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MATHEMATICS

INTRINSIC NILPOTENT APPROXIMATION TO FILTERED LIE ALGEBRAS

(approximate Hamiltonian action/asymptotic moment map/
phase space decomposition/microlocal analysis/
hypoellipticity)

by

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Symbols Used

1. F denotes boldface F (indicated in text only on first appearance of the given symbol).

1a. Ordinary underline denotes italics.

2. R in Theorem p.10 and Corollary p.11 should not be boldface.

3. ℓ (p.11 & 12) denotes lowercase italic form of L.

4. $\mathcal{S}$ (p.12) should be some distinguished form of S other than boldface.

5. The Proposition, p.9. includes the sentence: *Moreover, if, . . .* (on p. 10).

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Abstract

An intrinsic approximation process is introduced which arises in the context of non-isotropic perturbation theory for certain classes of differential or pseudodifferential operators P. The structure of P itself determines the minimal information that the initial approximation must contain. This may vary from point to point and requires corresponding approximating Lie algebras.

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There has been considerable work in recent years on using nilpotent Lie algebras for local and microlocal approximation of differential and pseudodifferential operators, in the context both of hypoellipticity and the construction of parametrices, (cf., for example, ref. 1,2,3). If we allow solvable algebras as well, these have also arisen in control theory, (cf., for example, ref. 4). In other guises, approximation by finite-dimensional Lie algebras figures in non-linear filtering, (cf., ref. 5).

In general the approximation has been non-intrinsic. Ordinarily the approximating Lie algebras, usually free nilpotents, are introduced externally. In (6) they are constructed internally, but still involve an explicit choice of generators. The approximation procedures have been viewed primarily as tools, albeit important ones, with the emphasis laid on the particular application.

Here we introduce a different approach, which puts the approximation issue in the forefront and provides an intrinsic construction. We shall present only an announcement of the main results and ideas. Details and other discussion appear in (7).

This approximation process is most naturally viewed from a seemingly abstract algebraic context, namely the "approximation" of certain infinite-dimensional filtered Lie algebras $\mathfrak{L}$ (over $\mathbb{R}$) at "points" p of "finite rank" by intrinsically constructed finite-dimensional graded (nilpotent) Lie algebras $\mathfrak{g}_p$, of rank equal to rank p. In particular, these nilpotent Lie algebras vary from "point" to "point"; this is in contrast to the external construction, where $\mathfrak{g}$ is deliberately chosen locally constant.

The construction of the $\mathfrak{g}_p$, and even the introduction of the "points" p involves a process of "localization", which requires an additional element of structure: namely, the Lie algebra L must be an F-module, where F is an R-algebra on which L acts as derivations, with $[X,fY] = X(f)Y + f[X,Y]$ for $X,Y \in L$ and $f \in F$. The "points" p correspond to the maximal ideals of F. We insist that the corresponding quotient fields of F must be $\cong$ R.

The process of constructing g_p is formal and, together with the above framework, naturally admits of various generalizations and refinements. We shall be concerned, however, with two concrete classes of examples: the local case, where L is a filtered Lie algebra of smooth vector fields on a manifold M, and $F = C^\infty_{\mathbb{R}}(M)$; the microlocal case, where L is a filtered Lie algebra of first-order pseudodifferential operators on M (more precisely, the associated algebra of principal symbols, with respect to Poisson brackets, in S^1_{hom}), and $F = S^0_{hom}$ (corresponding to principal symbols of zero-order pseudodifferential operators). Here $C^\infty_{\mathbb{R}}(M)$ denotes real-valued C^∞ functions on M, and S^i_{hom}, $i=0,1$, denotes real-valued C^∞ functions on $T^*M/0$ positive-homogeneous of degree i. In the first case the maximal ideal spectrum of F is M, and in the second case $T^*M/0$. The "finite rank" condition, which is manifested in the local algebra context as a stabilization property, becomes in the first case the Hörmander spanning condition (8), and in the second case the appropriate microlocal analogue, to be given below.

<u>Definition</u> A filtered Lie algebra L of vector fields at $x \in M$ consists of a sequence of subspaces L^i $i = 1,2,...$ with $L = \bigcup_{j=1}^{\infty} L^j$ such that

(i) $L^1 \subset L^2 \subset L^3 \subset \ldots$

(ii) $[L^i, L^j] \subset L^{i+j}$

(iii) Each L^i is an F-module, i.e., $FL^i \subset L^i$

(iv) As an $\dot{F}_x$ (i.e., $\dot{C}^\infty_x$) module each $\dot{L}^i_x$ is finitely generated.

Here $\dot{F}_x$, $\dot{L}^i_x$ denote germs at x. In practice one often uses a somewhat stronger local version of (iv). The simplest class of examples occurs when L^1 generates L, i.e., there are vector fields $X_1,\ldots,X_k$ such that L^1 = all $C^\infty(M)$ linear combinations of $X_1,\ldots,X_k$; $L^2 = L^1 + [L^1,L^1]$, etc.

We say that L is of <u>finite rank</u> at x if $\alpha_x(\dot{L}^r_x) = T_xM$ for some r, where α_x denotes evaluation of vector fields at x. Clearly finite rank is an open condition. The smallest such r is called the rank of L at x. Let $\dot{m}_x$ denote the maximal ideal of the local ring $\dot{F}_x$, i.e., the germs which vanish at x.

<u>Proposition</u> Let $\{L^i\}$ be of finite rank r at x. Let

$$g^i_x = \frac{\dot{L}^i_x}{\dot{L}^{i-1}_x + \dot{m}_x \dot{L}^i_x} \qquad (\text{where } \dot{L}^0_x = 0).$$

Then for $i > r$ $g^i_x = 0$, and for $i \leq r$ g^i_x is a finite-dimensional vector space over R. Let $\pi_i\colon L^i \to g^i_x$ be the canonical R-linear projection. Define $g_x = g^1_x \oplus \ldots \oplus g^r_x$. Then via the π_i g_x inherits canonically the structure of graded nilpotent Lie algebra over R.

The proof is straightforward, the vector space structure following from the fact that $\dot{F}_x/\dot{m}_x \cong R$. The main point to check is that $[\dot{L}^{i-1}_x + \dot{m}_x\dot{L}^i_x, \dot{L}^j_x]$

$\dot{L}_x^{i+j-1} + \dot{m}_x\dot{L}_x^{i+j}$. This uses the fact that the filtration begins with an L^1 term, so that $i + j-1 \geq i$.

The construction of g_x makes sense in the real-analytic and formal power series categories (and the lifting theorem, to be discussed below, is valid here also). Here (iv) automatically follows from Noetherian considerations.

The corresponding definitions and results are valid in the microlocal case, with $\dot{m}_x$ replaced by $\dot{m}_{(x,\xi)}$. In fact we get the pair $g_{(x,\xi)}$, η where $\eta \in g^*_{(x,\xi)}$, the dual space. The definition of finite rank at (x,ξ) is that $\exists f \in \dot{L}^r_{(x,\xi)}$ such that $f(x,\xi) \neq 0$.

Examples 1. Let $M = \mathbb{R}^2$, with L^1 generated by $\partial/\partial y$ and $y\ \partial/\partial t$, and L generated by L^1. Then $g_x \cong R^2$ for $x \neq (0,0)$, and $g_0 \cong h_1$, the Heisenberg algebra.

2. We introduce a constancy of rank condition modelled on (6), namely that $\dim \alpha_x(L^i)$, $i = 1,\ldots,r$ is independent of x. Then $\dim g_x^i$ is independent of x, and $\dim g_x = \dim M$. Furthermore, the g_x vary "smoothly" with x, but need not be isomorphic. Unlike (6), we do not assume that L^1 generates L. One can thus apparently extend to this context the hypoellipticity result in (9).

3. If $\{L^i\}, x$ is a filtered Lie algebra of vector fields, and $\{\tilde{L}^i\}$, (x,ξ) the associated filtered Lie algebra of symbols at $(x,\xi) \in T^*M/0$, then there

is a canonical surjective homomorphism $g_x \rightarrow g_{(x,\xi)} \rightarrow 0$.

4. Let Σ be a smooth conic submanifold of $T^*M/0$. Let $L^1 = \{u \in S^1_{hom} | u=0$ on $\Sigma\}$ and let $L^2 = L^3 = \ldots = S^1_{hom}$. Then $\{L^i\}$ is of rank 1 at any $(x,\xi) \in \Sigma$, and of rank 2 at any $(x,\xi) \in \Sigma$. At the former points $g_{(x,\xi)} \cong R$, and at the latter points $g_{(x,\xi)} \cong N(\Sigma)_{(x,\xi)} \oplus R$. Here $N(\Sigma)_{(x,\xi)}$ denotes the conormal space to Σ, and the Lie bracket is defined via $[(df_1,r_1), (df_2,r_2)] = (0, \{f_1,f_2\}|_{(x,\xi)})$. This filtered Lie algebra is closely related to the operator classes in (10).

A priori g_x may be too small because of the "collapsing" involved in its definition. That this is not the case is a consequence of Nakayama's lemma (11), a basic local algebra result. This implies that $g^i_x = 0 \Longleftrightarrow \dot{L}^{i-1}_x = \dot{L}^i_x$. An additional consequence is that stabilization of the sequence $\dot{L}^1_x \subset \dot{L}^2_x \subset \ldots$ rather than spanning is what is needed to define g_x. Similarly for $g_{(x,\xi)}$.

We need a way to relate g_x back to $\{L^i\},x$. In constrast to the external construction, we do not have available a partial homomorphism (12). The following weaker substitute is most natural in our setting, both local and microlocal.

<u>Definition</u>. Let $\{L^i\},x$ be a filtered Lie algebra and $\underset{\sim}{h} = \underset{\sim}{h}_1 \oplus \ldots \oplus \underset{\sim}{h}_s$ a graded nilpotent Lie algebra. A weak homomorphism γ (at x) from h into L is an R-linear map such that

(i) $\gamma : h_i \rightarrow \dot{L}^i_x$

(ii) For any Y_i, Y_j ε h_i, h_j, respectively,

$$\gamma([Y_i,Y_j]) - [\gamma(Y_i), \gamma(Y_j)] \in \dot{L}_x^{i+j-1} + \dot{m}_x\dot{L}_x^{i+j}$$

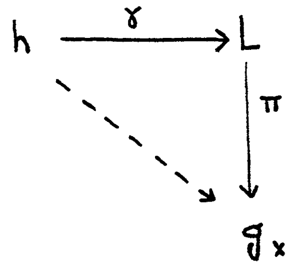
There are many weak homorphisms from g_x to $\{L^i\},x$. In fact, let β be any <u>cross-section</u> of the canonical projection π: $L \to g_x$. That is, β is any R-linear map such that β: $g_x^i \to \dot{L}_x^i$ and $\pi_i \circ \beta = I \quad \forall i$. Then β is automatically a weak homomorphism. The most important constructs are those which do not depend essentially on the choice of cross-section.

The following consequence of Nakayama's lemma is our basic technical tool. It is our substitute for an explicit a priori relation between g_x and a set of generators for $\{L^i\},x$.

<u>Lemma</u>. Let $\{L^i\},x$ be a finite rank r, and let $\{Y_\alpha\}$ be a graded basis for g_x; i.e., each $Y_\alpha \in g_x^i$ for some i, which we denote by $|\alpha|$. Let β be any cross-section. Then for every $i \leq r$ the $\{\beta(Y_\alpha)\}_{|\alpha|\leq i}$ span $\dot{L}_x^i$ as a $\dot{C}_x^\infty$-module.

The algebra g_x satisfies the following "universal" property, which shows that it is in some sense the minimal approximant to $\{L^i\}$ at x.

<u>Proposition:</u> Let $\{L^i\},x$ be of rank r, and $h = h_1 \oplus \dots \oplus h_s$ a graded nilpotent with γ: $h \to L$ a weak homomorphism. Then $\pi \cdot \gamma$ (i.e., $\pi_i \circ \gamma_i : h_i \to g_x^i$) is a homomorphism of graded Lie algebras.



Moreover, if, in particular, $\gamma(h_i)$ generates L^i over C^∞ modulo L^{i-1} for every $i \leq r$, then $\pi\circ\gamma$ is surjective.

It follows that if g is the external free nilpotent usually introduced, then g_x is a quotient of g.

One can define a notion of weak morphism between $\{L^i\}$, x, $\{K^i\}$,y, two filtered Lie algebras on possibly distinct manifolds, leading functorially to corresponding homomorphisms of graded nilpotents g_x, h_y. This shows, in particular, that the isomorphism class of g_x is invariant under diffeomorphisms. All the preceding considerations also hold in the microlocal case. In particular, $g_{(x,\xi)}$ is invariant under Fourier integral conjugation.

In the local case the following lifting theorem (cf. ref. 12) gives a precise sense in which g_x approximates $\{L^i\}$,x. Let β be a cross-section. In analogy with (12) define $W:C_x^\infty(M) \to C_0^\infty(g_x)$ via $(Wf)(u) = f(e^{\beta(u)}x)$, and a notion of local order at 0. Let $\tilde{Y}$ be the left-invariant vector field corresponding to Y.

Theorem Let $Y \in g_x^i$ and $X = \beta(Y)$. Then $WX = (\tilde{Y}+R)W$, where the vector field R is of local order $\leq$ i-1 at 0.

Corollary Let $X \in L^i$. Then $WX = (\pi_i(X)+S)W$, where S is of local order $\leq$ i-1 at 0.

Notice that although W depends on β, the element $\pi_i(X) \in g_x^i$ does not. The proof follows the general lines of (12), but requires additional work since we have only weak homomorphisms available. Also, in analogy with (13) one can prove a corresponding lifting theorem to a homogeneous space g_x/h_x of the same dimension as M. The graded subalgebra h_x has an intrinsic definition. In the constant rank case of Example 2 $h_x = 0$. This is a sharper statement than the corresponding case of (13) because of the minimality of g_x.

Helffer and Nourrigat (2,3) in conjunction with their work on maximal hypoellipticity introduce a set which we denote by $\Gamma^{\lambda}_{(x,\xi)} \subset g^*$, where g is an externally introduced free nilpotent. Because of the minimality of $g_{(x,\xi)}$ it is possible to adapt their construction to obtain an intrinsically defined set $\Gamma_{(x,\xi)} \subset g^*_{(x,\xi)}$, as shown below. It appears that one can adapt the arguments of (3) to the context of weak homomorphisms and prove the analogue of an approximation theorem of Nourrigat at irreducible representations corresponding to the points of $\Gamma_{(x,\xi)}$.

Let $\{L^i\}$, (x,ξ) be of finite rank, with $g_{(x,\xi)}$ the associated graded nilpotent, with standard dilations δ_t. Let β be a cross-section. A sequence $\{t_n, (x_n,\xi_n)\}$ with $t_n \in R^+$, $(x_n,\xi_n) \in T^*M/0$ such that $x_n \to x$, $|\xi_n| \to \infty$, and $\xi_n/|\xi_n| \to \xi/|\xi|$ is called β-admissible if $\exists \ell \in g^*_{(x,\xi)}$ such that lim $\beta(\delta_{t_n}Y)(x_n,\xi_n)$ exists and equals $\langle \ell, Y\rangle$ $\forall Y \in g_{(x,\xi)}$.

<u>Proposition</u>. Let β_1, β_2 be cross-sections. They they have the same admissible sequences, with the same corresponding limits $\ell \in g^*_{(x,\xi)}$. Let $\mathcal{A}_{(x,\xi)}$ denote the set of admissible sequences at (x,ξ), and define $\Phi_{(x,\xi)}: \mathcal{A}_{(x,\xi)} \to g^*_{(x,\xi)}$ by $\{(t_n, (x_n,\xi_n)\} \mapsto \ell$. Let $\Gamma_{(x,\xi)}$ denote its image. $\Gamma_{(x,\xi)}$ is a closed subset of $g^*_{(x,\xi)}$, closed under dilations δ^*_t, and closed under the coadjoint action of $\underline{G}_{(x,\xi)}$, the corresponding group. The Helffer-Nourrigat set $\Gamma^{\lambda}_{(x,\xi)}$ (defined primarily when L^1 generates) is the image of $\Gamma_{(x,\xi)}$ under the injection $0 \to g^*_{(x,\xi)} \to g^*$ corresponding to the natural surjection $g \to g_{(x,\xi)} \to 0$. Thus $\Gamma_{(x,\xi)}$ contains all the essential information.

To properly understand $\Gamma_{(x,\xi)}$ we must digress slightly. Given a Hamiltonian action (cf. ref. 14) of a Lie Group $\underline{G}$ on a symplectic manifold N, one obtains an intrinsically defined moment map $\Phi: N \to g^*$ which is G-equivariant. As a heuristic principle one expects the irreducible representations which enter into the "quantization" (if it exists) of the G-action on N to be those associated to the image of Φ. (In case G and N are compact this is given precise realization in (14).) In our context the original infinite-dimensional filtered Lie algebra L, already realized as an infinitesimal Hamiltonian action on $T^*M/0$, induces via a choice of cross-section an "approximate Hamiltonian action" of the finite-dimensional Lie group $G_{(x,\xi)}$ on the space $\mathcal{A}_{(x,\xi)}$, thereby making $\Phi_{(x,\xi)}$ $G_{(x,\xi)}$-equivariant. It is thus natural to regard $\Phi_{(x,\xi)}$ as an "asymptotic moment map" with image $\Gamma_{(x,\xi)}$.

The systematic investigation and development of the intrinsic approximation process as an analytical tool remains to be carried out. Nevertheless we believe it has significant potential. Precisely because of its abstract nature it may be pertinent to various contexts, such as (4) and possibly (5). For example, under appropriate conditions one can allow an L^0 term, and obtains a semi-direct sum $g^0 \circledS g$ with g^0 arbitrary and g graded nilpotent as before. Second, the g_x and $g_{(x,\xi)}$ are "invariants" of the filtered Lie algebra, and as such may be useful, for example, for canonical form results of the type in (15, Chap. 9). Particularly appropriate for "geometrical" microlocal contexts such as Example 4 is the fact that no canonical choice of generators is necessary, nor need L^1 generate. Moreover, although one may want to introduce g's externally as technical tools, the essential information already resides in the g_x and $g_{(x,\xi)}$.

The introduction of the filtered algebra $\{L^i\}$ is natural. It defines the class of operators we examine, namely the "enveloping" algebra $\underset{\sim}{U}(\underset{\sim}{L})$ (with C^∞ or zero-order pseudodifferential coefficients), and determines a filtration on U(L). A given operator P may lie in U(L) for various filtered algebras L. However, the relevant information about P qua element of a specific U(L) is contained in $\{\pi(\hat{P})\}$ where $\pi \in \Gamma_{(x,\xi)}$. (Given $P \in \underset{\sim}{U}^m(\underset{\sim}{L})$ one associates a "leading term" $\hat{P} \in \underset{\sim}{U}_m(g_{(x,\xi)})$, i.e., homogeneous of degree m. Strictly speaking this is not well-defined in general, though apparently $\pi(\hat{P})$ is well-defined for $\pi \in \Gamma_{(x,\xi)}$). We regard L as determining an intrinsically associated "Fourier analysis", or "phase space decomposition" in the case of $T^*M/0$, namely the decomposition into the relevant irreducible representations (i.e., those in $\Gamma_{(x,\xi)}$) of the approximating groups G_x or $G_{(x,\xi)}$. For example, one can define a notion of L-<u>hypoellipticity</u>

coinciding with maximal hypoellipticity when L^1 generates. The natural L-hypoellipticity conjecture is then that P is L-hypoelliptic $\Longleftrightarrow \pi(\dot{P})$ is left-invertible $\forall \pi \in \Gamma_{(x,\xi)}/0$. (Also, in (7) a provisional notion of L-wave front set is defined.) To put this in perspective: for L of rank 1 U(L) consists of the algebra of all differential (or pseudodifferential) operators, with the standard increasing filtration. This corresponds to standard elliptic theory and standard wave front set.

References

1. Rothschild, L.P. & Stein, E.M. (1977), Acta. Math. 137, 248-315.

2. Helffer, B. & Nourrigat, J. (1985), Hypoellipticite Maximale pour des Operateurs Polynomes de Champs de Vecteurs (Birkhäuser, Boston).

3. Nourrigat, J. (1984) in Contemporary Mathematics (American Mathematical Society, Providence), Vol. 27, pp. 199-204.

4. Crouch, P. (1984), SIAM J. Control and Optimization 22, 40-54.

5. Hazewinkel, M. & Willems, J.C. (1981), Stochastic Systems: The Mathematics of Filtering and Identification and Applications (Reidel, Dordrecht).

6. Metivier, G. (1976), Comm. in P.D.E. 1, 467-519.

7. Rockland, C. (1985), Technical Report LIDS-R-1482, Laboratory for Information and Decision Systems, M.I.T., Cambridge.

8. Hörmander, L. (1967), Acta. Math. 119, 147-171.

9. Rothschild, L.P. (1979), Comm. in P.D.E. 4(6), 645-699.

10. Boutet de Monvel, L., Grigis, A., & Helffer, B. (1976), Asterisque 34-35, 93-121.

11. Golubitsky, M. & Guillemin, V. (1973), Stable Mappings and Their Singularities (Springer, New York)

12. Goodman, R.W. (1976), Nilpotent Lie Groups (Lecture Notes in Mathematics 562) (Springer, Berlin)

13. Helffer, B. & Nourrigat, J. (1979), Arkiv For Matematik 2, 237-254.

14. Guillemin, V. & Sternberg, S. (1982), Invent. Math. 67, 515-538.

15. Trèves, F. (1980), Introduction to Peudodifferential and Fourier Integral Operators, Vol. 2 (Plenum, New York)