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Thermodynamics of Electrical Noise: A Frequency-Domain Inequality for Linear Networks

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Abstract

This paper addresses the frequency-domain characterization of electrical noise in linear time-invariant distributed networks. A new general inequality is derived. The flow of noise power from one source to another through a lossless coupling network is shown to obey a relation analogous to the second law of thermodynamics. The noise sources can be essentially any stationary random processes; in particular they need not be Gaussian and need not represent thermal noise. In this sense the inequality is quite general. Proofs are based on standard techniques from the theory of linear circuits and random signals; thermodynamic concepts are used only for motivation and interpretation.

I. Introduction

Unlike a block diagram for a control system or the flow chart of a computer program, a circuit diagram inevitably represents energy flow and dissipation as well as signal propagation. It is therefore not surprising that circuit theory and thermodynamics should, on occasion, join hands. One of the earliest examples is the Maxwell 'minimum heat' theorem [1], which describes the distribution of current in a linear resistive reciprocal network. Applied to a larger class of physical systems, this result has re-emerged as Prigogine's "principle of minimum entropy production" in nonequilibrium thermodynamics [2]. Another example is the classical Nyquist-Johnson model for thermal noise in a resistor [3, 4], which was derived using a mixture of circuit-theoretic and thermodynamic concepts.

We believe that some aspects of the noise behavior of electrical networks can best be understood using the concepts (but not necessarily the analytical techniques) of statistical physics, even in cases where the noise is not thermal in origin. More recent research on the thermodynamic behavior of noisy electrical networks includes [5-16].

The main result of the present paper is the general frequency-domain inequality (2.14) describing the steady-state flow of noise power in distributed linear networks. The noise signals can be essentially any stationary random processes, and the proof uses only standard concepts from the theory of random signals and linear systems. In the special case that the noise sources represent thermal noise in a linear resistor, the inequality has a natural interpretation in terms of the second law of thermodynamics (increase of entropy). Therefore one can view the result as a generalization, within the linear time-invariant framework, of the second law. The result also suggests that, if we view equivalent noise temperature as a driving force analogous to voltage, then lossless systems act as "passive" conductors of noise power.

II. Definitions, Assumptions, and Statement of the Main Result

2.1) Definitions and Assumptions

Definition 2.1

Let E be the expectation operator. A random process x is said to be wide-sense stationary [17] if its mean and variance are finite and time-independent and $E\{x(t)x(s)\}$ depends only on t-s. Two random processes x and y are said to be jointly wide-sense stationary if each is wide-sense stationary and $E\{x(t)y(s)\}$ depends only on t-s.

Definition 2.2

Let x and y be two jointly wide-sense stationary random processes. The cross-correlation function R_{xy} and the cross-spectral density S_{xy} are defined as usual by

$$R_{xy}(\tau) \triangleq E\{x(t+\tau)y(t)\}$$

$$S_{xy}(\omega) \triangleq \int_{-\infty}^{\infty} R_{xy}(\tau)e^{-j\omega\tau}d\tau = \mathcal{F}\{R_{xy}\} \tag{2.1}$$

where $\mathcal{F}$ is the Fourier transform. If x and y are identical, then R_{xx} is called the autocorrelation function and S_{xx} is called the power spectral density of x. We assume that all spectral densities encountered in this paper are well-defined, i.e. that the integral in (2.1) exists and is finite for each ω.

The standard Nyquist-Johnson model [3, 4] for the thermal noise in a linear resistor R at a temperature T degrees Kelvin is obtained by adding a noise voltage source n(t) in series with R as shown in Fig. 1a. The power spectral density S_{nn} is flat, at least out to optical frequencies, with magnitude $S_{nn}(\omega)$[1] = 2kTR, where k is Boltzmann's constant.

More general 1-ports can be assigned an "equivalent noise temperature" on the basis of a Thévenin equivalent network as shown in Fig. 1b. The following definition is standard [18].

Definition 2.3

For the 1-port in Fig. 1b assume that n is wide-sense stationary and $\mathrm{Re}\{Z(j\omega)\} > 0$ for all ω. The frequency-dependent equivalent noise temperature $T(\omega)$ is defined by

[1] The early references, e.g. [3, 4], define power spectral density only for $\omega \geq 0$ and hence write $S_{nn}(\omega) = 4kTR$.

$$T(\omega) = S_{nn}(\omega)/2kRe\{Z(j\omega)\}, \tag{2.2}$$

where k is Boltzmann's constant.[2]

This definition is physically reasonable, since for the Nyquist-Johnson model in Fig. 1a the equivalent noise temperature is equal to the physical resistor temperature, i.e. $T(\omega) \equiv T$, at least out to optical frequencies.

The results in this paper concern a linear time-invariant network consisting of noisy dissipative elements and noise-free lossless elements. The noisy dissipative elements have been extracted and displayed separately in Thévenin equivalent form in Fig. 2.

Let $\overline{P}_m$ represent the average power entering noise source #m in Fig. 2. If $S_{v_m i_m}(\cdot)$ is integrable, then

$$\begin{aligned} \overline{P}_m &= -E\{v_m(t)i_m(t)\} = -R_{v_m i_m}(0) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} -S_{v_m i_m}(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} -Re\left\{S_{v_m i_m}(\omega)\right\}d\omega, \end{aligned} \tag{2.3}$$

where the last equation holds because $Im\left\{S_{v_m i_m}(\omega)\right\}$ is an odd function and hence integrates to zero. This motivates the following definition.

Definition 2.4

The average power per unit bandwidth entering noise source #m is denoted $P_m(\omega)$ and defined by

$$P_m(\omega) \triangleq \frac{-1}{2\pi} Re\left\{S_{v_m i_m}(\omega)\right\}, \quad 1 \leq m \leq N. \tag{2.4}$$

[2]Since $k = 1.38 \times 10^{-16}$ erg/°K, $T(\omega)$ has units of degrees Kelvin provided we assign $Re\{Z(j\omega)\}$ the units of ohms.

Therefore, from (2.3),

$$\overline{P}_m = \int_{-\infty}^{\infty} P_m(\omega)\,d\omega. \tag{2.5}$$

Assumptions on the Noise Sources

The noisy dissipative elements (i.e. the noise sources) are represented in Thévenin form in Fig. 2, where the noise source characteristic impedances $Z_m(s)$, $1 \leq m \leq N$, appear in series with noisy independent voltage sources $n_m(t)$, $1 \leq m \leq N$. The voltage sources are zero mean, uncorrelated, wide-sense stationary random processes with autocorrelation functions $R_{n_m n_m}(\tau)$ and continuous[3] power spectral densities $S_{n_m n_m}(\omega)$. It is a standard fact that all power spectral densities are real and nonnegative: we assume further in this case that

$$S_{n_m n_m}(\omega) > 0, \forall\omega, 1 \leq m \leq N. \tag{2.6}$$

With the noise voltage source n_m set to zero, let $g_m(t)$ be the voltage response of dissipative element #m to an impulse of current applied at t = 0. In general $g_m(\cdot)$ may contain singularity functions and need not be causal, but we assume $g_m(\cdot)$ real and that for all $m \in \{1,\ldots,N\}$ the bilateral Laplace transform

$$Z_m(s) \triangleq \int_{-\infty}^{\infty} g_m(t)e^{-st}dt \tag{2.7}$$

[3]A number of technical assumptions in this section are used only in the proof of Prop. 1, which appears in the Appendix. Among them are i) the continuity of the power spectral densities $S_{n_m n_m}(\omega)$ and the cross-spectral densities $S_{v_m i_m}(\omega)$, ii) the existence of an impedance matrix for L and the assumption that it can be defined on D by analytic continuation, and iii) the assumption that the scattering matrix can be defined for all ω by analytic continuation. These restrictions have mathematical, but little or no engineering significance, and can be ignored in reading the body of this paper.

exists and is analytic in the closed right half plane $\mathrm{Re}\{s\} \geq 0$, and that

$$\mathrm{Re}\{Z_m(j\omega)\} > 0, \ \forall\omega, \ 1 \leq m \leq N. \tag{2.8}$$

Assumptions on L

In Fig. 2 all the lossless elements in the circuit have been collected in the lossless multiport L for convenience. Assume that L has a current - controlled representation,[3] and let $h_{ij}(t)$, $1 \leq i, j \leq N$ represent the voltage response at port #i of L to an impulse of current applied at port #j at $t = 0$ when all other ports are left open circuited, i.e. $\{h_{ij}(\cdot)\}$ is the impulse response matrix of L. In general $\{h_{ij}(\cdot)\}$ may contain singularity functions and need not be causal, but we assume $\{h_{ij}(\cdot)\}$ real, and that for all $i, j \in \{1, \ldots, N\}$ the bilateral Laplace transform

$$\ell_{ij}(s) \triangleq \int_{-\infty}^{\infty} h_{ij}(t)e^{-st}dt \tag{2.9}$$

exists and is analytic in some maximal open right half plane $\mathrm{Re}\{s\} > \sigma_0$. Let $\underset{\sim}{L}(s) \triangleq \{\ell_{ij}(s)\}$ be the impedance matrix of L. Since L is lossless we expect to encounter $j\omega$-axis poles, and hence typically $\sigma_0 \geq 0$. We assume,[3] however, that $\underset{\sim}{L}(\cdot)$ can be analytically continued to a region of the complex plane that includes a dense subset D of the $j\omega$-axis. By a slight abuse of notation we denote this portion of the analytic continuation of the impedance matrix by $\underset{\sim}{L}(j\omega), \forall j\omega \in D$.

Notation

Let $\underset{\sim}{M}^*$ denote the complex-conjugate tranpose of a matrix $\underset{\sim}{M}$, and $\{\underset{\sim}{M}\}_{ij}$ denote its (i, j) entry.

Definition 2.5

Our definition of losslessness for L is the standard frequency domain condition

$$\underset{\sim}{L}(j\omega) = -\underset{\sim}{L}^*(j\omega), \ \forall j\omega \in D. \qquad (2.10)$$

The Scattering Representation

Let $\underset{\sim}{Z}(s)$ be the diagonal N x N matrix containing the noise source impedances along its diagonal, and define $\underset{\sim}{R}(\omega)$ by

$$\underset{\sim}{R}(\omega) \triangleq \mathrm{Re}\{\underset{\sim}{Z}(j\omega)\}. \qquad (2.11)$$

The scattering matrix for L (normalized to the port impedances Z_m) is defined as usual [19] by

$$\underset{\sim}{S}(j\omega) = \underset{\sim}{R}^{1/2}(\omega)\left[\underset{\sim}{L}(j\omega) + \underset{\sim}{Z}(j\omega)\right]^{-1}\left[\underset{\sim}{L}(j\omega) - \underset{\sim}{Z}^*(j\omega)\right]\underset{\sim}{R}^{-1/2}(\omega). \qquad (2.12)$$

We show in the Appendix that $[\underset{\sim}{L}(j\omega) + \underset{\sim}{Z}(j\omega)]^{-1}$ exists for all $j\omega \in D$, so it follows that (2.12) defines $\underset{\sim}{S}(j\omega)$ throughout D.

Finally we assume[3] that $\underset{\sim}{S}(j\omega)$ as defined above can be analytically continued to some neighborhood of the $j\omega$-axis, that the entire network has evolved to a steady state in which the random processes $\underset{\sim}{v}$, $\underset{\sim}{i}$ and $\underset{\sim}{n}$ are jointly wide-sense stationary, and that the cross-spectral densities $S_{v_m i_m}(\omega)$, $1 \le m \le N$, exist and are continuous for all ω.

It would have been sufficient to impose the stronger requirement that $\underset{\sim}{L}(\cdot)$ and $\underset{\sim}{Z}(\cdot)$ be positive real, for then many of the specific assumptions listed in this section would have followed automatically. But the weaker assumptions given are actually all that is needed. Note that L and the noise source impedances need not be lumped, passive or causal, L need not be reciprocal, and the noise voltage sources need not be Gaussian in nature or thermal in origin.

2.2) The Main Result

Let the equivalent noise temperature of noise source #m in Fig. 2 be denoted $T_m(\omega)$, $1 \leq m \leq N$. Assumptions (2.6) and (2.8) imply that $T_m(\omega) > 0$, $\forall\omega$, $1 \leq m \leq N$. The main result of this paper is (2.14) below.

Theorem 1

For any network of the form shown in Fig. 2, under the assumptions given,

$$\sum_{m=1}^{N} P_m(\omega) = 0, \ \forall\omega \in \mathbb{R}. \tag{2.13}$$

$$\sum_{m=1}^{N} (P_m(\omega)/T_m(\omega)) \geq 0, \ \forall\omega \in \mathbb{R}. \tag{2.14}$$

III. Proof of Theorem 1

We first develop some preliminary machinery.

Definition 3.1

Let $\underset{\sim}{S}_{\underset{\sim}{n}\underset{\sim}{n}}(\omega)$ be the cross-spectral density matrix of the noise voltage sources, i.e.

$$\left\{\underset{\sim}{S}_{\underset{\sim}{n}\underset{\sim}{n}}(\omega)\right\}_{i,j} = S_{n_i n_j}(\omega). \tag{3.1}$$

Extending Def. 2.3, define the <u>equivalent temperature matrix</u> $\underset{\sim}{T}(\omega)$ by

$$\underset{\sim}{T}(\omega) \triangleq \frac{1}{2k} \underset{\sim}{S}_{\underset{\sim}{nn}}(\omega)\underset{\sim}{R}^{-1}(\omega), \tag{3.2}$$

where k is Boltzmann's constant, and $\underset{\sim}{R}(\omega)$ is as defined in (2.11). Since $\underset{\sim}{S}_{\underset{\sim}{nn}}$ and $\underset{\sim}{R}$ are diagonal, $\underset{\sim}{T}$ is also.

For $j\omega \in D$; we let $\underset{\sim}{S}(j\omega)$ denote the analytic continuation of the scattering matrix for L, as defined in (2.12) for $j\omega \in D$.

<u>Proposition 1</u>

Using the notation above, the average power per unit bandwidth entering port #m of L in Fig. 2, denoted $-P_m(\omega)$ for consistency with Def. 2.4, is given for <u>all</u> real ω by

$$-P_m(\omega) = \frac{k}{4\pi} \left\{\underset{\sim}{T}(\omega) - \underset{\sim}{S}(j\omega)\underset{\sim}{T}(\omega)\underset{\sim}{S}^*(j\omega)\right\}_{m,m}, \quad 1 \leq m \leq N. \tag{3.3}$$

The diagonal entries of $\underset{\sim}{T}(\omega)$ are the power spectral density analogs of the "incident waves" in scattering theory, and the diagonal entries of $\underset{\sim}{S}(j\omega)\underset{\sim}{T}(\omega)\underset{\sim}{S}^*(j\omega)$ are analogous to the reflected waves. This analogy can be used to arrive at a brief heuristic "derivation" of (3.3), but the rigorous proof, given in the Appendix, is somewhat involved.

<u>Notation</u>

The following notation and terminology apply to square matrices. If $\underset{\sim}{A} = \underset{\sim}{A}^*$, then $\underset{\sim}{A}$ is said to be <u>Hermitian</u>. If $\underset{\sim}{A}$ is invertible and $\underset{\sim}{A}^{-1} = \underset{\sim}{A}^*$, then $\underset{\sim}{A}$ is said to be <u>unitary</u>. Let $tr\{\underset{\sim}{A}\}$ denote the <u>trace</u>, i.e. the sum of the diagonal entries, of $\underset{\sim}{A}$, and $det\{\underset{\sim}{A}\}$ denote the determinant of $\underset{\sim}{A}$.

Proposition 2

Let $\underset{\sim}{P}$ and $\underset{\sim}{U}$ be complex N x N matrices. If $\underset{\sim}{P}$ is positive definite and Hermitian and $\underset{\sim}{U}$ is unitary, then

$$\operatorname{tr}\{\underset{\sim}{P}^{-1}\underset{\sim}{U}\underset{\sim}{P}\underset{\sim}{U}^*\} \geq N. \tag{3.4}$$

Proof of Proposition 2

It is a standard result in linear algebra that the trace of a matrix is the sum of its eigenvalues, and that the eigenvalues of a matrix are invariant under similarity transformations. Therefore

$$\operatorname{tr}\{\underset{\sim}{P}^{-1}\underset{\sim}{U}\underset{\sim}{P}\underset{\sim}{U}^*\} = \operatorname{tr}\{\underset{\sim}{P}^{1/2}(\underset{\sim}{P}^{-1}\underset{\sim}{U}\underset{\sim}{P}\underset{\sim}{U}^*)\underset{\sim}{P}^{-1/2}\} = \operatorname{tr}\{\underset{\sim}{P}^{-1/2}\underset{\sim}{U}\underset{\sim}{P}\underset{\sim}{U}^*\underset{\sim}{P}^{-1/2}\}. \tag{3.5}$$

But $\underset{\sim}{P}^{-1/2}\underset{\sim}{U}\underset{\sim}{P}\underset{\sim}{U}^*\underset{\sim}{P}^{-1/2} = (\underset{\sim}{P}^{-1/2}\underset{\sim}{U})\underset{\sim}{P}(\underset{\sim}{P}^{-1/2}\underset{\sim}{U})^*$ is Hermitian and positive definite, so its eigenvalues λ_i, $1 \leq i \leq N$, are real and positive. By the standard inequality of arithmetic and geometric means [20],

$$\frac{1}{N}\sum_{i=1}^{N}\lambda_i \geq \left(\prod_{i=1}^{N}\lambda_i\right)^{1/N}. \tag{3.6}$$

It is also a standard result in linear algebra that the product of the eigenvalues of a matrix is equal to its determinant, and that the determinant of a product of matrices equals the product of the determinants. Using these facts in (3.5) and (3.6), we have

$$\operatorname{tr}\{\underset{\sim}{P}^{-1}\underset{\sim}{U}\underset{\sim}{P}\underset{\sim}{U}^*\} \geq N(\det\{\underset{\sim}{P}^{-1}\underset{\sim}{U}\underset{\sim}{P}\underset{\sim}{U}^*\})^{1/N} = N,$$

since $\det\{\underset{\sim}{P}^{-1}\} = (\det\{\underset{\sim}{P}\})^{-1}$ and $\det\{\underset{\sim}{U}^*\} = \det\{\underset{\sim}{U}^{-1}\} = (\det\{\underset{\sim}{U}\})^{-1}$. ∎

Proof of Theorem 1

Since L is lossless, it is a standard fact of circuit theory [19] that $\underset{\sim}{S}(j\omega)$ is unitary[4], $\forall\omega \in \mathbb{R}$. Using this fact in (3.3) we have

$$\sum_{m=1}^{N} P_m(\omega) = \frac{-k}{4\pi}\left[\operatorname{tr}\{\underset{\sim}{T}(\omega)\} - \operatorname{tr}\{\underset{\sim}{S}(j\omega)\underset{\sim}{T}(\omega)\underset{\sim}{S}^*(j\omega)\}\right] = 0,$$

since the trace is invariant under similarity transformations. This proves (2.13).

Since the noise voltage sources are zero mean and uncorrelated, it follows from (2.6) that $\underset{\sim}{S}_{\underset{\sim}{nn}}(\omega)$ is diagonal and positive definite. And since $\underset{\sim}{R}(\omega)$ is diagonal and positive definite, it follows from (3.2) that $\underset{\sim}{T}(\omega)$ is also diagonal and positive definite. Referring to (3.3), we see that

$$\sum_{m=1}^{N} (P_m(\omega)/T_m(\omega)) = \frac{k}{4\pi} \operatorname{tr}\{\underset{\sim}{T}^{-1}(\omega)(\underset{\sim}{S}(j\omega)\underset{\sim}{T}(\omega)\underset{\sim}{S}^*(j\omega) - \underset{\sim}{T}(\omega))\}$$

$$= \frac{k}{4\pi} \operatorname{tr}\left\{\underset{\sim}{T}^{-1}(\omega)\underset{\sim}{S}(j\omega)\underset{\sim}{T}(\omega)\underset{\sim}{S}^*(j\omega) - \underset{\sim}{I}\right\} = \frac{k}{4\pi}\left[\operatorname{tr}\left\{\underset{\sim}{T}^{-1}(\omega)\underset{\sim}{S}(j\omega)\underset{\sim}{T}(\omega)\underset{\sim}{S}^*(j\omega)\right\} - N\right]. \quad (3.7)$$

Since $\underset{\sim}{S}(j\omega)$ is unitary, it follows from Prop. 2 that the right hand side of (3.7) is nonnegative. This proves (2.14). ∎

[4]The reader can also prove this fact directly from eqs. (2.10), (2.12) and (A.3).

IV Concluding Remarks

4.1) Interpretation of Theorem 1 and its Relation to Known Results

In Theorem 1, (2.13) merely says that the average power leaving L_1 through all its ports is zero in each portion of the spectrum. Since L_1 is lossless, this is merely a restatement in the stochastic domain of the well-known "zero average power" interpretation of losslessness [21]. It can also be viewed as a "per unit bandwidth" statement of the first law of thermodynamics (conservation of energy) for linear time-invariant systems.

Equation (2.14) is the new result in Theorem 1. It can be viewed as a generalization, within the linear time-invariant framework, of the second law of thermodynamics (increase of entropy). The physical grounds for this interpretation become apparent in the special case that each noise source is a Nyquist-Johnson model for resistor thermal noise, as in Fig. 1a. Then each noise source represents a resistor of value R_m in contact with a constant temperature reservoir at a temperature T_m. And from classical thermodynamics [22], $\overline{P}_m/T_m$ is the rate of increase of the entropy of the m-th constant temperature reservoir (see (2.3) and (2.5)). The sum of all N such terms is then the total rate of increase of entropy, and hence

$$\sum_{m=1}^{N} (\overline{P}_m/T_m) \geq 0 \qquad (4.1)$$

by the second law[5]. The inequality (2.14) is a <u>generalization</u> of (4.1) in that i) it holds at each point of the spectrum separately, and ii) it holds for non-thermal noise sources as well.

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[5]The electrical variables of the circuit are assumed to be in steady state, so the entropy contribution arising from electrical signals is independent of time. Therefore the second law asserts that the remaining entropy terms, i.e. the entropies of the constant temperature reservoirs, must increase as claimed.

In the special case N=2, Theorem 1 predicts that at each frequency noise power flows from the "hotter" source to the "colder" one. To see this, renumber the ports if necessary so that $T_1(\omega) > T_2(\omega)$, where ω is any frequency of interest. Equation (2.13) states that $P_2(\omega) = -P_1(\omega)$, and (2.14) then predicts that

$$\begin{aligned} &P_2(\omega)/T_2(\omega) + P_1(\omega)/T_1(\omega) \\ &= P_2(\omega)\left[1/T_2(\omega) - 1/T_1(\omega)\right] \geq 0, \end{aligned} \tag{4.2}$$

i.e. that $P_2(\omega) \geq 0$ whenever $T_1(\omega) > T_2(\omega) > 0$.

A particularly simple "passivity" interpretation is also possible in this case. If we multiply the left hand side of (4.2) by $T_1(\omega)\ T_2(\omega)$ and substitute $-P_1(\omega)$ for $P_2(\omega)$, etc., we have

$$T_1(\omega)(-P_1(\omega)) + T_2(\omega)(-P_2(\omega)) \geq 0,\ \forall\omega \in \mathbb{R}. \tag{4.3}$$

Recall from our sign conventions that $(-P_m(\omega))$ is the average power per unit bandwidth flowing <u>into</u> port #m of L. With this sign convention and the fact that $T(\omega)$ and $P(\omega)$ are real taken into account, (4.3) is analogous to the following necessary condition for passivity of an LTI dynamic 2-port:

$$\mathrm{Re}\{V_1(j\omega)I_1^*(j\omega) + V_2(j\omega)I_2^*(j\omega)\} \geq 0,\ \forall\omega.$$

In circuit theory terms the significance of (4.3) is that a difference in equivalent noise temperature acts as a "driving force" or an "across variable" for noise power flow separately at each frequency, just as a thermodynamic temperature difference acts as a driving force for heat flow. Furthermore the lossless (but possibly active) 2-port *L* acts as a "passive" conductor of noise power flow. But note that (4.3) cannot be extended to cases where $N > 2$. However in the general case (2.14) tells us that the N-port *L* acts as a "passive" system with respect to the driving forces $\{(-Tm(\omega))^{-1}\}$. Note that $(-Tm(\omega))^{-1}$ is a monotone increasing function of $Tm(\omega)$ for $Tm(\omega) > 0$.

Another result in the literature also suggests that lossless (but possibly active) networks act like passive systems with regard to equivalent temperature. Suppose we extract an arbitrary pair of nodes from *L* to form a new port, shown as port #0 in Fig. 3. Then it turns out that at each ω the equivalent noise temperature observed at port #0 will lie between the highest and the lowest of the noise source equivalent temperatures, i.e.

$$\min_{1 \leq m \leq N} \left\{T_m(\omega)\right\} \leq T_0(\omega) \leq \max_{1 \leq m \leq N} \left\{T_m(\omega)\right\}, \ \forall\omega.$$

This inequality follows immediately from results given in [23, 24], and the present interpretation was suggested in [25, p. 78]. It is identical in form to the classical "no gain" theorem [25, p. 42], [26] for networks of passive 2-terminal resistors.

4.2) Directions for Further Research

It should be possible to generalize this result to include systems in which equivalent temperatures can be negative. It might also be possible to allow for correlated noise voltage sources or coupled noise source impedances. If L is nonlinear and lossless it is plausible that (2.14) holds when integrated over the whole spectrum even though it may not hold for each ω separately. However, the Manley-Rowe formulas [27], appropriately generalized to the stochastic case, may enable one to establish bounds on the noise power flow at individual frequencies.

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Figure Captions

Fig. 1 a. The Nyquist-Johnson model for thermal noise in a resistor of value R at a temperature T°K. The voltage source is Gaussian white noise with the flat spectral density $S_{nn}(\omega) = 2\ kTR$.

b. A more general system in that the noise voltage spectrum need not be Gaussian or flat and the impedance need not be real or constant. The frequency-dependent "equivalent noise temperature" $T(\omega)$ is defined by analogy with the Nyquist-Johnson model.

Fig. 2 Figure for Theorem 1. Noise power flows from one noise source to another through the lossless N-port L, subject to the constraints of Theorem 1.

Fig. 3 The same network as in Fig. 2, but with an arbitrary pair of nodes extracted from L to form port #0. Existing results show that the equivalent noise temperature at port #0 is a convex combination of the noise source temperatures, establishing a "no gain" type condition on equivalent noise temperature.

Appendix: Proof of Proposition 1

A.1) Preliminary Results

Let $j\omega$ be an arbitrary point in D. To simplify notation we will write $\underset{\sim}{S}_{nn}$, $\underset{\sim}{Z}$, $\underset{\sim}{R}$, $\underset{\sim}{T}$, $\underset{\sim}{L}$, $\underset{\sim}{\mathcal{S}}$ and P_m in place of $\underset{\sim}{S}_{nn}(\omega)$, $\underset{\sim}{Z}(j\omega)$, etc. For a complex matrix $\underset{\sim}{M}$ we define $\underset{\sim}{M}_R \triangleq \text{Re}\{\underset{\sim}{M}\}$, $\underset{\sim}{M}_I = \text{Im}\{\underset{\sim}{M}\}$.

Fact 1

The matrix $(\underset{\sim}{L} + \underset{\sim}{Z})$ is nonsingular.

Proof

Define $\underset{\sim}{A} \triangleq \underset{\sim}{L} + \underset{\sim}{Z}$. The goal will be to show that $\underset{\sim}{B}\underset{\sim}{A} = \underset{\sim}{I}$, where the N x N matrix $\underset{\sim}{B}$ is defined by

$$\underset{\sim}{B}_R \triangleq \left(\underset{\sim}{A}_R + \underset{\sim}{A}_I^T\underset{\sim}{A}_R^{-1}\underset{\sim}{A}_I\right)^{-1}$$

$$\underset{\sim}{B}_I \triangleq -\underset{\sim}{B}_R\underset{\sim}{A}_I^T\underset{\sim}{A}_R^{-1}. \tag{A.1}$$

It then follows that $\det(\underset{\sim}{A})\det(\underset{\sim}{B}) = 1$ and hence that $\det(\underset{\sim}{A}) \neq 0$.

We must first establish that all the matrix-inverses appearing in (A.1) exist. By (2.10) $\underset{\sim}{L}_R$ is antisymmetric. And $\underset{\sim}{Z}_R = \underset{\sim}{R}$ is diagonal and positive definite by assumption. Therefore $\underset{\sim}{A}_R = (\underset{\sim}{L}_R + \underset{\sim}{Z}_R)$ is positive definite and hence nonsingular. Similarly, $\left(\underset{\sim}{A}_R + \underset{\sim}{A}_I^T\underset{\sim}{A}_R^{-1}\underset{\sim}{A}_I\right)$ is positive definite and hence nonsingular. Therefore $\underset{\sim}{B}_R$ and $\underset{\sim}{B}_I$ are well-defined.

To show that $\underset{\sim}{B}\underset{\sim}{A} = \underset{\sim}{I}$ we calculate

$$\text{Re}\{\underset{\sim}{B}\underset{\sim}{A}\} = \text{Re}\{(\underset{\sim}{B}_R + j\underset{\sim}{B}_I)(\underset{\sim}{A}_R + j\underset{\sim}{A}_I)\}$$

$$= \underset{\sim}{B}_R\underset{\sim}{A}_R - \underset{\sim}{B}_I\underset{\sim}{A}_I = \underset{\sim}{B}_R(\underset{\sim}{A}_R + \underset{\sim}{A}_I^T\underset{\sim}{A}_R^{-1}\underset{\sim}{A}_I)$$

$$= \underset{\sim}{B}_R\underset{\sim}{B}_R^{-1} = \underset{\sim}{I},$$

$$\mathrm{Im}\{\underset{\sim}{B}\underset{\sim}{A}\} = \mathrm{Im}\{(\underset{\sim}{B}_R + j\underset{\sim}{B}_I)(\underset{\sim}{A}_R + j\underset{\sim}{A}_I)\}$$

$$= \underset{\sim}{B}_I\underset{\sim}{A}_R + \underset{\sim}{B}_R\underset{\sim}{A}_I = -\underset{\sim}{B}_R\underset{\sim}{A}_I^T\underset{\sim}{A}_R^{-1}\underset{\sim}{A}_R + \underset{\sim}{B}_R\underset{\sim}{A}_I$$

$$= \underset{\sim}{B}_R(-\underset{\sim}{A}_I^T + \underset{\sim}{A}_I) = \underset{\sim}{0},$$

since $\underset{\sim}{A}_I = \underset{\sim}{L}_I + \underset{\sim}{Z}_I$ is symmetric.

Fact 2

$$\underset{\sim}{T} - \underset{\sim}{S}\underset{\sim}{T}\underset{\sim}{S}^* = 2\underset{\sim}{R}^{1/2}(\underset{\sim}{L} + \underset{\sim}{Z})^{-1}(\underset{\sim}{T}\underset{\sim}{L}^* + \underset{\sim}{L}\underset{\sim}{T})(\underset{\sim}{L} + \underset{\sim}{Z})^{-1*}\underset{\sim}{R}^{1/2} \qquad (A.2)$$

Proof

Using the definitions of $\underset{\sim}{T}$ and $\underset{\sim}{S}$ we expand the left hand side of (A.2) as follows:

$$\{\underset{\sim}{T} - \underset{\sim}{S}\underset{\sim}{T}\underset{\sim}{S}^*\} = \{\underset{\sim}{T} - \underset{\sim}{R}^{1/2}(\underset{\sim}{L} + \underset{\sim}{Z})^{-1}(\underset{\sim}{L} - \underset{\sim}{Z}^*)\underset{\sim}{R}^{-1/2}\underset{\sim}{T}\underset{\sim}{R}^{-1/2}(\underset{\sim}{L} - \underset{\sim}{Z}^*)^*(\underset{\sim}{L} + \underset{\sim}{Z})^{-1*}\underset{\sim}{R}^{1/2}\} =$$

$$\underset{\sim}{R}^{1/2}(\underset{\sim}{L} + \underset{\sim}{Z})^{-1}\{(\underset{\sim}{L} + \underset{\sim}{Z})\underset{\sim}{R}^{-1/2}\underset{\sim}{T}\underset{\sim}{R}^{-1/2}(\underset{\sim}{L} + \underset{\sim}{Z})^* - (\underset{\sim}{L} - \underset{\sim}{Z}^*)\underset{\sim}{R}^{-1/2}\underset{\sim}{T}\underset{\sim}{R}^{-1/2}(\underset{\sim}{L} - \underset{\sim}{Z}^*)^*\}(\underset{\sim}{L} + \underset{\sim}{Z})^{-1*}\underset{\sim}{R}^{1/2} =$$

(using the fact that $\underset{\sim}{R}^{-1/2}\underset{\sim}{T}\underset{\sim}{R}^{-1/2} = \underset{\sim}{T}\underset{\sim}{R}^{-1}$ since $\underset{\sim}{T}$ and $\underset{\sim}{R}$ are diagonal)

$$\underset{\sim}{R}^{1/2}(\underset{\sim}{L} + \underset{\sim}{Z})^{-1}\Big\{\underset{\sim}{L}\underset{\sim}{T}\underset{\sim}{R}^{-1}\underset{\sim}{L}^* + \underset{\sim}{L}\underset{\sim}{T}\underset{\sim}{R}^{-1}\underset{\sim}{Z}^* + \underset{\sim}{Z}\underset{\sim}{T}\underset{\sim}{R}^{-1}\underset{\sim}{L}^* + \underset{\sim}{Z}\underset{\sim}{T}\underset{\sim}{R}^{-1}\underset{\sim}{Z}^*$$

$$-\underset{\sim}{L}\underset{\sim}{T}\underset{\sim}{R}^{-1}\underset{\sim}{L}^* + \underset{\sim}{Z}^*\underset{\sim}{T}\underset{\sim}{R}^{-1}\underset{\sim}{L}^* + \underset{\sim}{L}\underset{\sim}{T}\underset{\sim}{R}^{-1}\underset{\sim}{Z} - \underset{\sim}{Z}^*\underset{\sim}{T}\underset{\sim}{R}^{-1}\underset{\sim}{Z}\Big\}(\underset{\sim}{L} + \underset{\sim}{Z})^{-1*}\underset{\sim}{R}^{1/2} =$$

(using the fact that $\underset{\sim}{Z}$, $\underset{\sim}{T}$ and $\underset{\sim}{R}$ are diagonal and hence $\underset{\sim}{Z}\underset{\sim}{T}\underset{\sim}{R}^{-1}\underset{\sim}{Z}^* = \underset{\sim}{Z}^*\underset{\sim}{T}\underset{\sim}{R}^{-1}\underset{\sim}{Z}$, and the fact that $\underset{\sim}{L}\underset{\sim}{T}\underset{\sim}{R}^{-1}(\underset{\sim}{Z} + \underset{\sim}{Z}^*) = \underset{\sim}{L}\underset{\sim}{T}\underset{\sim}{R}^{-1}(2\underset{\sim}{R}) = 2\underset{\sim}{L}\underset{\sim}{T}$, etc.,)

$$R^{1/2}(L + Z)^{-1}\{2TL^* + 2LT\}(L + Z)^{-1*}R^{1/2},$$

as claimed.

Notation

Define $X \triangleq \text{Im}\{Z\}$ and $M \triangleq (L + jX)$. Then $(M + R) = (L + Z)$ and hence is invertible by Fact 1.

Fact 3

$$R^{1/2}(M + R)^{-1}M = R^{-1/2}M(M + R)^{-1}R. \tag{A.3}$$

Proof

Begin with the identity

$$MR^{-1}M + M = MR^{-1}M + M,$$

factor both sides as follows

$$MR^{-1}(M + R) = (M + R)R^{-1}M,$$

multiply each side on the right and left by $(M + R)^{-1}$ to obtain

$$(M + R)^{-1}MR^{-1} = R^{-1}M(M + R)^{-1},$$

then multiply each side on the left by $R^{1/2}$ and on the right by R, yielding

$$R^{1/2}(M + R)^{-1}M = R^{-1/2}M(M + R)^{-1}R,$$

as claimed.

Fact 4

Define

$$
\begin{aligned}
P &\triangleq (L + Z)^{-1}RT(L + Z)^{-1*} \\
W &\triangleq R^{1/2}(L + Z)^{-1}XT(L + Z)^{-1*}R^{1/2} \qquad \text{(A.4)} \\
Q &\triangleq R^{1/2}(L + Z)^{-1}LT(L + Z)^{-1*}R^{1/2}.
\end{aligned}
$$

Then

$$Q = R^{-1/2}LPR^{1/2} + jXR^{-1/2}PR^{1/2} - jW \qquad \text{(A.5)}$$

Proof

Since $(M + R) = (L + Z)$, Q can be written as follows

$$
\begin{aligned}
Q &= R^{1/2}(M + R)^{-1}(M - jX)T(M + R)^{-1*}R^{1/2} \\
&= R^{1/2}(M + R)^{-1}MT(M + R)^{-1*}R^{1/2} \qquad \text{(A.6)} \\
&\quad -jR^{1/2}(L + Z)^{-1}XT(L + Z)^{-1*}R^{1/2}.
\end{aligned}
$$

We expand the first term on the right of (A.6):

$$R^{1/2}(M + R)^{-1}MT(M + R)^{-1*}R^{1/2} =$$

(using Fact 3)

$$\underset{\sim}{R}^{-1/2}\underset{\sim}{M}(\underset{\sim}{M} + \underset{\sim}{R})^{-1}\underset{\sim}{R}\underset{\sim}{T}(\underset{\sim}{M} + \underset{\sim}{R})^{-1*}\underset{\sim}{R}^{1/2} =$$

(substituting $\underset{\sim}{L} + j\underset{\sim}{X}$ for $\underset{\sim}{M}$, using $\underset{\sim}{M} + \underset{\sim}{R} = \underset{\sim}{L} + \underset{\sim}{Z}$, and noting that $\underset{\sim}{X}$ and $\underset{\sim}{R}^{-1/2}$ commute since both are diagonal)

$$\underset{\sim}{R}^{-1/2}\underset{\sim}{L}(\underset{\sim}{L} + \underset{\sim}{Z})^{-1}\underset{\sim}{R}\underset{\sim}{T}(\underset{\sim}{L} + \underset{\sim}{Z})^{-1*}\underset{\sim}{R}^{1/2}$$

$$+ j\underset{\sim}{X}\underset{\sim}{R}^{-1/2}(\underset{\sim}{L} + \underset{\sim}{Z})^{-1}\underset{\sim}{R}\underset{\sim}{T}(\underset{\sim}{L} + \underset{\sim}{Z})^{-1*}\underset{\sim}{R}^{1/2} =$$

(using (A.4))

$$\underset{\sim}{R}^{-1/2}\underset{\sim}{L}\underset{\sim}{P}\underset{\sim}{R}^{1/2} + j\underset{\sim}{X}\underset{\sim}{R}^{-1/2}\underset{\sim}{P}\underset{\sim}{R}^{1/2}.$$

Thus the first term on the right of (A.6) equals the sum of the first two terms on the right of (A.5). Since $-j\underset{\sim}{W}$, the last term on the right of (A.5), equals the last term on the right of (A.6), the proof is concluded.

<u>Fact 5</u>

Let $\underset{\sim}{F}$ and $\underset{\sim}{G}$ be complex N x N matrices with $\underset{\sim}{F}$ nonsingular and diagonal. Then

$$\{\underset{\sim}{F}\underset{\sim}{G}\underset{\sim}{F}^{-1}\}_{mm} = \{\underset{\sim}{G}\}_{mm}, \; 1 \leq m \leq N. \quad \text{(A.7)}$$

This is a straightforward calculation.

Notation

If f, g: $\mathbb{R} \to \mathbb{R}$, then $f*g$ represents convolution. If $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are real, n-dimensional wide-sense stationary random processes, then $\underset{\sim}{R}_{\underset{\sim}{x}\underset{\sim}{y}}$ is their cross-correlation matrix and $\underset{\sim}{S}_{\underset{\sim}{x}\underset{\sim}{y}}$ is their cross-spectral density matrix, i.e.

$$\left\{\underset{\sim}{R}_{\underset{\sim}{x}\underset{\sim}{y}}(\tau)\right\}_{ij} = E\left\{x_i(t+\tau)x_j(t)\right\}$$

$$\left\{\underset{\sim}{S}_{\underset{\sim}{x}\underset{\sim}{y}}(\omega)\right\}_{ij} = F\left\{\underset{\sim}{R}_{\underset{\sim}{x}\underset{\sim}{y}}(\tau)\right\}_{ij},$$

where F is the Fourier transform.

If $\underset{\sim}{A}(t)$ is an n x m matrix of time functions, then $\hat{\underset{\sim}{A}}(j\omega)$ represents the Fourier transform of $\underset{\sim}{A}(t)$, i.e.

$$\{\hat{\underset{\sim}{A}}(j\omega)\}_{ij} \triangleq \int_{-\infty}^{\infty} a_{ij}(t)e^{-j\omega t}dt.$$

Finally, if u, y : $\mathbb{R} \to \mathbb{R}^n$, and $\underset{\sim}{A}(\cdot)$ is an n x n matrix of real valued time functions, we use the notation

$$\underset{\sim}{y} = \underset{\sim}{A}*\underset{\sim}{u}$$

to mean $y_i = \sum_{j=1}^{n} a_{ij}*u_j$, $1 \leq i \leq n$.

Fact 6

Let $\underset{\sim}{u}$ be a real, n-dimensional, wide-sense stationary random process, and $\underset{\sim}{A}$, and $\underset{\sim}{B}$ be n x n matrices of real-valued time functions, possibly including singularity functions, such that $\underset{\sim}{x} \triangleq \underset{\sim}{A}*\underset{\sim}{u}$ and $\underset{\sim}{y} \triangleq \underset{\sim}{B}*\underset{\sim}{u}$ are jointly wide-sense stationary random processes. Then

$$S_{\underset{\sim}{xy}}(\omega) = \hat{A}(j\omega)S_{\underset{\sim}{uu}}(\omega)\hat{B}^*(j\omega),$$

provided all terms exist.

This is a straightforward calculation.

A.2) Conclusion of the Proof of Proposition 1

We will first show that Prop. 1 holds for each $j\omega \in D$, i.e. that

$$-P_m = \frac{k}{4\pi}\{T - STS^*\}_{m,m}, \quad 1 \leq m \leq N. \tag{A.10}$$

Let S_{vi} represent $S_{vi}(j\omega)$. Using Def. 2.4 we can restate (A.10) as follows

$$\mathrm{Re}\left\{S_{vi}\right\}_{mm} = \frac{k}{2}\left\{T - STS^*\right\}_{mm}, \quad 1 \leq m \leq N. \tag{A.11}$$

To prove that (A.11) holds, we first rewrite its left hand side as follows. Note from Fig. 2 that the transfer function mapping $n \mapsto i$ is $(L + Z)^{-1}$ and the transfer function mapping $n \mapsto v$ is $L(L + Z)^{-1}$. Therefore

$$\mathrm{Re}\left\{S_{vi}\right\}_{mm} =$$

(using Fact 6)

$$\mathrm{Re}\left\{L(L + Z)^{-1}S_{nn}(L + Z)^{-1*}\right\}_{mm} =$$

(using (3.2))

$$2k\,\mathrm{Re}\left\{\underset{\sim}{L}(\underset{\sim}{L} + \underset{\sim}{Z})^{-1}\underset{\sim}{R}\underset{\sim}{T}(\underset{\sim}{L} + \underset{\sim}{Z})^{-1*}\right\}_{mm} =$$

(using (A.4))

$$2k\,\mathrm{Re}\{\underset{\sim}{L}\underset{\sim}{P}\}_{mm}, \; 1 \leq m \leq N. \tag{A.12}$$

We now expand the right hand side of (A.11). Using the definition of $\underset{\sim}{Q}$ in (A.4), Fact 2, and the self-adjointness of $\underset{\sim}{T}$ we obtain the first equality below. (The second equality is trivial.)

$$\begin{aligned} \frac{k}{2}\{\underset{\sim}{T} - \underset{\sim}{S}\underset{\sim}{T}\underset{\sim}{S}^*\}_{mm} &= k\{\underset{\sim}{Q} + \underset{\sim}{Q}^*\}_{mm} \\ &= 2k\,\mathrm{Re}\{\underset{\sim}{Q}\}_{mm}. \end{aligned} \tag{A.13}$$

Using (A.12) and (A.13), we can restate (A.11) as follows:

$$\mathrm{Re}\{\underset{\sim}{L}\underset{\sim}{P}\}_{mm} = \mathrm{Re}\{\underset{\sim}{Q}\}_{mm}, \; 1 \leq m \leq N. \tag{A.14}$$

To show that (A.14) holds, consider the expansion of $\underset{\sim}{Q}$ given in (A.5). The third term is -j times the self-adjoint matrix $\underset{\sim}{W}$. Since the diagonal elements of any self-adjoint matrix are real, $\mathrm{Re}\left\{-j\underset{\sim}{W}\right\}_{mm} = 0$, $1 \leq m \leq N$. Now consider the second term on the right hand side of (A.5). The matrix $\underset{\sim}{P}$ is self-adjoint, so by Fact 5 the diagonal elements of $\underset{\sim}{R}^{-1/2}\underset{\sim}{P}\underset{\sim}{R}^{1/2}$ are real. Since $\underset{\sim}{X}$ is real and diagonal, $\mathrm{Re}\left\{j\underset{\sim}{X}\underset{\sim}{R}^{-1/2}\underset{\sim}{P}\underset{\sim}{R}^{1/2}\right\}_{mm} = 0$, $1 \leq m \leq N$.

Finally, the first term on the right hand side of (A.5) has the same diagonal elements as $\underset{\sim}{L}\underset{\sim}{P}$ by Fact 5. This proves (A.14) and hence (A.10). Therefore (3.3) holds for all $j\omega\varepsilon D$.

To see that (3.3) must hold for all $j\omega \notin D$ as well, note that $P_m(\cdot)$ is defined for all ω and continuous by assumption. Since $\underset{\sim}{R}(\cdot)$ is defined by analytic continuation on the entire $j\omega$-axis it is also continuous there, and the diagonal entries are strictly positive by assumption. Also $\underset{\sim}{S}_{\underset{\sim}{n}\underset{\sim}{n}}(\cdot)$ is defined and continuous on the entire $j\omega$-axis by assumption. Therefore $\underset{\sim}{T}(\omega) \triangleq \underset{\sim}{S}_{\underset{\sim}{n}\underset{\sim}{n}}(\omega)\ (2k\underset{\sim}{R}(\omega))^{-1}$ is defined and continuous on the $j\omega$-axis. Similarly $\underset{\sim}{\mathcal{S}}(j\omega)$ is defined by analytic continuation on the entire $j\omega$-axis, so it is also continuous. Therefore the right hand side of (3.3) is defined and continuous on the entire $j\omega$-axis. Since D is dense in the $j\omega$-axis and (3.3) holds on D, (3.3) must hold for all $j\omega$. ■

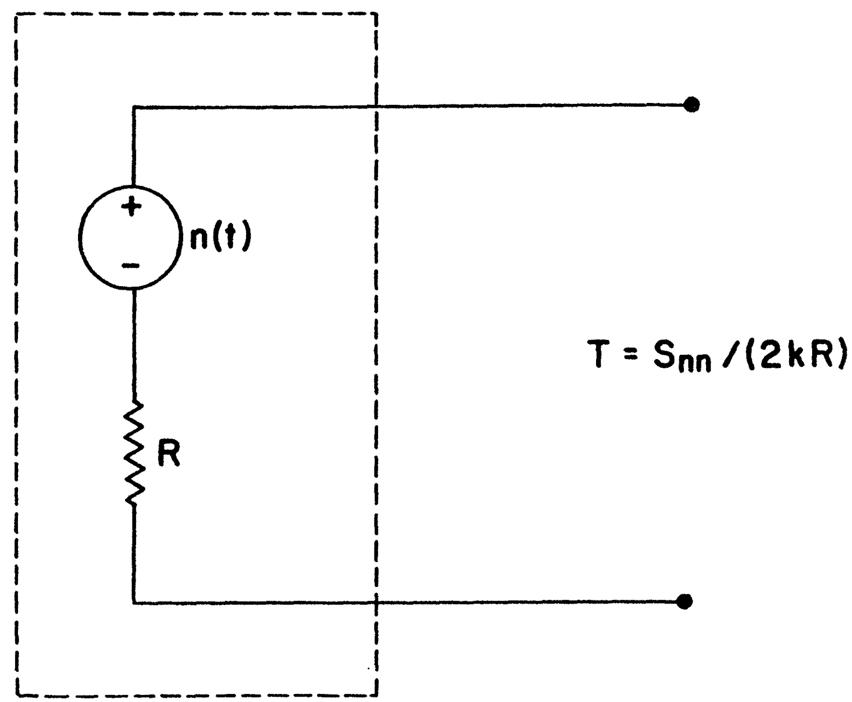


Fig. 1a

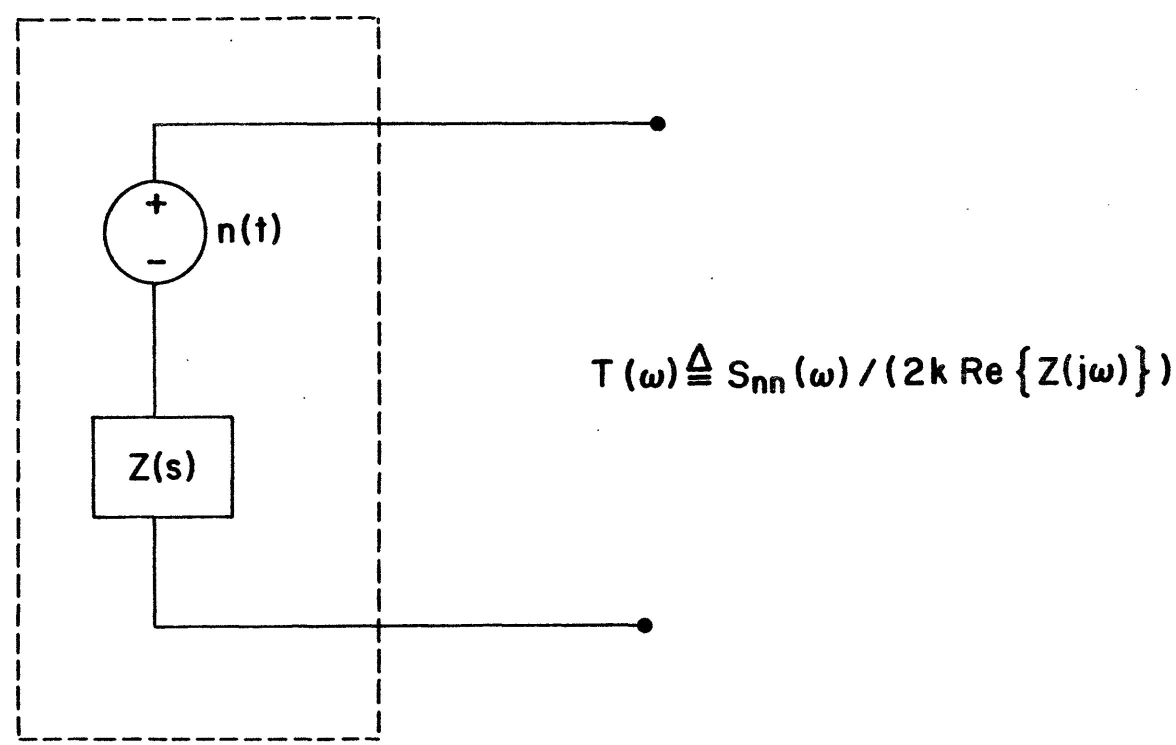
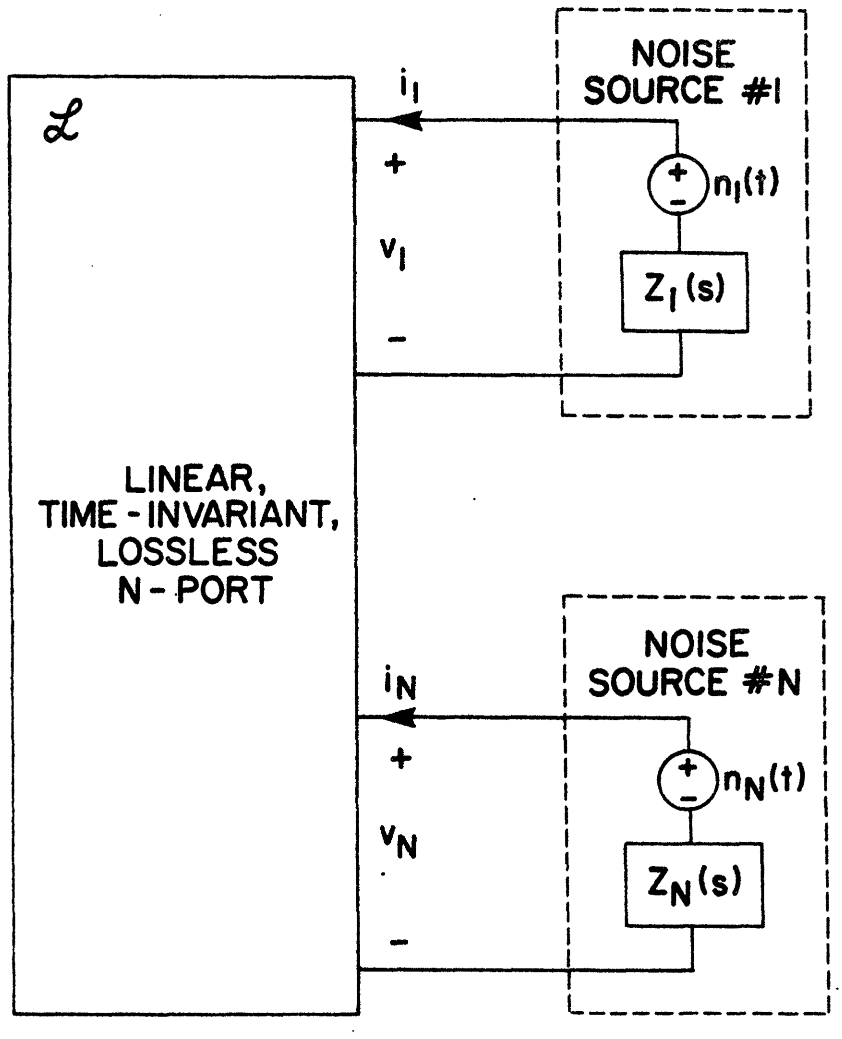


Fig. 1 b.



NOISE SOURCE #1
i_1
$\mathcal{L}$
+
$n_1(t)$
v_1
$Z_1(s)$
−
LINEAR, TIME-INVARIANT, LOSSLESS N-PORT
NOISE SOURCE #N
i_N
+
$n_N(t)$
v_N
$Z_N(s)$
−

Fig. 2

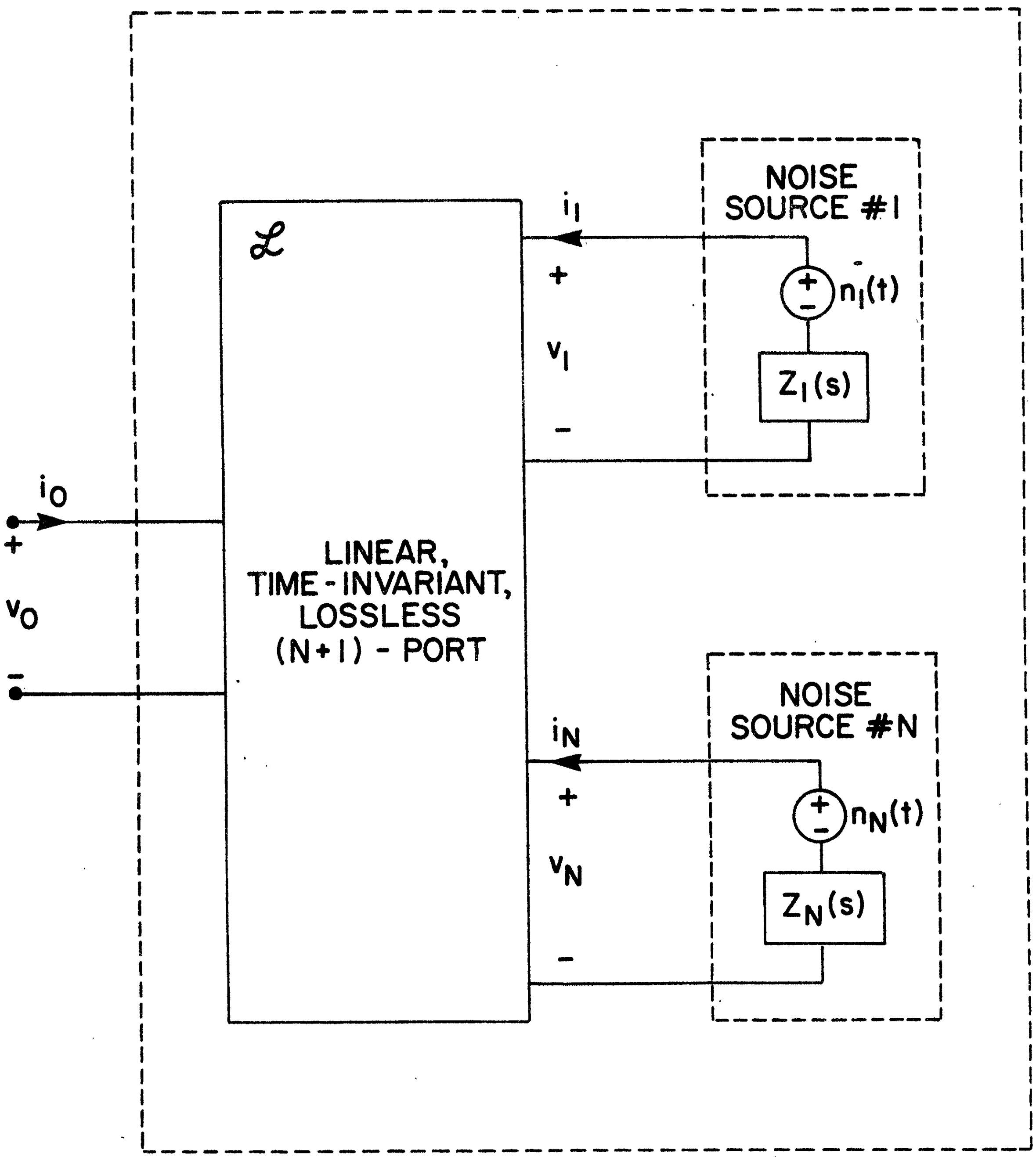

NOISE SOURCE #1
i_1
$\mathcal{L}$
+
v_1
−
$n_1(t)$
$Z_1(s)$
i_0
+
v_0
−
LINEAR, TIME-INVARIANT, LOSSLESS (N+1) - PORT
NOISE SOURCE #N
i_N
+
v_N
−
$n_N(t)$
$Z_N(s)$

Fig. 3