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CONIC SECTORS FOR SAMPLED-DATA FEEDBACK SYSTEMS*

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ABSTRACT

A multivariable sampled-data feedback system contains an analog plant controlled by a sampled-data compensator. Conic sectors that can be used to analyze sampled-data feedback systems are presented and then proved to be valid. Also included is the gain of the sampled-data operator.

Keywords: Sampled-data feedback systems
Multivariable feedback systems
Conic sector
Gain of operators
Closed loop stability
Robustness

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1. INTRODUCTION

The usefulness of conic sectors for analyzing feedback systems [1,2,3,4] depends on the determination of a particular conic sector for the feedback system of interest. Conic sectors that are useful for the analysis of sampled-data feedback systems are now described. These results are compared with previously reported results for analog feedback systems [3,4].

Preliminaries are presented in Section 2. Conic sectors useful for analog feedback system are reviewed in Section 3. The major results of this paper are the new conic sectors useful for sampled-data feedback systems presented in Section 4. Proofs of the new results are in Section 5, and a summary in Section 6.

2. PRELIMINARIES

The notation leading up to and including conic sectors is now defined. The use of conic sectors for determining closed loop stability and robustness is reviewed. For more detail see [1] to [5].

A relation K is any subset of the product space $L_{2e}^{r} \times L_{2e}^{m}$, where L_{2e}^{r} is the extended normed linear space of square integrable functions $\underset{\sim}{e}: R_{+} \to R^{r}$ (from the set of real numbers ≥ 0 to the set of r-dimensional vectors) that have finite truncated norms for all $\tau \in R_{+}$:

$$||\underset{\sim}{e}||_{\tau} \triangleq \left[\int_{0}^{\tau} ||\underline{e}(t)||_{E}^{2}\, dt\right]^{1/2} \qquad (1)$$

The subscript "E" indicates the Euclidean vector norm. In the limit as $\tau \to \infty$ the truncated norm is the L_2 function norm denoted by $||\underset{\sim}{e}||_{L_2}$

The inverse relation always exists and is defined by

$$K^{I} \triangleq \left\{(\underset{\sim}{u},\underset{\sim}{e}) \in L_{2e}^{m} \times L_{2e}^{r} \;\middle|\; (\underset{\sim}{e},\underset{\sim}{u}) \in K\right\} \qquad (2)$$

The gain of the relation K is defined by

$$||K||_{L_2} \triangleq \sup \frac{||K\underset{\sim}{e}||_{\tau}}{||\underset{\sim}{e}||_{\tau}} \qquad (3)$$

where the supremum is taken over all nonzero $\underset{\sim}{e}$ in the domain of K, all corresponding $K\underset{\sim}{e}$ in the range of K, and all $\tau \in R_{+}$. The relation K is L_{2e}-stable if $||K||_{L_2} < \infty$. An operator K is a special case of a relation

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NOTE TO PRINTER:

Use script boldface for symbols e, u etc.
Use roman boldface for symbols $\underline{\tilde{e}}$ $\underline{\tilde{A}}$, $\underline{H}$ etc.

that satisfies the two conditions (1) the domain is equal to L_{2e}^{r} and (2) for each $\underset{\sim}{e}$ in the domain there exists a unique $\underset{\sim}{u}$ in the range such that $(\underset{\sim}{e},\underset{\sim}{u}) \in K$.

Define K to be a relation and C and R to be operators. If

$$||\underset{\sim}{u} - C\,\underset{\sim}{e}||_{\tau}^{2} \leq ||R\,\underset{\sim}{e}||_{\tau}^{2} - \varepsilon||\underset{\sim}{e}||_{\tau}^{2} \tag{4}$$

for all $(\underset{\sim}{e},\underset{\sim}{u}) \in K$, $\tau \in R_{+}$, and some $\varepsilon > 0$ then K is <u>strictly inside cone (C, R)</u> with center C and radius R. On the other hand, if

$$||\underset{\sim}{e} - C\,\underset{\sim}{u}||_{\tau}^{2} \geq ||R\,\underset{\sim}{u}||_{\tau}^{2} \tag{5}$$

for all $(\underset{\sim}{u},\underset{\sim}{e}) \in -K^{I}$ and all $\tau \in R_{+}$, then $-K^{I}$ is <u>outside cone (C, R)</u>.

A general feedback system is shown in Figure 1. It is defined algebraically by

$$\left.\begin{array}{l} (\underset{\sim}{e},\underset{\sim}{u}) \in K \\ (\underset{\sim}{u},\underset{\sim}{y}) \in \tilde{G} \\ \underset{\sim}{e}=\underset{\sim}{r}-\underset{\sim}{y} \\ \text{where } \underset{\sim}{e},\underset{\sim}{r},\underset{\sim}{y} \in L_{2e}^{r} \text{ and } \underset{\sim}{u} \in L_{2e}^{m} \end{array}\right\} \tag{6}$$

Define the closed loop relations

$$\begin{array}{l} (\underset{\sim}{r},\underset{\sim}{e}) \in E \\ (\underset{\sim}{r},\underset{\sim}{u}) \in U \end{array} \tag{7}$$

The feedback system is closed loop stable if E and U are L_{2e}-stable. A sufficient condition for this to be true [1,3] is that a cone $(\mathcal{C},\mathcal{R})$ exists such that

$$\left.\begin{array}{ll} K & \text{is strictly inside cone } (\mathcal{C},\mathcal{R}) \\ -\tilde{G}^{I} & \text{is outside cone } (\mathcal{C},\mathcal{R}) \end{array}\right\} \qquad (8)$$

Alternatively, a sufficient condition for E and U to be L_{2e}-stable is that a cone (G,R) exist such that

$$\left.\begin{array}{ll} -K^{I} & \text{is outside cone } (G,R) \\ \tilde{G} & \text{is strictly inside cone } (G,R) \end{array}\right\} \qquad (9)$$

This completes the preliminary section. Now it is reviewed how conic sectors are applied to analog feedback systems.

3. ANALOG FEEDBACK SYSTEMS

For an analog feedback system the relations K and $\tilde{G}$ of Figure 1 are causal linear time invariant (LTI) operators, and hence can be represented by the Laplace transform matrices $\underline{K}(s)$ and $\underline{\tilde{G}}(s)$. Lemma 1 gives sufficient conditions for a LTI operator to be strictly inside of a cone. Lemma 2 (which can be considered a corollary of Lemma 1) gives the gain of a LTI operator, and Lemma 3 gives sufficient conditions for the inverse of a LTI operator to be outside of a cone.

Lemma 1 [4, Lemma A4] Define the LTI operators K, C, and R. Assume that R^I is also a LTI operator, and that K-C, R, and R^I are L_{2e}-stable. K is strictly inside cone (C,R) if

$$\sigma_{min}[\underline{R}(j\omega)] \geq \frac{1}{(1-\varepsilon)^{1/2}} \sigma_{max}[\underline{K}(j\omega)-\underline{C}(j\omega)] \tag{10}$$

for all ω and some $\varepsilon>0$.

Lemma 2 [e.g. 2] The gain of the LTI operator K is

$$||K||_{L_2} = \max_{\omega} \sigma_{max}[\underline{K}(j\omega)] \tag{11}$$

Lemma 3 [4, Lemma A2] Define the LTI operators K, C, and R. Assume that R^I is also a LTI operator, and that $K(I+CK)^I$, R, and R^I are L_{2e}-stable. Then, $-K^I$ is outside cone (C,R) if

$$\sigma_{max}[\underline{R}\ \underline{K}(\underline{I}+\underline{C}\ \underline{K})^{-1}(j\omega)] \leq 1 \qquad \text{for all } \omega \tag{12}$$

The minimum and maximum singular values of a complex-valued matrix are denoted by σ_{min} and σ_{max}, respectively. For scalars both σ_{min} and σ_{max} are equal to the absolute value. See [6,7] for more information about singular values.

In Lemma 1 it is not necessary that K be L_{2e}-stable, but $K-C$ must be. Stability is determined by the location of the open-loop poles. In Lemma 3 the open-loop stability requirement is replaced by the closed-loop stability requirement that $K(I+CK)^I$ is L_{2e}-stable. This is determined by the location of the closed loop poles or by the multivariable Nyquist criterion.

Lemmas 1,2, and 3 can be used to analyze analog feedback systems, in particular to determine multivariable robustness margins. See, for instance, [5]. Our attention now shifts to sampled-data feedback systems. Results similar to these three lemma are now presented.

4. SAMPLED-DATA FEEDBACK SYSTEMS

For sampled-data feedback systems the relation $\tilde{G}$ of Figure 1 is a causal LTI operator and the relation K is a causal sampled-data operator. A block diagram of K is shown in Figure 2. The prefilter and hold are represented by Laplace transform matrices $\underline{F}(s)$ and $\underline{H}(s)$, respectively; and the digital computer is represented by the z-transform matrix $\underline{D}(z)$. The star notation [8] is defined:[1]

$$\underline{D}^*(s) \triangleq \underline{D}(z)\Big|_{z=e^{j\omega T}}$$

$$[\underline{F}\ \underline{e}(s)]^* \triangleq \frac{1}{T}\sum_n \underline{F}\ \underline{e}(s-j\omega_s n);\quad \omega_s = \frac{2\pi}{T} \tag{13}$$

The input-output transformation of the sampled-data operator is

$$\underline{u}(s) = \underline{H}\ \underline{D}^*[\underline{F}\ \underline{e}(s)]^* \tag{14}$$

The sampled-data operator K is a linear time varying (LTV) operator. It cannot be represented by a transfer function matrix.

Results analogous to Lemmas 1,2, and 3 are now presented. Theorem 1 gives sufficient conditions for a sampled data operator to be strictly inside of a cone. Theorem 2 gives an upperbound for the gain, and Theorem 3 gives sufficient conditions for the inverse of a sample-data operator to be outside of a cone. The proofs are presented in Section 5.

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[1] The abbreviation $\sum_n$ is short for $\sum_{n=-\infty}^{\infty}$.

Theorem 1: Define the sampled-data operator $\mathcal{K}$ and the LTI operators $\mathcal{C}$ and $\mathcal{R}$. Assume that $\mathcal{R}^I$ is also a LTI operator, and that $\mathcal{K}, \mathcal{C}, \mathcal{R}$ and $\mathcal{R}^I$ are L_{2e}-stable. Then $\mathcal{K}$ is strictly inside cone $(\mathcal{C},\mathcal{R})$ if

$$\sigma_{min}[\underline{R}(j\omega)] \geq \frac{1}{(1-\varepsilon)^{1/2}} \left[\frac{1}{T^2} \sum_k \sum_{n \neq k} \sigma^2_{max}(\underline{H}_k \underline{D}^* \underline{F}_n) + \sum_k \sigma^2_{max}\left(\frac{1}{T^2} \underline{H}_k \underline{D}^* \underline{F}_k - \underline{C}_k\right)\right]^{1/2} \tag{15}$$

for all ω and some $\varepsilon > 0$

Furthermore, the choice of center $\underline{C} = \frac{1}{T} \underline{H}\, \underline{D}^* \underline{F}$ (called the "optimal center") minimizes the lower bound for $\sigma_{min}[\underline{R}(j\omega)]$.

Theorem 2: An upperbound for the gain of the sampled-data operator is

$$||\mathcal{K}||_{L_2} \leq \max_{0 \leq \omega \leq \frac{\pi}{T}} \left[\frac{1}{T^2} \sum_k \sum_n \sigma^2_{max}(\underline{H}_k \underline{D}^* \underline{F}_k) \right]^{1/2} \tag{16}$$

Furthermore, this upperbound equals the gain when $\underline{H}$, $\underline{D}^*$, and $\underline{F}$ are single-input single-output (SISO).

Theorem 3: Define the sample-data operator $\mathcal{K}$ and the LTI operators $\mathcal{C}$ and $\mathcal{R}$. Assume that $\mathcal{R}^I$ and $(I+\mathcal{C}\mathcal{K})^I$ are also operators, and that $\mathcal{K}(I+\mathcal{C}\mathcal{K})^I$, $\mathcal{R}$, and $\mathcal{R}^I$ are L_{2e}-stable. Then $-\mathcal{K}^I$ is outside cone $(\mathcal{C},\mathcal{R})$ if

$$\sigma_{max}[\underline{R}(j\omega)] \leq \left[\frac{1}{T^2} \sum_k \sum_n \sigma^2_{max}(\underline{H}_k \underline{D}^*_{c\ell} \underline{F}_k) \right]^{-1/2} \quad \text{for all } \omega \tag{17}$$

where

$$\underline{D}^*_{c\ell}(s) = \underline{D}^*[\underline{I} + (\underline{F}\, \underline{C}\, \underline{H})^* \, \underline{D}^*(s)]^{-1} \tag{18}$$

In Theorem 1 it is assumed that both $\mathcal{K}$ and $\mathcal{C}$ are L_{2e}-stable, which is an open-loop stability assumption. The center $\mathcal{C}$ is otherwise arbitrary, though poor choices result in large radii; and the optimal center minimizes the radius. In Theorem 3 the less restrictive assumption is made that $K(I+CK)^{I}$ is L_{2e}-stable, which is a closed loop stability assumption. This closed loop system has the input-output transformation

$$\underline{y}(s) = \underline{H}\ \underline{D}^{*}_{c\ell}[\underline{F}\ \underline{r}(s)]^{*} \qquad (19)$$

which is similar in structure to the open-loop sampled-data operator (14), and is stable if the poles of $\underline{D}_{c\ell}(z)$ are inside the unit circle.

After some thought (possibly after considerable thought) Theorems 1, 2, and 3 should appear to be natural extensions of Lemmas 1,2, and 3. This is easiest to visualize when $\underline{F}(j\omega)$ is bandlimited to $|\omega| < \frac{\pi}{T}$, in which case the lemma and theorems give the same results, except that for the latter the radius is periodic. When $\underline{F}(j\omega)$ is not bandlimited, then the radii and gain increase depending on the amount of aliasing.

5. PROOFS

Theorems 1,2, and 3 are proved in this section. Use will be made of the Laplace transform matrices

$$\underline{K}_n(s) = \begin{cases} \frac{1}{T}\,\underline{H}(s)\underline{D}^*(s)\underline{F}(s)-\underline{C}(s) & ; \quad n=0 \\ \frac{1}{T}\,\underline{H}(s)\underline{D}^*(s)\underline{F}(s-j\omega_s n) & ; \quad n\neq 0 \end{cases} \tag{20}$$

for the following manipulation of equation (14):

$$\underline{u}(s) - \underline{C}(s)\underline{e}(s) = \sum_n \underline{K}_n(s)\underline{e}(s-j\omega_s n) \tag{21}$$

A critical step used in the proofs is highlighted in the following lemma.

Lemma 4: Define the $\underline{K}_n(s)$ of (20) and assume that for $|\omega|$ sufficiently large that[1]

$$\sum_n ||\underline{K}_n(j\omega)||^2 \leq \frac{\alpha}{|\omega|^{1+\beta}} \quad \text{for some } \alpha,\beta > 0 \tag{22}$$

Then it follows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} ||\sum_n \underline{K}_n(j\omega)\underline{e}(j\omega-j\omega_s n)||_E^2\, d\omega$$
$$\leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_k \sum_n ||\underline{K}_n(j\omega-j\omega_s k)||^2\right] ||\underline{e}(j\omega)||_E^2\, d\omega \tag{23}$$

Proof of Lemma 4: Use is made of the Cauchy-Schwartz inequality [9, p.30] and Lebesque Dominated Convergence [10, p.44].

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[1] The matrix norm $||\underline{A}|| = \sigma_{max}(\underline{A})$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\|\sum_n \underline{K}_n(j\omega)\underline{e}(j\omega-j\omega_s n)\right\|_E^2 d\omega$$

$$\leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_n ||\underline{K}_n(j\omega)||\cdot||\underline{e}(j\omega-j\omega_s n)||_E\right]^2 d\omega \qquad (24)$$

$$\leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_n ||\underline{K}_n(j\omega)||^2\right]\left[\sum_k ||\underline{e}(j\omega-j\omega_s k)||^2\right] d\omega \qquad (25)$$

(By the Cauchy-Schwartz inequality. Define $a_n = \|\underline{K}_n(j\omega)\|$, $b_n = ||\underline{e}(j\omega-j\omega_s n)||_E$. Define $\underline{a}$ and $\underline{b}$ as ℓ_2 vectors with components a_n and b_n for all integers n. Then, $|\underline{a}^T\underline{b}|^2 \leq ||\underline{a}||_E^2 \, ||\underline{b}||_E^2$)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_n ||\underline{K}_n(j\omega)||^2\right]||\underline{e}(j\omega-j\omega_s k)||^2 d\omega \qquad (26)$$

[By Lebesque Dominated Convergence, which is guaranteed by the assumption (22)].

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_k \sum_n ||\underline{K}_n(j\omega+j\omega_s k)||^2\right]||\underline{e}(j\omega)||^2 d\omega \qquad (27)$$

This completes the proof.

Remark: The assumption (22) used to guarentee Lebesque Dominated Convergence is true if $\underline{F}(s)$ and $\underline{H}(s)$ each have at least a one-pole rolloff, i.e., for $|\omega|$ sufficiently large, $||\underline{F}(j\omega)||$ and $||\underline{H}(j\omega)||$ are bounded by $\frac{\alpha}{|\omega|}$ for some $\alpha>0$.

<u>Proof of Theorem 1</u>: The objective is to show that K is strictly inside cone (C,R). Except for the use of Lemma 4, this proof is similar to [4, Lemma A4]. Define the truncated function

$$\underline{e}_\tau(t) = \begin{cases} (R\,\underline{e})(t) \; ; & t \le \tau \\ 0 \; ; & t > \tau \end{cases} \tag{28}$$

For all $\underset{\sim}{e} \in L_{2e}$ and all $\tau \in R_t$:

$$||(K-C)\underset{\sim}{e}||_\tau^2 = ||(K-C)R^I \underset{\sim}{e}_\tau||_\tau^2 \tag{29}$$

$$\le ||(K-C)R^I \underset{\sim}{e}_\tau||_{L_2}^2 \tag{30}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} ||\sum_n \underline{K}_n \underline{R}^{-1} \underline{e}_\tau(j\omega)||_E^2 \, d\omega \quad \text{(by Parsevals' theorem)} \tag{31}$$

$$\le \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_k \sum_n ||\underline{K}_n(j\omega - j\omega_s k)||^2 \right] ||\underline{R}^{-1} \underline{e}_\tau(j\omega)||_E^2 \, d\omega \tag{32}$$

(by Lemma 4)

$$\le \frac{1}{2\pi} \int_{-\infty}^{\infty} (1-\varepsilon) ||\underline{e}_\tau(j\omega)||_E^2 \, d\omega \quad \text{[by (15)]} \tag{33}$$

$$= (1-\varepsilon) ||\underset{\sim}{e}_\tau||_{L_2}^2 \quad \text{(by Parsevals' theorem)} \tag{34}$$

$$\le ||R\,\underset{\sim}{e}||_\tau - \varepsilon' ||\underset{\sim}{e}||_\tau^2 , \quad \text{where } \varepsilon' = \varepsilon ||R||_{L_2} \tag{35}$$

This verifies the inequality (4), which determines strict inequality, and completes the proof.

Proof of Theorem 2: The objective is to verify the upperbound for the gain of the sampled-data operator and to show that the upperbound actually is the gain for the SISO case. The upperbound is a special case of Theorem 1 when the center $\underline{C}(s)=0$, and a separate proof is not included. For the SISO case the following input signal achieves the upperbound as the truncated time $\tau\to\infty$:

$$e(t) = \sum_n |a_n| \cos[(\omega_o - \omega_s n)t + \mathrm{Arg}(a_n)] \tag{36}$$

where

$$a_n = d^*(-j\omega_o) f(-j\omega_o + \omega_s n) \tag{37}$$

$$\omega_o = \text{frequency that maximizes (16)}$$

This complete the proof.

Remark: The gain of the multivariable sampled-data operator remains to be found. We conjecture that the gain is given by (16), but we have been unable as of yet to find a signal [i.e. a vector version of (36)] that achieves the upperbound given by (16).

Proof of Theorem 3: The objective is to show that $-K^I$ is strictly outside cone (C,R). This is true if and only if the composite operator $RK(I+CK)^I$ has gain ≤ 1. This proof shows the latter. The assumption that $(I+CK)^I$ guarantees that the feedback system with the closed loop operator $RK(I+CK)^I$ is well-posed [11]. Let

$$\underline{K}_n(s) = \frac{1}{T} \underline{R}(s)\underline{H}(s)\underline{D}^*_{c\ell}(s)\underline{F}(s-j\omega_s n) \qquad \text{for all } n \tag{38}$$

and

$$\underline{e}_\tau(t) = \begin{cases} \underline{e}(t) \; ; & t \leq \tau \\ \underline{0} \; ; & t > \tau \end{cases} \tag{39}$$

then for all $\underset{\sim}{e} \in L_{2e}$ and all $\tau \in R_t$:

$$||RK(I+CK)^{I}\underset{\sim}{e}||_{\tau}^{2} \leq ||RK(I+CK)^{I}\underset{\sim}{e}_{\tau}||_{2}^{2} \tag{40}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} ||\underline{K}_n(j\omega)\underline{e}_\tau(j\omega - j\omega_s n)||_E^2 \, d\omega \quad \text{(by Parsevals' theorem)} \tag{41}$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_k \sum_n ||\underline{K}_n(j\omega - j\omega_s n)||^2\right] ||\underline{e}_\tau(j\omega)||^2 d\omega \quad \text{(by Lemma 4)} \tag{42}$$

$$\leq \frac{1}{2} \int_{-\infty}^{\infty} ||\underline{e}_\tau(j\omega)||_E^2 \, d\omega \qquad \text{[by (17)]} \tag{43}$$

$$= ||\underset{\sim}{e}||_{\tau}^{2} \tag{44}$$

This completes the proof.

6. CONCLUSIONS

Conic sectors that are useful for the analysis of multivariable feedback systems are presented in this paper. The conic sectors are analogous to those for multivariable analog feedback systems, and are distinguished by the use of the frequency domain inequality of Lemma 4.

The usefulness of the new conic sectors is demonstrated in [5], where it is shown how they can be used to determine closed loop stability, robustness margins, and steady state response to commands in sampled-data designs.

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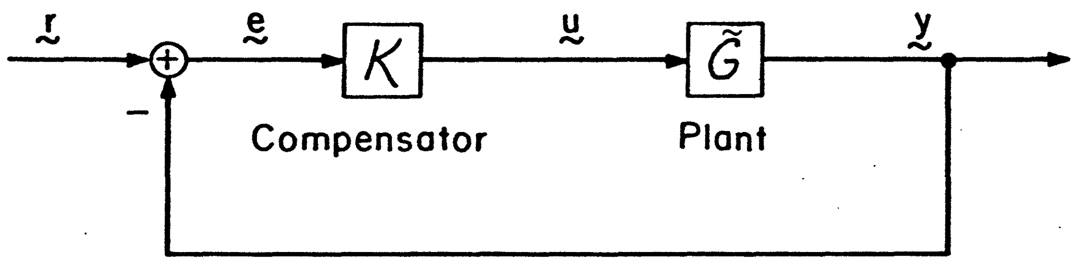

FIGURE 1: The general feedback system.

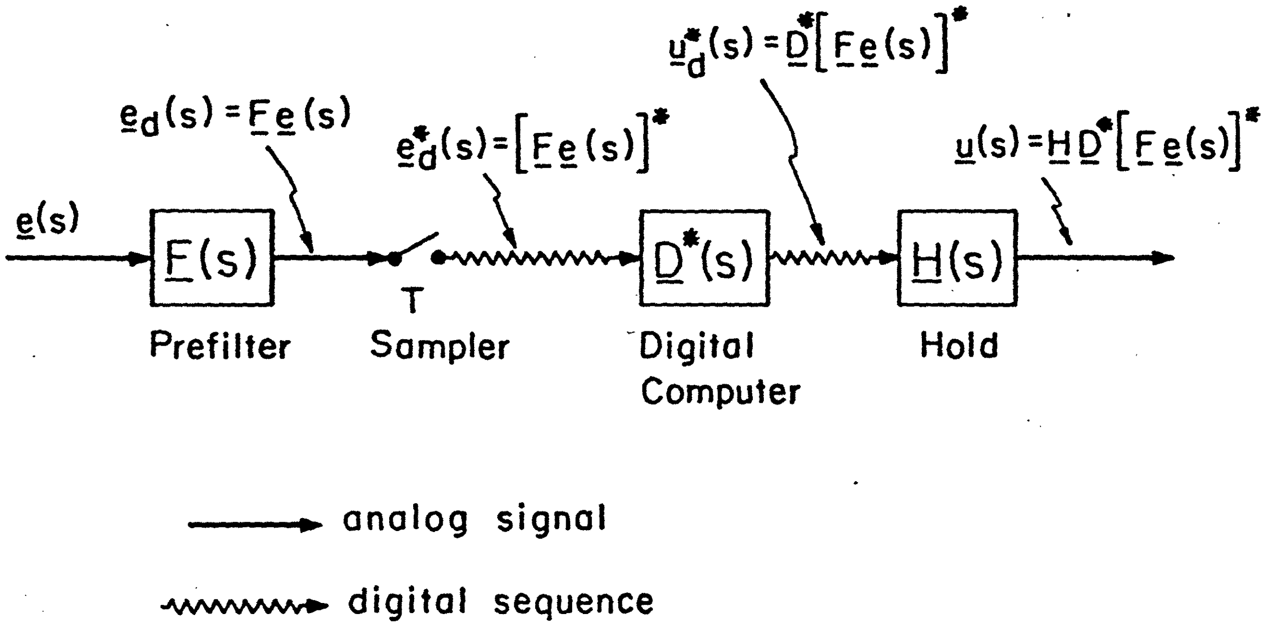

FIGURE 2: The hybrid compensator.