A compact moduli space for Cohen-Macaulay curves in projective space

by

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Abstract

We define a moduli functor parametrizing finite maps from a projective (locally) Cohen-Macaulay curve to a fixed projective space. The definition of the functor includes a number of technical conditions, but the most important is that the map is almost everywhere an isomorphism onto its image. The motivation for this definition comes from trying to interpolate between the Hilbert scheme and the Kontsevich mapping space. The main result of this thesis is that our functor is represented by a proper algebraic space. As an application we obtain interesting compactifications of the spaces of smooth curves in projective space. We illustrate this in the case of rational quartics, where the resulting space appears easier than the Hilbert scheme.

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Chapter 1

Introduction

In algebraic geometry one often has some "nice" class of varieties varying in an open parameter space. In order to study the parameter space some sort of compactness is needed, and there are various approaches to the problem of compactifying, i.e. how to decide what to add to the boundary.

Maybe the most natural object is the Hilbert scheme of Grothendieck which exists in a great generality,

Definition 1.0.0.1. Let S be a scheme and let $Hilb^{p(t)}_{\mathbb{P}^n_S}$ be the contravariant functor from schemes over S to sets that to a scheme T assigns all commutative diagrams

$$\begin{tikzcd} C_T \arrow[r, hook, "\iota"] \arrow[d, "p"'] & \mathbb{P}^n_T \arrow[dl] \\ T & \end{tikzcd}$$

where C_T is flat over T and ι is a closed immersion and all fibers of p have Hilbert polynomial $p(t)$.

Grothendieck showed in the early 1960s that $Hilb$ is represented by a projective scheme locally of finite presentation over S (see [FAG] or [HaMo]). In 1963 Hartshorne showed that the Hilbert scheme is connected (see [HartConn]).

The state of knowledge is also pretty sparse when it comes to detailed descriptions of specific Hilbert schemes, that is, for a given Hilbert polynomial $p(t)$ how does the corresponding Hilbert schemes "look like?" By this we could for example ask, how many components does it have? What are the dimensions of the components? Which components intersect each other etc. These seemingly innocent questions are in general very hard to attack.

Already for Hilbert polynomials of degree zero the situation becomes complicated, namely Iarrobino [Iarr] has shown that if $n \geq 3$ then there are Hilbert schemes that are not irreducible. In other words, there are zero-dimensional schemes that can not be smoothed to a collection of distinct points. For $n \leq 2$ Fogarty has shown that the Hilbert scheme is smooth, in particular irreducible since it is connected.

For Hilbert polynomials of degree one the only case studied in detail is the Hilbert scheme of twisted cubics, this scheme was completely described by Piene and Schlessinger [PiSc]; in this case there is one additional component that parametrizes plane cubics union one point. Both components are smooth and they intersect in a divisor.

In some sense the next simplest example is the case of rational quartics in $\mathbb{P}^4$, in this case there are 4 components and their dimensions are known. In addition [MDP] shows that the component parametrizing rational quartics is singular at certain points that do not lie on any other component, i.e. the component is intrinsically singular.

For Hilbert polynomials of degree 2 the author is not aware of any detailed published examples.

A more recent approach to the problem of compactifying spaces of curves is due to Kontsevich and it appeared in the early 1990s:

Definition 1.0.0.2. (Stable maps) Let $\bar{M}_g(\mathbb{P}^n, \beta)$ be the contravariant functor that to a scheme T over a field k_0 associates all flat families

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d, "p"'] & \mathbb{P}^n_T \arrow[dl] \\ T & \end{tikzcd}$$

Where for all $s \in S$ the map $C_s \to \mathbb{P}^n$ is a map from a nodal curve C_s of genus g such that the image of the curve under ι has a fixed cohomology class β, finally the map is required to have a finite automorphism group.

The functor $\bar{M}_g(\mathbb{P}^n, \beta)$ is not representable as a scheme, but has a proper coarse moduli space with normal quotient singularities. In more fancy language, the functor gives a smooth proper Deligne-Mumford stack (see [FuPa]).

In this thesis a new compactification, denoted CM (for Cohen-Macaulay), of spaces of curves is introduced; this compactification has similarities with the Hilbert scheme and with Kontsevich's mapping space, and grew out of our attempt to try to understand the boundary of the Hilbert scheme compactifications of Veronese surfaces and rational quartics in $\mathbb{P}^4$.

In analogy with the Hilbert scheme we fix a Hilbert polynomial of degree one and consider flat families of curves over a field k_0 where the fibers have the given Hilbert polynomial. In contrast to the Kontsevich space we require the map ι to be finite, in addition we require that for each fiber C_s (for $s \in S$) the induced map $\iota_s \colon C_s \to \mathbb{P}^n_{\kappa(s)}$ is an isomorphism onto its image away from at most finitely many points. Finally, we require all fibers to be Cohen-Macaulay schemes. Here is the main theorem in this thesis.

Theorem 1.0.0.3. *The functor CM described above is represented by a proper algebraic space.*

In Section 2 we define the functor and then we give an alternative description in terms of algebra. In Section 3 we go through the necessary background on algebraic spaces and then we state Artin's theorem about representing a functor by an algebraic space; the remainder of this section is devoted to verifying the conditions of Artin's theorem. Finally, in Section 4, we first apply the existence of CM to give a new proof of the existence of Macaulayfications for a variety over a field, secondly we illustrates some of the differences between the Hilbert scheme and CM in the case of rational quartics in $\mathbb{P}^4$.

Chapter 2

The Cohen-Macaulay functor

2.1 Definition of the Cohen-Macaulay functor

We will always work over a fixed field k_0 of of arbitrary characteristic and not necessarily algebraically closed. Let $p(t)$ be numerical polynomial of degree 1. For a scheme S we define $CM(S) := CM^{p(t)}_{\mathbb{P}^n}(S)$ to be the set of commutative diagrams

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d] & \mathbb{P}^n_S \arrow[dl] \\ S & \end{tikzcd}$$

up to isomorphism, where the following holds:

(1) ι is a finite morphism,

(2) C is flat over S,

(3) C locally of finite presentation over S,

(4) For all points $s \in S$ the fiber C_s is a Cohen-Macaulay scheme of pure dimension one,

(5) For all points $s \in S$ the morphism $C_s \longrightarrow \mathbb{P}^n_{\kappa(s)}$ is an isomorphism onto its image apart from a finite set of closed points in C_s,

(6) For all points $s \in S$ the ample invertible sheaf $\iota^*\mathcal{O}(1)_s$ on C_s has Hilbert polynomial $p(t)$.

Remarks 2.1.0.4. (i) CM is a functor: A morphism being finite or locally of finite presentation is stable under base change. The fibers of a base changed family are essentially the same except that the new fibers have coefficients from a field extension of the original fiber. In particular a fiber being Cohen-Macaulay or having a fixed Hilbert polynomial does not change.

(ii) A morphism $f\colon X \to Y$ is *locally of finite presentation* if for any point $x \in X$ there is an open affine neighborhood $V = \mathrm{Spec}(B)$ in X and $U = \mathrm{Spec}(A)$ in Y with $x \in V, f(x) \in U$ and $f|_V\colon V \to U$ makes B into an algebra of finite presentation over A, that is there exists integers m, n so that

$$B \cong A[y_1, \ldots, y_m]/(f_1, \ldots, f_n)$$

(iii) By Cohen-Macaulay in (4) we mean that all local rings are Cohen-Macaulay local rings.

(iv) We say that two elements

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d] & \mathbb{P}^n_S \arrow[dl] \\ S & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C' \arrow[r, "\iota'"] \arrow[d] & \mathbb{P}^n_S \arrow[dl] \\ S & \end{tikzcd}$$

are isomorphic if there is an isomorphism $C \longrightarrow C'$ making the following diagram commute.

$$\begin{tikzcd} & \mathbb{P}^n_S \arrow[ddl] \arrow[rr, phantom, "="] & & \mathbb{P}^n_S \arrow[ddl] \\ C \arrow[ur, "\iota"] \arrow[rr] \arrow[d] & & C' \arrow[ur, "\iota'"] \arrow[d] & \\ S \arrow[rr, phantom, "="] & & S & \end{tikzcd}$$

(v) In (5) we have

$$
\begin{tikzcd}
C_s \arrow[rr, "\iota_s"] \arrow[dr] & & \mathbb{P}^n_{\kappa(s)} \\
& \bar{C}_s \arrow[ur, hook] &
\end{tikzcd}
$$

where $\bar{C}_s$ is the scheme-theoretic image of C_s, the morphism $C_s \to \bar{C}_s$ is finite so it is equivalent to require ι_s to be an isomorphism away from a finite set of points of it's image.

(vi) If S is an affine scheme Spec(R) we will often use the notation $CM(R)$, with this notation CM is *covariant.*

2.2 Algebraic reformulation of the functor

For later use it will be convenient to reformulate the conditions on CM in terms of algebra.

2.2.1 Global case

Let S =Spec(R) be an affine scheme. Suppose we have a diagram

$$
\begin{tikzcd}
C \arrow[r, "\iota"] \arrow[d] & \mathbb{P}^n_{\mathrm{Spec}(R)} \arrow[dl] \\
\mathrm{Spec}(R) &
\end{tikzcd}, \tag{2.2.1.1}
$$

that defines an element of $CM(R)$. By [EGA] **II**, Cor. 6.1.11 a finite morphism is projective, by [EGA] **II**, Prop. 5.5.5 a composition of projective morphisms is projective. This shows that $C \longrightarrow S$ is projective, by [EGA] **II**, Prop. 5.5.1 it follows that C =Proj(A) where $A := \bigoplus_{d \geq 0} \Gamma(C, \iota^* \mathcal{O}(d))$. With this notation we can reformulate the conditions from the definition of CM as follows:

(1)' A is a finite module over $R[x_0, \dots, x_n]$.

(2)' A is flat as an R-module.

(3)' A is locally of finite presentation as an algebra over R.

(4)' For all prime ideals $\mathfrak{p}$ in R, the fiber $A \otimes_R \kappa(\mathfrak{p})$ is a Cohen-Macaulay ring after localizing (and taking degree zero) at any maximal relevant homogeneous prime ideal $\mathfrak{n} \subseteq A \otimes_R \kappa(\mathfrak{p})$.

(5)' For all prime ideals $\mathfrak{p}$ in R, let $\iota_{\mathfrak{p}}^{\#}$ denote the homomorphism

$$\iota_{\mathfrak{p}}^{\#} \colon \kappa(\mathfrak{p})[x_0, \dots, x_n] \longrightarrow A \otimes_R \kappa(\mathfrak{p}),$$

and let $\alpha_{\mathfrak{p}}$ denote the homomorphism

$$\alpha_{\mathfrak{p}} \colon \kappa(\mathfrak{p})[x_0, \dots, x_n]/\ker(\iota_{\mathfrak{p}}^{\#}) \hookrightarrow A \otimes_R \kappa(\mathfrak{p}).$$

Then for all but finitely many maximal relevant homogeneous prime ideals $\mathfrak{n}$ in $\kappa(\mathfrak{p})[x_0, \dots, x_n]/ker(\iota^{\#})$, the homomorphism $\alpha_{\mathfrak{p}}$ becomes an isomorphism after localizing at $\mathfrak{n}$ and taking degree zero.

(6)' For $d \gg 0$ the R-module A_d is locally free of rank $p(d)$.

Remark 2.2.1.1. If A is a graded R-algebra, with homogeneous generators $g_1, \dots, g_r$. A homogenous prime ideal $\mathfrak{p}$ is A is called *relevant* if it does not contain $(g_1, \dots, g_r)$.

Lemma 2.2.1.2. *Let $\phi \colon R \to R'$ be a finite injective homomorphism of 2-dimensional graded algebras over a field such that Proj(R') is Cohen-Macaulay. Then the following are equivalent.*

(1) For all but finitely maximal relevant homogeneous prime ideals $\mathfrak{p}$ in R, the homomorphism $R_{\mathfrak{p}} \to R'_{\mathfrak{p}}$ is an isomorphism.

(2) There exists an $f \in R$ homogeneous of positive degree such that $R[1/f] \to R'[1/f]$ is an isomorphism, and such that f is a nonzero divisor on each $R[1/h]$ resp. $R'[1/h]$ for any homogeneous element $h \in R$ of positive degree.

Proof. Assume (1) and let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the maximal homogeneous relevant primes of R for which $\phi_{\mathfrak{p}_i}$ fails to be an isomorphism. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the minimal primes for each irreducible component of $\mathrm{Proj}(R')$. Let J denote the ideal $\bigcap \mathfrak{p}_i$, by prime avoidance (see [Eisen] Lemma 3.3) we can find a homogeneous element $f \in J - \bigcup \mathfrak{q}_i$. The scheme $V(f)$ contains $V(\mathfrak{p}_1) \cup \dots \cup V(\mathfrak{p}_r)$, hence, by (1), $R[1/f] \cong R'[1/f]$. Conversely, assume we are in situation (2). If f is an element with the property of being a nonzero divisor on

$R[1/h] \subseteq R'[1/h]$ for all homogeneous h in R, then f is only contained in finitely many maximal relevant ideals, hence the desired localization property holds. $\square$

2.2.2 Affine case

In the sequel we will frequently use an affine description of a diagram

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d, "p"'] & \mathbb{P}^n_{\mathrm{Spec}(R)}, \arrow[dl] \\ \mathrm{Spec}(R), & \end{tikzcd}$$

that defines an element of $CM(R)$. To fix notation we recall: $\mathbb{P}^n_R$ is covered by affine schemes $\mathrm{Spec}(R[y_1, \ldots, y_n])$ where $y_j = x_j/x_i$, for all i and all j. Let $R_i := R[x_0/x_i, \ldots, x_n/x_i]$ be the ring arising from inverting x_i. Since the morphism ι is finite it follows that $\iota^{-1}(\mathrm{Spec}(R_i))$=$\mathrm{Spec}(A_i) \subseteq \mathrm{Proj}(A)$. Hence we have a diagram

$$\begin{tikzcd} A_i & R_i \arrow[l, "\iota^{\#}"'] \\ R \arrow[u] \arrow[ur] & \end{tikzcd}$$

The conditions (1)'-(6)' take the following form affine locally.

(1)" A_i is a finite module over R_i.

(2)" A_i is flat as an R-module.

(3)" A_i is finite algebra over R.

(4)" For all prime ideals $\mathfrak{p}$ in R, the fiber $A_i \otimes_R \kappa(\mathfrak{p})$ is a Cohen-Macaulay ring after localizing at any maximal ideal $\mathfrak{n} \subseteq A_i \otimes_R \kappa(\mathfrak{p})$.

(5)" For all prime ideals $\mathfrak{p}$ in R, let $i^{\#}_{\mathfrak{p}}$ denote the homomorphism

$$i^{\#}_{\mathfrak{p}} \colon \kappa(\mathfrak{p})[y_i, \ldots, y_n] \longrightarrow A_i \otimes_R \kappa(\mathfrak{p}),$$

and let $\alpha_{\mathfrak{p}}$ denote the homomorphism

$$\alpha_{\mathfrak{p}} \colon \kappa(\mathfrak{p})[y_1, \ldots, y_n]/\ker(i^{\#}_{\mathfrak{p}}) \hookrightarrow A_i \otimes_R \kappa(\mathfrak{p}).$$

Then for all but finitely many maximal ideals $\mathfrak{n}$ in $\kappa(\mathfrak{p})[y_1, \ldots, y_n]$, $\alpha_{\mathfrak{p}}$ becomes an isomorphism after localizing at $\mathfrak{n}$.

(6)" This condition is empty in the affine case.

The following observation will be useful later.

Lemma 2.2.2.1. *Let*

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d] & \mathbb{P}^n \arrow[dl] \\ Spec(k) & \end{tikzcd} \tag{2.2.2.1}$$

be an element of $CM(k)$, for k any field, and let $\bar{C} \subseteq \mathbb{P}^n$ be the scheme-theoretic image of C, then $\bar{C}$ is Cohen-Macaulay.

Proof. The question is local so we can assume we are in the following affine situation:

$$\begin{tikzcd} A & k[x_1, \ldots, x_n] \arrow[l, "\iota^{\#}"'] \arrow[dl] \\ B \arrow[u, hook] & \end{tikzcd}$$

where A is Cohen-Macaulay of dimension 1, $\iota^{\#}$ is finite, and

$$B = k[x_1, \ldots, x_n]/\ker(\iota^{\#}).$$

Spec(A) $\longrightarrow$Spec(B) is an isomorphism away from finitely many points so B has dimension 1. By [OldM] Proposition 9A, $\mathrm{Ass}_B(A) = \phi^a(\mathrm{Ass}_A(A))$, but A is Cohen-Macaulay of dimension 1 so $\mathrm{Ass}_A(A)$ consists of minimal primes. Clearly $\mathrm{Ass}_B(B) \subseteq \mathrm{Ass}_B(A)$, hence the associated primes of B are all minimal so B has no embedded primes, since B has dimension 1 this implies that B is Cohen-Macaulay. $\square$

Chapter 3

Representing the moduli functor

In Artin's seminal paper [Artin] several criteria for checking when a contravariant functor from schemes/k_0 to sets is an algebraic space is given. In this section we recall some basic definitions and results about algebraic spaces, then we state Artin's theorem, finally we relate the theorem to the functor CM.

3.1 Background on Algebraic Spaces

Our presentation on Algebraic Spaces follows [AlgS],[Knut] and [Mil].

To completely describe a scheme (say separated) X we need only know an open affine cover $\{U_i\}$, the gluing data $V_{ij} = U_i \cap U_j$ and the open immersions $V_{ij} \to U_i$ and $V_{ij} \to U_j$. Let U denote the disjoint union of the U_i, since $V_{ij} = U_i \times_X U_j$ we can write the disjoint union of the V_{ij} as $R = U \times_X U$. The canonical morphism $R \to U \times U$ identifies R as an equivalence relation on U. Let π_1 and π_2 be the two projections of $R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} U$, and $\pi\colon U \to X$ the covering map.

There is a diagram

$$R \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} U \xrightarrow{\pi} X$$

expressing the fact that π is the coequalizer of the maps π_1 and π_2 in the category of schemes. There is also a canonical injection $U \hookrightarrow R$ whose image is a component of R, isomorphic to U, the diagonal component.

For X any scheme we let $X^\bullet$ denote the (representable) functor $\mathrm{Hom}(-, X)$, via the Yoneda embedding this makes the category of schemes a full subcategory of the category of sheaves of sets. In the sequel we will abuse language and write X when we mean the associated functor of points. Finally one can show that X is a sheaf in the Zariski topology on the category of affine schemes, and that X is the quotient, in the category of sheaves, of the equivalence relation $R \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} U$.

Definition 3.1.0.2. (Étale topology) The *étale topology* on the category of schemes is the (Grothendieck) topology where the covering maps are étale morphism, i.e. if U is a scheme then there is a covering $(U_i \to U)$ where for any $u \in U$ there is an i such that u is in the image of $U_i \to U$ and each $U_i \to U$ is an étale morphism.

Let

$$F\colon \{Schemes/k_0\} \to \{Sets\}$$

be a contravariant functor.

Definition 3.1.0.3. (Sheaf axiom) F is a *sheaf* for the étale topology, if for every $U \to X$ étale and every étale covering $(U_i \to U)$ the following sequence is exact:

$$F(U) \xrightarrow{r} \prod_{i \in I} F(U_i) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{i,j \in I} F(U_i \times_U U_j)$$

Remark 3.1.0.4. A sequence of sets like the one in Definition 3.1.0.3 is said to be exact if the first map r is injective and its image is the set on which the two maps p_1, p_2 agree.

The following result is useful for checking when a functor is a sheaf.

Proposition 3.1.0.5. *A contravariant functor F from Schemes to sets is a sheaf in the étale topology if and only if the following two conditions hold:*

1. *For any scheme U, the restriction of F to the usual Zariski topology on U is a sheaf.*

2. *for any surjective étale $U' \to U$ with both U and V affine,*

$$F(U) \xrightarrow{r} F(U') \underset{p_2}{\overset{p_1}{\rightrightarrows}} F(U' \times_U U')$$

is exact.

Proof. See [Mil], Chapter II. Proposition 1.5. p.50. ∎

Definition 3.1.0.6. (Local representability) F is said to be *locally representable* if there exists a scheme U and a map of sheaves $U \to F$ such that for all schemes V, and morphisms $V \to F$, the sheaf fiber-product $U \times_F V$ is representable and the morphism $U \times_F V \to V$ is induced by an étale surjective morphism of schemes.

Definition 3.1.0.7. (Algebraic space) A sheaf F in the étale topology is an *algebraic space* if it is locally representable.

Definition 3.1.0.8. (Separated and locally separated) F is *locally separated (resp. separated)* if $R \to U \times U$ is an immersion (resp. closed immersion).

Definition 3.1.0.9. (Locally of finite presentation) F is *locally of finite presentation* over k_0 if the canonical map

$$\varinjlim F(A_i) \to F(\varinjlim(A_i))$$

is bijective for any filtering direct system of k_0-algebras $\{A_i\}$.

Theorem 3.1.0.10. *([Artin], Theorem 3.4) Let F be a contravariant functor from schemes/k_0 to sets. Then F is a locally separated algebraic space (respectively, a separated algebraic space) locally of finite type over k_0 if and only if the following conditions hold:*

(1) (sheaf axiom) F is a sheaf for the étale topology.

(2) (finiteness) F is locally of finite presentation.

(3) (effectively pro-representability) F is effectively pro-representable.

(4) (relative representability) Let X be an k_0-scheme of finite type and let $\xi, \eta \in F(X)$. Then the condition $\xi = \eta$ is represented by a subscheme (respectively, a closed subscheme) of X.

(5) (openness of formal étalness) Let X be an k_0-scheme of finite type, and let $\xi \colon X \to F$ be a map. If ξ is formally étale at a point $x \in X$, then it is formally étale in a neighborhood of x.

We explain more in detail what the terms used in the theorem above means:
Let F be a contravariant functor $F\colon (Sch/k_0) \longrightarrow (Sets)$. Let $k_0 \subseteq K$ be a field extension and $\xi_0 \in F(K)$ be an element. By *infinitesimal deformation* of ξ_0 we mean a pair (A, η)

where A is an artinian local k_0-algebra with residue field K, and $\eta \in F(A)$ is an element that induces $\xi_0 \in F(K)$ by functorality. A *formal deformation* of ξ_0 is a pair $(\bar{A}, \{\xi_n\})$, where $\bar{A}$ is a complete noetherian local k_0-algebra with residue field K such that

$$\xi_n \in F(\bar{A}/\mathfrak{m}^{n+1}), n = 0, 1, 2, \ldots$$

is a compatible system of elements (i.e. ξ_n induces ξ_{n-1} in $F(\bar{A}/\mathfrak{m}^n)$). Here $\mathfrak{m}_{\bar{A}}$ denotes the maximal ideal of $\bar{A}$. A pair $(\bar{A}, \bar{\xi})$, with $\bar{A}$ as above and $\bar{\xi} \in F(\bar{A})$ inducing ξ_0 will be called an *effective formal deformation.*

Let (B', η') be an infiniteismal deformation of ξ_0, and say that the $(n+1)$-st power of the maximal ideal of B' is zero. Let B be a quotient of B', and denote by $\eta \in F(B)$ the element induced by η'.

Suppose given a map $\bar{A} \to B$, say $\xi_n \mapsto \eta$. Then $\mathfrak{m}_{\bar{A}}$ maps to zero so $\bar{A}/\mathfrak{m}_{\bar{A}}^{n+1} \to B$ hence induces $F(\bar{A}/\mathfrak{m}_{\bar{A}}^{n+1}) \to F(B)$. We say that $(\bar{A}, \bar{\xi})$ is versal (resp. universal) if there exists (resp. exists uniquely) a map $\bar{A}/\mathfrak{m}_{\bar{A}} \to B'$ making the following diagram commutative

$$\begin{tikzcd}
\bar{A}/\mathfrak{m}^{n+1} \arrow[r] \arrow[dr] & B' \arrow[d] \\
 & B
\end{tikzcd}$$

sending

$$\begin{tikzcd}
\xi_n \arrow[r, mapsto] \arrow[dr, mapsto] & \eta' \arrow[d, mapsto] \\
 & \eta
\end{tikzcd}$$

Definition 3.1.0.11. (Effectively pro-representable) A sheaf F is said to be *effectively pro-representable* if every (K, ξ_0) has an effective formal deformation $(\bar{A}, \{\xi_n\})$ that is universal.

Let X be a scheme and $\xi \in F(X)$ then there is a natural transformation $\mathrm{Hom}(-, X) \to F$ defined via $T \to X$ maps to $\xi_T \in F(T)$, where ξ_T is the image of ξ under the map $F(X) \to F(T)$. We will abuse notation and write $\xi \colon X \to F$.

Definition 3.1.0.12. (Formally étale)
Let $\xi \in F(X)$, we say that ξ is *formally étale* at $x \in X$ if for every commutative diagram of solid arrows

$$\begin{tikzcd} X \arrow[d, "\xi"'] & Z_0 \arrow[l, "f_0"'] \arrow[d, hook] \\ F & Z \arrow[l] \arrow[ul, dotted, "f"'] \end{tikzcd}$$

where Z is the spectrum of a local artinian k_0-algebra, Z_0 is a closed subscheme of Z defined by a nilpotent ideal, and f_0 is a map sending Z_0 to x set-theoretically, there exists a unique dotted arrow making the diagram commute.

The following result will be used frequently.

Lemma 3.1.0.13. *The functor CM is locally of finite presentation.*

Proof. To show that CM is locally of finite presentation we need to show that the canonical map

$$\varinjlim CM(R_i) \longrightarrow CM(\varinjlim(R_i))$$

is a bijection for all directed systems of k_0-algebras $\{R_i\}$. Let $R := \varinjlim R_i$, and let $\alpha \in CM(R)$ be an element. Want to show that there is some element α' in $\varinjlim CM(R_i)$ mapping to α. Assume

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d] & \mathbb{P}^n_R \arrow[dl] \\ \mathrm{Spec}(R) & \end{tikzcd}$$

is an element of $CM(R)$. Since C is locally of finite presentation over Spec(R) and by [EGA] **IV$_3$** 8.8.2. and 11.2.6 there exists an λ and a scheme C_λ flat and of finite presentation over R_λ such that $C = C_\lambda \times_{\mathrm{Spec}(R_\lambda)} \mathrm{Spec}(R)$ and a finite morphism $\iota_\lambda \colon C_\lambda \to \mathbb{P}^n_{R_\lambda}$. Furthermore, by increasing λ if necessary, we can assume that all the $R[x_0, \dots x_n]$-module generators and relations of A lies in A_λ (where $\mathrm{Proj}(A) = C$ and $\mathrm{Proj}(A_l) = C_l$), i.e. A_λ is of finite presentation over $R_\lambda[x_0, \dots, x_n]$.

For injectivity: Assume $\alpha_\lambda \in CM(R_\lambda)$ and $\beta_\mu \in CM(R_\mu)$ are elements that map to the same element in $CM(R)$. Need to show that α_λ and β_μ maps to the same element in $\varinjlim CM(R_\nu)$. Let α_λ and β_μ be represented by diagrams

$$\begin{tikzcd} C_\lambda \arrow[r, "\iota_\lambda"] \arrow[d] & \mathbb{P}^n_{R_\lambda} \arrow[dl] \\ \mathrm{Spec}(R_\lambda) & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_\mu \arrow[r, "\iota_\mu"] \arrow[d] & \mathbb{P}^n_{R_\mu} \arrow[dl] \\ \mathrm{Spec}(R_\mu) & \end{tikzcd}$$

By assumption $C_\lambda \times_{\mathrm{Spec}(R_\lambda)} \mathrm{Spec}(R)$ and $C_\mu \times_{\mathrm{Spec}(R_\mu)} \mathrm{Spec}(R)$ are isomorphic. Pick some $l \geq \lambda, \mu$ then α and β can be naturally identified with $C_l := C_\lambda \times_{\mathrm{Spec}(R_\lambda)} \mathrm{Spec}(R_l)$ and $C'_l := C_\mu \times_{\mathrm{Spec}(R_\mu)} \mathrm{Spec}(R_l)$ in $CM(R_l)$. Then $\varinjlim C_\lambda \cong \varinjlim C'_\lambda \in CM(R)$, by [EGA] **IV₃**, Cor. 8.8.2.5 there exist an $m \geq l$ such that $C_m \cong C'_m$, i.e. $\alpha = \beta$. This finishes the proof of the lemma.

□

3.2 Automorphisms of the functor

Let (C, ι) be an element of $CM(R)$, for R any k_0-algebra, that is a diagram

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d] & \mathbb{P}^n_R \arrow[dl] \\ \mathrm{Spec}(R) & \end{tikzcd}$$

By definition, an automorphism of such an element is an automorphism of C that is compatible with ι, i.e. a commutative diagram,

$$\begin{tikzcd} & \mathbb{P}^n_R \arrow[dd] & \\ C \arrow[ur, "\iota"] \arrow[rr] \arrow[dr] & & C \arrow[ul, "\iota"'] \arrow[dl] \\ & \mathrm{Spec}(R) & \end{tikzcd}$$

Theorem 3.2.0.14. *Let R be any k_0-algebra, then the only automorphism of an element of $CM(R)$ is the identity.*

Proof. First we do the case where R is a field k. By choosing affine coverings (see 2.2.2) we can assume we are in the following algebraic situation:

$$
\begin{tikzcd}
 & & k[x_1,\dots,x_n] \arrow[dll, two heads] \arrow[ddl, "\iota^\#"'] \arrow[ddr, "\iota^\#"] \arrow[drr, two heads] & & \\
B \arrow[dr, hook] & & & & B \arrow[dl, hook'] \\
 & A \arrow[rr, "\phi"] & & A &
\end{tikzcd}
$$

where all diagrams commute, and $B = k[x_1, \dots, x_n]/ker(\iota)$. It is straightforward to check that $\phi|B \colon B \longrightarrow B$ is the identity. There exists a nonzero divisor f in B such that $B[1/f] \cong A[1/f]$ (see Lemma 2.2.1.2). Obviously ϕ induces the identity on $A[1/f]$ so hence ϕ is the identity. Next we do the case where R is any local artinian k_0-algebra with residue field k. By choosing affine covers (see 2.2.2) we are in the following algebraic situation:

$$
\begin{tikzcd}
 & & R[x_1,\dots,x_n] \arrow[dll, two heads] \arrow[ddl, "\iota"'] \arrow[ddr, "\iota"] \arrow[drr, two heads] & & \\
B \arrow[dr, hook] & & & & B \arrow[dl, hook'] \\
 & A \arrow[rr, "\phi"] & & A & \\
 & & R \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

where all diagrams commute, and $B = R[x_1, \dots, x_n]/ker(\iota)$. If we divide out by $\mathfrak{m}_R$ we know that there exists a one-dimensional subring, call it $\bar{B}$ of $A/\mathfrak{m}_R A$ such that (see Lemma 2.2.1.2)

1. $\bar{B}$ is in the image of B, and

2. there exists $\bar{f}$ in $\bar{B}$ that acts as a non-zero divisor on $A/\mathfrak{m}_R$ and $\bar{B}[1/\bar{f}] \cong A/\mathfrak{m}_R[1/\bar{f}]$. Furthermore, we see that also

3. any lifting, f, of $\bar{f}$ to B, acts as a non-zero divisor on A,

4. by Nakayamas Lemma $B[1/f]$ surjects onto $A[1/f]$.

From (3) we get that $B \longrightarrow B[1/f]$ and $A \longrightarrow A[1/f]$ are both injections; since ϕ is the identity on B it is the identity on $B[1/f]$; by (4) it is also the identity on $A[1/f]$, but since A is a subring of $A[1/f]$ it follows that ϕ is the identity. This finishes the Artin local case.

For the general case: first we may reduce to R a finitely generated k_0-algebra since CM is locally of finite presentation. Secondly, consider the R-module $M := A_d = \Gamma(C, \iota^*\mathcal{O}(d))$ for some $d \gg 0$ large enough so $\mathcal{O}(d)$ is very ample. Hence M is locally free of rank $p(d)$ (see Subsection 2.2.1). The automorphism ϕ acts on M and we want to show that $\phi - id_M = 0$. For this we use the following lemma.

Lemma 3.2.0.15. *Let R be noetherian and M a finite locally free R-module, and ϕ an R-module automorphism of M. If for all homomorphisms $R \to R'$ with R' local artinian we have that $\phi \otimes id_{R'}$ acts on $M \otimes_R R'$ as the identity, then $\phi = id_M$.*

Proof. For any nonzero non-unit element $r \in R$ there exists a maximal ideal $\mathfrak{m}_r$ such that $r \in \mathfrak{m}_r$, and $\cap_n \mathfrak{m}_r^n = 0$ since R is noetherian, hence there exists an m such that $r \notin \mathfrak{m}_r^m$. Clearly, $R_r := R/\mathfrak{m}_r^m$ is a local artinian ring. Define a map $R \hookrightarrow \prod_{r\in R} R_r$, by sending r to $\bar{r} \in R_{r'}$ for all $r' \in R$; when r is a nonunit r maps to something nonzero in R_r, if r is a unit it maps to a unit in $R_{r'}$ for all r'. It is clear that the map

$$R \hookrightarrow \prod_{r\in R} R_r$$

is injective (since nothing nonzero maps to zero) and since any artinian ring is a product of local artinian rings we can assume that each R_r is a local artinian ring. Since M is noetherian (more generally of finite presentation (see [Bour] ex. 9, p.43) it follows that $M \subseteq \prod_{r\in R}((R_r) \otimes_R M)$ and since ϕ is the identity on each $R_r \otimes_R M$ it follows that ϕ is the identity. □

This finishes the proof of Theorem 3.2.0.14. □

3.3 The functor is a sheaf for the étale topology

Proposition 3.3.0.16. *The functor CM is a sheaf for the étale-topology.*

Proof. We will use Proposition 3.1.0.5.

First we check that CM is a sheaf for the Zariski topology.

Assume $U = \bigcup U_i$, where each U_i is Zariski open in U, and $U \longrightarrow X$ is étale. Pick two elements

$$\begin{tikzcd}
C_U \arrow[r] \arrow[d] & \mathbb{P}^n_U \arrow[dl] \\
U &
\end{tikzcd}
\text{ and }
\begin{tikzcd}
C'_U \arrow[r] \arrow[d] & \mathbb{P}^n_U \arrow[dl] \\
U &
\end{tikzcd}$$

of $CM(U)$ that agree when pulled back to U_i, for all i. Since these elements agree when pulled back to U_i there are isomorphisms $\alpha_i \colon C_{U_i} \longrightarrow C'_{U_i}$ making the following diagram commute.

$$\begin{tikzcd}
& \mathbb{P}^n_{U_i} \arrow[ddl] \arrow[rr, equal] & & \mathbb{P}^n_{U_i} \arrow[ddl] \\
C_{U_i} \arrow[ur] \arrow[rr] \arrow[d] & & C'_{U_i} \arrow[ur] \arrow[d] & \\
U_i \arrow[rr, equal] & & U_i &
\end{tikzcd}$$

Being an open immersion is stable under base-change so we have an open immersion $C_{U_i} \hookrightarrow C_U$, and clearly the C_{U_i} cover C_U. If α_{ij} (resp. α_{ji}) denotes the restriction of α_i (resp. α_j) to $C_i \cap C_j$ then $\alpha_{ji}^{-1} \circ \alpha_{ij}$ defines an automorphism of $CM(U_i \cap U_j)$, by 3.2.0.14 it has to be the identity, hence $\alpha_{ij} = \alpha_{ji}$. It follows that we can define an isomorphism $\alpha \colon C \longrightarrow C'$ by using the $\alpha'_i s$. This shows that "restriction is injective" for Zariski opens.

Let $\alpha_i \in \prod_i \mathrm{CM}(U_i)$ be a set of elements that agree on overlaps, i.e. α_i and α_j induce the same element in $CM(U_i \cap U_j)$. We need to construct an element $\alpha \in CM(U)$ that restricts to each α_i on each U_i. Put C_U to be the glueing of all the C_{U_i} along their overlaps, this is possible, once again by the fact that the only automorphism is the identity. Furthermore C_U is flat over U because being flat is a local property. Being finite is a stable condition in the Zariski topology, likewise is being locally of finite presentation (See [Knut] Chapter I). The conditions about Cohen-Macaulayness and the Hilbert polynomial are both checked locally on the base so they hold automatically. The condition that each fiber should be almost everywhere an isomorphism is also automatic since each fiber of α occurs as a fiber of some α_i and hence the condition holds. All in all this shows that we can define an element $\alpha \in CM(U)$ and obviously α restricts to α_i on each U_i. The remaining conditions are left to the reader to check. This shows that CM is a sheaf for the Zariski topology.

Let $V \to U$ be a surjective étale morphism of affine schemes, we need to verify that

$$CM(U) \longrightarrow CM(V) \rightrightarrows CM(V \times_U V)$$

is exact. Let

$$\begin{tikzcd} C_U \arrow[r] \arrow[d] & \mathbb{P}^n_U \arrow[dl] \\ U & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C'_U \arrow[r] \arrow[d] & \mathbb{P}^n_U \arrow[dl] \\ U & \end{tikzcd}$$

be two elements of $CM(U)$ that agree when pulled back to V. This means that there is an isomorphism $C_V \to C'_V$ compatible with the morphism to $\mathbb{P}^n_V$. Being an isomorphism satisfies effective decent (in the étale topology), see [Knut] Chapter I, Proposition 4.12, p.66, so there is an isomorphism $C_U \to C'_U$, it's left to the reader to check that this isomorphism is compatible with the maps to $\mathbb{P}^n_U$, hence "restriction is injective in the étale topology."

For the "glueing" condition consider the cartesian diagram

$$\begin{tikzcd} V \times_U V \arrow[r, "p_2"] \arrow[d, "p_1"'] & V \arrow[d] \\ V \arrow[r] & U \end{tikzcd}$$

Let

$$p_{ij} \colon V \times_U V \times_U V \to V \times_U V$$

denote the various projection morphisms $p_{ij}(v_1, v_2, v_3) = (v_j, v_i)$ for $j > i$.

Theorem 3.3.0.17. *Let $V \to U$ be faithfully flat and quasi-compact. To give a scheme C_U affine over U is the same as giving a scheme C_V affine over V plus an isomorphism $\phi \colon p_1^* C_V \to p_2^* C_V$ satisfying the cocycle condition*

$$p_{31}^*(\phi) = p_{32}^*(\phi) p_{21}^*(\phi),$$

and in such a situation we get a cartesian diagram

$$\begin{tikzcd} C_V \arrow[r] \arrow[d] & C_U \arrow[d] \\ V \arrow[r] & U \end{tikzcd} \tag{*}$$

Proof. See [Mil] Theorem 2.23, p.19. □

Since an étale morphism is faithfully flat and quasi-compact we are allowed to use the previous theorem. Assume α_V is an element of $CM(V)$ that pulls back to the same element of $CM(V \times_U V)$ via the two projection maps p_1, p_2, we need to check that it "descends" to an element of $CM(U)$. That α descends is completely standard so we only sketch the argument. Let α_V be represented by the diagram

$$\begin{tikzcd} C_V \arrow[r, "\iota_V"] \arrow[d] & \mathbb{P}^n_V \arrow[dl] \\ V & \end{tikzcd}.$$

Since α pulls back to the same element over $V \times_U V$ means, by definition of CM, that there is an isomorphism $\phi\colon p_1^* C_V \to p_2^* C_V$ compatible with ι_V. If we pull back ϕ to $V \times_U \times_U V$ using the various p_{ij} then $p_{32}^*(\phi) p_{21}^*(\phi)(\alpha_V)$ and $p_{31}^*(\phi)(\alpha_v)$ corresponds to two automorphism of $\alpha_{V \times_U \times_U V}$, but by Theorem 3.2.0.14 any automorphism is the identity, hence $p_{31}^*(\phi) = p_{32}^*(\phi) p_{21}^*(\phi)$. By Theorem 3.3.0.17 we have showed the existence of a cartesian diagram $(*)$. We also get a commutative diagram

$$\begin{tikzcd} & \mathbb{P}^n_V \arrow[r] \arrow[dl] & \mathbb{P}^n_U \arrow[dl] \\ C_V \arrow[ur, "\iota_V"] \arrow[r] \arrow[d, "p_V"'] & C_U \arrow[ur, "\iota_U"'] \arrow[d, "p_U"'] & \\ V \arrow[r] & U & \end{tikzcd}$$

where the top and front diagrams are cartesian. Since $V \to U$ is a surjective étale morphism it follows from ([Knut] Chapter 1, Proposition 4.10 p. 65 and Proposition 4.11 p.65) that ι_U is finite since since ι_V is finite, p_U is locally of finite presentation since p_V is locally of finite presentation. The remaining conditions to check that α_U

$$\begin{tikzcd} C_U \arrow[r, "\iota_U"] \arrow[d, "p_U"'] & \mathbb{P}^n_U \arrow[dl] \\ U & \end{tikzcd}$$

is in $CM(U)$ are all OK since they follow from the fact that fibers of p_U and p_V are essentially the same (they are just defined over different fields). $\square$

3.4 Schlessinger's Criteria

Consider CM as a functor on the category (art) of local artinian k_0-algebras with fixed residue field $k \supseteq k_0$. Pick an element $\xi \in CM(k)$, and set

$$CM_\xi(R) := \{\tau \in CM(R) | \tau \text{ maps to } \xi \in CM(k)\}.$$

It is straightforward to see that CM_ξ is a subfunctor of CM. Let $R' \longrightarrow R$ and $R'' \longrightarrow R$ be morphisms in (art). We have a pullback diagram

$$\begin{tikzcd} R' \times_R R'' \arrow[r] \arrow[d] & R'' \arrow[d] \\ R' \arrow[r] & R \end{tikzcd}$$

that induces a commutative diagram

$$\begin{tikzcd} CM_\xi(R' \times_R R'') \arrow[r] \arrow[d] & CM_\xi(R'') \arrow[d] \\ CM_\xi(R') \arrow[r] & CM_\xi(R) \end{tikzcd}$$

By the universal property of cartesian products there is a unique map

$$CM_\xi(R' \times_R R'') \longrightarrow CM_\xi(R') \times_{CM_\xi(R)} CM_\xi(R''). \tag{3.4.0.1}$$

Schlessinger's theorem says that CM_ξ is pro-representable, i.e. isomorphic to $\mathrm{Hom}(\bar{R}, -)$ for some complete local noetherian ring $\bar{R}$, if the following holds:

(S1) (3.4.0.1) is a surjection whenever $R'' \longrightarrow R$ is a small extension.

(S2) (3.4.0.1) is a bijection when $R = k, R' = k[x]/(x^2)$.

(S3) (3.4.0.1) is a bijection for $R'' = R'$ and $R' \longrightarrow R$ a small extension.

(S4) CM_ξ has finite-dimensional tangent space.

Lemma 3.4.0.18. (3.4.0.1) *is (always) injective.*

Proof. Pick two familes $\mathrm{Proj}(A_1)$ and $\mathrm{Proj}(A_2)$ in $CM_\xi(R' \times_R R'')$ that agree in $CM(R') \times_{CM(R)} CM(R'')$ (see 2.2.1). This means that in high degrees $A_1 \otimes_{A^*} A'$ and $B_2 \otimes_{A^*} A'$ agree (resp $B_1 \otimes_{A^*} A''$ and $B_2 \otimes_{A^*} A''$ agree). Call the first graded ring A' (resp. A''). We have a diagram

$$\begin{tikzcd}
A_1 \text{ or } A_2 \arrow[rrr] \arrow[dr] & & & A'' \arrow[dl] \\
& A' \arrow[r] & A & \\
R^* \arrow[uu] \arrow[rrr] \arrow[dr] & & & R'' \arrow[uu] \arrow[dl] \\
& R' \arrow[uu] \arrow[r] & R \arrow[uu] &
\end{tikzcd}$$

By [Schl], Cor. 3.6 and Theorem 3.2.0.14 it follows that $A_1 \cong A_2 \cong A' \times_A A''$ in high degrees as R^*-modules. From the end of the proof of Theorem 3.5.1.2 it follows that $A_1 \cong A_2 \cong A' \times_A A''$ in high degrees as R^*-algebras . It follows that $\mathrm{Proj}(A_1) \cong \mathrm{Proj}(A_2)$. □

Lemma 3.4.0.19. (3.4.0.1) *is (always) surjective.*

Proof. Let $C' = \mathrm{Proj}(A') \longrightarrow \mathrm{Spec}(R')$ and $C'' = \mathrm{Proj}(A'') \longrightarrow \mathrm{Spec}(R'')$(see 2.2.1) be two families that agree when pulled back to $\mathrm{Spec}(R)$. We have the following diagram

$$\begin{tikzcd}
A^* \arrow[rrr] \arrow[dr] & & & A'' \arrow[dl] \\
& A' \arrow[r] & A & \\
R^* \arrow[uu] \arrow[rrr] \arrow[dr] & & & R'' \arrow[uu] \arrow[dl] \\
& R' \arrow[uu] \arrow[r] & R \arrow[uu] &
\end{tikzcd}$$

Where we define $R^* := R' \times_R R''$ and $A^* := A' \times_A A''$. From [Schl], Lemma 3.4 it follows that A^* is flat over R^* and that $A^* \otimes_{R^*} R' = A'$ and $A^* \otimes_{R^*} R'' = A''$. Since A' and A'' are graded it follows that A^* has a natural grading. Put $C^* :=\mathrm{Proj}(A^*)$. The criteria for $C^* \to \mathrm{Spec}(R^*)$ to be in $CM_\xi(R^*)$ now follows very easily from what we have already remarked and the fact that $\mathrm{Spec}(R^*)$ has just one point.

□

We have now proved S1,S2, and S3. Left to show that $CM_\xi(k[\epsilon])$ is a finite dimensional vector space.

Lemma 3.4.0.20. *$CM_\xi(k[\epsilon])$ is a finite-dimensional vector space.*

Proof. The Lemma is standard so we only sketch the proof. Let $f\colon X \to Y$ be a morphism. By [Illu] there exists a complex $L^\bullet_{X/Y}$ such that infinitesimal deformations of (X, f) are in one-to-one correspondence with the points of the k-vector space $\mathbb{E}xt^1_{\mathcal{O}_X}(L^\bullet_{X/Y}, \mathcal{O}_X)$. The following facts can also be found in [Illu]:
-The cohomology sheaves $\mathcal{H}^i$ of $L^\bullet_{X/Y}$ are in all in negative (including 0) degrees.
-If X and Y are of finite type over k then the cohomology sheaves are coherent.
-There exists a spectral sequence

$$Ext^j_{\mathcal{O}_X}(\mathcal{H}^{-i}(L^\bullet_{X/Y}), \mathcal{O}_X) \Longrightarrow \mathbb{E}xt^{i+j}_{\mathcal{O}_X}(L^\bullet_{X/Y}, \mathcal{O}_X)$$

Hence is suffices to show that
(1) $\dim_k Ext^1(\mathcal{H}^0(L^\bullet_{X/Y}), \mathcal{O}_X) < \infty$
(2) $\dim_k Ext^0(\mathcal{H}^{-1}(L^\bullet_{X/Y}, \mathcal{O}_X)) < \infty$
- (1) and (2) are OK if X is proper. □

This finishes the proof of Schlessingers criteria. By Grothendieck's Existence Theorem it follows that CM is effectively pro-representable.

3.5 Properness of the Functor

In this subsection we prove that CM is a proper algebraic space. The strategy is to prove separatedness and universally closedness using the valuative criterion, finally we show that CM is of finite type, hence establishing the properness.

3.5.1 Valuative Criterion

Proposition 3.5.1.1. *(Valuative criterion for properness) A map $f\colon X \longrightarrow S$ of algebraic spaces of finite type is proper iff for any valuation ring Spec(R) mapping to S and a lift of the map on the fraction field Spec(R) $\longrightarrow X$, there is a finite extension R' of R such that the induced map Spec(K') $\longrightarrow X$ may be extended uniquely to a map Spec(R') $\longrightarrow X$ covering the induced map Spec(R') $\longrightarrow S$.*

Proof. See [LaMB], Chapter 7. □

Since CM is locally of finite presentation we can assume that the valuation ring R is of finite type over k_0, in particular it is noetherian; furthermore we may assume R is a DVR (see [Falt] p. 366).

For a proof of the fact that CM is of finite type over k_0 see the next subsection.

Theorem 3.5.1.2. *The functor CM satisfies the valuative criterion, more precisely, suppose we are given a commutative diagram of solid arrows*

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & CM \arrow[d] \\ Spec(R) \arrow[r] \arrow[ur, dashed] & Spec(k_0) \end{tikzcd}$$

then there exists a unique dotted arrow making the diagrams commutative, in particular, the valuative criterion of Proposition 3.5.1.1 holds.

We will now start the proof of the Theorem 3.5.1.2.

Let R be a discrete valuation ring with field of fractions K and uniformizer π. Suppose given a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & CM \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(k_0) \end{tikzcd}$$

The top horizontal map corresponds to an element, call it α_K, of $CM(K)$,

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d] & \mathbb{P}^n_K \arrow[dl] \\ \mathrm{Spec}(K) & \end{tikzcd} \tag{3.5.1.1}$$

We need to construct an element

$$\begin{tikzcd} C_R \arrow[r, "\iota"] \arrow[d, "p"'] & \mathbb{P}^n_R \arrow[dl] \\ \mathrm{Spec}(R) & \end{tikzcd} \tag{3.5.1.2}$$

of $CM(R)$ that maps to α_K in $CM(K)$. Let $\bar{C}$ be the closure of C in $\mathbb{P}^n_K$ let $\bar{C}_R$ be the flat closure of $\bar{C}$ in $\mathbb{P}^n_R$. Choose a linear subspace L in $\mathbb{P}^n_R$ of dimension $n-2$ that misses $\bar{C}_R$, this is possible because it is possible if we pass to the special point of Spec(R). Let p be the projection of $\bar{C}_R$ to $\mathbb{P}^1_R$, then p is a finite morphism because no component of $\bar{C}_R$ is contracted. By passing to the affine situation (see Subsection 2.2.2 and Lemma 2.2.1) where $x_0 \neq 0$ we are in the following algebraic situation:

$$\begin{tikzcd}
 & & A[1/f] & & & \\
A \arrow[urr, hook] & & & & B[1/f] \arrow[ull, hook', "\cong"'] & \\
 & & B \arrow[ull, hook'] \arrow[urr, hook] & & \bar{B} \arrow[ll, hook'] & \\
K[x_1,\ldots,x_n] \arrow[uu] \arrow[urr, two heads] & & & & R[x_1,\ldots,x_n] \arrow[u, two heads] \arrow[llll, hook'] & R[x] \arrow[l, hook']
\end{tikzcd} \tag{3.5.1.3}$$

where Spec$(A) \subseteq C$, Spec$(B) \subseteq \bar{C}$, and Spec$(\bar{B}) \subseteq \bar{C}_R$ are the following open affines

A and $K[x_1,\ldots,x_n]$ comes from (3.5.1.1).

$$B = K[x_1,\ldots,x_n]/\ker(K[x_1,\ldots,x_n] \longrightarrow A)$$

$$\bar{B} = R[x_1,\ldots,x_n]/\ker(R[x_1,\ldots,x_n] \longrightarrow B).$$

Note: the f in the diagram (3.5.1.3) above is equal to f/x_0^{degf} in Lemma 2.2.1.
Furthermore, A is a finite $K[x_1,\ldots,x_n]$-module and $\bar{B}$ is a finite $R[x]$-module because p above is finite. We see that f not in any minimal prime (over π) of B, as f is not a zerodivisor. From Lemma 2.2.2.1 we know that B is Cohen-Macaulay of dimension 1.

Lemma 3.5.1.3. *B is torsion-free as an $R[x]$-module.*

Proof. Since $R[x] \longrightarrow \bar{B}$ is finite it follows that $K[x] \longrightarrow B$ is finite. Any nonzero element of $K[x]$ is not in any minimal prime of B (if so $K[x] \longrightarrow B$ would not be finite), hence any nonzero element of $R[x]$ is not a zerodivisor on B. $\square$

Since A is a finite B-module we can write $A = B[z_1,\ldots,z_l]$ where $z_i z_j = \sum_k b_{ijk} z_k$. We can find an integer N such that $\pi^N z_i \in \bar{B}[1/\pi^N f]$ and $\pi^N b_{ijk} \in \bar{B}$ for all i,j,k. Let

z_i' denote $\pi^N z_i$. Set $A' := \bar{B}[z_1', \dots, z_l']$, note that A' is a finite torsion free $R[x]$-module. By construction $A' \subseteq B[1/f] = A[1/f]$, set $\bar{A} = \mathrm{Hom}_{R[x]}(\mathrm{Hom}_{R[x]}(A', R[x]), R[x])$. From standard facts about double duals and reflexive modules (see for example [HuLe] Proposition 1.1.10 p.6 and Example 1.1.16 p. 9) we get the following properties:

($\bar{1}$) $\bar{A}$ is flat over R.

($\bar{2}$) $\bar{A}$ is a finite $R[x]$-module, hence also finite over $R[x_1, \dots, x_n]$ since $x = \sum r_i x_i$ for some $r_i \in R$.

($\bar{3}$) $\bar{A}$ is a reflexive $R[x]$-module, hence locally free.

($\bar{4}$) $\bar{A}$ is S_2, hence Cohen-Macaulay as an $R[x]$-module.

($\bar{5}$) The natural homomorphism $ev\colon A' \longrightarrow \bar{A}$ is an isomorphism after localizing at any ht 1 prime of $R[x]$.

($\bar{6}$) There exists finitely many maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_\rho$ in $R[x]$ such that ev is an isomorphism away from these maximal ideals.

($\bar{7}$) $A = B_K$, because Spec(A) and Spec(B) are birational.
$A'_K = \bar{B}_K[z_1, \dots, z_l] = B_K[z_1, \dots, z_l] = B[z_1, \dots, z_l] = A$.
$A'_K = \bar{A}_K$ because Spec(A') and Spec($\bar{A}$) are birational.

Finally we need the following well-known fact (sometimes called Hartog's Theorem)

Proposition 3.5.1.4. *If S is a noetherian normal scheme of dimension at least two, and $s_1, \dots, s_\rho$ are points of height 2, $\mathcal{F}$ a coherent locally free $\mathcal{O}_S$-module then* $\Gamma(S, \mathcal{F}) = \Gamma(S - \{s_1, \dots, s_\rho\}, \mathcal{F})$.

Proof. See [Eisen] Corollary 11.4 and remarks following Corollary 11.4. □

Finally, ($\bar{1}$) – ($\bar{7}$) and Proposition 3.5.1.4 imply also

($\bar{8}$) $\bar{A}$ is a ring, and $\bar{B} \to \bar{A}$ is a ring homomorphism.

Proof. This follows from Proposition 3.5.1.4 applied to S =Spec($R[x]$) and s_i correspond to $\mathfrak{m}_i$: Two elements $a_1, a_2 \in \bar{A}$ naturally correspond to two sections, call them a_1, a_2, in the sheaf $\mathcal{F}$ associated to the $R[x]$-module $\bar{A}$. Since A' is a ring the sheaf $\mathcal{F}'$ associated to A' is a sheaf of algebras. By $(\bar{5})$ we have that $\mathcal{F} \cong cF'$ away from $s_1, \dots, s_\rho$. So $a_1 a_2$ is defined as a section of $\mathcal{F} \cong \mathcal{F}'$ on $S - \{s_1, \dots, s_\rho\}$. □

$(\bar{9})$ $\bar{A} \subseteq A$.

Proof. Follows from (7) since $\bar{A} \subseteq \bar{A}_K$. □

To summarize we have constructed a diagram

$$\begin{tikzcd} \bar{A} & R[x_1, \dots, x_n] \arrow[l] \\ R \arrow[u] \arrow[ur] & \end{tikzcd}$$

We also see that conditions $(1)''$ and $(2)''$ of 2.2.2 are satisfied by $(\bar{2})$, respectively $(\bar{1})$. That $\bar{A}$ is a finite R-algebra follows since A' is a finite R-algebra so $(3)''$ is also OK. Condition $(4)''$ follows from $(\bar{4})$. For the final condition $(5)''$ we need to check at for the two points of $\operatorname{Spec}(R)$ the corresponding fibers of $\bar{A}$ are almost everywhere isomorphic to its image. For the generic fiber this is a part of the assumption. Let $\mathfrak{m}_R = (\pi)$ be the closed point of $\operatorname{Spec}(R)$. We need to show that there exists a nonzero divisor $f \in \bar{B} \otimes_R \kappa(\mathfrak{m}_R)$ such that

$$\bar{B} \otimes_R \kappa(\mathfrak{m}_R)[1/f] \cong \bar{A} \otimes_R \kappa(\mathfrak{m})[1/f].$$

The f used in diagram (3.5.1.3) stays a nonzero divisor also after dividing out with $\mathfrak{m}_R$, and since $\bar{A}$ is flat over R it follows that the f used in diagram (3.5.1.3) also works for the special fiber. All in all we have now showed that

$$\begin{tikzcd} \bar{A} & R[x_1, \dots, x_n] \arrow[l] \\ R \arrow[u] \arrow[ur] & \end{tikzcd}$$

satisfies the affine situation in Subsection 2.2.2.

Next, we show that $\bar{A}$ is unique in the class of $R[x_1, \dots, x_n]$-algebras satisfying the conditions of the affine situation in Subsection 2.2.2. More precisely, assume $\bar{\bar{A}}$ is an $R[x_1, \dots, x_n]$-algebra that satisfies the following conditions:

(1)" finite as an $R[x_1, \ldots, x_n]$-module,

(2)" flat over R,

(3)" finite as an R-algebra,

(4)" for all $\mathfrak{p} \in$Spec(R) the ring $\bar{\bar{A}} \otimes_R \kappa(\mathfrak{p})$ is Cohen-Macaulay after localizing at its maximal ideals,

(5)" for all $\mathfrak{p} \in$Spec(R) the map Spec$(\bar{\bar{A}} \otimes_R \kappa(\mathfrak{p})) \to$ Spec$(\kappa(\mathfrak{p})[x_1, \ldots, x_n])$ is so that Spec$(\bar{\bar{A}})$ is an isomorphism onto its image away from finitely many closed points.

Finally, assume

$$\bar{\bar{A}} \otimes_R K = A. \tag{3.5.1.4}$$

From $\bar{\bar{A}}$ we can make a diagram similar to diagram (3.5.1.3), this is so because everything in diagram (3.5.1.3) comes from $K[x_1, \ldots, x_n] \to A$ and by assumption $\bar{\bar{A}}$ satisfies (3.5.1.4). It follows that $\bar{B} \to \bar{\bar{A}}$ and $R[x] \to \bar{\bar{A}}$ are finite homomorphisms. Since CM is a functor it follows that $\bar{\bar{A}}/(\pi)\bar{\bar{A}}$ is Cohen-Macaulay after localizing at all of its maximal ideals $\mathfrak{m}$; since all maximal ideals in $R[x]$ is of the form $(\pi, \mathfrak{m})$ and π is a nonzero divisor on $\bar{\bar{A}}$ it follows that $\bar{\bar{A}}$ is a Cohen-Macaulay $R[x]$-module. Since $\bar{A}$ and $\bar{\bar{A}}$ have the same generic fiber over R it follows that $\bar{A}$ and $\bar{\bar{A}}$ differ only in codimension 2 as $R[x]$-modules. Since $\bar{A}$ and $\bar{\bar{A}}$ are S_2 they are reflexive, and since Spec$(R[x])$ has dimension 2 reflexive is the same as locally free, hence, by $\bar{(}8)$ we get that $\bar{A} = \bar{\bar{A}}$ as $R[x]$-submodules of A, but then it follows that $\bar{A} = \bar{\bar{A}}$ as rings. We leave it to the reader to check that the uniqueness of $\bar{A}$ above shows that affine locally we can glue, that is, repeat the construction in 3.5.1.3 for each i and glue the different $\bar{A}$. By the nonexistence of nontrivial automorphisms (Theorem 3.2.0.14) the morphisms Spec$(\bar{A}) \to$Spec$(R[x_0/x_i, \ldots, x_n/x_i]$ glue to give a unique element $(C, \iota) \in CM(R)$. This finishes the proof of the valuative criterion for CM.

3.5.2 Finite type

Let (C, ι) be an element of $CM(k)$, that is a diagram

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d, "p"'] & \mathbb{P}^n_k \arrow[dl] \\ \mathrm{Spec}(k), & \end{tikzcd} \tag{3.5.2.1}$$

where k is a field extension of k_0.

Definition 3.5.2.1. Let $\{\mathcal{F}_\alpha\}$ be a family of isomorphism classes of sheaves such that for each α the sheaf $\mathcal{F}_\alpha$ is a coherent $\mathcal{O}_{X\otimes_{k_0}k_\alpha}$-module. We say that the family is *bounded* if there exists a k_0-scheme S of finite type and a coherent $\mathcal{O}_{X\times S}$-module $\mathcal{F}$ such that for each α there exists a k_α-valued point s_α of S such that $\mathcal{F}_\alpha \cong \mathcal{F}_{s_\alpha}$.

Theorem 3.5.2.2. *(Kleiman) Let $\{\mathcal{F}_\alpha\}$ be a collection of coherent sheaves on a projective scheme X/k_0, all with the same Hilbert polynomial P. Then this family is bounded if and only if there exists constants $C_0, \dots, C_d$, where $d =$ degree of P, such that for every α there exists an $\mathcal{F}_\alpha$-regular sequence of hyperplane-sections $H_1, \dots, H_d$ such that $h^0(\mathcal{F}|\bigcap_{j\leq i} H_j) \leq C_i$.*

Proof. See [HuLe] Theorem 1.7.8. □

We are going to show that the collection of sheaves $\{\iota_*\mathcal{O}_C\}$ is bounded using Theorem3.5.2.2. Since ι is finite $\iota_*\mathcal{O}_C$ is coherent on $\mathbb{P}^n$ and the cohomology of $\mathcal{O}_C$ and $\iota_*\mathcal{O}_C$ is also the same, in particular the two sheaves have the same Hilbert polynomial.

In our case the conditions of Theorem 3.5.2.2 amounts to the following two conditions: For all $\iota_*\mathcal{O}_C$

1. Exists a constant C_0 such that $h^0(\iota_*\mathcal{O}_C|H) \leq C_0$ for some regular hyperplane section H on $\iota_*\mathcal{O}_C$, and
2. Exists a constant C_1 such that $h^0(\iota_*\mathcal{O}_C) \leq C_1$.

For (1) we can use $C_0 = d$ where d is the coefficient of the linear term in the Hilbert polynomial P. For (2) we observe that there is an exact sequence of $\mathcal{O}_{\mathbb{P}^n}$-modules

$$0 \longrightarrow \mathcal{O}_{\iota(C)} \longrightarrow \iota_*\mathcal{O}_C \longrightarrow Q \longrightarrow 0$$

where $\iota(C)$ denotes the scheme-theoretic image of C under ι and Q is a skyscraper sheaf of length l. By [ESchles] Theorem 3.2, the arithmetic genus of a locally Cohen-Macaulay curve of degree d in $\mathbb{P}^n$ is bounded from above by $1/2(d-1)(d-2)$. In our case the arithmetic genus of $\iota(C)$ is also bounded from below since the Hilbert polynomial of $\iota(C)$ is less than or equal to the Hilbert polynomial of C. It follows that there are only finitely many possibilities for the Hilbert polynomial of $\iota(C)$. Since $\iota_*\mathcal{O}_C$ has fixed Hilbert polynomial it follows that there are only finitely many possible values l can take. Since each $\iota(C)$ define a point in a finite list of Hilbert schemes $\underline{\mathrm{Hilb}}_{\mathbb{P}^n}^{p(t)-l}$, there is an integer N so that $H^1(\mathcal{O}_{\iota(C)}(N)) = 0$ for all $(C, \iota) \in CM(k)$ and all fields $k \supseteq k_0$.

Twisting 3.5.2 by N and taking the associated long-exact cohomology sequence yields that $h^0(\iota_*\mathcal{O}_C(N) = h^0(\mathcal{O}_{\iota(C)}(N)) + l$. It follows that $h^0(\iota_*\mathcal{O}_C(N))$ is bounded independent of C, this proves (2).

So far this shows that the set of sheaves coming from (3.5.2.1) is bounded, but since non-isomorphic algebras can have isomorphic underlying modules we are not quite done yet.

If we pick any particular (C, ι) then there is an N such that the sheaf $\mathcal{O}_C(N) = \iota^*\mathcal{O}_{\mathbb{P}^n}(N)$ is very ample. Next we show that such an N can be picked uniformly for all $(C, \iota) \in CM(k)$, all $k \supseteq k_0$.

Proposition 3.5.2.3. *There exists an integer N such that for all $(C, \iota) \in CM(k)$ the sheaf $\mathcal{O}_C(N)$ is very ample.*

Proof. Step1: First we remark that we can assume that $k = \bar{k}$ is algebraically closed since $\mathcal{O}_C(N)$ is very ample on C if and only if $\mathcal{O}_{C\otimes_{k_0}k}(N)$ is very ample on $C \otimes_{k_0} k$.
Step2: Assume k is algebraically closed. We will use [Hart] Chapter II, Proposition 7.3 to show that there is an N so that $\mathcal{O}_C(N)$ is very ample. Pick a closed point $p \in C$, we have an exact sequence

$$0 \to \mathfrak{m}_p \to \mathcal{O}_C \to \mathcal{O}_p \to 0 \qquad (3.5.2.2)$$

After taking push-forward we get an exact sequence

$$0 \to \iota_*\mathfrak{m}_p \to \iota_*\mathcal{O}_C \to \iota_*\mathcal{O}_p \to 0. \qquad (3.5.2.3)$$

The sheaf $\iota_*\mathcal{O}_C$ is a coherent $\mathcal{O}_{\mathbb{P}^n}$-module with Hilbert polynomial $p(t)$, hence the collection $\{\iota_*\mathcal{O}_C\}$ is bounded (by the existence of the Quot scheme). The coherent $\mathcal{O}_{\mathbb{P}^n}$-module $\iota_*\mathfrak{m}_p$ has Hilbert polynomial $p(t)-1$, and $H^0(\iota_*\mathfrak{m}_p|H) \subseteq H^0(\iota_*\mathcal{O}_C|H)$ for any hyperplane $H \subseteq \mathbb{P}^n$. It follows Theorem(3.5.2.2) that the collection $\{\iota_*\mathfrak{m}_p\}$ is bounded. In particular there exists an N such that for all $(\iota, C) \in CM(k)$ and for all $p \in C$ that $\iota_*(\mathfrak{m}_p)(N)$ is globally generated.

Fix such an N, and let $s_1, \dots s_{p(N)} \in \Gamma(\mathbb{P}^n, \iota_*\mathfrak{m}_p(N)) = \Gamma(C, \mathfrak{m}_p(N))$ be global sections that generate $\iota_*\mathfrak{m}_p$. If $q \in C$ is a closed point different from $p \in C$ then clearly not all s_i can vanish on q, so $\mathcal{O}_C$ separates points. Since $\mathfrak{m}_p(N)$ is globally generated it also follows that the s_i generate $\mathfrak{m}_p(N)/\mathfrak{m}_p(N)$, so $\mathcal{O}_C$ separates tangent vectors. By [Hart], Chapter II, Proposition 7.3, it follows that $\mathcal{O}_C(N)$ is very ample. $\square$

Fix such an N, this gives a closed immersion $C \hookrightarrow \mathbb{P}^{p(N)-1}$. Combined with ι we get a closed immersion $j\colon C \hookrightarrow \mathbb{P}^n \times \mathbb{P}^{p(N)-1}$. Fix the very ample sheaf $\mathcal{O}(1,1)$ on $\mathbb{P}^n \times \mathbb{P}^{p(N)-1}$. Now see that j defines a point of $\underline{\mathrm{Hilb}}^P_{\mathbb{P}^n \times \mathbb{P}^{p(N)-1}}$, for $P(t) := p((N+1)t)$.

Inside $\underline{\mathrm{Hilb}}^P_{\mathbb{P}^n \times \mathbb{P}^{p(N)-1}}$ the set consisting of Cohen-Macaulay schemes form an open subscheme (see [EGA] $\mathbf{IV_3}$, Théorème 12.2.1. (*vii*)). Let us call the open subscheme U, so $U \subseteq \underline{\mathrm{Hilb}}^P_{\mathbb{P}^n \times \mathbb{P}^{p(N)-1}}$. Next, let $V \subseteq U$ be the set of points corresponding to curves $C \subseteq \mathbb{P}^n \times \mathbb{P}^{p(N)-1}$ such that $pr_1\colon C \longrightarrow \mathbb{P}^n$ is finite.

Claim: V is open.

This the reader can see by looking at the universal projection $pr_1\colon \mathcal{C} \to \mathbb{P}^n_U$ where $\mathcal{C} \to U$ is the universal family over U. Namely, the set Z of points t in $\mathbb{P}^n_U$ where $pr_2^{-1}(t)$ has dimension ≥ 1 is closed, and then observe $V = U -$ image of Z in U. We now denote $\mathcal{C} \to V$ the universal family. Let $W \subseteq V$ be the set of points corresponding to $C \subseteq \mathbb{P}^n \times \mathbb{P}^{p(N)-1}$ such that $pr_1\colon C \to \mathbb{P}^n$ is an isomorphism away from finitely many points.

Claim: W is open.(If so then we are done.)

To prove this look at the universal map $\mathcal{C} \to \mathbb{P}^n_V$. By our choice of V this is finite. Let's consider the map of $\mathcal{O}_{\mathbb{P}^n_V}$-modules

$$\mathcal{O}_{\mathbb{P}^n_V} \to pr_{1*}(\mathcal{O}_{\mathcal{C}}). \tag{3.5.2.4}$$

The support T of the cokernel of the map in (3.5.2.4) is a closed subscheme of $\mathbb{P}^n_V$. The set W is the set of points $v \in V$ such that $\dim(T_v) \leq 0$, which is open.

All in all this shows that there is an open subscheme W of $\underline{\mathrm{Hilb}}(\mathbb{P}^n \times \mathbb{P}^{p(N)-1})$ such that $T \to CM$ and this map is a surjection on all k-points for all fields $k \supseteq k_0$. We have now shown the following.

Theorem 3.5.2.4. *CM is of finite type over k_0.*

3.6 Relative representability

Let S be a scheme over k_0, and fix two elements $\alpha_1 = (C_1, \iota_1), \alpha_2 = (C_2, \iota_2)$ in $CM(S)$. We need to show that the equality $\alpha_1 = \alpha_2$ is represented by a closed subscheme of S. Let $I := Isom_S((C_1, \iota_1), (C_2, \iota_2))$ denote the contravariant functor that to a scheme T over S assigns the set of all isomorphisms $f\colon C_{1,T} \longrightarrow C_{2,T}$ making the following diagram commute:

$$\begin{tikzcd}
 & \mathbb{P}^n_T \arrow[rr, phantom, "="] \arrow[ddl] & & \mathbb{P}^n_T \arrow[ddl] \\
C_{1,T} \arrow[ur, "\iota_{1,T}"] \arrow[rr, "f"] \arrow[d, "p"'] & & C_{2,T} \arrow[ur, "\iota_{2,T}"] \arrow[d, "p'"'] & \\
T \arrow[rr, phantom, "="] & & T &
\end{tikzcd}$$

We have a cartesian diagram of functors

$$\begin{tikzcd}
Isom(C_1, C_2) \arrow[r, "\alpha"] & Mor(C_1, \mathbb{P}^n_S) \times Mor(C_1, \mathbb{P}^n_S) \\
I \arrow[u] \arrow[r] & Mor_S(C_1, \mathbb{P}^n_S) \arrow[u, "\Delta"']
\end{tikzcd}$$

Where Δ is the diagonal map and α is the map that to an isomorphism

$$f_T\colon C_{1,T} \to C_{2.,T} \text{ of } Isom(C_1, C_2)(T)$$

assigns the element

$$(\iota_{1,T}, \iota_{1,T} \circ f_T^{-1}) \text{ of } Mor(C_1, \mathbb{P}^n_S)(T) \times Mor(C_1, \mathbb{P}^n_S)(T).$$

Since $Isom$ and Mor are represented by schemes locally of finite presentation over S (see [FAG], 221-19), it follows that I is represented by a scheme $\underline{\mathrm{I}} \to S$ locally of finite presentation.

From 3.2.0.14 we know that any object of $CM(R)$, for R any ring, has only the identity as an automorphism. In particualar this is true for R a field or the dual numbers. It follows that $j : \underline{\mathrm{I}} \to S$ is unramified. On the other hand, the valuative criterion of properness holds by Section 3.5. By [EGA] IV_4, Cor. 18.12.6 if we show that for each $s \in S$ the scheme $j^{-1}(s)$ is either empty or isomorphic to Spec($\kappa(s)$) then it follows that j is an closed immersion. Let $s \in S$ be any point, if $j^{-1}(s)$ is empty then we are done so assume the fiber is not empty. We have the following cartesian diagram:

$$\begin{tikzcd}
\underline{\mathrm{I}} \arrow[d] & \underline{\mathrm{I}}_s \arrow[l] \arrow[d] & \underline{\mathrm{I}}_{\bar{s}} \arrow[l] \arrow[d] \\
S & \mathrm{Spec}(\kappa(s)) \arrow[l] & \mathrm{Spec}(\overline{\kappa(s)}) \arrow[l]
\end{tikzcd}$$

where $\overline{\kappa(s)}$ is the algebraic closure of $\kappa(s)$ and $\bar{s}$ =Spec($\overline{\kappa(s)}$). The middle vertical map is an isomorphism if and only if the right map is an isomorphism. The right map is an isomorphism by the non-existence of nontrivial automorphisms. It follows that j is a closed immersion.

To summarize we have shown the following.

Theorem 3.6.0.5. *CM is relatively representable.*

3.7 Openness of formal étalness

In this subsection we prove the last criterion in Artin's theorem. Recall the following definition.

Definition 3.7.0.6. (formally étale)

Let $\xi \in CM(X)$, we say that ξ is *formally étale* at $x \in X$ if for every commutative diagram of solid arrows

$$\begin{tikzcd}
X \arrow[d, "\xi"'] & C_0 \arrow[l, "f_0"'] \arrow[d, hook] \\
F & C \arrow[l] \arrow[ul, dotted, "f"']
\end{tikzcd}$$

where Z is the spectrum of a local artinian k_0-algebra, Z_0 is a closed subscheme of Z defined by a nilpotent ideal, and f_0 is a map sending Z_0 to x set-theoretically, there exists a unique dotted arrow making the diagram commute.

Consider the following situation:

Set-up. Let X a scheme of finite type over k_0. Let $\xi \in CM(X)$ be a family

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d] & \mathbb{P}^n_X \arrow[dl] \\ X & \end{tikzcd}$$

then we have the following result.

Theorem 3.7.0.7. *(openness of formal étaleness) Let X be a k_0-scheme of finite type, and let $\xi\colon X \to CM$ be a map. If ξ is formally étale at a point $x_0 \in X$, then it is formally étale in a open neighborhood of x_0.*

Proof. The proof goes in several steps.

Let x_0 be a point of X.

Step(0): Clearly we may assume X is some affine scheme $\mathrm{Spec}(A)$.

Step(1): There exists an N such that $i^*\mathcal{O}_{\mathbb{P}^n}(N)$ is globally generated and very ample, let $s_0, \dots, s_{p(N)} \in \Gamma(C, \iota^*\mathcal{O}_{\mathbb{P}^n}(N))$ be a minimal generating set over $\Gamma(X, \mathcal{O}_X)$. If we localize at the prime ideal corresponding to x_0 the flat $\Gamma(X, \mathcal{O}_X)$-module $\Gamma(C, \iota^*\mathcal{O}_{\mathbb{P}^n}(N))$ becomes free, and by a standard "clearing annihilators" argument the s_i remain a basis even for an honest affine neighborhood, hence we may assume the module $\Gamma(C, \iota^*\mathcal{O}_{\mathbb{P}^n}(N))$ is free over $\Gamma(X, \mathcal{O}_X)$.

Step(2): Using the sections $\{s_i\}$ from Step(1) to produce a family

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathbb{P}^n_X \times_X \mathbb{P}^{p(N)-1}_X \arrow[dl] \\ X & \end{tikzcd}$$

and since this family is flat over X we get a morphism

$$\xi_H \colon X \to \underline{\mathrm{Hilb}}^P(\mathbb{P}^n_{k_0} \times \mathbb{P}^{p(N)-1}_{k_0}) =: \underline{\mathrm{Hilb}}.$$

Step(3): $\mathrm{PGL}_{p(N)}$ acts on $\mathbb{P}^{p(N)-1}$ and produces a morphism

$$\bar{\xi}\colon X \times PGL_{p(N)} \to \underline{\mathrm{Hilb}}$$

Proposition 3.7.0.8. *For any point $x \in X$ the following are equivalent.*

1. *$\bar{\xi}$ is formally étale at $(x, 1)$.*

2. *ξ is formally étale at x.*

Proof. Lets prove (i) implies (ii) first. We have a commutative diagram

$$\begin{tikzcd}
\mathbb{P}^n_X \arrow[dr] & C \arrow[l, "\iota"'] \arrow[d] & & \\
& X \arrow[d, "\xi"'] & Z_0 \arrow[l, "f_0"'] \arrow[d, hook] & \\
& CM & Z \arrow[l] \arrow[ul, dashed, "?"'] & \\
& & C_Z \arrow[u] \arrow[r, "\iota_C"'] & \mathbb{P}^n_Z \arrow[ul]
\end{tikzcd} \tag{3.7.0.5}$$

where Z is the spectrum of a local artinian ring, and $Z_0 \subseteq Z$ is a closed subscheme mapping to x_0 via f_0, finally let C_Z denote the pullback of C to Z. Pull back the $s_0, \ldots, s_{p(N)-1}$ that we picked in Step (1) to Z_0 (i.e. we are mapping ξ_H to ξ_{H,Z_0}), and call the pullbacks $s_i|_{Z_0}$. We can find $s_{0,Z}, \ldots, s_{p(N)-1,Z} \in \Gamma(C_Z, \iota^* \mathcal{O}_{\mathbb{P}^n_Z}(N))$ lifting the $s_i|_{Z_0}$, and by Nakayamas Lemma they still form a basis. Define a morphism

$$C_Z \stackrel{(\iota_Z; s_{i,Z})}{\longrightarrow} \mathbb{P}^n_Z \times_Z \mathbb{P}^{p(N)-1}_Z$$

this gives a Z-valued point $\xi_{H,Z}$ of $\underline{\mathrm{Hilb}}$ that reduces to the Z_0-valued point ξ_{H,Z_0}, i.e.

$$\begin{tikzcd}
Z_0 \arrow[r] & X \arrow[r] & X \times \{1\} \arrow[r] \arrow[dr, hook] & \underline{\mathrm{Hilb}} \\
& & & X \times PGL \arrow[u, "\bar{\xi}"']
\end{tikzcd}$$

By (1) this lifts to some morphism $\psi\colon Z \longrightarrow X \times PGL_{p(N)}$ that restricts to $f_0 \times 1$ on Z_0. The desired $\dashrightarrow$ in (3.7.0.5) is just the composition $Z \to X \times PGL \to X$.

For (ii) implies (i): we are given a solid diagram

$$\begin{tikzcd} X \times PGL_{p(N)} \arrow[r] & \underline{\text{Hilb}} \\ Z_0 \arrow[u, "{(f_0,g_0)}"] \arrow[r, hook] & Z \arrow[u] \arrow[ul, dashed] \end{tikzcd} \tag{3.7.0.6}$$

The map $Z \to \underline{\text{Hilb}}$ is given by a closed immersion $C_Z \to \mathbb{P}^n_Z \times_Z \mathbb{P}^{p(N)-1}_Z$ with properties so that when we project to the first factor we get an element of CM (fill in, clearify). Condition (ii) means that there exists a morphism $f\colon Z \to X$ such that $f|_{Z_0} = f_0$ and an isomorphism $f^*C \to C_Z$ compatible with the morphisms to $\mathbb{P}^n_Z$. The commutativity of the solid diagram (3.7.0.6) means that the following vector-relation hold:

$$(s_{i,Z}|_{Z_0}) \cong \tilde{g}_0 \cdot (f_0^* s_i),$$

where $\tilde{g}_0$ is some lift of g_0 to an element of $GL_{p(N}$. Since $(s_{i,Z})$ and (f^*s_i) for $i = 0, \ldots, p(N)-1$, are bases for $\Gamma(C_Z, i_Z^*\mathcal{O}(N)$ there is a unique matrix $\tilde{g}$ in $Gl_{p(N}$ such that the following vector-relation hold

$$(s_{i,Z}) = \tilde{g} \cdot (f^* s_i),$$

let g be the image of $\tilde{g}$ in PGL. We leave it to the reader to check that the morphism $(f,g)\colon Z \to X \times PGL_{p(N)}$ is the solution. □

To finish the proof of the openness of formal étaleness for ξ around x_0 we need only observe that $\bar{\xi}$ satisfies openness of étaleness around $(x_0, 1)$ because $X \times PGL_{p(N)}$ is a scheme of finite type over k_0. □

This ends the proof of the representability (as an algebraic space) of CM.

Remarks 3.7.0.9. The functor CM can be naturally defined over $\mathbb{Z}$, we have not checked all conditions, but we think the corresponding functor will be represented by an algebraic space in that case as well.

A natural question to ask: Is CM representable by a scheme? We have no counterexamples.

Chapter 4

Applications

In this section we include two applications of the representability of CM; first we show that for any variety there exists a Macaulayfication, secondly, we illustrate some of the differences between CM and $Hilb$, by looking at the Borel-fixed points of the two spaces.

4.1 Macaulayfication

In this section we prove a result about the existence of a Macaulayfication. More precisely, a *Macaulayfication* of a variety X is a proper birational morphism $X' \to X$ where X' is Cohen-Macaulay. Our result is not new, in fact it is already know to hold in greater generality: Kawasaki has shown [Kawa] that separated schemes of finite type over a Noetherian base ring with a dualizing complex has a Macaulayfication, but we think our proof is simpler. Actually, our proof is a straightforward adaption of some of the ideas used by de Jong in his seminal paper on alterations [deJong].

Theorem 4.1.0.10. *For any variety X over any field k there exists a Macaulayfication.*

Proof. Since we are proving something about birationality we can freely replace X by birational varieties that are proper over X. The proof is by a sequence of steps, each making the situation nicer.

Step(1): We work by induction on the dimension of X, the cases dim X=0 and 1 are OK by the existence of resolution of singularities.

Step(2): Let X be a variety of dimension d and assume the theorem is proved for dimensions less than d. By Chow's Lemma ([EGA] **II**, Théorème 5.6.1) we can assume the

X is quasi-projective.

Step(3): Let $X \subseteq \bar{X}$ be a projective completion, the theorem will follow for X if we can prove it for $\bar{X}$, hence we may assume that X is projective.

Step(4): Choose a very ample line bundle $\mathcal{O}(1)$ on X.

Step(5): Choose global sections s_i for $i = 0, \ldots, d-1$ of $\mathcal{O}(1)$ that are linearly independent and such that the common zero set has dimension 0. If k is a finite field this might be impossible, but it will always be possible after replacing $\mathcal{O}(1)$ with $\mathcal{O}(n)$ for some large n.

Step(6): Let $Bl(X) \to X$ be the blow-up of X in the common zero-set of the sections s_i. We obtain a dominant morphism $Bl(X) \to \mathbb{P}^{d-1}$, whose fibers are all one-dimensional.

Proof. The fibers are all one-dimensional since they are scheme-theoretically the intersections of $d-1$ sections of $\mathcal{O}(1)$, and upon cutting by one more section of $\mathcal{O}(1)$ you get something of dimension zero. □

Step(7): Replace X by $Bl(X)$. Hence we can assume we have a morphism $\pi \colon X \to \mathbb{P}^{d-1}$ as above. Choose a closed immersion $i \colon X \to \mathbb{P}^n$. Since π is generically flat and the generic fiber is reduced (because X is a variety, hence reduced, and the generic fiber is a limit of localizations of X), we see that over a nonempty open U of $\mathbb{P}^{d-1}$ we get a morphism

$$U \longrightarrow CM$$

so that the curve $\pi^{-1}(U) \to U$ is the pullback of the universal curve over CM.

Step(8): Note that the image of the the closed immersion $(\pi, i) \colon X \to \mathbb{P}^{d-1} \times \mathbb{P}^n$ is the Zariski closure of the flat family over U.

Step(9): Since CM is a proper algebraic space the closure W of the locally closed morphism $U \to \mathbb{P}^{d-1} \times CM$ is a proper algebraic space containing U as an open dense subvariety. Since W maps into CM we can pull back the universal family over CM to W, call it $C \to W$. Since C is flat over W, the restriction C_U of C to U is schematically dense in C and we conclude by Step 8 that C maps into X (thought of as a subvariety of $\mathbb{P}^{d-1} \times \mathbb{P}^n$).

Step(10): Now we may replace X by C and $\mathbb{P}^{d-1}$ by W, so that now X is a flat family of Cohen-Macaulay curves over a proper base variety. The only (slight) problem is that W is a proper algebraic space, and not a variety. This is not a serious issue since there is a Chow's Lemma for algebraic spaces (see Step(2)) that allows us to replace W by a projective variety.

Step(11): By the induction hypothesis there is a Macaulayfication $W' \to W$ with W' a Cohen-Macaulay variety and pull back the flat family to get $X' \to W'$ flat with Cohen-Macaulay fibers of dimension one, hence by ([BrHe] Proposition 2.1.16 p.63) X' is Cohen-Macaulay. □

4.2 A comparison of the Hilbert and Cohen-Macaulay functors for rational quartics

In this section we illustrate some of the differences between the two functors in the title by looking at the Borel-fixed ideals for each functor. In some sense this gives information about how "bad" degenerations that can occur on the "boundary of each functor."

4.2.1 The Hilbert scheme of rational quartics

Let H denote the Hilbert scheme of subschemes of $\mathbb{P}^n$ with Hilbert polynomial $4t+1$.

Proposition 4.2.1.1. *H has four irreducible components $H_1, \ldots, H_4$, they have generic points corresponding to the following subschemes of $\mathbb{P}^n$:*

(H1): rational quartic,

(H2): plane cubic disjoint union a line,

(H3): genus one quartic (in a $\mathbb{P}^3$) disjoint union a point,

(H4): plane quartic disjoint union 3 points.

Proof. See [MDP] □

Remark 4.2.1.2. We see that only H_1 and H_2 have generic points corresponding to subschemes of pure dimension.

In the remainder of this subsection we work over a basefield k_0 of characteristic zero, not because it is necessary, but to make the notation easier. Let $R := k_0[x_0, \ldots, x_n]$ be the polynomial ring in the variables $x_0, \ldots, x_n$ over k_0. Let G be a group of $n+1$ by $n+1$

matrices. An ideal $I \subseteq R$ is said to be *G-invariant* if $gI = I$ for all $g \in G$, where $g = (a_{ij})$ acts on R via the transformation $x_j \mapsto \sum_{k=0}^{n} a_{kj} x_k$. We let B denote the group of upper-triangular matrices and call it the *Borel group*, the group of diagonal matrices is denoted by D. The basic result about the structure of Borel-invariant ideals is the following.

Theorem 4.2.1.3. *Let I be an ideal in R. Then the following holds.*

1. *The ideal I is D-invariant if and only if it is generated by monomials.*

2. *A D-invariant ideal I is B-invariant if and only if the following condition is satified for all monomials $m \in I$: If x_j^r is the greatest power of x_j that divides m then for all $i < j$ and all $s < r$ $(\frac{x_i}{x_j})^s m \in I$.*

Proof. [Eisen] Chapter 15. $\square$

Example 4.2.1.4. Consider the polynomial ring $k_0[x_0, \ldots, x_4]$. Theorem 4.2.1.3 implies that, for example, if $x_1^2 \in I$, then x_0^2, x_0x_1 also has to be in I. It is then easy, but tediuos, to make the following list (in degree 2).

Monomial	Generated monomials	Monomial	Generated monomial
x_0^2	x_0^2	x_2^2	$x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2$
x_0x_1	x_0^2, x_0x_1	x_2x_3	$x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2,$ x_1x_3, x_2^2, x_2x_3
x_0x_2	x_0^2, x_0x_1, x_0x_2	x_2x_4	$x_0^2, x_0x_1, x_0x_2, x_0x_3, x_0x_4, x_1^2$ $x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_2x_4$
x_0x_3	$x_0^2, x_0x_1, x_0x_2, x_0x_3$	x_3^2	$x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2$ $x_1x_3, x_2^2, x_2x_3, x_3^2$
x_0x_4	$x_0^2, x_0x_1, x_0x_2, x_0x_3, x_0x_4$	x_3x_4	$x_0^2, x_0x_1, x_0x_2, x_0x_3, x_0x_4, x_1^2$ $x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_2x_4$ x_3^2, x_3x_4
x_1^2	x_0^2, x_0x_1, x_1^2	x_4^2	$x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_2x_4$ x_3^2, x_3x_4, x_4^2
x_1x_2	$x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2$		
x_1x_3	$x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2,$ x_1x_3		
x_1x_4	$x_0^2, x_0x_1, x_0x_2, x_0x_3, x_0x_4, x_1^2,$ x_1x_2, x_1x_3, x_1x_4		

To classify all B-invariant ideals $I \subseteq R$, generated in degree 2, is now easy; if m is a monomial generator in I, then I has to contain all the "generated" monomials (see table above in case $n = 4$). Hence, to find all B-invariant ideals, generated in degree 2, in $k[x_0, \ldots, x_4]$ we simply have to find all finite ideal sums of the ideals above. Equivalently, we can find all finite (distinct) unions of the sets of generators in the table above.

This method isn't very efficient so luckily there are other ways to find Borel-fixed ideals. In her thesis Allyson Reeves (see [Reeves]) has a program that can compute all Borel-fixed ideals with a given Hilbert polynomial, and as an example she computes all Borel-fixed ideals with Hilbert polynomial $4t + 1$ in $\mathbb{P}^4$. (Note that these are not always generated in degree 2.)

Example 4.2.1.5. (see [Reeves]) Borel-fixed ideals in $k_0[x_0, \ldots, x_4]$ with Hilbert polynomial

$4t+1$.

(Ideal 1): $(x_0, x_1, x_2^5, x_2^4x_3^3) = (x_0, x_1, x_2^4) \cap (x_0, x_1, x_2^5, x_3^3)$

(Ideal 2): $(x_0, x_1^2, x_1x_2, x_1x_3, x_2^5, x_2^4x_3^2) = (x_0, x_1, x_2^4) \cap (x_0, x_1, x_2^5, x_3^2) \cap (x_0, x_1^2, x_2, x_3)$

(Ideal 3): $(x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_1x_3, x_2^5, x_2^4x_3) = (x_0, x_1, x_2^4) \cap (x_0, x_1, x_2^5, x_3) \cap (x_0, x_1^2, x_2, x_3) \cap (x_0^2, x_1, x_2, x_3)$

(Ideal 4): $(x_0, x_1^2, x_1x_2, x_1x_3^2, x_2^5, x_2^4x_3) = (x_0, x_1, x_2^4) \cap (x_0, x_1^2, x_2, x_3^2) \cap (x_0, x_1, x_2^5, x_3)$

(Ideal 5): $(x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_1x_3^2, x_2^4) = (x_0, x_1, x_2^4) \cap (x_0, x_1^2, x_2, x_3^2) \cap (x_0^2, x_1, x_2, x_3)$

(Ideal 6): $(x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_2^3) = (x_0, x_1, x_2^3) \cap (x_0, x_1^2, x_2) \cap (x_0^2, x_1, x_2, x_3)$

(Ideal 7): $(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2)$

(Ideal 8): $(x_0, x_1^2, x_1x_2^2, x_1x_2x_3, x_1x_3^2, x_2^4) = (x_0, x_1, x_2^4) \cap (x_0, x_1^2, x_2, x_3^2) \cap (x_0, x_1^2, x_2^2, x_3)$

(Ideal 9): $(x_0, x_1^2, x_1x_2^2, x_1x_2x_3, x_2^3) = (x_0, x_1, x_2^3) \cap (x_0, x_1^2, x_2) \cap (x_0, x_1^2, x_2^2, x_3)$

(Ideal 10): $(x_0, x_1^2, x_1x_2, x_2^4, x_2^3x_3) = (x_0, x_1, x_2^3) \cap (x_0, x_1^2, x_2) \cap (x_0, x_1, x_2^4, x_3)$

(Ideal 11): $(x_0, x_1^2, x_1x_2, x_1x_3^3, x_2^4) = (x_0, x_1, x_2^4) \cap (x_0, x_1^2, x_2, x_3^3)$

(Ideal 12): $(x_0, x_1, x_2^6, x_2^5x_3, x_2^4x_3^2) = (x_0, x_1, x_2^4) \cap (x_0, x_1, x_2^5, x_3^2) \cap (x_0, x_1, x_2^6, x_3)$

It seems to be difficult to decide which Borel-fixed ideals lies on which component(s) of the Hilbert scheme. For example a rational quartic in $\mathbb{P}^4$ is given as the two by two minors of a two by four matrix with independent linear forms as entries, hence the rational quartic is given by 6 quadratic equations. Since cohomology is upper-semi continuous any (flat) degeneration of a rational quartic must have at least 6 quadratic equations in it's ideal. Furthermore, a plane degree 4 curve have Hilbert polynomial $4t-2$, hence a degeneration of a rational quartic to a curve with support in a plane will necessarily have 3 embedded point (not necessarily distinct). Furthermore these embedded points have to point out of the plane.

Combining these observations we see that we can rule out ideals with two linear equations, or ideals with less than 6 quadratic equations. In the list above this only excludes ideals 1 and 12. Only ideal 7 is Cohen-Macaulay.

4.2.2 The Cohen-Macaulay functor for rational quartics

A Borel-fixed point of $CM_{\mathbb{P}^4}^{4t+1}$ corresponds to a diagram

$$\begin{tikzcd} C \arrow[rr, "\iota"] \arrow[dr, "i"'] & & \mathbb{P}^4 \\ & \bar{C} \arrow[ur, hook] & \end{tikzcd}$$

Where C is Borel-fixed, and hence $\bar{C}$, the scheme-theoretic image, is Borel-fixed. $\bar{C}$ is a Cohen-Macaulay curve of degree 4 in $\mathbb{P}^4$.

Lemma 4.2.2.1. *The Hilbert polynomial of $\bar{C}$ is $4t + l$, where $l = -2, -1, 0, 1$.*

Proof. We have an exact sequence

$$0 \to \mathcal{O}_{\bar{C}} \to i_* \mathcal{O}_C \to \mathcal{K} \to 0$$

where $\mathcal{K}$ has zero-dimesnional support. It follows that $\bar{C}$ has Hilbert polynomial $4t+l$ where $l \leq 1$. Let $I_{\bar{C}}$ denote the (saturated) ideal of $\bar{C}$. Since $I_{\bar{C}}$ is saturated and Borel-fixed it follows that no generator of $I_{\bar{C}}$ has any x_4-factor. Furthermore, since $\bar{C}$ is Cohen-Macaulay, it follows that there does not exist any monomial m in $k_0[x_0, \dots, x_4]$ such that $x_3 m \in I_{\bar{C}}$, but m is not in $I_{\bar{A}}$. Hence $I_{\bar{C}}$ has generators only involving x_0, x_1, x_2. Consider the k_0-vector space $k_0[x_0, x_1, x_2]/I_{\bar{C}}$. This is of dimension 4 because $\bar{C}$ has degree 4, and it is easy to write down all possible bases it can have. Each choice of basis corresponds to a different $\bar{C}$ and an easy check shows that $l = -2$ is the minimal value. □

By using A. Reeves' software package 'Borel' one can find that the possible ideals in each case:

Case 1: Hilbert polynomial $4t - 2$

1. (x_0, x_1, x_2^4), this corresponds to a Cohen-Macaulay scheme.

Case 2: Hilbert polynomial $4t - 1$

1. $(x_0, x_1, x_2^5, x_2^4 x_3) = (x_0, x_1, x_2^4) \cap (x_0, x_1, x_2^5, x_3)$, corresponds to a non Cohen-Macaulay.
2. $x_0, x_1^2, x_1 x_2, x_1 x_3, x_2^4) = (x_0, x_1, x_2^4) \cap (x_0, x_1^2, x_2, x_3)$, corresponds to a non Cohen-Macaulay.

Case3: Hilbert polynomial $4t$

1. $(x_0, x_1, x_2^5, x_2^4x_3^2) = (x_0, x_1, x_2^4) \cap (x_0, x_1, x_2^5, x_3^2)$, corresponds to a non Cohen-Macaulay scheme.

2. $(x_0, x_1^2, x_1x_2, x_1x_3, x_2^5, x_2^4x_3) = (x_0, x_1, x_2^4) \cap (x_0, x_1, x_2^5, x_3) \cap (x_0, x_1^2, x_2, x_3)$, corresponds to a non Cohen-Macaulay.

3. $(x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_1x_3, x_2^4) = (x_0, x_1, x_2^4) \cap (x_0, x_1^2, x_2, x_3^2)$, corresponds to a non Cohen-Macaulay.

4. $(x_0, x_1^2, x_1x_2, x_1x_3^2, x_2^4) = (x_0, x_1, x_2^4) \cap (x_0, x_1^2, x_2, x_3^2)$, corresponds to a non Cohen-Macaulay.

5. $(x_0x_1^2, x_1x_2, x_2^3) = (x_0, x_1, x_2^3) \cap (x_0, x_1^2, x_2)$, corresponds to a non Cohen-Macaulay.

Case4: Hilbert polynomial $4t+1$. For the list of ideals see Example 4.2.1.5. Notice that only Ideal 7, i.e. the ideal $(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2)$ corresponds to a Cohen-Macaulay scheme.

The upshot of this analysis is that $\bar{C}$ is either $(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2)$, which corresponds to ι being a closed immersion, or (x_0, x_1, x_2^4), which corresponds to the image of ι has Hilbert polynomial $4t-2$.

Let us study this last possibility. So assume that $\bar{C} \subseteq \mathbb{P}^4$ is the scheme

$$\mathrm{Proj}(k_0[x_0, x_1, x_2, x_3, x_4]/(x_0, x_1, x_2^4) = \mathrm{Proj}(k_0[c, d, e]/c^4).$$

This curve has Hilbert polynomial $4t-2$. We want to find all possible finite maps $C \to \bar{C}$ that define B-invariant points of $CM_{\mathbb{P}^4}^{4t+1}$. Hence $\mathcal{K}$ in the proof of Lemma 4.2.2.1 has length 3. Lets indicate two cases. **Case 1:** Assume we add the elements c^3/d^2 and c^2/d to $k_0[c, d, e]/c^4$. More precisely, consider the ring extension $k_0[c, d, e]/(c^4) \subseteq k_0[c, d, e]/(c^4)[c^3/d^2, c^2/d] =: R$, where the latter ring is considered a subring of the total ring of fractions Q of $k_0[c, d, e]/(c^4)$. The Borel-group B acts on Q in a natural way induced by the action on $k_0[c, d, e]/(c^4)$.

Lemma 4.2.2.2. *The ring R is B-invariant as a subring of Q.*

Proof. Assume $c \mapsto \alpha_1 c$ and $d \mapsto \alpha_2 c + \alpha_3 d$ under the action of some element $\begin{pmatrix} \alpha_1 & \alpha_3 \\ 0 & \alpha_2 \end{pmatrix}$ of B. We need to show that c^3/d^2 and c^2/d stays inside R under the above transformation. We can assume $\alpha_1 = \alpha_2 = \alpha_3 = 1$ since multipling with a scalar does not change a fraction in any essential way. We have

1. $\frac{c^3}{d^2} \mapsto \frac{c^3}{(c+d)^2} = \frac{c^3}{d^2(1+\frac{c}{d})^2} \overset{c^4=0}{=} \frac{c^3}{d^2}(1 - \frac{c}{d} + \frac{c^2}{d^2} - \frac{c^3}{d^3}) \overset{c^4=0}{=} \frac{c^3}{d^2}$
2. $\frac{c^2}{d} \mapsto \frac{c^2}{c+d} = \frac{c^2}{d(1+\frac{c}{d})} \overset{c^4=0}{=} \frac{c^2}{d}(1 - \frac{c}{d} + \frac{c^2}{d^2} - \frac{c^3}{d^3}) \overset{c^4=0}{=} \frac{c^2}{d} + \frac{c^3}{d^2}$ which is OK by the item above.

$\square$

Let $u := c^3/d^2$ and $v := c^2/d$. Using the sections $x_0 = c, x_1 = d, x_2 = e, x_3 = u, x_4 = v$ we get an embeding of C into $\mathbb{P}^4$ as

$$C = \operatorname{Proj}(k_0[x_0, x_1, x_2, x_3, x_4]/(x_0x_4, x_3^2, x_3x_4, x_4^2, x_1x_3 - x_0^2, x_0x_3 - x_1x_4)).$$

This defines $C \to \operatorname{Proj}(k_0[x_0, x_1, x_2, x_3.x_4]/(x_0, x_1, x_2^4) \subseteq \mathbb{P}^4$ as a B-invariant point of $CM^{4t+1}_{\mathbb{P}^4}$.

Case 2: Assume we add the element $u := c^3/d^3$, that is, consider the ring extension $k_0[c, d, e]/(c^4) \subseteq k_0[c, d, e]/(c^4)[\frac{c^3}{d^3}] := R'$. Where R' is considered a subring of Q.

Lemma 4.2.2.3. *The ring R' is B-invariant as a subring of Q.*

Proof. The proof is the same as the proof Lemma 4.2.2.2. $\square$

Using the sections $x_0 = c, x_1 = d, x_2 = e, x_3 = du, x_4 = eu$ then we have quadratic relations

$$x_3^2, x_3x_4, x_4^2, x_0x_3, x_0x_4, x_1x_4 - x_2x_3$$

and one qubic relation

$$x_1^2x_3 - x_0^3.$$

Hence we get we get an embeding of C into $\mathbb{P}^4$ as

$$C = \operatorname{Proj}(k_0[x_0, x_1, x_2, x_3, x_4]/(x_3^2, x_3x_4, x_4^2, x_0x_3, x_0x_4, x_1x_4 - x_2x_3, x_1^2x_3 - x_0^3))$$

and

$$C \to \mathrm{Proj}(k_0[x_0, x_1, x_2, x_3.x_4]/(x_0, x_1, x_2^4) \subseteq \mathbb{P}^4$$

defines a B-invariant point of $CM_{\mathbb{P}^4}^{4t+1}$.

We think the two cases considered above are the only ones. If so, it means that $CM_{\mathbb{P}^4}^{4t+1}$ has precisely two Borel-fixed points whereas $\underline{\mathrm{Hilb}}_{\mathbb{P}^4}^{4t+1}$ has 12.

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