

## Radon-space detection and estimation

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### Abstract

The problem of reconstructing a multi-dimensional field from noisy, limited projection measurements is approached using an object-based stochastic field model. Objects within a cross-section are characterized by a finite-dimensional set of parameters, which are estimated *directly* from the projection measurements using maximum likelihood estimation. The computational structure, performance and robustness of the ML estimation procedure are investigated.

### Introduction

The problem of reconstructing an  $n$ -dimensional function from its  $(n-1)$ -dimensional projections arises, typically in the context of cross-sectional imaging, in a diversity of disciplines. In the two dimensional version of this problem, let  $f(x)$  represent the value of the cross-sectional function (for example x-ray attenuation coefficient) at the point  $x = (x_1, x_2)$ . The *projection* of  $f(x)$  at any angle  $\theta$  is a one-dimensional (1D) function given by

$$g(t, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \delta(t - \underline{x}'\underline{\theta}) dx_1 dx_2 = \int_{\underline{x}'\underline{\theta}=t} f(x) ds$$
$$(t, \theta) \in Y = \{(t, \theta) : -\infty < t < \infty, 0 \leq \theta < \pi\} \quad (1)$$

where  $\delta(t)$  is the Dirac delta function and  $\underline{\theta}$  is the unit vector  $(\cos\theta \ \sin\theta)'$ . Considered as a function,  $g: Y \rightarrow \mathbb{R}$  is called the *Radon transform* of  $f(x)$ .

Success in reconstruction from projections in radio astronomy, electron microscopy, medical CAT scanning and other fields has recently led to suggestions to apply reconstruction techniques to a number of novel and technologically demanding tasks, such as real-time monitoring of high production rate manufacturing processes, mesoscale oceanographic thermal mapping, quality control nondestructive testing, and "stop action" internal imaging of very rapidly changing media.<sup>1,2</sup> In virtually all problems involving the processing of projection measurements, however, the *ultimate* goal is often far more modest than obtaining high-resolution cross-sectional imagery. Rather, imaging is usually an intermediate step, and the ultimate goal involves extracting specific information, typically related to objects, regions or boundaries within the cross-section. Such objects include, for example, organs, tumors, bone and metallic surgical clips in medical CAT scanning, high-contrast thermal regions such as cold-core rings and the Gulf stream in oceanography,<sup>1</sup> and interior cracks and flaws in materials in non-destructive testing.<sup>2</sup>

Typically, object-related information (e.g. object location, size, or detailed boundary information) is extracted from projection measurements by post-processing a reconstructed image, either visually or by automated techniques. The success of such an approach relies upon reconstructed imagery that is accurate, artifact-free and of high-resolution. Such imagery is known to require abundant low-noise projection data taken over a wide viewing angle; if the projection measurements are limited in number or view angle, or have high noise levels, the inverse problem is ill-posed and/or has a numerically sensitive or noisy solution.<sup>3</sup> The acquisition of abundant, wide-angle measurements, however, is not always practical or possible, due to time, economic, environmental or physical constraints that limit the total measurement viewing angle or time, or limit the number or sensitivity of measurement transducers.

In this paper, we focus our attention on the processing of projection measurements when (1) the overall goal is to extract object-related information about the cross-section, and (2) severe limitations on the total number, SNR, or overall view angle of the projection measurements preclude the formation of accurate and artifact-free reconstructed imagery. We propose and investigate an alternative to full image reconstruction by processing limited, noisy projection (Radon transform) measurements *directly*, in order to detect, locate and characterize one or more objects within the cross-section. Specifically, a stochastic object-based field model is introduced, in which a single object is represented by a finite number of parameters, characterizing, say, the object location, size, boundary shape, contrast, and/or detailed internal density variations. These parameters are estimated, for the case of limited projection measurements corrupted by additive white Gaussian noise, by maximum likelihood (ML) parameter estimation.<sup>4</sup> In this paper, we present a brief overview of this object-based formulation and the associated analysis, and illustrate the results by way of a simple example.

### Cross-section and measurement model

Consider a 2D cross-section containing a single object

$$f(x) = f_b(x) + d \cdot f_o(x-c; \gamma) \quad (2)$$

where  $f_b(x)$  is a known background and  $d \cdot f_o(x-c; \gamma)$  is a randomly located object having known *density* or contrast  $d$  [where  $f_o(x; \gamma) = 1$ ] and unknown location  $c \in \mathbb{R}^2$ ;  $\gamma$  is a possibly unknown vector of parameters characterizing, for example, the size, shape and/or orientation of the object. By the linearity of (1), the Radon transform of  $f(x)$  is the sum of two components,

$$g(t, \theta) = \int_{x' \underline{c} - t} f_b(x) ds + d \cdot \int_{x' \underline{c} - t} f_o(x - c; \gamma) ds \cdot$$

$$\Delta g_b(t, \theta) + d \cdot g_o(t - \underline{c}' \underline{c}, \theta; \gamma) \quad (3)$$

where  $g_o(t, \theta; \gamma)$  is the Radon transform of  $f_o(x; \gamma)$ , the unit-contrast object located at the origin. Note that because the location of the object  $c \in \mathbb{R}^2$  is unknown, the component of the Radon transform due to the object is characterized by an unknown *sinusoidal* shift  $\underline{c} = c_1 \cos \theta + c_2 \sin \theta$  in the  $t$  variable.

Let the noisy projection measurements be given by\*

$$y(t, \theta) = \left[ d \cdot g_o(t - \underline{c}' \underline{c}, \theta; \gamma) \cdot h(t) \right] + w(t, \theta)$$

$$= s(t, \theta; c, \gamma) + w(t, \theta) \quad (t, \theta) \in S_y \subset Y \quad (4)$$

where  $\cdot$  denotes one-dimensional convolution in the  $t$  variable,  $h(t)$  is a 1D measurement aperture function, and  $w(t, \theta)$  is a zero-mean Gaussian noise process.\*\* In terms of the present notation, the object-based detection and estimation problem may be stated as: *given* noisy, limited measurements of the Radon transform on the set  $S_y$  as shown in (4), *detect* whether an object is present or not, and if so, *estimate* the location  $c \in \mathbb{R}^2$  and parameters  $\gamma$  of the object. It should be noted that with the exception of the density parameter  $d$ , the parameters characterizing the object enter the problem nonlinearly, and lead to a nonlinear estimation problem of small dimensionality. This is in contrast to full image reconstruction, in which a linear estimation problem of high dimensionality is solved.

At this point, in order to illustrate this perspective toward processing projection measurements, we focus on the specific problem of employing maximum likelihood (ML) techniques to estimate the *location* of a single object which is situated at some unknown point within the cross-section, but is otherwise completely known (i.e.  $\gamma$  is assumed to be known). This problem allows us (1) to develop insight into the structure of the parameter estimation computations, and (2) to demonstrate the quantitative tools that can be utilized in critically evaluating the estimator performance and robustness to modeling errors. This problem serves to establish a framework within which more sophisticated algorithms may be developed which take into account, for example, detailed a priori information about unknown object shapes or the presence of multiple objects.<sup>5</sup>

#### ML object localization

In this section, we consider the special case of *full-view* measurements (i.e.,  $S_y = Y$ ), with  $w(t, \theta)$  a 2D zero-mean Gaussian noise process with covariance  $E[w(t, \theta)w(\tau, \phi)] = \frac{N_0}{2} \delta(t - \tau, \theta - \phi)$ .† The *maximum-likelihood* (ML) location estimate  $\hat{c}_{ML}$  is that value of the parameter  $c$  that maximizes the log likelihood function. <sup>4</sup>

$$l(c) = \frac{2}{N_0} \int_0^\pi \int_{-\infty}^\infty y(t, \theta) s(t, \theta; c) dt d\theta - \frac{1}{N_0} \int_0^\pi \int_{-\infty}^\infty s^2(t, \theta; c) dt d\theta \quad (5)$$

where  $\gamma$  is known in the localization problem and has been suppressed. The first term in (5) corresponds to a matched filtering operation in Radon space (this operation maps the Radon-space measurements into a function on  $\mathbb{R}^2$ ); the second term in (5) compensates for the energy in the Radon space matched filtering template. Since  $s(t, \theta; c)$  depends on  $c$  only via a shift in the  $t$  variable, the second term in (5) is  $c$ -independent and can be dropped, as can the  $\frac{2}{N_0}$  scaling factor, to yield

$$L(c) = \int_0^\pi \int_{-\infty}^\infty y(t, \theta) s(t - \underline{c}' \underline{c}, \theta; 0) dt d\theta \quad (6)$$

The log likelihood function for this problem is seen to be obtained by a *convolution back-projection* (CBP) operation (such as that used in conventional image reconstruction<sup>3</sup>), where the generally  $\theta$ -dependent and nonsymmetric convolving kernel  $s(t, \theta; 0)$  has been specified in the solution to the optimal object localization problem.

#### Performance analysis - an example

The ML location estimate  $\hat{c}_{ML}$  is that value of the location parameter  $c$  maximizing the log likelihood function in (6). Noise in the measurements  $y(t, \theta)$  leads to errors in the location estimate. These errors may be examined by substituting the signal and noise measurement components in (4) into (6), and writing the log likelihood function as the sum of a 2D deterministic *ambiguity function* and a 2D zero-mean correlated random field. By examining the effects of both *local* errors (Cramer-Rao bound analysis) and *global* errors (anomaly analysis) in the estimation procedure, the covariance of the estimation error may be approximately quantified.<sup>5,6</sup>

As an example, consider a constant-density *disk object* of radius  $R$ ,

$$f_o(x; R) = \begin{cases} 1 & \text{if } \|x\| < R \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

which is located within a circular region of radius  $T \gg R$ , and let the measurement aperture function  $h(t)$  correspond to an ideal spatial low-pass filter of fixed bandwidth. In this full-view case the error analysis is circularly-symmetric, and the error covariance matrix is  $\sigma^2$  times a  $2 \times 2$  identity matrix. Figure 1 is a plot of  $(\sigma/T)^{-2}$ , the inverse of the normalized error variance, versus the normalized object size  $R/T$ , for several values of the ratio of contrast squared to noise level. This figure indicates a definite threshold behavior -- for given values of the object contrast  $d$  and measurement noise level  $N_0$ , there exists a smallest object size for reliable localization.

### Robustness analysis - an example

In the previous discussion, a number of assumptions were made in order to simplify the analysis so that insight could be more easily obtained; in particular, it was assumed that (1) the object function  $f_o(x)$  is known precisely, and (2) the cross-sectional field consists of at most one object superimposed on a background that is known exactly. The sensitivity of the performance to each of these assumptions may be evaluated by considering the effect of specific modeling errors, and in this way, the degradation in local performance (Cramer-Rao analysis) and/or global performance (anomaly analysis) due to modeling errors may be studied.<sup>5,6</sup> Here, we illustrate the robustness of ML object localization by considering global performance degradation due to several modeling errors. A global error, or *anomaly*, corresponds to the event that an ML location estimate is obtained that is not in the vicinity of the actual object location (*vicinity* here corresponds to that region of the plane close enough to the actual object location so that linearized Cramer-Rao error analysis is valid).

Let  $P_0$  represent the probability that an anomaly does *not* occur, i.e.,  $P_0$  represents the probability that the ML location estimate is in the vicinity of the actual object location. Insight into the robustness of ML localization to modeling errors may be obtained by evaluating an approximation to, or bound on, the probability  $P_0$  in the presence of modeling errors. As a simple example of this analysis applied to object size and density modeling errors, consider the problem of attempting to locate the disk object of radius  $R$  in (7) with contrast  $d$ , when the actual field consists of a disk object of radius  $R_a$  and contrast  $d_a$ . Because the Radon transform in (3) is linear in object contrast, the localization performance depends on  $d_a$  only through the measurement signal energy, which for the full-view case being considered is  $E_a = \frac{16}{3} \pi d_a^2 R_a^3$ . A lower bound on the probability  $P_0$  is plotted in Figure 2 versus the measurement signal-to-noise ratio (SNR)  $E_a/N_o$ , for several values of radius ratio  $R/R_a$ .

As indicated by this figure, the best global performance is obtained with perfect knowledge of the object size, and the performance is quite robust to moderately-sized modeling errors. Even when the modeled object size (cross-sectional area) is in error by a factor of two, the measurement SNR must be increased by only about 2 dB to overcome the performance degradation caused by the size mismatch. It should be noted that the robustness of local and global performance may be evaluated in a similar way for other types of modeling errors, e.g. errors in the detailed boundary shape of the actual object;<sup>5,6</sup> such analyses indicate that these ML localization procedures are very robust to a variety of modeling errors.

These techniques may also be employed to evaluate the performance robustness to the presence of multiple unmodeled objects contained within the background field. In particular, let the modeled field be given by the object in (7) with a contrast  $d$ , but let the actual field consist of the object in (7) with contrast  $d$ , plus the superposition of  $N$  nonoverlapping disk objects in the background, where each of these objects has the same contrast  $\bar{d}$  and radius  $\bar{R}$ , and has a random location point. As a final example, a lower bound on the probability  $P_0$  (the probability that the ML location estimate occurs in the vicinity of the actual object location) was computed for the case of  $N=20$  unmodeled background field objects. These background "objects" may be thought of, in a sense, as corresponding to random fluctuations in the background field about its nominal value, which thus far has been assumed to be known perfectly.

Figure 3 is a plot of a lower bound on  $P_0$ , versus the unmodeled object contrast ratio  $\bar{d}/d$ , for two values of object radius ratio  $R/\bar{R}$  and a constant value of  $E/N_o = \frac{16}{3} \pi d^2 R^3 = 20$  dB. As indicated by this figure, the global localization performance is extraordinarily robust to the presence of smaller unmodeled objects, even if there are many of them and they are more dense than the object whose location is being estimated.

### Detailed object boundary estimation

Up to this point we have considered the problem of using noisy projection measurements to locate an object in a cross-section when its density profile  $f_o(x;\gamma)$  is known, but its location  $c$  is unknown. One may similarly consider the problem of estimating the finite-dimensional geometry parameter vector  $\gamma$  from noisy projection data, by forming a log likelihood function for  $\gamma$  (assuming the object location is known), and finding those geometry parameter values at which the maximum occurs.

Various geometry parameterizations are possible. One parameterization capturing information about object size, elongation or eccentricity; and principal orientation (if elongated) has been investigated,<sup>5</sup> including a study of the robustness of ML geometry estimation to errors in the assumed object location.

### Conclusions

A framework has been presented for detecting, locating and characterizing objects in a cross-section by using noisy projection measurements *directly*, rather than post-processing a reconstructed image. Within this framework, the performance, robustness and computational structure of ML estimation procedures have been investigated for both object location and geometry parameters.

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\* By the stated assumptions, the Radon transform of the background,  $g_b(t, \theta)$ , is known and its effect has been subtracted from the measurements.

\*\* The problem where  $y(t, \theta)$  is a counting process with a rate that depends on  $s(t, \theta; c, \gamma)$  may also be considered; such a model is appropriate, for example, in very low-dose x-ray problems.

+ Incomplete measurement cases, in which views are available over only a limited view angle or at a finite number of views, are treated similarly. <sup>5</sup>

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FIGURE CAPTIONS

Fig. 1. Localization performance versus disk object size.

Fig. 2.  $P_0$  lower bound versus measurement SNR in the presence of size modeling error.

Fig. 3.  $P_0$  lower bound versus contrast ratio  $\tilde{d}/d$ ; 20 unmodeled objects.





