Tight Contact Structures on Small Seifert Spaces

by

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Abstract

In this thesis, we discuss the relation between the Euler number of a tight contact small Seifert space and the contact framing of Legendrian vertical circles in it, and apply this relation to classify up to isotopy tight contact structures on small Seifert spaces with $e_0 \neq 0, -1, -2$.

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Chapter 1

Introduction to 3-dimensional Contact Topology

1.1 Definitions and First Examples

Definition 1.1.1. A contact structure ξ on a 3-manifold M is a nowhere integrable tangent plane distribution, i.e., near any point of M, ξ is defined locally by a 1-form α, s.t., $\alpha \wedge d\alpha \neq 0$. A manifold M with a given contact structure ξ is called a contact 3-manifold, and is denoted by (M, ξ).

Note that the orientation of M given by $\alpha \wedge d\alpha$ depends only on ξ, not on the choice of α. So any contact manifold is orientable. If the manifold M is oriented, i.e. comes with a native orientation, then a contact structure ξ on M is called positive if the orientation given by ξ agrees with the native orientation, and negative if otherwise. A contact structure ξ is said to be co-orientable if ξ is defined globally by a 1-form α. Clearly, an co-orientable contact structure is orientable as a plane distribution, and a choice of α determines an orientation of ξ. Unless otherwise specified, all manifolds in this thesis will be oriented, and all contact structures in this thesis will be positive and co-oriented, i.e., with a prescribed up to positive scaling defining form α such that $\alpha \wedge d\alpha > 0$.

Example 1.1.2. The standard contact structure ξ_0 on $\mathbb{R}^3$ is defined by the 1-form $\alpha_0 = dz - ydx$, where (x, y, z) are the standard Cartesian coordinates of $\mathbb{R}^3$.

Example 1.1.3. Let (r, θ, z) be the standard cylindrical coordinates of $\mathbb{R}^3$, and $\alpha_1 = dz + r^2 d\theta$. Then the tangent plane distribution ξ_1 defined by α_1 is also a contact structure on $\mathbb{R}^3$.

Example 1.1.4. (r, θ, z) are still the standard cylindrical coordinates of $\mathbb{R}^3$. Let $\alpha_2 = \cos r dz + r \sin r d\theta$. Then the tangent plane distribution ξ_2 defined by α_2 is again a contact structure on $\mathbb{R}^3$.

Example 1.1.5. Let S^3 be the unit 3-sphere in $\mathbb{R}^4$, and (x_1, y_1, x_2, y_2) the standard Cartesian coordinates of $\mathbb{R}^4$. The standard contact structure ξ_{st} of S^3 is defined by the 1-form $\alpha = (x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)|_{S^3}$.

Definition 1.1.6. Two contact structures on a 3-manifold are called homotopic if they are homotopic as tangent plane distributions. They are called isomorphic if there is a self-diffeomorphism of the ambient 3-manifold that map one of them to the other. They are called isotopic if there is a homotopy between them through contact structures.

Remark 1.1.7. By Gray's Theorem, two contact structures η_0 and η_1 on a closed 3-manifold are isotopic if and only if there is an isotopy $\{\varphi_t\}$ of the ambient manifold such that $\varphi_0 =$ identity and $(\varphi_1)_*(\eta_0) = \eta_1$.

Example 1.1.8. Define $\varphi_t(x, y, z) = (x, \frac{y}{1+t}, z - \frac{txy}{1+t})$ for $0 \leq t \leq 1$. Then, $\{\varphi_t\}$ is an isotopy of $\mathbb{R}^3$, s.t., $\varphi_0 = id$ and $(\varphi_1)^*(\alpha_1) = \alpha_0$. So ξ_0 and ξ_1 are isotopic.

Define $\beta_t = \cos[(t-1)r]dz + r^2 \frac{\sin[(t-1)r]}{(t-1)r} d\theta$ for $1 < t \leq 2$. Then β_t is nowhere vanishing for $1 < t \leq 2$, $\beta_2 = \alpha_2$ and $\lim_{t\to 1} \beta_t = \alpha_1$. So ξ_1 and ξ_2 are homotopic.

We will explain later why ξ_2 is not isotopic to ξ_0 and ξ_1.

Since we are mainly interested in the isotopy classes of contact structures, we will sometimes call $(\mathbb{R}^3, \xi_1)$ the standard contact 3-space too.

Example 1.1.9. If we remove one point from S^3, then the remaining part is diffeomorphic to $\mathbb{R}^3$, and the restriction of ξ_{st} to that part is isotopic to ξ_0.

An important question about contact structures is "Does every closed oriented 3-manifold admit a contact structure?" Lutz [37] and Martinet [39] answered it affirmatively.

Theorem 1.1.10 ([37, 39]). *Every homotopy class of tangent plane distributions on a closed oriented 3-manifold contains a contact structure.*

1.2 Legendrian Knots

Definition 1.2.1. A smooth embedding of S^1 in a 3-manifold is called a knot. A Legendrian knot in a contact 3-manifold is a knot that is everywhere tangent to the contact structure.

Definition 1.2.2. Two Legendrian knots are called Legendrianly isotopic if they are isotopic through Legendrian knots.

The contact planes give a Legendrian knot a framing, called contact framing. This is a basic invariant of Legendrian knots under Legendrian isotopy. If a Legendrian knot L admits some sort of canonical framing $\mathcal{F}_r$, then its contact framing can be represented by its index relative to the canonical framing. We denote this index by $t(L, \mathcal{F}_r)$. The following is an important special case.

Definition 1.2.3. If a Legendrian knot L in a contact 3-manifold (M, ξ) is null-homologous, i.e., bounds a Seifert surface Σ, then the index of the contact framing along L relative to the framing given by Σ is called the Thurston-Bennequin number of L relative to Σ, and is denoted by $tb(L, \Sigma)$.

In this case, there is another numerical invariant $r(L, \Sigma)$ of (L, Σ) called the rotation number, which is defined to be the obstruction to the extension of the tangent vectors of L to a non-vanishing section of $\xi|_{\Sigma}$.

Remark 1.2.4. The Thurston-Bennequin number and rotation number of a null-homologous Legendrian knot L depend on the relative homology class of the chosen Seifert surface in $H_2(M, L)$. But, in the special situation where $H_1(M) = H_2(M) = 0$, all the Seifert surfaces are relatively homologous. And, hence, the Thurston-Bennequin number and rotation number depend only on L. In this case, we will denote them by $tb(L)$ and $r(L)$, respectively.

The most studied example of such contact manifolds is $(\mathbb{R}^3, \xi_0)$ (or, if you like, (S^3, ξ_{st})). The Thurston-Bennequin and rotation numbers of Legendrian knots in $(\mathbb{R}^3, \xi_0)$ can be calculated combinatorially from their projections onto certain planes.

Example 1.2.5. Legendrian projections – projections to the xy-plane
The projection of a Legendrian knot in $(\mathbb{R}^3, \xi_0)$ onto the xy-plane is an immersed curve, and is called the Legendrian projection of the Legendrian knot. A Legendrian knot is uniquely determined by its Legendrian projection up to translation parallel to the z-axis. By Stoke's theorem, the oriented region bounded of the Legendrian projection has area 0. After a slight Legendrian isotopy, we make the Legendrian projection into a regular knot diagram. At each crossing, one can determine which branch is on top by the fact $\triangle z = \int y dx$. Then the Thurston-Bennequin number equals the self-intersection number (or writhe) of the diagram, and the rotation number equals the degree of the Gauss map of the diagram.

Example 1.2.6. Front diagrams – projections to the xz-plane
The projection of a Legendrian knot in $(\mathbb{R}^3, \xi_0)$ onto the xz-plane is an immersed curve with cusps, and is called the front diagram of the Legendrian knot. A Legendrian knot is uniquely determined by its front diagram. Such front diagrams have no tangent lines parallel to the z-axis. At each crossing, the branch with less slope is on top. A cusp is positive if the diagram passes downward (i.e., in the $(-z)$-direction) near it, and is negative if otherwise. Then the Thurston-Bennequin number equals the self-intersection number of the diagram minus half of the total number of cusps, and the rotation number equals half of the number of positive cusps minus half of the number of negative cusps.

1.3 Tightness and Overtwistedness

In [1], Bennequin proved that any Legendrian knot L in $(\mathbb{R}^3, \xi_0)$ satisfies the following Bennequin inequality:

$$tb(L) + |r(L)| \leq -\chi(\Sigma), \tag{1.1}$$

where Σ is any Seifert surface of L. He also demonstrated that there are Legendrian unknots in $(\mathbb{R}^3, \xi_2)$ that do not satisfy this inequality. This implies ξ_0 and ξ_2 are not isotopic.

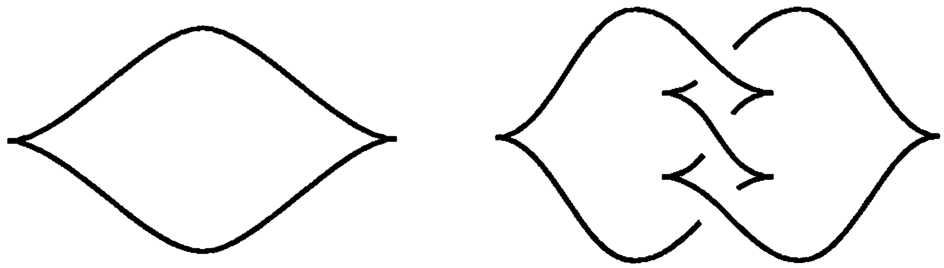


Figure 1-1: The front projections of a Legendrian unknot and a Legendrian left hand trefoil knot

Generalizing this idea, Eliashberg gave the following dichotomy of contact structures on 3-manifolds in [10].

Definition 1.3.1. A contact structure ξ on a 3-manifold M is called overtwisted if there exists an embedded 2-disk $D \subset M$ such that ∂D is Legendrian, but D is transverse to ξ along ∂D. Such a disk D is called an overtwisted disk. A contact structure is called tight if it is not overtwisted.

Eliashberg proved following fundamental results on 3-dimensional contact topology.

Theorem 1.3.2 ([6]). *Every homotopy class of tangent plane distributions on a closed oriented 3-manifold contains an overtwisted contact structure. Two homotopic overtwisted contact structures on a closed oriented 3-manifold are isotopic.*

Theorem 1.3.3 ([9, 10]). *Let (M, ξ) be a tight contact manifold.*

- *If Σ is a closed oriented surface embedded in M, then*

$$| < e(\xi), [\Sigma] > | \leq max\{-\chi(\Sigma), 0\}. \tag{1.2}$$

- *If L is a Legendrian knot in M with Seifert Surface Σ, then*

$$tb(L, \Sigma) + |r(L, \Sigma)| \leq -\chi(\Sigma). \tag{1.3}$$

The first part of Theorem 1.3.2 is just a re-interpretation of Theorem 1.1.10 since all the contact structures constructed in the proof of Theorem 1.1.10 are overtwisted. The second part of Theorem 1.3.2 shows that overtwisted contact structures are very "soft", i.e., isotopy classes of overtwisted contact structures are in one-to-one correspondence with homotopoy classes of tangent planes distributions. So the isotopy theory of these structures do not reveal more properties of the manifold than the homotopy theory.

On the other hand, Theorem 1.3.3 shows that tight contact structures demonstrate interesting rigidity. Thus, the theory of tight contact structures is the focus of 3-dimensional contact topology.

Since a covering map is a local diffeomorphism, any contact structure on the base manifold can be lifted to a contact structure on the covering manifold. Universal tightness and virtual overtwistedness are introduced to describe certain behaviors of tight contact structures under such liftings. It's not known whether every tight contact structure is either universally tight or virtually overtwisted.

Definition 1.3.4. Let ξ be a tight contact structure on a 3-manifold M. ξ is said to be universally tight if its lifting to the universal covering of M is tight. ξ is said to be virtually overtwisted if there exists a finite covering of M so that the lifting of ξ to it is overtwisted.

1.4 Convex Surfaces

The principal tool Eliashberg used to prove Theorem 1.3.3 is the characteristic foliation of an oriented surface embedded in a contact manifold.

Definition 1.4.1. Let (M, ξ) be a contact 3-manifold, and Σ an oriented surface embedded in M. Then the characteristic foliation Σ_ξ of Σ is the singular foliation of Σ generated by the singular tangent line distribution $T\Sigma \cap \xi|_\Sigma$.

To make things more precise, we pick an area form ω of Σ. Let α be a defining 1-form of ξ, and Y the section of $T\Sigma$ such that $i_Y\omega = \alpha|_\Sigma$. Then Σ_ξ is the singular foliation of Σ by the flow lines of Y. The singularities of Σ_ξ occur precisely at the points where $\alpha|_\Sigma = 0$, or, equivalently, $\xi = T\Sigma$. Clearly, if Σ_ξ has a closed leaf that bounds a disk, then ξ is overtwisted. Eliashberg proved Theorem 1.3.3 by showing that, if the inequalities are not true, then one can manipulate the surface to create a closed leaf bounding a disk in its characteristic foliation.

In [20], Giroux defined convex surfaces in contact manifolds and the dividing sets of convex surfaces. The dividing sets encode all the essential information of characteristic foliations, and are much easier to visualize and manipulate than the characteristic foliations.

Definition 1.4.2 ([20, 26, 33]). Let (M, ξ) be a contact 3-manifold, and Σ an embedded closed oriented surface or an embedded compact oriented surface with Legendrian boundary. Σ is called convex if there exists a contact vector field v that is transverse to Σ. Here a contact vector field means a vector field whose flow preserves the contact structure.

The dividing set of a convex surface Σ is the set $\Gamma_\Sigma = \{p \in \Sigma | v(p) \in \xi_p\}$.

Remark 1.4.3. The dividing set of a convex surface is a properly embedded multi-curve, i.e., a union of finitely many disjoint properly embedded 1-manifolds in Σ. We often call the dividing set the dividing curves, and a component of the dividing set a dividing curve. Proposition 1.4.4 below explains the meaning of the word "dividing", and gives dividing curves a canonical orientation. As an oriented multi-curve, the dividing set is independent of the contact vector field in the definition up to isotopy in Σ. Actually, when talking about the dividing curves, we are often referring to the isotopy class of the dividing curves.

Proposition 1.4.4 ([20]). *The dividing set divides the characteristic foliation in the following sense:*

(1) Γ_Σ is transverse to Σ_ξ.

(2) $\Sigma \backslash \Gamma_\Sigma = \Sigma_+ \sqcup \Sigma_-$, where Σ_+ (resp. Σ_-) is the set of points p where $L_Y(\omega) > 0$ (resp. $L_Y(\omega) < 0$).

From now on, we orient Γ_Σ as the boundary of Σ_+.

It is easy to determine the contact framing of a Legendrian knot contained in a convex surface.

Proposition 1.4.5 ([26, 33]). *Let L be a Legendrian knot contained in a convex surface Σ in a contact 3-manifold. Then $t(L, T\Sigma) = -\frac{1}{2}\#(L \cap \Gamma_\Sigma)$.*

In the special case when Σ is the Seifert surface of L, we have $tb(L, \Sigma) = -\frac{1}{2}\#(L \cap \Gamma_\Sigma)$, and $r(L, \Sigma) = \chi(\Sigma_+) - \chi(\Sigma_-)$.

The following theorem, known as Giroux's Criterion, gives a simple method to check whether a convex surface is contained in a tight contact manifold.

Theorem 1.4.6 (Giroux's Criterion [20, 26]). *Let Σ be a convex surface in a contact 3-manifold (M, ξ). If $\Sigma \neq S^2$, then Σ has a tight neighborhood if and only if Γ_Σ has no components that bound disks in Σ. If $\Sigma = S^2$, then Σ has a tight neighborhood if and only if $\Gamma_\Sigma \approx S^1$.*

Clearly, a C^∞-small perturbation of a convex surface is still convex. So the two propositions below show that convex surfaces are generic in some sense.

Proposition 1.4.7 ([20]). *A closed oriented embedded surface in a contact 3-manifold can be deformed by a C^∞-small isotopy into a convex surface.*

Proposition 1.4.8 ([26]). *Let (M, ξ) be a contact manifold, and Σ an embedded compact oriented surface with Legendrian boundary. Assume that $t(\gamma, T\Sigma) \leq 0$ for all components γ of $\partial\Sigma$. There exit a C^0-small perturbation near the boundary $\partial\Sigma$ that fixes $\partial\Sigma$ and puts an annulus neighborhood A of $\partial\Sigma$ into the standard form, and a subsequent C^∞-small perturbation of the perturbed surface that fixes an annulus neighborhood $A' \subset A$ of $\partial\Sigma$, and make the whole surface convex.*

Here, the standard form means the convex annulus depicted in Figure 1-2. And a convex surface whose Legendrian boundary has a neighborhood consists of convex annuli in standard form is said to have collared Legendrian boundary.

The following proposition gives a criterion to determine convexity from the characteristic foliation.

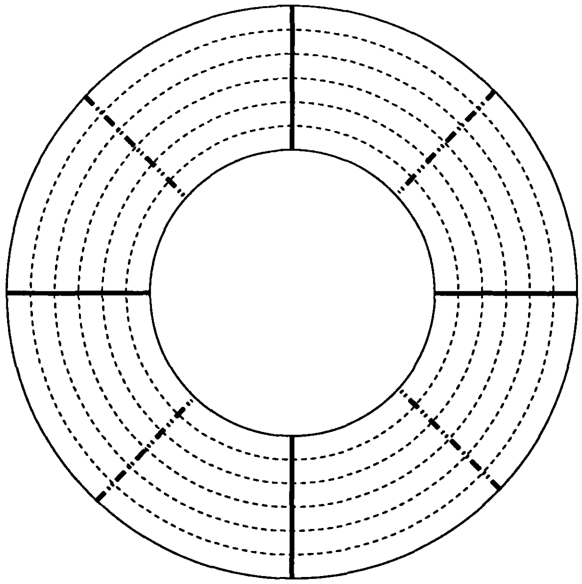


Figure 1-2: The standard form of convex annulus

Proposition 1.4.9 ([20, 33]). *Let (M, ξ) be a contact 3-manifold, and Σ an embedded closed oriented surface or an embedded compact oriented surface with Legendrian boundary. Then Σ is convex if and only if there exists an oriented multi-curve that divides its characteristic foliation in the sense of Proposition 1.4.4.*

The next theorem is known as Giroux's Flexibility Theorem which shows that all the essential information contained in the characteristic foliation is captured by the dividing set. Corollary 1.4.11 is a very useful technical result known as the Legendrian Realization Principle.

Theorem 1.4.10 (Giroux's Flexibility Theorem [20, 26]). *Let Σ be a convex surface in a contact 3-manifold (M, ξ) with characteristic foliation Σ_ξ, transversal contact vector field v and dividing set Γ_Σ. If $\mathcal{F}$ is another foliation on Σ divided by Γ_Σ in the sense of Proposition 1.4.4, then there exists an isotopy $\{\phi_s\}$ of Σ such that $\phi_0 = id$, $\phi_1(\Sigma)_\xi = \phi_1(\mathcal{F})$, ϕ_s fixes Γ_Σ, $\Gamma_{\phi_s(\Sigma)} = \Gamma_\Sigma$, and $\phi_s(\Sigma)$ is transverse to v for $0 \leq s \leq 1$.*

Corollary 1.4.11 (Legendrian Realization Principle [20, 26]). *Let Σ be a convex surface in a contact 3-manifold (M, ξ) with dividing set Γ_Σ. Assume C is a non-isolating multi-curve that intersects Γ_Σ transversally, where C being non-isolating means that each component of $\Sigma \setminus C$ intersects Γ_Σ nontrivially. Then there exists an isotopy $\{\phi_s\}$ of Σ such that $\phi_0 = id$, $\phi_1(C)$ is Legendrian, and ϕ_s fixes Γ_Σ, and $\Gamma_{\phi_s(\Sigma)} = \Gamma_\Sigma$ for $0 \leq s \leq 1$.*

The following discussion of convex torus is of particular importance for our study.

Example 1.4.12 (Convex tori in standard form). Let T^2 be a convex torus in a contact manifold (M, ξ). Suppose that the dividing set of T^2 consists of even number of disjoint closed simple curves that do not bound disks in T^2. (This is always true when (M, ξ) is tight.) After some identification of T^2 to $\mathbb{R}^2/\mathbb{Z}^2$, we may assume that the dividing curves are $2n$ parallel circles of slope $s \in \mathbb{Q} \cup \{\infty\}$. In each component of $T^2 \setminus \Gamma_{T^2}$, pick a embedded circle parallel to boundary. Denote by $\mathcal{L}$ the union of these circles. Let r be a rational number (including ∞) not equal to s, and $\widetilde{\mathcal{F}}$ the foliation of T^2 by embedded circles of slope r that intersect $\Gamma_{T^2} \cup \mathcal{L}$ efficiently. We modify $\widetilde{\mathcal{F}}$ into a singular foliation $\mathcal{F}$, s.t., the singular set of $\mathcal{F}$ is $\mathcal{L}$, leafs of $\mathcal{F}$ are components of leafs of $\widetilde{\mathcal{F}}$ with intersections with $\mathcal{L}$ removed, each leaf of $\mathcal{F}$ starts at a singularity in T^2_+ and end at a singularity in T^2_-. Then $\mathcal{F}$ is a singular foliation of T^2 divided by Γ_{T^2}. And we can isotope T^2 in a small neighborhood of it so that the dividing curves remain the same and the characteristic foliation of the perturbed torus is $\mathcal{F}$.

A convex torus with dividing curves and characteristic foliation of this form is said to be in standard form. A component of $\mathcal{L}$ is called a Legendrian divide, and a Legendrian circle in T^2 of slope r formed by a union of leafs of $\mathcal{F}$ is called a Legendrian ruling. In the special case that $n = 1$, i.e., T^2 has only two dividing curves, T^2 is said to be a minimal convex torus.

From the discussion above, we have:

Lemma 1.4.13. *Let ξ be a contact structure on $D^2 \times S^1$ with convex boundary. If the dividing curves of $\partial(D^2 \times S^1)$ are homotopic to the meridians, then ξ is overtwisted. In particular, after isotoping the boundary into a standard form, the Legendrian divides bound overtwisted disks.*

The following proposition shows how to merge two convex surfaces intersecting transversally along a common boundary Legendrian curve.

Proposition 1.4.14 (Edge-rounding, [26]). *Let Σ_1 and Σ_2 be two convex surfaces with collared Legendrian boundary intersecting transversally along a boundary Legendrian curve L. Then the points of $L \cap \Gamma_{\Sigma_1}$ and $L \cap \Gamma_{\Sigma_2}$ lie alternatingly along L. We can isotope Σ_1 and Σ_2 slightly near L to merge them into a new smooth convex surface Σ, and the dividing curves of Σ are obtained as following:*

Let $\Theta(< \pi)$ be the planer angle formed by Σ_1 and Σ_2 along L, and n_i be the unit normal vector of Σ_i pointing out of Θ. Walk along L (in either direction) with your head pointing to the $(n_1 + n_2)$-direction. Every time you meet a dividing curve coming from the surface on your right hand side, connect it to the next dividing curve you meet coming from the surface on your left hand side. Finally, smooth out the edge L to get Σ, and smooth the multi-curve created above by a slight isotopy. Then the smoothed multi-curve is isotopic to the dividing set of Σ.

Bypass adding is a special kind of isotopy of convex surfaces, and is invented by Honda [26] to manipulate the dividing curves of convex surfaces. We first give the definition of bypasses.

Definition 1.4.15 (Bypass [26]). Let Σ be a convex surface in a contact 3-manifold (M, ξ). A bypass for Σ is an oriented embedded convex half-disk D with Legendrian boundary, satisfying the following:

(1) ∂D is the union of two smooth Legendrian arcs γ_1, γ_2 intersecting transversally at their endpoints.

(2) D intersects Σ transversally along γ_1, and stands on the positive side of Σ.

(3) The singularities of D_ξ are following:

- positive elliptic points at the endpoints of γ_1 (= endpoints of γ_2),
- one negative elliptic point in the interior of γ_1,
- positive singularities along γ_2, alternating between elliptic and hyperbolic,
- there are no other singularities on D.

(4) γ_1 intersects Γ_Σ exactly at the three singularities of D on γ_1.

Remark 1.4.16. Although the definition of a bypass seems very strong, it's actually quite easy to find a bypass. Indeed, if Σ' is a convex surface intersecting Σ transversally along a Legendrian curve L, and has a dividing curve which co-bounds a disc in Σ' with L, then we can use Giroux's Flexibility Theorem to isotope Σ' to obtain a bypass attached to Σ or $-\Sigma$. From Proposition 1.4.5, it's easy to observe the following Imbalance Principle, which gives a common scheme to construct bypasses.

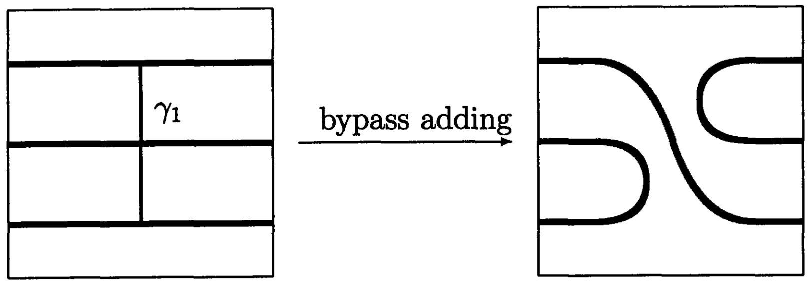
Proposition 1.4.17 (Imbalance Principle [26]). *Let A be a convex annulus with Legendrian boundary, and L_1, L_2 the two boundary components of A. If $t(L_1, TA) < t(L_2, TA) \leq 0$, then there exists a component of Γ_A that co-bounds a disk in A with L_1. Such a component is called a ∂-parallel dividing curve of A on the L_1 side.*

Now we describe the bypass adding procedure.

Let Σ and D be as in the definition. Since both of them are convex, we can find small oriented thickenings $\Sigma \times I$ and $D \times I$ of them such that ξ is I-invariant in these thickenings. We can assume that $(\Sigma \times I)$ intersects $(D \times I)$ only in a small tubular neighborhood of γ_1, and $(D \times I) \backslash (\Sigma \times I)$ is connected. The subset $U = (\Sigma \times I) \cup (D \times I)$ of M has two boundary components. One is $\Sigma \times \{-1\}$, which is contact isotopic to Σ. The other is a convex surface with edges that isotopic (but not contact isotopic) to Σ. Denote by Σ' this surface after edge-rounding. We say that Σ' is obtained from Σ by adding the bypass D. The following proposition illustrates how dividing curves change after a bypass adding.

Proposition 1.4.18 (Bypass adding [26]). *Let the bypass D be attached to Σ in the neighborhood depicted in (a) of Figure 1-3. Then the dividing set of Σ' looks like (b) of Figure 1-3 in the corresponding neighborhood, and is identical to that of Σ outside this neighborhood.*

For the purpose of our study, it's important to know how a bypass adding affects the dividing set of a convex torus. Before we can state the result, we need first



(a) Γ_Σ (b) $\Gamma_{\Sigma'}$

Figure 1-3: Bypass adding

introduce the Farey tessellation of the hyperbolic unit disk $\mathbb{H}^2 = \{(x,y)|x^2+y^2 \leq 1\}$. We start by labelling $(1,0)$ as $0 = \frac{0}{1}$, and $(-1,0)$ as $\infty = \frac{1}{0}$. Next we inductively label points on the upper half of $S^1 = \partial\mathbb{H}^2$ with positive rational numbers as following: Suppose we have already labelled $0 \leq \frac{p}{q} < \frac{p'}{q'} \leq \infty$, where (p,q) and (p',q') are pairs of relatively prime integers such that $pq' - p'q = \pm 1$, then label the midpoint of the arc $[\frac{p}{q}, \frac{p'}{q'}]$ in the upper half of S^1 as $\frac{p+p'}{q+q'}$. The points on the lower half of S^1 are then labelled as following: If a point in the upper half of S^1 is labelled by $\frac{p}{q} > 0$, then we label its mirror image across the x-axis as $-\frac{p}{q} > 0$. Now let $\frac{p}{q}$ and $\frac{p'}{q'}$ be any two labelled points on S^1, where (p,q) and (p',q') are pairs of relatively prime integers. We connect $\frac{p}{q}$ and $\frac{p'}{q'}$ by a hyperbolic geodesic inside $\mathbb{H}^2$ if $pq' - p'q = \pm 1$. The interiors of these hyperbolic geodesics are disjoint. These hyperbolic geodesics divide $\mathbb{H}^2$ into infinitely many triangles. And this is called the Farey tessellation.

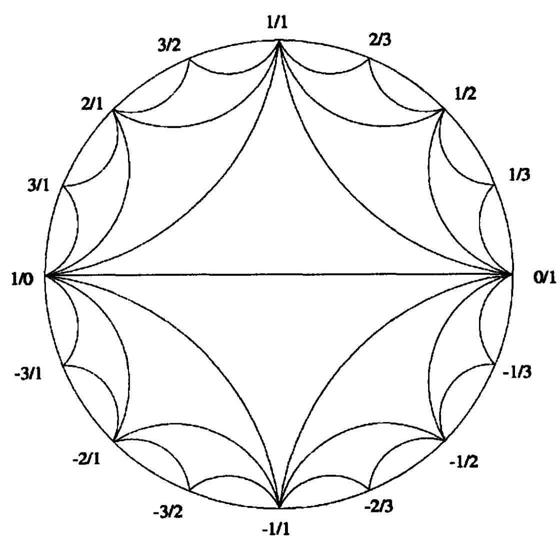


Figure 1-4: The Farey tessellation

Proposition 1.4.19 ([26]). *Let T^2 be a standard convex torus with $2n$ dividing curves embedded in a contact manifold (M,ξ). Identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$, and denote the slopes of the dividing curves and Legendrian rulings of T^2 by s and r, respectively. Assume a bypass is attached to T^2 along a Legendrian ruling. After the bypass adding:*

(1) If $n > 1$, then the new convex torus has exactly $2n-2$ dividing curves of slope s.

(2) If $n = 1$, then the new convex torus has exactly 2 dividing curves of slope s', where s' is the point closest to r on the counterclockwise arc in S^1 starting from r ending in s that is connected to s by a hyperbolic geodesic in the Farey tessellation.

1.5 Fillability and Legendrian Surgeries

We first introduce three notions of fillability for 3-dimensional contact manifolds.

Definition 1.5.1. Let (M,ξ) be a contact 3-manifold.

(1) (M,ξ) is said to be weakly fillable if there exists a symplectic manifold (W,ω) with $\partial W = M$ and $\omega|_\xi > 0$. Such a (W,ω) is called a weak filling of (M,ξ).

(2) (M,ξ) is said to be strongly fillable if there exists a symplectic manifold (W,ω) with $\partial W = M$ such that ω is exact near M, and there exists a primitive α of ω near M such that $\alpha|_M$ defines ξ and $d\alpha|_\xi > 0$. Such a (W,ω) is called a strong filling of (M,ξ).

(3)(M,ξ) is said to be holomorphically fillable if there exists a Stein surface W which has M as its strictly pseudo-convex boundary and such that ξ is the field of complex tangencies to M. Such a W is called a holomorphic filling of (M,ξ).

Remark 1.5.2. Clearly, a holomorphic filling is also a strong filling, and a strong filling is also a weak filling. So, holomorphically fillable $\Rightarrow$ strongly fillable $\Rightarrow$ weakly fillable.

Example 1.5.3. The standard contact 3-sphere (S^3,ξ_{st}) is holomorphically filled by the unit 4-ball with standard complex structure.

Theorem 1.5.4 (Gromov, Eliashberg). *Weakly fillable contact structures are tight.*

Indeed, although examples of non-fillable tight contact structures have been constructed, fillable contact structures are still the main sources of tight contact structures. The reason we like fillable contact structures is because they behave well under certain surgeries of 3-manifolds.

Theorem 1.5.5 ([7, 16, 43]). *Let L be a Legendrian knot in a weakly (resp. strongly, holomorphically) fillable contact 3-manifold (M,ξ), and V a small tubular neighborhood of L. Identify ∂V with $\mathbb{R}^2/\mathbb{Z}^2$ so that the meridian of V corresponds to $(1,0)^T$ and the contact framing of L corresponds to $(0,1)^T$. Define $\varphi : \partial V \to \partial V$ by*

$$\varphi = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Let $\widetilde{M} = (M \setminus V) \cup_\varphi V$. *Then, up to isotopy, there is only one way to extend* $\xi|_{M\setminus V}$ *to a contact structure* $\widetilde{\xi}$ *on* $\widetilde{M}$, *and this contact structure* $\widetilde{\xi}$ *is weakly (resp. strongly, holomorphically) fillable.*

The above procedure is called a Legendrian surgery along L.

Theorem 1.5.5 shows that we can construct new fillable contact 3-manifolds from known ones. In [22], Gompf did an extensive study of construction of holomorphically fillable structures from (S^3, ξ_{st}). We are particularly interested in these constructions.

Let L be a Legendrian link in (S^3, ξ_{st}) consists of components L_1,..., L_m, and (M, ξ) the holomorphically fillable contact 3-manifold obtained by performing Legendrian surgery along L, i.e., by performing Legendrian surgery along every component of L. Clearly, the surgery coefficient of each component L_i is $tb(L_i) - 1$. So the topological knot type of L and the Thurston-Bennequin numbers of all the components determine the ambient manifold M. The following two theorems show that, when the topological knot type and the Thurston-Bennequin numbers of all the components are fixed, the different choices of the rotation numbers of the components lead to non-isotopic holomorphically fillable contact structures on the same ambient manifold.

Proposition 1.5.6 ([22]). *Let* L, (M, ξ) *be as above,* X *the* 4*-dimensional handle body obtain from* B^4 *by adding the* 2*-handles corresponding to the Legendrian surgery along* L, *and* J *the Stein structure on* X *induced from the handle adding. Then* (X, J) *is a holomorphic filling of* (M, ξ), *and the Chern class* $c_1(J) \in H^2(X)$ *is represented by a cocycle whose value on the two handle corresponding to* L_i *is* $r(L_i)$.

Theorem 1.5.7 ([34, 35, 41]). *Let* X *be a smooth* 4*-manifold with boundary, equipped with two Stein structures* J_1, J_2 *with associated Spin*^{c} *structures* $\mathfrak{s}_1$, $\mathfrak{s}_2$, *and let* ξ_1, ξ_2 *be the induced contact structures on* $M = \partial X$. *If the Spin*^{c} *structures* $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *are not isomorphic, then the two contact structures* ξ_1 *and* ξ_2 *are non-isotopic. More precisely, they have distinct Ozsváth-Szabó invariants.*

1.6 Tight Contact Structures on Basic Building Blocks

The first results on classification of tight contact structures are given by Eliashberg in [10], in which he proved the uniqueness of tight contact structures on S^3 and B^3 up to isotopy. The precise statements of these results are below.

Theorem 1.6.1 ([10]). *(1) Any tight contact structure on* S^3 *is isotopic to* ξ_{st} *defined in Example 1.1.5.*

(2) Two tight contact structures on B^3 *that coincide in a neighborhood of* ∂B^3 *are isotopic relative to* ∂B^3.

Remark 1.6.2. Since $\mathbb{R}^3$ is S^3 with one point removed, part (1) of Theorem 1.6.1 implies that any tight contact structure on $\mathbb{R}^3$ is isotopic to ξ_0 defined in Example 1.1.2. See [11] for a detailed proof of this.

In [26], Honda classified tight contact structures on solid torus and thickened torus, and, in [27], he classified tight contact structures on pair-of-pants times S^1 that satisfy certain boundary conditions. These will be the basic building blocks in our effort to understand tight contact structures on small Seifert spaces. In the rest of this section, we introduce the part of his results that is relevant to our study.

Before stating these results, we need introduce our convention of the continued fractions.

Notation 1.6.3. Let s be a rational number. Then there is a unique way to express s as

$$s = r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cdots \cfrac{1}{r_{k-1} - \cfrac{1}{r_k}}}}, \tag{1.4}$$

where r_i is an integer, and is ≤ -2 when $i \geq 1$. We denote by $< r_0, r_1, \cdots, r_k >$ the expression on the right hand side of equation (1.4).

Theorem 1.6.4 ([26]). *Identify the boundary T^2 of $D^2 \times S^1$ with $\mathbb{R}^2/\mathbb{Z}^2$ so that the meridian is identified with $(1,0)^T$. Consider the tight contact structures on $D^2 \times S^1$ with minimal convex boundary (see Example 1.4.12), for which the slope of the dividing curves is $s \leq -1$. Up to isotopy fixing T^2, there exist exactly $|(r_0+1)(r_1+1)\cdots(r_{k-1}+1)r_k|$ tight contact structures on $D^2 \times S^1$ with the given boundary condition. Here, $r_0, \cdots,$ r_k are the coefficients in the continued fraction expansion $s =< r_0, r_1, \cdots, r_k >$.*

To state Honda's results on thickened torus, we need define minimal twisting tight contact structures on thickened torus.

Definition 1.6.5. Identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$. Let ξ be a tight contact structure on $T^2 \times I$ so that $T^2 \times \{0\}$ and $T^2 \times \{1\}$ are both convex, and the slopes of their dividing curves are s_0 and s_1, respectively. ξ is called minimal twisting if the slope of the diving curves of any convex torus in $T^2 \times I$ isotopic to $T^2 \times \{0\}$ lies on the counterclockwise arc in $\partial\mathbb{H}^2$ from s_1 to s_0 in the Farey tessellation.

Theorem 1.6.6 ([26]). *Identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$. Consider the minimal twisting tight contact structures on $T^2 \times I$ such that $T^2 \times \{0\}$ and $T^2 \times \{1\}$ are both minimal convex tori (see Example 1.4.12), and the slopes of their diving curves are -1 and $s < -1$, respectively. Up to isotopy fixing the boundary, there exist exactly $|(r_0+1)(r_1+1)\cdots(r_{k-1}+1)r_k|$ such tight contact structures on $T^2 \times I$. Here, $r_0, \cdots,$ r_k are the coefficients in the continued fraction expansion $s =< r_0, r_1, \cdots, r_k >$.*

Let Σ be a pair-of-pants, and $-\partial\Sigma \times S^1 = T_1 + T_2 + T_3$, where the "$-$" sign means reversing the orientation. We identify T_i to $\mathbb{R}^2/\mathbb{Z}^2$ by identifying the corresponding component of $-\partial\Sigma \times \{\text{pt}\}$ to $(1,0)^T$, and $\{\text{pt}\} \times S^1$ to $(0,1)^T$. An embedded circle in $\Sigma \times S^1$ is said to be vertical if it's isotopic to $\{\text{pt}\} \times S^1$. Such a circle admits a native framing $\mathcal{F}$ from the product structure.

Then we have following results.

Proposition 1.6.7 ([27]). *Isotopy classes relative to boundary of tight contact structures on $\Sigma \times S^1$ such that all three boundary components of $\Sigma \times S^1$ are minimal convex with vertical dividing curves are in 1-1 correspondence with isotopy classes of embedded multi-curves on Σ with 2 fixed end points on each component of $\partial\Sigma$ that have no homotopically trivial components.*

The correspondence here is given by mapping mapping a tight contact structure to the dividing set of a dividing set minimizing convex surface isotopic to $\Sigma \times \{pt\}$.

Proposition 1.6.8 ([27]). *Consider the tight contact structures on $\Sigma \times S^1$ such that T_1, T_2 and T_3 are all minimal convex tori, and the slopes of dividing curves of T_1, T_2, T_3 are $s_1, s_2, s_3 \in \mathbb{Z}$, respectively. Then we have following:*

1. *A tight contact structure with a vertical Legendrian circle L that has $t(L, \mathcal{F}) = 0$ admits a factorization $\Sigma \times S^1 = L_1 \cup L_2 \cup L_3 \cup (\Sigma' \times S^1)$, where L_1, L_2, L_3 are disjoint thickened tori with minimal twisting and convex minimal boundary $\partial L_i = T_i' - T_i$, and all the components of $-\partial\Sigma' \times S^1 = T_1' + T_2' + T_3'$ have vertical dividing curves.*

2. *A universally tight contact structure with a vertical Legendrian circle L with $t(L, \mathcal{F}) = 0$ admits a unique extension to $\Sigma'' \times S^1$ obtained by gluing tight contact thickened tori L_1'', L_2'', L_3'' with minimal twisting and minimal boundary to $\Sigma \times S^1$ along T_1, T_2, T_3, so that all the components of $-\partial\Sigma'' \times S^1 = T_1'' + T_2'' + T_3''$ have vertical dividing curves. Two such universally tight contact structures on $\Sigma \times S^1$ are isotopic relative to boundary if and only if the minimal configurations of dividing curves on properly embedded convex surfaces in $\Sigma'' \times S^1$ properly isotopic to $\Sigma'' \times \{pt\}$ are the same for these two contact structures.*

3. *If $s_1 + s_2 + s_3 \leq -2$, then:*
   (i) Any tight contact structure satisfying the boundary condition admits a vertical Legendrian circle L with $t(L, \mathcal{F}) = 0$.
   (ii) Up to isotopy relative to boundary, the number of virtually overtwisted contact structures on $\Sigma \times S^1$ satisfying the boundary condition is 2 if $s_1 + s_2 + s_3 < -3$, 1 if $s_1 + s_2 + s_3 = -3$, and 0 if $s_1 + s_2 + s_3 = -2$.

4. *If $s_1 + s_2 + s_3 > -2$, then there are exactly $2 + s_1 + s_2 + s_3$ tight contact structures on $\Sigma \times S^1$ that satisfy the boundary condition and admit no vertical Legendrian circles L with $t(L, \mathcal{F}) = 0$.*

Remark 1.6.9. In the theorems above, we only specified the dividing curves, but did no specify the characteristic foliation of the boundary. This is because that, if two characteristic foliations of the boundary are both divided by the same dividing curves, then there is a 1-1 correspondence between the relative isotopy classes of tight contact structures inducing these foliations on boundary. So, when applying these theorems, one can assume the characteristic foliation of the boundary to be any singular foliation divided by the dividing curves specified.

Lemma 1.6.10. *Let L be a Legendrian knot in a tight contact 3-manifold with a given framing $\mathcal{F}$, and $t(L,\mathcal{F}) = n \leq 0$. Then there is a tubular neighborhood U of L with minimal convex boundary ∂U whose dividing curves have slope $\frac{1}{n}$ if we identify ∂U with $\mathbb{R}^2/\mathbb{Z}^2$ such that the meridian corresponds to $(1,0)^T$ and $\mathcal{F}$ corresponds to $(0,1)^T$. Such a neighborhood of L is called a standard neighborhood of L.*

The following is a special case of Theorem 1.6.4, which implies that standard neighborhood of a Legendrian knot is unique up to isotopy.

Corollary 1.6.11. *Let ξ be a tight contact structure on $D^2 \times S^1$ such that $T^2 = \partial(D^2 \times S^1)$ is minimal convex with dividing curves of slope $\frac{1}{n}$, where $n \in \mathbb{Z}^{\leq 0}$, and T^2 is identified to $\mathbb{R}^2/\mathbb{Z}^2$ such that the meridian corresponds to $(1,0)^T$ and $\{pt\} \times S^1$ corresponds to $(0,1)^T$. Then there exists a Legendrian knot in $D^2 \times S^1$ isotopic to $\{pt\} \times S^1$ that has twisting number n with respect to the product framing. Any two such contact structures are isotopic relative to boundary.*

The simplest tight contact thickened tori are called basic slices. These are the basic building blocks in the study of tight contact structures on thickened torus.

Definition 1.6.12. Let $(T^2 \times I, \xi)$ be a tight contact thickened torus with minimal twisting and minimal convex boundary. Identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$. Denote by s_0 and s_1 the slopes of dividing curves of $T^2 \times \{0\}$ and $T^2 \times \{1\}$. If s_0 and s_1 are connected by a hyperbolic geodesic in the Farey tessellation, then $(T^2 \times I, \xi)$ is called a basic slice.

As a special case of Theorem 1.6.6, we have:

Proposition 1.6.13. *Identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$. Let s_0, s_1 be two rational numbers connected by a hyperbolic geodesic in the Farey tessellation, and v_0, v_1 shortest integer vectors of slopes s_0, s_1 forming an oriented $\mathbb{Z}$-basis for $\mathbb{Z}^2$. Then there are exactly two basic slices such that the slope of dividing curves of $T^2 \times \{0\}$ and $T^2 \times \{1\}$ are s_0 and s_1. These two basic slices are distinguished by their relative Euler classes, whose Poincaré duals are represented by the two circles in $T^2 \times I$ corresponding to $\pm(v_1 - v_0)$.*

Remark 1.6.14. We call the basic slice with Euler class dual to $(v_1 - v_0)$ positive, and the other one negative. This convention depends on the choice of v_1. We will specify our choices of this vector when we use this convention.

With standard neighborhoods and basic slices defined, we now introduce a procedure, called "stabilization", that reduces the twisting number of a Legendrian knot by 1.

Proposition 1.6.15. *Let L be a Legendrian knot in a tight contact manifold with a given framing $\mathcal{F}$, twisting number $t(L,\mathcal{F}) = n \leq 0$, and a standard neighborhood U. Identify ∂U with $\mathbb{R}^2/\mathbb{Z}^2$ as in Lemma 1.6.10. There exist two decompositions of U:*

$$U = U_+ \cup (T^2_+ \times I)$$

and
$$U = U_- \cup (T^2_- \times I),$$
where $U_\pm$ is a standard neighborhood of a Legendrian knot $L_\pm$ isotopic to L with twisting number $t(L_\pm, \mathcal{F}) = n-1$, and $T^2_\pm \times I$ is a $\pm$ basic slice with $T^2_\pm \times \{0\} = \partial U_\pm$ and $T^2_\pm \times \{1\} = \partial U$. Here, the sign convention is given by $v_1 = (n,1)^T$.

The Legendrian knot $L_\pm$ is called a $(\pm)$-stabilization of L.

By Proposition 1.4.19, we know that adding a bypass along some Legendrian arc to the boundary of a standard neighborhood acts as the inverse of a stabilization.

Corollary 1.6.16. *Let L be a Legendrian knot in a tight contact manifold with a given framing $\mathcal{F}$, twisting number $t(L,\mathcal{F}) = n < 0$, and a standard neighborhood U. Identify ∂U with $\mathbb{R}^2/\mathbb{Z}^2$ as in Lemma 1.6.10. Assume there is a bypass D attached to ∂U along a Legendrian ruling of slope r, where $\frac{1}{r} \geq n+1$. Then there exists a Legendrian knot $\widetilde{L}$ isotopic to L with twisting number $t(\widetilde{L},\mathcal{F}) = n+1$ contained in a neighborhood of $U \cup D$.*

Any tight contact thickened torus with minimal convex boundary can be factorized into basic slices. Sometimes, we will have the freedom to shuffle the signs of several adjacent basic slices without affecting the contact structure. Such a block is called a continued fraction block. And, as suggested by the name, a maximal continued fraction block corresponds to a coefficient in the continued fraction expansion.

Definition 1.6.17. Let $(T^2 \times I, \xi)$ be a tight contact thickened torus with minimal twisting and minimal convex boundary. Identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$. Denote by s_0 and s_1 the slopes of dividing curves of $T^2 \times \{0\}$ and $T^2 \times \{1\}$. If there exists a rational number s that is connected to both s_0 and s_1 by hyperbolic geodesics in the Farey tessellation, then $(T^2 \times I, \xi)$ is called a continued fraction block.

Proposition 1.6.18. *Let $(T^2 \times I, \xi)$ be a continued fraction block. Identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$. Let s, s_0 and s_1 be defined as in Definition 1.6.17. Then s, s_0 and s_1 uniquely determine $m \in \mathbb{Z}^{>0}$ and $s_{\frac{m-1}{m}}, \cdots, s_{\frac{1}{m}} \in \mathbb{Q}$ that lie in a counterclockwise order on the counterclockwise open arc in $\partial \mathbb{H}^2$ from s_1 to s_0, so that $s_{\frac{i}{m}}$ is connected to $s_{\frac{i+1}{m}}$ and s by hyperbolic geodesics in the Farey tessellation. And we can factorize $T^2 \times I$ into $T^2 \times I = (T^2 \times [0, \frac{1}{m}]) \cup \cdots \cup (T^2 \times [\frac{m-1}{m}, 1])$, where each $T^2 \times [\frac{i}{m}, \frac{i+1}{m}]$ is a basic slice so that the slope of dividing curves of $T^2 \times \{\frac{i}{m}\}$ is $s_{\frac{i}{m}}$.*

Pick a shortest integer vector of slopes s_1. This fixes a sign convention for these basic slices. Two continued fraction blocks with the same boundary condition are isotopic relative to boundary if and only if the numbers of positive slices in these two continued fraction blocks are equal.

Remark 1.6.19. Specially, the proposition means that we can shuffle the signs of basic slices within a continued fraction block without changing the tight contact structure.

The following proposition from [26] gives the existence of convex tori with certain dividing sets in a tight contact thickened torus, which provide us with a convenient technical tool.

Proposition 1.6.20. *Let $(T^2 \times I, \xi)$ be tight contact with convex boundary. Identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$. Denote by s_0 and s_1 the slopes of dividing curves of $T^2 \times \{0\}$ and $T^2 \times \{1\}$. Given any s that lies on the counterclockwise arc in $\partial\mathbb{H}^2$ from s_1 to s_0, there is a minimal convex torus T in $T^2 \times I$ isotopic to $T^2 \times \{0\}$ whose dividing curves have slope s.*

Specially, this proposition implies:

Corollary 1.6.21. *Let $(T^2 \times I, \xi)$ be a non-minimal twisting tight contact thickened torus with convex boundary. Identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$. Then given any $s \in \mathbb{Q}$, there is a minimal convex torus T in $T^2 \times I$ isotopic to $T^2 \times \{0\}$ whose dividing curves has slope s.*

Combining Propositions 1.6.15 and 1.6.20, we have:

Corollary 1.6.22. *Let ξ be a tight contact structure on $D^2 \times S^1$ such that $\partial(D^2 \times S^1)$ is minimal convex with dividing curves of slope s. Here, $\partial(D^2 \times S^1)$ is identified with $\mathbb{R}^2/\mathbb{Z}^2$ as in Corollary 1.6.11. Then Given any s' that lies on the counterclockwise open arc in $\partial\mathbb{H}^2$ from s to 0, there is a minimal convex torus T in $D^2 \times S^1$ isotopic to $\partial(D^2 \times S^1)$ whose dividing curves have slope s'.*

We introduce an effective procedure to verify if several tight contact thickened tori can be glued together to from a new tight contact thickened torus.

Lemma 1.6.23. *Let ξ be a contact structure on $T^2 \times [0,2]$ such that each $N_0 = T^2 \times [0,1]$ and $N_1 = T^2 \times [1,2]$ are both basic slices. Identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$. Let s_i be the slope of dividing curves of $T^2 \times \{i\}$. Assume s_2, s_1, s_0 are the three vertices of a triangle in the Farey tessellation placed in a counterclockwise order. Pick a shortest vector v_2 of slope s_2. This gives a sign convention for these two slices. Then ξ is tight if and only if N_0 and N_1 have the same sign. In this case, we call N_0 and N_1 compatible.*

Proposition 1.6.24 ([26]). *Let ξ be a contact structure on $T^2 \times [0,n]$ such that each $N_i = T^2 \times [i, i+1]$ is a basic slice. Identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$. Let s_i be the slope of dividing curves of $T^2 \times \{i\}$. Assume $s_n, s_{n-1}, \cdots, s_1, s_0$ lie on the counterclockwise arc in $\partial\mathbb{H}^2$ from s_n to s_0 in a counterclockwise order. Clearly, s_i is connected to s_{i+1} by a hyperbolic geodesic in the Farey tessellation. Then ξ is tight if and only if, after we merge all compatible adjacent pairs of basic slices, and delete the corresponding s_i from the sequence, the remaining sequence is the shortest counterclockwise sequence from s_n to s_0 such that each element is connected to the next by a hyperbolic geodesic in the Farey tessellation.*

1.7 Tight Contact Structures on More General Manifolds

As we said before, tight contact structures demonstrate interesting rigidity. For example, the first part of Theorem 1.3.3 implies that, for a closed 3-manifold M, only finitely many elements of $H^2(M)$ can be represented as the Euler classes of tight contact structures. In [4, 5], Colin, Giroux and Honda strengthened this finiteness result.

Theorem 1.7.1 ([4, 5]). *Let M be a closed oriented 3-manifold.*

- *Only finitely many homotopy classes of tangent plane distributions on M carry tight contact structures.*
- *The number of isotopy classes of tight contact structures is finite if M is atoroidal.*

Also, in [30], Honda, Kazez and Matić proved:

Theorem 1.7.2. *On a toroidal 3-manifold, there are infinitely many tight contact structures up to isomorphism.*

Clearly, their works gave complete answer to the finiteness problem about tight contact structures on closed 3-manifolds (up to homotopy, isomorphism or isotopy).

Another fundamental problem is the existence of tight contact structures. We do not have complete understanding of this problem yet. But, by observing the close relation between tight contact structures and taut foliations, there are several good results in this area derived from similar results in foliation theory.

In [13], Eliashberg and Thurston proved:

Theorem 1.7.3. *Contact structures that are C^0-close to a taut foliation are weakly fillable and universally tight.*

Remark 1.7.4. In their original book, they only proved that these contact structures are weakly semi-fillable. But, from [12], we now know that semi-fillability is equivalent to fillability.

Then, by Gabai's result on the existence of taut foliations in [17], we have the following theorem, which is also proved directly in [29] by Honda, Kazez and Matić using convex decompositions.

Theorem 1.7.5. *Let M be a closed, oriented, connected, irreducible 3-manifold with $H_2(M) \neq 0$. Then M carries a universally tight contact structure.*

In [30], Honda, Kazez and Matić also used convex decompositions to prove a similar result on 3-manifolds with boundary.

Theorem 1.7.6. *Let (M, γ) be an oriented, compact, connected, irreducible, sutured 3-manifold which has nonempty boundary, is taut, and has annular sutures. Then (M, γ) carries a universally tight contact structure. In particular, any oriented, compact, connected, irreducible 3-manifold with nonempty boundary admits a universally tight contact structure.*

Remark 1.7.7. In the existence theorems above, we only discussed irreducible 3-manifolds. But there is a 1-1 correspondence between the isotopy classes of tight contact structures on a connected sum and the tuples of isotopy classes of tight contact structures on its summands (See, e.g., [3, 10, 38]). So the theorems above also give some sufficient conditions to the existence of tight contact structures on reducible 3-manifolds.

With the above theorems in hand, we can start to look for closed oriented 3-manifolds that admit no tight contact structures. For example, from Theorems 1.7.2 and 1.7.5, we have:

Corollary 1.7.8. *If a Seifert fibred manifold admits no tight contact structures, then it is a small Brieskorn homology sphere, i.e., a homology sphere that's Seifert fibred over S^2 with 3 singular fibers.*

Remark 1.7.9. In [15], Etnyre and Honda found the first example of such manifolds. They showed that the Poincaré homology sphere with the reversed orientation admits no tight contact structures. Later, in [36], Lisca and Stipsicz generalized their result and found an infinite family of small Brieskorn homology spheres with certain orientations that admit no tight contact structures. From Remark 1.7.7, one can see that, if an oriented 3-manifold M does not admit tight contact structures, then $M\#(-M)$ does not admit tight contact structures in either orientation, where $-M$ means M with reversed orientation.

Up to now, classification of tight contact structures remains elusive. It is only known for limited classes of 3-manifolds. Except for the examples listed in section 1.6, we also know classification of tight contact structures on lens spaces (Honda, [26]), T^3 (Kanda, [32]), more generally, T^2-bundles over S^1 and S^1-bundles over closed surfaces (Honda, [27]), Seifert fibred spaces over T^2 with one singular fiber (Ghiggini, [18]), several small Seifert spaces, and a few compact manifolds with boundary and certain boundary conditions. We also know the classification of tight contact structures with maximal Euler classes on hyperbolic 3-manifolds fibred over S^1 with pseudo-Anosov holonomy (Honda, Kazez and Matić, [31]).

Since we have complete classification of tight contact structures on thickened torus and solid torus, and also have some understanding of pair-of-pants times S^1. It is easier to work with Seifert fibred manifolds than hyperbolic manifolds. But still, even if the genus or the number of singular fibers increases only by 1, the difficulty to achieve classification increases tremendously.

In the rest of this thesis, we will study the contact framing of Legendrian knots in tight contact small Seifert spaces isotopic to regular fibers. And, then, apply the results we get to give a complete classification of tight contact structures on small Seifert spaces with Euler number $e_0 \neq 0, -1, -2$.

Chapter 2

The Contact Framing of Legendrian Vertical Circles in Small Seifert Spaces

2.1 Introduction and Statement of Results

A small Seifert space is a 3-manifold Seifert fibred over S^2 with 3 singular fibers. Any regular fiber f in a small Seifert space $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ admits a canonical framing given by pulling back an arc in the base S^2 containing the projection of f. An embedded circle in $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ is said to be vertical if it is isotopic to a regular fiber. Any vertical circle inherits a canonical framing from the canonical framing of regular fibers. We call this framing $\mathcal{F}r$.

Definition 2.1.1. Let ξ be a contact structure on a small Seifert space $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, and L a Legendrian vertical circle in (M, ξ). Define the twisting number $t(L)$ of L to be $t(L) = t(L, \mathcal{F}r)$.

In [5], Colin, Giroux and Honda divided the tight contact structures on a small Seifert space into two types: those for which there exists a Legendrian vertical circle with twisting number 0, and those for which no Legendrian vertical circles with twisting number 0 exist. It is proven in [5] that, up to isotopy, the number of tight contact structures of the first type is always finite, and, unless the small Seifert space is also a torus bundle, the number of tight contact structures of the second type is finite too. Their work gives in principle a method to estimate roughly the upper bound of the number of tight contact structures on a small Seifert space. In this chapter, we demonstrate that most small Seifert spaces admit only one of the two types of tight contact structures. To make our claim precise, we need the Euler number. (See, e.g., [22].)

Definition 2.1.2. For a small Seifert space $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, define the Euler number of M to be $e_0(M) = \lfloor \frac{q_1}{p_1} \rfloor + \lfloor \frac{q_2}{p_2} \rfloor + \lfloor \frac{q_3}{p_3} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer not greater than x.

Clearly, $e_0(M)$ is an invariant of M, i.e., it does not depend on the choice of the representatives $(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$. Now we can formulate our claim precisely in the following two theorems.

Theorem 2.1.3. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space. If $e_0(M) \geq 0$, then every tight contact structure on M admits a Legendrian vertical circle with twisting number 0.*

Theorem 2.1.4. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space. If $e_0(M) \leq -2$, then no tight contact structures on M admit Legendrian vertical circles with twisting number 0.*

Remark 2.1.5. In particular, Theorem 2.1.4 means that, for any small Seifert space $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, either M or $-M$ does not admit tight contact structures for which there exists a Legendrian vertical circle with twisting number 0, where $-M$ is M with reversed orientation. This is because that $e_0(M) + e_0(-M) = -3$, and, hence, one of $e_0(M)$ and $e_0(-M)$ has to be less than or equal to -2.

It turns out that the case when $e_0(M) = -1$ is the most difficult. Only very weak partial results are known. For example, in [19], Ghiggini and Schönenberger proved that, when $r \leq \frac{1}{5}$, no tight contact structures on the small Seifert space $M(r, \frac{1}{3}, -\frac{1}{2})$ admit Legendrian vertical circles with twisting number 0.

We have following results about the case $e_0(M) = -1$.

Theorem 2.1.6. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space such that $0 < q_1 < p_1$, $0 < q_2 < p_2$ and $-p_3 < q_3 < 0$.*

(1) If $\frac{q_1}{p_1} + \frac{q_3}{p_3} \geq 0$ or $\frac{q_2}{p_2} + \frac{q_3}{p_3} \geq 0$ or $\frac{q_1}{p_1} + \frac{q_2}{p_2} \geq 1$, then every tight contact structure on M admits a Legendrian vertical circle with twisting number 0.

(2) If $q_3 = -1$, $\frac{q_1}{p_1} < \frac{1}{2p_3 - 1}$ and $\frac{q_2}{p_2} < \frac{1}{2p_3}$, then no tight contact structures on M admit Legendrian vertical circles with twisting number 0.

(3) If $q_1 = q_2 = 1$ and $p_1, p_2 > -2\lfloor \frac{p_3}{q_3} \rfloor$, then no tight contact structures on M admit Legendrian vertical circles with twisting number 0.

In the rest of this thesis, we let Σ be a pair-of-pants, and $-\partial\Sigma \times S^1 = T_1 + T_2 + T_3$, where the "$-$" sign means reversing the orientation. We identify T_i to $\mathbb{R}^2/\mathbb{Z}^2$ by identifying the corresponding component of $-\partial\Sigma \times \{\text{pt}\}$ to $(1,0)^T$, and $\{\text{pt}\} \times S^1$ to $(0,1)^T$. Also, for $i = 1,2,3$, let $V_i = D^2 \times S^1$, and identify ∂V_i with $\mathbb{R}^2/\mathbb{Z}^2$ by identifying a meridian $\partial D^2 \times \{\text{pt}\}$ with $(1,0)^T$ and a longitude $\{\text{pt}\} \times S^1$ with $(0,1)^T$.

2.2 The $e_0 \geq 0$ Case

The $e_0 \geq 0$ case is the simplest case. Theorem 2.1.3 is a special case of Lemma 2.2.2, which also implies part (1) of Theorem 2.1.6.

The following lemma is purely technical.

Lemma 2.2.1. *Let ξ be a tight contact structure on $\Sigma \times S^1$. Assume that each T_i is minimal convex with dividing curves of slope s_i. Then there exist collar neighborhoods $T_1 \times I$ and $T_2 \times I$ of T_1 and T_2, and a properly embedded vertical convex annulus A in $(\Sigma \times S^1) \setminus (T_1 \times I \cup T_2 \times I)$ connecting $T_1 \times \{1\}$ to $T_2 \times \{1\}$ with Legendrian boundary satisfying that following:*

1. *$T_1 \times I$ and $T_2 \times I$ are mutually disjoint and disjoint from T_3;*
2. *for $i = 1, 2$, $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is minimal convex with dividing curves of slope $s_i' \leq s_i$;*
3. *A has no ∂-parallel dividing curves, and the Legendrian boundary of A intersects the dividing sets of $T_1 \times \{1\}$ and $T_2 \times \{1\}$ efficiently.*

Proof. If both s_1 and s_2 are ∞, then we can isotope T_1 and T_2 slightly to have vertical Legendrian divides. Connect a Legendrian divide of T_1 to a Legendrian divide of T_2 by a properly embedded vertical convex annulus A. Then we are done. If $s_1 = \infty$, but s_2 is finite, then we make T_1 to have vertical Legendrian divides, and T_2 to have vertical Legendrian rulings. Connect a Legendrian divide of T_1 to a Legendrian ruling of T_2 by a properly embedded vertical convex annulus B. Then no dividing curves of B intersects $B \cap T_1$. And we can decrease s_2 to ∞ by isotoping T_2 across the dividing curves of B starting and ending on $B \cap T_2$ through bypass adding. We can keep T_2 disjoint from both T_1 and T_3 through out the isotopy since bypass adding can be done in a small neighborhood of the bypass and the original surface. Then we are back to the case when s_1 and s_2 are both ∞.

Assume $s_i = \frac{q_i}{p_i}$ is finite for $i = 1, 2$, where $p_i > 0$. First, we isotope T_1 and T_2 slightly so that they have vertical Legendrian rulings. Note that the Legendrian rulings always intersect dividing curves efficiently. Then connect a Legendrian ruling of T_1 to a Legendrian ruling of T_2 by a properly embedded vertical convex annulus A in $\Sigma \times S^1$. If A has no ∂-parallel dividing curves, then we are done. If A has a ∂-parallel dividing curve, say on the T_1 side, then, after possibly isotoping A slightly, we can assume there is a bypass of T_1 on A. Adding this bypass to T_1, we get a minimal convex torus T_1' in $\Sigma \times S^1$ that co-bounds a collar neighborhood of T_1. We can make T_1' to have vertical Legendrian ruling. By Proposition 1.4.19, we have that the slope of the dividing curves of T_1' is $s_1' = \frac{q_1'}{p_1'} < s_1$, where $0 \leq p_1' < p_i$. Now we delete the thickened torus between T_1 and T_1' from $\Sigma \times S^1$, and repeat the procedure above. This whole process will stop in less than $p_1 + p_2$ steps, i.e, we can either find the collar neighborhoods and the annulus with the required properties, or force one of s_1 and s_2 to decrease to ∞. But the lemma is proved in the latter case. This finishes the proof. $\square$

Lemma 2.2.2. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space such that $\frac{q_1}{p_1}, \frac{q_2}{p_2} > 0$ and $\frac{q_1}{p_1} + \frac{q_3}{p_3} \geq 0$. Then every tight contact structure on M admits a Legendrian vertical circle with twisting number 0.*

Proof. Choose $u_i, v_i \in \mathbf{Z}$ such that $0 < u_i < p_i$ and $p_i v_i + q_i u_i = 1$ for $i = 1, 2, 3$. Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \to T_i$ by

$$\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & v_i \end{pmatrix}.$$

Then

$$M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3).$$

Let ξ be a tight contact structure on M. We first isotope ξ to make each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $n_i < 0$, i.e., ∂V_i is minimal convex with dividing curves of slope $\frac{1}{n_i}$ when measured in the coordinates of ∂V_i. Let s_i be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \frac{-n_i q_i + v_i}{n_i p_i + u_i} = -\frac{q_i}{p_i} + \frac{1}{p_i(n_i p_i + u_i)} < -\frac{q_i}{p_i}.$$

By Lemma 2.2.1, we can thicken V_1 and V_2 to V_1' and V_2' such that

1. V_1', V_2' and V_3 are pairwise disjoint;

2. for $i = 1, 2$, $T_i' = \varphi_i(\partial V_i')$ is minimal convex with dividing curves of slope $s_i' = -\frac{q_i'}{p} \leq s_i$, where $p, q_i' > 0$;

3. there exists a properly embedded vertical convex annulus A connecting T_1' to T_2' that has no ∂-parallel dividing curves, and the Legendrian boundary of A intersects the dividing sets of these tori efficiently.

If none of the dividing curves of A is an arc connecting the two components of ∂A, then, by the Legendrian Realization Principle, we can isotope A to make a vertical circle L on A disjoint from the dividing curves Legendrian. Note that A gives the canonical framing of L, and the twisting number of $\xi|_L$ relative to $TA|_L$ is 0 by Proposition 1.4.5. So $t(L) = 0$.

If there are dividing curves connecting the two components of ∂A. Cut $M \setminus (V_1' \cup V_2' \cup V_3)$ open along A. We get an embedded thickened torus $T_3 \times I$ in M such that $T_3 \times \{0\} = T_3$, and $T_3 \times \{1\}$ is minimal convex with dividing curves of slope $s_3' = \frac{q_1' + q_2' - 1}{p}$. Note that

$$s_3' = \frac{q_1' + q_2' - 1}{p} \geq \frac{q_1'}{p} \geq -s_1 > \frac{q_1}{p_1} \geq -\frac{q_3}{p_3} > s_3.$$

According to Proposition 1.6.20, there exists a convex torus T in $T_3 \times I$ parallel to T_3 with vertical dividing curves. We can then isotope T to make it in standard form. Then a Legendrian divide of T is a Legendrian vertical circle with twisting number 0. $\square$

Proof of Theorem 2.1.3 and Theorem 2.1.6(1). If $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ satisfies that $e_0(M) \geq 0$, then we can assume that $\frac{q_i}{p_i} > 0$ for $i = 1, 2, 3$. It's then clear that $\frac{q_1}{p_1}, \frac{q_2}{p_2} > 0$ and $\frac{q_1}{p_1} + \frac{q_3}{p_3} > 0$. Thus, Lemma 2.2.2 implies Theorem 2.1.3.

Now we assume $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ is a small Seifert space such that $0 < q_1 < p_1$, $0 < q_2 < p_2$ and $-p_3 < q_3 < 0$. If $\frac{q_1}{p_1} + \frac{q_3}{p_3} \geq 0$ or $\frac{q_2}{p_2} + \frac{q_3}{p_3} \geq 0$, then Lemma 2.2.2 applies directly. If $\frac{q_1}{p_1} + \frac{q_2}{p_2} \geq 1$, we apply Lemma 2.2.2 to $M = M(\frac{q_1}{p_1}, \frac{q_3}{p_3} + 1, \frac{q_2}{p_2} - 1)$. This proves Theorem 2.1.6(1). □

2.3 The $e_0 \leq -2$ Case

Definition 2.3.1. Let ξ be a contact structure on $\Sigma \times S^1$. ξ is said to be inappropriate if ξ is overtwisted, or there exists an embedded $T^2 \times I$ with convex boundary and I-twisting at least π such that $T^2 \times \{0\}$ is isotopic to one of the T_i's. ξ is called appropriate if it is not inappropriate.

Lemma 2.3.2. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space, and ξ a tight contact structures on M. Suppose that V_1, V_2, V_3 are tubular neighborhoods of the three singular fibers, and $\Sigma \times S^1 = M \setminus (V_1 \cup V_2 \cup V_3)$. Then $\xi|_{\Sigma \times S^1}$ is appropriate.*

Proof. Without loss of generality, we assume ∂V_i is identified with T_i by the diffeomorphism φ_i. $\xi|_{\Sigma \times S^1}$ is clearly tight. If it is inappropriate, then there exists an embedded $T^2 \times I$ with convex boundary and I-twisting at least π such that $T^2 \times \{0\}$ is isotopic to one of the T_i's. Let's say $T^2 \times \{0\}$ is isotopic to T_1. $T^2 \times I$ has I-twisting at least π implies that, for any rational slope s, there is a convex torus T_0 contained in $T^2 \times I$ isotopic to T_1 that has dividing curves of slope s. Specially, we let m be a meridian of ∂V_1, and s the slope of $\varphi_1(m)$. Then the above fact means that we can thicken V_1 so that ∂V_1 has dividing curves isotopic to its meridians, which implies that the thickened V_1 is overtwisted. This contradicts the tightness of ξ. Thus, $\xi|_{\Sigma \times S^1}$ must be appropriate. □

Lemma 2.3.3 ([15], Lemma 10). *Let ξ be an appropriate contact structure on $\Sigma \times S^1$ such that all three boundary components of $\Sigma \times S^1$ are minimal convex with vertical dividing curves. If Σ_0 is a properly embedded convex surface properly isotopic to $\Sigma \times \{pt\}$ with Legendrian boundary that intersects the dividing set of $\partial\Sigma \times S^1$ efficiently, then the dividing set of Σ_0 consists of three properly embedded arcs, each of which connects a different pair of components of $\partial\Sigma_0$.*

The following lemma from [19] plays a key role in the proof of Theorem 1. For the convenience of readers, we give a detailed proof here.

Lemma 2.3.4 ([19], Lemma 36). *Let ξ be an appropriate contact structure on $\Sigma \times S^1$. Suppose that T_1, T_2 and T_3 are minimal convex and such that T_1 and T_2 have vertical Legendrian rulings and dividing curves of slope $-\frac{1}{n}$, where $n \in \mathbf{Z}^{>0}$, and T_3 has vertical dividing curves. Let $T_1 \times I$ and $T_2 \times I$ be collar neighborhoods of T_1 and T_2 that are mutually disjoint and disjoint from T_3, and such that, for $i = 1, 2$, $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is minimal convex with vertical dividing curves. If*

$\xi|_{T_1\times I}$ and $\xi|_{T_2\times I}$ are both isotopic to a given minimal twisting tight contact structure η on $T^2 \times I$ relative to the boundary, then there exists a properly embedded convex vertical annulus A with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap T_1) \cup (A \cap T_2)$ intersects the dividing curves of T_1 and T_2 efficiently.

Proof. Let $\Sigma' \times S^1 = (\Sigma \times S^1) \setminus [(T_1 \times [0,1)) \cup (T_2 \times [0,1))]$, and Σ'_0 a properly embedded convex surface in $\Sigma' \times S^1$ properly isotopic to $\Sigma' \times \{\text{pt}\}$ that has Legendrian boundary intersecting the dividing set of $\partial\Sigma' \times S^1$ efficiently. Since $\xi|_{\Sigma' \times S^1}$ is appropriate, the dividing set of Σ'_0 consists of three properly embedded arc, each of which connects a different pair of boundary components of Σ'_0. Up to isotopy relative to $\partial\Sigma'_0$, there are infinitely many such multi-arcs on Σ'_0. But, up to isotopy of Σ'_0 which leaves $\partial\Sigma'_0$ invariant, there are only two, each represented by a diagram in Figure 2-1. Such an isotopy of Σ'_0 extends to an isotopy of $\Sigma' \times S^1$ which, when restricted on a component of $\partial\Sigma' \times S^1$, is a horizontal rotation. Thus, up to isotopy of $\Sigma' \times S^1$, which, when restricted on a component of $\partial\Sigma' \times S^1$, is a horizontal rotation, there are only two appropriate contact structures on $\Sigma' \times S^1$. Now let Φ_t be such an isotopy of $\Sigma' \times S^1$ changing $\xi|_{\Sigma' \times S^1}$ to one of the two standard appropriate contact structures. We extend Φ_t to an isotopy $\widetilde{\Phi}_t$ of $\Sigma \times S^1$, which fixes a neighborhood of $T_1 \cup T_2$, and leaves $T_1 \times I$, $T_2 \times I$ and $\Sigma' \times S^1$ invariant. Note that the relative Euler class of $\xi|_{T_i \times I}$ is $(2k-n, 0)^T$, where k is the number of positive basic slices contained in $(T^2 \times I, \eta)$, and is invariant under $\widetilde{\Phi}_t|_{T_i \times I}$. So $\xi|_{T_i \times I}$ and $\widetilde{\Phi}_{1*}(\xi)|_{T_i \times I}$ have the same relative Euler class, and are both continued fraction blocks satisfying the same boundary condition. According to the classification of tight contact structures on $T^2 \times I$, $\xi|_{T_i \times I}$ and $\widetilde{\Phi}_{1*}(\xi)|_{T_i \times I}$ are isotopic relative to boundary. So $\widetilde{\Phi}_{1*}(\xi)$ satisfies the conditions given in the lemma, and is of one of the two standard form. Thus, up to isotopy fixing T_1, T_2 and leaving T_3 invariant, there are only two appropriate contact structures on $\Sigma \times S^1$ satisfying the given conditions. Rotating the diagram on the left of Figure 2-1 by 180° induces a self-diffeomorphism of $\Sigma \times S^1$ mapping T_1 to T_2 and changing the dividing set of Σ'_0 on the left of Figure 2-1 to the one on the right. So this self-diffeomorphism is isotopic to a contactomorphism between the two standard appropriate contact structures on $\Sigma \times S^1$. Hence, up to isomorphism, there is only one such appropriate contact structure on $\Sigma \times S^1$. Thus, we can show the existence of an annulus with the required properties by explicitly constructing such an annulus in a model contact structure on $\Sigma \times S^1$ which satisfies the given conditions.

Consider the minimal twisting tight contact structure η on the thickened torus $T^2 \times I$. Note that the vertical Legendrian rulings of $T^2 \times \{0\}$ intersect its dividing curves efficiently. Without loss of generality, we assume that $T^2 \times \{1\}$ has horizontal Legendrian rulings and two vertical Legendrian dividings. We further assume that, for a small $\varepsilon > 0$, $\eta|_{T^2 \times [0,\varepsilon]}$ is invariant in the I direction. This is legitimate since $T^2 \times \{0\}$ is convex. So $T^2 \times \{\frac{\varepsilon}{2}\}$ is also a convex torus with vertical Legendrian rulings and dividing curves of slope $-\frac{1}{n}$. Let L be a Legendrian ruling of $T^2 \times \{\frac{\varepsilon}{2}\}$. Since the twisting number of $\eta|_L$ relative to the framing given by $T^2 \times \{\frac{\varepsilon}{2}\}$ is $-n$, we can find a standard neighborhood U of L in $T^2 \times (0, \varepsilon)$ such that ∂U is convex with vertical Legendrian ruling and two dividing curves of slope $-\frac{1}{n}$. Now, we set $\Sigma \times S^1 = (T^2 \times I) \setminus U$, where $T_1 = T^2 \times \{0\}$, $T_2 = \partial U$ and $T_3 = T^2 \times \{1\}$, and let

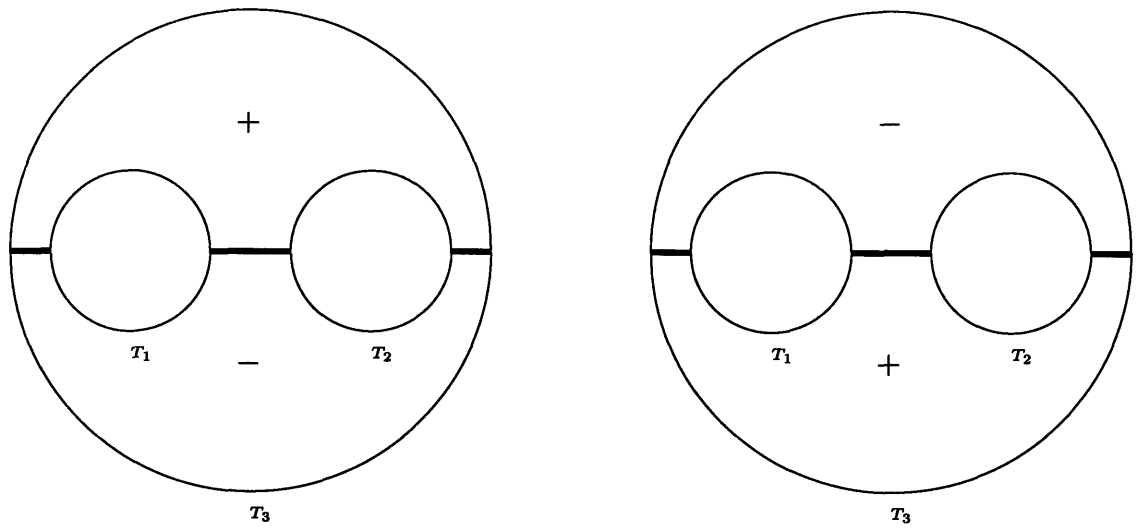


Figure 2-1: The two possible configurations of dividing curves on Σ_0'.

$\xi = \eta|_{\Sigma\times S^1}$. Since η is tight, so is ξ. And there are no embedded thickened tori in $\Sigma \times S^1$ with convex boundary isotopic to T_2 and I-twisting at least π. Otherwise, L would have an overtwisted neighborhood in $T^2 \times I$, which contradict the tightness of η. Also, since the I-twisting of η is less than π, there exists no embedded thickened tori in $\Sigma \times S^1$ with convex boundary isotopic to T_1 or T_3 and I-twisting at least π. Thus, ξ is appropriate. Now we choose a vertical convex annulus A_1 in $\Sigma \times S^1$ connecting a Legendrian ruling of T_1 to a Legendrian dividing of T_3, and a vertical convex annulus A_2 in $\Sigma \times S^1$ connecting a Legendrian ruling of T_2 to the other Legendrian dividing of T_3 such that $(T_1 \cup A_1) \cap (T_2 \cup A_2) = \phi$. The dividing set of A_i consists of n arcs starting and ending on $A_i \cap T_i$. For $i = 1, 2$, we can find a collar neighborhood $T_i \times I$ of T_i, for which $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is convex with dividing set consisting of two circles of slope ∞, by isotoping T_i to engulf all the dividing curves of A_i through bypass adding. Since bypass adding can be done in a small neighborhood of the original surface and the bypass, we can make $T_1 \times I$ and $T_2 \times I$ mutually disjoint and disjoint from T_3. Note that both $T_1 \times I$ and $T_2 \times I$ are minimal twisting. So they are continued fraction blocks satisfying the boundary conditions specified above. Let k_i be the number of positive slices in $T_i \times I$, and $B_i = A_i \cap (T_i \times I)$. Then $2k_i - n = \chi((B_i)_+) - \chi((B_i)_-) = \chi((A_i)_+) - \chi((A_i)_-)$. But $\chi((A_1)_+) - \chi((A_1)_-) = 2k - n$, where k is the number of positive basic slices contained in $(T^2 \times I, \eta)$. So $k_1 = k$. And, since $\eta|_{T^2\times(0,\varepsilon)}$ is I-invariant, we can extend A_2 to a vertical annulus $\widetilde{A}_2$ in $T^2 \times I$ starting at a Legendrian ruling of T_1 and such that $\chi((\widetilde{A}_2)_+) - \chi((\widetilde{A}_2)_-) = \chi((A_2)_+) - \chi((A_2)_-)$. Clearly, $2k - n = \chi((\widetilde{A}_2)_+) - \chi((\widetilde{A}_2)_-)$. So $k_2 = k$. Thus, $k_1 = k_2 = k$. But the isotopy type of a continued fraction block is determined by the number of positive slices in it. Thus, $\xi|_{T_1\times I}$, $\xi|_{T_2\times I}$ and η are isotopic relative to boundary. So our $(\Sigma \times S^1, \xi)$ is indeed a legitimate model. Now we connect a Legendrian ruling of T_1 and a Legendrian ruling of T_2 by a vertical convex annulus A which is contained in $(T^2 \times [0, \varepsilon]) \setminus U$. Then ∂A intersects the dividing sets of T_1 and T_2 efficiently. If A has ∂-parallel diving curves, then $(T^2 \times [0, \varepsilon])$ has non-zero I-twisting, which contradicts our choice of the slice $(T^2 \times [0, \varepsilon])$. Thus, A has no ∂-parallel diving curves. $\square$

Now we are in position to prove Theorem 2.1.4.

Proof of Theorem 2.1.4. Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space with $e_0(M) \leq -2$. Without loss of generality, we assume that $p_1, p_2, p_3 > 1$, $0 < q_1 < p_1$, and $q_2, q_3 < 0$. Choose $u_i, v_i \in \mathbf{Z}$ such that $p_i v_i + q_i u_i = 1$ for $i = 1, 2, 3$. Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \to T_i$ by

$$\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & v_i \end{pmatrix}.$$

Then

$$M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3).$$

Assume that ξ is a tight contact structure on M for which there exits a Legendrian vertical circle L in M with twisting number $t(L) = 0$. We first isotope ξ to make $L = \{\mathrm{pt}\} \times S^1 \subset \Sigma \times S^1$, and each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $n_i < 0$, i.e., ∂V_i is minimal convex with dividing curves of slope $\frac{1}{n_i}$ when measured in the coordinates of ∂V_i. Let s_i be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \frac{-n_i q_i + v_i}{n_i p_i + u_i} = -\frac{q_i}{p_i} + \frac{1}{p_i(n_i p_i + u_i)}.$$

From our choice of p_i and q_i, one can see that $-1 \leq s_1 \leq 0$ and $0 \leq s_2, s_3 < \infty$. Now, without affecting the properties of L and V_i asserted above, we can further isotope the contact structure ξ to make the Legendrian rulings of T_i to have slope ∞ when measured in the coordinates of T_i.

Pick a Legendrian ruling $\widetilde{L}_i$ on each T_i, and connect L to $\widetilde{L}_i$ by a vertical convex annulus A_i such that $A_i \cap A_j = L$ when $i \neq j$. Let Γ_{A_i} be the dividing set of A_i. Since A_i gives the canonical framing Fr of L, we know that the twisting number of $\xi|_L$ relative to $TA_i|_L$ is 0. This means that $\Gamma_{A_i} \cap L = \phi$. But $\Gamma_{A_i} \cap \widetilde{L}_i \neq \phi$. There are dividing curves of A_i starting and ending on $\widetilde{L}_i$. According to Proposition 1.4.19, we can find an embedded minimal twisting slice $T_i \times I$ in $\Sigma \times S^1$, for which $T_i \times \{0\} = T_i$, $T_i \times \{1\}$ is convex with two vertical dividing curves, by isotoping T_i to engulf all the dividing curves of A_i starting and ending on $\widetilde{L}_i$ through bypass adding. Since bypass adding can be done in a small neighborhood of the bypass and the original surface, and the bypasses from different A_i's are mutually disjoint, we can make $T_i \times I$'s pairwise disjoint. By Proposition 1.6.20, we can find a minimal convex torus in $T_i \times (0, 1)$ isotopic to T_i with dividing curves of the slope -1. Without loss of generality, we assume that this torus is $T_i \times \{\frac{1}{2}\}$. Moreover, for $i = 2, 3$, we can find another minimal convex torus, say $T_i \times \{\frac{1}{4}\}$, in $T_i \times (0, \frac{1}{2})$ isotopic to T_i with dividing curves of slope 0.

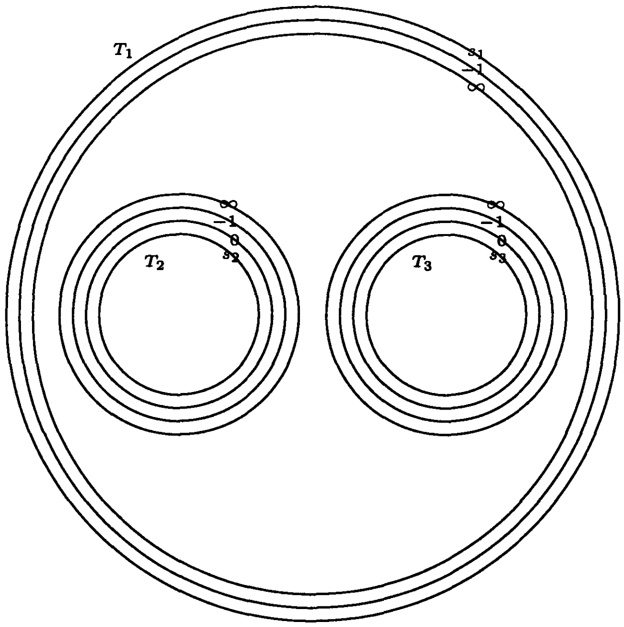


Figure 2-2: Layout of the tori mentioned in the proof of Theorem 2.1.4 with slopes of dividing curves marked on them.

Since the slice $T_i \times I$ is minimal twisting, so is any of its sub-slices. Let's consider the thickened tori $T_i \times [\frac{1}{2}, 1]$. All of these have the same boundary condition, and are minimal twisting. There are only two such tight contact structures up to isotopy relative to boundary. So two of these have to be isotopic relative to boundary. There are 3 cases.

Case 1. $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ are isotopic. We apply Lemma 2.3.4 to

$$\Sigma' \times S^1 = (\Sigma \times S^1) \setminus (T_1 \times [0, \frac{1}{2}) \cup T_2 \times [0, \frac{1}{2}) \cup T_3 \times [0, 1)).$$

Then there exists a vertical convex annulus A connecting $T_1 \times \{\frac{1}{2}\}$ and $T_2 \times \{\frac{1}{2}\}$ with no ∂-parallel dividing curves that has Legendrian boundary intersecting the dividing sets of these tori efficiently. We can extend A across $T_2 \times [\frac{1}{4}, \frac{1}{2}]$ to a convex annulus $\widetilde{A}$ connecting $T_1 \times \{\frac{1}{2}\}$ and $T_2 \times \{\frac{1}{4}\}$ with Legendrian boundary intersecting the dividing sets of these two tori efficiently. Since $T_2 \times [\frac{1}{4}, \frac{1}{2}]$ is minimal twisting, $\widetilde{A} \setminus A$ has no ∂-parallel dividing curves. Thus, $\widetilde{A}$ has no ∂-parallel dividing curves either. Cut $(\Sigma \times S^1) \setminus (T_1 \times [0, \frac{1}{2}) \cup T_2 \times [0, \frac{1}{4}) \cup T_3 \times [0, 1))$ along $\widetilde{A}$, and round the edges. We get a thickened torus $T_3 \times [1, 2]$ embedded in $\Sigma \times S^1$ with minimal convex boundary, where the dividing curves of $T_3 \times \{2\}$ have slope 0. Now we can see that the thickened torus $T_3 \times [0, 2]$ has I-twisting at least π since the dividing curves of $T_3 \times \{\frac{1}{4}\}$ and $T_3 \times \{2\}$ have slope 0 and those of $T_3 \times \{1\}$ have slope ∞. Thus, $\Sigma \times S^1$ is inappropriate. This is a contradiction.

Case 2. $T_1 \times [\frac{1}{2}, 1]$ and $T_3 \times [\frac{1}{2}, 1]$ are isotopic. The proof for this case is identical to that of Case 1 except for interchanging the subindexes 2 and 3.

Case 3. $T_2 \times [\frac{1}{2}, 1]$ and $T_3 \times [\frac{1}{2}, 1]$ are isotopic. Similar to Case 1, we can find a vertical convex annulus B connecting $T_2 \times \{\frac{1}{2}\}$ and $T_3 \times \{\frac{1}{2}\}$ with no ∂-parallel dividing curves that has Legendrian boundary intersecting the dividing sets of these tori efficiently. Extend B across $T_2 \times [\frac{1}{4}, \frac{1}{2}]$ and $T_3 \times [\frac{1}{4}, \frac{1}{2}]$ to a convex annulus $\widetilde{B}$ connecting $T_2 \times \{\frac{1}{4}\}$ and $T_3 \times \{\frac{1}{4}\}$ with Legendrian boundary intersecting the dividing sets of these two tori efficiently. For reasons similar to above, neither component of $\widetilde{B} \setminus B$ has ∂-parallel dividing curves. Thus, $\widetilde{B}$ has no ∂-parallel dividing curves. Cut $(\Sigma \times S^1) \setminus (T_1 \times [0, 1) \cup T_2 \times [0, \frac{1}{4}) \cup T_3 \times [0, \frac{1}{4}))$ along $\widetilde{B}$, and round the edges. We get a thickened torus $T_1 \times [1, 2]$ embedded in $\Sigma \times S^1$ with minimal convex boundary, where the dividing curves of $T_1 \times \{2\}$ have slope -1. Now we can see that the thickened torus $T_1 \times [0, 2]$ has I-twisting at least π since the dividing curves of $T_1 \times \{\frac{1}{2}\}$ and $T_1 \times \{2\}$ have slope -1 and those of $T_1 \times \{1\}$ have slope ∞. Thus, $\Sigma \times S^1$ is inappropriate. This is again a contradiction.

Thus, $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ admits no tight contact structures for which there exists a Legendrian vertical circle with twisting number 0. $\square$

2.4 The $e_0 = -1$ Case

Since part (1) of Theorem 2.1.6 is already proven, we will concentrate on parts (2) and (3) of Theorem 2.1.6. We will refine the method used in the $e_0 \leq -2$ case to prove these results. Lemmata 2.4.1 and 2.4.2 will be the main technical tools used in the proof.

Lemma 2.4.1. *Let ξ be an appropriate contact structure on $\Sigma \times S^1$. Suppose that T_1, T_2 and T_3 are minimal convex, and such that dividing curves of T_1 and T_2 have slope -1, and T_3 has horizontal dividing curves. Assume that there are pairwise disjoint collar neighborhoods $T_i \times I$ of T_i in $\Sigma \times S^1$ for $i = 1, 2, 3$, such that $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is minimal convex with vertical dividing curves. Then $(T_1 \times I, \xi|_{T_1 \times I})$, $(T_2 \times I, \xi|_{T_2 \times I})$ and $(T_3 \times I, \xi|_{T_3 \times I})$ are all basic slices, and the signs of these basic slices can not be all the same, where the sign convention of $(T_i \times I, \xi|_{T_i \times I})$ is given by associating the vector $(0, 1)^T$ to $T_i \times \{1\}$.*

Proof. Since ξ is appropriate, each $(T_i \times I, \xi|_{T_i \times I})$ is minimal twisting. From the boundary condition of these slices, we can see these are all basic slices. Assume that all these basic slices have the same sign. Then we have that $(T_1 \times I, \xi|_{T_1 \times I})$ and $(T_2 \times I, \xi|_{T_2 \times I})$ are isotopic relative to boundary. We isotope T_1 and T_2 slightly so that they have vertical Legendrian rulings. By Lemma 2.3.4, we can then find a properly embedded convex vertical annulus A with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap T_1) \cup (A \cap T_2)$ intersects the dividing sets of T_1 and T_2 efficiently. Cut $\Sigma \times S^1$ open along A, we get a thickened torus $T_3 \times [0, 2]$ such that $T_3 \times \{0\}$, $T_3 \times \{1\}$ and $T_3 \times \{2\}$ are minimal convex, and the slopes of their dividing curves are 0, ∞ and 1, respectively. Note that the slice $(T_3 \times [1, 2], \xi|_{T_3 \times [1,2]})$ has the sign opposite to that of $(T_1 \times I, \xi|_{T_1 \times I})$, and the slice $(T_3 \times [0, 1], \xi|_{T_3 \times [0,1]})$ has the same sign as that of $(T_1 \times I, \xi|_{T_1 \times I})$. So $\xi_{T_3 \times [0,2]}$ is overtwisted. This is a contradiction. Thus, the signs of the basic slices $(T_1 \times I, \xi|_{T_1\times I})$, $(T_2 \times I, \xi|_{T_2\times I})$ and $(T_3 \times I, \xi|_{T_3\times I})$ can not be all the same. $\square$

The following lemma is a special case of Lemma 37 of [19]. Its proof is quite similar to that of Lemma 2.3.4 ([19], Lemma 36). We will only give a sketch of it.

Lemma 2.4.2 ([19], Lemma 37). *Let ξ be an appropriate contact structure on $\Sigma \times S^1$. Suppose that T_1, T_2 and T_3 are minimal convex and such that T_1 has vertical Legendrian rulings and dividing curves of slope $-\frac{1}{n}$, where $n \in \mathbf{Z}^{>0}$, T_2 has vertical Legendrian rulings and dividing curves of slope $\frac{1}{n}$, and T_3 has vertical dividing curves. Let $T_1 \times I$ and $T_2 \times I$ be collar neighborhoods of T_1 and T_2 that are mutually disjoint and disjoint from T_3, and such that, for $i = 1, 2$, $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is minimal convex with vertical dividing curves. If basic slices of $(T_1 \times I, \xi|_{T_1\times I})$ and $(T_2 \times I, \xi|_{T_2\times I})$ are all of the same sign, then there exists a properly embedded convex vertical annulus A with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap T_1) \cup (A \cap T_2)$ intersects the dividing sets of T_1 and T_2 efficiently.*

Sketch of proof. Similar to the proof of Lemma 2.3.4, we can show that, if we prescribe the sign of the basic slices of $(T_1 \times I, \xi|_{T_1\times I})$ and $(T_2 \times I, \xi|_{T_2\times I})$, then up to isotopy that fixes T_1, T_2 and leaves T_3 invariant, there are at most two appropriate contact structures on $\Sigma \times S^1$ that satisfies the given conditions each corresponding one of the two diagrams in Figure ??. Since the two layers $T_1 \times I$ and $T_2 \times I$ are not contactomorphic, we can not find a contactomorphism between these two possible appropriate contact structures as before. Instead, we will construct an appropriate contact structure corresponding to each of these two diagrams, and show that each of these admit an annulus with the required properties.

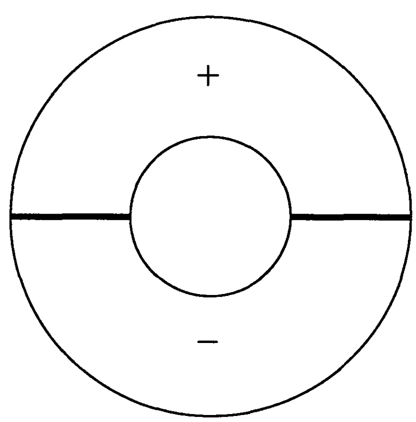


Figure 2-3: Dividing curves on B.

Now consider the tight contact thickened torus $(T_2 \times I, \xi|_{T_2\times I})$. Like in the proof of Lemma 2.3.4, we can construct an appropriate contact structure on $\Sigma \times S^1$ satisfying the conditions in the lemma that admits an annulus A with the required properties by "digging out" a vertical Legendrian ruling of a torus in an I-invariant neighborhood of $T_2 \times \{0\}$ parallel to the boundary. Indeed, both of the possible appropriate contact structures can be constructed this way. To see that, we isotope $T_2 \times \{0\}$ and $T_2 \times \{1\}$ lightly to T_2' and T_3' with the same dividing curves and horizontal Legendrian rulings. Then connect a Legendrian ruling of T_2' and a Legendrian ruling of T_3' by a horizontal convex annulus B. The dividing curves of B is given in Figure 2-3. We can choose the vertical Legendrian ruling to be dug out to intersect one of the two dividing curves of B. These two choices correspond to the two possible configurations of the dividing curves on Σ_0' in Figure ??, and, hence, gives the two possible appropriate contact structures on $\Sigma \times S^1$ satisfying the given conditions. □

Proof of (2) and (3) of Theorem 2.1.6. Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space such that $0 < q_1 < p_1$, $0 < q_2 < p_2$ and $-p_3 < q_3 < 0$. Choose $u_i, v_i \in \mathbf{Z}$ such that $0 < u_i < p_i$ and $p_i v_i + q_i u_i = 1$ for $i = 1, 2, 3$. Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \to T_i$ by

$$\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & v_i \end{pmatrix}.$$

Then

$$M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3).$$

Assume that ξ is a tight contact structure on M for which there exits a Legendrian vertical circle L in M with twisting number $t(L) = 0$. We first isotope ξ to make $L = \{\mathrm{pt}\} \times S^1 \subset \Sigma \times S^1$, and each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $n_i < 0$, i.e., ∂V_i is minimal convex with dividing curves of slope $\frac{1}{n_i}$ when measured in the coordinates of ∂V_i. Let s_i be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \frac{-n_i q_i + v_i}{n_i p_i + u_i} = -\frac{q_i}{p_i} + \frac{1}{p_i(n_i p_i + u_i)}.$$

Then $-1 \leq s_1, s_2 \leq 0$ and $0 \leq s_3 < 1$. Now, without affecting the properties of L and V_i asserted above, we can further isotope the contact structure ξ to make the Legendrian rulings of T_i to have slope ∞ when measured in the coordinates of T_i. As before, we can find pairwise disjoint collar neighborhoods $T_i \times I$'s in $\Sigma \times S^1$ of T_i's, such that $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is minimal convex with vertical dividing curves.

We now prove part (2).

Assume that $q_3 = -1$, $\frac{q_1}{p_1} < \frac{1}{2p_3-1}$ and $\frac{q_2}{p_2} < \frac{1}{2p_3}$. By choosing $n_i \ll -1$, we can make $-\frac{1}{2p_3-1} < s_1 < -\frac{q_1}{p_1}$, $-\frac{1}{2p_3} < s_2 < -\frac{q_2}{p_2}$ and $\frac{1}{p_3+1} < s_3 < \frac{1}{p_3}$. So there is a minimal convex torus in $T_i \times I$ parallel to the boundary, say $T_i' = T_i \times \{\frac{1}{2}\}$, that has dividing curves of slope $-\frac{1}{2p_3-1}$, $-\frac{1}{2p_3}$ and $\frac{1}{p_3+1}$ for $i = 1, 2$ and 3, respectively. Let's consider the layers $T_i \times [\frac{1}{2}, 1]$. $T_1 \times [\frac{1}{2}, 1]$ is a continuous fraction block consisting of $2p_3 - 1$ basic slices. $T_2 \times [\frac{1}{2}, 1]$ is a continuous fraction block consisting of $2p_3$ basic slices. $T_3 \times [\frac{1}{2}, 1]$ consists of 2 continuous fraction blocks, each of which is a basic slice. We can find a minimal convex torus $T_i'' = T_i \times \{\frac{3}{4}\}$ in $T_i \times [\frac{1}{2}, 1]$ parallel to boundary with dividing curves of slope -1 for $i = 1, 2$, and 0 for $i = 3$.

Let the sign of the basic slice $T_3 \times [\frac{3}{4}, 1]$ be $\sigma \in \{+, -\}$. Note that, when $q_3 = -1$, then diffeomorphism $\varphi_3 : \partial V_3 \to T_3$ is given by

$$\varphi_3 = \begin{pmatrix} p_3 & p_3 - 1 \\ 1 & 1 \end{pmatrix}.$$

So the slopes 0 and $\frac{1}{p_3+1}$ of the dividing sets of T_3'' and T_3' correspond to twisting numbers -1 and -2 of Legendrian circles isotopic to the $-\frac{1}{p_3}$-singular fiber. And the basic slice $T_3 \times [\frac{1}{2}, \frac{3}{4}]$ corresponds to a stabilization of a Legendrian circle isotopic to the $-\frac{1}{p_3}$-singular fiber. Since we can freely choose the sign of such a stabilization, we can make the sign of the basic slice $T_3 \times [\frac{1}{2}, \frac{3}{4}]$ to be σ, too.

According to Lemma 2.4.1, the sign of the basic slices $T_i \times [\frac{3}{4}, 1]$ can not be all the same. Note that we can shuffle the signs of basic slices in a continuous fraction block. So at least one of $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ consists only of basic slices of sign $-\sigma$.

Case 1. Assume that all the basic slices in $T_1 \times [\frac{1}{2}, 1]$ are of the sign $-\sigma$. If $T_2 \times [\frac{1}{2}, 1]$ contains p_3 basic slices of the sign $-\sigma$, then we shuffle these signs to the p_3 slices closest to $T_2 \times \{1\}$. Consider the thickened tori $T_1 \times [\frac{5}{8}, 1]$ and $T_2 \times [\frac{5}{8}, 1]$ formed by the unions the p_3 basic slices closest to $T_1 \times \{1\}$ and $T_2 \times \{1\}$ in $T_1 \times I$ and $T_2 \times I$, respectively. Remove from M the solid tori bounded by $T_1 \times \{\frac{5}{8}\}$, $T_2 \times \{\frac{5}{8}\}$ and $T_3 \times \{1\}$. We apply Lemma 2.3.4 to the resulting $\Sigma \times S^1$ and the thickened tori $T_1 \times [\frac{5}{8}, 1]$ and $T_2 \times [\frac{5}{8}, 1]$. Then there exists a properly embedded convex vertical annulus A in $\Sigma \times S^1$ with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap (T_1 \times \{\frac{5}{8}\})) \cup (A \cap (T_2 \times \{\frac{5}{8}\}))$ intersects the dividing curves of $T_1 \times \{\frac{5}{8}\}$ and $T_2 \times \{\frac{5}{8}\}$ efficiently. Cutting $\Sigma \times S^1$ open along A and round the edges, we get a minimal convex torus $\widetilde{T}_3$ isotopic to T_3 with dividing curves of slope $\frac{1}{p_3}$. This means there exists a thickening $\widetilde{V}_3$ of V_3 with convex boundary $\partial\widetilde{V}_3$ that has two dividing curves isotopic to a meridian. Then $\xi|_{\partial\widetilde{V}_3}$ is overtwisted. This contradicts the tightness of ξ.

If $T_2 \times [\frac{1}{2}, 1]$ contains $p_3 + 1$ basic slices of the sign σ, then we shuffle all these signs to the $p_3 + 1$ slices closest to $T_2 \times \{1\}$. Let $T_2 \times [\frac{5}{8}, 1]$ be the union of these $p_3 + 1$ basic slices. Remove from M the solid tori bounded by $T_1 \times \{1\}$, $T_2 \times \{\frac{5}{8}\}$ and $T_3 \times \{\frac{1}{2}\}$. Apply Lemma 2.4.2 to the resulting $\Sigma \times S^1$ and the thickened tori $T_2 \times [\frac{5}{8}, 1]$ and $T_3 \times [\frac{1}{2}, 1]$. Then there exists a properly embedded convex vertical annulus A in $\Sigma \times S^1$ with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap (T_2 \times \{\frac{5}{8}\})) \cup (A \cap (T_3 \times \{\frac{1}{2}\}))$ intersects the dividing curves of $T_2 \times \{\frac{5}{8}\}$ and $T_3 \times \{\frac{1}{2}\}$ efficiently. Cutting $\Sigma \times S^1$ open along A, we get a thickened torus $T_1 \times [1, 2]$ embedded in $\Sigma \times S^1$ that has minimal convex boundary such that $T_1 \times \{1\}$ has vertical dividing curves and $T_1 \times \{2\}$ has dividing curves of slope $-\frac{1}{p_3+1}$. Then the thickened torus $T_1 \times [\frac{1}{2}, 2] = (T_1 \times [\frac{1}{2}, 1]) \cup (T_1 \times [1, 2])$ has I-twisting at least π. This again contradicts the tightness of ξ.

But $T_2 \times [\frac{1}{2}, 1]$ is a continuous fraction block consisting of $2p_3$ basic slices. So it either contains p_3 basic slices of the sign $-\sigma$, or contains $p_3 + 1$ basic slices of the sign σ. So, the basic slices in $T_1 \times [\frac{1}{2}, 1]$ can not be all of the sign $-\sigma$.

Case 2. Assume that all the basic slices in $T_2 \times [\frac{1}{2}, 1]$ are of the sign $-\sigma$. If $T_1 \times [\frac{1}{2}, 1]$ contains either p_3 basic slices of the sign $-\sigma$ or $p_3 + 1$ basic slices of the sign σ, then there will be a contradiction just like in Case 1. So the only possible scenario is that $T_1 \times [\frac{1}{2}, 1]$ contains $p_3 - 1$ basic slices of the sign $-\sigma$ or p_3 basic slices of the sign σ. Now we shuffle all the $-\sigma$ signs in $T_1 \times [\frac{1}{2}, 1]$ to the $p_3 - 1$ basic slices closest to $T_1 \times \{1\}$. Let $T_1 \times [\frac{5}{8}, 1]$ and $T_2 \times [\frac{5}{8}, 1]$ be the unions the $p_3 - 1$ basic slices closest to $T_1 \times \{1\}$ and $T_2 \times \{1\}$ in $T_1 \times I$ and $T_2 \times I$. Remove from M the solid tori bounded by $T_1 \times \{\frac{5}{8}\}$, $T_2 \times \{\frac{5}{8}\}$ and $T_3 \times \{1\}$, and apply Lemma 2.3.4 to the resulting $\Sigma \times S^1$ and the thickened tori $T_1 \times [\frac{5}{8}, 1]$ and $T_2 \times [\frac{5}{8}, 1]$. Then there exists a properly embedded convex vertical annulus A in $\Sigma \times S^1$ with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap (T_1 \times \{\frac{5}{8}\})) \cup (A \cap (T_2 \times \{\frac{5}{8}\}))$ intersects the dividing sets of $T_1 \times \{\frac{5}{8}\}$ and $T_2 \times \{\frac{5}{8}\}$ efficiently. Cutting $\Sigma \times S^1$ open along A and round the edges, we get a minimal convex torus $\widetilde{T}_3$ isotopic to T_3 with dividing curves of slope $\frac{1}{p_3-1}$. This means we can thicken V_3 to a standard neighborhood $\widetilde{V}_3$ of a Legendrian circle isotopic to the $-\frac{1}{p_3}$-singular fiber with twisting number 0. Stabilize this Legendrian circle twice. We get a thickened torus $\widetilde{T}_3 \times [\frac{1}{2}, 2]$ with minimal convex boundary such that $\widetilde{T}_3 \times \{2\} = \widetilde{T}_3$, $\widetilde{T}_3 \times \{\frac{3}{4}\}$ is minimal convex with dividing curves of slope 0, and $\widetilde{T}_3 \times \{\frac{1}{2}\}$ has dividing curves of slope $\frac{1}{p_3+1}$. Since we can choose the signs of these stabilizations freely, we can make both basic slices $\widetilde{T}_3 \times [\frac{1}{2}, \frac{3}{4}]$ and $\widetilde{T}_3 \times [\frac{3}{4}, 2]$ to have the sign $-\sigma$. There exists a minimal convex torus, say $\widetilde{T}_3 \times \{1\}$, in $\widetilde{T}_3 \times [\frac{3}{4}, 2]$ parallel to boundary with vertical dividing curves. Use $\widetilde{T}_3 \times \{1\}$, we can thicken $T_1 \times [\frac{1}{2}, \frac{5}{8}]$ to $\widetilde{T}_1 \times [\frac{1}{2}, 1]$, such that $\widetilde{T}_1 \times [\frac{1}{2}, \frac{5}{8}] = T_1 \times [\frac{1}{2}, \frac{5}{8}]$, and $\widetilde{T}_1 \times \{1\}$ is convex with two vertical dividing curves. Since the basic slice $\widetilde{T}_3 \times [\frac{3}{4}, 2]$ has the sign $-\sigma$, all the basic slices in $\widetilde{T}_1 \times [\frac{5}{8}, 1]$ have the sign σ. Also note that all the basic slices in $\widetilde{T}_1 \times [\frac{1}{2}, \frac{5}{8}] = T_1 \times [\frac{1}{2}, \frac{5}{8}]$ have the sign σ. So we are now in a situation where the basic slices $\widetilde{T}_3 \times [\frac{1}{2}, \frac{3}{4}]$ and $\widetilde{T}_3 \times [\frac{3}{4}, 1]$ both have the sign $-\sigma$, and all the basic slices in $\widetilde{T}_1 \times [\frac{1}{2}, 1]$ have the sign σ. After we thicken $T_2 \times [\frac{1}{2}, \frac{5}{8}]$ to $\widetilde{T}_2 \times [\frac{1}{2}, 1]$, where $\widetilde{T}_2 \times \{1\}$ is minimal convex with vertical dividing curves, we are back to Case 1, which is shown to be impossible. Thus, the basic slices in $T_2 \times [\frac{1}{2}, 1]$ can not be all of the sign $-\sigma$ either.

But, as we mentioned above, one of $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ have to consist only of basic slices of sign $-\sigma$. This is a contradiction. Thus, no such ξ exists on M, and, hence, we proved part (2) of Theorem 2.1.6.

It remains to prove part(3) now.

Assume that $q_1 = q_2 = 1$ and $p_1, p_2 > 2m$, where $m = -\lfloor \frac{p_3}{q_3} \rfloor$. By choosing $n_i \ll -1$, we can make $-\frac{1}{2m} < s_1 < -\frac{1}{p_1}$, $-\frac{1}{2m} < s_2 < -\frac{1}{p_2}$, and $0 < s_3 < -\frac{q_3}{p_3}$. Similar to the proof of part (2), we can find convex a torus $T_i' = T_i \times \{\frac{1}{2}\}$ in $T_i \times I$ parallel to boundary with two dividing curves such that have slope $-\frac{1}{2m}$ for $i - 1, 2$, and 0 for $i = 3$. Then each of $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ is a continued fraction block consists of $2m$ basic slices. And $T_3 \times [\frac{1}{2}, 1]$ is a basic slice. Let the sign of the basic slice $T_3 \times [\frac{1}{2}, 1]$ be $\sigma \in \{+, -\}$. For reasons similar to above, one of $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ can not contain basic slices of the sign σ. Without loss of generality, we assume that all basic slices in $T_1 \times [\frac{1}{2}, 1]$ are of the sign $-\sigma$. We now consider the signs of the basic slices in $T_2 \times [\frac{1}{2}, 1]$.

Case 1. Assume that $T_2 \times [\frac{1}{2}, 1]$ contains m basic slices of the sign $-\sigma$. Then we shuffle these signs to the m basic slices in $T_2 \times [\frac{1}{2}, 1]$ closest to $T_2 \times \{1\}$. Denote by $T_1 \times [\frac{3}{4}, 1]$ and $T_2 \times [\frac{3}{4}, 1]$ the unions of the m basic slices in $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ closest to $T_1 \times \{1\}$ and $T_2 \times \{1\}$, respectively. Remove from M the solid tori bounded by $T_1 \times \{\frac{3}{4}\}$, $T_2 \times \{\frac{3}{4}\}$ and $T_3 \times \{1\}$. We apply Lemma 2.3.4 to the resulting $\Sigma \times S^1$ and the thickened tori $T_1 \times [\frac{3}{4}, 1]$ and $T_2 \times [\frac{3}{4}, 1]$. Then there exists a properly embedded convex vertical annulus A in $\Sigma \times S^1$ with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap (T_1 \times \{\frac{3}{4}\})) \cup (A \cap (T_2 \times \{\frac{3}{4}\}))$ intersects the dividing sets of $T_1 \times \{\frac{3}{4}\}$ and $T_2 \times \{\frac{3}{4}\}$ efficiently. Cutting $\Sigma \times S^1$ open along A and round the edges, we get a thickened torus $T_3 \times [1, 2]$ with minimal convex boundary such that $T_3 \times \{1\}$ has dividing curves of slope ∞, and $T_3 \times \{2\}$ has dividing curves of slope $\frac{1}{m}$. Note that $\frac{1}{m} \leq -\frac{q_3}{p_3}$. If $\frac{1}{m} = -\frac{q_3}{p_3}$, then, as above, the existence of $T_3 \times [1, 2]$ means that we can thicken V_3 to $\widetilde{V}_3$ such that $\xi|_{\widetilde{V}_3}$ is overtwisted, which contradicts the tightness of ξ. If $\frac{1}{m} < -\frac{q_3}{p_3}$, we can choose s_3 so that $\frac{1}{m} < s_3 < -\frac{q_3}{p_3}$. Then the thickened torus $T_3 \times [0, 2] = (T_3 \times I) \cup (T_3 \times [1, 2])$ has I-twisting greater than π, which again contradicts the tightness of ξ. So $T_2 \times [\frac{1}{2}, 1]$ can not contain m basic slices of the sign $-\sigma$.

Case 2. Assume that $T_2 \times [\frac{1}{2}, 1]$ contains $m + 1$ basic slices of the sign σ. We shuffle one of the σ to the basic slice in $T_2 \times [\frac{1}{2}, 1]$ closest to $T_2 \times \{1\}$. Denote by $T_2 \times [\frac{3}{4}, 1]$ this basic slice. Similar to the proof of Theorem 2.1.4, we can find a convex vertical annulus A in M satisfying:

1. A has no ∂-parallel dividing curves;
2. $\partial A = (A \cap (T_2 \times \{\frac{3}{4}\})) \cup (A \cap (T_3 \times \{\frac{1}{2}\}))$, which is Legendrian and intersects the dividing sets of $T_2 \times \{\frac{3}{4}\}$ and $T_3 \times \{\frac{1}{2}\}$ efficiently;
3. A is disjoint from T_1 and the interior of the solid tori in M bounded by $T_2 \times \{\frac{3}{4}\}$ and $T_3 \times \{\frac{1}{2}\}$.

Note that, since $q_1 = 1$, the diffeomorphism $\varphi_1 : \partial V_1 \to T_1$ is given by

$$\varphi_1 = \begin{pmatrix} p_1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Remove from M the interior of the solid tori in M bounded by $T_2 \times \{\frac{3}{4}\}$ and $T_3 \times \{\frac{1}{2}\}$, and cut it open along A. We get a thickening $\widetilde{V}_1$ of V_1, whose boundary is convex with two dividing curves of slope ∞. Then $\widetilde{V}_1$ is a standard neighborhood of a Legendrian circle isotopic to the $\frac{1}{p_1}$-fiber with twisting number 0. We stabilize this Legendrian circle once. This gives a thickened torus $\widetilde{T}_1 \times [0, 2]$ with minimal convex boundary such that $\widetilde{T}_1 \times \{2\} = \partial \widetilde{V}_1$ has dividing curves of slope 0, and $\widetilde{T}_1 \times \{0\}$ has dividing curves of slope $-\frac{1}{p_1 - 1}$, where the slopes are measured in the coordinates of T_1. Since we can choose the sign of the stabilization, we can make the sign of this basic slice σ. Since $-\frac{1}{p_1-1} \geq -\frac{1}{2m}$, we can find minimal convex tori $\widetilde{T}_1 \times \{\frac{1}{2}\}$ and $\widetilde{T}_1 \times \{1\}$ in $\widetilde{T}_1 \times [0,2]$ parallel to the boundary such that $\widetilde{T}_1 \times \{\frac{1}{2}\}$ has dividing curves of slope $-\frac{1}{2m}$, and $\widetilde{T}_1 \times \{1\}$ has dividing curves of slope ∞. Note that $\widetilde{T}_1 \times [\frac{1}{2}, 1]$ is now a continued fraction block consisting of $2m$ basic slices of the sign σ. Now use $\widetilde{T}_1 \times \{1\}$ to thicken $T_2 \times [\frac{1}{2}, \frac{3}{4}]$ to $\widetilde{T}_2 \times [\frac{1}{2}, 1]$ such that $\widetilde{T}_2 \times [\frac{1}{2}, \frac{3}{4}] = T_2 \times [\frac{1}{2}, \frac{3}{4}]$, and $\widetilde{T}_2 \times \{1\}$ is minimal convex with vertical dividing curves. Note that $\widetilde{T}_2 \times [\frac{1}{2}, 1]$ is a continued fraction block that contains at least m basic slices of the sign σ. Now, similar to Case 1, we can find a contradiction. Thus, $T_2 \times [\frac{1}{2}, 1]$ can not contain $m+1$ basic slices of the sign σ either.

But $T_2 \times [\frac{1}{2}, 1]$ contains $2m$ basic slices. So either m of these are of the sign $-\sigma$, or $m+1$ of these are of the sign σ. This is a contradiction. Thus, no such ξ exists on M, and, hence, we proved part (3) of Theorem 2.1.6. $\square$

Chapter 3

Counting Tight Contact Structures

3.1 Statement of Results

In this chapter, we apply Theorems 2.1.3 and 2.1.4 to classify, up to isotopy, tight contact structures on small Seifert spaces with $e_0 \neq 0, -1, -2$. More precisely, we have the following:

Theorem 3.1.1. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, e_0 + \frac{q_3}{p_3})$ be a small Seifert space, where p_i, q_i and e_0 are positive integers, s.t., $p_i > q_i$ and g.c.d.$(p_i, q_i) = 1$. Assume that, for $i = 1, 2, 3$, $-\frac{p_i}{q_i} =< b_0^{(i)}, b_1^{(i)}, \cdots, b_{l_i}^{(i)} >$, where all $b_j^{(i)}$'s are integers less than or equal to -2. Then, up to isotopy, there are exactly $|\prod_{i=1}^{3} b_0^{(i)} \prod_{j=1}^{l_i}(b_j^{(i)}+1)|$ tight contact structures on M. All these tight contact structures are constructed by Legendrian surgeries of (S^3, ξ_{st}), and are therefore holomorphically fillable contact structures distinguished by their Ozsváth-Szabó invariants.*

Theorem 3.1.2. *Let $M = M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$ be a small Seifert space, where p_i and q_i are integers, s.t., $p_i \geq 2$, $q_i \geq 1$ and g.c.d.$(p_i, q_i) = 1$. Assume that, for $i = 1, 2, 3$, $-\frac{q_i}{p_i} =< a_0^{(i)}, a_1^{(i)}, \cdots, a_{m_i}^{(i)} >$, where all $a_j^{(i)}$'s are integers, $a_0^{(i)} = -(\lfloor \frac{q_i}{p_i} \rfloor + 1) \leq -1$, and $a_j^{(i)} \leq -2$ for $j \geq 1$. Then, up to isotopy, there are exactly $|(e_0(M) + 1)\prod_{i=1}^{3}\prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ tight contact structures on M. All these tight contact structures are constructed by Legendrian surgeries of (S^3, ξ_{st}), and are therefore holomorphically fillable contact structures distinguished by their Ozsváth-Szabó invariants.*

3.2 Continued Fractions

In this section, we establish some properties of continued fractions, which will be used to prove Theorems 3.1.2 and 3.1.1.

Lemma 3.2.1. *Let a_0, $a_1, \cdots$, a_m be real numbers such that $a_0 \leq -1$, and $a_j \leq -2$ for $1 \leq j \leq m$. Define $\{p_j\}$ and $\{q_j\}$ by*

$$\begin{cases} p_j = -a_j p_{j-1} - p_{j-2}, \ j = 0, 1, \cdots, m, \\ p_{-2} = -1, \ p_{-1} = 0, \end{cases}$$

$$\begin{cases} q_j = -a_j q_{j-1} - q_{j-2}, \ j = 0, 1, \cdots, m, \\ q_{-2} = 0, \ q_{-1} = 1. \end{cases}$$

Then, for $1 \le j \le m$, we have

1. $-\frac{q_j}{p_j} = < a_0, a_1, \cdots, a_j >$,
2. $p_j \ge p_{j-1} > 0$, $q_j \ge q_{j-1} > 0$,
3. $p_j q_{j-1} - p_{j-1} q_j = 1$,
4. $-\frac{q_j + (a_0+1)p_j}{q_{j-1} + (a_0+1)p_{j-1}} = < a_j, a_{j-1}, \cdots, a_2, a_1 + 1 >$.

Proof. By the definitions of $\{p_j\}$ and $\{q_j\}$, we have $p_0 = 1$, $q_0 = -a_0$, $p_1 = -a_1$, and $q_1 = a_0 a_1 - 1$. Then it's easy to check that the lemma is true for $j = 1$. Assume that the lemma is true for $j - 1 \ge 1$. Then,

$$\begin{aligned} < a_0, a_1, \cdots, a_j > &= < a_0, a_1, \cdots, a_{j-1} - \frac{1}{a_j} > \\ &= -\frac{-(a_{j-1} - \frac{1}{a_j})q_{j-2} - q_{j-3}}{-(a_{j-1} - \frac{1}{a_j})p_{j-2} - p_{j-3}} \\ &= -\frac{(a_j a_{j-1} - 1)q_{j-2} + a_j q_{j-3}}{(a_j a_{j-1} - 1)p_{j-2} + a_j p_{j-3}} \\ &= -\frac{a_j(a_{j-1}q_{j-2} + q_{j-3}) - q_{j-2}}{a_j(a_{j-1}p_{j-2} + p_{j-3}) - p_{j-2}} \\ &= -\frac{-a_j q_{j-1} - q_{j-2}}{-a_j q_{j-1} - q_{j-2}} \\ &= -\frac{q_j}{p_j}. \end{aligned}$$

Also, since $q_{j-1} \ge q_{j-2} > 0$ and $-a_j \ge 2$, we have $q_j = -a_j q_{j-1} - q_{j-2} \ge 2q_{j-1} - q_{j-2} \ge q_{j-1} > 0$, and, similarly, $p_j \ge p_{j-1} > 0$.

Furthermore, by definitions of $\{p_j\}$ and $\{q_j\}$,

$$\begin{aligned} p_j q_{j-1} - p_{j-1} q_j &= (-a_j p_{j-1} - p_{j-2})q_{j-1} - p_{j-1}(-a_j q_{j-1} - q_{j-2}) \\ &= p_{j-1} q_{j-2} - p_{j-2} q_{j-1} \\ &= 1. \end{aligned}$$

Finally,

$$\begin{aligned} -\frac{q_j + (a_0+1)p_j}{q_{j-1} + (a_0+1)p_{j-1}} &= -\frac{(-a_j q_{j-1} - q_{j-2}) + (a_0+1)(-a_j p_{j-1} - p_{j-2})}{q_{j-1} + (a_0+1)p_{j-1}} \\ &= \frac{a_j(q_{j-1} + (a_0+1)p_{j-1}) + (q_{j-2} + (a_0+1)p_{j-2})}{q_{j-1} + (a_0+1)p_{j-1}} \\ &= a_j - \frac{1}{< a_{j-1}, \cdots, a_2, a_1 + 1 >} \\ &= < a_j, a_{j-1}, \cdots, a_2, a_1 + 1 > . \end{aligned}$$

This shows that the lemma is also true for j. □

Remark 3.2.2. In the proof of Theorem 3.1.2 and 3.1.1, all the a_j's will be integers, and so will the corresponding p_j's and q_j's be. Then, property (3) in Lemma 3.2.1 implies that $g.c.d.(p_j, q_j) = 1$.

3.3 The $e_0 \geq 1$ Case

The following lemma is a reformulation of Proposition 1.6.8.

Lemma 3.3.1. *Let ξ be an appropriate contact structure on $\Sigma \times S^1$ with minimal convex boundary that admits a vertical Legendrian circle with twisting number* 0. *Assume that dividing curves of T_1, T_2 and T_3 are of slopes -1, -1, $-n$, respectively, where n is an integer greater than* 1. *Then there is a factorization $\Sigma \times S^1 = L_1 \cup L_2 \cup L_3 \cup (\Sigma' \times S^1)$, where L_i's are embedded thickened tori with minimal twisting and minimal convex boundary $\partial L_i = T_i' - T_i$, s.t., dividing curves of T_i' have slope ∞. The appropriate contact structure ξ is uniquely determined by the signs of the basic slices L_1, L_2 and L_3. The sign convention here is given by associating $(0,1)^T$ to T_i'.*

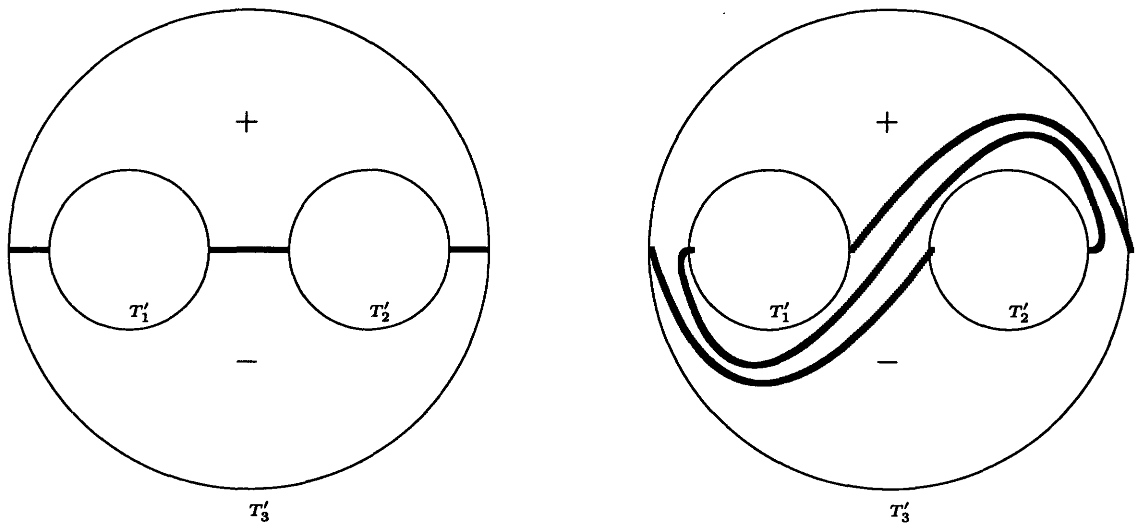


Figure 3-1: Possible configurations of dividing curves on Σ_0'

Proof. We only prove the last sentence. The rest is just part 1 of Proposition 1.6.8. Let Σ_0 be a properly embedded pair-of-pants in $\Sigma \times S^1$ isotopic to $\Sigma \times \{pt\}$, and $\Sigma_0' = \Sigma_0 \cap (\Sigma' \times S^1)$. We isotope Σ_0 so that Σ_0 and Σ_0' are convex with Legendrian boundaries that intersect the dividing curves of $\partial\Sigma \times S^1$ and $\partial\Sigma' \times S^1$ efficiently. Then each component of $\partial\Sigma_0'$ intersects the dividing curves of Σ_0' twice. Since ξ is appropriate, Σ_0' has no ∂-parallel dividing curves. This implies that, up to isotopy relative to boundary and Dehn twists parallel to boundary components, there are only two configurations of dividing curves on Σ_0'. (See Figure 3-1.) Thus, there are only two tight contact structure on $\Sigma' \times S^1$, up to isotopy relative to boundary and full horizontal rotations of each boundary component.

Let $A_i = \Sigma_0 \cap L_i$. Then the dividing set of each of A_1 and A_2 consists of two arcs connecting the two boundary components. And the dividing set of A_3 consists of two arcs connecting the two boundary components and $n-1$ ∂-parallel arcs on the T_3 side. From the relative Euler class of $\xi|_{L_3}$, one can see that the half discs bounded by these ∂-parallel arcs must be pairwise disjoint and of the sign opposite to that of L_3. By isotoping Σ_0 relative to $\Sigma_0' \cup \partial\Sigma_0$, we can freely choose the holonomy of the non-∂-parallel dividing curves of each A_i. This implies that, up to isotopy relative to boundary, there are only two possible configurations of dividing curves on Σ_0 when the signs of L_i's are given. (See Figure 3-2.)

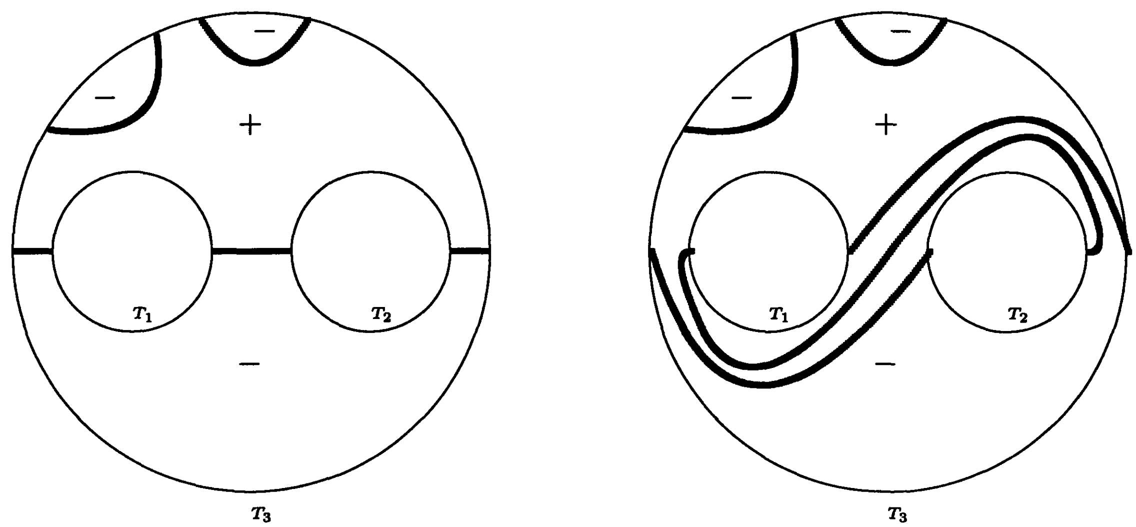


Figure 3-2: Possible configurations of dividing curves on Σ_0. Here, $n = 3$, and the layer L_3 is positive

When the signs of L_i's are mixed, we can extend $(\partial\Sigma \times S^1, \xi)$ to a universally tight contact manifold $(\partial\Sigma'' \times S^1, \xi'')$ by gluing to T_i a basic slice L_i'' of the same sign as L_i for each i, where L_i'' has minimal convex boundary $\partial L_i'' = T_i - T_i''$, and the dividing curves of T_i'' are vertical. Extend Σ_0 across L_i'' to Σ_0'' so that Σ_0'' is convex with Legendrian boundary intersecting the dividing curves of T_i'' efficiently. For $i = 1, 2$, the dividing set of $\Sigma_0'' \cap L_i''$ consists of 1 ∂-parallel arcs on each boundary component. From the relative Euler class of $\xi''|_{L_i''}$, we can see that the half discs on $\Sigma_0'' \cap L_i''$ bounded by these ∂-parallel arcs are of the same sign as the basic slice L_i. The dividing set of $\Sigma_0'' \cap L_3''$ consists of n ∂-parallel arcs on the T_3 side and 1 ∂-parallel arcs on the T_3'' side. From the relative Euler class of $\xi''|_{L_3''}$, we can see that the half discs on $\Sigma_0'' \cap L_3''$ bounded by these ∂-parallel arcs are pairwise disjoint and of the same sign as the basic slice L_3.

Now, one can see that, after the extension, the two possible configurations of dividing curves on Σ_0 become the same minimal configuration of dividing curves on Σ_0''. (See Figure 3-3.) By Proposition 1.6.8, the two configurations correspond to

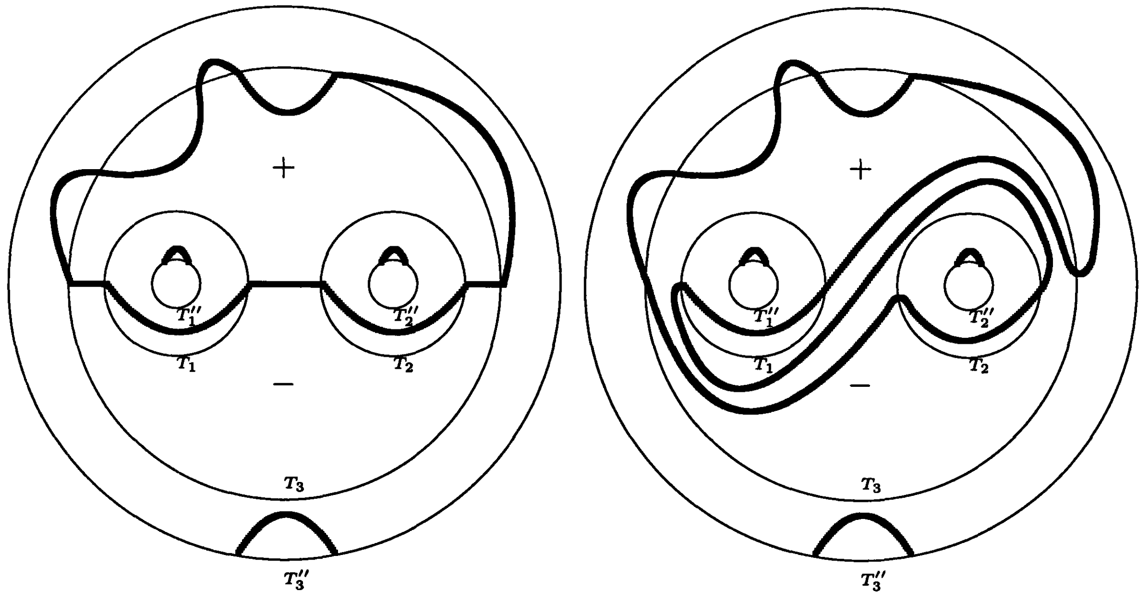


Figure 3-3: After extending to Σ_0'', the two possible configurations become the same. Here, $n = 3$, and the signs of the layers L_1, L_2 and L_3 are $-$, $-$, $+$, respectively.

the same universally tight contact structure on $\Sigma \times S^1$. This shows that, when the signs of L_i's are mixed, ξ is uniquely determined by the signs of L_i's. When all the L_i's have the same sign, ξ is virtually overtwisted, and the isotopy type relative to boundary of such a contact structure is determined by the action of the relative Euler class on Σ_0, which is, in turn, determined by the sign of L_3. Thus, when all the L_i's have the same sign, this common sign determines ξ. $\square$

Proof of Theorem 3.1.1. Define $\{p_j^{(i)}\}$ and $\{q_j^{(i)}\}$ by

$$\begin{cases} p_j^{(i)} = -b_j^{(i)} p_{j-1}^{(i)} - p_{j-2}^{(i)}, \ j = 0, 1, \cdots, l_i, \\ p_{-2}^{(i)} = 0, \ p_{-1}^{(i)} = 1, \end{cases}$$

$$\begin{cases} q_j^{(i)} = -b_j^{(i)} q_{j-1}^{(i)} - q_{j-2}^{(i)}, \ j = 0, 1, \cdots, l_i, \\ q_{-2}^{(i)} = -1, \ q_{-1}^{(i)} = 0. \end{cases}$$

By Lemma 3.2.1 and Remark 3.2.2, we have $p_i = p_{l_i}^{(i)}$ and $q_i = q_{l_i}^{(i)}$. Let $u_i = -p_{l_i-1}^{(i)}$ and $v_i = -q_{l_i-1}^{(i)}$. Then $p_i v_i - q_i u_i = 1$.

Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \to T_i$ by

$$\varphi_i = \begin{cases} \begin{pmatrix} p_i & -u_i \\ -q_i & v_i \end{pmatrix}, & i = 1, 2; \\ \begin{pmatrix} p_3 & -u_3 \\ -q_3 - e_0 p_3 & v_3 + e_0 u_3 \end{pmatrix}, & i = 3. \end{cases}$$

Then

$$M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, e_0 + \frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3).$$

Let ξ be a tight contact structure on M. By Theorem 2.1.3, ξ admits a vertical Legendrian circle L with twisting number 0. We first isotope ξ so that there is a vertical Legendrian circle with twisting number 0 in the interior of $\Sigma \times S^1$, and each V_i is a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $t_i < 0$, i.e., ∂V_i is minimal convex with dividing curves of slope $\frac{1}{t_i}$ when measured in the coordinates of ∂V_i. Let s_i be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \begin{cases} \frac{-t_i q_i + v_i}{t_i p_i - u_i} = -\frac{q_i}{p_i} + \frac{1}{p_i(t_i p_i - u_i)}, & i = 1, 2; \\ \frac{-t_3(q_3 + e_0 p_3) + (v_3 + e_0 u_3)}{t_3 p_i - u_3} = -e_0 - \frac{q_i}{p_i} + \frac{1}{p_i(t_i p_i - u_i)}, & i = 3. \end{cases}$$

We choose $t_i << -1$ so that $\frac{1}{b_0^{(i)}+1} < s_i < -\frac{q_i}{p_i}$ for $i = 1, 2$, and $-e_0 + \frac{1}{b_0^{(3)}+1} < s_3 < -e_0 - \frac{q_3}{p_3}$. Using the vertical Legendrian circle L, we can thicken V_i to V_i', s.t., V_i''s are pairwise disjoint, and $T_i' = \partial V_i'$ is a minimal convex torus with vertical dividing curves when measured in coordinates of T_i. By Proposition 1.6.20, there exits a minimal convex torus T_i'' in the interior of $V_i' \setminus V_i$ isotopic to T_i that has dividing curves of slope $\frac{1}{b_0^{(i)}+1}$ for $i = 1, 2$, and $-e_0 + \frac{1}{b_0^{(3)}+1}$ for $= 3$. Let V_i'' be the solid torus bounded by T_i'', and $\Sigma'' \times S^1 = M \setminus (V_1'' \cup V_2'' \cup V_3'')$.

Now we count the tight contact structures on $\Sigma'' \times S^1$ and V_i'' that satisfy the given boundary condition. First, we look at V_i''. In the coordinates in ∂V_i, the dividing curves of $T_i'' = \partial V_i''$ have slope $\frac{(b_0^{(i)}+1)q_i + p_i}{(b_0^{(i)}+1)v_i + u_i}$. By part 4 of Lemma 3.2.1 and the definitions of u_i, v_i, we have that $\frac{(b_0^{(i)}+1)q_i + p_i}{(b_0^{(i)}+1)v_i + u_i} =< b_{l_i}^{(i)}, b_{l_i-1}^{(i)}, \cdots, b_2^{(i)}, b_1^{(i)} + 1 >$. Thus, by Theorem 1.6.4, there are exactly $|\prod_{j=1}^{l_i}(b_j^{(i)} + 1)|$ tight contact structures on each V_i'' satisfying the given boundary condition. Then we look at $\Sigma'' \times S^1$. The thickened torus L_i bounded by $T_i' - T_i''$ is a continued fraction block consisting of $|b_0^{(i)} + 1|$ basic slices. Let L_i' be the basic slice in L_i closest to T_i', and $\partial L_i' = T_i' - T_i'''$. Note that T_i''' is a minimal convex torus with dividing curves of slope -1 for $i = 1, 2$, and $-e_0 - 1$ for $i = 3$. Let $\Sigma' \times S^1 = M \setminus (V_1' \cup V_2' \cup V_3')$. By Lemma 3.3.1, the tight contact structure on $(\Sigma' \times S^1) \cup L_1' \cup L_2' \cup L_2'$ is uniquely determined by the signs of the basic slices L_i'. But we can shuffle the signs of the basic slices within a continued fraction block. Let's shuffle all the positive signs in L_i to the basic slices closest to T_i'. Then the sign of L_i' is uniquely determined by the number of positive slices in L_i, and so is the number of positive slices in $L_i \setminus L_i'$. Thus, the tight contact structures on $(\Sigma' \times S^1) \cup L_1' \cup L_2' \cup L_2'$ and $L_i \setminus L_i'$ are uniquely determined by these three numbers. But there are only $|b_0^{(1)} b_0^{(2)} b_0^{(3)}|$ ways to choose these three numbers. So there are at most $|b_0^{(1)} b_0^{(2)} b_0^{(3)}|$ on $\Sigma'' \times S^1$ that satisfy the given boundary condition. Altogether, there are at most $|\prod_{i=1}^{3} b_0^{(i)} \prod_{j=1}^{l_i}(b_j^{(i)} + 1)|$ tight contact structures on M.

It remains to construct $|\prod_{i=1}^{3} b_0^{(i)} \prod_{j=1}^{l_i}(b_j^{(i)} + 1)|$ tight contact structures on M by Legendrian surgeries of (S^3, ξ_{st}). We begin with the standard surgery diagram of

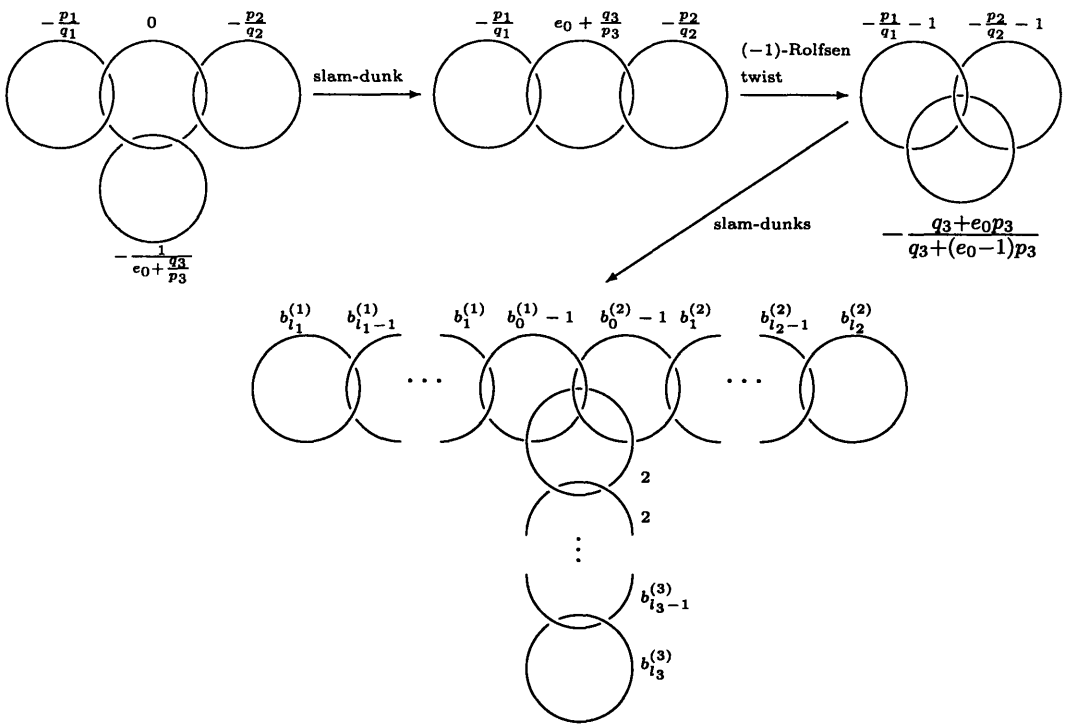


Figure 3-4: Construction of tight contact structures on $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, e_0 + \frac{q_3}{p_3})$

$M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, e_0 + \frac{q_3}{p_3})$. Then, perform a slum-dunk between the 0-component and the $\frac{-1}{e_0+\frac{q_3}{p_3}}$-component, after which the $\frac{-1}{e_0+\frac{q_3}{p_3}}$-component disappears and the original 0-component becomes a $(e_0 + \frac{q_3}{p_3})$-component. Next we perform a (-1)-Rolfsen twist on the $(e_0+\frac{q_3}{p_3})$-component, after which the three components remain trivial and have coefficients $-\frac{p_1}{q_1} - 1$, $-\frac{p_2}{q_2} - 1$ and $-\frac{q_3+e_0p_3}{q_3+(e_0-1)p_3}$. But we have

$$-\frac{p_1}{q_1} - 1 =< b_0^{(1)} - 1, b_1^{(1)}, \cdots, b_{l_1}^{(1)} >,$$

$$-\frac{p_2}{q_2} - 1 =< b_0^{(2)} - 1, b_1^{(2)}, \cdots, b_{l_2}^{(2)} >$$

and

$$-\frac{q_3 + e_0p_3}{q_3 + (e_0 - 1)p_3} =< -2, \cdots, -2, b_0^{(3)} - 1, b_1^{(3)}, \cdots, b_{l_3}^{(3)} >,$$

where, on the right hand side of the last equation, there are $e_0 - 1$ many -2's in front of $b_0^{(3)} - 1$. Now, we perform (inverses of) the slam-dunks corresponding to these three continued fractions here, which lead us to the diagram at the bottom of Figure 3-4. Note that all components in this diagram are trivial. Since the maximal Thurston-Bennequin number of an unknot in (S^3, ξ_{st}) is -1, it's easy to see that there are $|\prod_{i=1}^3 b_0^{(i)} \prod_{j=1}^{l_i}(b_j^{(i)} + 1)|$ ways to realize this diagram by Legendrian surgeries. According to Proposition 1.5.6 and Theorem 1.5.7, these Legendrian surgeries give $|\prod_{i=1}^{3} b_0^{(i)} \prod_{j=1}^{l_i}(b_j^{(i)}+1)|$ pairwise non-isotopic holomorphically fillable contact structures on M distinguished by their Ozsváth-Szabó invariants. $\square$

3.4 The $e_0 \leq -3$ Case

Proof of Theorem 3.1.2. Define $\{p_j^{(i)}\}$ and $\{q_j^{(i)}\}$ by

$$\begin{cases} p_j^{(i)} = -a_j^{(i)} p_{j-1}^{(i)} - p_{j-2}^{(i)}, \ j = 0, 1, \cdots, m_i, \\ p_{-2}^{(i)} = -1, \ p_{-1}^{(i)} = 0, \end{cases}$$

$$\begin{cases} q_j^{(i)} = -a_j^{(i)} q_{j-1}^{(i)} - q_{j-2}^{(i)}, \ j = 0, 1, \cdots, m_i, \\ q_{-2}^{(i)} = 0, \ q_{-1}^{(i)} = 1. \end{cases}$$

By Lemma 3.2.1 and Remark 3.2.2, we have $p_i = p_{m_i}^{(i)}$ and $q_i = q_{m_i}^{(i)}$. Let $u_i = p_{m_i-1}^{(i)}$ and $v_i = q_{m_i-1}^{(i)}$. Then $p_i \geq u_i > 0$, $q_i \geq v_i > 0$, and $p_i v_i - q_i u_i = 1$.

Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \to T_i$ by

$$\varphi_i = \begin{pmatrix} p_i & u_i \\ q_i & v_i \end{pmatrix}.$$

Then

$$M = M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3).$$

Let ξ be a tight contact structure on M. We first isotope ξ to make each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $t_i < -2$, i.e., ∂V_i is minimal convex with dividing curves of slope $\frac{1}{t_i}$ when measured in the coordinates of ∂V_i. Let s_i be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \frac{t_i q_i + v_i}{t_i p_i + u_i} = \frac{q_i}{p_i} + \frac{1}{p_i(t_i p_i + u_i)}.$$

The fact $t_i < -2$ implies that $\lfloor \frac{q_i}{p_i} \rfloor < s_i < \frac{q_i}{p_i}$.

After a possible slight isotopy supported in a neighborhood of $T_i = \partial V_i$, we assume that T_i has Legendrian ruling of slope ∞ when measured in the coordinates of T_i. For each i, pick a Legendrian ruling L_i on T_i. Choose a convex vertical annulus $A \subset \Sigma \times S^1$, such that $\partial A = L_1 \cup L_2$, and the interior of A is contained in the interior of $\Sigma \times S^1$. By Theorem 2.1.4, ξ does not admit Legendrian vertical circles with twisting number 0. So there must be dividing curves of A that connect the two boundary components of A. We isotope T_1 and T_2 by adding to them the bypasses corresponding to the ∂-parallel dividing curves of A. Since bypass adding is done in a small neighborhood of the bypass and the original surface, we can keep V_i's disjoint during this process. Also T_i remains minimal convex after each bypass adding. After we depleted all the ∂-parallel dividing curves of A, each of the remaining dividing curves connects the two boundary components of A. So the slopes of the dividing curves of T_1 and T_2 after the isotopy are $s_1' = \frac{k_1}{k}$ and $s_2' = \frac{k_2}{k}$, where $k \geq 1$ and $g.c.d.(k, k_i) = 1$ for $i = 1, 2$. Since $\lfloor \frac{q_i}{p_i} \rfloor < s_i$, We have that, for $i = 1, 2$, $s_i' \geq \lfloor \frac{q_i}{p_i} \rfloor \geq 0$, and, hence $k_i \geq 0$. This is because that, by Proposition 1.4.19, $s_i' < \lfloor \frac{q_i}{p_i} \rfloor$ implies $s_i' = \infty$ which contradicts Theorem 2.1.4. Now, cut M open along $A \cup T_1 \cup T_2$ and round the edges. We get a minimal convex torus isotopic to T_3 with dividing curves of slope $-\frac{k_1+k_2+1}{k}$ when measure in the coordinates of T_3. When measured in the coordinates of ∂V_3, these dividing curves have slope $-\frac{kq_3+(k_1+k_2+1)p_3}{kv_3+(k_1+k_2+1)u_3}$. It's easy to check that $-\frac{kq_3+(k_1+k_2+1)p_3}{kv_3+(k_1+k_2+1)u_3} < -\frac{q_3}{v_3} < 0$. So, by Corollary 1.6.22, we can isotope ∂V_3 so that it becomes minimal convex with dividing curves of slope $-\frac{q_3}{v_3}$. Measured in the coordinates of T_3, the slope is 0. This implies that the maximal twisting number of a Legendrian vertical circle is -1.

After an isotopy of ξ, we can find a Legendrian vertical circle L in the interior of $\Sigma \times S^1$ with twisting number -1, and, again, make each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $t_i < -2$. As before, we can assume that T_i has Legendrian ruling of slope ∞ when measured in the coordinates of T_i. Let L_i be a Legendrian ruling of T_i. For each i, we choose a convex vertical annulus $A_i \subset \Sigma \times S^1$, s.t., $\partial A_i = L \cup L_i$, the interior of A_i is contained in the interior of $\Sigma \times S^1$, and $A_i \cap A_j = L$ when $i \neq j$. A_i has no ∂-parallel dividing curves on the L side since $t(L)$ is maximal. So the dividing set of A_i consists of two curves connecting L to L_i and possibly some ∂-parallel curves on the L_i side. We now isotope T_i by adding to it the bypasses corresponding to these ∂-parallel dividing curves, and keep V_i's disjoint in this process. After this isotopy, we get a convex decomposition

$$M = M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3)$$

of M, where each T_i is minimal convex with dividing curves of slope $\lfloor \frac{q_i}{p_i} \rfloor$ when measured in the coordinate of T_i. When measured in coordinates of ∂V_i, the slope of the dividing curves becomes $-\frac{q_i - \lfloor \frac{q_i}{p_i} \rfloor p_i}{v_i - \lfloor \frac{q_i}{p_i} \rfloor u_i} = -\frac{q_i + (a_0^{(i)}+1)p_i}{v_i + (a_0^{(i)}+1)u_i}$.

By part 4 of Proposition 1.6.8, there are exactly $2 + \lfloor \frac{q_1}{p_1} \rfloor + \lfloor \frac{q_2}{p_2} \rfloor + \lfloor \frac{q_3}{p_3} \rfloor = |e_0(M)+1|$ tight contact structures on $\Sigma \times S^1$ satisfying the boundary condition and admitting no Legendrian vertical circles with twisting number 0. By Theorem 1.6.4 and part 4 of Lemma 3.2.1, there are exactly $|\prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ tight contact structures on V_i satisfying the boundary condition. Thus, up to isotopy, there are at most $|(e_0(M) + 1)\prod_{i=1}^{3}\prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ tight contact structures on M.

It remains to construct $|(e_0(M)+1)\prod_{i=1}^{3}\prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ tight contact structures on M by Legendrian surgeries of (S^3, ξ_{st}). We begin with the standard surgery diagram of $M = M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$. Then, for each i, perform an $a_0^{(i)}$-Rolfsen twist on the $\frac{p_i}{q_i}$-component. Since $a_0^{(1)} + a_0^{(2)} + a_0^{(3)} = e_0(M)$ and $\frac{p_i}{q_i + a_0^{(i)}p_i} = < a_1^{(i)}, \cdots, a_{m_i}^{(i)} >$, the new surgery coefficients of the four components are $e_0(M)$, $< a_1^{(1)}, \cdots, a_{m_1}^{(1)} >$, $< a_1^{(2)}, \cdots, a_{m_2}^{(2)} >$, and $< a_1^{(3)}, \cdots, a_{m_3}^{(3)} >$. Now, we perform (inverses of) the slam-dunks

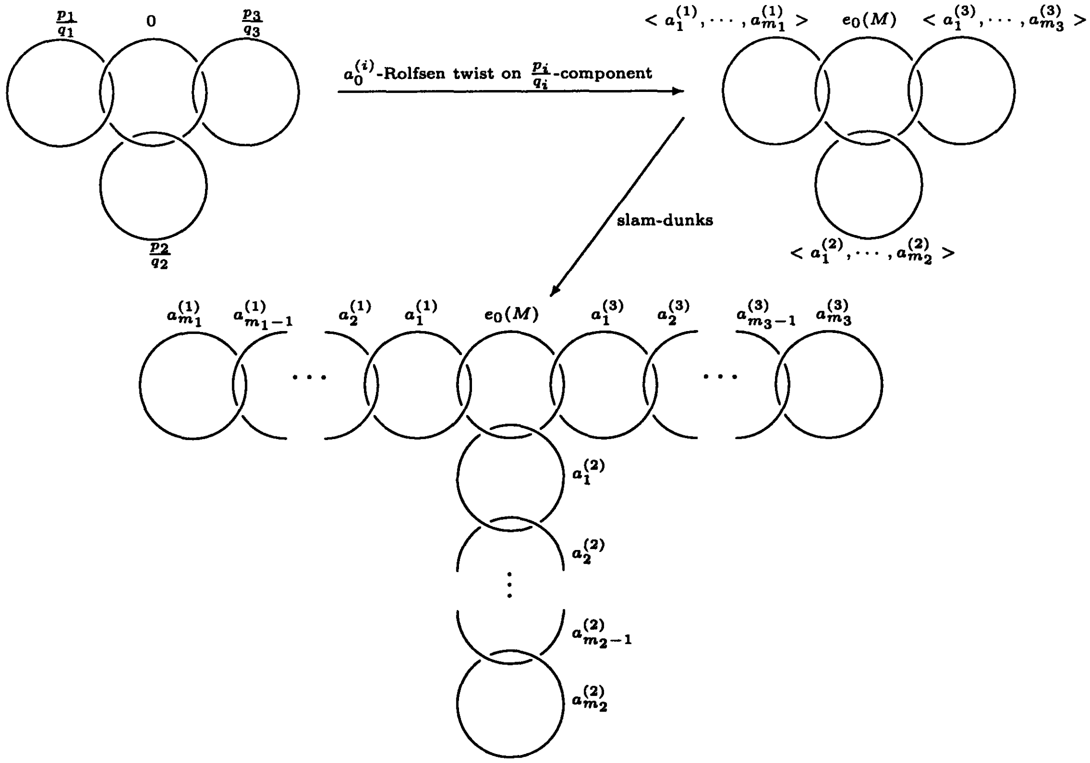


Figure 3-5: Construction of tight contact structures on $M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$

corresponding to the three continued fractions here, which lead us to the diagram at the bottom of Figure 3-5. Since the maximal Thurston-Bennequin number of an unknot in (S^3, ξ_{st}) is -1, there are $|(e_0(M)+1)\prod_{i=1}^{3}\prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ ways to realize this diagram by Legendrian surgeries. According to Proposition 1.5.6 and Theorem 1.5.7, these Legendrian surgeries give $|(e_0(M)+1)\prod_{i=1}^{3}\prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ pairwise non-isotopic holomorphically fillable contact structures on M distinguished by their Ozsváth-Szabó invariants. $\square$

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