Nilpotent Orbits and Multiplicity-free Representations

by

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Abstract

Let G be a semisimple Lie group and K be a maximal compact subgroup of G. The Lie algebra of G (denoted by $\mathfrak{g}$) has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $K_{\mathbb{C}}$ denote the complexification of K. This is a complex reductive group. Let $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$ and set $\mathfrak{n}_{\mathfrak{p}} = $ nilpotent elements in $\mathfrak{p}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$. $K_{\mathbb{C}}$ acts on $\mathfrak{n}_{\mathfrak{p}}$. It is a celebrated result of Kostant that there is only a finite number of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{n}_{\mathfrak{p}}$.

Let $\mathcal{O}$ be one such orbit. So $\mathcal{O} = K_{\mathbb{C}} \cdot E$, $E \in \mathcal{O} \subseteq \mathfrak{n}_{\mathfrak{p}}$, and is isomorphic to $K_{\mathbb{C}}/K_{\mathbb{C}}^{E}$, a homogeneous space. Here $K_{\mathbb{C}}^{E}$ is the stabilizer of E in $K_{\mathbb{C}}$. So the regular functions on the orbit can be realized as functions on the associated homogeneous space. We are interested in those nilpotent orbits that give rise to multiplicity-free representations. We formulate the problem as follows.

Let (π, V) be an irreducible representation of $K_{\mathbb{C}}$. The multiplicity of π in functions on the orbit $\mathcal{O}$ is equal to $\dim \left(V^{K_{\mathbb{C}}^{E}} \right) < \dim \pi < \infty$ $(E \in \mathcal{O})$.

In this dissertation we are interested in those orbits $\mathcal{O}$ such that the multiplicity of every π is 0 or 1. We shall see that this problem is intimately related to the notion of spherical homogeneous spaces.

Thesis Supervisor: David Vogan
Title: Professor

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All the way my Saviour leads me, what have I to ask beside

Can I doubt His tender mercy, who through life has been my guide

Heavenly peace, divinest comfort, here by faith in Him to dwell

For I know whatever befall me, Jesus doeth all things well

All the way my Saviour leads me, cheers each winding path I tread

Gives me grace for every trial, feeds me with the living bread

Though my weary steps may falter, and my soul a-thirst may be

Gushing from the Rock before me; lo! a spring of joy I see

All the way my Saviour leads me, oh the fullness of His love

Perfect rest to me is promised in my Father's house above

When my spirit, clothed immortal, wings its flight to realms of day

This my song through endless ages, Jesus led me all the way

"Blessed is he whose help is in the God of Jacob, whose hope is in the LORD his God, the Maker of heaven and earth, the sea, and everything in them- the LORD, who remains faithful forever... Praise the LORD." Psalm 146:5-10

Contents

Chapter 1

Introduction

Let G be a semisimple Lie group and K be a maximal compact subgroup of G. The Lie algebra of G (denoted by $\mathfrak{g}$) has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $K_{\mathbb{C}}$ denote the complexification of K. This is a complex reductive group. Let $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$ and set $\mathfrak{n}_{\mathfrak{p}} = $ nilpotent elements in $\mathfrak{p}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$. $K_{\mathbb{C}}$ acts on $\mathfrak{n}_{\mathfrak{p}}$. It is a celebrated result of Kostant that there is only a finite number of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{n}_{\mathfrak{p}}$.

Let $\mathcal{O}$ be one such orbit. So $\mathcal{O} = K_{\mathbb{C}} \cdot E$, $E \in \mathcal{O} \subseteq \mathfrak{n}_{\mathfrak{p}}$, and is isomorphic to $K_{\mathbb{C}}/K_{\mathbb{C}}^E$, a homogeneous space. Here $K_{\mathbb{C}}^E$ is the stabilizer of E in $K_{\mathbb{C}}$. So the regular functions on the orbit can be realized as functions on the associated homogeneous space. We are interested in those nilpotent orbits that give rise to multiplicity-free representations. We formulate the problem as follows.

Let (π, V) be an irreducible representation of $K_{\mathbb{C}}$. The multiplicity of π in functions on the orbit $\mathcal{O}$ is equal to $\dim \left(V^{K_{\mathbb{C}}^E} \right) < \dim \pi < \infty$ $(E \in \mathcal{O})$.

In this dissertation we are interested in those orbits $\mathcal{O}$ such that the multiplicity of every π is 0 or 1. We shall see that this problem is intimately related to the notion of spherical homogeneous spaces introduced by Brion, Luna and Vust in [B1]. We say that the subgroup $K_{\mathbb{C}}^E$ or the orbit $\mathcal{O}$ is spherical if $\mathcal{O}$ is a spherical homogeneous space. This definition will be recalled later.

We will now give an outline for the whole thesis. First we show that $K_{\mathbb{C}}^E$ spher-

ical implies $R\left(K_{\mathbb{C}}/K_{\mathbb{C}}^E\right)$ is multiplicity-free. E embeds into a $\mathfrak{sl}(2)$-triple $\{H, E, F\}$ ($H \in \mathfrak{k}$, $F \in \mathfrak{p}$) and this gives rise to $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$; $\mathfrak{g}_i$ is the i-eigenspace of $\operatorname{ad} H$ and is θ-stable. We show that if $\mathfrak{k}_i = \mathfrak{k}_i^E$ for $i \geq 1$ and when $\mathfrak{k}_0^E$ is spherical in $\mathfrak{k}_0$, then $R\left(K_{\mathbb{C}}/K_{\mathbb{C}}^E\right)$ is multiplicity-free. (Here $\mathfrak{k}_i$ is the compact part of $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$). We will then show that the irreducible $\mathfrak{sl}(2)$-representations that can occur in $\operatorname{ad}(\mathfrak{g})|_{\mathfrak{sl}(2)}$ must be of dimension 1, 2, 3. The 5-dimensional representation can occur only if its highest weight is in $\mathfrak{k}$.

Using the above criteria we compute H explicitly for the real forms of $Sp(n, \mathbb{C})$ as an illustration.

Chapter 2

Sphericality and Multiplicity-free Representations

In this chapter, let G be complex and all subgroups be algebraic.

Let M be a closed subgroup of a reductive group G. Then G/M is a homogeneous space.

Theorem. *If there exists a Borel subgroup B of G such that B has an open orbit on G/M, then the functions on G/M (denote by $R(G/M)$) are multiplicity-free.*

Proof. Let V_μ be any irreducible highest weight module of $R(G/M)$ with highest weight μ. Let $v \in V_\mu$ be a highest weight vector. Thus $b \cdot v = \mu(b)v$, $b \in B$. But $(b \cdot v)(z) = v(b^{-1} \cdot z)$, $z \in G/M$.

$$
\begin{aligned}
\therefore \quad v(b \cdot z) &= \left(b^{-1} \cdot v\right)(z) \\
&= \mu\left(b^{-1}\right) v(z).
\end{aligned}
$$

On the open orbit $B \cdot x$, $v(b \cdot x) = \mu\left(b^{-1}\right) v(x)$. So $v|_{\text{open orbit } B \cdot x}$ is only determined by $v(x)$ which is one-dimensional. Therefore $R(G/M)$ is multiplicity-free. $\square$

Definition. *Let G/M be a homogeneous space. We say M is spherical in G or G/M is*

spherical if there exists a Borel subgroup B of G such that B has an open orbit on G/M.

We are especially interested in the case when $G = K_{\mathbb{C}}$ and $M = K_{\mathbb{C}}^E$ where E is a nilpotent element in $\mathfrak{p}_{\mathbb{C}}$.

Given a general nilpotent E in $\mathfrak{g}$ and with $M = G^E$, by the Jacobson-Morozov theorem there exists a $\mathfrak{sl}(2)$-triple $\{H, E, F\}$ containing E. By the representation theory of $\mathfrak{sl}(2)$,

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \quad \mathfrak{g}_i = \{X \in \mathfrak{g} | \, [H, X] = iX\}.$$

We have the following theorem relating the sphericality of $\mathfrak{g}_0^E$ in $\mathfrak{g}_0$ and the sphericality of $\mathfrak{g}^E$ in $\mathfrak{g}$.

Theorem. *$\mathfrak{g}_0^E$ is spherical on $\mathfrak{g}_0$ and $\mathfrak{g}_i^E = \mathfrak{g}_i$, $i > 0 \Rightarrow \mathfrak{g}^E$ is spherical in $\mathfrak{g}$.*

Proof. Choose $\mathfrak{b}_0 \subset \mathfrak{g}_0$ Borel subalgebra such that $\mathfrak{b}_0 + \mathfrak{g}_0^E = \mathfrak{g}_0$; this is possible since $\mathfrak{g}_0^E$ is spherical in $\mathfrak{g}_0$. Define $\mathfrak{b} = \mathfrak{b}_0 + \bigoplus_{i<0} \mathfrak{g}_i$. This is a Borel subalgebra in $\mathfrak{g}$, since $\mathfrak{g}_0 + \bigoplus_{i<0} \mathfrak{g}_i$ is a parabolic subalgebra in $\mathfrak{g}$. So,

$$
\begin{aligned}
\mathfrak{b} + \mathfrak{g}^E &= \left(\mathfrak{b}_0 + \bigoplus_{i<0} \mathfrak{g}_i \right) + \left(\mathfrak{g}_0^E \oplus \bigoplus_{i>0} \mathfrak{g}_i^E \right) \\
&= \left(\mathfrak{b}_0 + \mathfrak{g}_0^E \right) + \bigoplus_{i<0} \mathfrak{g}_i + \bigoplus_{i>0} \mathfrak{g}_i \\
&= \mathfrak{g}_0 + \bigoplus_{i<0} \mathfrak{g}_i + \bigoplus_{i>0} \mathfrak{g}_i \\
&= \mathfrak{g}.
\end{aligned}
$$

Thus $\mathfrak{g}^E$ is spherical in $\mathfrak{g}$. $\qquad\square$

Theorem. *$\mathfrak{g}^E$ spherical in $\mathfrak{g} \Rightarrow \mathfrak{g}_0^E$ spherical in $\mathfrak{g}_0$.*

Proof. We shall prove the following equivalent statement: G_0^E not spherical in $G_0 \Rightarrow G^E$ not spherical in G. By a result of Brion in [B1], G_0^E not spherical in G_0 implies that there exists 1-dimensional character $\lambda_0 : G_0^E \to \mathbb{C}^\times$ and an irreducible representation Z of G_0 such that Z appears more than once in $\mathrm{Ind}_{G_0^E}^{G_0}(\lambda_0)$ (*i.e.*, λ_0 appears more than once

in $Z|_{G_0^E}$). Now $G_0 \cdot U$ is parabolic in G, so

$$\hat{G} \hookrightarrow \hat{G}_0 \quad \text{via} \quad V \to V^U.$$

Case 1: Z is in the image of this map.

Then we have a representation V of G such that $V^U = Z$. Define a character λ_0 of $G^E = G_0^E \cdot U^E$ by a trivial extension, namely by making U^E act trivially on λ_0. Then

$$
\begin{aligned}
\text{Hom}_{G^E}\left(\mathbb{C}_{\lambda_0}, V\right) &= \text{Hom}_{G_0^E}\left(\mathbb{C}_{\lambda_0}, V^{U^E}\right), \text{ since } U^E \text{ acts trivially.} \\
&\supseteq \text{Hom}_{G_0^E}\left(\mathbb{C}_{\lambda_0}, V^U\right) \\
&= \text{Hom}_{G_0^E}\left(\mathbb{C}_{\lambda_0}, Z\right),
\end{aligned}
$$

and this space has dimension ≥ 2.

Case 2: Z is not in the image.

In this case we choose a 1-dimensional character ϕ of G_0 such that $\langle \phi, \alpha \rangle$ is sufficiently large for all roots α of a fixed Cartan subalgebra in $\mathfrak{u} = \text{Lie}(U)$. Replace λ_0 in Case 1 by $\lambda_0 \otimes \phi|_{G_0^E}$ and Z by $Z \otimes \phi$. The ϕ thus chosen ensures that the highest weight of $Z \otimes \phi$ is G-dominant. Now proceed as in Case 1 to complete the proof.

$\square$

Proposition. $\mathfrak{g}_i = \mathfrak{g}_i^E$, $i \geq 1 \Leftrightarrow$ *only irreducible $\mathfrak{sl}(2)$-representations of dimensions 1, 2, 3 can occur in* $\text{ad}(\mathfrak{g})|_{\mathfrak{sl}(2)}$.

Proof. $\mathfrak{g}_i = \mathfrak{g}_i^E = \mathfrak{g}_i \cap \mathfrak{g}^E$, $i \geq 1 \Leftrightarrow \mathfrak{g}_i \subset \mathfrak{g}^E$, $i \geq 1$. It is clear that such representations can occur. We show that these are the only ones. Recall that an irreducible representation of $\mathfrak{sl}(2)$ of dimension n has weights $n-1, n-3, \ldots, -(n-3), -(n-1)$. For representations of dimension $n \geq 4$, $\text{ad}E : \mathfrak{g}_{n-3} \to \mathfrak{g}_{n-1} \neq 0$. Thus $\mathfrak{g}_{n-3} \not\subset \mathfrak{g}^E$. $\square$

Lemma. *If the irreducible $\mathfrak{sl}(2)$-representations are 1, 2 or 3 dimensional, then $\mathfrak{g}_0^E$ is a symmetric subalgebra of $\mathfrak{g}_0$ (i.e., $\mathfrak{g}_0^E$ is the centralizer of an involution of $\mathfrak{g}_0$).*

Proof. We will construct one such involution. Let $\phi : SL(2) \to G$ be the homomorphism such that $d\phi \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = H$. Consider $\theta = \mathrm{Ad}\left[\phi\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right]$. Then

$$
\begin{aligned}
\theta^2 &= \mathrm{Ad}\left[\phi\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right]\mathrm{Ad}\left[\phi\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right] \\
&= \mathrm{Ad}\left[\phi\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\phi\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right] \\
&= \mathrm{Ad}\left[\phi\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^2\right] \\
&= \mathrm{Ad}\phi\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \\
&= \mathrm{Ad}\phi\left(\begin{smallmatrix} e^{i\pi} & 0 \\ 0 & e^{-i\pi} \end{smallmatrix}\right) \\
&= \mathrm{Ad}\phi\left[\exp\left(\begin{smallmatrix} i\pi & 0 \\ 0 & -i\pi \end{smallmatrix}\right)\right] \\
&= \mathrm{Ad}\left[\exp d\phi\left(\begin{smallmatrix} i\pi & 0 \\ 0 & -i\pi \end{smallmatrix}\right)\right] \\
&= \mathrm{Ad}\exp(i\pi H) \\
&= \exp \mathrm{ad}(i\pi H) \\
&= 1 \text{ on } \mathfrak{g}_0 \text{ since } [H, X] = 0 \text{ for } X \in \mathfrak{g}_0.
\end{aligned}
$$

Thus θ is an involution on $\mathfrak{g}_0$. We need to show $\mathfrak{g}_0^E = $ fixed points of θ in $\mathfrak{g}_0$. Now $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ acts by 1 on the trivial representation. There is no 0-weight space for the 2-dimensional representation. $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ acts by -1 on 0-weight space of the 3-dimensional representation. We are done. $\qquad\square$

Corollary. *If the irreducible $\mathfrak{sl}(2)$-representations are 1, 2 or 3 dimensional, then B_0 has an open orbit on G_0/G_0^E.*

Proof. B_0 has open orbit on $G_0/G_0^E \Leftrightarrow G_0^E$ has open orbit on G_0/B_0. Thus it suffices to show that G_0^E is the set of fixed points of an involution of G_0. But this is precisely the preceding lemma. $\qquad\square$

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We will now summarize our results in the following theorem.

Theorem. *Consider the following conditions on E.*

(1) The $\mathfrak{sl}(2)$-representations occurring in $\mathrm{ad}(\mathfrak{g})|_{\mathfrak{sl}(2)}$ have dimensions 1, 2 or 3.

(2) $\mathfrak{g}_0^E$ spherical in $\mathfrak{g}_0$ and $\mathfrak{g}_i^E = \mathfrak{g}_i$, $i \geq 1$.

(3) $\mathfrak{g}^E$ spherical in $\mathfrak{g}$.

(4) $R\left(G/G^E\right)$ is multiplicity-free.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4).

Remarks. The implication (4) $\Rightarrow$ (3) comes from the theorem of Brion stated below. The orbit closure $\overline{G \cdot E}$ is an affine algebraic variety (closed in the vector space $\mathfrak{g}$), so $G \cdot E$ is quasiaffine (open in an affine variety).

Theorem. *If G/M is quasi-affine, then $\mathfrak{m}$ is spherical in $\mathfrak{g}$ if and only if $R(G/M)$ is multiplicity-free.*

Proof. [B2]. □

Remarks. D. Panyushev in [P] explicitly classified all spherical nilpotent orbits in $\mathfrak{sl}(V)$, $\mathfrak{sp}(V)$ and $\mathfrak{so}(V)$ for complex finite dimensional vector spaces V. However, his method does not carry over for real V. Our method applies to real V as well. This is the subject of the next chapter.

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Chapter 3

Main Results and Illustrations

Now fix a Cartan involution θ of $\mathfrak{g}_\mathbb{C}$. Let $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C}$ be the corresponding Cartan decomposition.

Let $E \in \mathfrak{p}_\mathbb{C}$ be a nilpotent element. By Kostant-Rallis [K-R] we can find a standard $\mathfrak{sl}(2)$-triple $\{H, E, F\}$ with $H \in \mathfrak{k}_\mathbb{C}$ and $F \in \mathfrak{p}_\mathbb{C}$. This triple is determined by E up to conjugation by $K_\mathbb{C}^E$. By the representation theory of $\mathfrak{sl}(2)$, $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, where $\mathfrak{g}_i = \{X \in \mathfrak{g} \mid \operatorname{ad}_H X = [H, X] = iX\}$.

Set $\mathfrak{k}_i = \mathfrak{g}_i \cap \mathfrak{k}$ and $\mathfrak{p}_i = \mathfrak{g}_i \cap \mathfrak{p}$. Since $\theta H = H$, θ preserves each eigenspace $\mathfrak{g}_i$ of $\operatorname{ad}(H)$; so $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{k}_i \oplus \mathfrak{p}_i$. Note that

$$\operatorname{ad}(E) : \mathfrak{k}_i \quad \rightarrow \quad \mathfrak{p}_{i+2}$$

$$\operatorname{ad}(E) : \mathfrak{p}_i \quad \rightarrow \quad \mathfrak{k}_{i+2}.$$

Recall that we are interested in the multiplicity-free $K_\mathbb{C}$ nilpotent orbits on $\mathfrak{p}_\mathbb{C}$. The following analogous theorem holds.

Theorem. *Consider the following conditions on E.*

1. *The irreducible $\mathfrak{sl}(2)$-representations coming from the $\mathfrak{sl}(2)$-triple $\{H, E, F\}$ have dimensions 1, 2, 3 or 5. The 5-dimensional representation can appear only if its highest weight is in $\mathfrak{k}$ and only if $\mathfrak{k}_0^E$ is spherical in $\mathfrak{k}_0$.*

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2. *There exists a Borel subgroup B of $K_{\mathbb{C}}$ such that $B_0 = B \cap (K_{\mathbb{C}})_0$ has an open orbit on $(K_{\mathbb{C}})_0 / \left(K_{\mathbb{C}}^E \right)_0$ and $\mathfrak{k}_i = \mathfrak{k}_i^E$ for $i \geq 1$.*

3. *B has an open orbit on $K_{\mathbb{C}}/K_{\mathbb{C}}^E$.*

4. *$R \left(K_{\mathbb{C}}/K_{\mathbb{C}}^E \right)$ is multiplicity-free.*

Then $(1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4)$.

Remarks. Only the equivalence of (1) and (2) needs comment as the others follow from a previous theorem.

Proof. Let $\{H, E, F\}$ be a standard $\mathfrak{sl}(2)$-triple and θ be a Cartan involution, so that

$$\theta E = -E, \qquad \theta F = -F \qquad \text{and} \qquad \theta H = H.$$

Let $G_{\mathbb{C}}^\theta$ be the group generated by $G_{\mathbb{C}}$ and θ (*i.e.*, $G_{\mathbb{C}}^\theta = \langle G_{\mathbb{C}}, \theta \rangle = G_{\mathbb{C}} \rtimes \mathbb{Z}/2\mathbb{Z}$; here $\mathbb{Z}/2\mathbb{Z} = \{1, \theta\}$). Let us look at representation of $SL(2, \mathbb{C})^\theta$ on $\mathfrak{g}$. The irreducible representations differ very little from $SL(2, \mathbb{C})$, namely it has exactly 2 irreducible representations for each fixed positive dimension n. (The highest weight of each representation is in $\mathfrak{k}$ or in $\mathfrak{p}$. In $\mathfrak{k}$ $\theta = +1$ and in $\mathfrak{p}$ $\theta = -1$.) Recall that we want $\mathfrak{k}_i \subset \mathfrak{k}^E$, $i \geq 1$. Now

$$\mathrm{ad}E \quad : \mathfrak{k}_i \rightarrow \mathfrak{p}_{i+2}$$
$$: \mathfrak{p}_i \rightarrow \mathfrak{k}_{i+2},$$

since $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. It is easy to see that when the dimensions of the irreducible representations of $SL(2, \mathbb{C})^\theta$ are 1, 2, 3, $\mathfrak{k}_i$ is killed by E for $i \geq 1$.

When $\dim n = 4$:

Case 1: Highest weight $= 3$ is in $\mathfrak{k}$:

$$[\mathfrak{k}_3, E] = 0.$$

Next smaller $\mathfrak{k}$-weight ≤ 3 is -1. So $\mathfrak{k}_i$ is killed by E for $i \geq 1$.

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Case 2: Highest weight $= 3$ is in $\mathfrak{p}$:

$$[E, \mathfrak{t}_1] \subseteq \mathfrak{p}_3 \neq 0.$$

So this case is not allowed by hypothesis (2).

At a first glance Case 1 seems to be admissible. A closer look suggests otherwise. Let $V_4^\pm$ be the four-dimensional irreducible representation of $SL(2, \mathbb{C})^\theta$ in which θ acts by ± 1 on the top weight space. Then $\left(V_4^\pm\right)^* \cong V_4^\mp$.

Since $\mathfrak{g} \cong \mathfrak{g}^*$ by the Killing form, a representation occurs in $\mathfrak{g}$ if and only if its dual occurs in $\mathfrak{g}$. Since V_4^- is ruled out by Case 2, so is V_4^+. Therefore $\mathfrak{t}_i \subset \mathfrak{t}^E (i \geq 1) \Rightarrow$ there is no 4-dimensional representation of $SL(2)$ in $\mathrm{ad}(\mathfrak{g})|_{\mathfrak{sl}(2)}$.

When dimension $n = 5$:

Case 1: Highest weight is in $\mathfrak{t}$:

$$[\mathfrak{t}_4, E] = 0.$$

Next lower $\mathfrak{t}$-weight is 0. Therefore $\mathfrak{t}_i \subset \mathfrak{t}^E$ for $i \geq 1$. So this case is admissible.

Case 2: Highest weight is in $\mathfrak{p}$:

$$\mathrm{ad}E : \mathfrak{t}_2 \to \mathfrak{p}_4 \neq 0.$$

So this case is inadmissible. Thus there is no 5-dimensional irreducible representation of $SL(2)$ when the highest weight is in $\mathfrak{p}$.

When dimension $n \geq 6$:

Case 1: Highest weight $(n - 1)$ is in $\mathfrak{t}$:

$$\mathfrak{t}_{n-5} \neq 0 \quad \text{and} \quad \mathrm{ad}E : \mathfrak{t}_{n-5} \to \mathfrak{p}_{n-3} \neq 0.$$

Case 2: Highest weight $(n - 1)$ is in $\mathfrak{p}$:

$$\mathrm{ad}E : \mathfrak{t}_{n-3} \to \mathfrak{p}_{n-1} \neq 0.$$

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Hence there are no $SL(2)$-representations with dimensions ≥ 6.

To summarize, $\mathfrak{k}_i \subset \mathfrak{k}^E$ ($i \geq 1$) if and only if the $SL(2)$ representations have dimensions 1, 2, 3 or 5. The 5-dimensional representation occurs only if its highest weight is in $\mathfrak{k}$. $\square$

To help us visualize the theorem consider the following example.

Example. $G = SL(2,\mathbb{R})$, $K = SO(2)$, $K_{\mathbb{C}} = SO(2,\mathbb{C}) \cong \mathbb{C}^{\times}$

$\mathfrak{p} = $ symmetric 2×2 trace zero matrices.

Nilpotent elements in $\mathfrak{p}$ are $\left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} : a^2 + b^2 = 0, a, b \in \mathbb{C} \right\}$.

There are 3 nilpotent classes: $0, \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$.

Case 1: $E = 0$

$K_{\mathbb{C}}^E = K_{\mathbb{C}} \cong \mathbb{C}^{\times}$. Therefore,

$$K_{\mathbb{C}}/K_{\mathbb{C}}^E \cong \{1\}$$
$$R(K_{\mathbb{C}}/K_{\mathbb{C}}^E) \cong \mathbb{C}.$$

Only the trivial representation occurs, and it has multiplicity 1.

Case 2: $E = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$

$$K_{\mathbb{C}}^E \cong \{\pm 1\}$$
$$K_{\mathbb{C}}/K_{\mathbb{C}}^E \cong \mathbb{C}^{\times}/\{\pm 1\}$$
$$R(K_{\mathbb{C}}/K_{\mathbb{C}}^E) \cong R(\mathbb{C}^{\times}/\{\pm 1\}) \cong \mathbb{C}[z^2, z^{-2}]$$
$$= \oplus_{n \in \mathbb{Z}} \mathbb{C}z^{2n}.$$

$\mathrm{ad}(\mathfrak{sl}(2)) \cong \mathfrak{sl}(2)$, the 3 dimensional representation of $\mathfrak{sl}(2)$.

Let us now examine how we can make use of this theorem in concrete cases.

Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a Cartan subalgebra in $\mathfrak{k}$. Set

$$\mathfrak{h} = \mathfrak{g}^{\mathfrak{t}}$$
$$= (\mathfrak{k} \oplus \mathfrak{p})^{\mathfrak{t}}$$

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$$= \mathfrak{t} \oplus \mathfrak{p}^{\mathfrak{t}}$$

$\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$.

Let $H \in \mathfrak{t}$. Choose a θ-stable system of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{h})$ which makes H dominant. Let $\Pi = \{\alpha_1, \ldots, \alpha_m\}$ be a set of simple roots in this root system. Then $\alpha(H) \geq 0$ for any $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})$. The integer $m_i = \alpha_i(H)$, $\alpha_i \in \Pi$ are known as the numerical labels of H. This labeling of the Dynkin diagram is known as the weighted Dynkin diagram of H (or of a $\mathfrak{sl}(2)$-triple $\{H, E, F\}$ or of the nilpotent orbit associated to E.) All these are independent of the choice of triples (see [S.S]) modulo the choice of $\Delta^+(\mathfrak{g}, \mathfrak{h})$.

Since the simple roots $\{\alpha_1, \ldots, \alpha_m\}$ form a base of the root system, we can write the highest root $\beta = \sum_{i=1}^m n_i \alpha_i$. Since the $\mathfrak{sl}(2)$-representations we need have dimensions 1, 2, 3, or 5,

$$\beta(H) \leq 4, \qquad \beta(H) \neq 3.$$

In terms of the numerical labels of H, the condition is

$$\sum_{i=1}^m m_i n_i \leq 4, \qquad \sum_{i=1}^m m_i n_i \neq 3.$$

This is the first condition we need to check.

Recall that we also require the 4-eigenspace of H to be contained in $\mathfrak{t}$. Write

$$\mathrm{Cent}(H) = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \cdots \oplus \mathfrak{l}_r \oplus \mathrm{center},$$

where $\mathfrak{l}_i$'s are the simple factors of $\mathrm{Cent}(H)$.

Lemma. *Suppose $\beta(H) = 4$. The 4-eigenspace of H is in $\mathfrak{t}$ if and only if the highest weight vector X_β is in $\mathfrak{t}$ and those $\mathfrak{l}_i$'s in $\mathrm{Cent}(H)$ not commuting with X_β are in $\mathfrak{t}$.*

Proof. It is obvious that X_β has to be in $\mathfrak{t}$. We now prove the equivalent statement

that if $\mathfrak{l}_i \cap \mathfrak{p} \neq 0$, then $[\mathfrak{l}_i, X_\beta] = 0$. Now $[\mathfrak{l}_i \cap \mathfrak{p}, X_\beta] \subseteq$ 4-eigenspace of $H \subseteq \mathfrak{k}$.

$$\mathfrak{l}_i \cap \mathfrak{p} \subseteq \mathfrak{p} \quad \text{and} \quad X_\beta \in \mathfrak{p} \Rightarrow [\mathfrak{l}_i \cap \mathfrak{p}, X_\beta] \subseteq \mathfrak{p}.$$

Therefore $[\mathfrak{l}_i \cap \mathfrak{p}, X_\beta] = 0$. But since $\mathfrak{l}_i$ is simple and $\mathfrak{l}_i \cap \mathfrak{p} \neq 0$, $\mathfrak{l}_i$ is generated by $\mathfrak{l}_i \cap \mathfrak{p}$ under $[\ , \ ]$. Hence $[\mathfrak{l}_i, X_\beta] = 0$. $\qquad\square$

As an illustration to the theory, let us consider the real groups G such that its complexification is $G_{\mathbb{C}} = Sp(n, \mathbb{C})$. We will classify the multiplicity-free nilpotent $K_{\mathbb{C}}$-orbits by specifying H in the $\mathfrak{sl}(2)$-triple ($\{H, E, F\}$). This suffices because of the conjugacy theorems of Kostant and Malcev (see [C-M]). $\qquad\square$

Recall that $Sp(p, q) = \left\{ g \in \mathrm{Aut}(\mathbb{H}^{p+q}) | g \text{ preserves } \sum_{i=1}^{p} |z_i|^2 - \sum_{i=p+1}^{p+q} |z_i|^2 \right\}$.

Consider an n-dimensional representation of $SL(2, \mathbb{R})$ over $\mathbb{H}$. As in the representation theory of $SL(2, \mathbb{R})$ over $\mathbb{R}$ or $\mathbb{C}$, there is precisely one irreducible representation having quaternionic dimension n for each $n \in \mathbb{Z}^+$.

Let $\mathbb{H}^n$ be a $\mathbb{H}$-representation of $SL(2, \mathbb{R})$.

Proposition. $\mathbb{H}^n$ *is isomorphic to* $\bigoplus_{m=1}^{\infty} \mathbb{R}^m \otimes_{\mathbb{R}} \mathrm{Hom}_{SL(2,\mathbb{R})}(\mathbb{R}^m, \mathbb{H}^n)$ *as a right $\mathbb{H}$-vector space.*

Proof. Let $T \in \mathrm{Hom}_{SL(2,\mathbb{R})}(\mathbb{R}^m, \mathbb{H}^n)$ and $v \in \mathbb{R}^m$. Consider the map $v \otimes T \mapsto T(v)$. The two spaces are isomorphic as real vector spaces by the $SL(2, \mathbb{R})$ representation theory over $\mathbb{R}$. A routine verification will show that this map respects the $\mathbb{H}$-action. $\qquad\square$

It is a classical fact that for m even $\mathbb{R}^m$ carries an $SL(2, \mathbb{R})$-invariant symplectic form; for m odd (say $m = 2j + 1$) $\mathbb{R}^m$ carries a $SL(2, \mathbb{R})$-invariant quadratic form with signature $(j + 1, j)$.

We will now decompose the n dimensional representation of $SL(2, \mathbb{R})$ over $\mathbb{H}$ into irreducibles. Write $\mathbb{H}^n = \mathbb{R} \otimes \mathbb{H}_{a_1} \oplus \mathbb{R}^2 \otimes \mathbb{H}_{a_2} \oplus \cdots$, where $\mathbb{H}_{a_i} = \mathrm{Hom}_{SL(2,\mathbb{R})}(\mathbb{R}^i, \mathbb{H}^n)$ and $\mathbb{R}^i = i$-dimensional irreducible representation of $SL(2, \mathbb{R})$ and $n = a_1 + 2a_2 + 3a_3 + \cdots$.

The form on $\mathbb{H}^n$ and the form on $\mathbb{R}^i$ together induce an $\mathbb{H}$-valued form $\langle \ , \ \rangle_{a_i}$ on $\mathbb{H}_{a_i}$,

characterized by

$$\langle T(v), S(w)\rangle = \langle v, w\rangle\langle T, S\rangle_{a_i}, \quad v, w \in \mathbb{R}^i, \quad T, S \in \mathrm{Hom}_{SL(2,\mathbb{R})}\left(\mathbb{R}^i, \mathbb{H}^n\right).$$

It follows that

$$\langle T, S\rangle_{a_i} = (-1)^{i+1}\langle \overline{S, T}\rangle_{a_i}.$$

So $\langle \, , \, \rangle_{a_{2j+1}}$ has a well defined signature (p_j, q_j), and $\mathbb{R}^{2j+1} \otimes \mathbb{H}_{a_{2j+1}}$ has signature

$$\left((j+1)p_j + jq_j, jp_j + (j+1)q_j\right).$$

Similarly $\mathbb{R}^{2j} \otimes \mathbb{H}_{a_{2j}}$ has signature $\left(ja_{2j}, ja_{2j}\right)$. The form on $\mathbb{H}_{a_{2j}}$ is uniquely determined up to isomorphism. Therefore we see that the form on $\mathbb{H}^n$ has signature

$$\sum_{2j+1} \left[(j+1)p_j + jq_j, (j+1)q_j + jp_j\right] \quad + \quad \sum_{2j} \left(ja_{2j}, ja_{2j}\right).$$

Thus $\sum \left[(j+1)p_j + jq_j\right] + \sum ja_{2j} = p$ and $\sum \left[(j+1)q_j + jp_j\right] + \sum ja_{2j} = q$. With this information on hand we are able to write down the semisimple elements H that can occur as $\mathfrak{sl}(2)$-triples $\{H, E, F\}$ satisfying our main theorem.

To this end let us go back to our case when $G_\mathbb{C} = Sp(n, \mathbb{C})$. We first consider a simple case when $\mathbb{H}^n = \mathbb{R}^m \otimes \mathbb{H}_1$.

Case 1: m is even.

H has eigenvalues $m-1, m-3, \ldots, -(m-3), -(m-1)$ on $\mathbb{H}^n$ and the form on $\mathbb{H}^n$ has signature $\left(\frac{m}{2}, \frac{m}{2}\right)$. So $H = (m-1, m-3, \ldots, 1)(m-1, m-3, \ldots 1)$. The $\mathfrak{sl}(2)$-triple defined by this H corresponds to the homomorphism $\phi : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sp}\left(\frac{m}{2}, \frac{m}{2}\right)$ with $\phi\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = H$.

Case 2: m is odd, say $m = 2j + 1$.

For simplicity and to see things more clearly, consider $m = 3$. Then $\mathbb{R}^{2,1} \otimes_\mathbb{R} \mathbb{H} = \mathbb{H}^{2,1}$ is a representation of $SL(2, \mathbb{R})$. Here $\mathbb{R}^{2,1}$ denotes the 3-dimensional irreducible

representation of $SL(2,\mathbb{R})$ and superscript $(2,1)$ denotes the signature. Then we have

$$SL(2,\mathbb{R}) \longrightarrow Sp(2,1)$$
$$\cup \qquad\qquad \cup$$
$$SO(2) \longrightarrow Sp(2) \times Sp(1).$$

We have a $SO(2)$-invariant decomposition of $\mathbb{R}^{2,1} \cong \mathfrak{sl}(2,\mathbb{R})$ with the Killing form. So $\mathbb{R}^{2,1} \cong \mathfrak{k} \oplus \mathfrak{p}$ ($SO(2)$-invariant decomposition). Form is negative-definite on $\mathfrak{k}$ and positive-definite on $\mathfrak{p}$. Hence $SO(2)$ has a 2-dimensional action on $\mathfrak{p}_\mathbb{C}$, with weights 2 and -2. On $\mathfrak{k}_\mathbb{C}$ $SO(2)$ acts with weight 0. Since the weight -2 can be conjugated to weight 2 (by an element of $Sp(2)$), the possible H are $(2,2)(0)$ or $(0)(2,2)$. Their corresponding homomorphisms are

$$\phi : \mathfrak{sl}(2) \to \mathfrak{sp}(2,1) \quad\text{and}\quad \phi : \mathfrak{sl}(2) \to \mathfrak{sp}(1,2),$$

mapping $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to the corresponding H.

We are now ready to see the general picture. The highest root $\beta = (2,0,\dots,0)$ with respect to the standard roots for $Sp(2n,\mathbb{C})$. Since we need $\beta(H) \le 4$, the largest coordinate of H must be at most 2. Thus irreducible representations of $SL(2,\mathbb{R})$ over $\mathbb{H}$ having dimensions greater than 3 do not occur in the decomposition of $\mathbb{H}^n$ into irreducibles.

Therefore $\mathbb{H}^n = \mathbb{R}^1 \otimes \mathbb{H}_{a_1} \oplus \mathbb{R}^2 \otimes \mathbb{H}_{a_2} \oplus \mathbb{R}^3 \otimes \mathbb{H}_{a_3}$, with $n = a_1 + 2a_2 + 3a_3$.

Case 1: $\mathbb{H}_{a_3}$ has positive definite form and $a_3 \ne 0$.

Therefore possible $H = \underbrace{(2,2)(0)}_{a_3 \text{ pairs}} \underbrace{(0\dots 0}_{a_1^+} \underbrace{(0\dots 0)}_{a_1^-} \underbrace{(1,1)}_{a_2 \text{ pairs}}$.

Dominant $H = (\underbrace{2\dots 2}_{2a_3}\underbrace{1\dots 1}_{a_2}\underbrace{0\dots 0}_{a_1^+})(\underbrace{1\dots 1}_{a_2}\underbrace{0\dots 0}_{a_3+a_1^-}), \quad a_1 = a_a^+ + a_1^-.$

Since $\beta(H) \ne 3$ (recall that the 4-dimensional $\mathfrak{sl}(2)$ representation does not occur)

and $a_3 \neq 0$, therefore $a_2 = 0$ (*i.e.*, no 1's).

Therefore possible dominant $H = (\underbrace{2\ldots2}_{2a_3}\underbrace{0\ldots0}_{a_1^+})(\underbrace{0\ldots0}_{a_3+a_1^-})$.

These are elements in $sp\left(2a_3 + a_1^+, a_3 + a_1^-\right)$.

Case 2: $\mathbb{H}_{a_3}$ has negative definite form and $a_3 \neq 0$.

Possible dominant $H = (\underbrace{0\ldots0}_{a_3+a_1^+})(\underbrace{2\ldots2}_{2a_3}\underbrace{0\ldots0}_{a_1^-})$,

which are in $sp\left(a_3 + a_1^+, 2a_3 + a_1^-\right)$.

Case 3: $a_3 = 0$.

In this case the 1's can appear since there is no danger that $\beta(H) = 3$.

Therefore dominant $H = (\underbrace{1\ldots1}_{a_2}\underbrace{0\ldots0}_{a_1^+})(\underbrace{1\ldots1}_{a_2}\underbrace{0\ldots0}_{a_1^-})$,

which are in $sp\left(a_2 + a_1^+, a_2 + a_1^-\right)$.

Case 4: $\mathbb{H}_{a_3}$ has an indefinite form of signature (a_3^+, a_3^-), with $a_3^+ a_3^- \neq 0$.

In this case, $H = (\underbrace{2\ldots2}_{2a_3^+}\underbrace{0\ldots0}_{a_1^+})(\underbrace{2\ldots2}_{2a_3^-}\underbrace{0\ldots0}_{a_1^-})$. Therefore, H has eigenvalue 4 on the noncompact root $(1,0,\ldots,0)(1,0,\ldots,0)$ and this case is never admissible.

Remarks. Recall that if the 5-dimensional $sl(2)$-representation occurs in $\mathrm{ad}(\mathfrak{g})|_{sl(2)}$, K_0/K_0^E is not symmetric in general, which implies that K_0^E is not spherical in general. Only in Case 1 and 2 can $\beta(H)$ equal 4. In these cases, the dominant H has the form $(\underbrace{2\ldots2}_{2a_3}\underbrace{0\ldots0}_{a_1^+})(\underbrace{0\ldots0}_{a_3+a_1^-})$. So

$$K_0 \cong U(2a_3) \times Sp(a_1^+) \times Sp(a_3 + a_1^-)$$

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and $K_0^E \cong Sp(a_3) \times Sp(a_1^+) \times Sp(a_1^-)$, which imbeds diagonally into K_0. Hence K_0^E is spherical in K_0 if and only if $Sp(a_3) \times Sp(a_1^-)$ is spherical in $U(2a_3) \times Sp(a_3 + a_1^-)$.

Proposition. *Let $m = a_3$ and $n = a_1^-$. Then K_0^E is not spherical when $m^2 - n^2 + 2mn - 2m - n > 0$.*

Proof.

$$\dim Sp(m) \times Sp(n) = 2m^2 + m + 2n^2 + n$$
$$\dim U(2m) \times Sp(m + n) = 4m^2 + 2m^2 + 2n^2 + 4mn + m + n$$
$$\dim B_0 \text{ for } U(2m) \times Sp(m + n) = 2m^2 + m + (m + n)^2 + (m + n).$$

Therefore,

$$\dim (U(2m) \times Sp(m + n))/B_0 = 3m^2 + n^2 + 2mn - m$$
$$> \dim Sp(m) \times Sp(n)$$
$$\text{when } m^2 - n^2 + 2mn - 2m - n > 0.$$

For such cases, K_0^E is not spherical. $\qquad\qquad\qquad\qquad\qquad\square$

Remarks. When the above inequality does not hold, refer to [B2] to check if K_0^E is spherical.

We summarize our results in the following theorem.

Theorem. *All admissible H are of the form*

$$(\underbrace{2\ldots2}_{2a_3}\underbrace{0\ldots0}_{a_1^+})(\underbrace{0\ldots0}_{a_3+a_1^-}), \quad or \quad (\underbrace{0\ldots0}_{a_3+a_1^+})(\underbrace{2\ldots2}_{2a_3}\underbrace{0\ldots0}_{a_1^-})$$

$$or \quad (\underbrace{1\ldots1}_{a_2}\underbrace{0\ldots0}_{a_1^+})(\underbrace{1\ldots1}_{a_2}\underbrace{0\ldots0}_{a_1^-}), \quad a_1 + 2a_2 + 3a_3 = n.$$

The first element does not appear if $a_3^2 - (a_1^-)^2 + 2a_3a_1^- - 2a_3 - a_1^- > 0$ and the second element does not appear if $a_3^2 - (a_1^+)^2 + 2a_3a_1^+ - 2a_3 - a_1^+ > 0$.

Bibliography

[B1]   Brion, M., Luna, D., Vust, Th.: Espaces homogènes sphériques, *Invent. Math.* **84**, 617–632 (1986).

[B2]   Brion, M: Classification des espaces homogènes sphériques, *Compositio Math.* **63**, 189–208 (1987).

[C-M] Collingwood, D., McGovern, W.: Nilpotent orbits in semisimple Lie algebras, Van Nostrand Reinhold Mathematics Series (1992).

[H]   Humphreys, J.: Introduction to Lie Algebras and Representation Theory, *Graduate Texts in Math.* **9**, Springer-Verlag, New York (1972).

[K-R] Kostant, B., Rallis, S.: Orbits and representations associated with symmetric spaces, *Amer. J. Math.* **93**, 753–809 (1971).

[P]   Panyushev, D.: Complexity and nilpotent orbits, Max-Planck-Institut für Math., Bonn (1993).

[S-S] Springer, T., Steinberg, R.: Conjugacy classes, in *Seminar on Algebraic Groups and Related Finite Groups, Lecture Notes in Math.* **131**, Springer-Verlag, New York, 167–266 (1970).