

# Maps and localizations in the category of Segal spaces

by

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B.A. (Hons.), University of Cambridge (1999)

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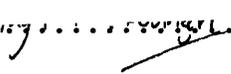
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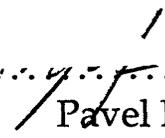
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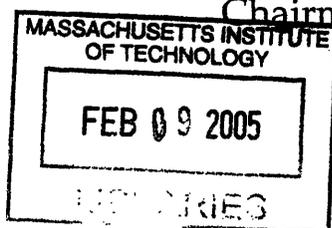
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## Abstract

The category of Segal spaces was proposed by Charles Rezk in 2000 as a suitable candidate for a model category for homotopy theories. We show that Quillen functors induce morphisms in this category and that the morphisms induced by Quillen pairs are “adjoint” in a useful sense. Quillen’s original total derived functors are then obtained as a suitable localization of these morphisms within the category of Segal spaces.

As an application, we consider a construction of “homotopy fibres” within a homotopy theory modelled by a Segal space and show that the homotopy fibre of a map is preserved by a localization which remembers only the homotopy category plus the automorphism groups of objects.

Thesis Supervisor: Haynes R. Miller  
Title: Professor of Mathematics

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# Chapter 1

## Preliminaries

We will use this chapter to reiterate basic definitions which can be found in [Rezk]. We will then go on to establish some essential technical results which will be required in subsequent chapters.

### 1.1 Notation and nomenclature

By a “space” we will always mean a simplicial set, which will almost always be Kan fibrant. The model structure on bisimplicial sets will be the Reedy–Kan model structure; thus we will generally only want to consider one of the simplicial directions levelwise, justifying our preferred term “simplicial space”. In conformity with this point of view, note the distinction between a *constant* simplicial space (one in which all the face and degeneracy maps between levels are isomorphisms) and a *discrete* simplicial space (one in which each level is the union of points).

The  $i$ th simplicial level of a simplicial space  $\mathcal{X}$  is denoted  $\mathcal{X}_i$ . Thus the 0th level (the “space of objects”, if  $\mathcal{X}$  is a Segal space which we are thinking of as behaving like a category-object in spaces) will be  $\mathcal{X}_0$ ; if we wish to refer instead to the “underlying discrete category” which is the simplicial set obtained by taking the 0-simplices of each simplicial level, we will write instead  $|\mathcal{X}|$ . (Here  $|(-)|$  is intended to suggest an extension of the notion of the right-exact functor “set of points”, not the left-exact functor “realization”.) We take the attitude that an “element” of an

object is a map from the terminal object, so when we write  $x \in X$  where  $X$  is a space we understand that  $x$  is an 0-simplex. We will try not to write  $x \in \mathcal{X}$  where  $\mathcal{X}$  is a simplicial space, but if we were to, it would mean the same as  $x \in \mathcal{X}_0$ .

Our guiding principles of notation will be to use script capitals for simplicial spaces. Lower-case Latin, resp. Greek letters, will denote points of the zeroth, resp. first, simplicial level of a simplicial space; since we will generally be trying to think of a simplicial space, particularly a Segal space, as some kind of generalized category we may refer to these as internal “objects”, resp. “morphisms”. Lower-case bold Latin and Greek letters will correspondingly denote “diagrams” and “natural transformations” indexed by a simplicial space. Thus bold Latin letters will represent maps between simplicial spaces: a “natural transformation” will be defined below to be a particular kind of map, so notational confusion may still ensue.

We will tend to refer to categories as “discrete categories” to distinguish them from our simplicial spaces. When we need to notate discrete categories and functors we will use respectively script capitals and roman capitals. If that isn’t confusing enough, a few “small” simplicial spaces which will be used for indexing diagrams will also be indicated by roman capitals, in order to be consistent with the notation in [Rezk].

We will not distinguish between a category and its nerve. All our model categories have functorial cofibrant and fibrant replacements.

## 1.2 Basic definitions

We here recapitulate some fundamental definitions and results, most of which can be found in [Rezk].

By  $F(n)$  and  $G(n)$  we will mean the simplicial spaces which corepresent respectively the two functors  $F(n)(\mathcal{X}) = \mathcal{X}_n$  and  $G(n)(\mathcal{X}) = \mathcal{X}_1 \times_{\mathcal{X}_0} \cdots \times_{\mathcal{X}_0} \mathcal{X}_1$  (the product having  $n$  factors). We will sometimes denote  $F(0)$  by  $*$ . We will also occasionally use  $\mathbf{d}^i: F(n-1) \rightarrow F(n)$  the map of simplicial spaces induced by the

corresponding face map.  $E$  will be the nerve of the category which contains two objects and has exactly one morphism between every pair of objects, viewed as a discrete simplicial space. (Thus  $E_n$  is the discrete space with  $2^{n+1}$  points). When we need the equivalent of  $F(n)$  for the other simplicial direction—that is, the constant simplicial space with value the  $n$ -simplex—we will call it  $\Delta(n)$ .

**Definition.** If  $\mathcal{X}$  and  $\mathcal{Y}$  are simplicial spaces, then:

- $\mathcal{X} \times \mathcal{Y}$  is the usual Cartesian product on bisimplicial sets, that is, the level-wise Cartesian product of sets in both simplicial directions.
- $\text{Map}_{\text{SS}}(\mathcal{X}, \mathcal{Y})$  is the space whose  $i$ -simplices are maps  $\mathcal{X} \times \Delta(i) \rightarrow \mathcal{Y}$ .
- $\mathcal{Y}^{\mathcal{X}}$  is the simplicial space whose  $j$ th simplicial level is  $\text{Map}_{\text{SS}}(\mathcal{X} \times F(j), \mathcal{Y})$ . This is a Cartesian closure for the category of simplicial spaces.
- $\mathcal{X}^{\text{op}}$ , the *opposite* simplicial space is the usual opposite of a simplicial object, that is, that induced by the involution of the simplex category  $\Delta$  which interchanges  $d_i, d_{n-i}: [n] \rightarrow [n+1]$  (and hence also  $s_i, s_{n-i}: [n] \rightarrow [n-1]$ ).

**Definition.** A *Segal space* is a Reedy–Kan fibrant simplicial space which is local with respect to  $G(i) \hookrightarrow F(i)$  for each  $i \geq 1$ . A *complete* Segal space is a Segal space which additionally is local with respect to the inclusion (of either point)  $* \hookrightarrow E$ .

**Definition.** Let  $\mathcal{X}$  be a Reedy fibrant simplicial space and let  $Y \subseteq \mathcal{X}_0$  be a (Kan) fibrant subspace. We define the simplicial subspace *induced by*  $Y$  to be the simplicial subspace  $\mathcal{Y} \subseteq \mathcal{X}$  where each  $\mathcal{Y}_k = (\prod_{0 \leq i \leq k} d_i^k)^{-1}(Y \times Y \times \cdots \times Y)$ , the  $d_i^k$  denoting the  $(k+1)$  possible  $k$ -fold face maps  $\mathcal{X}_k \rightarrow \mathcal{X}_0$ . (This is a right Kan extension of  $Y \hookrightarrow \mathcal{X}_0$  to the category of simplicial spaces over  $\mathcal{X}$ .) We say that a simplicial subspace  $\mathcal{Y} \subseteq \mathcal{X}$  is *full* if  $\mathcal{Y}$  is induced by some subspace of  $\mathcal{X}_0$ .

**Proposition.** *In the situation above, if  $\mathcal{X}$  is a Segal space then so is  $\mathcal{Y}$ . If in addition  $Y \hookrightarrow \mathcal{X}_0$  is a homotopy monomorphism (that is, an inclusion of path-components up to homotopy, see [Rezk §12.2]) then if  $\mathcal{X}$  is a complete Segal space then so is  $\mathcal{Y}$ .*

*Proof.* The matching map  $\mathcal{Y}_n \rightarrow L_n \mathcal{Y}_\bullet$  in the Reedy fibrancy condition is (for each  $n$ ) a pullback of the corresponding map for  $\mathcal{X}$ , and hence is a fibration because the latter is. Similarly the map  $\mathcal{Y}_n \xrightarrow{\sim} \mathcal{Y}_1 \times_{\mathcal{Y}_0} \mathcal{Y}_1 \times_{\mathcal{Y}_0} \cdots \times_{\mathcal{Y}_0} \mathcal{Y}_1$  in the Segal space condition is a pullback of the corresponding map for  $\mathcal{X}$ , and hence is an acyclic fibration for the same reason.

If  $Y \hookrightarrow \mathcal{X}_0$  is a homotopy monomorphism then since  $\mathcal{X}$  is Reedy fibrant, so is every  $i_n: \mathcal{Y}_n \hookrightarrow \mathcal{X}_n$ . Now we have  $i_1 s_0^{\mathcal{Y}} = s_0^{\mathcal{X}} i_0$  where  $i_0, i_1$  and  $s_0^{\mathcal{X}}$  are homotopy monomorphisms and hence so is  $s_0^{\mathcal{Y}}$ . Thus  $\mathcal{Y}$  is a complete Segal space.  $\blacksquare$

We now give some definitions and results, mostly from [Rezk], motivated by our intention to treat Segal spaces as some kind of “category up to homotopy”.

**Definition.** If  $\mathcal{X}$  is a simplicial space and  $x, y \in \mathcal{X}_0$ , then the (*internal*) *mapping space* from  $x$  to  $y$  is given by the double fibre map  $\text{map}_{\mathcal{X}}(x, y) := (d_0 \times d_1)^{-1}(x, y) \subseteq \mathcal{X}_1$ . We may refer to points of this space as (*internal*) *morphisms* from the (*internal*) *object*  $x$  to  $y$ .

**Definition.** If  $\mathcal{X}$  is a Segal space,  $x, y \in \mathcal{X}_0$ ,  $\phi \in \text{map}_{\mathcal{X}}(x, y)$  and  $\psi \in \text{map}_{\mathcal{X}}(y, z)$  then the space of *compositions* of  $\phi$  and  $\psi$  is the double fibre of  $\psi \times \phi$  in  $d_2 \times d_0: \mathcal{X}_2 \xrightarrow{\sim} \mathcal{X}_1 \times_{\mathcal{X}_0} \mathcal{X}_1$ . A *composite* of  $\phi$  and  $\psi$ , denoted  $\psi \circ \phi$ , is any element of the image of this space under  $d_1$ . Since the space of compositions is always contractible, any two composites of the same maps are *homotopic*, that is, lie in the same component of  $\text{map}_{\mathcal{X}}(x, z)$ . We say  $\phi$  is a *homotopy equivalence* if it has both a left inverse (that is, some  $\psi$  such that there is a composition  $\psi \circ \phi \simeq s_0 d_0 \phi$ ) and a right inverse (guess). We say  $\phi$  is *strictly invertible* if the corresponding morphism of Segal spaces  $F(1) \rightarrow \mathcal{X}$  factors through  $F(1) \hookrightarrow E$ .

**Proposition.** *If  $\mathcal{X}$  is a Segal space, then:*

- *the homotopy type of  $\text{map}_{\mathcal{X}}(x, y)$  depends only on the choice of component of  $(x, y) \in \mathcal{X}_0 \times \mathcal{X}_0$ .*
- *choosing a lift of  $d_2 \times d_0: \mathcal{X}_2 \xrightarrow{\sim} \mathcal{X}_1 \times_{\mathcal{X}_0} \mathcal{X}_1$  and applying  $d_1$  gives a map of spaces*

$$\text{map}_{\mathcal{X}}(x, y) \times \text{map}_{\mathcal{X}}(y, z) \rightarrow \text{map}_{\mathcal{X}}(x, z),$$

*the homotopy type of which does not depend on the choice of lift.*

*Proof.* See [Rezk §5]. ■

The following important result gives the relation between homotopy equivalences and strictly invertible morphisms:

**Theorem.** *The inclusion (of subspaces of  $\mathcal{X}_1$ ) of the space of strictly invertible morphisms into that of homotopy equivalences is a weak equivalence. If  $\phi, \psi$  are homotopic and  $\phi$  is a homotopy equivalence, then so is  $\psi$ .*

*Proof.* See [Rezk §11]. ■

### 1.3 Diagrams indexed by Segal spaces

In this section we will prove some essential technical lemmas that will enable us to treat maps between Segal spaces as functors and to define transformations between them. For example, we will need to verify that the vertical composition of “natural transformations” and the inverse of “natural homotopy equivalences” are suitably coherent.

**Definition.** If  $f, g: \mathcal{X} \rightarrow \mathcal{Y}$  are maps of simplicial spaces then a *natural transformation*  $\tau: f \rightarrow g$  is a map  $\tau: \mathcal{X} \times F(1) \rightarrow \mathcal{Y}$  satisfying  $\tau \circ (\mathcal{X} \times \mathbf{d}^0) = f$  and  $\tau \circ (\mathcal{X} \times \mathbf{d}^1) = g$ . If  $\tau x$  is a homotopy equivalence for every  $x \in \mathcal{X}_0$  then we say  $\tau$  is a natural transformation *through homotopy equivalences*. If  $f, g, h: \mathcal{X} \rightarrow \mathcal{Y}$  and  $\alpha: f \rightarrow g, \beta: g \rightarrow h, \gamma: f \rightarrow h$ , then a *natural (vertical) composition*  $\kappa: \beta \circ \alpha \simeq \gamma$  is a map  $\kappa: \mathcal{X} \times F(2) \rightarrow \mathcal{Y}$  satisfying with the obvious notation  $\kappa \circ (\mathcal{X} \times \mathbf{d}^2) = \alpha$ ,  $\kappa \circ (\mathcal{X} \times \mathbf{d}^0) = \beta$ ,  $\kappa \circ (\mathcal{X} \times \mathbf{d}^1) = \gamma$ . In a similar way, a *natural n-fold composition* of  $\tau_i: f_{i-1} \rightarrow f_i, 1 \leq i \leq n$  is a map  $\kappa: \mathcal{X} \times F(n) \rightarrow \mathcal{Y}$  whose restriction to the  $i$ th 1-face of  $F(n)$  in the obvious sense is  $\tau_i$ . A *homotopy* between  $f$  and  $g$  is a map  $\mathcal{X} \times \Delta(1) \rightarrow \mathcal{Y}$ , with the obvious condition on the endpoints. If such a map exists then  $f$  and  $g$  are *homotopic*.

In order to manipulate maps between Segal spaces, it would be nice to know, in particular—since composition of “morphisms” is not unique—that one can always find a natural vertical composition of two “natural transformations” (albeit not uniquely determined.) The following lemma shows that this is true when the target is a Segal space.

**Invaluable Lemma.** *If  $\mathcal{X}$  is a simplicial space and  $\mathcal{Y}$  is a Segal space, then any map  $\mathcal{X} \times G(n) \rightarrow \mathcal{Y}$  factors through the natural inclusion  $\mathcal{X} \times G(n) \hookrightarrow \mathcal{X} \times F(n)$ .*

*Proof.* It would suffice that  $\mathcal{X} \times G(n) \rightarrow \mathcal{X} \times F(n)$  were an acyclic cofibration in Rezk’s “Segal space model category structure”, in which  $\mathcal{Y}$  is a fibrant object. But this is immediate from compatibility of this model structure with the cartesian closed structure [Rezk §2.5, §7.1]. ■

Finally, we prove a coherent version of the theorem ending the previous section, which related homotopy equivalences to strictly invertible morphisms:

**Indispensable Lemma.** *Let  $f, g: \mathcal{X} \rightarrow \mathcal{Y}$  be morphisms of Segal spaces. Suppose  $\varepsilon: f \rightarrow g$  is a natural transformation such that  $\varepsilon x \in \mathcal{Y}_{\text{hoequiv}}$  for all  $x \in \mathcal{X}_0$ . Then:*

- (i) *there is  $\tilde{\varepsilon}: \mathcal{X} \rightarrow \mathcal{Y}$  homotopic to  $\varepsilon$  such that  $\tilde{\varepsilon}$  is a transformation through strictly invertible morphisms;*
- (ii) *there is a natural transformation  $\varepsilon^{-1}: g \rightarrow f$  such that compositions exist  $\varepsilon^{-1} \circ \varepsilon \simeq s_0 f$  and  $\varepsilon \circ \varepsilon^{-1} \simeq s_0 g$ ; and*
- (iii) *if  $\mathcal{Y}$  is complete then  $f$  and  $g$  are homotopic.*

*Proof.* Define the simplicial space  $\mathcal{Y}^{\text{hoequiv}}$  by  $(\mathcal{Y}^{\text{hoequiv}})_i = (\mathcal{Y}^{F(i)})_{\text{hoequiv}}$ . Notice that  $(\mathcal{Y}^{F(i)})_{\text{hoequiv}} \subseteq (\mathcal{Y}^{F(i)})_1 = (\mathcal{Y}^{F(1)})_i$  so  $\mathcal{Y}^{\text{hoequiv}}$  includes into  $\mathcal{Y}^{F(1)}$ . Moreover, since  $(\mathcal{Y}^{\text{hoequiv}})_i$  consists precisely of those components of  $(\mathcal{Y}^{F(1)})_i$  whose 0-faces lie in  $(\mathcal{Y}^{\text{hoequiv}})_0 = \mathcal{Y}_{\text{hoequiv}}$ , we find that  $\mathcal{Y}^{\text{hoequiv}}$  is a full subspace of  $\mathcal{Y}^{F(1)}$ , and in particular is a Segal space. Moreover,  $\mathcal{Y}^E \hookrightarrow \mathcal{Y}^{F(1)}$  factors through  $\mathcal{Y}^{\text{hoequiv}} \hookrightarrow \mathcal{Y}^{F(1)}$ , and applying [Rezk] Theorem 6.2 to each simplicial level, we find that the inclusion  $\iota_{\mathcal{Y}}: \mathcal{Y}^E \hookrightarrow \mathcal{Y}^{\text{hoequiv}}$  is a levelwise weak equivalence, which is a Reedy

weak equivalence between Reedy cofibrant–fibrant objects and hence a homotopy equivalence.

Now consider  $\varepsilon$  as a map  $\mathcal{X} \rightarrow \mathcal{Y}^{F(1)}$ . By hypothesis  $\varepsilon$  factors as  $\mathcal{X} \xrightarrow{\varepsilon'} \mathcal{Y}^{Hoequiv} \hookrightarrow \mathcal{Y}^{F(1)}$ . Then we can find  $\rho: \mathcal{X} \rightarrow \mathcal{Y}^E$  with  $\iota_{\mathcal{Y}}\rho$  homotopic to  $\varepsilon'$  via  $\mathbf{h}: \mathcal{X} \times \Delta(1) \rightarrow \mathcal{Y}^{Hoequiv}$ . We may not have  $\mathbf{d}_i \iota_{\mathcal{Y}}\rho$  equal to  $\mathbf{d}_i \varepsilon'$  ( $i = 0, 1$ ), but they will be homotopic via  $\mathbf{d}_i \mathbf{h}$ . This latter is a homotopy  $\mathcal{X} \times \Delta(1) \rightarrow \mathcal{Y}$ : to get a natural transformation through strictly invertible morphisms, take the homotopy lift in:

$$\begin{array}{ccc}
 \mathcal{X} & \xrightarrow{\iota_{\mathcal{Y}}\rho} & \mathcal{Y}^E \\
 i_0 \downarrow \sim & \nearrow \cdots & \downarrow \mathbf{d}_0 \times \mathbf{d}_1 \\
 \mathcal{X} \times \Delta(1) & \xrightarrow{(\mathbf{d}_0 \times \mathbf{d}_1)\mathbf{h}} & \mathcal{Y} \times \mathcal{Y}.
 \end{array}$$

and define  $\tilde{\varepsilon}: \mathcal{X} \rightarrow \mathcal{Y}^E$  to be this lift restricted to the vertex  $1 \in \Delta(1)$ .

Parts (ii) and (iii) follow immediately. ■

Part (ii) of the preceding lemma justifies the notation  $\tau^{-1}$ , which we will use in the sequel when  $\tau$  is a natural transformation through homotopy equivalences: it denotes any choice of inverse, just as  $\circ$  denotes any choice of composite.

# Chapter 2

## Adjunctions for Segal spaces

In this chapter we will define an “adjunction” between maps of Segal spaces and prove some desirable consequences of this definition.

**Definition.** A Segal space *adjunction* is a pair of Segal space morphisms  $\mathcal{X} \begin{matrix} \xrightarrow{\mathbf{f}} \\ \xleftarrow{\mathbf{g}} \end{matrix} \mathcal{Y}$  together with natural transformations  $\eta: 1_{\mathcal{X}} \rightarrow \mathbf{g}\mathbf{f}$  and  $\varepsilon: \mathbf{f}\mathbf{g} \rightarrow 1_{\mathcal{Y}}$ , such that there exist natural compositions  $\rho: \mathbf{g}\varepsilon \circ \eta\mathbf{g} \simeq 1_{\mathbf{g}}$  and  $\lambda: \varepsilon\mathbf{f} \circ \mathbf{f}\eta \simeq 1_{\mathbf{f}}$ .

The adjoint of a morphism, if it exists, is unique up to homotopy:

**Proposition.** Let  $\mathbf{f}: \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism of Segal spaces where  $\mathcal{X}$  is complete. If  $(\mathbf{f}, \mathbf{g}, \eta, \varepsilon)$  and  $(\mathbf{f}, \mathbf{g}', \eta', \varepsilon')$  are adjunctions, then  $\mathbf{g}, \mathbf{g}': \mathcal{Y} \rightarrow \mathcal{X}$  are homotopy equivalent.

*Proof.* We will show that  $\mathbf{g}'\varepsilon \circ \eta'\mathbf{g}: \mathbf{g} \rightarrow \mathbf{g}'$  and  $\mathbf{g}\varepsilon' \circ \eta\mathbf{g}': \mathbf{g}' \rightarrow \mathbf{g}$  are mutual homotopy inverses, and then the fact that they are equivalences will follow from completeness of  $\mathcal{X}$ .

The squares in the diagram below commute up to homotopy by naturality; applying  $\mathbf{g}\lambda'\mathbf{g}$  and then  $\rho$  to the toprightmost path through the diagram, this shows that some-hence-every choice of quadruple composite  $\mathbf{g}\varepsilon' \circ \eta\mathbf{g}' \circ \mathbf{g}'\varepsilon \circ \eta'\mathbf{g}$  is homotopic to the identity transformation of  $\mathbf{g}$ . A similar argument with the primes redistributed shows the same for the other composite, thus establishing the proposition.

$$\begin{array}{ccccc}
g & \xrightarrow{\eta g} & gfg & & \\
\eta' g \downarrow & & gfn'g \downarrow & & \\
g'fg & \xrightarrow{\eta g'fg} & gfg'fg & \xrightarrow{g\varepsilon'fg} & gfg \\
g'\varepsilon \downarrow & & gfg'\varepsilon \downarrow & & g\varepsilon \downarrow \\
g' & \xrightarrow{\eta g'} & gfg' & \xrightarrow{g\varepsilon'} & g
\end{array}$$

■

From this proof of uniqueness of right adjoints one can easily deduce uniqueness of left adjoints by taking opposite Segal spaces: it is easy to see that  $(-)^{op}$  takes right adjoints into left adjoints and vice versa.

The uniqueness of adjoints will be important in chapter 5. In the remainder of this chapter we will relate adjunctions of Segal spaces to adjunctions on their homotopy categories which induce weak equivalences of mapping spaces. In particular we will see that imposing an extra condition on our adjunctions, analogous to that which characterizes equivalences among adjunctions of discrete categories, will give precisely those adjunctions which are mutually inverse pairs of “Dwyer–Kan equivalences”, defined in [Rezk] as morphisms of Segal spaces which induces equivalence both of homotopy categories and of corresponding mapping spaces.

**Proposition.** *Let  $L$  be a localization of the model category of Segal spaces which is compatible with the cartesian closure ([Rezk §2.5]). Then a Segal space adjunction  $\mathcal{X} \begin{smallmatrix} \xrightarrow{f} \\ \xleftarrow{g} \end{smallmatrix} \mathcal{Y}$  induces adjunction between the localized maps  $L\mathcal{X} \begin{smallmatrix} \xrightarrow{Lf} \\ \xleftarrow{Lg} \end{smallmatrix} L\mathcal{Y}$ .*

*Proof.* Let  $Lf$  and  $Lg$  be the functorial localizations of  $f$ ,  $g$ . Now writing  $H$  for  $F(0) \sqcup F(1)$ , we have by compatibility that  $\mathcal{X} \times F(1) \sqcup_{\mathcal{X} \times H} L\mathcal{X} \times H \hookrightarrow L\mathcal{X} \times F(1)$  is

an L-equivalence. Hence we can define  $\mathbf{L}\eta$  by taking the homotopy extension in

$$\begin{array}{ccc}
\mathcal{X} \times F(1) \sqcup_{\mathcal{X} \times H} \mathbf{L}\mathcal{X} \times H & \longrightarrow & \mathbf{L}\mathcal{X} \\
\downarrow & \nearrow \cdots & \downarrow \\
\mathbf{L}\mathcal{X} \times F(1) & \longrightarrow & *
\end{array}$$

where the top map is  $\eta \sqcup (\text{id}_{\mathcal{X}}, (\mathbf{L}\mathbf{g})(\mathbf{L}\mathbf{f}))$ . We form  $\mathbf{L}\varepsilon$  in the same way.

To extend  $\lambda$ , let us define  $M(2)$  to be the simplicial space corepresenting the functor  $\mathcal{X} \mapsto M_2\mathcal{X} = \lim(\mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_1 \rightrightarrows \mathcal{X}_0 \times \mathcal{X}_0 \times \mathcal{X}_0)$  which gives the second matching space of  $\mathcal{X}$ . This time we will be taking the extension in

$$\begin{array}{ccc}
\mathcal{X} \times F(2) \sqcup_{\mathcal{X} \times M(2)} \mathbf{L}\mathcal{X} \times M(2) & \longrightarrow & \mathbf{L}\mathcal{Y} \\
\downarrow & \nearrow \cdots & \downarrow \\
\mathbf{L}\mathcal{X} \times F(2) & \longrightarrow & *
\end{array}$$

where the map  $\mathbf{L}\mathcal{X} \times M(2) \rightarrow \mathbf{L}\mathcal{Y}$  is defined by gluing together the three maps  $\mathbf{L}\mathcal{X} \times F(1) \rightarrow \mathbf{L}\mathcal{Y}$  given by  $(\mathbf{L}\varepsilon)(\mathbf{L}\mathbf{f})$ ,  $\mathbf{L}\mathbf{f} \circ \text{pr}_{\mathbf{L}\mathcal{X}}$  and  $(\mathbf{L}\mathbf{f})(\mathbf{L}\eta)$ , and this map is then glued with  $\lambda$  to give the top map in the diagram. Again the left vertical arrow is a cofibration which is an L-equivalence by compatibility, so the homotopy extension gives a suitable  $\mathbf{L}\lambda$ . We obtain  $\mathbf{L}\rho$  similarly.  $\blacksquare$

**Corollary.** A Segal space adjunction  $\mathcal{X} \begin{smallmatrix} \xrightarrow{\mathbf{f}} \\ \xrightleftharpoons{\mathbf{g}} \\ \xrightarrow{\mathbf{g}} \end{smallmatrix} \mathcal{Y}$  induces an adjunction (in the usual sense)

$$\text{ho } \mathcal{X} \begin{smallmatrix} \xrightarrow{\text{ho } \mathbf{f}} \\ \xrightleftharpoons{\text{ho } \mathbf{g}} \\ \xrightarrow{\text{ho } \mathbf{g}} \end{smallmatrix} \text{ho } \mathcal{Y}.$$

*Proof.* Using the results of chapter 4, this is just the preceding Proposition applied to the localization  $\mathbf{L}_S$  considered there. Alternatively, it is easy to prove this statement directly by emulating the proof of the Proposition using the functor  $\text{ho}$  in place of the localization.  $\blacksquare$

**Corollary.** Let  $(\mathbf{f}, \mathbf{g}, \eta, \varepsilon)$  be a Segal space adjunction. Then  $\eta$  and  $\varepsilon$  are both transformations through homotopy equivalences iff  $\mathbf{f}$  is a Dwyer–Kan equivalence (hence also iff  $\mathbf{g}$

is).

*Proof.* ( $\Rightarrow$ ) Localize in the complete Segal space model structure: by the Proposition, this gives an adjunction  $(f', g', \eta', \varepsilon')$ . Then  $\eta'$  and  $\varepsilon'$  are transformations through homotopy equivalences, since everything in their image is homotopic to something in the image of  $\eta$  and  $\varepsilon$  respectively, But applying the Indispensable Lemma shows that the localized maps are inverse homotopy equivalences of Segal spaces, so the original  $f$  and  $g$  were also weak equivalences in the localized model structure. Hence by [Rezk] Theorem 7.7,  $f$  and  $g$  are Dwyer–Kan equivalences. ( $\Leftarrow$ ) By Rezk’s definition of Dwyer–Kan equivalence, the induced adjunction on homotopy categories is an equivalence. Hence the images of  $\eta$  and  $\varepsilon$  in the homotopy category are natural isomorphisms, so  $\eta$  and  $\varepsilon$  were transformations through homotopy equivalences.  $\blacksquare$

**Proposition.** A Segal space adjunction  $\mathcal{X} \xrightleftharpoons[g]{f} \mathcal{Y}$  induces mutually inverse classes of homotopy equivalences

$$b: \text{map}_{\mathcal{Y}}(fa, b) \longrightarrow \text{map}_{\mathcal{X}}(a, gb) \text{ and } \sharp: \text{map}_{\mathcal{X}}(a, gb) \longrightarrow \text{map}_{\mathcal{Y}}(fa, b)$$

for each  $a \in \mathcal{X}_0, b \in \mathcal{Y}_0$ .

*Proof.* Let us use the notation  $\text{map}_{\mathcal{X}}(x, y, z) := (d_1 d_2, d_0 d_2, d_0 d_0)^{-1}(x, y, z)$  for  $x, y, z \in \mathcal{X}_0$ , that is,  $\text{map}_{\mathcal{X}}(x, y, z) \subset \mathcal{X}_2$  is the set of compositions of pairs of maps  $x \rightarrow y \rightarrow z$ ; and the still more *ad hoc* convention that for any given  $m \in \text{map}_{\mathcal{X}}(x, y)$  we write  $\text{map}_{\mathcal{X}}(x_m y, z) := \text{map}_{\mathcal{X}}(x, y, z) \cap d_2^{-1} m$ , the set of compositions  $x \xrightarrow{m} y \rightarrow z$ .

Note that  $d_0: \text{map}_{\mathcal{X}}(x_m y, z) \rightarrow \text{map}_{\mathcal{X}}(y, z)$  is an acyclic fibration because it is the pullback of one:

$$\begin{array}{ccc} \text{map}_{\mathcal{X}}(x_m y, z) & \longrightarrow & \mathcal{X}_2 \\ \sim \downarrow & & \sim \downarrow \\ \text{map}_{\mathcal{X}}(y, z) & \xrightarrow{m \times -} & \mathcal{X}_1 \times_{\mathcal{X}_0} \mathcal{X}_1 \end{array}$$

Thus we may take  $\flat$  by choosing any lifting

$$\flat: \text{map}_{\mathcal{Y}}(\mathbf{fa}, \mathbf{b}) \xrightarrow{\mathbf{g}} \text{map}_{\mathcal{X}}(\mathbf{gfa}, \mathbf{gb}) \xleftarrow{\sim} \text{map}_{\mathcal{X}}(\mathbf{a}_{\eta_a} \mathbf{gfa}, \mathbf{gb}) \xrightarrow{d_1} \text{map}_{\mathcal{X}}(\mathbf{a}, \mathbf{gb})$$

and similarly for  $\sharp$  we have

$$\sharp: \text{map}_{\mathcal{X}}(\mathbf{a}, \mathbf{gb}) \xrightarrow{\mathbf{f}} \text{map}_{\mathcal{Y}}(\mathbf{fa}, \mathbf{fgb}) \xleftarrow{\sim} \text{map}_{\mathcal{Y}}(\mathbf{fa}, \mathbf{fgb}_{\varepsilon_b} \mathbf{b}) \xrightarrow{d_1} \text{map}_{\mathcal{Y}}(\mathbf{fa}, \mathbf{b}).$$

That  $\sharp\flat$  is homotopic to the identity follows by considering the following diagram:

$$\begin{array}{ccccccc}
 & & \text{map}(\mathbf{gfa}, \mathbf{gb}) & \xleftarrow{\mathbf{g}} & \text{map}(\mathbf{fa}, \mathbf{b}) & \xrightarrow{\varepsilon_{S_0}} & \text{map}(\mathbf{fgfa}_{\varepsilon_a} \mathbf{fa}, \mathbf{b}) \\
 \sim \nearrow & & \mathbf{f} \downarrow & & \varepsilon_{S_1} \downarrow & & \downarrow & \nwarrow \sim \\
 \text{map}(\mathbf{a}_{\eta_a} \mathbf{gfa}, \mathbf{gb}) & \text{map}(\mathbf{fgfa}, \mathbf{fgb}) & \xleftarrow{\sim} & \text{map}(\mathbf{fgfa}, \mathbf{fgb}_{\varepsilon_b} \mathbf{b}) & \rightarrow & \text{map}(\mathbf{fgfa}, \mathbf{b}) & \text{map}(\mathbf{fa}_{\eta_a} \mathbf{fgfa}_{\varepsilon_a} \mathbf{fa}, \mathbf{b}) \\
 \downarrow & \mathbf{f} \searrow & \sim \uparrow & \sim \uparrow & \sim \uparrow & \sim \uparrow & \swarrow \checkmark \\
 \text{map}(\mathbf{a}, \mathbf{gb}) & \text{map}(\mathbf{fa}_{\eta_a} \mathbf{fgfa}, \mathbf{fgb}) & \xleftarrow{\sim} & \text{map}(\mathbf{fa}_{\eta_a} \mathbf{fgfa}, \mathbf{fgb}_{\varepsilon_b} \mathbf{b}) & \rightarrow & \text{map}(\mathbf{fa}_{\eta_a} \mathbf{fgfa}, \mathbf{b}) & \\
 \mathbf{f} \searrow & \downarrow & & \downarrow & & \downarrow & \\
 & \text{map}(\mathbf{fa}, \mathbf{fgb}) & \xleftarrow{\sim} & \text{map}(\mathbf{fa}, \mathbf{fgb}_{\varepsilon_b} \mathbf{b}) & \rightarrow & \text{map}(\mathbf{fa}, \mathbf{b}) & 
 \end{array}$$

Here the composite  $\sharp\flat: \text{map}(\mathbf{fa}, \mathbf{b}) \rightarrow \text{map}(\mathbf{fa}, \mathbf{b})$  is defined to be that around the left-hand edge of the diagram from top to bottom, choosing any lifts through the acyclic fibrations. By properties of Segal spaces and by naturality of  $\varepsilon$  the diagram is homotopy commutative and so this composite is homotopic to that along the right-hand edge of the diagram. But the latter factors through the internal triple composition map

$$d_1 d_1 = d_1 d_2: \text{map}(\mathbf{fa}_{\eta_a} \mathbf{fgfa}_{\varepsilon_a} \mathbf{fa}, \mathbf{b}) \rightarrow \text{map}(\mathbf{fa}, \mathbf{b}),$$

and if one were to replace this by  $d_0 d_0 = d_0 d_1$ , the face map which simply extracts the last edge, one would instead get the identity as composite. So it suffices to

show that  $d_1 d_1 \sim d_0 d_0$ . But these maps factor as

$$\text{map}(fa_{f\eta_a} fgfa_{\epsilon_a} fa, b) \xrightarrow{\sim} \text{map}(fa, b) \xleftarrow{\sim} \text{map}(fa_{f\eta_a} fgfa_{\epsilon_a} fa, b) \xrightarrow{d_1} \text{map}(fa_{id_{fa}} fa, b) \rightrightarrows \text{map}(fa, b)$$

where the subscript  $\lambda_a$  denotes restriction to  $d_3^{-1}\lambda_a$ , and with the rightmost arrow being  $d_0$  and  $d_1$  respectively. These latter are homotopic since  $s_0: \text{map}(fa, b) \rightarrow \text{map}(fa_{id_{fa}} fa, b)$  is a homotopy inverse to each of them.

The proof for  $b\sharp$  is similar. ■

# Chapter 3

## Derived Segal morphisms

The “classification diagram” construction  $N$  associates a simplicial space to any category with weak equivalences. However, a functor between such categories need not give a morphism between the associated simplicial spaces except in the very special case where the functor preserves weak equivalences.

The aim of this chapter is to show, firstly, that a Quillen functor between model categories induces (up to homotopy) a map between their classification diagrams, and secondly, that the maps so induced by a Quillen pair of adjoint functors are themselves adjoint in the sense defined in the previous chapter. We will see that this can be viewed as a generalization of Quillen’s construction of the total derived functor of a Quillen functor ([Qui §I.4]).

**Proposition 1.** *(compare [Qui] §I.4, Proposition 1) Let  $F: \mathcal{C} \rightarrow |\mathcal{B}|$  be a map of simplicial sets, where  $\mathcal{C}$  is a model category and  $\mathcal{B}$  is a Segal space. Suppose that  $F$  carries weak equivalences in  $\mathcal{C}_c$  into homotopy equivalences in  $\mathcal{B}$ . Then there is a map of Segal spaces  $f: N\mathcal{C} \rightarrow \mathcal{B}$  together with a natural transformation  $\varepsilon: |f|_{\mathcal{C}} \rightarrow F$  such that for any such pair  $(g, \zeta)$  there is a natural transformation  $\theta: f' \rightarrow f$ , unique up to homotopy, such that  $(|\theta|_{\mathcal{C}}) \cdot \varepsilon = \zeta$ .*

*Proof.* Let  $\lambda: L \rightarrow \text{id}_{\mathcal{C}}$  denote a functorial cofibrant replacement on  $\mathcal{C}$ . Now  $FL: \mathcal{C} \rightarrow |\mathcal{B}|$  takes weak equivalences to homotopy equivalences, hence it induces (using the functorial Reedy fibrant replacement) a morphism  $f: N\mathcal{C} \rightarrow \mathcal{B}$  which satisfies

$FL \simeq |f|_{\mathcal{C}}$ . Define  $\varepsilon: |f|_{\mathcal{C}} \rightarrow F$  to be the natural transformation  $F\lambda$ . We verify its universal property: suppose  $\zeta: |g|_{\mathcal{C}} \rightarrow F$  where  $g: NC \rightarrow \mathcal{B}$ . Define  $\theta: |g|_{\mathcal{C}} \rightarrow |f|_{\mathcal{C}}$  as the composite

$$|g|_{\mathcal{C}} \xrightarrow{|g|_{(\iota_{\mathcal{C}}\lambda)^{-1}}} |g|_{\mathcal{C}L} \xrightarrow{\zeta_L} FL = |f|_{\mathcal{C}};$$

since  $\iota_{\mathcal{C}}\lambda$  is a transformation through homotopy equivalences, the notation  $(\iota_{\mathcal{C}}\lambda)^{-1}$  is justified by the Indispensable Lemma applied to diagrams indexed by  $\text{discnerv } \mathcal{C}$ . Thus we have  $\theta: \text{discnerv } \mathcal{C} \rightarrow \mathcal{B}^{F(1)}$ , or equivalently  $\theta: \mathcal{C} \rightarrow |\mathcal{B}^{F(1)}| = |\mathcal{B}|^{F(1)}$  which takes weak equivalences to homotopy equivalences and hence descends to a map of Segal spaces  $\theta: NC \rightarrow \mathcal{B}^{F(1)}$ , which is a natural transformation  $g \rightarrow f$ . The uniqueness of  $\theta$  up to homotopy is clear since it is determined by  $\zeta$  on  $N(\mathcal{C}_{\mathcal{C}})$ , which is a deformation retract of  $NC$ . ■

**Theorem 3.** (compare [Qui] §I.4, Theorem 3) *Let  $\mathcal{C}$  and  $\mathcal{C}'$  be model categories and let*

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{C}'$$

*be a pair of adjoint functors,  $F$  being the left and  $G$  the right adjoint functor. Suppose that  $F$  preserves cofibrations and that  $F$  carries weak equivalences in  $\mathcal{C}_{\mathcal{C}}$  into weak equivalences in  $\mathcal{C}'$ . Also suppose that  $G$  preserves fibrations and that  $G$  carries weak equivalences in  $\mathcal{C}'_{\mathcal{C}'}$  into weak equivalences in  $\mathcal{C}$ . Then the induced maps*

$$NC \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} NC'$$

*are adjoint.*

*Proof.* Let  $\lambda: L \rightarrow 1_{\mathcal{C}}$  be the cofibrant replacement on  $\mathcal{C}$  and  $\rho: 1_{\mathcal{C}'} \rightarrow R$  the fibrant replacement on  $\mathcal{C}'$ . Let  $\eta: 1_{\mathcal{C}} \rightarrow GF$  be the unit and  $\varepsilon: FG \rightarrow 1_{\mathcal{C}'}$  the counit of the adjunction. Now the natural transformations (viewed as functors from  $- \times 2$ )  $\hat{\eta} := g\rho f \circ \eta: \mathcal{C} \times 2 \rightarrow \mathcal{C}$  and  $\hat{\varepsilon} := \varepsilon \circ f\lambda g: \mathcal{C}' \times 2 \rightarrow \mathcal{C}'$  take weak equivalences to weak equivalences and so by the proposition they induce  $\hat{\eta}: NC \times F(1) \rightarrow NC$  and  $\hat{\varepsilon}: NC' \times F(1) \rightarrow NC'$ . We define  $\eta$  and  $\varepsilon$  by choosing natural composites  $\eta \simeq \hat{\eta} \circ \lambda^{-1}$  and  $\varepsilon \simeq \rho^{-1} \circ \hat{\varepsilon}$ , where  $\lambda$  and  $\rho$  extend to the classifying diagrams because

they preserve all weak equivalences, and then taking their inverse is justified by the Indispensable Lemma.

Now in the diagrams of functors

$$\begin{array}{ccc}
 LG & \xrightarrow{\lambda G} & G \\
 \eta LG \downarrow & & \eta G \downarrow \\
 GFLG & \xrightarrow{GF\lambda G} & GFG \xrightarrow{G\varepsilon} G \\
 G\rho FLG \downarrow & & G\rho FG \downarrow \quad G\rho \downarrow \\
 GRFLG & \xrightarrow{GRF\lambda G} & GRFG \xrightarrow{GR\varepsilon} GR
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 F & \xleftarrow{F\lambda} & FL \\
 \downarrow F\eta & & \downarrow FL\eta \\
 F & \xleftarrow{\varepsilon F} & FGF \xleftarrow{F\lambda GF} FLGF \\
 \downarrow \rho F & & \downarrow FG\rho F \quad \downarrow FLG\rho F \\
 RF & \xleftarrow{\varepsilon RF} & FGRF \xleftarrow{F\lambda GRF} FLGRF
 \end{array}$$

the squares commute by naturality and the triangles commute by the characterizing property of a unit and counit. Hence the diagrams commute, whence we get natural composites  $GR\varepsilon \circ \eta GR \simeq (GR\rho)^{-1} \circ G\rho R$  and  $\varepsilon FL \circ FL\eta \simeq F\lambda \circ (FL\lambda)^{-1}$ , in which both right-hand sides are natural transformations through weak equivalences and thus homotopic to the identity. So  $f = FL$  and  $g = GR$  are adjoint with  $\eta$  and  $\varepsilon$  as above. ■

# Chapter 4

## The homotopy category as a localization

In this chapter we will show that there exists a set  $\mathcal{S}$  of cofibrations between Segal spaces which has the property that the unit  $\mathcal{X} \rightarrow \mathbf{N} \text{ ho } \mathcal{X}$  of the adjunction  $\text{ho: Seg} \rightleftarrows \text{Cat: N}$  between the category of Segal spaces and that of discrete categories is an  $\mathcal{S}$ -localization. (Here, of course,  $\mathbf{N}$  denotes only the classifying diagram of a discrete category, not the more general classification diagram of a model category.)

Unless otherwise stated, assertions which need to be interpreted in the context of a model category structure on simplicial spaces refer to Rezk's "complete Segal space" structure.

We will use the following construction, which we will call  $\gamma$  because of its resemblance to one introduced by that name in [Sim97 §2.4]. (Simpson works in a slightly different category, that of so-called *Segal categories*, but opportunity for confusion should be minimal.)

**Definition.** Let  $X$  be a space. Define  $\widehat{\Upsilon}(X)$  to be the semi-simplicial space (that is, diagram of spaces indexed by the subcategory  $\Delta_\delta$  consisting of all face maps of  $\Delta$

the simplicial indexing category) given by

$$\begin{cases} \widehat{\Upsilon}(X)_0 = S^0 \\ \widehat{\Upsilon}(X)_1 = X \\ \widehat{\Upsilon}(X)_i = \emptyset \text{ for } i > 1 \end{cases}$$

where  $d_0, d_1: \widehat{\Upsilon}(X)_1 \rightarrow \widehat{\Upsilon}(X)_0$  are constant maps to the two respective components of  $S^0$ . Now set  $\Upsilon(X)$  to be the simplicial space which is the left Kan extension to  $\Delta$  of  $\widehat{\Upsilon}(X)$ .

We remark that  $\Upsilon$  is functorial in an obvious way, and that when  $X$  satisfies the Kan condition, it is clear that  $\Upsilon(X)$  is a complete Segal space.

The utility of the construction  $\Upsilon(X)$  for us lies in the following

**Proposition.** *Let  $f: A \hookrightarrow B$  be a Kan cofibration, i.e., an inclusion of simplicial sets. Then a Segal space  $\mathcal{X}$  is  $\Upsilon(f)$ -local iff every mapping space of  $\mathcal{X}$  is  $f$ -local.*

*Proof.* The inclusions  $\emptyset \hookrightarrow A \hookrightarrow B$  induce a diagram

$$\begin{array}{ccccc} \coprod_{(x,y)} \text{Map}(B, \text{map}(x, y)) & \longrightarrow & \text{Map}_{\text{SS}}(\Upsilon(B), \mathcal{X}) & \longrightarrow & \text{Map}_{\text{SS}}(\Upsilon(\emptyset), \mathcal{X}) = \mathcal{X}_0 \times \mathcal{X}_0 \\ \downarrow & & \downarrow & & \downarrow \\ \coprod_{(x,y)} \text{Map}(A, \text{map}(x, y)) & \longrightarrow & \text{Map}_{\text{SS}}(\Upsilon(A), \mathcal{X}) & \longrightarrow & \text{Map}_{\text{SS}}(\Upsilon(\emptyset), \mathcal{X}) = \mathcal{X}_0 \times \mathcal{X}_0, \end{array}$$

where the coproducts take one representative  $(x, y)$  of each component of  $\mathcal{X}_0 \times \mathcal{X}_0$ . By the last sentence of [Rezk] Theorem 7.2,  $\mathcal{X}$  is  $\Upsilon(f)$ -local iff the middle vertical map is a weak equivalence. But each  $\text{map}(x, y)$  is  $f$ -local iff the left-hand vertical map is a weak equivalence. Since the rows are fibration sequences and the other vertical map is an isomorphism, the result follows. ■

The localization in which we will be particularly interested has as its set of generating cofibrations

$$\mathcal{S} = \left\{ \Upsilon(S^i) \hookrightarrow \Upsilon(D^{i+1}) \mid i \geq 1 \right\},$$

that is, our functor  $\Upsilon$  applied to the inclusion of the  $i$ -sphere into the  $(i + 1)$ -ball for each  $i$ .

**Lemma 1.** *Let  $\{f_i\}_i$  be a set of Kan cofibrations. Then localization of the category of Segal spaces (resp. complete Segal spaces) with respect to the set  $\{\Upsilon(f_i)\}_i$  exists and is compatible with the cartesian closure.*

*Proof.* Existence follows from Theorem 4.1.1 in [Hir], since the Reedy–Kan structure on simplicial sets is a left proper cellular model category and since the category of Segal spaces (resp. complete Segal spaces) is defined as a left Bousfield localization thereof.

Using [Rezk] Prop. 9.2 together with Thm. 7.2, for compatibility with the cartesian closure it suffices to show that whenever  $\mathcal{X}$  is an  $\Upsilon(f_i)$ -local Segal space,  $\mathcal{X}^{F(1)}$  is also  $\Upsilon(f_i)$ -local. Thus we must check that for such an  $\mathcal{X}$ ,  $(\mathcal{X}^{F(1)})_1 = \text{Map}(F(1) \times F(1), \mathcal{X}) \xrightarrow{d_0 \sqcup d_1} \text{Map}(F(1) \sqcup F(1), \mathcal{X}) = (\mathcal{X}^{F(1)})_0 \times (\mathcal{X}^{F(1)})_0$  has  $f_i$ -local fibres. This factors as

$$\text{Map}(F(1) \times F(1), \mathcal{X}) \rightarrow \text{Map}(Z(3), \mathcal{X}) \rightarrow \text{Map}(F(1) \sqcup F(1), \mathcal{X})$$

where the fibre of the right-hand map over  $(\phi, \psi) \in \text{Map}(F(1) \sqcup F(1), \mathcal{X})$  is just  $\text{map}_{\mathcal{X}}(d_0\phi, d_1\psi)$  which is  $f_i$ -local. Thus the proposition will follow if we can show that

$$\text{Map}(F(2), \mathcal{X}) \xrightarrow{d_0, d_1} \text{Map}(F(1)_{d^0} \sqcup_{F(0)} d^0 F(1), \mathcal{X})$$

(which is a fibration by the Reedy fibrancy condition) has  $f_i$ -local fibres. But every fibre of the latter is also the homotopy fibre of

$$\text{map}_{\mathcal{X}}(x, y) \times \text{map}_{\mathcal{X}}(y, z) \rightarrow \text{map}_{\mathcal{X}}(x, z) \times \text{map}_{\mathcal{X}}(y, z)$$

which is a map of  $f_i$ -local spaces. Hence it is  $f_i$ -local as required. ■

**Lemma 2.** *The classifying diagram  $N(C)$  of a discrete category  $C$  is  $\mathcal{S}$ -local. In particular, for any Segal space  $\mathcal{X}$ ,  $N \text{ ho } \mathcal{X}$  is  $\mathcal{S}$ -local.*

*Proof.* The fibres  $\text{map}(a, b)$  are discrete spaces and therefore are  $\{S^i \hookrightarrow D^{i+1}\}$ -local for  $i \geq 1$ . The result follows from the Proposition above.  $\blacksquare$

**Lemma 3.** *Every  $\mathcal{S}$ -equivalence between complete Segal spaces induces an equivalence of homotopy categories.*

*Proof.* That the result holds for weak equivalences between complete Segal spaces follows from [Rezk] Prop. 7.6 and Theorem 7.2, so it suffices to prove it for  $\mathcal{S}$ -acyclic cofibrations. But any such map is  $\mathcal{S}$ -cellular (i.e., a transfinite composite of pushouts of maps in  $\mathcal{S}$ ), and since the result is easily verified for each map in  $\mathcal{S}$  it holds for all  $\mathcal{S}$ -cellular maps because  $\text{ho}$  preserves small colimits.  $\blacksquare$

**Proposition.** *If  $\mathcal{X}$  is  $\mathcal{S}$ -local then  $\eta_{\mathcal{X}}: \mathcal{X} \rightarrow \text{N ho } \mathcal{X}$  is a weak equivalence.*

*Proof.* *Claim 1:*  $\eta_{\mathcal{X}}$  induces equivalence on  $-_0$ . For any complete Segal space  $\mathcal{A}$  and any object  $a \in \mathcal{A}_0$ , define  $\mathcal{A}_a^{\text{B}}$  to be the component of  $\mathcal{A}_0$  containing  $a$ , and  $\mathcal{A}_a^{\text{E}} = \mathcal{A}_{\text{hoequiv}} \cap d_1^{-1}a \subset \mathcal{A}_1$ . Then  $\mathcal{A}_a^{\text{E}} \simeq *$  since  $d_1$  is a fibration and a weak equivalence on this component of  $\mathcal{A}_1$  (because it is a left inverse to the weak equivalence  $s_0$ ). Define  $\mathcal{A}_a^{\text{F}}$  to be the fibre over  $a$  of the fibration  $d_0: \mathcal{A}_a^{\text{E}} \rightarrow \mathcal{A}_a^{\text{B}}$ . ( $\mathcal{A}_a^{\text{F}}$  is thus the space  $\text{haut}(a) \subset \text{map}(a, a)$  of homotopy automorphisms of  $a$ .) Now for each 0-simplex  $x \in \mathcal{X}_0$ ,  $\eta_{\mathcal{X}}$  induces a diagram of fibre sequences

$$\begin{array}{ccccc} \mathcal{X}_x^{\text{F}} & \longrightarrow & \mathcal{X}_x^{\text{E}} \simeq * & \longrightarrow & \mathcal{X}_x^{\text{B}} \\ \downarrow & & \downarrow & & \downarrow \\ (\text{N ho } \mathcal{X})_x^{\text{F}} & \longrightarrow & (\text{N ho } \mathcal{X})_x^{\text{E}} \simeq * & \longrightarrow & (\text{N ho } \mathcal{X})_x^{\text{B}} \end{array}$$

in which the first two vertical arrows are weak equivalences—the first because  $\text{N ho}$  takes each mapping space to its set of components whereas we know from Lemma 2 that  $\mathcal{X}_x^{\text{F}}$  is discrete, the second because both spaces are contractible—so the rightmost arrow must also be an equivalence. Since  $\eta_{\mathcal{X}}$  is bijective on the 0-simplices of  $-_0$ , every component of  $(\text{N ho } \mathcal{X})_0$  can be expressed as  $(\text{N ho } \mathcal{X})_x^{\text{B}}$  for some  $x \in \mathcal{X}_0$ , so  $(\eta_{\mathcal{X}})_0$  is an equivalence.

Claim 2:  $\eta_{\mathcal{X}}$  induces equivalence on  $-_1$ . This time define

$$\mathcal{X}_{x,y}^E = (d_0 \times d_1)^{-1}(\mathcal{X}_x^B \times \mathcal{X}_y^B)$$

and

$$\mathcal{X}_{x,y}^F = (d_0 \times d_1)^{-1}(x \times y)$$

for any  $x, y \in \mathcal{X}_0$ . Then we have

$$\begin{array}{ccccc} \mathcal{X}_{x,y}^F & \longrightarrow & \mathcal{X}_{x,y}^E & \longrightarrow & \mathcal{X}_x^B \times \mathcal{X}_y^B \\ \downarrow & & \downarrow & & \downarrow \\ (\mathrm{N} \mathrm{ho} \mathcal{X})_{x,y}^F & \longrightarrow & (\mathrm{N} \mathrm{ho} \mathcal{X})_{x,y}^E & \longrightarrow & (\mathrm{N} \mathrm{ho} \mathcal{X})_x^B \times (\mathrm{N} \mathrm{ho} \mathcal{X})_y^B \end{array}$$

in which the rows are fibration sequences. Now Claim 1 implies that the rightmost vertical arrow is an equivalence, and the leftmost vertical arrow is an equivalence for the same reason it was so in the proof of Claim 1, so it follows that the centre arrow is also an equivalence. Since every component of  $\mathcal{X}_1$  is included in some  $\mathcal{X}_{x,y}^E$ , Claim 2 is proved.

From the fact that the homotopy type of each space  $\mathcal{A}_i$  of a Segal space  $\mathcal{A}$  is determined by that of  $\mathcal{A}_0$  and  $\mathcal{A}_1$ , it follows that any map between Segal spaces which is equivalence on the zeroth and first spaces is an equivalence. Thus the theorem is a consequence of Claims 1 and 2.  $\blacksquare$

**Theorem.** For any complete Segal space  $\mathcal{X}$ , the map  $\mathcal{X} \rightarrow \mathrm{N} \mathrm{ho} \mathcal{X}$  is an  $\mathcal{S}$ -localization.

*Proof.* Given  $\mathcal{X}$ , by Lemma 1 there must exist a localization  $L_{\mathcal{X}}: \mathcal{X} \rightarrow L_{\mathcal{S}}\mathcal{X}$ , that is, an  $\mathcal{S}$ -equivalence to an  $\mathcal{S}$ -local object. Functoriality of  $\mathrm{N} \mathrm{ho}$  gives us a diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & L_{\mathcal{S}}\mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{N} \mathrm{ho} \mathcal{X} & \longrightarrow & \mathrm{N} \mathrm{ho} L_{\mathcal{S}}\mathcal{X} \end{array}$$

and by the preceding Proposition, the composite  $\mathcal{X} \rightarrow L_{\mathcal{S}}\mathcal{X} \rightarrow \mathrm{N} \mathrm{ho} L_{\mathcal{S}}\mathcal{X}$  is also

an  $\mathcal{S}$ -localization of  $\mathcal{X}$ . To conclude that the left-hand vertical arrow is an  $\mathcal{S}$ -localization, it would suffice to show that  $\mathrm{Nho} L_{\mathcal{X}}: \mathrm{Nho} \mathcal{X} \rightarrow \mathrm{Nho} L_{\mathcal{S}} \mathcal{X}$  is an  $\mathcal{S}$ -equivalence. But by Lemma 3,  $\mathrm{ho} L_{\mathcal{X}}$  is an equivalence of categories, so (obviously or by [Rezk] Theorem 7.7)  $\mathrm{Nho} L_{\mathcal{X}}$  is a weak equivalence. ■

# Chapter 5

## $\mathcal{S}$ -localization and homotopy fibres

In this chapter we will consider Segal spaces with an internal “zero object”. This will enable us to consider a special class of composable pair of internal morphisms, viz. those whose composite is zero; thus we can define a notion of a “fibre sequence” which has a universal property among such pairs.

**Definition.** A *pointed Segal space* is a Segal space  $\mathcal{X}$  together with a chosen zero object  $0 \in \mathcal{X}_0$  which satisfies  $\text{map}(x, 0) \simeq * \simeq \text{map}(0, x)$  for every  $x \in \mathcal{X}_0$ .

Since (all choices of) composites of homotopic maps are homotopic, it is clear that for objects  $x, y$  in a pointed Segal space  $\mathcal{X}$  there is a unique *zero component* of  $\text{map}_{\mathcal{X}}(x, y)$ , which we may denote  $\text{map}_{\mathcal{X}}^0(x, y)$ , which contains all maps which factor through 0.

**Definition.** For a pointed Segal space  $\mathcal{X}$ , let  $\mathcal{X}_{\uparrow} \subseteq \mathcal{X}_1$  be given by

$$\mathcal{X}_{\uparrow} = \bigcup_{x, y \in \mathcal{X}_0} \text{map}_{\mathcal{X}}^0(x, y)$$

and let  $\mathcal{X}_{\downarrow} \subseteq \mathcal{X}_2$  denote  $d_1^{-1} \mathcal{X}_{\uparrow}$ . Now define  $\widehat{\mathcal{X}^{\text{F}(2)}} \subseteq \mathcal{X}^{\text{F}(2)}$  to be the full Segal subspace of  $\mathcal{X}^{\text{F}(2)}$  induced by  $\mathcal{X}_{\downarrow} \hookrightarrow \mathcal{X}_2$ .

**Definition.** The map  $\mathcal{X}^{\text{d}^0}: \mathcal{X}^{\text{F}(2)} \rightarrow \mathcal{X}^{\text{F}(1)}$  (which forgets the first morphism in any composition) restricts to a map we will call  $\mathbf{f}: \widehat{\mathcal{X}^{\text{F}(2)}} \rightarrow \mathcal{X}^{\text{F}(1)}$ . If there is some

$\mathbf{g}: \mathcal{X}^{\mathbf{F}(1)} \rightarrow \widehat{\mathcal{X}^{\mathbf{F}(2)}}$  right adjoint to  $\mathbf{f}$  (in the sense of chapter 2) we say that  $\mathbf{g}\phi$ , or by abuse of language  $\mathcal{X}^{\mathbf{d}^2}\mathbf{g}\phi$  or just  $\mathcal{X}^{\mathbf{d}^1\mathbf{d}^2}\mathbf{g}\phi$ , is a (*homotopy*) *fibre* of  $\phi \in (\mathcal{X}^{\mathbf{F}(1)})_0 = \mathcal{X}_1$ . The analogous construction in  $\mathcal{X}^{\text{op}}$  is of course a (*homotopy*) *cofibre*. (We may omit the word “homotopy” because as with other constructions on Segal spaces, there is no other meaningful possibility.)

In the remainder of this chapter we will consider localization  $\mathbf{L}_{\mathcal{S}}$  with respect to the set of maps  $\mathcal{S}$  from Chapter 4. However, we will now be starting from Rezk’s “Segal space model structure” rather than his “complete Segal space model structure”: that is, we will not be taking  $\mathbf{F}(0) \hookrightarrow \mathbf{E}$  to be acyclic. Thus our localized structure will be “less local” than the one from Chapter 4.

By the first Proposition from the preceding Chapter,  $\mathcal{S}$ -local Segal spaces are precisely those for which all mapping spaces are discrete. However, unlike in the preceding Chapter,  $\mathcal{X} \hookrightarrow \mathbf{L}_{\mathcal{S}}\mathcal{X}$  is always a homotopy equivalence in level zero since this is true for all the generating acyclic cofibrations and preserved by pushouts. Hence the effect of  $\mathbf{L}_{\mathcal{S}}$  is to keep the the “space of homotopy automorphisms”  $\mathcal{X}_0$  intact while replacing the internal mapping spaces (the fibres of  $\mathcal{X}_1 \rightarrow \mathcal{X}_0 \times \mathcal{X}_0$ ) with discrete spaces.

Our goal is the following result, in which we show that although we obviously cannot expect it to preserve arbitrary homotopy Kan extensions,  $\mathbf{L}_{\mathcal{S}}$  still preserves *fibres* of maps when it is applied to complete Segal spaces (i.e. those for which the homotopy automorphism data is “correct” to begin with.)

**Theorem.** *Let  $\mathcal{X}$  be a complete pointed Segal space and suppose that the restriction  $\mathbf{f}: \mathcal{X}^{\widehat{\mathbf{F}(2)}} \rightarrow \mathcal{X}^{\mathbf{F}(1)}$  described above has right adjoint  $\mathbf{g}$ . Then the analogous construction on the  $\mathcal{S}$ -localized space,  $\mathbf{f}': (\mathbf{L}_{\mathcal{S}}\mathcal{X})^{\widehat{\mathbf{F}(2)}} \rightarrow (\mathbf{L}_{\mathcal{S}}\mathcal{X})^{\mathbf{F}(1)}$ , has a right adjoint  $\mathbf{g}'$  with the property that  $\mathbf{g}$  factors as  $\mathbf{g}' \circ (\mathcal{X} \hookrightarrow \mathbf{L}_{\mathcal{S}}\mathcal{X})^{\mathbf{F}(1)}$ . In particular the fibre of every internal morphism of  $\mathcal{X}$  is homotopy equivalent to the fibre of its image in  $\mathbf{L}_{\mathcal{S}}\mathcal{X}$ .*

*Proof.* Recall that  $\mathcal{X}^{\mathbf{F}(1)}$  and  $\mathcal{X}^{\mathbf{F}(2)}$  are complete Segal spaces, which follows from Corollary 7.3 of [Rezk].

First we note that  $\varepsilon\phi$  is a homotopy equivalence for all  $\phi \in (\mathcal{X}^{F(1)})_0$ . To see this, it suffices to show that the adjunction induced on homotopy categories has counit which is a natural isomorphism. On the level of homotopy categories, we have

$$\text{ho } \varepsilon\phi = \underset{(\text{ho } f)a \rightarrow \phi}{\text{colim}} ((\text{ho } f)a \rightarrow \phi) \cong \text{id}_\phi$$

since the colimit is over a category which has a terminal object (obtained e.g. by precomposing  $\phi$  with a morphism from 0). So every  $\varepsilon\phi$  and hence every  $g\varepsilon\phi$  is a homotopy equivalence and thus so is its one-sided inverse  $\eta g\phi$ .

Let us denote by  $\mathcal{X}^{\widetilde{F(2)}} \subseteq \mathcal{X}^{F(2)}$  the full Segal subspace induced by the inclusion of those components of  $(\mathcal{X}^{F(2)})_0$  which are hit by  $g$ . Then since  $\mathcal{X}^{\widetilde{F(2)}}$  and  $\mathcal{X}^{F(1)}$  are complete, it follows from the previous paragraph and the Indispensable Lemma that  $f$  and  $g$  induce mutually inverse homotopy equivalences  $\mathcal{X}^{\widetilde{F(2)}} \rightleftarrows \mathcal{X}^{F(1)}$ .

Now we have a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{X}^{\widetilde{F(2)}} & \xrightleftharpoons{f} & \mathcal{X}^{F(1)} \\ \downarrow & & \downarrow \\ (\mathbf{L}_S \mathcal{X})^{\widetilde{F(2)}} & \xrightarrow{f'} & (\mathbf{L}_S \mathcal{X})^{F(1)} \\ \downarrow & & \downarrow \\ \mathcal{X} \times \mathcal{X} \times \mathcal{X} & \rightarrow & \mathcal{X} \times \mathcal{X} \end{array}$$

and we wish to find  $g'$  adjoint to  $f'$ . Since the localization  $\mathbf{L}_S$  acts componentwise on each fibre of  $\mathcal{X}^{F(2)} \rightarrow \mathcal{X} \times \mathcal{X} \times \mathcal{X}$  and  $\mathcal{X}^{F(1)} \rightarrow \mathcal{X} \times \mathcal{X}$ , in order to be able to find a  $g'$  making the diagram homotopy commutative, it would suffice for  $g$  to take components of fibres to components of fibres up to homotopy, or equivalently, that

$$\text{hofib}(\mathcal{X}^{d^0 \times d^1}, (y, z)) \hookrightarrow \mathcal{X}^{F(1)} \rightarrow \mathcal{X}^{\widetilde{F(2)}} \rightarrow \mathcal{X} \times \mathcal{X} \times \mathcal{X}$$

be homotopically trivial on each component, for each  $(y, z) \in \mathcal{X} \times \mathcal{X}$ . That  $g$  preserves the last two factors  $\mathcal{X}$  is immediate because its inverse  $f$  does so by definition. For the first factor  $\mathcal{X}$ , take  $\psi \in \mathcal{X}^{F(1)}$  representing a component of

$(\mathcal{X}^{d^0 \times d^1})^{-1}(y, z)$ —we may denote the full Segal subspace corresponding to the component by  $\mathcal{X}^\psi$ —and define  $\phi \in \mathcal{X}^{F(1)}$  by  $\phi = \mathcal{X}^{d^2} \mathbf{g}\psi$ , that is, the fibre of  $\psi$ . Now  $\phi$  determines a constant diagram  $\phi: \mathcal{X}^\psi \rightarrow \mathcal{X}^{F(1)}$  which is composable with the restriction to  $\mathcal{X}^\psi$  of  $\mathbf{g}$ , because  $\mathbf{d}^1$  of one and  $\mathbf{d}^0$  of the other are both equal to the constant diagram determined by  $y$ . Moreover any choice  $\mathbf{h}: \mathcal{X}^\psi \rightarrow \mathcal{X}^{F(2)}$  for their composition lies entirely within  $\mathcal{X}^{\widetilde{F(2)}}$  because  $\mathcal{X}^{\widetilde{F(2)}} \subseteq \mathcal{X}^{F(2)}$  is an inclusion of path-components and  $\mathcal{X}^\psi$  is connected by definition. Now we have (where all maps are restricted to  $\mathcal{X}^\psi$ )

$$\mathcal{X}^{d^0 \times d^1 \times d^2} \mathbf{g} = \mathcal{X}^{d^0 \times d^1 \times d^2} \mathbf{g}\mathbf{f}\mathbf{h} \simeq \mathcal{X}^{d^0 \times d^1 \times d^2} \mathbf{h}$$

which is a constant map as required.

Similarly, for  $\eta: \mathcal{X}^{\widetilde{F(2)}} \rightarrow (\mathcal{X}^{\widetilde{F(2)}})^{F(1)}$  to extend to  $\eta': (\mathbf{L}_S \mathcal{X})^{\widetilde{F(2)}} \rightarrow ((\mathbf{L}_S \mathcal{X})^{\widetilde{F(2)}})^{F(1)}$  requires that  $\eta$  take components of fibres to components of fibres, in the sense that

$$\text{hofib}(\mathcal{X}^{d^0 \times d^1 \times d^2}, (x, y, z)) \hookrightarrow \mathcal{X}^{\widetilde{F(2)}} \rightarrow (\mathcal{X}^{\widetilde{F(2)}})^{F(1)} \rightarrow \mathcal{X}^{F(1)} \times \mathcal{X}^{F(1)} \times \mathcal{X}^{F(1)}$$

be homotopically trivial on each component, for each  $(x, y, z)$ . The same argument applies as for extending  $\mathbf{g}$ : the last two factors are preserved because  $\eta$  is the homotopy equivalence between the identity and  $\mathbf{g}\mathbf{f}$ , and the first factor is preserved because the relevant morphism in the composition can be assumed to be constant without loss of generality.

A similar argument shows that  $\varepsilon$  can be extended to  $\varepsilon'$ . Then  $(\mathbf{g}'\varepsilon' \circ \eta'\mathbf{g}')\psi$  and  $(\varepsilon'\mathbf{f}' \circ \mathbf{f}'\eta')\phi$  are homotopic to identity maps for all  $\psi, \phi$ , so since mapping spaces in  $(\mathbf{L}_S \mathcal{X})^{F(1)}$  and  $(\mathbf{L}_S \mathcal{X})^{\widetilde{F(2)}}$  are discrete (by compatibility of  $\mathbf{L}_S$  with the cartesian closure; see Chapter 4, Lemma 1) it is possible to choose compositions so that the composites are the identity, as required.  $\blacksquare$

# Bibliography

- [BK] A. K. Bousfield, D. M. Kan. *Homotopy limits, completions and localizations*. Springer Lecture Notes in Mathematics **304** (1972).
- [Bor] F. Borceux. *Handbook of categorical algebra*. Cambridge University Press, Cambridge, 1994.
- [CP] J.-M. Cordier, T. Porter. Homotopy-coherent category theory. *Trans. Amer. Math. Soc.* **349** (1997), 1–54.
- [DHKS] W. G. Dwyer, P. S. Hirschhorn, D. M. Kan, J. H. Smith. *Homotopy limit functors on model categories and homotopical categories*. AMS Mathematical Surveys and Monographs **113** (2004).
- [Hir] P. S. Hirschhorn. *Model categories and their localizations*. AMS Mathematical Surveys and Monographs **99** (2003).
- [Qui] D. G. Quillen. *Homotopical Algebra*. Springer Lecture Notes in Mathematics **43** (1967).
- [Ree] C. L. Reedy. Homotopy theory of model categories. Preprint 1974.
- [Rezk] C. Rezk. A model for the homotopy theory of homotopy theory. *Trans. Amer. Math. Soc.* **353** (2001), 973–1008.
- [Seg] G. Segal. Categories and cohomology theories. *Topology* **13** (1974), 293–312.
- [Sim97] C. Simpson. Limits in n-categories. Preprint alg-geom/9708010.
- [Sim99] C. Simpson. A Giraud-type characterization of the simplicial categories associated to closed model categories as  $\infty$ -pretopoi. Preprint math.AT/9903167.