March 1983

LIDS-P-1290

Presented at the 1983 American Control Conference

FAILURE DETECTION WITH UNCERTAIN MODELS†

Xi-Cheng Lou

Alan S. Willsky

George C. Verghese

I. Introduction

In this paper we consider the issue of robust failure detection. In one way or another all failure detection methods (such as those surveyed in [1,4]) generate signals which tend to highlight the presence of particular failures if they have actually occurred. However, if any model uncertainties have affects on the observables which are at all like those of one or more of the failure modes, these will also be accentuated. Consequently the problem of robust failure detection is concerned with generating signals which are maximally sensitive to some effects (failures) and minimally sensitive to others (model errors).

The initial impetus for our approach to this problem came from the work reported in [5, 13] which document the first and to date by far most successful application and flight testing of a failure detection algorithm based on advanced methods which use analytic redundancy. The singular feature of that project was that the dynamics of the aircraft were decomposed in order to analyze the relative reliability of each individual of potentially useful failure detection information.

In [2] we presented the results of our initial attempt to extract the essence of the method used in [9, 13] in order to develop a general approach to robust failure detection. As discussed in that reference and in others (such as [3, 7-9]), all failure detection systems are based on exploiting analytical redundancy relations or (generalized) parity checks. These are simply functions of the temporal histories of the measured quantities which have the property that they are small (ideally zero) when the system is operating normally. In [2] we present one criterion for measuring the reliability of a particular redundancy relation and use this to pose an optimization problem to determine the most reliable relation. The particular measure chosen, however, leads to an extremely complex optimization problem. Moreover, if one is interested in obtaining a list of redundancy relations in order from most to least reliable, one must essentially solve a separate (and progressively more complicated) optimization problem for each relation in the list.

In this paper we look at an alternative measure of reliability for a redundancy relation. Not only does this alternative have a helpful geometric interpretation, but it also leads to a far simpler optimization procedure involving a singular value decomposition. In addition, it allows us in a natural and computationally feasible way to consider issues such as scaling, relative merits of alternative sensor sets, and explicit tradeoffs between detectability and robustness.

II. Redundancy Relations

In this paper we focus attention on linear, undriven, discrete-time systems, and in this section we consider the noise-free model

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

$$y(k) = Cx(k) \tag{2}$$

where x is n-dimensional, u is m-dimensional, y is r-dimensional, and A and B are perfectly known. A redundancy relation for this model is some linear combination of present and lagged values of y which should be identically zero if no changes (i.e. failures) occur in (1), (2). As discussed in [2, 3], redundancy relations can be specified mathematically in the following way. The subspace of (p+1) r-dimensional vectors given by

$$G = \left\{ \omega \;\middle|\; \omega' \begin{bmatrix} C \\ CA \\ \vdots \\ CA^p \end{bmatrix} = 0 \right\} \tag{3}$$

is called the space of parity relations of order p. Note that if $\omega \in G$, then the parity check

$$r(k) = \omega' \begin{bmatrix} y(k) \\ \vdots \\ y(k+p) \end{bmatrix} \tag{4}$$

is identically zero.

An important observation for the approach described in the sequel is that G is the orthogonal complement of the range Z of the matrix

$$\begin{bmatrix} C \\ \vdots \\ CA^p \end{bmatrix} \tag{5}$$

and thus a complete set of independent parity relations of order p is given by the orthogonal projection of the window of observations y(k), y(k+1),...,y(k+p) onto G.

III. An Angular Measure of Robustness

In this section we begin by focussing on a noise-free model with uncertain parameters:

$$x(k+1) = A(\eta)\, x(k) \tag{6}$$

$$y(k) = C(\eta)\, x(k) \tag{7}$$

where η is a vector of unknown parameters taking values in a specified set K. Referring to the comment at the end of the preceding section, we note that it is impossible to find parity checks which are perfect for all possible values of K. That is, in general we cannot find a subspace G which is orthogonal to

$$Z(\eta) = \text{Range} \begin{bmatrix} C(\eta) \\ \vdots \\ C(\eta)\, A(\eta)^p \end{bmatrix} \tag{8}$$

for all η.

What would seem to make sense in this case is to choose a subspace G which is "as orthogonal as possible" to all possible $Z(\eta)$. Several possible ways in which this can be done are described in detail in [3]. In this paper we focus on the one approach which leads to the most complete picture of robust redundancy and which is computationally the simplest. To do this, however, we must make the assumption that K, the set

† This research was supported in part by the Office of Naval Research under Grant N00014-77-C-0224 and by NASA Ames and NASA Langley Research Centers under Grant NGL-22-009-124.

of possible values of η is finite. Typically what this would involve is choosing representative points out of the actual, continuous range of parameter values. Here "representative" means spanning the range of possible values and having density variations reflecting any desired weightings on the likelihood or importance of particular sets of parameter values. However this is accomplished, we will assume for the remainder of this paper that η takes on a discrete set of values η-1,..., L, and will use the notation A_i for $A(\eta=i)$, Z_i for $Z(\eta=i)$, etc.

To obtain a simple computational procedure for determining robust redundancy relations we first compute an <u>average</u> observation subspace Z_o which is as close as possible to all of the Z_i, and we then choose G to be the orthogonal complement of Z_o. To be more precise, note first that the Z_i are subspaces of possibly differing dimensions (dim $Z_i = \nu_i$) embedded in a space of dimension $N = (p+1)r$ (corresponding to histories of the last p+1 values of the r-dimensional output). We will find it convenient to use the same symbols $Z_1,\ldots,Z_L$ to denote matrices of sizes $N\times\nu_i$, $i=1,\ldots,L$, whose columns form orthonormal bases for the corresponding subspaces. Letting $M = \nu_1+\ldots+\nu_L$, we define the NxM matrix

$$Z = [Z_1 \vdots \cdots \vdots Z_L] \tag{9}$$

Thus the columns of Z span the possible directions in which observation histories may lie under normal conditions.

We now suppose that we wish to determine the s best parity checks (so that dim G=s). Thus we wish to determine a subspace Z_o of dimension N-s. The optimum choice for this subspace is taken to be the span of the (not necessarily orthogonal) columns of the matrix Z_o which minimizes

$$\| Z - Z_o \|_F^2 \tag{10}$$

subject to the constraint that rank Z_o = N-s. Here $\|\cdot\|_F$ denotes the Frobenius norm:

$$\| D \|_F^2 = \sum_j \sum_i |d_{ij}|^2 \tag{11}$$

Thus the matrix Z_o is chosen so that the sum of the squared distances between the columns of Z and of Z_o have only N-s linearly independent columns.

There are several important reasons for choosing this criterion, one being that it does produce a space which is as close as possible to a specified set of directions. A second is that the resulting optimization problem is easy to solve. In particular, let the singular value decomposition of Z [14, 15] be given by

$$Z = U \Sigma V \tag{12}$$

where U and V are orthogonal matrices, and

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 & \vdots & \\ & \ddots & & \vdots & 0 \\ 0 & & \sigma_n & \vdots & \end{bmatrix} \tag{13}$$

Here $\sigma_1 \le \sigma_2 \le \ldots \le \sigma_N$ are the singular values of Z ordered by magnitude. Note we have assumed $N \le M$. If this is not the case we can make it so without changing the optimum choice of Z_o by padding Z with additional columns of zeros. It is readily shown that the matrix Z_o minimizing (10) is given by

$$Z_o = U \begin{bmatrix} 0 & & & 0 & \vdots & \\ & \ddots & & & \vdots & \\ & & 0 & \sigma_{s+1} & \vdots & 0 \\ & & & \ddots & \vdots & \\ 0 & & & \sigma_N & \vdots & \end{bmatrix} V \tag{14}$$

Moreover, since the columns of V are orthonormal, we immediately see that the orthogonal complement of the range of Z_o is given by the first s left singular vectors of Z_{10}, i.e. the first s columns of U. Consequently the columns of the matrix

$$G = [u_1 \vdots \cdots \vdots u_s] \tag{15}$$

are the optimum redundancy relations.

There is an alternative interpretation of this choice of G which provides some very useful insight. Specifically, recall that what we wish to do is to find a G whose columns are as orthogonal as possible to the columns of the Z_i; that is, we would like to choose G to make each of the matrices $Z_i'G$ as close to zero as possible. In fact, as shown in [3], the choice of G given in (15) minimizes

$$J(s) = \sum_{i=1} \| Z_i' G \|_F^2 \tag{16}$$

yielding the minimum value

$$J(s) = \sum_{j=1}^{s} \sigma_i^2 \tag{17}$$

There are two important points to observe about the result (16), (17). The first is that we can now see a straightforward way in which to include unequal weightings on each of the terms in (16). Specifically, if the w_i are positive numbers, then

$$\sum_{i=1} w_i \| Z_i' G \|_F^2 = \sum_{i=1} \| \sqrt{w_i}\, Z_i' G \|_F^2 \tag{18}$$

so that minimizing this quantity is accomplished using the same procedure described previously but with Z_i replaced by $\sqrt{w_i}\, Z_i$. As a second point note that the optimum value (17) provides us with an interpretation of the singular values as measures of robustness <u>and</u> with an ordered sequence of parity relations from most to least robust: u_1 is the most reliable parity relation with σ_1^2 as its measure of robustness, u_2 is the next best relation with σ_2^2 as its robustness measure, etc. Consequently from a single singular value decomposition we can obtain a <u>complete</u> solution to the robust redundancy relation problem for a <u>fixed</u> value of p, i.e. for a fixed length time history of output values. To compare relations for different values of p it is necessary to solve a singular value decomposition for each.

IV. <u>Several Important Extensions</u>

In this section we address several of the drawbacks and limitations of the result of the preceding section and obtain modifications to this result which overcome then at no fundamental increase in complexity.

4.1 <u>Scaling</u>

A critical problem with criteria of the preceding section is that <u>all</u> vectors in the observation spaces Z_i are treated as being equally likely to occur. If there are differences in scale among the system variables this may lead to poor solutions for the optimum parity relations. To overcome this drawback we proceed as follows. Suppose that we are given a scaling matrix P so that with the change of basis

$$\xi = Px \tag{19}$$

one obtains a variable ξ which is equally likely to lie in any direction. Such a matrix could, for example, be obtained from covariance analysis.

As a next step, recall that what we would ideally like to do is to choose a matrix G (whose columns represent the desired parity relations) so that

$$G' \begin{bmatrix} C_i \\ C_i A_i \\ \vdots \\ C_i A_i^P \end{bmatrix} x = G' \begin{bmatrix} C_i P^{-1} \\ C_i A_i P^{-1} \\ \vdots \\ C_i A_i^p P^{-1} \end{bmatrix} \xi \overset{\Delta}{=} G' \overline{C}_i \xi \tag{20}$$

is as small as possible. In the preceding section we considered all directions in Z_i = Range $(\overline{C}_i)$ to be on equal footing and arrived at the criterion (16). Since all directions for ξ are on equal footing, we are led naturally to the following criterion for choosing an NxS matrix G which takes scaling into account

$$J(s) = \min_{G'G=I} \sum_{i=1}^{L} ||\overline{C}_i' G||_F^2 \tag{21}$$

The solution to this problem is obtained in exactly the same way as in the previous section: We perform a singular value decomposition of the matrix

$$\overline{C} = [\overline{C}_1 \vdots \overline{C}_2 \vdots \cdots \vdots \overline{C}_2] = U\Sigma V \tag{22}$$

where $\sigma_1^2 \leq \sigma_2^2 \leq \cdots \leq \sigma_N^2$ and $U = [u_1 \vdots u_2 \vdots \cdots \vdots u_N]$. Then u_1 is the best parity relation with σ_1^2 as its measure of robustness, u_2 is the next best, etc., and $J^*(s)$ is given by (17). Finally, in anticipation of the next subsection, suppose that we use the stochastic interpretation of ξ, i.e. that

$$E[\xi\xi'] = I \tag{23}$$

In this case if we define the parity check vector

$$\mu_i = G'\overline{C}_i \xi \tag{24}$$

Then

$$E[||\mu_i||^2] = ||\overline{C}_i' G||^2{}_F \tag{25}$$

4.2 Observation and Process Noise

In addition to choosing parity relations which are maximally insensitive to model uncertainties it is also important to choose relations which suppress noise. Consider then the model

$$x(k+1) = A_i x(k) + D_i w(k) \tag{26}$$

$$y(k) = C_i x(k) + v(k) \tag{27}$$

where w and v are independent, zero-mean white noise processes with covariances Q and R, respectively.

Let

$$u = G' \begin{bmatrix} y(k) \\ \vdots \\ y(k+D) \end{bmatrix} \tag{28}$$

Then using the interpretation provided in (25), we obtain the following natural generalization of the criterion (21):

$$J(s) = \sum_{i=1} E_i [||\mu||^2] \tag{29}$$

where E_i denotes expectation assuming that the ith model is correct. Assuming that $\xi(k) = Px(k)$ has the identity as its covariance, using the whiteness of w and v, and performing some algebraic manipulations we obtain [3]

$$J(s) = \sum_{i=1} ||C_i' G||_F^2 + ||S'G||_F^2 \tag{30}$$

where S is defined by the following:

$$\overline{D}_i = \begin{bmatrix} 0 & 0 \cdots\cdots & 0 \\ C_i D_i & 0 & \vdots \\ C_i A_i D_i & C_i D_i & \vdots \\ \vdots & \vdots & 0 \\ C_i A_i^{p-1} D_i & C_i A_i^{p-2} D_i \cdots & C_i D_i \end{bmatrix} \tag{31}$$

$\overline{Q}$ = diag $(Q,\ldots,Q)$ (p times), $\overline{R}$ = diag $(R,\ldots,R)$

((p+1) times) (32)

$$N = \sum_{i=1}^{L} \overline{D}_i \overline{Q} \overline{D}_i' + L\overline{R} = SS' \tag{33}$$

From (30) we see that the effect of the noise is to specify another set of directions, namely the columns of S, to which we would like to make the columns of G as close to orthogonal as possible. From this it is evident that the optimum choice of G is computed by performing a singular value decomposition on the matrix

$$[\overline{C}_1 \vdots \cdots \vdots \overline{C}_L \vdots S] \tag{34}$$

As before (64) provides a complete set of parity relations ordered in terms of their degrees of insensitivity to model errors and noise.

4.3 Detection Versus Robustness

The methods described to this point involve measuring the quality of redundancy relations in terms of how small the resulting parity checks are under normal operating conditions. That is, good parity checks are maximally insensitive to modeling errors and noise. However, in some cases one might prefer to use an alternative viewpoint. In particular there may be parity checks which are not optimally robust in the senses we have discussed but are still of significant value because they are extremely sensitive to particular failure modes. In this subsection we consider a criterion which takes such a possibility into account. For simplicity we focus on the noise-free case. The extension to include noise as in the previous subsection is straightforward.

The specific problem we consider is the choice of parity checks for the robust detection of a particular failure mode. We assume that the unfailed model of the system is

$$x(k+1) = A_u(\eta)\, x(k) \tag{35}$$

$$y(k) = C_u(\eta)\, x(k) \tag{36}$$

while if the failure has occurred the model is

$$x(k+1) = A_F(\eta)\ x(k) \tag{37}$$

$$y(k) = C_F(\eta)\ x(k) \tag{38}$$

In this case one would like to choose G to be "as orthogonal as possible" to $Z_u(\eta)$ and "as parallel as possible" to $Z_f(\eta)$.

Assume again that η takes on one of a finite set of possible values, and let $\bar{C}_{ui}$ and $\bar{C}_{fi}$ denote the counterparts of $\bar{C}_i$ in (20) for the unfailed and failed models, respectively. A natural criterion which reflects our objective is

$$J(s) = \min_{G'G=I} \sum_{i=1}^{L} \{||C_{ui}'\ G||_F^2 - ||C_{Fi}'\ G||_F^2\} \tag{39}$$

If we define the matrix

$$H = [\underbrace{C_{u1} \vdots C_{u2} \vdots \cdots \vdots C_{uL}}_{M_1 \text{ columns}} \vdots \underbrace{C_{f1} \vdots C_{f2} \vdots \cdots \vdots C_{FL}}_{M_2 \text{ columns}}] \tag{40}$$

$$J(s) = \min_{G'G=I} \ tr\{G'HSH'G\} \tag{41}$$

where

$$S = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{matrix} M_1 \\ M_2 \end{matrix} \quad (\text{columns } M_1,\ M_2) \tag{42}$$

To solve this problem we perform an eigenvector-eigenvalue analysis on the matrix

$$HSH' = U\ \Lambda\ U' \tag{43}$$

where $U'U = I$ and

$$\Lambda = \text{diag}\ (\lambda_1, \ldots, \lambda_N) \tag{44}$$

with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ and $U = [u_1 \vdots \cdots \vdots u_N]$. Then the optimum choice of G is

$$G = [u_1 \vdots \cdots \vdots u_s] \tag{45}$$

and the corresponding value of (73) is

$$J^*(s) = \sum_{i=1}^{s} \lambda_i \tag{46}$$

Note that in general some of the λ_i are negative. In fact the parity check based on u_i is likely to have larger values under failed than failed conditions only if $\lambda_i < 0$. Thus we immediately see that the maximum number of useful parity relations for detecting this particular failure mode equals the number of negative eigenvalues of HSH'.

References

1. A.S. Willsky, "A Survey of Design Methods for Failure Detection," Automatica, Vol. 12, 1976, pp. 601-611.

2. E.Y. Chow and A.S. Willsky, "Issues in the Development of a General Design Algorithm for Reliable Failure Detection," Proc. IEEE Conf. on Decision and Control, Albuquerque, New Mexico, Dec. 1980.

3. X.-C. Lou, "A System Failure Detection Method: The Failure Projection Method," S.M. Thesis. M.I.T. Dept. of Elec. Eng. and Comp. Sci.; also Report. LIDS-TH-1203, M.I.T. Lab. for Inf. and Dec. Sys., June 1982.

4. A.S. Willsky, "Failure Detection in Dynamic Systems," AGARD Lecture Series No. 109 on Fault Tolerance Design and Redundancy Management Techniques, "Athens, Rome, and London, Oct. 1980.

5. J.C. Deckert, M.N. Desai, J.J. Deyst, and A.S. Willsky, "F8-DFBW Sensor Failure Detection and Identification Using Analytic Redundancy," IEEE Trans. Aut. Control, Vol. AC-22, No. 5, Oct. 1977, pp. 795-803.

6. G.C. Leininger, "Sensor Failure Detection with Model Mismatch," 1981 Joint Automatic Control Conf., Charlottesville, Virginia, June 1981.

7. S.R. Hall, "Parity Vector Compensation for FDI," S.M. Thesis, Dept. of Aero. and Astro., M.I.T., Feb. 1981.

8. J.E. Potter and M.C. Suman, "Thresholdless Redundancy Management with Arrays of Skewed Instruments," in AGARDOGRAPH-224 on Integrity in Electronic Flight Control Systems, 1977, pp. 15-1 to 15-25.

9. M. Desai and A. Ray, "A Failure Detection and Isolation Methodology," 20th IEEE Conf. on Decision and Control, San Diego, California.

10. F.C. Schweppe, Uncertain Dynamic Systems, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973.

11. J.R. Winkelman, "Multi-time-scale Analysis of a Power System," Automatica, Vol. 16, 1980, pp. 35-43.

12. A.S. Willsky, E.Y. Chow, S.B. Gershwin, C.S. Greene, P.K. Houpt, and A.L. Kurkjian," Dynamic Model-Based Techniques for the Detection of Incidents on Freeways," IEEE Trans. Aut. Control, Vol. AC-25, No. 3, June 1980, pp. 347-360.

13. J.C. Deckert, "Flight Test Results for the F-8 Digital Fly-by-Wire Aircraft Control Sensor Analytic Redundancy Management Technique," Proc. AIAA Guidance and Control Conference; Albequerque, New Mexico, Aug. 1981, pp. 491-499.

14. V.C. Klema and A.J. Laub, "The Singular Value Decomposition: Its Computation and Some Applications," IEEE Trans. Aut. Control, Vol. AC-25, No. 2, April 1980, pp. 164-176.

15. B. Noble, Applied Linear Algebra, Prentice-Hall, Inc., Englewood Cliffs, N.J. 1969.

16. A. Bjorck and G.H. Golub, "Numerical Methods for Computing Angles Between Linear Subspaces," Math. Comput., Vol. 27, 1973, pp. 579-594.