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AN AGGREGATE MODEL OF LARGE POWER NETWORKS AND THE FEASIBILITY SET†

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ABSTRACT

An aggregate model of large power networks that is based on a continuous DC model is presented. The DC load flow problem is formulated as a boundary value problem for a partial differential equation. The model is then used to derive an aggregated feasibility set defined in the space of lumped loads.

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1. THE FEASIBILITY SET

The feasibility set has been defined [1] as the set of substation loads that can be served in steady-state with the available generation resources without overloading the transmission lines or transformers. This feasibility set takes the form of a convex polyhedron in the space of substation load vectors, when the distribution of power flow through the network is represented either by the DC load flow [2] or by the transportation flow model. Thus, it can be described by a system of inequalities which are linear in the real bus loads. They take the following form:

$$\sum_{i\in A}(tP_i^*)L_i - \sum_{i\in A}(tP_i^*)^+ S_i \le \sum_{m\in B} \Psi_m\, |t|\, \left|b_m(P_j^* - P_k^*) + \alpha^*(z_r) - \alpha^*(z_h)\right| \qquad (1)$$

where

(1) L_i is the real bus load at bus i (in megawatts), S_i is the maximum generating capacity at bus i (in megawatts), Ψ_m is the maximum phase angle bound across branch m.

(2) A is a subset of the set of node indices and B is a subset of the set of branch indices.

(3) For each compatible choice of A and B, the coefficients P_i^* and $\alpha^*(z_r)$ are determined as the solution of a linear algebraic system; furthermore, the P_i^* coefficients constitute the solution of a DC load flow in the network with the branches of B removed.

(4) The factor t is arbitrary and can be positive or negative;

(5) the notation $(x)^+$ refers to the positive part, i.e.,

$$(x)^+ = \frac{x + |x|}{2}$$

The only assumption needed is that the power system network can be represented by a planar graph, so that the dual graph can be introduced. The coefficients $\alpha^*(z_r)$ refer to the nodes z_r of the dual graph, while the coefficients P_i refer to the nodes i of the power network graph.

A detailed derivation and interpretation of inequalities (1) is given in [1].

2. AGGREGATE FEASIBILITY SET CONCEPT

The goal of this work is to replace the exact description (1) of the feasibility set by an approximate, aggregate description, for large systems. Indeed, once system size increases substantially, the number of combinatorial variants is the choice of the sets A and B will make the above method intractable.

The aggregate description sought is of the following form:

$$\sum_{k=1}^{K}(t\bar{P}^k)\bar{L}_k - \sum_{k=1}^{K}(t\bar{P}^k)^+ \bar{S}_k \le F \qquad (2)$$

where the area covered by the power network has been divided into K subareas, $\bar{L}_k$ and $\bar{S}_k$ denote the lumped load and generating capacity corresponding to subarea k and the right-hand side F has to be determined. The aggregate feasibility set will be therefore a set in the space R^K, the dimesnion of which is the number of subareas.

The method for arriving at an aggregate description of the feasibility set uses a continuum model of the power network, where the discrete, algebraic network equations of the DC load flow model are replaced by a partial differential equation. Complete results have been obtained only in rather special cases where some "homogeneity" and "isotropy" properties arise. The analysis performed here will be confined to these cases.

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3. THE CONTINUUM VIEWPOINT

The aggregation method involves substituting a geographical description of the power system for the topological (i.e., network-based) one. To that end, the geographical domain over which the power system extends, denoted by D, is covered by a square grid. The spacings of the grid have a length denoted by Δ, and the elementary meshes of the grid are referred to as Δ-cells. On each Δ-cell, the cell variables are defined in terms of the networks variables that refer to branches and nodes contained in the cell. The detailed description in terms of network variables is thereby replaced by an aggregate description in terms of cell variables. The more nodes and branches each Δ-cell contains, the less detailed the description. One can define the *aggregation level* as the average number of nodes in a Δ-cell. Also, the *scale* can be defined as the ratio h/Δ of h(a one-dimensional measure of the extent of the domain D) to the cell side Δ. The scale is an indicator of the number of Δ-cells needed to cover the power system and therefore of the number of aggregate variables.

This approach is applicable primarily to very large systems where, simultaneously, the scale is large and the aggregation level is reasonably high. Because of the large scale, the ratio Δ/h will be small and linear expansions will be allowed. On the other hand, having more than one node in each Δ-cell will entail an economy when network variables are aggregated and replaced by cell variables.

All data pertaining to an electrical power system that are of interest here refer to network elements. The cell variables, obtained by aggregating the network variables over the respective cells, can be viewed as resulting from the discretization of continuous point functions. Namely, instead of dealing with a table of numbers - one number for each Δ-cell - one can deal with a continuous function (x,y) whose average the k^{th} Δ-cell is the k^{th} entry of the tableau. This is referred to as the continuum viewpoint: it substitutes continuous functions for discrete variables and partial differential equations for algebraic equations.

The procedure to arrive at a continuum description involves first-order expressions in Δ/h. To simplify the notation, however, the length h is taken as the unit of distance. This is equivalent to representing the power system on a map in such a way that a physical length h corresponds to a unit on the map. First-order expassions in Δ are then carried out after this normalization.

4. DC MODEL FROM THE CONTINUUM VIEWPOINT

By relying on physical analogy, it is possible to define continuous variables from the discrete voltage phase angles and power flows, by means of a cell-by-cell aggregation: in this way one obtains a continuous voltage phase angle density $v(x,y)$ and a power flow vector density $\underline{i}(x,y)$. Given loads and generation outputs, modeled respectively by a load density $\ell(x,y)$ and a generation density $g(x,y)$ (in megawatts per square mile), the voltage phase angle density $v(x,y)$ is obtained as the solution of the following boundary value problem:

$$\mathrm{div}(\underline{\sigma}\ \underline{\mathrm{grad}}\ v) = \ell(x,y) - g(x,y) \quad (x,y)\in D^{o} \tag{3}$$

$$\underline{\mathrm{grad}}\ v^{T}\ \underline{\sigma}\ \underline{n} = 0 \qquad (x,y)\in\partial D \tag{4}$$

and the power flow density $\underline{i}$ is obtained as

$$\underline{i} = -\ \underline{\sigma}\ \underline{\mathrm{grad}}\ v \tag{5}$$

The matrix $\underline{\sigma}$ is a location - dependent conductance, constructed by aggregation from the discrete branch susceptances b_m [2]. The geographical domain which contains the power system is denote by D (with interior D^o, boundary ∂D, and outward normal to the boundary $\underline{n}$). The problem (3), (4) in solvable if and only if the compatibility condition

$$\iint_D [\ell(x,y) - g(x,y)]\ dxdy = 0 \tag{6}$$

expressing the equality between total load and generation, holds; this is in complete analogy with the discrete model.

The discrete DC load flow equations [2] approach (3), (4) in the limit as $\Delta \to 0$, which means in the limit of an infinitely large system (the number of Δ-cells contained in the unit square of the map goes infinity). On the other hand, for fixed Δ, a comparison can be established between the discrete and the continuous solution; which shows that the difference goes to zero with Δ. This is so because the discrete equations constitute a finite-difference approximation of the partial differential equation. Of special interest is the homogeneous-isotropic case, where $\underline{\sigma}(x,y)$ is a constant multiple σ of the identity matrix. The partial differential equation of the DC load flow (3) then becomes the Poisson equation:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = (1/\sigma)\ [\ell(x,y) - g(x,y)] \tag{7}$$

and the boundary condition (4) is simply

$$\frac{\partial v}{\partial n} = 0$$

A straightforward example of a network that leads to a homogeneous-isotropic equation is provided by a square grid with equal branch susceptances. More general networks can also give rise to isotropic - and sometimes homogeneous - continua, when some statistical properties occur, which refer to the density of the transmission (in miles per square mile) and the orientation of the transmission lines. These properties involve: (a) the variation of the transmission density around a constant as the aggregation level increases; (b) uniformity in the distribution of branch orientations. In the homogeneous-isotropic case, the error between the discrete and the continuous DC load flow solution can be bounded explicitly in terms of Δ:

$$|v(x_i,y) - v_i| \leq c\ \Delta^2 |\log\Delta| \tag{8}$$

where c is a constant independent of Δ.

5. DERIVIATION OF THE AGGREGATE FEASIBILITY SET

Application of the continuum viewpoint to the DC load flow problem, eqs. (3), (4), makes it possible to derive an aggregate description for the feasibility set, eq. (2), which approximates the exact description (1) for large systems. To this end, the quantities occuring on both sides of (1) must be modeled from the continuum viewpoint.

The phase angle bounds Ψ_m are modeled by means of a function $\Psi(x,y)$, where

$$\Psi(x,y) = \max_{\text{over } \underline{n}} \Psi(x,y,\underline{n}) \qquad (9)$$

and $\Psi(x,y,\underline{n})$ is the ratio of the phase angle bound Ψ_m for the branch m in direction $\underline{n}$ to the length ℓ_m of that branch. The P_i and $\alpha(z_r)$ coefficients are modeled by functions $P(x,y)$ and $\bar{\alpha}(x,y)$, which are solutions of continuous DC load flow problems that approximate the discrete load flow problem of which P_i and $\alpha(z_r)$ are, respectively, solutions. From now on, homogeneity and isotropy are assumed to hold. The bound (8) then applies to the error between P_i and $P(x,y)$, or to that between $\alpha(z_r)$ and $\alpha(x,y)$. An explicit assumption is also made about the scale and the aggregation level: (1) the *scale* is *large*, so that Δ is small as compared to the extent of the domain D; (2) the *aggregation level* is *high* enough so that the length ℓ of a branch is smaller than Δ. These assumptions, reasonable for a large system, imply that the two extremities of a branch of B can be modeled as infinitely close to each other (Fig. 1). As a result, the function $P(x,y)$ becomes the potential created by a dipole [3] located at the limit location ζ and directed along the straight line from source to sink. In the case where B consists of just one branch, the following expression results for $P(x,y)$:

$$P(x,y) = \frac{M}{2\pi\sigma} \frac{\cos\theta}{r} + C_2 \qquad (10)$$

where M is the moment of the dipole, C_2 is an arbitrary additive constant and θ,r are, respectively, the angle as defined in Fig. 1 and the distance from (x,y) to ζ.

Also, since $P(x,y)$ is a harmonic function (except at the singularity ζ, location of the dipole) there exists [4] an analytic function of the complex variable z=x+iy of which the imaginary part is $\sigma P(x,y)$.

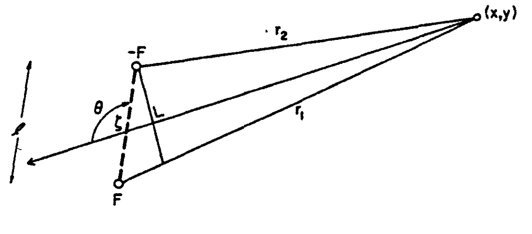

Figure 1 Branch of B as a Dipole

It turns out that

$$A(z) = \alpha(x,y) + i\,\sigma\,P(x,y) \qquad (11)$$

and, from (10), one can infer that

$$A(z) = \frac{M}{2\pi} \frac{e^{i\gamma}}{z-\zeta} + C \qquad (12)$$

where, in (12), i is the imaginary unit, C is a complex constant* and γ is an angle that determines the dipole orientation. For a number of dipoles greater than one, A(z) is obtained by the principle of superposition.

Finally, the feasibility conditions are modeled as follows from the continuum viewpoint (the determination of the right-hand side requires some complex analysis):

$$\iint\{\ell(x,y)\ t\,P(x,y) - g(x,y)[tP(x,y)]^+\}dxdy$$
$$\leq \sum_{k=1}^{K} \Psi(\zeta_k)|M_k| \quad \text{for } t > 0 \text{ and } t < 0 \qquad (13)$$

One such pair of inequalities arises for each choice of:

- an integer ν not larger then the number of generator nodes;
- a node subset $\bar{A}$ with ν generator nodes where P is required to vanish.
- a distribution of ν dipoles located at $\zeta_1,\dots,\zeta_\nu$ with given orientation $\gamma_1,\dots,\gamma_\nu$ and moments $M_1,\dots,M_\nu$ to be determined.

Given $\bar{A}$ and the dipole distributions, the function $P(x,y)$ is obtained as

$$P(x,y) = (1/2\pi\sigma)\sum_{k=1}^{\nu} M_k \frac{\cos\theta_k}{r_k}$$
$$= (1/2\pi\sigma)\sum_{k=1}^{\nu} M_k \,\mathrm{Im}\, \frac{e^{i\gamma_k}}{z-\zeta_k} + C_2 \qquad (14)$$

The function P is completely determined -up to a factor - by (14) and the requirement the P vanish on the ν points of $\bar{A}$.

The *aggregate* feasibility set, then, can be defined by the following inequalities in the lumped loads $\bar{L}_k$ and generating capacities $\bar{S}_k$:

$$\sum_{k=1}^{K} t\,\bar{P}^k\,\bar{L}_k - \sum_{k=1}^{K} (t\bar{P}^k)^+\,\bar{S}^k$$
$$\leq \sum_{j=1}^{\nu} \Psi(\zeta_j)\ |M_j|\ |t| \qquad (15)$$

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*$i^2 = -1$; also, the notation ζ refers to the complex number representation of the point ζ.

where $\bar{P}^k$ is the average of the function P(x,y) over sub-area k. Thus, the use of the continuum viewpoint has made possible a procedure for deriving aggregate inequalities of the type (2) to approximate the inequalities (1). The bound (8) implies that, when there is no aggregation (i.e., each subarea k contains just one node), (15) gets nearer to (1), and the more so, the larger the system is.

6. NEIGHBORHOODS OF INFLUENCE

From (14) it is clear that, when all dipoles are far enough from (x,y) (so that $r_k \to \infty$), P(x,y) goes to a constant. If, furthermore, the points of $\bar{A}$ are also far enough from the dipoles, then P(x,y) vanishes since, for any a_i in $\bar{A}$,

$$P(a_i) \simeq P(\infty) = C_2$$

but, by definition of $\bar{A}$,

$$P(a_i) = 0$$

Accordingly, if the load is located in the domain E, then, for a given choice of $\bar{A}$, it is not necessary to consider conditions induced by dipoles located far enough from the domain E and from $\bar{A}$, because these conditions, in which the left-hand side of (13) is nearly zero, are trivially satisfied. From the explicit expression for P(x,y), it is possible to define precisely what is meant by "far enough", i.e., to define neighborhoods of influence around a point of $\bar{A}$ and a domain E, to which dipoles can be restricted.

The approximation of P_k and $\alpha(z_r)$ by P(x,y), as derived from (12), has been tested satisfactorily on a numerical example ([5], [6]).

7. REFERENCES

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