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A PARAMETRIZATION OF ALL STABLE STABILIZING COMPENSATORS FOR SINGLE-INPUT-OUTPUT SYSTEMS*

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ABSTRACT

A characterization is given of all stable compensators that stabilize a given scalar plant. In the process, a simple proof is given of an interpolation lemma due to Youla et al.

Key words: strong stabilization, interpolation by units

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1. INTRODUCTION

In this paper, we study the following problem: Given a lumped linear scalar plant described by its transfer function $p(\cdot)$, find all stable compensators $c(\cdot)$ (if any) such that the transfer function $p/(1+cp)$ is stable. Previously, Youla *et al*. [1] have given necessary and sufficient conditions for the *existence* of a stable stabilizing compensator for a given plant (if such a compensator can be found, the plant is said to be *strong stabilizable* [1]). However, at present no characterization is available of *all* stable stabilizing compensators. This is given in the present paper. In the process, we also obtain an exceedingly simple proof of an interpolation lemma first proved in [1]. Finally, the results given here can also be used to characterize the set of all compenators that simultaneously stabilize each of two given plants p_1 and p_2.

2. LOGARITHMS IN THE DISC ALGEBRA

Throughout this paper, the main technique used is that of embedding the problem in a Banach algebra and then taking logarithms. Hence it is appropriate to begin with a discussion of this topic.

The disc algebra A consists of functions that are analytic on the open unit disc and continuous on the closed unit disc [2,p. 77]. If we define the norm of a function f in A by

$$||f|| = \max_{|z|\leq 1} |f(z)| \tag{1}$$

and the product of two functions in A to be their point-wise product, then A becomes a commutative Banach algebra with identity. We let A_s denote the subalgebra of A consisting of <u>symmetric</u> functions; i.e.

$$A_s = \{f \in A: \overline{f}(\overline{z}) = f(z) \quad \forall\ z \in D\} \tag{2}$$

where D is the closed unit disc and the bar denotes conjugation. It is important to note that A is viewed as an algebra over the field of complex numbers, whereas A_s is an algebra over the reals.

Suppose a linear scalar time-invariant plant has the transfer function p(s). We say that the plant is <u>stable</u> if the function $z \to p(z) = \tilde{p}[(1+z)/(1-z)]$ is in A_s. The bilinear transformation is used to map the closed right half-plant plus the point at infinity (which is usually considered to be the unstable region for continuous-time systems) into the closed unit disc D. In the case of discrete-time systems, a plant with transfer function p(z) is stable if $p \in A_s$. The plant is <u>lumped</u> if p or $\tilde{p}$, as appropriate, is a rational function.

A <u>unit</u> of A_s is an element that has an inverse in A_s. It is easy to see that $f \in A_s$ is a unit if and only if $f(z) \neq 0$ for all $z \in D$.

An element of $f \in A_s$ is said to <u>have a logarithm</u> in A_s if there exists a $g \in A_s$ such that

$$f = \exp(g) = \sum_{i=0}^{\infty} g^i/i\,! \tag{3}$$

If f has a logarithm f, then f exp(-g) = 1, so that f must be a unit. The converse question, namely whether every unit in A_s has a logarithm, is answered next.

<u>Proposition 1</u> [3, p.14, Theorem (1.4.10)] Suppose B is a Banach algebra with identity, and let L denote the set of units in B that have a logarithm. Then the group (under multiplication) generated by L coincides with the connected component of the set of units containing the identity. In particular, if B is commutative, then L is a group, so that L coincides with this component.

<u>Proposition 2</u>. A unit f in A_s has a logarithm in A_s if and only if $f(z) > 0$ for all $z \in [-1,1]$.

<u>Proof</u>. "only if" Suppose $f = \exp(g)$, where $g \in A_s$. Then $g(z)$ is real for all $z \in [-1,1]$, so that $f(z) = \exp[g(z)] > 0$ for all $z \in [1,1]$.

"if" Suppose a unit f satisfies $f(z) > 0$ for all $z \in [-1,1]$. Then, in particular, we have that $f(0) > 0$. Now define

$$h(r,z) = f(rz) \quad \forall\, r \in [0,1], \quad \forall z \in D. \tag{4}$$

Then h is a continuous map from $[0,1] \times A_s$ into A_s. Moreover, since

$f(z) \neq 0$ $\forall z \in D$, it is easy to see that $h(r,z) \neq 0$ $\forall z \in D$, $\forall r \in [0,1]$. Hence $h(r,.)$ is a unit for every r in $[0,1]$. Finally, $h(0,.) = f(0)$, $h(1,.) = f(\cdot)$, so that f is homotopic to the constant function $f(0)$. Since $f(0) > 0$, it is homotopic to the identity via the map

$$w(r,z) = rf(0) + (1-r), \quad \forall z \in D, \forall r \in [0,1] \tag{4}$$

This shows that f is homotopic to the identity. By Proposition 1, it now follows that f has a logarithm in A_s.

Though it is not needed for this paper, we state another proposition in the interests of completeness.

Proposition 3. Every unit in A has a logarithm in A.

Proof. If f is a unit of A, then $f(z) \neq 0$ $\forall z \in D$. Now define

$$h(r,z) = \begin{cases} f[(1-2r)z], & r \in [0,1/2] \\ \exp[1-2r]a, & r \in [1/2,1] \end{cases} \qquad \forall\, z \in D \tag{4}$$

where $\exp(a) = f(0) \neq 0$. Then h provides a homotopy from f to 1 within the set of units of A, so that f has a logarithm in A, by Proposition 1.

In particular, every unit of A_s (which is also a unit of A) has a logarithm in A, but not necessarily in A_s. The difference arises because the set of nonzero complex numbers is connected, while the set of nonzero real numbers is not.

Proposition 4. Suppose a unit f in A_s has a logarithm in A_s. Then the logarithm is unique.

Proof. Let g_1, g_2 be logarithms of f in A_s. Then we have successively

$$\exp(g_1(z)) = \exp(g_2(z)) \qquad \forall\ z \in D \tag{5}$$

$$g_1(z) = g_2(z) + j2\pi m(z) \qquad \forall\ z \in D \tag{6}$$

where m(z) is an integer for all $z \in D$. Since $m(z) = [q_1(z) - g_2(z)]/j2\pi$, it is analytic in D and therefore continuous in D. Now a continuous function that assumes only discrete values must be constant. Hence there is an integer m such that

$$g_1(z) = g_2(z) + j2\pi m, \quad \forall z \in D \tag{7}$$

Finally, since g_1, $g_2 \in A_s$, the quantity $g_1(z) - g_2(z)$ is real whenever $z \in [1,1]$. This shows that $m = 0$.

The same argument shows that logarithms in A are unique to within integer multiples of the constant function $j2\pi$.

3. A SIMPLE PROOF OF AN INTERPOLATION LEMMATT

In this section, we give a simple proof of an interpolation lemma from [1] which forms the basis of all the results on strong stabilizability given in [1].

Suppose we are given a finite collection $\{z_1,\ldots,z_\ell\}$ of points in D, together with nonnegative integers $m_1,\ldots,m_\ell$ and complex numbers c_{ij}, $0 \le j \le m_i$, $1 \le j \le \ell$. The objective is to determine whether there exists a unit $u \in A_s$ such that $u^{(j)}(z_i) = c_{ij}$ for all i,j, where as usual the zeroth order derivative is the function itself. Since u is required to be in A_s, all the required interpolation conditions must occur in complex conjugate pairs; that is, if z_i is real, then c_{ij} must be real for all j, and if $z_i = \bar{z}_k$, then $m_i = m_k$ and $c_{ij} = \bar{c}_{kj}$ for all j. Let us assume without loss of generality that these elementary consistency conditions are met, and that the z_i are so numbered that $z_1,\ldots,z_r$ are real $z_{r+1},\ldots,z_\ell$ have nonreal.

Proposition 5. With the above notation, there exists a unit $u \in A_s$ such that $u^{(j)}(z_i) = c_{ij}$ for all i,j if and only if $c_{10},\ldots,c_{r0}$ all have the same sign.

Proof "only if" If such a u exists, then u(z) is real and does not change sign as z varies over [-1,1]. In particular, $u(z_1) = c_{10},\ldots,$ $u(z_r) = c_{r0}$ are all of the same sign.

"if" we may assume without loss of generality that $c_{10},\ldots,c_{r0}$ are all positive; if not, we construct a unit $v \in A_s$ such that $v^{(j)}(z_i) =$ $-c_{ij}, \forall i,j$, and let $u = -v$. The problem is solved if we can construct a function $f \in A_s$ such that

$$\frac{d^j}{dz^j} \exp(f(z))\Big|_{z=z_i} = c_{ij}, \quad \forall i,j \tag{8}$$

Let us translate the conditions in (8) into conditions on the values of f and its derivatives. First, we get

$$f(z_i) = \log c_{i0}, \quad i=1,\ldots,\ell \tag{9}$$

The point to note is that when z_i is real, c_{i0} is real and positive, so that $\log c_{i0}$ is (more precisely, can be taken to be) real. When z_i is nonreal, $\log c_{i0}$ may be nonreal, but this is no problem. For higher derivatives, we get

$$\begin{aligned} f'(z_i) &= c_{i1}/c_{i0} \\ f''(z_i) &= \{c_{i2}-[f'(z_i)]^2\}/c_{i0} \end{aligned} \tag{10}$$

<u>et cetera.</u> Since $c_{i0} \neq 0$ for all i, the expressions in (10) are all well-defined. Moreover, the quantities $f^{(j)}(z_i)$ are all real whenever z_i is real, and occur in complex conjugate pairs. Thus the original interpolation problem is reduced to one of constructing a symmetric analytic function meeting prespecified interpolation conditions. This is trivial to solve: in fact, f can always be chosen to be a polynomial.

<u>Example 1</u>. Suppose it is required to construct a unit in A_s satisfying

$$u(1) = 2, \quad u(0) = 1, \; u'(0) = -1, \; u(j) = 1 + 2j$$

Since $u(1)$ and $u(0)$ are both positive, such a unit exists. To solve the problem, let $u = \exp(f)$. Then f must satisfy

$$f(1) = \log 2 \simeq 0.7$$

$$f(0) = 0$$

$$f'(0) = -1$$

$$f(j) = \log(1+2j) \simeq 0.8 + j\,1.1$$

Using Lagrange interpolation, one can readily find a polynomial satisfying the above conditions, namely:

$$\begin{aligned} f(z) &= 0.7 \cdot z^2(z^2+1)/2 + z(z-1)(z^2+1) \\ &\quad + (0.8 + j1.1)\ z^2(z-1)(z+j)/(2+2j) \\ &\quad + (0.8 - j1.1)\ z^2(z-1)(z-j)/(2-2j) \\ &= 2.3\ z^4 - 2.1\ z^3 + 1.5\ z^2 - z \end{aligned}$$

It should be noted that the procedure described above does not result in a <u>rational</u> unit function, in contrast with the procedure in [1]. The existence of a rational unit is a consequence of the next result.

<u>Proposition 6</u>. Suppose d, n are rational functions in A_s with no common zeros in D. Then there exists a rational c in A_s such that d+cn is a unit if and only if the values of d at the real zeros of n in [-1,1] are all of the same sign (or equivalently, if the number of zeros of d (counting multiplicity) between any pair of real zeros of n in [-1,1] is even).

<u>Proof</u>. Let $z_1,\ldots,z_\ell$ be the zeros of n in D, with multiplicities $m_1,\ldots,m_\ell$. Then a function g in A_s is of the form d+cn for some c in A_s if and only if

$$g^{(j)}(z_i) = d^{(j)}(z_i), \quad 0 \le j \le m_i - 1;\ 1 \le i \le \ell \tag{11}$$

"only if" Suppose there exits a rational c such that d+cn is a unit. Let u denote d+cn; then u satisfies (11). By Proposition 5, this implies that $d(z_1),\ldots,d(z_r)$ are all of the same sign, where $z_1,\ldots,z_r$ are the real zeros of n in D.

"if" Suppose $d(z_1),\ldots,d(z_r)$ are all of the same sign. Then by Proposition 4 there exists a (not necessarily rational) unit u_1 in A_s such that $u_1 = d+c_1n$ for some c_1 in A_s. If c_1 is not rational, find a rational function c in A_s (e.g. a polynomial) such that $||c-c_1|| < 1/[||u_1^{-1}||\cdot||n||]$. Let $u = d+cn$. Then $||u-u_1|| \leq ||c-c_1||\cdot||n|| < 1/||u_1^{-1}||$, so that u is also a unit.

Now we return to Proposition 5 and examine the existence of a rational unit u satisfying $u^{(j)}(z_i) = c_{ij}$ for all i,j. Let d be any polynomial in A_s such that $d^{(j)}(z_i) = c_{ij}$ $\forall i,j$, and let

$$n(z) = \prod_{i=1}^{\ell} (z-z_i)^{m_i} \tag{12}$$

The condition $c_{i0} > 0$ for $i=1,\ldots,r$ insures that d and n satisfy the hypotheses of Proposition 6. Thus there exists a <u>rational</u> $c \in A_s$ such that u = d+cn is a unit. Clearly u is also rational and satisfies $u^{(j)}(z_i) = c_{ij}$ $\forall$ i,j.

4. STABLE STABILIZING COMPENSATORS

Given rational functions d,n in A_s, the problem of finding all $c \in A_s$ such that $d + cn$ is a unit is equivalent to finding all units $u \in A_s$ such that $d-u$ is a multiple of n. Let $z_1,\ldots,z_\ell$ be the zeros of n in D, with multiplicities $m_1,\ldots,m_\ell$. Then the problem is one of finding <u>all</u> units $u \in A_s$ that satisfy

$$u^{(j)}(z_i) = d^{(j)}(z_i), \quad 0 \leq j \leq m_i-1; \quad 1 \leq i \leq \ell \tag{13}$$

Suppose u_1 and u_2 are two units that <u>each</u> satisfy (13), and let v denote the unit $u_2u_1^{-1}$. Then a routine calculation shows that

$$v(z_i) = 1, \quad 1 \leq i \leq \ell \tag{14}$$

$$v^{(j)}(z_i) = 0, \quad 1 \leq j \leq m_i-1; \quad 1 \leq i \leq \ell \tag{15}$$

Equivalently, v is of the form $1 + fn$ for some $f \in A_s$, since v interpolates the function "1" and its derivatives at the zeros of n. This leads to the next result.

<u>Proposition 7</u>. Given d, $n \in A_s$, let u be any unit in A_s satisfying (13). Then the set of <u>all</u> units in A_s satisfying (13) is given by $\{uv : v \in A_s$ is a unit of the form $1+fn$ for some $f \in A_s\}$.

For a $g \in A_s$, let $U(g)$ denote the set of all units of the form $1+fg$ for some $f \in A_s$. If we can parametrize $U(n)$, then Proposition 6 enables us to find all units that satisfy (13). Then the set of all $c \in A_s$ that stabilize the plant n/d is just $\{(d-u)/n : u$ satisfies $(13)\}$.

The next two results give an explicit description of $U(g)$ when $g \in A_s$. It is necessary to treat separately the cases where g has real zeros in D and where it does not. To aid to the presentation of the results, some notation is introduced. Suppose $g \in A_s$ and has only a finite number of zeros in D. Let $z_1,\ldots,z_s, \bar{z}_1,\ldots,\bar{z}_s,\sigma_1,\ldots,\sigma_r$ denote the distinct zeros of g in D, where $z_1,\ldots,z_s$ are nonreal and $\sigma_1,\ldots,\sigma_r$ are real. For convenience, let $z_1,\ldots,z_{2s+r}$ denote the same sequence, and let μ_i denote the multiplicity of z_i as a zero of g. Select polynomials $p_1(z),\ldots,p_{2s}(z)$ such that

$$p_i(z_i) = 1, \; p_i^{(j)}(z_i) = 0 \quad \text{for } j = 1,\ldots, \mu_i-1 \tag{16}$$

$$p_i^{(j)}(z_k) = 0 \text{ for } j = 0,\ldots, \mu_k-1 \text{ if } k\neq i$$

Since the zeros of g occur in complex conjugate pairs, we may suppose that $p_i(z) = \bar{p}_{i+s}(\bar{z})$ for $i = 1,\ldots,s$. Finally, define the polynomials

$$\phi_i(z) = j\,[p_i(z) - p_{i+s}(z)], \quad i = 1,\ldots, s \tag{17}$$

and observe that $\phi_i \in A_s$ for all i.

Proposition 8. Suppose $g \in A_s$ has only a finite number of zeros in D, and that g has at least one real zero in D. Then every unit in A_s of the form $1 + fg$, $f \in A_s$, can be expressed as exp(v), where $v\in A_s$ has the form

$$v = hg + \sum_{i=1}^{s} 2\pi m_i \phi_i \tag{18}$$

where $h \in A_s$, $m_1,\ldots,m_s$ are arbitrary integers, and ϕ_i are defined in (17). Conversely every unit exp(v) where v is of the form (18) can also be written as 1+fg for some $f \in A_s$. In summary,

$$U(g) = \{\exp(v): v \text{ is of the form (18)}\} \tag{19}$$

Proof. Suppose $u = 1 + fg$ is a unit. Since g has at least one real zero in D, u(z) equals one for some real z in D. Hence $u(z) > 0$ for

$z \in [-1,1]$, and by Proposition 2, has a logarithm v in A_s. Since $g(z_i) = 0$ for $i = 1,\ldots, 2s+r$, it follows that $u(z_i) = 1\ \forall i$, which in turn implies that

$$v(z_i) = j2\pi m_i,\ m_i \text{ an integer, for } i = 1,\ldots,2s+r \tag{20}$$

Since $v(z) = \overline{v}(\overline{z})$, it is immediate from (20) that $m_i = -m_{i+s}$ for $i=1,\ldots,s$, and that $m_{2s+1} = \ldots. = m_{2s+r} = 0$. At multiple zeros of g, successive higher derivatives of g vanish, which implies that the corresponding derivatives of u and v also vanish. Thus

$$v^{(j)}(z_i) = 0 \quad \text{for } j = 1,\ldots,\mu_i-1; \quad i=1,\ldots,2s+r \tag{21}$$

Now (20) and (21) lead to the conclusion that g divides the function $v - \sum_{i=1}^{2s} j2\pi m_i p_i$, which equals $v - \sum_{i=1}^{s} 2\pi m_i \phi_i$. This is precisely (18).

Conversely, suppose v is of the form (18). Then v satisfies (20) and (21), which in turn implies that $\exp(v) - 1$ is divisible by g. Hence $\exp(v)$ is of the form $1+fg$ for some $f \in A_s$.

In practice, the assumption that g has at least one real zero in D does not pose a major restriction, because if g is the "numerator" of a strictly proper plant, then $g(1) = 0$. Also, note that g is not required to be rational - merely to have only a finite number of zeros in D.

We state the next result in the interests of completeness. The proof is omitted, as it closely follows that of Proposition 8.

Proposition 9. Suppose $g \in A_s$ has only a finite number of zeros in D, all of them nonreal. Then every unit $u \in A_s$ of the form $1+fg$, $f \in A_s$

can either be written as exp(v) where

$$v = hg + \sum_{i=1}^{s} 2\pi m_i \phi_i; \quad m_1, \ldots, m_s \text{ arbitrary integers, } h \in A_s \tag{22}$$

or as -exp(v) where $v \in A_s$ has the form

$$v = hg + \sum_{i=1}^{s} (2m_i+1)\pi\phi_i; \quad m_1, \ldots, m_s \text{ arbitrary integers, } h \in A_s \tag{23}$$

Conversely, every unit exp(v) where v is of the form (22) can be written as 1+fg for some $f \in A_s$; every unit -exp(v) where v is of the form (23) can be written as $1+fg$ for some $f \in A_s$.

<u>Example</u>. Consider the problem of determining all stable stabilizing compensators for the plant whose transfer function is $(s^2+1)/[(s+2)^2(s-3)]$. After substituting $s = (1+z)/(1-z)$, this becomes

$$\frac{n(z)}{d(z)} = \frac{(z^2+1)(1-z)}{(z-3)^2(2z-1)}$$

where n and d denote the numerator and denominator polynomials, respectively. Thus the problem is one of finding all $c \in A_s$ such that d+cn is a unit of A_s. This can be solved using Propositions 7 and 8. First, we construct a unit u_0 such that $d-u_0$ is a multiple of n. Since the zeros of n inside the unit disc are at 1, j, -j, we require $u_0(z)$ to equal d(z) at these points. Thus we must have

$$u_0(1) = 4, \quad u_0(j) = 4 + 22j$$

If $u_0 = \exp f_0$, then f_0 must satisfy

$$f_0(1) = \log 4 \cong 1.4, \quad f_0(j) = \log (4+22j) \cong 3.1 + j1.4$$

Using Lagrange interpolation, one can find

$$f_0(z) = -1.55\ z^2 + 1.4z + 1.55$$

Therefore, <u>one</u> stabilizing compensator is given by

$$c_0(z) = [d(z) - \exp f_0(z)]/n(z)$$

To find <u>all</u> stabilizing compensators, we compute the set U(n). Again using Lagrange interpolation, one can find polynomials to satisfy (16), namely

$$p_1(z) = \frac{(z-1)(z+j)}{2+2j}\ , \quad p_2(z) = \overline{p}_1(\overline{z}), \quad p_3(z) = \frac{z^2+1}{2}$$

$$\phi_1(z) = j[p_1(z) - p_2(z)] = -\frac{\pi}{2}(z-1)^2$$

Thus

$$U(n) = \{\exp u, \quad \text{where } v = hn + \pi m(z-1)^2, \ m \text{ an integer}, \ h \in A_s\}$$

Thus the set of <u>all</u> stable stabilizing compensators is given by

$$C = \{[d(z) - \exp f(z)]/n(z), \quad \text{where}$$

$$f(z) = f_0(z) + h(z)n(z) + \pi m(z-1)^2, \ m \text{ an integer}, \ h \in A_s\}$$

5. MULTIVARIABLE PLANTS

Suppose $D \in A_s^{p\times p}$, $N \in A_s^{q\times p}$ are rational and right-coprime. Then necessary and sufficient conditions for the existence of a rational $C \in A_s^{p\times q}$ such that D+CN is a unit of $A_s^{p\times p}$ are given in [1]. Now the problem of finding all $C \in A_s^{p\times q}$ that stabilize ND^{-1} is equivalent to the problem of finding all units U in $A_s^{p\times p}$ such that D-U is a left multiple of N. It turns out that Proposition 7 can be radily extended to the multivariable case. Define

$$U(N,D) = \{U \in A_s^{p\times p}: U \text{ is a unit and } U = D+CN \text{ some } C \in A_s^{p\times q}\} \tag{24}$$

Let U(N) be a shorthand for U(N,I). Thus

$$U(N) = \{U \in A_s^{p\times p}: U \text{ is a unit and } U = I+FN \text{ for some } F \in A_s^{p\times q}\} \tag{25}$$

Proposition 10: Let U be any element of U(N,D). Then

$$U(N,D) = \{UV : V \in U(N)\} \tag{26}$$

Proof. Suppose $D-U = C_0N$. We show first that $UV \in U(N,D)$ for all $V \in U(N)$, and then we show that every $W \in U(N,D)$ is of the form UV for some $V \in U(N)$. So suppose first that $V \in U(N)$, and let $V = I + FN$. Then $D - UV = (U + C_0 N) - U(I+FN) = (C_0-F)N$ is a left multiple of N, so that $UV \in U(N,D)$. Conversely, suppose $w \in U(N,D)$, and let $D-W = CN$ for some $C \in A_s^{p\times q}$. Define $F = U^{-1}(C-C_0)$ and $V = I+FN$. Then $W = D+CN = D+C_0N + (C-C_0)N = U + UFN = UV$. Since W and U are units, so is V; i.e., $V \in U(N)$.

The main difficulty in extending Propositions 8 and 9 to the multivariable case is that the set of units that have logarithms is not necessarily a group under multiplication. Hence, even if a unit is homotopic to the identity, it need not have a logarithm. As a result, it is not possible to characterize all stable stabilizing compensators, since it is not possible to characterize all units of the form $I+FN$ for a given N. However, it is possible to generate an infinite family of such compensators.

Suppose $N \in A_s^{pxq}$. Then every function $\exp(HN)$, $H \in A_s^{qxp}$ is a unit of A_s^{qxq} of the form $I+FN$; in fact

$$\exp(HN) = I + \left[\sum_{i=0}^{\infty} \frac{(HN)^i}{(i+1)!} H \right] N \tag{24}$$

Thus, for any $H \in A_s^{qxp}$, the function

$$F = \sum_{i=0}^{\infty} \frac{(HN)^i}{(i+1)!} H \tag{25}$$

belongs to $U(N)$. In this way it is possible to generate infinitely many C such that D+CN is a unit, using Proposition 10.

6. CONCLUDING REMARKS

In this paper, we have given a characterization of all stable compensators that stabilize a given scalar plant. In the process, we have also given a simple proof of an interpolation lemma due to Youla et al. [1]. In the case of multivariable plants, we have given a procedure for generating infinitely many (but not necessarily all) stable stabilizing compensators.

It is shown in [4] that the problem of simultaneously stabilizing two plants is equivalent to stabilizing an auxiliary system using a stable compensator. Thus the results of this paper can also be used to determine all compensators that simultaneously stabilize each of two given scalar plants, and to determine an infinite number of compensators that simultaneously stabilize each of two given multivariable plants.

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