proof of the proposition 3

5.1 A resolution of $i_{GM}(\sigma)$

Recall we have fixed $M \in \mathcal{L}, M \neq G$ and an irreducible unitary cuspidal representation σ of M. Let $M^0 = \bigcap_{\chi \in X^*(M)} \mathrm{kernel}|\chi|$ where $X^*(M)$ is the set of F-rational characters of M. Then M^0 is the subgroup generated by all compact subgroups of M and $M/M^0 \cong \mathbf{Z}^n$ for some $n > 0$. Let $\{t_1, \ldots, t_n\}$ be a basis for M/M^0. Then $B = \mathbf{C}[M/M^0] = \mathbf{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is the ring of regular functions on the algebraic torus $\Psi(M)$. Let $\chi_{un} : M \to B$ be the natural representation of M on B given by the translation.

Lemma 9 *$\chi_{un} \otimes \sigma$ is a projective M-module.*

Proof. It's easy to see $\chi_{un} \cong \mathrm{ind}^M_{M^0}(1) = \{f : M \to \mathbf{C} | f(m_0 m) = f(m)$ and support of f is compact modulo $M^0\}$ and $\chi_{un} \otimes \sigma \cong \mathrm{ind}^M_{M^0}(\sigma|_{M^0})$. Since the center of M^0 is compact, by the same argument given at the end of last section, any cuspidal representation of M^0 is projective. And $\mathrm{ind}^M_{M^0}$, being left adjoint to the restriction functor $(\cdot)|_{M^0}$, maps projective M^0-modules to projective M-modules. So $\chi_{un} \otimes \sigma$ is projective. Q.E.D.

Let $\epsilon : B \to \mathbf{C}$ be the ring homomorphism given by $f \mapsto f(1)$. $\mathbf{C}$ can be viewed as a B-module via ϵ and the Koszul complex gives a resolution of $\mathbf{C}$ by free B-modules. More precisely, let $e_1, \cdots, e_n$ be symbols and let $\bigwedge^l B$ be the free B-module generated by $e_{i_1} \wedge \cdots \wedge e_{i_l}$ $(i_1 < \cdots < i_l)$.

The boundary map $d_l : \bigwedge^{l+1} B \to \bigwedge^l B$ is the B-linear map given by

$$e_{i_1} \wedge \cdots \wedge e_{i_{l+1}} \mapsto \sum_{j=1}^{l+1} (-1)^{j-1}(1 - t_{i_j}) e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_{l+1}}$$

$0 \to \bigwedge^n B \to \cdots \to \bigwedge^0 B = B \to \mathbf{C} \to 0$ is a free resolution of the B-module $\mathbf{C}$. Tensoring by σ, we get a projective resolution of σ. And then applying i_{GM} we get a projective resolution of $i_{GM}(\sigma)$ by the following proposition and its corollary. First, recall we have fixed a minimal parabolic subgroup P_0 and i_{GM}, r_{MG} are defined relative to the parabolic subgroup $P = MP_0$ [9]. Let $\bar{r}_{MG}$ be the Jacquet functor defined in the same way as r_{MG} but this time relative to the opposite parabolic subgroup $\bar{P} = M\bar{P}_0$.

Proposition 4 ([5]) *$\bar{r}_{MG}$ is exact and right adjoint to the functor i_{GM}, i.e. for any G-module π and an M-module τ,*

$$Hom_G(i_{GM}(\tau), \pi) = Hom_M(\tau, \bar{r}_{MG}(\pi))$$

Corollary 2 *i_{GM} maps projective M-modules to projective G-modules.*

For some technical reasons we must use the local ring B_m instead of B, where m is the kernel of the ring homomorphism $\epsilon : B \to \mathbf{C}$. This is justified by the following proposition and its corollary. The M-module structure of B_m is induced from that of B.

Proposition 5 *For any M-module τ of finite length and $j > 0$,*

$$Ext^j_M(B_m \otimes \sigma, \tau) = 0$$

Proof. Using induction on the length of τ and the long exact sequence for Ext, we may assume τ is irreducible. Since σ splits the category $\mathcal{M}$ of M-modules, if τ is not of the form $\psi\sigma$ for some

$\psi \in \Psi(M)$ then $\mathrm{Ext}^j_M(B_m \otimes \sigma, \tau) = 0$. More precisely, $\mathcal{M}$ is the direct sum of two full subcategories $\mathcal{M} = \mathcal{M}(\Sigma) \times \mathcal{M}(\Sigma)^\perp$ where $\mathcal{M}(\Sigma)$ is the category of M-modules whose irreducible subquotients are of the form $\psi\sigma$ and $\mathcal{M}(\Sigma)^\perp$ is the category of M-modules none of their subquotients are of such forms [9, Theorem 2.44]. If $\pi_1 \in \mathcal{M}(\Sigma)$ and $\pi_2 \in \mathcal{M}(\Sigma)^\perp$ then $\mathrm{Ext}^j_M(\pi_1, \pi_2) = \mathrm{Ext}^j_M(\pi_2, \pi_1) = 0$ for all j.

The Bernstein center $\mathcal{C}_M$ of the category $\mathcal{M}$ is the product $\mathcal{C}_{\mathcal{M}(\Sigma)} \times \mathcal{C}_{\mathcal{M}(\Sigma)^\perp}$ of centers of subcategories $\mathcal{M}(\Sigma)$ and $\mathcal{M}(\Sigma)^\perp$. $\mathcal{C}_{\mathcal{M}(\Sigma)}$ can be identified with the ring of regular functions on the algebraic variety $\Psi(M)/S_\sigma$, where $S_\sigma = \{\psi \in \Psi(M) | \psi\sigma \cong \sigma\}$ is a finite subgroup of $\Psi(M)$ [4, Theorem 2.13]. $B \otimes \sigma$ (and $B_m \otimes \sigma$) is a B-module in a natural way and σ has a B-module structure via $\epsilon \otimes 1 : B \otimes \sigma \to \sigma$. If we identify $\mathcal{C}_{\mathcal{M}(\Sigma)}$ with a subring D of B, then the action of $\mathcal{C}_{\mathcal{M}(\Sigma)}$ on $B_m \otimes \sigma$ is the same as that of D on it. The same is true for σ. Now suppose $\tau = \psi\sigma \ncong \sigma$. Then we can choose $z \in D$ such that $z(\sigma) \neq z(\psi\sigma)$, so $z - z(\psi\sigma) \notin m$. Since $z - z(\psi\sigma)$ is in the center, the actions on $\mathrm{Ext}^j_M(B_m \otimes \sigma, \psi\sigma)$ induced from the actions of $z - z(\psi\sigma)$ on the first and the second variables are the same. And this must be zero since $z - z(\psi\sigma)$ acts by zero on $\psi\sigma$. On the other hand, multiplication by $(z - z(\psi\sigma))^{-1}$ defines an M-module endomorphism of $B_m \otimes \sigma$, hence induces an endomorphism of $\mathrm{Ext}^j_M(B_m \otimes \sigma, \psi\sigma)$ which must be the inverse to the endomophism induced from $z - z(\psi\sigma)$. This is impossible unless $\mathrm{Ext}^j_M(B_m \otimes \sigma, \psi\sigma) = 0$.

Finally, assume $\tau = \sigma$. Since S_σ is contained in $T_l = \{x \in \Psi(M) | x^l = 1\}$ for some l, we have $D \supset \mathbf{C}[t_1^{\pm l}, \ldots, t_n^{\pm l}]$. As above $E = \mathrm{Ext}^j_G(B_m \otimes \sigma, \sigma)$ has a natural $B - B_m$ bimodule structure induced from the actions of B_m and B on $B_m \otimes \sigma$ and σ respectively. (For simplicity, we consider E as a left B-module and as a right B_m-module. Since these rings are commutative, we can write in any way.) Since the action of D on E from both sides is the same, $e \mapsto t_i e t_i^{-1}, i = 1, \ldots, n$ defines a representation of T_l on E. Here we viewed T_l as the quotient of the free abelian group with base $\{t_1, \ldots, t_n\}$ modulo the relations $t_1^l = \ldots = t_n^l = 1$. We have a canonical decomposition of E as $E = \oplus_{\chi \in \widehat{T}_l} E_\chi$ where $\widehat{T}_l$ is the set of irreducible representations of T_l and E_χ is the subspace on which T_l acts by χ. We claim that $E_\chi = 0$ if $\chi \neq 1$. If $\chi \neq 1$, then $\chi(t_i) = \zeta \neq 1$ for some t_i. For $e \in E_\chi$, $t_i e t_i^{-1} = \chi(t_i)e = \zeta e$. So $(t_i - 1)e = e(\zeta t_i - 1)$. Since the B-module structure of E is induced from that of σ, we have $(t_i - 1)e = 0$. Since $\zeta t_i - 1$ is a unit in B_m, this implies $e = 0$ and $E_\chi = 0$.

Now we consider $E_1 = E$ more carefully. Let η be an M-module in $\mathcal{M}(\Sigma)$. We say that η is a (B, M)-module if η has a B-module structure which commutes with the action of M and the action of the center D on η is the same as the action as the subring of B. Let $\mathcal{M}(B, \Sigma)$ be the subcategory of $\mathcal{M}(\Sigma)$ whose objects are (B, M)-modules and morphisms are M-module morphisms which are also B-linear. $\mathcal{M}(B, \Sigma)$ is equivalent to the category of modules over $B \otimes_D \mathcal{H}e_\sigma$, where e_σ is the idempotent in the center $\mathcal{C}_M$ of the category $\mathcal{M}$ such that $e_\sigma \equiv 1$ on $\mathcal{M}(\Sigma)$ and $e_\sigma \equiv 0$ on $\mathcal{M}(\Sigma)^\perp$.

Lemma 10

$$Ext^j_{\mathcal{M}(B,\Sigma)}(B_m \otimes \sigma, \sigma) = Ext^j_M(B_m \otimes \sigma, \sigma)$$

Proof. Let $\cdots \to B^{I_1} \to B^{I_0} \to B_m \to 0$ be a free resolution of the B-module B_m. Tensoring by σ we get $\cdots \to P^{I_1} \to P^{I_0} \to B_m \otimes \sigma \to 0, (P = B \otimes \sigma)$, a projective resolution of the G-module $B_m \otimes \sigma$. Note the boundary maps are also B-linear. E is the j-th cohomology group of the complex $0 \to H^{(0)} \to H^{(1)} \to \cdots$, where $H^{(k)} = \mathrm{Hom}_M(P^{I_k}, \sigma)$. Like E, $H^{(k)}$ is a $B - B$ bimodule such that the action of D from both sides is the same. Hence $H^{(k)}$ is a T_l-module. Since the boundary maps $P^{I_{i+1}} \to P^{I_i}$ are B-linear, the maps $H^{(k)} \to H^{(k+1)}$ are T_l-linear and the T_l-module structure of E comes from that of $H^{(j)}$. This means that $E_1 = E$ is the j-th cohomology group of the complex $0 \to H_1^{(0)} \to H_1^{(1)} \to \cdots$, where $H_1^{(k)}$ is the subspace of $H^{(k)}$ on which T_l acts trivially. Clearly,

$H_1^{(k)} = \mathrm{Hom}_{\mathcal{M}(B,\Sigma)}(P^{I_k}, \sigma)$. It remains to show that $P = B \otimes \sigma$ is a projective object in $\mathcal{M}(B, \Sigma)$. It's enough to show that for an epimorphism $\eta' \to \eta \to 0$ of (B, M)-modules, the induced map $\mathrm{Hom}_{\mathcal{M}(B,\Sigma)}(P, \eta') \to \mathrm{Hom}_{\mathcal{M}(B,\Sigma)}(P, \eta)$ is surjective. This follow from the same argument as above. Really, $\mathrm{Hom}_M(P, \eta') \to \mathrm{Hom}_M(P, \eta)$ is a surjective morphism of T_l-modules, and so is the map $\mathrm{Hom}_M(P, \eta')_1 \to \mathrm{Hom}_M(P, \eta)_1$. The lemma is proven.

Let $\mathcal{M}(B_m, \Sigma)$ be the subcategory of $\mathcal{M}(\Sigma)$ of (B_m, M)-modules defined in the same way as $\mathcal{M}(B, \Sigma)$. Let $F : \mathcal{M}(B, \Sigma) \to \mathcal{M}(B_m, \Sigma)$ be the functor given by $F(\eta) = B_m \otimes_B \eta$. Here the action of M on $B_m \otimes_B \eta$ is defined by $m(b \otimes v) = b \otimes mv$. It's easy to see that this is well defined and the action of D as the center and as the subring of B is the same. Obviously, F is exact and is left adjoint to an exact functor, the restriction-of-scalar functor. Since σ is a (B_m, M)- module, we have $\mathrm{Ext}^j_{\mathcal{M}(B,\Sigma)}(B_m \otimes \sigma, \sigma) = \mathrm{Ext}^j_{\mathcal{M}(B_m,\Sigma)}(F(B_m \otimes \sigma), \sigma)$. It's easy to see that $F(B_m \otimes \sigma) = B_m \otimes \sigma$ and $F(B \otimes \sigma) = B_m \otimes \sigma$. Since $B \otimes \sigma$ is projective in$\mathcal{M}(B, \Sigma)$, $F(B \otimes \sigma)$ is projective in $\mathcal{M}(B_m, \Sigma)$ and $\mathrm{Ext}^j_{\mathcal{M}(B_m,\Sigma)}(F(B \otimes \sigma), \sigma) = 0$. By the above lemma, the proposition is now proven. Q.E.D.

Corollary 3 *For any G-module π of finite length and $j > 0$,*

$$Ext^j_G(i_{GM}(B_m \otimes \sigma), \pi) = 0$$

Proof. By Proposition 4, $\mathrm{Ext}^j_G(i_{GM}(B_m \otimes \sigma), \pi) = \mathrm{Ext}^j_M(B_m \otimes \sigma, \bar{r}_{MG}(\pi)) = 0$.

5.2 Extension of the intertwining operators

For each $w \in R_\sigma$ we have the normalized intertwining operator $A(w, \sigma) : i_{GM}(\sigma) \to i_{GM}(\sigma^w)$. We want to extend this to a morphism of the complexes

$$\begin{array}{ccccccccccc} 0 & \to & i_{GM}(\tau_n) & \to & \cdots & \to & i_{GM}(\tau_0) & \to & i_{GM}(\sigma) & \to & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \to & i_{GM}(\tau_n^w) & \to & \cdots & \to & i_{GM}(\tau_0^w) & \to & i_{GM}(\sigma^w) & \to & 0 \end{array}$$

where $\tau_l = \bigwedge^l B_m \otimes \sigma$. First, we have to prove the following

Proposition 6 *Let $\tau = \tau_0 = B_m \otimes \sigma$. For each $w \in R_\sigma$, there exists an operator $A_w : i_{GM}(\tau) \to i_{GM}(\tau^w)$ such that the following diagram commutes and A_w intertwines the actions of both G and B_m on $i_{GM}(\tau)$ and $i_{GM}(\tau^w)$.*

$$\begin{array}{ccc} i_{GM}(\tau) & \overset{A_w}{\to} & i_{GM}(\tau^w) \\ \downarrow \epsilon & & \downarrow \epsilon^w \\ i_{GM}(\sigma) & \overset{A(w,\sigma)}{\to} & i_{GM}(\sigma^w) \end{array}$$

where we wrote ϵ for $i_{GM}(\epsilon \otimes 1)$.

Proof. For $\psi \in \Psi(M)$, let $\pi_\psi = i_{GM}(\psi\sigma)$. It's well known that all π_ψ act on the space V of $\mathrm{ind}_{K_0 \cap P}^{K_0}(\sigma|_{K_0 \cap M})$ where K_0 is the good maximal compact subgroup chosen before. Similarly, $\pi_\psi^w = i_{GM}((\psi\sigma)^w)$ acts on the space V' of $\mathrm{ind}_{K_0 \cap P}^{K_0}(\sigma^w|_{K_0 \cap M})$. Let K be a sufficiently small congruence subgroup of K_0 and let $m = \dim \pi_\psi^K = \dim V^K$. Then $\dim(\pi_\psi^w)^K = \dim V'^K = m$ since $\pi_\psi^w \cong \pi_\psi$. Clearly, $i_{GM}(\tau) = i_{GM}(B_m \otimes \sigma)$ acts on the space $B_m \otimes V$ and $i_{GM}(\tau)^K = B_m \otimes V^K$ is a free B_m-module of rank m. The normalized intertwining operator $A(w, \psi) = A(w, \psi\sigma) : V^K \to V'^K$ can be viewed as a linear map which intertwines the actions of the Hecke algebra $\mathcal{H}_K = \mathcal{H}(G, K)$

on π_ψ^K and $(\pi_\psi^w)^K$. It's known that the matrix coefficients of $A(w,\psi)$ are rational functions in $\psi \in \Psi(M)$ [1, Theorem 2.1]. In other words, by fixing bases for V^K and V'^K, $A(w,\psi)$ is given by an $m \times m$ matrix whose matrix coefficients are in the field F of rational functions on the algebraic variety $\Psi(M)$. Let $A_w \in M_{m\times m}(F)$ be this matrix. Let S be the multiplicatively closed subset of B generated by denominators of entries of A_w and let $S^{-1}B$ be the localization of B with respect to S. Since $A(w,\psi)$ is holomorphic at $\psi = 1$, $S^{-1}B \subset B_m$ and $A_w \in M_{m\times m}(S^{-1}B)$. A_w defines a $S^{-1}B$-linear map $S^{-1}B \otimes V^K \to S^{-1}B \otimes V'^K$. We claim this $S^{-1}B$-linear map intertwines the actions of $\mathcal{H}_K$ on $S^{-1}B \otimes V^K = i_{GM}(S^{-1}B \otimes \sigma)^K$ and $S^{-1}B \otimes V'^K = i_{GM}((S^{-1}B \otimes \sigma)^w)^K$. Let $h \in \mathcal{H}_K$. Since the action of $S^{-1}B$ on $i_{GM}(S^{-1}B \otimes \sigma)$ commutes with the action of G, h defines an $S^{-1}B$-linear map on $S^{-1}B \otimes V^K$, hence is given by a matrix in $M_{m\times m}(S^{-1}B)$ which is also denoted by h. Similarly, the action of h on $S^{-1}B \otimes V'^K$ gives a matrix in $M_{m\times m}(S^{-1}B)$ which is denoted by h'. Clearly, the action of h on π_ψ^K (respectively, on $(\pi_\psi^w)^K$) is given by $h(\psi)$ (respectively, by $h'(\psi)$). We know that for ψ in a Zariski dense subset of $\Psi(M)$, $A(w,\psi)h(\psi) - h'(\psi)A(w,\psi) = 0$. So $A_w h - h' A_w = 0$, as desired. So the same A_w defines a B_m -linear map $B_m \otimes V^K \to B_m \otimes V'^K$ which intertwines the actions of $\mathcal{H}$. Q.E.D.

Recall that we have fixed an isomorphism $\alpha_w : \sigma^w \xrightarrow{\sim} \sigma$ for each $w \in R_\sigma$. It induces an isomorphism $i_{GM}(\sigma^w) \xrightarrow{\sim} i_{GM}(\sigma)$, which will be denoted by the same α_w. We have $I(w,\sigma) = \alpha_w A(w,\sigma) : i_{GM}(\sigma) \to i_{GM}(\sigma)$. Define a $\mathbf{C}$-linear map $\iota_w : B \to B$ by $m \mapsto wmw^{-1}$ (recall $B = \mathbf{C}[M/M^0]$). Then ι_w gives an isomorphism of M-modules $\chi_{un}^w \xrightarrow{\sim} \chi_{un}$ which extends to an isomorphism $B_m^w \xrightarrow{\sim} B_m$. Hence $(\iota_w) \otimes \alpha_w$ is an isomorphism of M-modules $(B_m \otimes \sigma)^w \xrightarrow{\sim} B_m \otimes \sigma$ and induces an isomorphism of G-modules $i_{GM}(\tau^w) \xrightarrow{\sim} i_{GM}(\tau)$, which will be denoted by β_w. Let

$I_w = \beta_w A_w : i_{GM}(\tau) \to i_{GM}(\tau)$. Then the following diagram is commutative

$$\begin{array}{ccc} i_{GM}(\tau) & \stackrel{I_w}{\to} & i_{GM}(\tau) \\ \downarrow \epsilon & & \downarrow \epsilon \\ i_{GM}(\sigma) & \stackrel{I(w,\sigma)}{\to} & i_{GM}(\sigma) \end{array}$$

and for $b \in B$,

$$I_w b = b^w I_w, \text{ where } b^w = \iota_w(b) \; .$$

(By some abuse of notations, we denoted by the same b, the endomorphism of $i_{GM}(B_m \otimes \sigma)$ induced from the multiplication by b.)

Define $d_l^w : \bigwedge^{l+1} B \to \bigwedge^l B$ in the same way as d_l but using $t_i^w = \iota_w(t_i)$ in place of t_i ($i = 1, \ldots, n$). To get an endomorphism of the complex $0 \to i(\tau_n) \to \cdots \to i(\tau_0) \to i(\sigma) \to 0$ extending the endomorphism $I(w, \sigma)$ of $i_{GM}(\sigma)$, it turns out that we need to find B_m-module homomorphisms $\phi_l : \bigwedge^{l+1} B_m \to \bigwedge^l B_m$ $(l = 0, \ldots, n)$ completing the following diagram.

$$\begin{array}{ccccccccccc} 0 & \to & \bigwedge^n B_m & \stackrel{d^w}{\to} & \cdots & \stackrel{d^w}{\to} & \bigwedge^0 B_m & \stackrel{\epsilon}{\to} & \mathbf{C} & \to & 0 \\ & & \downarrow \phi_n & & & & \downarrow \phi_0 = id & & \| & & \\ 0 & \to & \bigwedge^n B_m & \stackrel{d}{\to} & \cdots & \stackrel{d}{\to} & \bigwedge^0 B_m & \stackrel{\epsilon}{\to} & \mathbf{C} & \to & 0 \end{array}$$

And we also need to know the alternating sum of traces of these ϕ_l. ($\bigwedge^l B_m$ is a free B_m-module of finite rank, so the trace of ϕ_l is well-defined.) Both are more or less well known. (See for example [13]) It's easy to see that there exist $b_{ij} \in B$ such that

$$1 - t_i^w = \sum_{j=1}^{n} b_{ij}(1 - t_j)$$

Define a B_m-linear map $\phi_1 : \bigwedge^1 B_m \to \bigwedge^1 B_m$ by $e_i \mapsto \sum_j b_{ij} e_j$. (Recall that $e_1, \ldots, e_n$ are symbols forming a base for $\bigwedge^1 B_m$.) And let $\phi_l = \phi_1 \wedge \ldots \wedge \phi_1 : \bigwedge^l B_m \to \bigwedge^l B_m$. Then $\phi_0 = id, \phi_1, \ldots, \phi_n$ complete the above diagram and

$$\sum_{j=0}^{n} (-1)^j \mathrm{tr}(\phi_j) = \det(1-b), \text{ where } b = (b_{ij}) \in M_{m\times m}(B) .$$

If $t_i^w = t_1^{l_{i1}} \cdots t_n^{l_{in}} (l_{ij} \in \mathbf{Z})$, then

$$\begin{aligned} \epsilon(b_{ij}) &= b_{ij}|_{t_1=\cdots=t_n=1} = \tfrac{\partial}{\partial t_j}\left(\textstyle\sum_{k=1}^n b_{ik}(t_k-1)\right)|_{t_1=\cdots=t_n=1} \\ &= \tfrac{\partial}{\partial t_j}(t_i^w - 1)|_{t_1=\cdots=t_n=1} = l_{ij} \end{aligned}$$

So $\epsilon(\det(1-b)) = \det(1-l)$ where $l = (l_{ij}) \in M_{n\times n}(\mathbf{Z})$. We claim that $\det(1-l) = \det(1-w)_{\mathbf{a}_M} = d(w)$. Clearly, l is the matrix of the endomorphism $m \mapsto m^w$ of the lattice M/M^0 and $\mathbf{a}_M = \mathrm{Hom}(X^*(M)_F, \mathbf{R}) = \mathrm{Hom}(X^*(A_M)_F, \mathbf{R}) = X(A_M)^\vee \otimes_{\mathbf{Z}} \mathbf{R}$ where $X(A_M)^\vee$ is the dual lattice. On the other hand, we have $(M/M^0)\otimes_{\mathbf{Z}} \mathbf{R} = (A_M/A_M^0)\otimes_{\mathbf{Z}} \mathbf{R}$ and a canonical isomorphism $A_M/A_M^0 \cong X(A_M)^\vee$

By some abuse of notations, we write ϕ_l for the endomorphism of $i_{GM}(\bigwedge^l B_m \otimes \sigma) = i_{GM}(\tau_l)$ induced from the M-module homomorphism $\phi_l \otimes 1 : \bigwedge^l B_m \otimes \sigma \to \bigwedge^l B_m \otimes \sigma$. We have $i_{GM}(\bigwedge^l B_m \otimes \sigma) = \bigwedge^l i_{GM}(B_m \otimes \sigma) = i_{GM}(B_m \otimes \sigma) \otimes \bigwedge^l \mathbf{C}$, which means that elements of $i_{GM}(\bigwedge^l B_m \otimes \sigma)$ are linear combinations of $f e_{i_1} \wedge \cdots \wedge e_{i_l}$ with $f \in i_{GM}(B_m \otimes \sigma)$. The G-module endomorphism I_w of $i_{GM}(B_m \otimes \sigma)$ induces an endomorphism of $i_{GM}(\bigwedge^l B_m \otimes \sigma) = i_{GM}(B_m \otimes \sigma) \otimes \bigwedge^l \mathbf{C}$ which is denoted by $I_w^{(l)}$.

Proposition 7 *The following diagram commutes.*

$$\begin{array}{ccccccccccc} 0 & \to & i_{GM}(\tau_n) & \to & \cdots & \to & i_{GM}(\tau_0) & \to & i_{GM}(\sigma) & \to & 0 \\ & & \downarrow \phi_n I_w^{(n)} & & & & \downarrow \phi_0 I_w^{(0)} & & \downarrow I(w,\sigma) & & \\ 0 & \to & i_{GM}(\tau_n) & \to & \cdots & \to & i_{GM}(\tau_0) & \to & i_{GM}(\sigma) & \to & 0 \end{array}$$

Proof. This follows directly from the definition of ϕ_l and the following observations.

1. The boundary map $\partial : i(\tau_{l+1}) \to i(\tau_l)$ is given by

   $fe_{i_1} \wedge \cdots \wedge e_{i_{l+1}} \mapsto \sum_{j=1}^{l+1} (-1)^{j-1}(1 - t_{i_j}) fe_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_{l+1}}.$

2. $I_w b = b^w I_w$ for all $b \in B$.

3. $d_l \phi_l = \phi_{l-1} d_l^w$

For $w \in \tilde{R}_\sigma$, we can define $\tilde{I}_w$ (and $\tilde{I}_w^{(l)}$) in the same way as $\tilde{I}(w,\sigma)$ using the same $\xi_\sigma : \tilde{R}_\sigma \to \mathbf{C}^*$.

5.3 Calculation of the trace

First, we need the following

Lemma 11

$$Hom_M(B_m \otimes \sigma, \sigma) = \mathbf{C}$$

Proof. As in the proof of Proposition 5, $\mathrm{Hom}_M(B_m \otimes \sigma, \sigma)$ is a $B - B_m$ bimodule hence is a T_l-module. Also the arguments there show that T_l acts trivially on this space. In other words, any $\phi \in \mathrm{Hom}_M(B_m \otimes \sigma, \sigma)$ is B-linear hence is B_m-linear. $\mathrm{Hom}_M(B \otimes \sigma, \sigma)$ is spanned by α_ψ : $B \otimes \sigma \overset{\epsilon(\psi)}{\to} \psi\sigma \overset{\sim}{\to} \sigma$ for $\psi \in S_\sigma$. Among these, only $\alpha_1 = \epsilon$ is B-linear. So $\phi|_{B\otimes\sigma}$ is unique up to scalar. And clearly, ϕ is determined by $\phi|_{B\otimes\sigma}$. Q.E.D.

Proposition 8

$$\dim_{\mathbf{C}} Hom_G(i_{GM}(B_m \otimes \sigma), i_{GM}(\sigma)) = |R_\sigma|$$

Proof. We have $\mathrm{Hom}_G(i_{GM}(B_m \otimes \sigma), i_{GM}(\sigma)) = \mathrm{Hom}_M(B_m \otimes \sigma, \bar{r}_{MG}i_{GM}(\sigma))$. $\bar{r}_{MG}i_{GM}(\sigma)$ has a canonical filtration whose quotients are isomorphic to $\sigma^w, w \in W^M$ [9, Theorem 5.2]. Since $\mathrm{Ext}^1_M(B_m \otimes \sigma, \pi) = 0$ for any M-module π of finite length, from the long exact sequence for Ext we have $\dim \mathrm{Hom}_M(B_m \otimes \sigma, \bar{r}_{MG}i_{GM}(\sigma)) = \sum_{w \in W^M} \dim \mathrm{Hom}_M(B_m \otimes \sigma, \sigma^w)$. The proof of Proposition 5 and the previous lemma show that $\dim \mathrm{Hom}_M(B_m \otimes \sigma, \sigma^w)$ is 0 if $\sigma^w \ncong \sigma$ and is 1 if $\sigma^w \cong \sigma$. So $\dim_{\mathbf{C}} \mathrm{Hom}_G(i_{GM}(B_m \otimes \sigma), i_{GM}(\sigma)) = |\{w \in W^M | \sigma^w \cong \sigma\}| = |W_\sigma|$. It is not difficult to show that if $\tilde{R}_{\sigma,reg} \neq \emptyset$ then $W^\circ_\sigma = \{e\}$ and $W_\sigma = R_\sigma$ (See the remarks following [2, Proposition 3.1]). Q.E.D.

$\{I(w,\sigma) | w \in R_\sigma\}$ is a base for $\mathrm{End}_G(i_{GM}(\sigma))$. (See §3.1.) Since the canonical homomorphism $\epsilon : i_{GM}(B_m \otimes \sigma) \to i_{GM}(\sigma)$ induced from $\epsilon \otimes 1 : B_m \otimes \sigma \to \sigma$ is surjective, this combined with the last proposition implies that $\{I(w,\sigma)\epsilon = \epsilon I_w | w \in R_\sigma\}$ is a base for $\mathrm{Hom}_G(i_{GM}(B_m \otimes \sigma), i_{GM}(\sigma))$. Let's calculate the trace of $\tilde{I}^*_{w^{-1}}\tilde{I}(w,\sigma)_*$ on $\mathrm{Hom}_G(i_{GM}(B_m \otimes \sigma), i_{GM}(\sigma))$ given by $\alpha \mapsto \tilde{I}(w,\sigma)\alpha\tilde{I}_{w^{-1}}$. Since $\epsilon\tilde{I}_w = \tilde{I}(w,\sigma)\epsilon$, this is the same as the trace of $\tilde{I}(w^{-1})^*\tilde{I}(w)_*$ on $\mathrm{End}_G(i_{GM}(\sigma))$ where we wrote $\tilde{I}(w)$ for $\tilde{I}(w,\sigma)$. For $t \in \tilde{R}_\sigma$, let $\bar{t}$ be the its image in R_σ under $\tilde{R}_\sigma \to R_\sigma$. Recall that $\tilde{I}(w) = \xi_\sigma(w)^{-1}I(\bar{w})$ for a function $\xi_\sigma : \tilde{R}_\sigma \to \mathbf{C}^*$ such that $\xi_\sigma(zx) = \chi_\sigma(z)\xi_\sigma(x)$ for $z \in Z_\sigma$. Let $s \in R_\sigma$ and let $t \in \tilde{R}_\sigma$ such that $\bar{t} = s$. $\tilde{I}(w)I(s)\tilde{I}(w^{-1}) = \xi_\sigma(t)\tilde{I}(w)\tilde{I}(t)\tilde{I}(w^{-1}) = \xi_\sigma(t)\tilde{I}(wtw^{-1}) = \xi_\sigma(t)\xi_\sigma(wtw^{-1})^{-1}I(\bar{w}s\bar{w}^{-1})$. So if $\bar{w}s\bar{w}^{-1} \neq s$ then $I(s)$ does not contribute to the trace of $\tilde{I}^*_{w^{-1}}\tilde{I}(w,\sigma)_*$ on $\mathrm{Hom}_G(i_{GM}(B_m \otimes \sigma), \sigma)$. If $\bar{w}s\bar{w}^{-1} = s$, then $wtw^{-1} = zt$ for some $z \in Z_\sigma$. z preserves the conjugacy class of w in $\tilde{R}_\sigma$ since $zw = t^{-1}wt$. Recall the triplete (M, σ, w) is assumed to be *essential* in the sense that χ_σ is trivial on $\{z \in Z_\sigma |\ z$ preserves the conjugacy class of $w\ \}$.

So $\xi_\sigma(t)\xi_\sigma(wtw^{-1})^{-1} = \xi_\sigma(t)\chi_\sigma(z)^{-1}\xi_\sigma(t)^{-1} = 1$. We have shown that

$$\mathrm{tr}\left(\mathrm{End}_G(i_{GM}(\sigma)); \tilde{I}(w^{-1})^*\tilde{I}(w)_*\right) = |\{s \in R_\sigma | \bar{w}s\bar{w}^{-1} = s\}| = |R_{\sigma,\bar{w}}|$$

It's easy to show $|R_{\sigma,\bar{w}}| = |\tilde{R}_{\sigma,w}||O_w|^{-1}$ (recall that O_w is the Z_σ-orbit of w in $\{\tilde{R}_\sigma\}$, the set of conjugacy classes in $\tilde{R}_\sigma$).

If $u \in \tilde{R}_\sigma$ isn't conjugated to any of zw with $z \in Z_\sigma$, then there is no $s \in R_\sigma$ such that $\bar{u}s\bar{w}^{-1} = s$. Above argument shows $\mathrm{tr}(\mathrm{End}(i_{GM}(\sigma)); \tilde{I}(w^{-1})^*\tilde{I}(u)_*) = 0$. We have proven the following lemma.

Lemma 12 *1. tr* $\left(Hom_G(i_{GM}(\tau), i_{GM}(\sigma)); \tilde{I}^*_{w^{-1}}\tilde{I}(w,\sigma)_*\right) = |\tilde{R}_{\sigma,w}||O_w|^{-1}$

2. tr $\left(Hom_G(i_{GM}(\tau), i_{GM}(\sigma)); \tilde{I}^*_{w^{-1}}\tilde{I}(u,\sigma)_*\right) = 0$ *if* $u \in \tilde{R}_\sigma$ *isn't conjugated to any of* zw *with* $z \in Z_\sigma$.

$\mathrm{Hom}_G(i_{GM}(\tau_l), i_{GM}(\sigma))$ is a direct sum of copies of $\mathrm{Hom}_G(i(\tau), i(\sigma))$. Considering the actions of ϕ_l^*, $(\tilde{I}^{(l)}_{w^{-1}})^*$ and $\tilde{I}(w)_*$ on $\mathrm{Hom}_G(i(\tau_l), i(\sigma))$, it's easy to see that the trace of $(\phi_l\tilde{I}^{(l)}_{w^{-1}})^*\tilde{I}(w)_*$ is the product of the trace of $\tilde{I}^*_{w^{-1}}\tilde{I}(w)_*$ on $\mathrm{Hom}_G(i(\tau), i(\sigma))$ and $\epsilon(\mathrm{trace}(\phi_l))$. So we have

$$\begin{aligned}
&\sum_j (-1)^j \mathrm{tr}\left(\mathrm{Ext}^j_G(i_{GM}(\sigma), i_{GM}(\sigma)); \tilde{I}(w^{-1})^*\tilde{I}(w)_*\right) \\
&= \sum_j (-1)^j \mathrm{tr}\left(\mathrm{Hom}_G(i(\tau_j), i(\sigma)); (\phi_j\tilde{I}^{(j)}_{w^{-1}})^*\tilde{I}(w)_*\right) \\
&= \sum_j (-1)^j \epsilon(\mathrm{tr}(\phi_j))\mathrm{tr}\left(\mathrm{Hom}_G(i(\tau), i(\sigma)); \tilde{I}^*_{w^{-1}}\tilde{I}(w,\sigma)_*\right) \\
&= d(w)|\tilde{R}_{\sigma,w}||O_w|^{-1}
\end{aligned}$$

And if we replace $\tilde{I}(w)_*$ in the above equation by $\tilde{I}(t)_*$ where $t \in \tilde{R}_\sigma$ is not conjugated to any of $Z_\sigma w$ then we see that the left hand side of the equation equals to zero. Since $\mathbf{a}^w_M = 0$, w is a rotation of the Euclidean space. So $d(w) = \det(1-w)|_{\mathbf{a}_M} > 0$ and $|d(w)| = d(w)$. The Proposition 3 is now proven.

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