Expressive Mathematics:
Learning by Design

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Abstract

Since the writings of Francis Parker and John Dewey, educators have been excited by the possibilities of learning through design activities. At their best, design activities make it possible to learn about important ideas in a creative and compelling way. The introduction of computational media to education made this idea only more appealing, as thoughtful educators saw how computers could make it possible to explore even more areas of human understanding in an open-ended, design-based environment.

This dissertation explores the computer as an expressive medium for learning mathematics by taking as a model the architectural design studio, where expression and expressive activity are at the center of the learning process.

The thesis presents a theory or model of expression which looks at expression as an important component of mathematical activity, and suggests that, like schooling, expression is a complex, systemic phenomenon. The thesis also presents three empirical studies: a small-scale experiment, a longer-term observational study, and an in-depth experiment on mathematics learning in an expressive context.

The results of these studies show: (1) The design studio is a "learning system" that relies open-ended nature design challenges, iterative work, public presentations, and structured collaborative conversations known as "desk crits." (2) Students can and do learn mathematics through design, and they like math more as a result of working in a studio setting. (3) Mathematics learning takes place during design as a result of structured interactions in desk crits and pin-ups, the expressive nature of the design activities, and the underlying mathematical structure of the computational tools used.

While there is clearly further research to be done, these results suggests that expressive activities are a promising venue for thinking about deep mathematical ideas, and that the design studio is a useful model for structuring such activities.

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Expressive Mathematics

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Introduction

When a student learns to draw a human face in a life-drawing studio class, it is quite likely that before starting he or she will look at the work of masters of the craft: a face drawn by Leonardo, one by Van Gogh, by Picasso, by Matisse. A student may look at ten, twenty, or even a hundred such master works. When he or she sits down to draw a face, the problem is richer, but in every respect just as challenging and as genuine as it was before the student looked at the examples. In fact, even if the student were a perfect mimic and could recreate the masters' drawings perfectly, the drawing of a new face would still be a creative process.

Traditional mathematics problems are not presented to students in a way that makes them creative and expressive in this sense (Wilensky 1995, Boaler 1996, Noss and Hoyles 1996). Typical mathematics questions expect one or a very limited set of "right" answers and offer learners few meaningful choices about how to frame and solve the problem (Bottge and Hasselbring 1993). Research at the Media Lab has taken steps to change this by showing that expressive activities can be a powerful context for learning mathematics (Harel and Papert 1991, Kafai and Harel 1991b, Wilensky 1995, Kafai 1996, Noss and Hoyles 1996, Papert 1996, Shaffer in press), but fundamental questions about the relationship between expression and mathematical learning have not yet been answered by this body of work. This thesis looks at three unanswered questions about computation, mathematics learning, and expression:

(1) How do changes in mathematical thinking take place during expressive activity?

(2) What aspects of computational media help make computers expressive devices for thinking about mathematical questions?

(3) What aspects of the learning environment help students learn mathematics through expressive activities using computational media?

The thesis that follows presents a series of investigations into and meditations on these questions. The empirical and theoretical work presented is both independent and cumulative: that is, each chapter stands on its own (indeed, all are journal articles at some stage in the production-submission-revision-publication process), but each chapter builds on or supports the work in the other chapters.

The first chapter presented here is actually the last of the pieces chronologically. It is presented first because it provides an overview of expression as a psychological process. This chapter, in a sense, introduces the ideas raised and explored in the work that follows, looking at the issues of expression, mathematics, computation, and learning from a theoretical perspective. The chapter describes in some detail a "theory" or "model" of expression, looks at expression as an important component of mathematical activity, and then explores some specific ways in which computational media make mathematics (and other traditional subjects) more expressive. Finally, the chapter explores the idea that, like schooling, expression is a complex, systemic phenomenon—and that merging the two thus requires a systemic, rather than ad hoc, approach.

The remaining chapters of the thesis proceed in chronological order, beginning with a small-scale experiment, moving on to a longer-term observational study, and concluding with an in-depth experiment on mathematics learning in an expressive context.

The first of these investigations is a qualitative study of the idea of learning mathematics in a design studio. The experiments described were designed to determine whether the "mathematics studio" was a viable idea—could students learn mathematics through computer-mediated design activities?—and if so, what were some of the important characteristics of learning mathematics in an expressive way. The results of that early work showed that students can and do learn mathematics through design, and also that learning in a computational studio setting had important affective consequences in the learning process. That is, the kids liked math and liked learning math a lot more as a result of participating in the experiments.

Next is an observational study of a traditional architecture design studio. The goal of the study was to identify the key elements of the design studio as a learning environment, so that in subsequent experiments it would be possible test the design model more fully and faithfully—and also to make sure that in a larger-scale and longer-term experiment, the environment would be maximally effective in supporting both interesting design and interesting mathematics learning. The observational study identified a few key features of the design studio, including the open-ended nature of the design challenges posed, the iterative structure of design work, the importance of exemplars, the role of public presentations of work, and the central importance of structured collaborative conversations known as "desk crits." The observational study also made it clear that these features did not work in isolation, but functioned together as a holistic learning "system."

Third is an experimental study (this time both qualitative and quantitative) of mathematics learning through design in a computational studio setting. The project was large-scale, with 12 students working for a total of 80 hours over 4 weeks, and led up to a 6-week exhibit at the MIT Museum designed, created, and installed by the students. During the course of this study, detailed "design histories" were collected for each of the students, recording their design activity over time, including both records of sketches (process drawings) and final products, as well as detailed accounts of the students' own thinking about and descriptions of their designs and design process. These design histories were analyzed using qualitative and quantitative techniques to uncover some of the important cognitive processes that turned creative activity into reflective thinking about mathematics during the experiment.

The idea that personal expression is a powerful context for learning is certainly not new. Francis Parker, John Dewey, and Seymour Papert have written about the role of expression in learning, but did not propose specific hypotheses about how expression functions in learning (Parker 1894/1969, Dewey 1915, Dewey 1938, Dewey 1958, Papert 1980, Papert 1993, Papert 1996). Investigators are looking at the potential impact of expression in mathematical learning, but not necessarily at the underlying relationship between expressive activity and cognitive processes (Harel and Papert 1991, Kafai and Harel 1991b, Wilensky 1995, Kafai 1996, Noss and Hoyles 1996). Theorists have looked at applying methods of art and design learning to other disciplines, but have not looked at the role of computation in this process, or at the systemic nature of design learning (Sizer 1984, Eisner 1986, Gardner 1991, Sizer 1992, Willett 1992, Gardner 1993, Perkins and Blythe 1994).

What this thesis adds to this prior work on expression and learning is a systematic and integrated approach to thinking about expression and expressive activity, focused on but not limited to the domain of mathematics learning, in the context of computational media. The approach is based on the tradition of architectural design and the design studio, which provides a working pedagogical system for learning through creative activity. The approach here is thus both theoretically sound, and practically-grounded, and the results of this work suggest that expressive mathematics is a potentially-transformative way to think about new media and learning.

One word of warning is that the chapters of this thesis were originally composed as independent papers, and are included here in more or less their original form to preserve the unfolding nature of the projects they describe. The disadvantage of this approach is that some of the material—particularly theoretical background presented at the beginning of the original papers—gets repeated. It would have been possible to remove this repetition; however, I felt it was useful to make clear the development of these ideas over the course of the investigations, rather than hide the changes that happened over time under the cloak of post hoc editing. I hope that decision makes the collection a richer picture of the projects and the issues they explore.

Expressive Mathematics:
towards a new psychology of mathematics for a computational world

Introduction

Expression? *Expression.* Expression??? *Expression!!!*

The same word, repeated four times. Change a few marks of punctuation, add some italics, and a conversation is born, a dialog between a skeptic and a believer, an exchange that grows quickly in intensity, and in emotion. Our ability to express ourselves is an essential part of what it means to be human. Recent theories suggest that the advance that separated human from primate cognition was the development of an ability to recognize another person's communicative intent, and that the subsequent growth of human culture and society has been driven to a large extent by development of a wide array of visual and symbolic languages for expressing internal experience (see Donald 1991).

One of the most important collections of these symbolic languages goes by the name of mathematics. Scholars have pointed out the key role that mathematics has played (and continues to play) in human cognitive evolution (Donald 1991, Shaffer and Kaput in preparation), and as Papert suggests, the root word "math" originally referred to any kind of learning—a meaning preserved in the word "polymath" (Papert 1980). The roots of mathematics are deeply connected to the fundamental human act of expression; and yet, somewhere in the long development of human culture and of the science of mathematics, expressiveness and mathematics parted company in the popular understanding of what it means to think mathematically. Snow's description of two cultures (Snow 1959) and Pirsig's attempt to reconcile the technological and the aesthetic (Pirsig 1974) both describe a split that can be found by asking any high-school student what mathematics means (or does not mean) in his or her life (see Boaler 1996).

This paper is an attempt to reunite mathematics and expression from a psychological perspective. In particular, the paper addresses three questions:

1) What is expression and what role does it play in cognitive development?

2) What are the implications of a theory of expression and development for mathematics and mathematics learning?

3) What roles can new computational media play in thinking about mathematics and mathematics learning as expressive endeavors?

The answers to these questions suggest that a new pedagogy of expressive mathematics, combined with the power of computational media, has the potential to make deep understanding and appreciation of important mathematical ideas accessible to a wider range of students than has been reached using more traditional methods.

Background

There is a long tradition in the field of education of exploring how students can learn through creative activity. In the nineteenth century, practitioners such as Froebel, Montessori, and Parker explored ways in which children can learn through self-directed exploration of the world around them (Parker 1894/1969, Montessori 1984, Brosterman and Togashi 1997). More recently, theorists including Dewey, Bruner, and Papert have all discussed the nature of expressive activity and its role in learning (Dewey 1915, Dewey 1938, Dewey 1958, Papert 1980, Papert 1993, Bruner 1996, Papert 1996).

More recently, a series of practical experiments have begun to explore the nature of mathematics learning in the context of artistic activities as a way to understand the interaction of expression and cognitive development in the domain of mathematics. These include teaching methods such as Loeb's Design Science curriculum (Loeb 1993), Willett's comparative study of elementary mathematics learning in the context of arts-based activities versus traditional elementary mathematics instruction (Willett 1992), and a collection of projects known as "Escher's World" that look at how students learn transformational geometry in the context of graphic design (Shaffer 1996a, Shaffer 1996c, Shaffer 1997b, Shaffer, Kaput et al. 1997, Shaffer in preparation, Shaffer in press).

The "Escher's World" work, in particular, suggests that the expressive nature of arts-based activities play an important role in the development of mathematical understanding (Cossentino and Shaffer 1997, Shaffer in preparation). This paper both builds upon and supports this body of previous work by describing a psychological theory of expression, and based on that theory, suggesting an alternative way to conceptualize mathematics and mathematics learning.

What is expression?

Incepts, Excepts, and Expression

Incepts

In contrast to the behaviorist program (Skinner 1971), where the human mind is treated as a "black box" whose internal workings are unknowable and ultimately unimportant for understanding human behavior, theorists from a wide range of perspectives on development beginning with Piaget and Vygotsky and continuing through the present, suggest that the mind has significant internal frameworks of knowledge and understanding (see, for example, Bruner 1973, Vygotsky 1978, Fischer 1980, Papert 1980, Gardner 1982, White 1991, Gruber and Voneche 1995). There are things going on "in there" that have coherence and structure, and whose relationships can be meaningfully explored.

There is a rich literature of research on exactly how the "things inside" are acquired, stored, and processed on a physical level. Specific areas of the brain have been associated more or less closely with specific mental processes or types of information (Block 1981, Gardner 1982, Gardner 1983, Kosslyn and Koenig 1992). But even without going into the details of how the "things inside" the mind are stored and processed at the neural level, we can still talk meaningfully about mental events. Theorists debate distinctions between concepts represented in words and images, or between percepts and concepts (Arnheim 1969, Bruner 1973, Block 1981). For the moment, however, let us suspend concerns about these precise delineations. There are clearly some salient differences among these various kinds of mental structures and events. But equally clearly, they are all examples of internal mental events that in some way represent pieces of our understanding of the world.

For lack of a better word, and following the lead of psychological terms for mental constructs such as "percept" and "concept," I propose that we refer to internal mental events with the general term "incept." This provides a general way to describe internal events, whether they are cognitive or affective in nature. As a general term, the word incept allows us to recognize that thoughts and feelings are intertwined in our minds: we associate feelings with our ideas, and we think about our emotions. Incept is a broad term referring to a loose collection of idea/feeling/sensation that we in some way group together as a more or less distinguishable mental state.

Excepts

If the incepts of our mental life are many and varied, so too are the products of mental activity. As in the case of incepts, we can make distinctions among the many ways in which humans communicate. On the most basic level, Donald (Donald 1991) argues that the permanent nature of writing, drawing, and other recorded media makes them fundamentally different than spoken language, music, or dance; but Dewey (Dewey 1958) argues that while our interactions with the media are more or less fluid in relation to time, but they use the same basic psychological processes. Langer makes a distinction between "discursive" and "presentational" means of communication—purely symbolic and explicit modes such as mathematical formulae, versus media like drawing, that support iconic representations and shades of meaning (see Gardner 1982). But theorists such as Nelson Goodman and Howard Gardner argue that representations are not either metaphorical (iconic) or abstract (symbolic), but always more or less of one or the other (Goodman 1968, Gardner 1982).

There clearly exist different systems for sharing our thoughts and feelings, each with its own particular characteristics. Indeed, it seems likely that there are so many means of communication precisely because they differ in ways that make some better than others for sharing particular kinds of ideas or emotions. At the same time, it is clear that all means of human communication are similar in that they provide a way to reflect or represent in some external way parts of our internal mental lives. If our internal mental events are "incepts," then these external reflections (in whatever form) might be called "excepts." Excepts, in effect, are external actions—such as movement, vocalization, and the creation of objects—that, however loosely, correspond to some aspect of an internal mental state.

Expression

In his *Talks on Pedagogics* Francis Parker (Parker 1894/1969) outlined a "theory of concentration" in which he defined human expression as the manifestation of thought and emotion through the body. This formulation is clearly incomplete as a description of the complex processes of expression; but it does provide us with a starting place to think about expressive activity. In particular, it suggests that we should be wary of the tendency—especially in discussions of the arts—to associate "expression" with "expression of emotion" (Dewey 1958, Goodman 1968). It is revealing, for example, that in the same work (Gardner 1983) Gardner uses the term "expression" to refer to emotive qualities when discussing dance (p. 221), but later uses it in the more general sense of reflecting thought or emotion when discussing the development of symbolic competence (p. 309).

The more general form of expression, including the manifestation of both thoughts and emotions, is supported by Papert, who in his most recent book (Papert 1996) refers to expressive activity as creative activity—whether mathematical or artistic or both—which is personally meaningful. Similarly, in describing the development of aesthetic understanding Parsons (Parsons 1987) refers to an "expressive stage" where observers interpret a work of art in terms of its ability to convey a complete experience—that is, more than just an emotion or set of emotions.

Expression, then, is not just about the communication of emotion, but about the externalization of internal states. Using the more technical terminology developed above, we might say that at its most basic, expression is the creation of an except which, when perceived by its creator, matches an incept. Expression is a process whereby a person creates something (a gesture, a sound, an object) that he or she believes in some way reflects a part of him or herself (a sensation, an emotion, an idea). Expression is an alignment between internal and external understandings, between incept and except.

Alignment

The problem of representation

As Noss and Hoyles write in their recent book on mathematical meanings (Noss and Hoyles 1996), the issue of representation is an intellectual minefield. On the one hand, it seems likely that the collection of neurons that make up the brain function as a "complex system" (Hofstadter 1979); it may be as difficult to explain thought and feeling from the "bottom up" as it is to predict global weather patterns from the motions of individual air molecules. On the other hand, trying to explain how we make "meaning" from the world without referring to our existence as physical beings is like trying to lift oneself up by pulling on one's own belt loops; the cycles of infinite reference have to ground out somehow in our own bodily experiences and the operations of our neurons (Hofstadter 1979, Lakoff and Johnson 1980, Ornstein 1991). On yet another hand, it is clear that meanings do not emerge in a vacuum. Rather, we make sense of our experiences using culturally determined signs (Geertz 1973, Vygotsky 1978). And on a final hand, if we choose to walk away from the problem of representation, we are left with the nihilism of behaviorism (Skinner 1971), denying the importance of the thoughts and feelings that we all experience.

Surely this is a difficult problem if we so quickly run out of hands to describe it! But if we were to hold every psychological theory hostage to the solution of the problem of representation, we would never get anywhere. As Geertz so wisely said: "It is not necessary to know everything in order to understand something" (Geertz 1973, p. 20). So rather than try to solve the entire problem of representation here, it seems to make sense to shed some light on the pieces of the problem that relate to expression, and leave some issues for further research.

The map analogy and the structure of incepts

Let us imagine for the moment that a map of the world represents a particular person's mental states: that is, it is a somewhat stylized depiction of two of the dimensions of one person's multi-dimensional mental world. The points of the map — the various places one might go, whether a specific street corner in a big city or an empty spot in the great plains of Kansas — represent concrete images or emotions.[1]

One of the first things to notice about the map is that it is hard to describe locations without projecting onto the map a framework of identifiers: the names of specific locations, the identities of regions. A reference to "an empty spot in the great plains of Kansas" assumes that, at the very least, there is a part of the landscape known as Kansas.

This suggests that what we ordinarily think of as abstract ideas or concepts are like the lines of a political map drawn on top of the map of the physical (concrete) landscape. Concepts represent parts of the mental landscape that we think of as similar in some way. Of course, these parts need not be contiguous (as in Michigan). The lines can shift over time (as in Yugoslavia). Borders can overlap (as in Kashmir, which is claimed by both Pakistan and India), and form hierarchies (such as the counties within a state). And perhaps most compelling about this image of thinking, I can fly from state to state or country to country, jumping clearly from one concept to another. But if I hike in the back country, moving through concrete images, the landscape looks less and less like Massachusetts and more and more like Vermont as I move north; but it is often hard to know exactly when I have crossed the hypothetical line between one abstract concept and another.

Our thinking about the geographical is intertwined with our thinking about the political. If someone starts telling a story about Arizona, we may refer to a generic image

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[1] For purposes of this example, let us ignore for a moment that one does not experience a specific emotion or image in isolation any more than an electron has a specific location orbiting the nucleus of an atom. We might think of our mind as being in a constant swarm of motion around and near one place on the map or another as we scan parts of a mental image, play through bits of a song running through our mind, or experience our feelings swell and recede. The following section of the paper ("Two objections") discusses this issue in more detail.

we have of the state as we listen. We don't necessarily think about all of the complex detail we know about Arizona, but rather about a representative piece of it—some mesas, perhaps, with cactus and sagebrush. For the sake of argument, let us refer to this representative image of Arizona as the "capital" of the state. The claim here is that some, perhaps much, of our thinking in concepts is like moving from state capital to state capital, using concrete images to guide what appears to be abstract thought. We can, of course, explore a concept in more detail when we need to, describing the parts of the state in terms of counties, or watersheds, or other conceptual constructs. But much of the time we simply call up our stock image and use that.

This kind of prototype theory of cognition is well established in the literature (see Arnheim 1969, Lakoff and Johnson 1980, Block 1981, Bruner 1996). Research on design, for example, fairly consistently identifies how designers use archetypes as concrete containers for abstract knowledge of building forms and design rules (Schon 1988a). In one particularly compelling discussion, Jansson (Jansson 1993) describes in great detail the conceptual category of "gears" as a gradient from the "ideal" (most gear-like) image of a gear through a series of increasingly weak examples, eventually trailing off out of the conceptual space of gearhood. Lakoff and Johnson (Lakoff and Johnson 1980) describe the way we use linguistic hedges ("sort of a gear", "like a gear in some ways") to chart the this progression in the conceptual framework as specific examples and images become less and less like the prototype.

Prototype theory, then, describes a landscape where abstract concepts are more or less organized clusters of concrete images which we often (though not always) refer to using an "ideal concrete example" of the concept. What the image of the map adds to this picture is the notion that the concepts themselves are also organized in mental space. Thus, when I move through the concrete examples of one concept, I am also moving closer or further away from neighboring concepts. I can move around in New York, always staying in New York, but getting closer to or further from Massachusetts.

Incepts thus have structure. Concepts are related to concepts. Images are related to images within a concept. As in prototype theory, concepts are related to the images they "contain." And moreover, images are related to the other concepts and images in the space of incepts. Concepts and images are interconnected; the abstract and the concrete share a common structure. Sacks points out that one way of understanding autism is as the result of a breakdown of this kind of internal mental structure (Sacks 1970/1990). Using William James' terminology, Sacks describes autism as the disabling result of experience reduced to a "multiverse" of unrelated particulars. The concept of a universe implies connectedness and structure to space. Without structure—without a mental universe—we are quite literally lost in the details of mental life.

Fields of meaning

In mathematical terms, a "field" is a set of elements under closed binary operations satisfying particular axioms of commutativity and associativity. More generally, though, we might think of a field as referring to a set of instances with underlying relations among elements of the set. In this more general sense, the "mental landscape" described above is a "mental field" of incepts and the relationships among them. Interestingly, this terminology is used by other authors such as Vygotsky (Vygotsky 1978), who refers to thought as movement in a "field of meaning" and Vergnaud (Vergnaud 1983), who writes about mathematical understanding in terms of "conceptual fields."[2]

As the mental universe has structure, so do various media of expression. The terms "constraint" and "affordance" have long since degenerated into buzzwords. But they do describe an essential feature of media: that different media are not all equivalent. In particular, any medium has things that can and can not be done within it—or that can be done with more or less ease. This idea is in some senses trivially obvious: clay has different physical properties than paint. But notion that different media function differently is a point that theorists of art and expression consistently make (Dewey 1958, Eisner 1986, Mitchell and McCullough 1991, Mitchell 1994). If the structured landscape of incepts is an internal, mental field, then the media in which we express those concepts are external fields with their own constraints and relations—with their own structures.

Our interactions, in other words, take place between an internal field and an external field, both of which have structure. A host of authors have written about how we make meaning by creating mappings that reflect the structure of the external world in the structure of our mental field[3] (March and Steadman 1974, Hofstadter 1979, Fischer 1980, Lakoff and Johnson 1980, Eisner 1986, Stevens 1990, Mitchell and McCullough 1991, Jansson 1993, Norman 1993, Golomb 1994, Mitchell 1994). The general mathematical term for this kind of mapping—a mapping that preserves structure—is a homomorphism. Some authors (see Hofstadter 1979) suggest that the mapping between world and mental field is an isomorphism: a mapping that preserves identity as well as structure. But isomorphism is probably too strong a condition for meaning-making. The world is not *the same* as our mental representation of it. But pieces do function in similar ways. If we move from one place to another in the field of incepts, this motion can correspond in useful ways to action in a medium of expression.

We never get, in other words, an exact match between internal and external fields. Nor do we get a complete match. But we can find local mappings between incepts and excepts that preserve some of the structures between them. The more common terms for such mappings are homologue, analogy or metaphor. All of these terms refer to ways of understanding one thing in terms of another. A homologue, analogy or metaphor highlights certain relevant features and shows how the relationships between features of

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[2] One must, of course, be somewhat wary of taking this similarity of terminology too far. Vygotsky "uses" the word "field" only in translation.

[3] To be more precise, we should perhaps say that we make meaning by creating a mapping that reflects some part of the structure of some part of the external world in some part of the structure of some part of our mental field. This distinction is taken up in the following paragraphs.

one field correspond to the relationships between features in another field. As they say in standardized tests: "A is to E" as "B is to C"—first is to second as first is to second.[4]

Returning for a moment to the domain of cartography, if I look at a map and move my finger along a red line towards the top of the map, that has the same function as driving north on the "equivalent" road on the surface of the earth (assuming the map is in standard orientation). Of course these are not the same operation. But the results of changes in one field (moving my finger north along Interstate 5 until I touch the spot marked Sacramento, CA) correspond to changes in the other (driving north until I reach Sacramento). From the point of view of expression, we can understand representation in terms of a metaphorical mapping that reflects a part of the structure of the field of incepts in a part of the structure of the field of excepts. That is, an incepts and excepts are "aligned" if they share some significant pattern of relationships.

Two objections

The next section of the paper looks more closely at the process by which incepts and excepts are aligned—that is, the process through which expression takes place. Before moving on, however, I would like to take a moment to offer some brief thoughts on two possible and important objections to the view of representation described in the preceding section.

A first objection is that the theory presented here avoids entirely the difficult question of where the "structures" of incepts and media come from. The model seems to posit an objective, external reality and a separate, internal reality. As Geertz and others (Geertz 1973, Vygotsky 1978, Lakoff and Johnson 1980, Gardner 1982) point out, however, the structures associated with media of expression are to a large extent culturally determined; if they are part of an "external reality," that reality is a social construct rather than a literal one. One can thus discuss structures external to the individual without believing in an "objective" reality.

A second and more troubling objection is that this "model" of mind as overlapping physiographical (concrete) and political (abstract) maps is hopelessly underspecified. In particular, the model waves its metaphorical hands with great vigor at the problem of exactly what the various points on the map are. Is there a specific "image" at the corner of 3rd Street and Elm Drive on my mental map? In what sense is this continuous with a fragment of a song that might be located right next to it? More generally, the map metaphor plays fairly fast and loose with the problems of adjacency, dimensionality, and so forth.

The obvious response to this objection is that the map exemplar is necessarily and deliberately underspecified. The map is a homologue for understanding some claims about how the mind works. It is illustrative rather than denotative. Any metaphor

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[4] That is, first and second in the set of vowels are related in the same way (first and second in alphabetical order) as first and second in the set of consonants.

highlights some salient features while obscuring places where the images do not match. As Lakoff and Johnson point out (Lakoff and Johnson 1980), if I describe the vice president as the spare tire on the car of government, it is beside the point to argue that the metaphor does not make sense because the vice president is not round and made of rubber. The map exemplar is meant to give a concrete way of thinking about and talking about the idea that incepts exist within a structure that is simultaneously abstract and concrete, and abut the notion that we make meaning by constructing mappings that preserve structure between the field of incepts and media of expression.

At the same time arguing that "it's just an illustrative metaphor" is like eating one's cake and having it too. The point of the illustration is to make a case for a particular way of looking at mental activity, and there clearly is a problem in understanding just what it means to talk about specific points on the map in this model. No amount of descriptive language about mental events mental states concrete images iconic representations concrete symbols or the like can cover up the fact that no one has yet come up with a compelling and coherent way to describe non-conceptual incepts—which in some sense are inherently wordless—using words.

If pressed, I would suggest that the "points" in the mental field of concrete thought are probably in some sense *actions*. There is ample evidence that thinking and perceiving are intimately linked with doing (see, e.g., Kosslyn and Koenig 1992, Gruber and Voneche 1995, Sacks 1995). It has been suggested, for example, that we do not "see" mental images as a whole, but rather paint them in our mind's eye much as we would draw them on paper; then we "look" at them much as we would look at any image (Kosslyn and Koenig 1992). We organize parts of this landscape of action into bigger pieces through the process Piaget and others describe as reflection (Fischer 1980, Gruber and Voneche 1995). Then we access the bigger pieces as units, even though the underlying processing may still function as a sequence of discrete actions carried out so quickly as to escape notice.

Of course, that reply would not satisfy me if I were looking for a complete accounting of how the mind works. Thus my real response to the objection is to plead guilty. The model of expression presented here is imperfect. But that does not necessarily make it useless. The model and the implications that follow from it make claims about how representation is related to expression. Preliminary observations suggest that there is some validity to these claims. More important, despite its imperfections, the model presented is useful in that it provides a basis for asking empirical questions whose answers can be used to develop a better theoretical understanding of expression, computation, representation, and the connections between them. The perfect, in this case, could be the enemy of the good.

Cycles of feedback and reflection

The process of alignment

If expression is thus the creation of external representations which have internal relationships that map in some meaningful way to the internal relationships of a person's thoughts, sensations, and emotions—that is, if expression is the alignment of excepts with incepts—we might reasonably ask how this mapping, this alignment, takes place.

We can start to unravel this process by noting that the very nature of an except is that it is external, and thus observable by its creator and by others. As Figure 1 below shows, this creates a feedback loop, where action turns an incept into an except, and observation of the resulting except potentially changes the nature of the incept. Put in simpler terms, the "alignment" of incept and except takes place through trial-and-error, or more accurately trial-and-adjustment. Expression in this model is the result of a cycle of creation and observation (what Parker referred to as "expression" and "attention"; see Parker 1894/1969), where an initial external expression is compared to the original idea, leading to a second expression, a second comparison, and so on, until the expressive act is deemed complete.

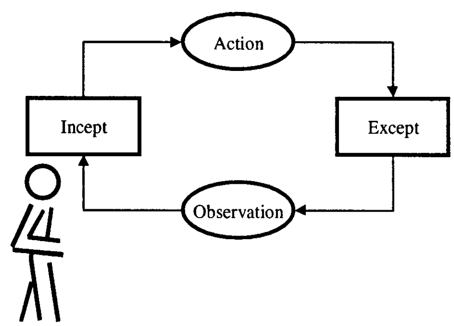

Figure 1: The feedback loop of action and observation.

Papert and Bruner have both suggested that one of the important mechanisms in this feedback loop is the extent to which external representations make internal states available for conscious reflection (Papert 1980, Bruner 1996). This is shown schematically in Figure 2 below, where an incept has both conscious and unconscious aspects—explicit and implicit understandings. These lead to an except that has both intentional and unintentional aspects, both of which are available for observation. Reflection on the creative product thus provides a mechanism through which implicit understanding can be turned into explicit insights via reflective activity.

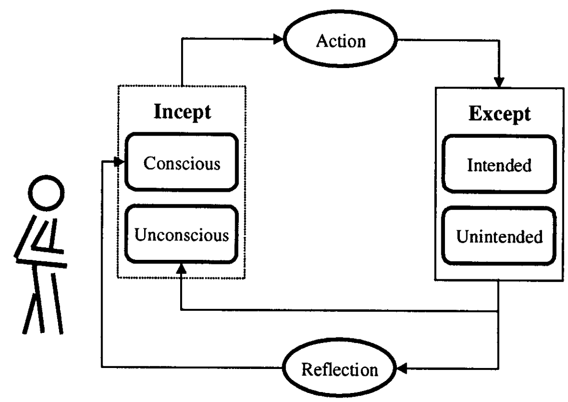

Figure 2: The role of reflection in the expressive process.

The basic feedback loop of expression not only provides an opportunity to transform tacit knowledge into explicit knowledge, it also provides a mechanism through which individual development can interact with external, social resources. As illustrated in Figure 3 below, the act of expression in a social context produces two different but related streams of input: reflection on one's own except, and the responses of those who interpret the work. We learn from others, in this model, by comparing our interpretations to theirs—or, rather, by comparing our interpretations of what we did to our interpretations of how they respond to what we did.

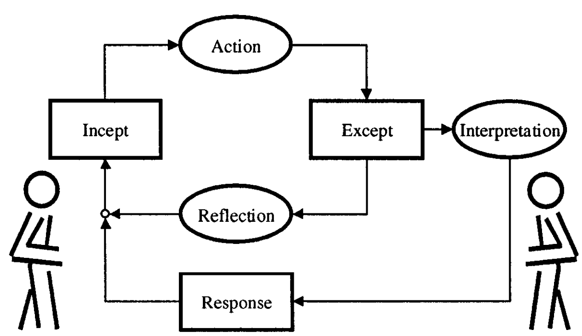

Figure 3: Social interaction plays an important role in expression.

Expression and cognitive development

This paper is certainly not the first place where it has been suggested that expressive activity plays a role in the process by which culture shapes individual development. In his most recent book, Bruner suggests that externalizing makes cognitive activity public, and thus an object for discussion (Bruner 1996). A fairly rich literature on collaborative work describes the role of computational media and other artifacts as focal points for collaborative conversations (CSCL 1997). However, analyzing the mechanisms of expressive activity as described above makes it possible to understand better the underlying relationship between reflective thinking and social interaction—between individual and social development.

We can start by observing that any interaction takes place within a medium (or within several media). I express an idea in writing; you read it and respond with a phone call. I send you a drawing; you reply with a letter. I express an except; you interpret my creation and express one in return. Any interpersonal interaction is thus a cycle of expression and interpretation. But, as we have seen above, expression itself is a cyclical process of creation and interpretation. A conversation, therefore, is one large cycle of interaction that contains two smaller cycles of expressive activity:

This basic model of social learning works for a process like language acquisition, where a child speaks and compares the responses of others to his or her desired effects. It also fits with the model of design learning in an art studio proposed by Donald Schon (Schon 1985). Schon suggests that the basic process of design learning is in the interaction between master and student, with the products of the student's work as the focal point of the conversation. The master poses a design problem. The student begins to work. The master observes the sketches in progress and gives feedback, sometimes with words and gestures, sometimes with sketches of his own. The student compares the master's interpretations with what he or she was trying to achieve in his or her own solution to the problem and adjusts the expressive work accordingly. Learning proceeds, in other words, through cycles of expression and feedback, as in the model above.

There are (at least) three immediate implications of this expressive model of social learning. The first is that there are clearly a number of places where the model can break down. Young children have not mastered the cultural domains of expression (symbol systems of language, drawing, gesture, and so on). When the conversation breaks down, it is not clear whether the problem was in the idea itself (the incept), the expression of the idea (the except), or in the interpretation of the response (the percept).

Studies of childrens' production of language and artifacts shows that, in general, comprehension is easier than production (Daneman and Case 1981, Crowe and Hurtt 1986), and thus it is not surprising that most children understand more language more than they can use in speaking. This suggests that a child learning to express him or her self would first have to gain some rudimentary confidence in his or her ability to interpret a particular cultural system of expression (some subset of the symbol system). Only then could he or she begin to refine his or her expressions in that system. And only after he or she had a reasonable level of confidence in his or her expressions would it make sense to begin questioning the underlying incept. Why question what you think when you can't be sure that you said what you meant?

This expressive model of social learning, in other words, "predicts" a period of time after the emergence of social communication (with language, for example) where the child's ideas (particularly ideas that are difficult to express) remain stable while the child gains basic mastery of cultural systems of expression. That is, there should be something that might look in some domains like a "stage" — a plateau of development — while the child masters the tools he or she needs to continue developing within a culture.

Because development of symbolic mastery (to steal Gardner's phrase) does not necessarily take place in quantum jumps, the "plateau" just described would be flat only from the broadest perspective. On a closer view, progress would be incremental in the sense that as a child became more competent with his or her use of a medium, he or she would be able to express more with it. This, in turn, would mean that he or she could get more feedback from others about work in that medium — which would make it possible to master more of that symbolic domain — which would make it possible to get more feedback — and so on. In other words, a second implication of an expressive model of social learning is that enculturation follows a bootstrapping process not unlike that in the Piagetian model of development. This bootstrapping quality of development contrasts with (though does not contradict) Gardner's image of "stream-like" development within a given symbolic domain (Gardner 1983).

Finally, development does not take place in isolation in any given symbolic domain. Developments in one system of expression seem to "spill over" into other areas of expression. Gardner (Gardner 1983) describes this as "waves" of symbolization (following the stream metaphor), and the model of expression we have been considering suggests why this might happen. In particular, notice that the except I create and the feedback I receive do not have to be within the same symbolic system: I can draw and get feedback in words, or sing and get feedback from gestures, or dance and get feedback from a diagram. In general, I can use growing competence in one domain to develop my abilities in other domains.

In sum, the model of expression as an alignment of incept and except provides a useful tool for examining learning from a cultural perspective. In particular, it helps explain how some of the features of development viewed from an individual perspective, such as general plateaus of cognitive growth or the cyclical nature of development, make sense within a cultural model. The incept/except model also helps explain a puzzling feature of development in the cultural model: namely why at some times we observe a general coordination of development across distinct cultural systems of expression.

The discussion above describes how the incept/except or alignment model of expression might help us understand cognitive development from an individual perspective. The key idea is of a loop or loops of creation and interpretation. The creation of an external representation (an except) of some internal state (an incept) provides a means by which an individual can get feedback about his or her own ideas or emotions. That feedback might come from his or her own reflection. Feedback might also come from another person's interpretation of the except. And, of course, part of the learning process might be in comparing these two types of feedback.

Qualities of expressive activity

The description of expression given above clearly applies broadly to human activity, in a sense to all human activity. Apart from solitary, practically immobile contemplation — and perhaps sleep — there is little that we can do that does not affect the world around us; conscious activity, in a sense, is the continuous production of excepts, and thus a continually expressive endeavor. But at the same time it is equally clear that not all activities are equally expressive. Some things (painting, or singing, for example) seem to provide richer opportunities to externalize internal states than others (doing mathematics homework problems from a typical textbook, or working on an assembly line, for instance). As above, this is not just that some activities are more emotional than others: chess/architectural design/bridge-playing and poetry/modern dance/jazz are all expressive, though the former set may be more cerebral and less emotive than the latter. All activity is expressive, but different activities may involve different qualities or kinds of expression.

When we talk about expression, in other words, the question is not: *Is activity X expressive?* Rather, the meaningful question is: *what type of expression* does a given activity, situation, or context tend to support? In effect, we can think of the range of all activity as a "space" of expressive contexts, with each point in the space—each location—representing a particular activity and its relationship to other kinds of expressive activity. Understanding expression does not mean defining the conditions under which an activity is or is not expressive; rather, understanding the expressive nature of an activity means locating the activity in relation to other activities along the significant dimensions of the space of expressive endeavor. Understanding expression means exploring the axes of expressive activity.

The next section of the paper begins the task of describing the salient features of expressive activity—the axes or continua through which we can compare the expressive potential of different contexts. But before describing these axes of expression, it is worth pointing out that no activity can be categorized on these scales in absolute terms. If expression is an alignment of incept and except, then the expressive qualities of a given situation depend as much on the person doing the expression as on the surrounding context. Expression is not exclusively a property of a medium—of a set of possible excepts. Expression is a relationship between a set of incepts and a set of excepts. Expression emerges from the interaction of an individual and his or her environment.

Axes of expression

Thought and emotion

The discussion above suggests that the distinction between thought and emotion is something of a straw man. It makes as little sense to ask whether expression is about thought or about emotion as it does to suggest that we can completely separate cognitive and affective processes. How we think about something is intimately intertwined with how we feel about it. But if any expressive act involves thinking as well as feeling (if any incept has cognitive and affective components), it is also the case that some expressive contexts support the expression of thoughts more than the expression of emotions, and others support emotions more effectively than thoughts. In some sense, it seems that the most effective environments for learning through expressive activity support both: that is, the most complete expressive experience is the most transformative.

Explicit intent

Dewey and Gardner (Dewey 1958, Gardner 1982) argue that expression has to be a conscious act, a deliberate shaping of experience. Otherwise, it is merely revealing of internal state, but not the creation of meaning. Dewey describes the process of having a genuine experience in any domain as a course of action "between the poles of aimlessness and mechanical efficiency... in which through successive deeds there runs a sense of growing meaning... toward an end that is felt as accomplishment of a process" (Dewey 1958, p. 39). Similarly, Gardner describes an artist as a person who consciously manipulates a system of symbols to produce aesthetic effects (Gardner 1982). In this view, expression is not simply a process of giving in to impulses; expression involves the deliberate transformation of a medium (Dewey 1958). A twitch that comes from nervousness may be revealing, but is not expressive. The fluid movement of a dancer's arms may hide his or her anxiety about being on stage at the same time that it expresses the graceful motion of a ribbon blowing in the breeze.

As the model of expression described above makes clear, however, there is always both a conscious or deliberate component to expressive action and an unconscious, accidental, or serendipitous component. The ratio between the two changes from expressive act to expressive act, but both parts play a role in the expressive process. I look at the conscious part of my expressive act to see how successful I have been in externalizing my idea (and of course feedback from others helps in this process); I understand my thinking in a deeper way by seeing it outside me. But I also learn from the expressive act by observing things that happened that were not intended. Are the unintended consequences the result of other ideas I have but that I have not yet become aware of? Or are they a necessary consequence of the ideas I am consciously trying to externalize? In either case, they present an opportunity for reflection and an invitation to further refinement.

It is an open question as to what the "optimal" ratio of explicit to implicit intent in the expressive process. And, of course, different degrees of explicit intent support different goals in terms of expression and also in terms of learning through expression. Schon's description of learning in a design studio supports the idea that explicit expressive intent does play an important role in design learning (Schon 1985). Similarly, empirical work on mathematics learning during design activity suggests that explicit design goals support the development of reflective insights during creative activity (Shaffer in preparation).

Declarative and tacit knowledge

If expressive activity can be more or less explicit, so too the developmental results of expression can be both declarative (conscious and describable) and tacit (unconscious and intuitive). Parker argued that "attention" (reflection) and expression were inseparably intertwined (Parker 1894/1969). Similarly, Papert writes about the kinds of conversations that arise around the creation of meaningful artifacts (Papert 1980), and empirical work suggests that a key feature of learning through design is how these conversations turn creative activity into reflective thinking (Shaffer in preparation).

On the other hand, it isn't clear that deliberate reflection is required for learning to take place from expression. I may not be able to explain clearly all of the engineering principles I have used in creating a working model—but I may have developed a good intuitive sense of the behavior of materials and of design principles nonetheless. Schon (Schon 1985) describes this process as reflection-in-action—the progressive development in the design process of understanding "how to do something" rather than how to explain it. Schon writes, for example: "to become skillful in the use of a tool is to learn to appreciate, as it were, directly, the qualities of materials that we apprehend *through* the tacit sensations of the tool in our hand" (Schon 1985, p. 23). Similarly, Dreyfus and Dreyfus discuss at great length the importance of intuition in explaining expert behavior in a variety of domains (Dreyfus and Dreyfus 1986).

In a sense, the argument here is that expressive activity can develop intuition as well as declarative knowledge. Reflection is a useful component of development when it happens. But reflection does not happen automatically with expression, nor is it necessarily required for expression to result in learning.

Self-directed and other-directed

Dewey argues strenuously that expression is not the same thing as communication (Dewey 1958). Expression is for its own sake; communication a happy by-product. When I write in my journal, for example, it is expressive (externalizes my thoughts and feelings) but is not intended for others to see. However, in another sense, when I write in my journal I do intend it for another to see. I intend it for myself at a later time. Even if a later time is only a moment later when I read over the previous sentence before continuing with the next—or even as I say the words to myself in the very act of writing—I am communicating from one part of myself to another through the external representation. Even without adopting a view of the mind as a collection of autonomous processing units (see, e.g., Minsky 1988, Ornstein 1991), it seems clear that there is some value in passing ones own communication with oneself through an external medium.

If expression is about communication, the degree to which this communication is self-directed (as in a private diary) or other-directed varies from expressive act to expressive act. As the model of expression described above suggests, both self-reflection and the reactions of others provide feedback that makes expression possible and makes it possible to learn though expressive activity. In fact, the model suggests that in many contexts, the richest opportunities for learning come from expressive activity that is *both* self- and other-directed. Self-and-other-directed activities provide the most feedback, and make it most compelling to think reflectively in a social context.

Iterations

The model of expressive activity described above clearly suggests that an important feature of expression is the inherently iterative nature of the process of aligning incept and except. The feedback loop works best when the positive feedback can develop over a series of cycles of action followed by reflection and/or response. It is hardly surprising, then, that one key feature of design learning is that students work on projects in a series of design cycles or episodes. The nature of these cycles is the subject of continuing research and theorizing (see Schon 1985, Akin 1986, Shaffer 1996c, Simon 1996), but in general research suggests that design episodes let learners revisit issues and explore them in depth over time.

In any event, it is clear that some media make iteration and revision easier than others. Clay is a relatively forgiving material—mistakes can be reworked as long as the piece has not yet been fired. Etching, in contrast, records every mark of the stylus, and even the tiniest stray mark becomes an indelible part of the final product. Similarly, the context of an expressive activity can make iterations of the design process more or less likely: limitations of time and money often preclude free experimentation. The model of expression described above suggests that from a developmental perspective, media that constrain iterative design may lead to more careful planning, but media that allow for a large number of design cycles maximize the potential for reflective and responsive feedback.

Investment

Perhaps it goes without saying that a salient dimension of activity is the extent to which someone is invested in—cares about—what they are doing. Expressive activity is no exception. It is possible to paint, draw, or dance in a way that resonates with one's entire being… and it is possible to do the same activities in a relatively mechanical and uninspired way. Both of these modes are expressive in the sense that some incept is being matched to an except. But clearly the depth of the incept, the extent to which it is important and interesting to the person doing the expression, and the degree to which the expression of that incept is valued, all play a role in determining the quality of the expressive activity.

At the same time, research suggests that one of the advantages of explicitly emphasizing expressive activity as a context for learning is that expression is in some sense an *invitation* to care. Expression—particularly in the context of a design studio—gives students a sense of control over their own activities: control over the pace of their work, over the kind and context of the feedback they get, as well as control over the content of their explorations (see Shaffer 1997b). It may well be that investment and expression are self-reinforcing: a positive feedback of a different kind that helps make expressive activity a rich context for development.

Means and ends

Another tautology about human endeavor is that any activity has both means and ends. Neither exists without the other, just as neither completely justifies the other. And yet it is equally clear that not every activity focuses equally on means and ends. In traditional school activities—in a conventional mathematics curriculum, for example—there is a long period of "cognitive apprenticeship," where students learn the *mechanisms* of symbolic thinking with little attention being paid to the things those mechanisms will ultimately be used for. The classic student question is: "What is this good for?" And the equally classic answer: "You'll need it later in calculus physics economics." On the other hand, there are learning environments, like the design studio, that focus on helping students find and communicate things that they care about. In another domain, Pirsig (Pirsig 1974) describes a writing class where the goal was to produce "quality"—essentially a way of talking about communicating a meaningful idea. The particulars of the representational system (prose) were less important than the goal of meaningful communication.

There is, then, a continuum between "means" and "ends" in expressive activities. Some activities are primarily about means—how an idea is reflected in some medium. Of course, there has to be some idea to be represented (an end), but the end is less important than the way that end is conveyed. In other contexts, the means is increasingly subordinated to the end of meaningful communication. The means are still present, of course, and still important, but the focus is very different: like the difference between learning to write by making sure every word is spelled correctly versus focusing on the content of the story; the difference between learning history as a set of facts versus as a way to use information to make an argument about what people did and why; the difference between using the "right" method for factoring a polynomial versus finding a good model for showing interesting connections in a set of data; the difference between practicing scales versus learning to play an instrument by playing songs.

The point is not that means are less important than ends--or the reverse. And the point is not that some activities have ends without means or means without ends. The point is that there is a continuum of activities that focus more on one or the other of means and ends. Expression tends to become richer and more interesting as the continuum moves towards emphasizing the importance of ends.

Choices

A final axis worth considering is the nature of the choices that a particular medium presents. This is important for two reasons. In the first place, it seems clear from the model of expression described above that the feedback loops in the expressive process depend on the availability of options at each point in the design cycle. Other work (Dewey 1958, Shaffer 1997a) has described the fundamental role that "choice" plays in the process of expression. Musicians determine the mood of a piece through subtle variations of pitch and timing in a score (Drake and Palmer 1993). Architects solve a practical and aesthetic problems through a series of Schon's "design nodes"—key choice points in the design process (Schon 1985, Schon 1988a). In general, the alignment of internal vision and external product takes place through a series of choices.

In order to make the expressive feedback loop work, these choices have to be multiple, valid, and accessible. "Multiple" is an obvious criteria for expressive choices. If you only have 1 choice, then it isn't really a choice. Two choices divides the world with a very broad distinction. Having more choices makes it possible to reflect more variety.

Having multiple options is only useful if they are "valid." If there are three possible paths to take towards a solution, but only one will go anywhere productive, then this "choice" feels more limiting than expressive. It says, in effect: "You may think it makes sense to do X or Z, but only Y works!" Of course, these multiple valid options may be multiple valid solutions, multiple valid ways of framing the problem, or multiple valid decisions at any point in the problem solving process. Thus, some aspect of a building is an expressive act for the architect when it "works" (satisfies the constraints of the design problem) *and* does "something else" as well. The "something else" is the parts of the building that reflect the architect's particular, personal understanding of the design situation.

That the options should be "accessible" is also a fairly clear criteria, although important nonetheless. Having multiple options is not particularly helpful if for some reason I can't act on them—either because I don't have the skills, or tools, or for whatever reason. In a sense this is a point made already above, and also suggested by Noss and Hoyles (Noss and Hoyles 1996): a medium or tool is neither more or less expressive by itself—expression emerges from the relationships among tool, user, and task.

Media that present a richer set of multiple, valid, accessible, and meaningful choices to a particular person will thus seem more expressive. And ideally, an expressive domain or problem presents a branching series of such choices—that is, an expressive problem presents not just one expressive choice, but a continuing series of expressive choices through which the solution becomes a meaningful reflection of the thought and emotions of the person(s) wrestling with the task. As described above, to the extent that the distinctions among these choices have explicit meaning for a particular person, the choices seem more expressive still.

In this context, it is worth pointing out, perhaps, that continuous, analog, iconic media (versus discrete, denotative, symbolic media)[5] are particularly good contexts for expression, precisely because they make available a large number of choices which in many cases can be interpreted as differing in meaningful ways. Goodman argues forcefully that the distinction between "science" and "art" is between discrete and continuous symbolic systems—or, rather, the distinction is between a tendency and striving for discrete vs continuous systems, since symbol systems are rarely absolutely one or the other (Goodman 1968). But the larger point is that either system can be expressive; it just may be that continuous systems provide a richer and easier context for expression in some circumstances.

Expression vs. external representation

Before closing this discussion of expression—both the working model described above and the various "axes" along which expressive experience varies—it is worth pointing out that the description of expression given above shares some common features with the idea of "external representations." In particular, both refer to the creation of external products (excepts or representations) that have some correspondence to internal, mental structures or events (incepts or mental models). Some theorists suggest, directly or indirectly, that expression and external representation are essentially the same phenomenon (see, e.g., Parker 1894/1969, Arnheim 1969).

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[5] This distinction is discussed in more detail above in the discussion of excepts.

However, there do seem to be some significant differences between the two concepts. The literature on external representations tends to focus on cognitive processes—the expression of ideas rather than of emotions or sensations (see, e.g., Tufte 1983, Norman 1993). The literature on external representations also tends to focus more on how the transformation of external representations supports (and in some cases substitutes for) internal mental processes, rather than on the (expressive) process by which those representations are created and interpreted (see, e.g., Norman 1993). Finally, the literature on external representations does not focus as much on the communicative and social aspects of the process of externalizing internal states. The concept of external representation focuses on the mapping of individual incepts to individual excepts, rather than on understanding in a more systemic way the process of aligning internal and external worlds as an essential part of the human experience.

Expression and Learning

The preceding section argues that expression is a homomorphic (structure-preserving) or metaphorical alignment of incepts and excepts that takes place through a recurring feedback loop of action and observation, which can be more or less about thought or emotion, more or less explicit in intent, more or less about the development of declarative or tacit knowledge, more or less self- or other-directed, more or less iterative, more or less motivating, more or less about means or ends, and more or less about having a rich array of meaningful choices.

This section of the paper uses this theory of expression as a fundamental process and property of activity to explore mathematical thinking in general, and to explore the role of computational media and expressive activities as contexts for mathematics learning.

Expression and mathematics

The question of exactly what constitutes "mathematics" and "mathematical thinking" is an important topic in mathematics education. Some theorists argue that mathematics is a collection of ways of looking at the world—a collection of "habits of mind" (Cuoco, Goldenberg et al. 1996). Following Polya's work (Polya 1957/71), the National Council of Teachers of Mathematics (NCTM) Standards for mathematics education focus on mathematics as a collection of problem solving skills (NCTM Commission on Standards for School Mathematics 1989, NCTM Commission on Standards for School Mathematics 1991). Still others argue that mathematical abstraction is a way of making increasingly rich connections amidst patterned information (Wilensky 1991, Noss and Hoyles 1996). More traditional views of mathematics emphasize the role of proof as the defining element of mathematical activity (Hanna 1995, Fitzgerald 1996, Wu 1996, see also Davis and Hersh 1982). And situated cognition theory argues in its most extreme form that mathematics is not a generalizable skill at all: rather, it is a collection of context-specific techniques for responding appropriately in a narrow range of social settings (see Lave 1988, Light and Butterworth 1992, Boaler 1993).

What many of these perspectives ignore or under-emphasize is the extent to which a key feature of "mathematics" is that it is an activity intimately connected with a symbolic language—or, more precisely, with a set of symbolic languages including number representations, algebraic representations, and a variety of visual representations including figures, graphs, and diagrams. This point has been made by a number of theorists, particularly those who look at language and cognition in general, and by those who are exploring the implications of new representational forms (new "languages") for mathematics learning (see, e.g., Donald 1991, Kaput 1992, Shaffer and Kaput in preparation, Kaput and Roschelle in press). More suggestively, Papert has argued that language learning provides an effective metaphor for the development of mathematical understanding (Papert 1980).

In this view, mathematics is fundamentally a *medium of expression*: a way of communicating about the world, a set of structured excepts that can be used to reflect important aspects of internal experience. The "strong form" of this position—for which I am *not* explicitly arguing here—is that mathematics is inherently an expressive activity, and that, in fact, symbolic representations using the languages of mathematics are not mathematical *unless they are expressive.* The strongest form of the argument that mathematics must be expressive is that memorizing the multiplication tables, learning long division as an empty algorithm, or following a "recipe" of steps for solving a system of equations are not just uninteresting mathematics—they aren't mathematics at all.

Even if we do not go as far as defining mathematical activity in terms of expression, there is a weaker form of this position—which this paper *does* explicitly support—that expressiveness is an important component of mathematical activity, and that mathematics is richer, more interesting, more vital, and more meaningful when it is used as a medium of expression. Boaler, for example, argues that what makes a "real world" example or situation compelling as a context for mathematical thinking is the extent to which students become involved in "discussion, negotiation, and interpretation" (Boaler 1993, p. 15). Boaler suggests that the key question about the value of a problem situation for mathematics learning is not whether mathematical symbols refer to "real" objects or events, but whether the mathematical representations are objects of meaning—whether they reflect an incept—or not.

Boaler's discussion does not consider the processes by which such meanings are created through interpretation and interaction, but clearly this position supports the idea that symbolic competence develops through cycles of expression where internal states (incepts) are brought into structural alignment with external representations (excepts). Put perhaps more simply, expression is an important component of mathematics because doing mathematics involves interrogative thinking about mathematical ideas: asking questions and talking with others about mathematical issues.

The preceding section of the paper argued that all activity is expressive, but that different activities are at different points along the "axes" of the spaces of expression. We might similarly consider the idea that all activity is in some sense mathematical—our activities almost universally involve repetition, pattern, or scale in some way or other—but that different activities have different qualities of mathematical engagement. By this line of thinking, reflective, interrogative, expressive uses of mathematics are a more interesting and more developmentally rich kind of mathematical activity than computation or empty symbol manipulation. This argument is similar to that made by Noss and Hoyles (Noss and Hoyles 1996) and Wilensky (Wilensky 1991), who suggest that mathematical ideas are not either abstract or concrete, but rather move in a continuum from more isolated (concrete) to more deeply interconnected (abstract).

Recent discourse about mathematics and mathematics learning has tended to focus on concepts such as motivation (McLeod 1992, Boaler 1993), on cognitive processes such as strategies for problem solving (Schoenfeld 1994a, Schoenfeld 1994b), or on aspects of mathematical problems such as "connections to the real world" (Lave 1988, Light and Butterworth 1992, Boaler 1993). These elements of mathematical activity are offered as keys to understanding the conditions that create a compelling context for mathematical thinking. A theory of expression suggests that "high quality" mathematical thinking is intimately connected with "high quality" expressiveness. In particular, features of expressive activity such as explicitness, reflection, investment, and so on are important components of mathematics learning. Similarly, uninteresting mathematics is often associated with less powerfully expressive contexts. In the strongest form of the argument—which research does not yet compellingly support—there is not "expressive" and "non-expressive" mathematics, but rather mathematics is an expressive activity at its core, and must be approached in that way.

Expression and computation

A theory of expression thus supports the idea that mathematics can (and perhaps must) be thought of as an expressive activity. This next section of the paper uses the model of expression described earlier to explore the ways in which computational media support expressive activity. In effect, this section explores how, by making it possible to move in positive directions along the axes of expressive activity, computational media can potentially help learners engage with mathematical ideas in a more expressive—and thus potentially deeper—way. Of course, there are other, non-computational means to make mathematics more expressive; the point here is, rather, that computational media provide one set of affordances for making mathematics more meaningful to students.

In the first place, it is worth noting that the idea that computational representations provide a context for collaborative discussions is well established in the research literature (see Papert 1980, Doerr 1996, Noss and Hoyles 1996, Bell 1997, Enyedy, Vahey et al. 1997, Shaffer in preparation). These studies show that one of the benefits of working with a computer is that students construct—and often co-construct—external representations of their thinking, which then become external "objects" for discussion, interrogation, and exploration. In other words, computational media can be and are actually used for expressive purposes. More to the point, though, the "professional look" of many computational artifacts (see Kalmbach 1996, Shaffer 1997b), and the inherently public nature of the computer as it is currently used in classrooms (because of its scarcity, or because of its prominent physical presence) seems to help encourage students to be more other-directed in their activities—which, as we have seen above, is both an important axis of expression and an important means of integrating activity within a social and symbolic context.

Several researchers argue that one important feature of computational media that makes computational representations so useful for expressive purposes is the relative ease of revising actions in a computational environment. Noss and Hoyles (Noss and Hoyles 1996) refer to the processes of "flagging" and "adjusting" as important parts of the creative process in a computational medium. They describe how computational media make it possible to make an initial representation and then progressively adjust it. Shaffer (Shaffer 1997b) describes how the ability to "undo" mistakes gives students a sense of control over their work, and also makes it possible for them to explore a large number of alternative solutions with relative ease. Students in such an environment come to regard mistakes as opportunities for growth and development, rather than roadblocks to be avoided at all costs. The ubiquitous "undo" feature of computational environments thus has important expressive consequences by raising the level of iterativity—the ease with which students can conduct expressive explorations, and thus the number of expressive cycles they can engage in for a given activity.

It is, of course, well established that computers are an inherently "motivating" context for some students, particularly for adolescent males (Grossman 1981, MacGregor 1985, Mohamedali and Others 1987, Levesque 1989, Durndell and Others 1995, Hammett 1997, Reinen and Plomp 1997, Shashaani 1997, Valenza 1997). In general, however, we should note that one of the affordances of computational media is that they allow students—and researchers—to explore new classes of phenomena, as well as old topics in new ways. Computational media make it possible to model complex systems such as ant colonies and traffic jams (Resnick 1994b), or to explore data-intensive subjects like global weather patterns or water quality (Bassingthwaighte 1985, Tinker 1985, Bellomonte and Sperandeo-Mineo 1993, Mellar 1994, Weiss 1996). Some of these topics are simply not within the scope of traditional modeling techniques, and certainly not within the scope of techniques accessible to most students (see Shaffer and Kaput in preparation, Kaput and Roschelle in press). Computational media also make it possible to approach old topics in new ways, from learning mathematics through graphic design or video game construction (Kafai and Harel 1991b, Kafai 1996, Shaffer 1997b, Shaffer in preparation) to learning history, biology, or political science though exploring pre-constructed simulations (Levesque 1989, Tonks and Long 1989, Frye and Frager 1996). It is not, in other words, just computers themselves that move students in a positive direction along the expressive axis of motivation. Computational media open up new and potentially more interesting topics for investigation, and make it possible to approach traditional subjects in new and more compelling ways.

In general, computational media create what Turkle and Papert (Turkle and Papert 1990) refer to as "epistemological pluralism": the ability to construct understanding from a variety of developmental approaches, and in particular, the ability to approach traditionally abstract ideas from concrete explorations. Computational media open up an array of new ways to solve problems. This plurality of means makes it possible—in some senses necessary—to make the process of problem solving an explicit object of discussion. At the same time, computational media's diversity of *means* focuses attention on the *ends* of activity. There is no longer a "right way" to solve a particular class of problems. Computational media thus help students move to a place on the means/ends axis of expression where both are given serious attention in the creative process.

Noss and Hoyles (Noss and Hoyles 1996) have coined the somewhat cumbersome term "autoexpressive" for media whose behavior reflects underlying structure. They suggest that, consistent with the model of expression described above, learning takes place as students work within a computer microworld through exploration of the structure of the computational environment. In a recent empirical study, Shaffer (Shaffer in preparation) showed that when students work in a computational environment whose behavior reflects the underlying structure (or lack of structure) of students' constructions, students are encouraged to think about their explicit design intent. Computational media can be designed so as to reward explicit attention to expressive goals and the means of achieving those goals; that is, to move students toward the more productive end of the explicit/implicit axis of expressive activity.

A final point worth making about computational media and expression is that computational media blur the distinction between continuous and discrete representations. Modern computers are, of course, digital devices, and thus fundamentally discrete in their representation of information. However, the increasing speed and power of computation makes it possible to create digital artifacts that have analog properties: the audio CD and image-editing software being only two obvious examples of digital representations that we perceive as continuous. As discussed above, both discrete and continuous symbol systems can be used in expressive ways. But the subtle shadings of meaning that arise in a continuous medium lend themselves to expressive activity by providing a rich range of expressive choices. The ability of computers to bridge the gap between analog and digital representations suggests that they may help us experience the expressive aspects of discrete, "scientific" symbol systems more easily (see also Shaffer and Kaput in preparation).

Computational media thus support expressive activity by supporting other-directed, iterative, personally-meaningful, means-and-ends focused, explicit explorations of spaces rich in underlying structure and expressive choices. Computational media help students move along the axes of expressive space towards activities that are richly expressive and thus provide contexts for meaningful engagement and deep reflection.

In a more general way, though, we might say that computational media lower the "skill barrier" to expressive activity. An essential component in making a work of art expressive is the artists' expertise with his or her medium. Expressiveness is a property of the relationship between person and medium (or tool)—a relationship we describe with the term "skill" (see also Fischer 1980). Without skill, it is impossible to use the choices available in a medium to align incepts and excepts. A violin is extremely expressive to a concert violinist—but I can not get a violin to produce much more than random noise. Noss and Hoyles (Noss and Hoyles 1996) argue that particular tools are not inherently more or less expressive than others. But surely some media are easier for a novice to work in and some more difficult. For example, both a guitar and violin require some skill to use. But the presence of frets on a guitar means that less precise fingering is required to produce desired notes than on a violin. In a similar way, computational media provide affordances to expression, with some computational tools being more disposed to expressive activity than others.

Expression and the studio

Expression is clearly a complex phenomenon. It is ubiquitous, but varies along a number of dimensions (axes) that make some expressive activities richer, more satisfying, and better contexts for cognitive and other kinds of development to take place. Creating effective environments for expressive learning is thus no mean feat—as is attested to by problems in Parker's Cook County Normal School, the progressive education movement, and alternative education more generally over the last century.

Part of the problem is that, as Papert (Papert 1980) and Tyack and Cuban (Tyack and Cuban 1996) suggest, schools are stable systems: they are highly resistant to change, and tend to absorb—and in many cases co-opt or corrupt—innovations without changing their fundamental structure. This stable state is the result of a complex set of demands on schooling as a means of education. Whether this state is optimal is doubtful, and in any case open for debate at the very least. But what is clear is that the complex system of education as it currently exists has certainly *not* been optimized around the idea of expression and expressive activity as a context for learning.

As we have seen in the discussion above, computational media present opportunities to extend the scope and depth of learning that can take place in the context of expressive activity. To do so effectively, however, means looking for some kind of alternative to traditional schooling. One possibility (though not a promising one) is to tinker at the margins of classroom activity, introducing computational media and trying to construct contexts for activities that are thoughtful, explicit, self- and other-directed, iterative, motivating, means-and-ends directed, and rich with meaningful choices. But given the complex inter-relationships within the system of schooling and within the system of expressive activity, the prospect of aligning the two in a sustainable way seems dubious (see also Cossentino and Shaffer 1997).

An alternative approach would be to look not for marginal changes, but rather to find an alternative *system* for learning, ideally one that has been constructed around (if not optimized for) fostering expression and turning creative activity into reflective, cognitive development. Fortunately, at least one such system exists, and somewhat ironically has already been the basis—in a piecemeal way—for education reform. The alternative system is arts-based education, particularly the "studio" approach to teaching and learning, and specifically the well-developed architectural design studio. Pieces of the studio tradition—particularly portfolio and performance assessment—have been proposed and in some places tried as alternatives to traditional educational practice (Perkins and Blythe 1994). But these implementations have been less than completely successful, at least partly because these elements of the studio model of learning were taken out of their original—expressive—context (see Cossentino and Shaffer 1997).

The studio is, at its core, an environment for developing expressive potential and for thinking, and talking, and working explicitly with and for expressive goals. Theorists of studio learning have looked in some detail at the processes of studio learning (see

Kepes 1965, Garrett 1975, Chafee 1977, Allen 1980, Gaines and Cole 1980, Levy 1980, Millet 1980, Ledewitz 1982, Schon 1985, Akin 1986, Crowe and Hurtt 1986, Eisner 1986, Anthony 1987, Davies 1987, Rowe 1987, Schon 1988a, Schon 1988b, Goldschmidt 1989, Elfland 1990, Frederickson and Anderton 1990, Mitchell and McCullough 1991, Branki 1993, Coyne 1993, Jansson 1993, Mitchell 1994, Johnson 1995). In general, the studio is characterized by an open learning environment anchored by specific pedagogical structures that help students turn creative activity into reflective thinking. Thus, open tasks, ample space, and unscheduled time are grounded by public presentation and critique of work, as well as structured collaborative conversations over emerging designs (see Schon 1985, Shaffer 1996c, Shaffer in preparation). The studio presents opportunities for an iterative sequence of design cycles, where students return to an ongoing project from a number of perspectives, with cycles punctuated by discussion and explanation of design goals and processes.

The structures of the studio simultaneously encourage young designers to convey both thought and emotion; to be explicit in their expressive intent; to develop both declarative and tacit knowledge about and through their designs; to create designs that are satisfying to themselves as well as to their peers and to the design world (and the public) at large. The studio supports and encourages — in some ways demands — that young designers work in a highly-iterative fashion and that they think simultaneously and explicitly about the ends of their design process and the means of achieving those ends. And finally, essential to the design process is a set of design challenges and design materials that provide a rich array of viable, meaningful, and interesting alternative choices to the designer.

The design studio, in other words, provides a coherent, powerful, alternative system for learning based on expressive activity. The expressive potential of computational media described above suggests that the design studio could be a powerful model for learning environments using computers, and the importance of expression in mathematical thinking suggests that the combination of computation and design is a potentially rich context for mathematics learning. This combination has been explored in empirical work (see Shaffer 1997b, Shaffer in preparation, also Kafai and Harel 1991b, Kafai 1996) which shows that students can and do learn mathematical ideas in the context of creative activity using computers.

Conclusion

This paper began with three basic questions: What is the nature of expressive activity? What role does expression play in learning and doing mathematics? And: What role do computational media play in an expressive mathematics?

The first part of the paper presented a somewhat detailed discussion of the processes of expression. Expression is a basic human activity, and in some ways a property of every human activity. Expression is the process by which internal mental states (incepts, or thoughts, emotions, sensations) are aligned with external reality (excepts, or actions, words, objects). Alignment, in this sense, is a mapping between internal and external states that in some way preserves the underlying structure of the "field" of incepts and the properties of the excepts. This alignment takes place though progressive cycles of action and observation, incorporating both self-reflection and feedback from others.

While this basic expressive process takes place in all activity, it is not the case that all activity is "equivalently" expressive. We can think of the range of possible activities as existing in a "space" of expression, whose dimensions include: expression of thought vs. emotion, intentionality, development of explicit vs. implicit knowledge, self- vs. other-directed communication, iterativity, motivation, means vs. ends directed activity, and also the abundance or lack of choices in the medium of expression. Of course, the location of a particular activity or context on these dimensions is not absolute. Rather, the qualities of expressive activity emerge from the relationships among the medium of expression, the expressive context, and the person doing the expressing.

This basic model of expression offers a theoretical framework for examining the role of expression in mathematics: it suggests expression is an important element of "high quality" mathematical activity—if not an essential component of what it means to do mathematics. Rich contexts for learning and doing mathematics have many of the same characteristics as rich contexts for expressive activity of any kind: in particular, both are characterized by explicit intent, explicit thinking, discussion, and questioning of ideas and actions, as well as high levels of motivation and communicative goals. In a similar way, we can look at the relationship between computational media and expressive activity, where the basic model of expression suggests that new media support "high quality" expression along many of the important dimensions of expressive activity.

This overlap of mathematics, expression, and computation suggests that expressive activity provides a compelling context for learning mathematics using computational media. Furthermore, the complex, multi-dimensional process of expression suggests that we should not just use "pieces" of expressive activity in creating such environments, but rather look for an alternative "system" for learning through creative activity. Other work has explored the architectural design studio as one such "learning system" for expressive mathematics (Shaffer 1997b, Shaffer in preparation). What this paper suggests is that such experiments are not only practically successful, but a theoretically sound approach creating alternative learning environments.

A theoretical model of expression suggests that the "studio" is an appropriate environment for learning through creative, computationally-mediated activity. Any domain—mathematics, science, history, literature—where learning is enhanced by intentional, motivated, communicative behavior could potentially benefit from more systematic thinking about the role of expression in learning and the affordances of computational media in an expressive setting.

Learning Mathematics through Design:
the anatomy of Escher's World

Introduction

Since the time of John Dewey and Francis W. Parker at the turn of the century, educators have understood that the arts can play a fundamental role in education (Sidelnick 1995). This paper explores one role for arts learning made possible by recent technological advances: helping students learn mathematics by making art in a computer-based "mathematics studio." The paper presents a detailed qualitative study of one successful intervention in which students learned mathematics through design activities. The focus is on three key questions:

* What specific aspects of a "mathematics studio" make it a valuable environment for learning mathematics?

* What role does the computer play in the success of a "mathematics studio"?

* What implications does the idea of a "mathematics studio" have for mathematics pedagogy or for our understanding of the process of learning more generally?

The answers to these questions suggest that computational media in conjunction with arts learning can be used profitably in mathematics education.

Background

Arguments for art in formal education have historically (and logically) have called for two distinct, though not necessarily exclusive, places for art in the curriculum. One role is as a distinct discipline. Citing beneficial effects on students' study skills, use as a set of job skills, or promotion of self-awareness and the understanding of others, educators have argued that students should take art as a formal course of study in addition to the "academic" subjects of mathematics, history, English, science, and foreign languages (Brown and Korzenik 1993). The other position, which goes back at least as far as the turn of the century, is that art as a mode of expression is a key component in the process of all learning. Following the lead of Colonel Parker's Cook County Normal School, proponents of this view argue that arts learning should be integrated into all of the traditional disciplines (Sidelnick 1995).

This second position—that arts learning should be integrated across the curriculum—presents a tantalizing vision for mathematics educators. The idea of "learning math by making art" appeals to those who feel disenfranchised by the "traditional" mathematics curriculum and pedagogy. People get excited by the idea of learning mathematics with the freedom and joy associated with art-making. Since the publication of the National Council of Teachers of Mathematics "Curriculum and Evaluation Standards for School Mathematics" in 1989, key components of this vision have also been sanctioned by a section of the professional community of mathematics educators. The Standards call for the introduction of extended projects, group work, and discussions among students—elements of a learning environment that overall seems more similar to a studio course in design or architecture than to a traditional mathematics class (NCTM Commission on Standards for School Mathematics 1989, NCTM Commission on Standards for School Mathematics 1991).

In recent years, there have been several examples of interventions that demonstrate the power of this vision of a "mathematics studio." In a quantitative study, Leslie Willett demonstrated that mathematics learning is more effective in the context of arts-based lessons than with standard mathematics pedagogy at the elementary school level (Willett 1992). Arthur Loeb's visual mathematics curriculum (Loeb 1993) has not been studied formally, but substantial anecdotal evidence supports his approach to the study of the formal mathematics of symmetry through a design studio as an effective learning environment for undergraduate students.

This paper examines another successful attempt to create a "mathematics studio" where students learn about mathematics and art simultaneously in an art studio-like environment. The Escher's World research project at the Massachusetts Institute of Technology Media Laboratory brought twelve high-school students from public schools in Boston, Massachusetts to the Media Laboratory for brief but intensive workshops during the spring and summer of 1995. In these workshops, students created posters and worked on other design projects using mathematical ideas of mirror and rotational symmetry. Using a qualitative research methodology based on observations of student behavior and structured interviews, this paper documents the process by which mathematics learning took place in the Escher's World design studio: the anatomy, as it were, of a learning environment. The central question is: What aspects of the design studio made it a good venue for learning mathematics?

The focus of this paper is on a detailed understanding of one specific instance of the "mathematics studio" idea, but the goal of this research is to elaborate a framework for thinking more generally about design activities as a context for learning. In particular, this paper looks at the role of expression in design activities, as well as the role that computational media can play in the process of learning.

Methods

Participants

The Escher's World project conducted two workshops during the spring and summer of 1995. At each workshop, six ninth- and tenth-grade students from Boston public schools came to the Media Lab for twelve hours over two or three days. The students had volunteered for "a workshop about mathematics and art" that they had heard about from their mathematics teachers or through a local after-school enrichment program. Students came from a range of backgrounds and ability levels. Five of the twelve students in the workshops were foreign-born, and eleven of the twelve students were persons of color, including two Latino students; there were no Asian or Asian-American students. Five of the students were male and seven were female.

Workshop Activities

Students spent twelve hours at the Media Lab. Workshops were divided into two sections, each lasting approximately six hours.[6] The first section was organized around the concept of mirror symmetry; the second was organized around the concept of rotational symmetry.

Each section of the workshop began with a "warm-up" activity, which lasted approximately 1/2 hour and presented students with a short mathematical game or puzzle which was described as an opportunity to "stretch their minds." The games were chosen to help create a relaxed atmosphere, and to help students meet each other and become more comfortable working together. At the end of each day, there was time for reflection on the day's activities lasting approximately 1/2 hour. Students wrote in their workshop journals in response to specific questions about the content and structure of the workshop. There was also time to discuss as a group any problems or concerns that came up during the day. The emphasis throughout the workshop was on creating an open, studio-like atmosphere for learning. Students were encouraged to sit and work where they liked, to use media of their own choosing, to collaborate or work alone as they wished, to eat, take breaks, go to the bathroom, and change projects at their own discretion. The majority of the day was spent on investigations and explorations of the concepts of mirror and rotational symmetry.

Investigations

Investigations lasted approximately one hour, with students working on short problems on their own or in small groups. Students wrote entries in their workshop journals, and discussed their observations. In the first day of the workshop, students began their investigation of mirror symmetry by making name-tags that read normally when viewed in a mirror. This was followed by a search for words that look the same when viewed in a mirror, and from there to the classification of the letters of the alphabet by their mirror lines. Students worked on each of these problems individually or in small groups at their own discretion, with the whole group discussing the "results" of each problem. Students conducted a similar sequence of investigations involving rotational symmetry using a telidescope in the second section of the workshop.

Explorations

Based on their investigations, students spent two to three hours working on extended projects in design on their own or with a partner. Students worked on one shorter project (approximately one hour), and then presented their work to the group for discussion, questions, and comments. Following this "peer review," students began a more ambitious project (approximately two hours), integrating ideas about symmetry,

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[6] In the first workshop, twelve hours of activities were spread over three days rather than two. Sections were still roughly equal in duration, with the break coming in the middle of the second day.

principles of design, and feedback from their presentation. In the first day of the workshop, students made a design using mirror symmetry. After discussing their designs, students worked for the remainder of the day creating designs that had mirror symmetry but did not place the focus of the composition in the middle of the design (see Figure 4, page 42). In the second section of the workshop, students tried to make designs that used rotational symmetry but presented a lopsided or unbalanced composition.

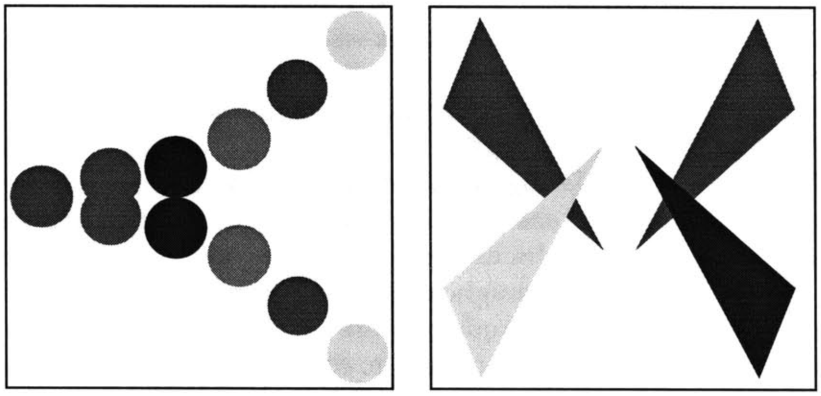


Figure 4: Student work from Escher's World: explorations of composition and symmetry

Workshop Facilities

Workshops took place in a conference room at the Media Lab that had been modified to resemble an art studio. Works of art by students and professional artists were placed on the walls, and a variety of artistic media were available for students' use. In addition to the author, who acted as workshop leader for both workshops, there were one or two other adults in the studio as a resource for students. Macintosh computers were provided throughout the workshops, with one computer available for every two or three students. The computers were connected by a network to flatbed scanners, color printers, and a large format color plotter. Computers were equipped with commercial drawing and image-manipulation programs (Aldus Corporation 1993, Knoll, Hamburg et al. 1993) and with the Geometer's Sketchpad (Jackiw 1995). During the investigation portion of the workshops, students were introduced to some of the basic functionality of these programs. Students were able to work on the computers or with traditional materials during their explorations; all of the students chose to use a computer for some portion of their work.

Data Collection

The main source of data for the Escher's World workshops was structured interviews conducted with each of the workshop participants before and after the workshop, as well as a shorter follow-up interview with each student from two to five months after the completion of the workshop. These interviews were supported by videotapes of the workshops and field notes from the workshop leader and other facilitators. Student sketches and designs from the workshops and student journals were preserved for review and analysis. Students in the second workshop were also given a brief written survey about their feelings towards mathematics, art, and computers before and after the workshop.

Pre- and post-interviews were divided into three components. The first part was a series of questions about mathematics and art, focusing particularly on attitudes towards these disciplines. In post-interviews, this first section of the interview also contained questions about the workshop and students' experiences during workshop activities. The second section of each interview was a detailed discussion of four works of art from a set of seven images. The final section of the interviews consisted of two to four mathematics problems from a set of 16 problems. Follow-up interviews (conducted two to five months after the workshop) asked students to describe their attitudes towards mathematics and art, focusing particularly on changes students experienced as a result of the workshop.

The surveys given to students before and after the second workshop asked students to rate how strongly they agreed or disagreed with a series of statements about mathematics, art, and computers. Ratings ranged from 5 (agree strongly) to 1 (disagree strongly).

Data Analysis

Interviews from the workshop were transcribed and broken into excerpts, where each excerpt represented one complete answer to a question, including any follow-up questions or clarification by the student. This was done to preserve the coherence of students' thinking as reflected in their responses, and resulted in somewhat longer excerpts that often were coded in multiple categories. This helped identify ways in which different themes in the students' experience of the workshop were related.

Each section of the interviews (general questions, image descriptions, and word problems) was coded separately. General questions were first coded for key concepts used by students. These concepts were then aggregated into larger analytic categories based on fundamental theoretical concepts in the literature of learning theory, including concepts such as Control, Expression, and Interaction (for a more detailed description of these categories, see the results section below). Excerpts were also coded for students' like or dislike of mathematics and art.

A more detailed description of the coding process for image descriptions and word problems is provided in a previous paper on the Escher's World project (Shaffer 1996a). Briefly, image descriptions were coded as comments about "Form," "Color," "Symmetry," and "Composition." Within each category, comments were further divided into "General" and "Analytical" comments. One important measure of the success of the project was students' development of mathematical or analytical language for the evaluation of images, reflected in the presence of "Symmetry/Analytical" references such as "it's four time radial symmetric" in students' description of images in interviews.

Word problems were coded for students' use of a visual representation during some portion of the problem-solving process (usually some form of sketch of the problem). Following Rieber, the term visual representation was used broadly to refer to "representations of information consisting of spatial, non-arbitrary (i.e., 'picture-like' qualities resembling actual objects or events), and continuous... characteristics," including both internal and external representations [Rieber, 1995 #361]. Problems were also coded for correct or incorrect answers to the problem, where "correct" answers included answers that fit the stated conditions of the problem even if a student's solution was not the "expected" answer.

Results

The results section that follows presents two analyses of the Escher's World project. The first is a summary of the "effects" of the workshop. These results are presented in more detail in a previous paper (Shaffer 1996a). Following this summary is a more detailed analysis of the particular factors in the workshop environment that led to its "success."

Mathematics Learning in Escher's World

During the Escher's World workshops, students developed their understanding of the mathematical concept of symmetry. Students also began to use visual thinking, and began to like mathematics more as a result of the workshop.

Understanding Symmetry

The criteria used for "understanding" mathematics in this analysis is "the ability to use ideas in appropriate contexts, to apply ideas to new situations, to explain ideas, and to extend ideas by finding new examples" (Gardner 1991, Gardner 1993, Sierpinska 1994, Shaffer 1996a). By this definition students were able to develop their understanding of symmetry as a result of the Escher's World workshops. During the workshops all of the students (12/12) were able to make designs using mirror symmetry, and 83% of the students (10/12) were able to make designs using rotational symmetry. Only 1 of 12 students was able to use and explain ideas about symmetry before the workshop, whereas 11 of 12 students were able to do so after completing the workshop. After the workshop students were able to find new examples of symmetry in the world around them: 75% of the students (9/12) reported thinking about symmetry beyond the context of the workshop in post interviews or follow-up interviews. Students reported seeing symmetry in drawings, chairs, wallpaper, rugs, video games, flowers, and clothing.

Students also clearly developed their ability to apply the concept of symmetry to the analysis of images. Before the workshop, students made analytical references to symmetry an average of 0.5 times while looking at 4 images in structured interviews. After the workshop, mean analytical references to symmetry rose to 4.3 references over 4 images (see Figure 5; mean change +3.8, $p<0.01$).

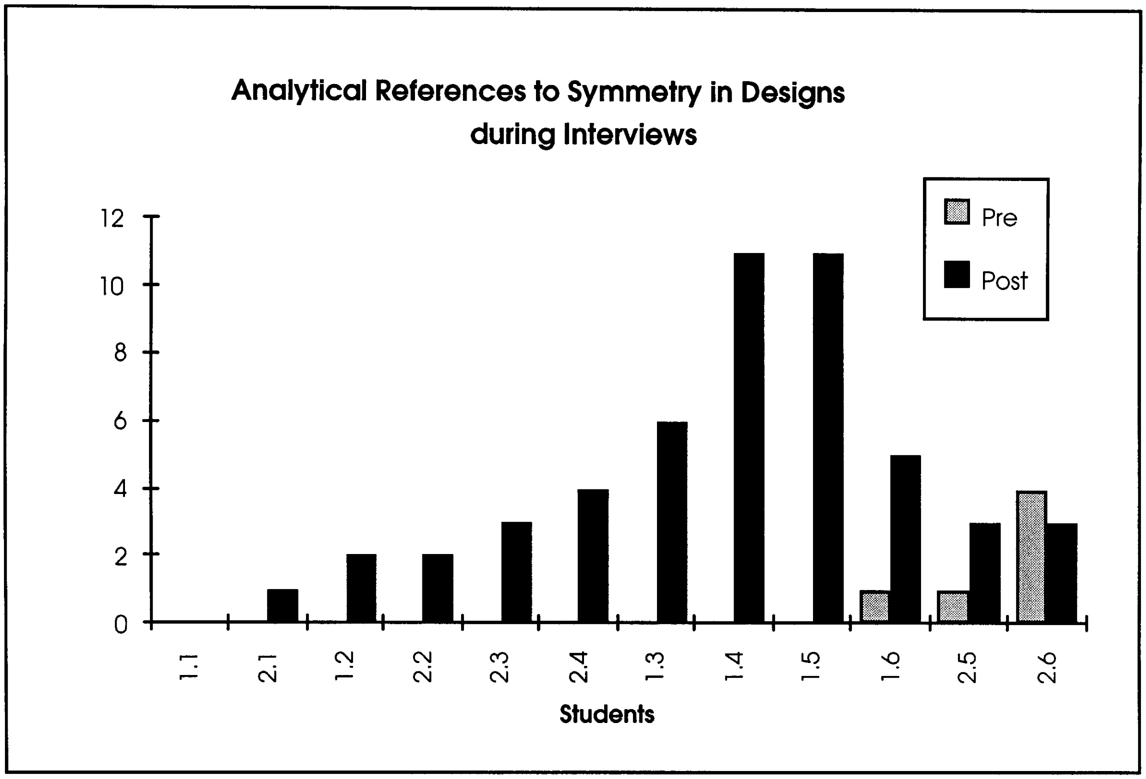


Figure 5: Students learned to use symmetry to analyze images. Students 1.1–1.6 attended the first workshop; students 2.1–2.6 attended the second workshop. Students have been ordered for clarity of presentation.

This change is statistically significant even with the small sample size of the Escher's World project (n=12). But perhaps more striking is the change in the overall tone of students' responses to the works of art they were looking at during interviews. Students used a richer, more formal, more analytical, and more mathematical vocabulary to describe images after the workshop. In a pre-interview, for example, one student described a quilt:

Mmmm, that's like a square with a square inside of it and another square inside of it. And inside the third square, there's some—like paint blotches or something, I'm not sure what they are. They almost look like chicken wings or chicken legs. And then, there's two—I don't know—gears, I guess, in the middle. That's pretty much what it looks like.

After the workshop, the same student described the same image:

This is a box with a red border, and then a smaller blue border around inside that, and then a box inside. And then a yellow box. And then it goes back. And then it uses again the red and blue colors and it just starts making little shapes on the inside that are sort of curvy with the blue. And then it makes red shapes that are connecting the blue. And then you go in further. And then there's sort of a blue circle with spikes, and inside of that there's a red circle with spikes, and inside of that there's a yellow dot. And you're sort of drawn to that dot. Meaning your eye is drawn to it. That's like the focus of the picture. And I guess I was going to say that it was, um, symmetrical, but then I noticed these little, little blue lines coming out of these little red designs—whatever they are—and I realized that it was, um, angular symmetry. Well, what it looks like is almost they took—they made— the person whoever made this, or wherever you got this, could have just started out with a block that had—that had, um, on two sides the red and blue base, and then on the rest of it just made the yellow, and these little—these blue dots and red dots and the little line right there. And then made part of the circle, or one fourth of those circles. And then just use like the way we did with the computer and made four versions of it with different angles, and then just moved them together.

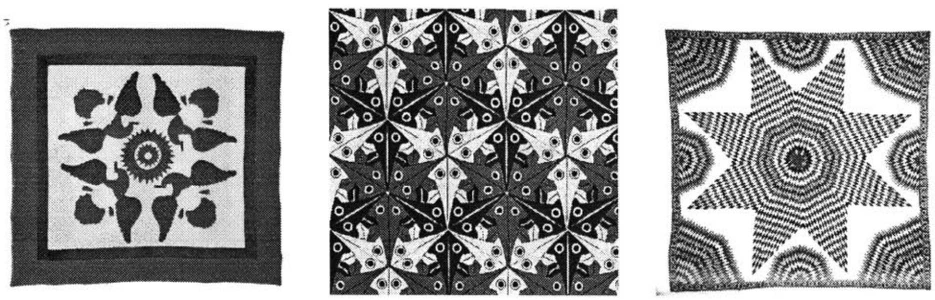


Figure 6: Some of the images students described during workshop interviews.

Visual Thinking in Mathematics

One of the most surprising and dramatic results of the Escher's World workshops was that students started to use visual problem solving strategies after the workshop. Before the workshop, only 33% of the students (4/12) used visual representations to solve word problems in interviews. After the workshop 75% of the students (9/12) did so. For example, in Figure 7 the student did not use a visual representation to solve the problem: "One day, Julie decides to go for a walk. She leaves her home and walks for 2 miles due north. Then she turns right and walks for 3 miles due east. After Julie turns right again and walks for another 2 miles, she decides to go home. How far does she have to go to get back to her home?" After the workshop, the same student working on a similar problem used a visual representation of the problem situation. Across all students, use of visual representations for word problems after the workshop was correlated with success in problem solving during post interview problems (r=0.83).

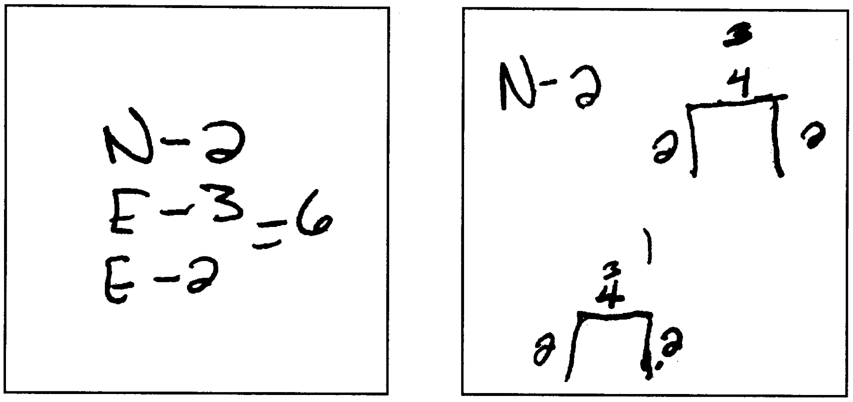


Figure 7: One student's notes while solving a problem during interviews. In the pre-interview (left image), the student did not use a visual representation. While solving a similar problem during her post-interview (right image), the student represented the problem visually and produced a correct solution.

Students Like Mathematics More

In post-interviews and follow-up interviews, 67% of students (8/12) reported feeling more positive about mathematics as a result of the workshop. This reported change was supported by survey data (collected for 6/12 participants). In the survey, students responded to 4 prompts about mathematics:

"I like math class/I don't like math class."

"I like doing math problems/I don't like doing math problems."

"I like thinking about math/I don't like thinking about math."

"I understand math/I don't understand math."

Students marked a scale from 5 (most positive) to 1 (least positive). Total rating for the 4 mathematics questions went up for 67% of students surveyed (4/6). Change for the "I like math class/I don't like math class" prompt (mean +0.67; $p<0.01$) was particularly striking.

As before, however, these statistics only confirm changes that were obvious to the students themselves and were reflected in their comments about the workshop. In post interviews and follow-up interviews, students said things like: "I love symmetry—whenever I see a drawing with these type of shapes I always look at it, just to see the shapes or see the symmetry in it." Or: "Math doesn't seem as complicated like before... I used to always think that [math] was so hard." Other students reported returning to school to find that mathematics became their favorite class, or that they thought of mathematics as more useful or interesting.

Understanding the Success of Escher's World

Students learned about the mathematical idea of symmetry in the Escher's World workshops, and learned to apply visual thinking skills to mathematical problem solving. At the same time they discovered they liked mathematics and liked this new kind of learning environment. One student said simply: "If school was like this, attendance would be perfect!" Certainly something good must have been going on if students were willing to give up time on their spring break or summer vacation to work for four to six hours at a time learning math with a kind of focus rarely seen in school class rooms. The question is: What made this learning environment "work" for students?

The remainder of the results section describes in some detail students' experiences during the workshop based on their comments in interviews. In particular, the analysis focuses on students' experience of "control" over their learning and learning environment, and on the relationship between control and the expressive nature of the activities in Escher's World.

Terms and Conditions

Students' comments about the Escher's World workshops were coded in five major analytic categories: Control, Expression, Interaction, Novelty, and Computer. Novelty and Computer need little explanation. Comments about computers (hardware, peripherals, or software) were coded accordingly. (Students used tools other than computers during the workshop, but did not particularly talk about other tools during interviews.) Students did not use the word "novelty" in any of the interviews, but they often talked about things being "new" or "different"; these comments were coded for Novelty. The other categories (Control, Expression, and Interaction) have a history in the literature of education theory. There is not space here to describe their etymologies in full, but the sections below outline the context in which these ideas apply in the Escher's World mathematics studio.

Control

John Dewey wrote in great detail about the role of freedom and social control in students' development, suggesting, in particular, that "freedom" is a necessary (though not sufficient) condition for the development of self-control. By "freedom" Dewey meant not only the physical freedom to move in space, but also the more important freedom to make decisions, to "frame purposes" and to exercise judgment (Dewey 1938). Other theorists similarly emphasize the extent to which learners must control their learning experiences (Sizer 1984, Papert 1991, Gardner 1993, Prawat 1995). For purposes of this analysis, excerpts were coded for Control when students referred to freedom of physical as well as intellectual movement, when they talked about making their own choices, judgments, or decisions—in short, when they described in a positive or negative way the effects of their own control (or lack of it) in their learning experience.

Expression

In his "Talks on Pedagogics," Francis Parker articulated the idea that "expression" is a process fundamental across many disciplines of thought (Parker 1894/1969). Rather than emphasizing individual modes of expression such as singing, writing, dance, or painting, Parker suggested that all of the means of expression are essential vehicles through which ideas in any domain are learned. In Parker's vision, the arts were integrated throughout the curriculum, much as writing is used in many different disciplines today. The key component of Parker's idea of expression is that it represents "the manifestation of thought and emotion" (Sidelnick 1995). That is, expression is the process of taking some part of ones internal being (an idea, a feeling, an impression) and representing it externally. Student comments were coded for Expression when students talked about making things that reflected their own ideas or preferences—or to times and places when they are prevented from doing so.

Interaction

One of the major themes of Lev Vygotsky's work is the idea that social interaction is a critical component in cognitive development. In particular, Vygotsky argued that the immediate potential for cognitive development (what he called the zone of proximal development) could only be fully realized with adequate adult guidance or peer collaboration (Vygotsky 1978). Other theorists similarly argue that an essential part of learning to think is learning to think with others (Bruner 1986, Bruner 1990, Pea 1993). For purposes of this analysis, excerpts were coded for Interaction when students referred to ways in which their learning experience was affected by the active participation of others (or lack thereof). This includes descriptions of help given to or received from adults or peers, collaborative work, public presentations of ideas or work, conversations or other "purely social" interactions—in short, Interaction refers to the range of students' relations to other people as it connects to their learning experiences.

A Framework for Thinking about the Mathematics Studio

The question, then, is how to understand the particular combination of control, interaction, expression, novelty, and use of computers that made the Escher's World a powerful learning environment for the twelve students who participated in the workshops.

Control and Interaction

The relative frequency of student comments in the various categories shows that Control and Interaction were the most important aspects of the Escher's World mathematics studio for these students (see Figure 8, page 50). Interestingly, comments about these aspects of the workshop showed significant overlaps, with 36 joint references to control and interaction. Student comments about interaction were correlated with comments about control with r=0.79.

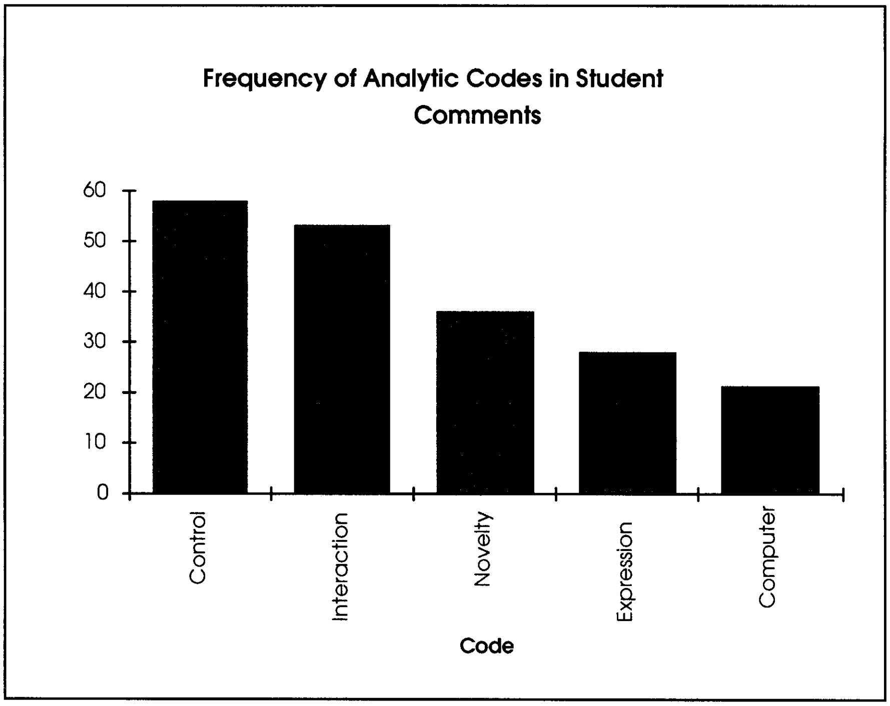


Figure 8: Frequencies of student comments, by analytic category.

Students talked about their ability to decide for themselves when to work alone, when to work with a peer, and when to consult with an adult: "[In the workshop] if I don't know something, I just ask you or other friends to sit by me. In class [at school] you can't talk." Similar sentiments were echoed in two-thirds of the comments where students talked about both control and interaction. In almost all of the comments about working with peers (16/19) and about getting help from adults (16/18) students talked about the fact that in the workshop they were in control of how and when these interactions took place.

Students also talked about control and interaction as independent categories. Most of the comments about interaction that did not have to do with control focused on public discussions about students' work or about ideas in the workshop. The majority of comments about control that did not deal with interaction were about expression.

Expression

Student comments about expression were strongly linked to the issue of control. Two thirds (67% or 19/28) of the comments about expression reflect the fact that the expressive arts-based activities of the workshop put students in control of their own learning. But perhaps more remarkable, the frequency with which students refer to expression is directly correlated to the frequency of comments about control overall—with r=0.94!

This control-through-expression was manifested in two ways: control over goal and control over effects. Students were able to adjust, for example, the level of difficulty of the projects they took on (control over goal). "You gave us an idea," one student said, "but we basically made up [the specific problem]. So it could be as hard as you want it to be or as simple as you want it to be." Students were able to "decide what we wanted [and] make it any which way we wanted." Or, as another student put it: "If I didn't like the way I was going I could stop and change it around [or] start over.... You said, 'I want something with angular symmetry.' That still leaves a lot of different ways you can go."

Students also had a sense of control over the emerging products of their work (control over effects). As one student said: "I can change where people's eyes go, and where the focus is, and even change how they look at it.... Before I would draw something, and if I didn't like it, I would just forget it and try something totally new. Whereas now, I know if I change something minor and change it drastically then it can become a whole new picture, but still very similar."

The cumulative effect of this sense of control-through-expression was obvious when students tried to make images that had rotational symmetry but were not balanced in their overall composition. Students were able to produce designs that fit the constraints of the problem after a short time, but they continued to work despite growing frustration with the difficulty of the problem. They were looking for a solution that *they* liked. One student explained: "When you do it for yourself, you can take any time, or you could do it any way you want to, because it's yourself, your own stuff." It may also be worth noting that the frequency of student comments about expression shows some correlation to students performance on word problems ($r=0.56$ for the correlation of frequency of comments about expression and percentage of correct answers on post-interview word problems).

Computers and Novelty

From a practical point of view, the "novelty" of the Escher's World workshop and its effect on students is problematic. Clearly some portion of students' positive response to Escher's World can be attributed to the fact that the workshop was a very different experience from learning in school—if only because the workshop took place in a different space, for only a few days, using powerful computers, with video cameras, interviews, and all of the other trappings of research. Escher's World would surely be at least somewhat less exciting if it were a regular part of every students' education.

It is clear from interviews that for these students, the most novel aspect of the Escher's World workshops was the use of new media technology, for which students used the generic term "computer." Students who used computers extensively during the workshop made an average of 0.8 comments about the novelty of the workshop, compared to an average of 0.4 comments for students who did not use computers extensively ($p<0.01$). The number of students' comments about computers overall is correlated with the number of comments about novelty ($r=0.55$).

At the same time, a close look at students' comments about novelty shows that while there was a significant "novelty" effect in students' reactions to the workshop related to computers, there were also more "deep structure" effects of media technology in the workshop environment. For example, only a third of the comments about computers overall referred to their novelty (35% or 6/17), and only three comments about computers referred to their novelty alone.

Not surprisingly, students' experiences of computers in the Escher's World workshops reflected to some extent the particular software they were using. Most students who used the computers to any great extent during the workshop used the Geometer's Sketchpad program. The Geometer's Sketchpad allows students to create basic geometric figures such as circles, lines, and arbitrary polygons, and change their size, orientation, and color. More important, students can define mathematical relationships between theses objects: ratios, angles, and geometric transformations. So, for example, a student could create a line and a polygon, and then create the reflection of the polygon in the line.

When objects are moved on the screen in the Geometer's Sketchpad, mathematical relationships are preserved. The display is updated in real time as students "drag" points, lines, and figures on the screen. In this way, students can explore the effects of various mathematical constraints and relations quickly and easily, looking for solutions to mathematical problems that have aesthetic appeal. Sketchpad also preserves a record of a student's actions during a given session with the program. This lets students "undo" their actions; they can step back to and through previous states in their exploration rapidly.

Students commented on the ease with which they could play with designs on the computer using the Geometer's Sketchpad. They described the program's ability to hold an image constant, to let them make very precise changes, and to let them explore the consequences of those changes. In other words, the computer helped students control their explorations:

The computer just made everything easy. You didn't have to hold everything right— [the computer] just it did it for you, so... you could concentrate on actually what you were seeing instead of just [thinking:] 'Well, I think I saw that, let me try that again and see if I see the same thing.'

You drew that dog, and then when you got the mirror on the screen you [moved] it around so that you could get a duplicate of it.... When we did it on the computer... I could actually move the mirror around the screen, move it in closer, and make like one picture out of the two, and move further apart.

The infinite undo feature of the Geometer's Sketchpad also gave students a sense of control over their exploration. As one student said: "The computers helped because it was like easier [than working] on paper [if] you'd have to erase it, or start again. You could just undo it, and then try something differently. That was easier because it was much quicker." Overall six of the seven students who worked extensively with computers commented about one or more of these ways in which the computer increased their ability to control their explorations of the mathematical problems of design.

Differences among students in Escher's World

In collecting qualitative data from interviews and close observations of students at work, one develops a picture of the students as individual learners, rather than as faceless "subjects" or identical elementary particles in a high-energy physics experiment. So for the Escher's World researchers, "student 1.3" is a sophomore in high school in a low mathematics track. But she says that she loves mathematics because she is good at it and because she knows it will help her become a doctor some day. For student 1.3, the workshop was eye-opening—something she and her friends from school who were also in the workshop think about often. They joke about symmetry in the T-shirts they wear to school. Student 1.3 was one of the students whose descriptions of images changed the most as a result of the workshop. Before the workshop she thought the pictures in the interview were "weird," and after the pre-interview it wasn't clear that she wanted to come to a workshop that was about strange things like that! After the workshop, she laughed at the same images, happily talking about symmetry, delighted to have a way to "understand" what had seemed so strange only days before. Student 1.3 was also one of the students who began to use visual representations in her problem solving as a result of the workshop, finding the interview problems less frustrating and more fun as a result.

"Student 2.6," on the other hand, is a junior in high school who likes mathematics and has always been confident in his mathematical abilities. For student 2.6, the workshop felt like it was more about art than about mathematics. He used concepts of symmetry to describe interview images and used visual representations in his problem solving both before and after the workshop, and felt like the workshop was interesting, but mostly review. "I do think it's a good workshop and it could help," he said at one point, "just not me." At the same time, student 2.6 did get some things out of the workshop. He talked in post-interviews about seeing symmetry in his work at the Computer Museum, and about how thinking visually helps him in his advanced physics course. He was also clearly affected by his explorations of the compositional concept of balance, using the idea to talk about all kinds of images and situations in the world around him, including the solution to mathematics problems.

Still, for all this sense of workshop participants as distinct individuals, 12 hours of workshop and three interviews are not enough to determine with certainty why some students responded with such enthusiasm while others were less forthcoming in their praise of the workshop. There was, however, at least one pattern among the students worth noting as a direction for further research. Overall, female students seemed to respond more to the workshop than males. Females used symmetry to analyze interview images substantially more in post interviews than males (females added an average of 5.4 references to symmetry between pre and post interviews, males added an average of 1.4 references, $p<0.01$). And in post interviews females used visual representations more often than males (females used visual representations in 57% of problems, males used visual representations in 30% of problems, $p<0.01$).

Discussion

The data described above in the Results section suggest several areas worth hilighting in the Escher's World project thus far. This section of the paper discusses the role that expression plays in empowering students, and the affective and expressive role of computers in learning.

Empowerment and Expression

The idea of "empowering" students is the topic of much discussion in educational circles, but there is not a consensus as to exactly what student empowerment means and how to achieve it. The results from Escher's World suggest a framework for thinking about empowerment in terms of students' sense of control over their learning environment and sense of control over their learning activities. Students could control their learning environment in the Escher's World workshops by getting help from adults and peers in the amount, manner, and time of their own choosing. Students were able to work collaboratively with peers and adults if and when they chose to. And students had the freedom to organize their activities in space and time—"freedom of movement" as Dewey described, but also the freedom to regulate the pace and timing of their work.

Escher's World also gave students control over the learning activities they were engaged in. The expressive nature of the activities meant that students were able to decide for themselves how to address the mathematical and artistic problems presented in the workshop. They were able to direct their problem solving process, and to decide what constituted a desirable solution. But this control did not mean that students were simply "doing whatever they wanted to do." Adults and the community of peers formed the context in which students evaluated their efforts, and the public presentation and critique of student work thus created a mechanism through which students could learn to regulate themselves.

Other authors since Dewey have described a similar relationship between freedom and control on the part of learners and the power of a learning environment. What the Escher's World research emphasizes is the role that personal expression and expressive activities play in creating a sense of responsible control for learners.

One concrete way to see how expressive activities help students feel empowered is to look at the role of mistakes in Escher's World. Despite reassurances from teachers, students know that in a traditional class room, mistakes are bad. As one student said: "If you make a mistake in school they made it seem like you didn't know what you were doing, and everybody else did.... If you made a mistake you were singled out, and the class had to slow down so you could catch up." In contrast, student's in Escher's World felt that mistakes were an opportunity rather than a liability—as one student put it, "you can learn from your wrongs." Students' control over their learning environment meant that "it was okay to make mistakes because you could always get help." The expressive nature of the activities of Escher's World helped students see that mistakes are not necessarily "wrong"; they can also be steps on the road to finding a better answer. "We could learn from our mistakes," said one student, "do it totally different if we wanted to.... There wasn't any one way we could do it. We could do it differently. We could make a whole bunch of the same thing, or different things.... We could change [a design], and it didn't have to look exactly like a house, or exactly like a dog, or something like that." Similarly, the computers helped create a more constructive role for mistakes: "The computers helped because it was [easier to deal with mistakes]. On paper you'd have to erase it, or start again. [On the computer] you could just undo it, and then try something different."

Students' feelings of control over their learning environment and learning activities can create a powerful context for learning. The results from Escher's World show, moreover, that expressive activities provide a particularly powerful vehicle for empowering students in this way. The ideas of "expression" and "expressive activities" thus help elaborate a framework for understanding the success of design activities as a context for learning by suggesting that an important component of design and construction activities is the way they provide an opportunity for students to be expressive.

Computers and Expressive Learning

A number of authors have written about specific aspects of computational media that make them effective in the process of learning (Perkins and Unger 1994, Jackson, Stratford et al. 1996). One trait commonly described is the dynamic quality of computer representations—that is, their ability to reflect change in a continuous and immediate fashion. As Albert Michotte argued in his monograph "The Perception of Causality" (Michotte 1963), when objects respond directly to each other's motions in time and space we attribute causality as a primary percept. That is, a sufficiently powerful dynamic representation can create direct intuitions about underlying causal relationships. It is not surprising, then that dynamic representations help students understand complex ideas.

What the Escher's World data shows, however, is that dynamic representations also have an affective component for learners. For students in the workshops, the ability to change a design quickly, easily, and in a continuous fashion contributed not only to the development of intuition, but also to a sense of control over their work. Students interpreted representations not only as being dynamic, but also flexible, and the process of continuous and immediate change was important in part because it gave students the ability to manage their own explorations of mathematical and artistic relationships. This sense of control was reinforced by the "infinite undo" feature of the Geometer's Sketchpad. Students could change their minds—and change mistakes—quickly and easily. The computer provided a forgiving environment in which students could explore freely.

This combination of affective and expressive functions for the computer clearly goes beyond the mere "novelty" of the computer equipment used, and it suggests a richer view of the potential role of computers in education than more cognitively-oriented theories suggest. At the same time, data from Escher's World makes it clear that computers were neither necessary nor sufficient for the success of this learning environment. Some students were able to learn without using the computers extensively—just as some students who used computers extensively showed less dramatic changes in their approach to mathematics after the workshop.

A more realistic view of the role of computers in an environment such as Escher's World might be as a "stimulating" (rather than necessary or sufficient) condition. Computers can play a significant role in empowering students, especially in the service of expressive activities. But computers alone are not enough to empower students, and they certainly can be used in ways that do not help students exercise effective control of their own learning process.

Conclusion: Towards an Expressive Mathematics

All of the above suggests that expressive activities are a powerful context for learning, and that one of the benefits of combining art and mathematics education in an environment such as Escher's World is that students are able to think about mathematical ideas in an expressive way. The larger question, however, is how to extend this idea beyond the arts realm. Part of the point of the Escher's World project is to show that mathematics and arts learning can be combined in a way that is productive for both domains. But ultimately the lessons of that "collaboration" should transfer back into each domain. A subsequent paper will explore the nature of arts learning in Escher's world, and within that context the role of analytical thinking in the arts. Here the logical question to ask is: What does it mean to have an "expressive" mathematics problem independent of the arts?

Perhaps an example from arts education will be suggestive. When a student learns to draw a human face in a life-drawing studio class, it is quite likely that before starting he or she will look at the work of masters of the craft: a face drawn by Leonardo, one by Van Gogh, by Picasso, by Matisse. A student may look at ten, twenty, or even a hundred such master works. When he or she sits down to draw a face, the problem is richer, but in every respect just as challenging and as genuine as it was before the student looked at the examples. In fact, even if the student were a perfect mimic and could recreate the masters' drawings perfectly, the drawing of a new face would still be a creative process.

This example from the arts suggests a working definition for an "expressive problem" in any domain. An expressive problem is one that: (1) has multiple valid solutions, where (2) the solutions are more or less interesting, pleasing, beautiful, socially, economically, or politically desirable, or otherwise preferable according to some set of personal criteria. In other words, an expressive problem is a problem where one can say: "Both of these answers are correct, but I prefer the first one because...." The results from Escher's World described above show that students saw the problems and activities of the Escher's World workshops as expressive by this definition—and also that these aspects of the problems and activities contributed directly to students' experience of the workshop as an environment where they were able to control their own learning.

This, in turn, suggests two questions for further study. The first question is whether expressive problems from the arts and from other domains could be the focus of a compelling mathematics education. The second, and perhaps more fundamental question, is about the nature of expressive activities in general—and about the relationship of expression to mathematical thinking in particular.

The data collected thus far from Escher's World shows that expressive activities help students control their own learning. But are there other connections between expression and mathematical thinking? Why, for example, do students begin to use visual thinking in their problem solving as a result of working in Escher's World? Is it just increased familiarity with visual representations as a result of art-making? Or are there some deeper, structural connections between expressive activity and problem solving in mathematics?

In other words, the results from Escher's World show that expression and expressive activities have a noticeable affective influence on the process of learning, and that therefore, expression can play a positive role in mathematics education by helping students control their own learning. More speculative evidence from Escher's World, such as the correlation between expression and students' success in problem solving, suggests that expressive activities may have a cognitive effect on learning as well—and that an investigation of the cognitive aspects of expression would be a productive subject for further research.

Understanding Design Learning:
the design studio as a model for education

Introduction

Why look at design?

In recent years, educators have become increasingly interested in the relationship between the arts and other disciplines. Some reformers suggest that traditional elements of arts education, such as assessment through performance or exhibition, should be included in the pedagogy of other disciplines (Sizer 1992, Perkins and Blythe 1994). Others describe "ideal" or "desirable" learning environments that emphasize long-term projects with a variety of alternative solutions that are creatively and personally engaging for learners (NCTM Commission on Standards for School Mathematics 1989, Fuson 1992). Such environments sound more like a design studio than a traditional classroom. Donald Schon (Schon 1985) suggests explicitly that the architectural design studio is a potentially valuable model for education reform.

At the same time, the literature on learning in the design studio is somewhat sparse. Omer Akin (Akin 1986) reviewed the range of studies on design and design learning and found it wanting. In general studies of design focus more on the psychological processes of design than on the learning environment of the design studio (Akin 1986, Davies 1987, Rowe 1987, Schon 1988a, Goldschmidt 1989, Branki 1993, Coyne 1993, Jansson 1993, Mitchell 1994). Articles on the design studio tend to focus on particular aspects of the studio process (Crowe and Hurtt 1986, Anthony 1987, Frederickson and Anderton 1990), or assume a level of intimacy with design traditions that most educators who work in other disciplines do not have (Gaines and Cole 1980, Levy 1980, Ledewitz 1982).

This paper looks at design education with the specific goal of understanding the design studio as a potential model for education in other disciplines. Following a description of one semester in an architecture studio course, the paper offers an analysis of the studio process on three levels. The first level is the "surface structure" of the studio, including somewhat obvious but nonetheless significant details such as studio space, resources, and scheduling. The second level of analysis is the "pedagogy" of the studio, including the role of feedback and assessment, and their relation to the range of media available in the studio. Finally, and perhaps most important, is the "substantive" underpinning of the studio: namely, the role of expression and expressive activity in design learning.

What will be clear, hopefully, from this analysis, is that the design studio is a coherent and powerful educational setting with a long history. The design studio has both strengths and weaknesses, and a great deal to offer as a coherent educational setting for reformers interested in changing existing relationships between learners, teachers, and intellectual domains. But adapting an arts-based model for education is a more complicated process than simply incorporating arts assessment or the language of arts education into other disciplines.

Overview of the study

This paper reports on a study of a design studio course taught at the Massachusetts Institute of Technology School of Architecture and Planning in the fall of 1996. The course was a mid-level architecture studio for undergraduate and graduate students, taught by a member of the school's junior faculty. For the study, an observer (this paper's author) was present for roughly one quarter of the studio's teaching hours, taking field notes on the studio as a whole. Observations focused in more detail on the work of five students in the studio, and as the semester progressed, one student agreed to participate in an in-depth observation of his learning process. Direct observations of the studio were supplemented by detailed interviews with students and teaching staff.

This paper, then, provides an analysis of *one particular* design studio as a learning environment, and the descriptions and discussions will, hopefully, make it clear where events or interactions reported were unique to this studio or to these individuals. In general, though, this studio was reasonably representative of the studio process as a whole, and many of the specific observations made in the study and reported here are supported by more general discussions of design and design learning.

Portrait of a Studio

The lay of the land

Walking into a design studio is quite unlike walking into a university lecture hall or seminar room, or into a class room in a typical elementary school, middle school, or high school. A well-equipped science lab, with its open plan and benches for student experiments, captures some of the flavor of a studio space. But even this misses some of the essential elements of the design environment.

Most apparent, perhaps, to a casual visitor is the total amount of space allocated to a design studio. In the MIT studios, 11 students have more space for their own individual drafting areas than most high schools provide for a class of 25-30 students (see Figure 9, page 61). In addition to this space for individual drafting, the studio class uses a meeting space easily the size of a room used for a typical seminar or section in a college course. The students also have access to a number of computers and printers within the studio (as well as access to woodworking machinery in a nearby room not shown on the diagram). The hall outside is connected to the studio space at MIT by large rolling doors; the hall outside can thus be used easily to post students' work for discussion and comment. And finally the studio has access to a large open space for more formal presentations of student work:

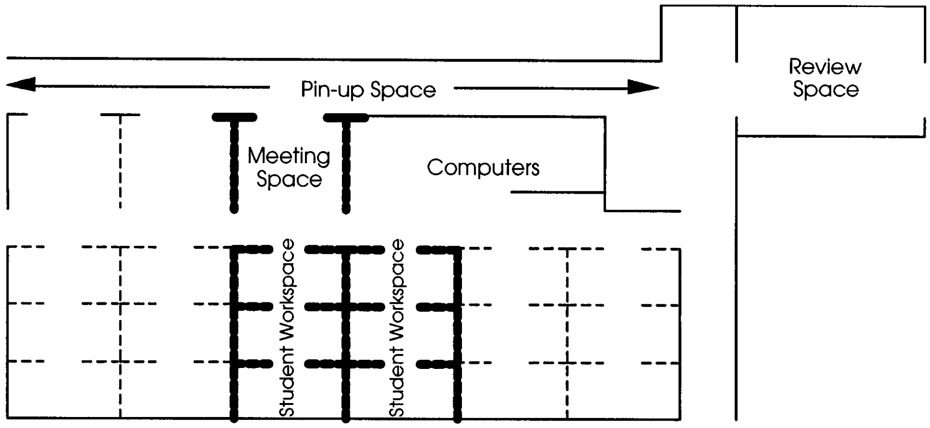


Figure 9: Diagram of studio space at MIT; students each get their own workspace in the design studio

Of course, all of these spaces (except the student's individual workspaces) are shared by other studio classes, and the studio space as a whole accommodates a number of studio courses simultaneously. But the organization of the studio is such that different spaces are available on a regular basis for a variety of studio activities.

As important as the total amount of space available, and the variety of spaces available, is the fact that each student has his or her own workspace. In a typical science class, or in many school art rooms, students share workspaces, and a significant amount of lab time is spent setting up before experiments or projects and cleaning up afterwards. In the studio, students work on a single project over a long period of time in the same space. They are able to adapt their work area to their own needs and working style—and to leave work in progress rather than start anew each time they come to class.

The other feature that immediately strikes a visitor to the studio is that the pace of work is quite unlike a traditional class. Studios at MIT meet from 2-6 pm three days a week. But these are more rough guidelines than a fixed schedule. Students and teaching staff routinely come to studio before or after 2 pm depending on the work they have to do on a particular day. Students and teachers often come in at night or on weekends as project deadlines approach. At any given time during "official" studio hours a class may be meeting around a seminar table discussing projects. Or students may be working at their desks. Or checking email. Or stepping out for a cup of coffee. Or meeting with faculty.

This informal approach to time in the studio makes it difficult, sometimes, to organize activities. Students may not all be present for a class discussion, and even major events in the semester, like final reviews, routinely start late and have participants drifting in and out. Problems of time management also come up for students; work is routinely left until the last minute and sometimes suffers as a result. But the large blocks of time allotted to the studio and the flexibility of the routine also make it possible for different studios to share workspaces. And the relative autonomy of students in the studio makes it possible for teaching staff to spend concentrated blocks of time with some students while others are working on their own.

The assignment cycle

The focus of the architecture studio observed for this project was the design of a new business school for Oxford University in Britain. The project brief (the design specifications) for the class was taken from a closed competition held to generate prospective plans for the new school. The proposed site for new school is on the edge of the urban development of Oxford. Thus the themes in the project centered on the relationship of the building and site to its larger context, and particularly on the relationship between urban and rural landscape. Student designs also addressed the problem of creating layers public and private spaces within the school itself. The project as a whole was quite complex, and as the studio's professor remarked, probably about as difficult a project as students at this level could handle.

The semester was divided into a series of six assignments, taking students through progressively more detailed examinations of the issues of the proposed Oxford business school. The first assignment asked students to examine a particular precedent for the project: a building or building complex that showed how other architects had dealt with issues of context and community in similar settings. Students chose from a list of buildings to research and then represent in a conceptual model. For example, one student researched the Strawberry Vale elementary school and made a model of its irregular central corridor, showing the informal gathering places created by the corridor's zigs and zags (see Figure 10).

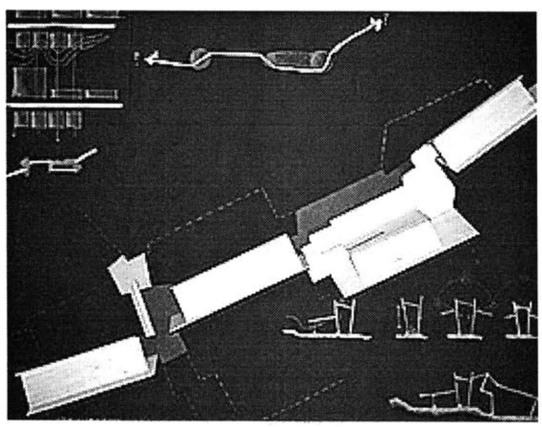


Figure 10:Exploratory model of the Strawberry Vale School corridor

The second assignment asked students to research the city of Oxford and represent in diagrams their understanding of the city and its surrounding landscape. Students also produced a model showing their general plan for dealing with these issues in the design of the business school. One student, for example, looked at the growth of Oxford's quadrangles, and saw his site as a way to end Oxford's urban development gracefully (see Figure 11). Another student focused integrating the community and the university (the "town and gown") through a non-hierarchical building. Students were then given the brief for the competition, including the program (list of requirements) for the building. They were asked to produce a large-scale (meaning small size) plan and model of the school, showing their general strategy for addressing the school's requirements within their conceptual scheme.

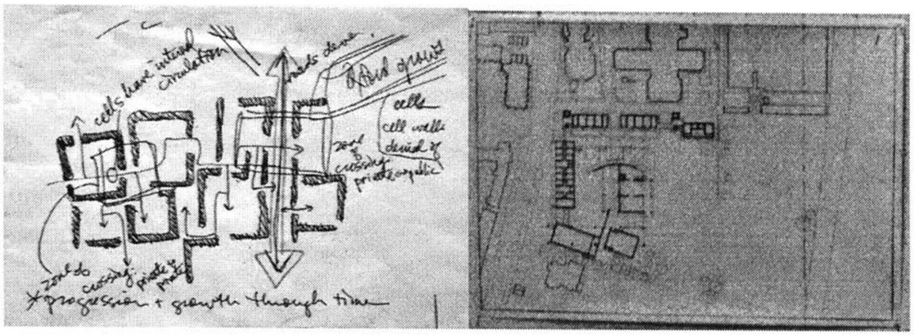


Figure 11: One student's concept and plan for the Oxford business school

The next stage of the class focused on developing this basic model in more detail. Students drew sections and elevations showing how the building functioned spatially. This led to a smaller-scale examination of one or more details of the building, focusing on how the different specific elements of the building related to the overall design strategy. For some students, this meant making a more detailed model of the school's learning resource center (library). Others focused on the circulation corridors, or on informal gathering places in the building. The final assignment of the studio was to bring these various investigations of the project—from the context of Oxford as a whole down to the detail of one part of the building—into a presentation that showed both a proposal for the project and the thinking process by which the student came to that proposal.

For purposes of this discussion, the specific details of the architectural explorations of the Oxford studio are less important than understanding that each of these exercises represented a cycle of design and criticism that culminated in a presentation of some kind. The first assignment (the exploration of an architectural precedent) led to a "pin-up": a group discussion of student work where individuals literally pin their working drawings and models on the wall, describe their work in progress, and get feedback from the teaching staff and from other students. The conceptual strategy assignment led to a "guest crit": essentially the same process as a pin-up, but more formal, and with outside professors and other professionals in the field of architecture invited to comment on the students' work. And, finally, two of the assignments (and the studio as a whole) led to much more formal "reviews," again with guest critics, but now demanding a much higher level of preparation and organization in the presentations:

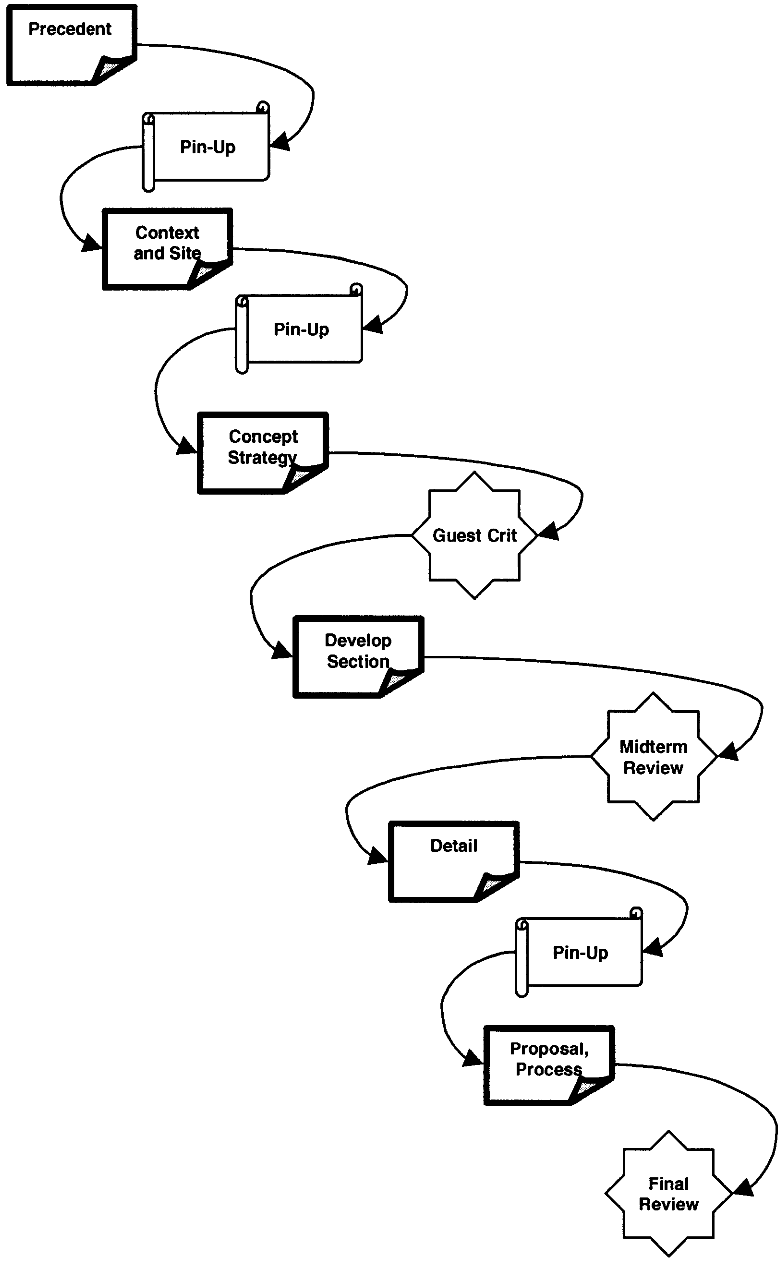


Figure 12: The assignments of the Oxford studio as a series of presentations

This alternation of assignment and presentation was cumulative: each assignment built both on the work that was prepared for the previous assignment and on the feedback that came from presenting that work. The assignment on building details, for example, continued the development of the project presented at the midterm. Many students were criticized at the midterm review for focusing too much on the form of their buildings and not enough on the relationship of the building to the site. For these students, development of the building at a smaller scale focused on the part or parts that were essential elements in the relation of the building to its context.

This alternation of presentation and feedback was similarly present *within* each of the assignment-and-presentation cycles. A typical assignment was a page of written description accompanied by some discussion and clarification from the professor. This would include a summary of the assignment's requirements, explanation of the reason for the particular assignment, description of the professor's expectations, and almost always discussion of examples of work for students to use as models. The assignment on site and context, for example, was accompanied by a discussion of Henry Foster's sketch diagrams. The professor described in some detail how he hoped students would use a similar technique to create a logical presentation of the architectural issues they saw in the Oxford business school.

After this initial introduction, students began working on their response to the assignment—as always in the leisurely manner of the studio. As questions came up, as students ran into problems in their emerging designs, or when students finished some coherent stage of their design process, they would sign up for individual conferences with the professor or with a teaching assistant.

These conferences are known as "desk crits," and are in a sense the heart of the studio process. Crits can be long or short, though they usually lasted somewhere between 20 and 40 minutes in the Oxford studio. During a crit, a student describes his or her work to the professor, including areas of particular interest or concern in the design. The professor probes the design, asking for clarification where needed, and then isolating potential problem areas. Looking at a drawing of a building in section, the professor might ask how a student plans to handle the problem of vertical movement between circulation corridors—that is, where the stairs go relative to the hallways—or what the student plans to do with the roof. As students present possible solutions, the professor explores the implications of various design choices, suggesting alternative possibilities, or offering ways for the student to proceed in his or her exploration of the problem.

Based on this feedback, the student returns to his or her project, perhaps signing up for a desk crit again before the presentation of the assignment, perhaps asking for a desk crit with a teaching assistant, perhaps working out some of the details of the problem in a crit with another student. In any event, the basic process of the design studio is this cycle where expression of an architectural idea leads to feedback leads to expression leads to feedback, eventually producing a design for presentation:

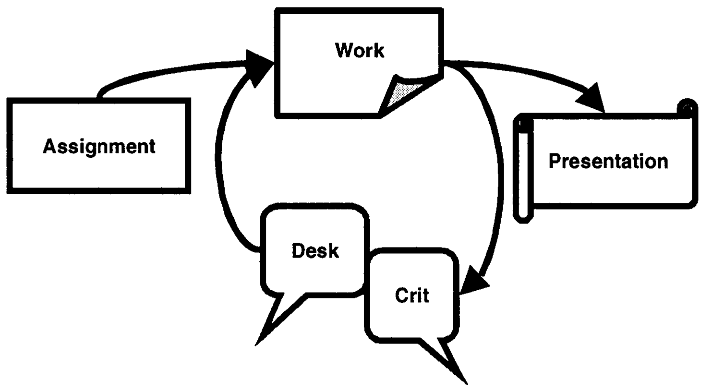


Figure 13: The cycle of expression and feedback in the design studio

And, of course, the presentation generates more response, leading to the next assignment and to a larger cycle of production and reflection that culminates in a final presentation—and hopefully in meaningful learning about the process and practice of design.

"Which do you prefer, models or drawings?"

No description of a studio would be complete without at least mentioning the variety of media used by students and teachers—and the role that these media play in shaping the generation and expression of design ideas. The idea that different representations are good for approaching different problems (or different aspects of a single problem) is a proposition that is more or less taken for granted in design. Even if one does not push this point to its extreme and claim, as some authors do, that the selection of an appropriate representation *is* the solution to a problem (Akin 1986), it is certainly true that an important part of learning to design is learning to chose and use different representations effectively.

In the Oxford studio, assignments generally asked students to work in more than one medium, most often a model and drawings, but sometimes plans and sections or a set of drawings at different scales. In this way, at any given point students were looking at their design from several points of view, figuratively if not literally. It is worth pointing out that in design, a change in scale often accomplishes the same function as a change in medium: both changes shift the designer's focus from one part of the problem to another (see Figure 14). A wall that appears as a thin line on a plan at 1:500 scale has dimension and weight—and thus can feel ponderous or playful, and can be seen to bear load or require structural support—at a smaller scale of 1:100:

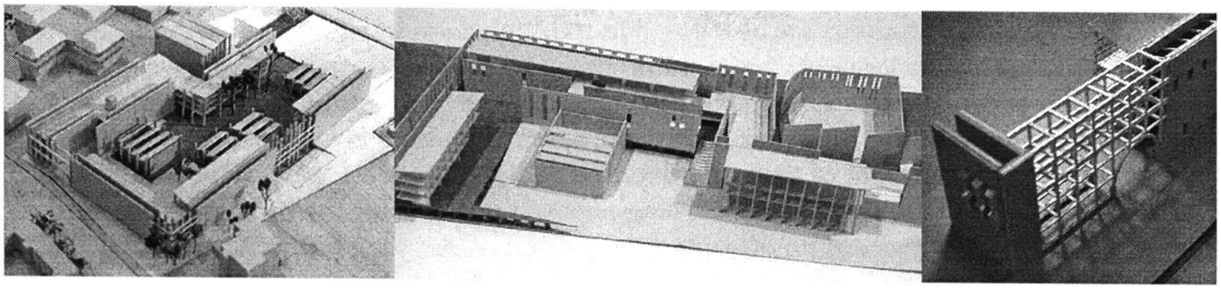


Figure 14: Three models of the same project at 1:500, 1:200, and 1:100 scale show how a change in scale focuses on different aspects of a design problem

The assignments in the Oxford studio asked students to investigate their projects at progressively smaller scales—and thus at increasing levels of detail. Partly this helped reinforce the idea that in good design, the same underlying ideas govern the architecture at all levels. If the design is about creating a bridge between greenscape and cityscape, then this idea should be reflected in the position of the building on the site as much as in the relationship between form and materials of the building itself.

The variety of representations across assignments also made it possible to structure the assignments of the studio in a cyclical fashion. Each assignment asked student to look at the same basic design problem, but at a different scale—and therefore from a slightly different point of view. Successive assignments allowed students to revisit their design choices and incorporate feedback from reviewers from a new perspective. In this way, the class was able to move forward even as circled the same set of fundamental design questions, developing an ever richer understanding of the problem of the Oxford business school and the principles of good design.

Discussion: Levels of Interpretation

The portrait above shows the design studio as a learning environment characterized by a particularly fluid concept of the time and space needed for learning, by assignments that create cycles of production and reflection leading to a series of presentations, and by a range of media used to revisit a central design problem from multiple perspectives over the course of a semester. As with any portrait, though, a deep understanding of the design studio comes from our interpretation of what we see. In the case of the design studio, this interpretation shows three levels of meaning at which the studio can be understood as a learning environment: surface structure, pedagogy, and substance.

Surface structure

At the most accessible level of interpretation, it is clear that the logistical organization of the design studio is substantially different from a class (K-12 or college) in a traditional subject. Not only is the total amount of space per student greater than in a traditional class, but students are also given individual space to work on their projects from class to class. The large blocks of studio time (4 hours a day, three days a week), and the fact that students and teaching staff can—and do—go to the studio at other times as well, provides enough time to work on large projects, but also allows teaching staff to spend a substantial amount of time in individual conferences (desk crits) with students. Finally, the informal approach to scheduling makes it possible for the professor to tailor the class to his or her perception of students' needs on an ongoing basis.

It is, of course, hard to know how essential the spatial and temporal characteristics of the studio are. One could imagine, for example, that students might be able to learn design without having a desk of their own. But one can also see that having a place to leave work from class to class creates a sense of continuity and helps students focus on the same project for months at a time. In more theoretical terms, one of the principles of architecture is that the ordering of space affects the quality of human life and activity in that space. Put another way, we might think of the organization of time and space as one of the "media" of activity in a learning environment—and as Marshall McLuhan (McLuhan 1964) argued: the medium is the message. Or in this case, the ordering of a learning environment is part of the message to students that design is a process that evolves over time rather than a series of quick answers to short problems.

If a particular approach to space and time are important parts of the organization of the studio, then we should also consider the organization of resources for the studio, particularly the resources of professor, teaching assistants, an other design "experts." The Oxford studio had a single professor and two teaching assistants for 11 students, creating a student/teacher ratio of just over 3-to-1. In addition to the presence of these "experts" on a regular basis, at three different points in the Oxford studio reviewers from outside came to give feedback on students' work. These reviews were usually 4-5 hours long, taking up most of an afternoon and evening, and demanded from the critics high levels of stamina, of architectural sophistication, and of sensitivity in giving feedback on student designs.

The model of architectural learning through critical review has been a mainstay of architectural training since the foundation of the ateliers at the Ecole des Beaux-Arts in Paris (Chafee 1977). "Crits" and reviews were brought to the United States with the founding of the first architecture schools in the nineteenth century. It was clear from watching the Oxford studio that not everyone makes a good reviewer; indeed, the problems with the review process are discussed openly in the architecture community (Anthony 1987, Frederickson and Anderton 1990). At the same time, everyone trained as an architect has gone through the review process and sees it as a crucial part of learning to design. Every studio professor knows that he or she has to serve as a critic in order to get critics for his or her own studio. This tradition of obligations means that the design studio has a pool of potential reviewers as a resource that would take other disciplines time and effort to develop (seeFigure 15).

Figure 15: Professors and design professionals from outside the studio critique a student's work

Deciding to adopt a "studio model" for learning, then, means more than simply giving longer projects, or adopting "performance" as a means of assessment. There is a collection of specific structural features of the studio—particularly its organization of time and learning space, its small class size, and its access to potential participants in the review process—that make the studio model work.

Pedagogy

Even if one could construct a studio-like space, organize an appropriate schedule, and recruit enough teachers and critics, these steps alone would not recreate the design studio. As we saw in the portrait above, these structural features of the studio do not exist as ends in themselves. Rather, they exist to support a particular approach to design learning.

At the heart of the pedagogy of the design studio is the "desk crit." In each assignment—and often several times during an assignment—a student meets with members of the teaching staff for a detailed discussion of his or her work During these discussions, the critic works to understand what the student is trying to do with his or her design, and then help him or her develop that design idea. This "scaffolding" can take many forms, including offering suggestions, pointing out potential problems, or referring to examples of work by other architects that have addressed similar problems. Often critic and student will "design together," with the critic sketching quickly a series of design possibilities, showing the student the consequences of possible design choices. In doing so, the critic both offers design ideas and models design thinking. In many cases a crit will end with a specific suggestion from the critic, not of a particular design direction to take, but of a way to think productively about the questions raised: not "why don't you cantilever the lecture hall over the courtyard," but "why don't you try drawing that section at a smaller scale to see how the lecture hall relates to the courtyard."

Donald Schon (Schon 1985) has written at some length and with perceptive insight about the process of the desk crit, focusing particularly on the way in which the desk crit is central to the cycle of expression and feedback that develops a student's ability to design thoughtfully. Schon refers to this ability as "reflection-in-action." From a pedagogical perspective one might think of the desk crit as an instantiation of Lev Vygotsky's idea of a zone of proximal development (Vygotsky 1978). Vygotsky suggests that development is a process whereby learners progressively internalize processes they can first do only with the help of others. So, for example, we learn to cook first by helping someone else. Then we follow a recipe on our own (still relying on the external aid, but not needing direct guidance). Then finally we are able to invent new dishes and improvise on our own. The zone of proximal development is the set of things we can do with the help of others, but not quite do on our own.

The desk crit works in this way, with the professor or teaching assistant providing the student with design skills and knowledge that the student lacks. The student is designing, as it were, beyond his or her reach with the help of the professor or teaching assistant. The understanding is that as the student becomes more sophisticated as a designer, the feedback will move to a higher level, always showing the next steps on the path. And so the design process goes. The student expresses his or her design ideas. The professor or teaching assistant responds with feedback to those ideas to help the student achieve a higher level of design. The student incorporates that feedback in a new expression.

This cycle of student work—desk crit—student work—desk crit culminates in a presentation. The individual assignments and presentations of the design studio in turn form a larger cycle of production and reflection, with students responding to the feedback from one presentation as they prepare for the next. As described above, the cycles of expression and feedback take place through different media, and thus focus on different aspects of the design problem with each iteration of the cycle. The scale at which students are working moves from broad questions of context and strategy to more detailed questions of architectural language within individual parts of the building.

As the design moves from large to small scale, the student's attachment to his or her basic design idea grows. The midterm review in the Oxford studio, for example, was designed explicitly to get students to commit to a design strategy which could then be worked out in more detail in the second half of the semester. In general, the presentations move from pin-up to guest crit to review; they increase in both their formality and the expectations they have for the quality of student work. With each loop around the cycle of production and reflection, students are asked to make a deeper and more public statement of their architectural ideas (see

Figure 12, page 64).

What makes this cycle possible is partly the "coaching" nature of the desk crit. The professor or teaching assistant helps the student develop his or her ideas for presentation by modeling design behavior and by providing missing or incomplete design skills. Partly the cycle of production and reflection is made possible by changes in media and changes in scale that let students approach the same design issues from different points of view.

And partly the cyclic nature of design learning is made possible by the nature of the feedback given during presentations.

The distinction between "constructive" and "non-constructive" criticism is perhaps somewhat shopworn, but it does point to an important distinction between feedback that gives directions for positive change and feedback that merely pronounces work as "good" or "bad." In particular, most of the feedback students receive from formal evaluations in a typical classroom is *summative* rather than *generative*. That is, even if it makes suggestions for improvement, the intent is to evaluate more than to stimulate further work. In contrast, criticism in pin-ups, guest crits, and reviews in a design studio is given with the explicit idea that a student will be able to use the advice to improve his or her design. The final review is an exception in this sense, and it comes as no surprise that students, professors, and practicing architects feel overall that interim crits are more useful learning experiences than final reviews (Anthony 1987).

What we can see, then, is that the surface structure of the design studio — the spatial and temporal organization of design activity, as well as the role of professor, teaching assistant and outside reviewers — exists to support a specific set of pedagogical ideas. These ideas include: the "scaffolding" nature of the desk crit, where the professor or teaching assistant demonstrates design skills that the student will internalize over time; the cyclical nature of the learning process, where students revisit the same basic design questions on different levels and from different perspectives; and the generative nature of feedback from presentations, where students are expected to use feedback directly in their subsequent work, rather than taking feedback as a measure of success or failure and moving on.

Substance

Again it is clear that a learning environment with appropriate time, space and personnel, and with a pedagogy of scaffolding, cyclical assignments, and generative feedback would still not necessarily be a design studio. Just as the temporal and spatial organization of the design studio supports the pedagogy, the pedagogy in turn supports the more substantive agenda of the design studio. Part of what makes a design studio "work" is the very nature of design itself. In particular, the surface structure and pedagogy of the design studio depend on the expressive nature of design.

In psychological terms (see Shaffer 1996b), we can understand "expression" as the process by which thoughts, emotions, or sensations (internal mental states, or incepts) are reflected in words, gestures, or physical creations (external representations, or excepts). In design, the internal state to be represented can be thought of as an "architectural idea": an understanding about the creation of space based on a particular design problem. The existence of such underlying ideas in design is common both in the literature on design (see Hewitt 1985, Davies 1987, Rowe 1987), and in the language and activities of the design studio. Professors and critics in the Oxford studio spoke often about the need to "develop an attitude," "develop an architectural idea," "find a valid architectural proposition," "decide on a strategy," "take a stand," "take a stance," "develop your criteria" — all expressions of the need to find an underlying idea or ideas to govern the development of a student's solution to the design problem. Similarly, the early assignments of the Oxford studio were designed to help students develop such an idea about their project.

A critical feature of architectural ideas is that they are expressive. In particular, this means they are ideas that a student has chosen as reflecting his or her own interpretation of a design problem. Students in the Oxford studio were presented with a problem that has an infinite number of potential solutions. Their task in the semester was to develop a solution, to understand that solution, and to convey in words, diagrams and models how the solution they chose met the demands of the original problem. The idea they developed, in other words, was one of their own choosing. As the professor in the Oxford studio said eloquently to one undergraduate in an early desk crit: "You're in control. Make it whatever size you want. Then I'll ask: Why is it that size? And you'll say: Because its doing this job. And you'll develop your argument."

The activities of a design studio are organized around the core of an expressive idea. The conversations in desk crits are about developing the idea. The representations presented in pin-ups and reviews are attempts to express the idea. Criticism and feedback go to understanding the idea, to refining it, to developing it further. Because the idea is unique to the student, the professor or teaching assistant (and critics) can refer to examples from the work of other architects—or even other students—and still leave the student free to develop his or her own thinking. And because the idea is one of the students' own choosing, it is possible to spend a semester refining a single vision and turning it into architecture.

The design studio, then, is an environment designed to foster students' ability to develop and express their ideas about architecture. The pedagogical tools of the desk crit, cyclical assignments, and generative feedback are all means to this end—they are the processes through which expressive learning takes place. Similarly, the spatial organization and the scheduling of the studio serve to make extended desk crits, ongoing projects, and guest reviews possible. The surface structures and pedagogical tools of the design studio "work" because they are set in the context of expressive activity.

Conclusions: Towards a Studio Model

As Seymour Papert (Papert 1993) points out, school is a coherent system that has developed over time to optimize its responses to a range of social, economic, and cultural pressures—as well as the occasional pedagogical concern. This idea has two implications for any consideration of the design studio as a learning environment.

The first implication is that the design studio must be seen, and interpreted, as a coherent whole. The elements of the design studio—whether schedule, assessment technique, or curriculum—can not simply be adopted *a la carte* into other educational settings. It seems likely that the design studio could be adapted for other disciplines, but such adaptation would require an understanding of the whole and not merely the piecemeal adoption of elements of the design model.

The second implication of the idea that school is a well developed system is that educators interested in creating more effective learning environments might do well to consider alternative environments such as the design studio. Many optimized systems (school among them) tend to return to a state of stable equilibrium after minor perturbations. This suggests that reforms that changing pedagogy in small increments will tend to leave the system in or close to its original state. Design education may provide, in a sense, a pre-optimized alternative system. If we can successfully import a variation of the "design studio model" to another discipline, we might reasonably hope that the system would stabilize in a new—and substantially different—state.

This idea of transplanting a studio model into other educational settings may not be as strange as it sounds. Educators are already looking to the arts as a model for educational practice (Sizer 1992, Perkins and Blythe 1994). If, as described above, the design studio is an environment designed "to foster students' ability to develop and express their ideas about architecture," one can certainly imagine thinking of a "math studio" as an environment to *foster students' ability to develop and express their ideas about mathematics*. A similar statement could be made about a "history studio" or a "science studio."

This kind of change in perspective would require both a substantial commitment of resources and a new understanding of traditional domains. That resources would be required are clear, if only to develop the infrastructure required for studio learning. On the conceptual front, new computational media are starting to make it possible to work in traditional domains in more expressive ways (see Papert 1996, Shaffer 1996a). In mathematics, for example, students can now explore geometric concepts in dynamic and intuitive ways using a computer. And the World Wide Web makes it possible for students to get information to research topics of a type and scale that is not possible using more traditional means in most school libraries.

A careful analysis of the design studio as learning environment suggests that any attempt to make these kinds of changes would have to start on a small scale. In particular, the design studio will clearly need to be "adapted" for younger students, and it is not immediately clear what changes can and should be made. Perhaps more problematic, it will take time to develop a core group of experts who can serve as knowledgeable, skilled, and sensitive critics for mathematics, science, or history reviews.

The point here is not that the design studio is an "ideal" learning environment. Like any product of human endeavor, it has flaws, such as problems of time management for students, or in finding critics who can be both subjective and constructive. But characteristics of the design studio match with proposals for educational reform. This suggests that an experiment in "studio education" is worth conducting. The argument presented here is that the potential success of such an experiment depends on a deep understanding of the design studio as a coherent system of education. The studio system is built on the foundation of expressive activity—and its application in other settings requires thoughtful adaptation of that system to promote expressive learning in other domains.

Society of Design:
the development of mathematical thinking in a digital design studio

Introduction

Recent experiments with computational media and mathematics learning suggest that arts-based design activities, supported by appropriate computational tools, provide a rich context for learning mathematics. (Papert 1991, Willett 1992, Loeb 1993, Shaffer 1997b). A recent project (Shaffer 1997b), for example, showed that learning about geometry through projects in graphic design gave students a sense of control over their own learning, led to more positive attitudes towards mathematics, and helped students develop a rich understanding of fundamental ideas about symmetry and composition.

Cossentino and Shaffer (Cossentino and Shaffer in press) have argued that the success of such experiments in creating "mathematics studios" depends on the extent to which they present students with the chance to engage in activities that are both genuinely mathematical and genuinely artistic. If this is the case, it quite naturally leads to questions about the relationship between mathematics learning and arts-based activity: How do "genuine" design activities help students understand mathematics? And what role do computational environments play in turning artistic explorations into mathematical insight?

This paper explores these questions by presenting a qualitative study of one example of "digital mathematics studio" as a learning environment. The paper focuses on four key issues:

1. **The role of design knowledge in mathematics learning.** Design activities naturally involve the development and use of knowledge about design. Do students use design concepts such as color, balance, negative space, or form to help them understand mathematical ideas in a studio setting?
2. **The role of pedagogical structures of the design studio in mathematics learning.** The design studio has a long history as an environment for learning to design, with specific practices such as pin-ups, desk crits, and design juries. Do these pedagogical practices also help students to develop mathematical knowledge in a studio setting?
3. **The role of expressive activity in mathematics learning.** In learning mathematics through design, students explore mathematical ideas within the framework of the design studio tradition, but they are also explore these ideas in the context of personally-meaningful, expressive projects. Does the expressive nature of arts-based activities effect the way students learn mathematics in a studio setting?
4. **The role of digital design tools in mathematics learning.** Learning mathematics through design clearly does not depend on learning with computational media (see Loeb 1993, Shaffer 1997b). But are there particular

properties of computational media that help students develop mathematical understanding through the design process?

This paper thus explores some of the specific attributes of the digital design studio environment that contribute to the development of mathematical understanding—the "natural history" of students' mathematical learning as they work on design projects using computers in a studio setting. The results suggest that collaborative work and expressive activity in a digital studio help students think reflectively about their work. This reflective thinking, in turn, appears to help students understand the mathematical aspects of their designs.

Context

This paper reports on a project at the MIT Media Laboratory known as "Escher's World." In Escher's World, middle- and high-school students from the Boston area come to MIT to learn about mathematics in a digital studio through projects in graphic design. The project has run several of these digital studios, and this paper describes the results of the most recent, and most extended, of these experiments: a summer project where 12 middle-school students worked for 80 hours over the course of 4 weeks to design and install an exhibit for the MIT Museum on mathematics and art.

The Escher's World project brings together a number of theoretical perspectives on cognition and learning. The most obvious of these is a long tradition of exploring how students can learn through creative activity. Theorists and practitioners such as Rousseau, Froebel, Montessori, Parker, and Dewey all suggested in different ways that children learn through self-directed exploration of the world around them (Parker 1894/1969, Dewey 1938, Rousseau 1966, Montessori 1984, Brosterman and Togashi 1997). Parker and Dewey argued that self-expression is an intrinsic part of learning through exploration, and in more recent years, Papert has extended this line of thinking suggesting that new media provide a particularly rich context for building understanding in the context of expressive activity (Papert 1980, Papert 1993, Papert 1996).

Escher's World grounds this theoretical perspective in the pedagogical tradition of the architectural design studio: that is, the Escher's World project explores how the structures of design learning can support the use of computational media to help students develop understanding in other domains. The design studio can trace its roots back more than a century to the Ecole des Beaux-Arts in France (Chafee 1977), where young architects learned their craft from a master who acted as the "patron" of an independent studio, or atelier. The impact earlier this century of the Bauhaus (Wingler 1978) and its focus on specific areas of content knowledge about materials, engineering, and manufacturing added a range of "content" courses to contemporary design education. But the focus of a designer's training is still on work in a studio, under the direction of a master architect, leading to projects that are reviewed by a design jury.

This studio model has been studied in some depth by scholars of design education. This work (see particularly Schon 1985, Shaffer 1996c) suggests that the key features of work in a studio are open-ended projects and a variety of structured, collaborative conversations, including desk crits, pin-ups, and design reviews, all of which culminate in a public presentation of work. (These specific aspects of the design studio are discussed in more detail in the Methods section below.) In the Escher's World project, students learn transformational geometry by working on projects in graphic design using these traditional design practices of design education.

From a cognitive perspective, the Escher's World project combines work on the social construction of knowledge with a constructivist perspective on individual development. Theorists such as Vygotsky and Bruner have described with considerable insight the importance of interactions with others—peers as well as "experts—in helping learners develop understanding (Vygotsky 1934/1986, Vygotsky 1978, Bruner 1996). Escher's World explores the relationship of this work to the seminal theories of Piaget, who described development as a "constructive" process of active engagement with the world through which an individual progressively builds and refines his or her understanding of phenomena first through actions on objects and later through mental actions with internal representations (Gruber and Voneche 1995). Escher's World examines the processes by which this development takes place in the context of the design studio, exploring in particular how expressive activity lets students draw on social resources (experts and peers) to help them make sense of new phenomena.

The Escher's World project—and this analysis—thus brings together a variety of theoretical perspectives. One of the powerful "lessons" of the Escher's World project is that design learning is a "systemic" phenomenon that has to be analyzed from multiple points of view simultaneously in a holistic way. This idea will be the subject of a subsequent paper about the Escher's World project that will explore from a theoretical perspective the relationships among expressive activity, design, and mathematics learning.

The purpose of this paper is to present some specific results from the latest Escher's World experiment. At times this analysis will naturally draw on various ideas about learning, cognitive development, and pedagogy. But the focus of this paper is less on exploring various theoretical positions than it is on explaining the experiences of twelve students as they learned about geometry by creating a museum exhibit. The goal is to use an understanding of those experiences to gain insight into some of the important aspects of how students learn mathematics through design.

Methods

Participants

The most recent experiment of the Escher's World project was a four-week Summer Program for middle-school students in the Boston area. The program recruited students by mailing flyers to local youth organizations and school departments advertising a summer workshop in computers, mathematics, and design leading up to an exhibit at the MIT Museum. A total of approximately 300 flyers were distributed, the first 8 applications from males and 7 applications from females between the ages of 12 and 14 years were accepted (there were considerably more male than female applications). In the end, 12 students (7 males and 5 females) attended the entire program. Three students dropped out in the first few days, one for health reasons, two because their families had travel plans and apparently had not intended to complete the program.

Although the applications were not purposefully selected to create a diverse group of participants, the program wound up with a mix of students from urban (6/12) and suburban (6/12) areas in and around Boston. There were two African-American participants, two Asian participants, two Latino participants, and six white participants. Ages ranged from 12 to 14 years, with participants having just completed grade 6, 7 or 8 in a local public school. One participant was home-schooled, one appeared to have an undiagnosed learning difference (dysgraphia) and none of the participants had yet taken a formal course in Euclidean geometry.

The program was designed and run by the author, with assistance from a graduate student in the MIT Department of Architecture and an MIT undergraduate student majoring in design. Two of these program leaders were male, and the other female. All three workshop leaders were present throughout the program.

Program Description

Participants attended the summer program from 9am to 1pm Monday through Friday for 4 weeks. Except for field trips and work at the MIT Museum, all activities took place in one of the graduate design studios at the MIT School of Architecture and Planning. Of the 20 days participants were in the program, six days were spent on field trips (to the MIT Media Lab and to tour the MIT Museum), on installing the final museum exhibit, or on other non-design activities. The activities of the remaining days were modeled on the practice of a traditional architectural design studio course, with some additional structures—such as morning and afternoon group meeting, check-in times, and a snack break—provided in the day because of the age of the participants.

The studio model

As described above, the design studio has been studied in some depth by scholars of design education (see especially Schon 1985, Schon 1988a, Schon 1988b, but also Gaines and Cole 1980, Akin 1986, Crowe and Hurtt 1986, Anthony 1987, Rowe 1987, Frederickson and Anderton 1990, Mitchell and McCullough 1991, Coyne 1993, Mitchell 1994, Shaffer 1996c). This work (see particularly Schon 1985, Shaffer 1996c) suggests several key features of work in a "studio" setting. The first feature of the studio is that students work on projects that they explore (a) in depth, (b) over time, (c) for which they exercise significant control over timing, location, and direction. These open-ended explorations are more like the projects described in recent work on mathematics education reform (see NCTM Commission on Standards for School Mathematics 1989, NCTM Commission on Standards for School Mathematics 1991) than the problem sets of a traditional mathematics curriculum.

Extended projects in a studio are supported by specific pedagogical devices. The most notable is the "desk crit": a loosely-structured conversation between a student and a "critic" (an instructor or peer) over the student's emerging design. In this conversation, the critic works to understand what the student is trying to do with his or her design, and then help him or her develop that design idea by offering suggestions, pointing out potential problems, referring to examples of work by other architects, or working on a piece of the design together. In addition to these individual crits, students in the design studio regularly present their work to their peers for comment and suggestion in "pin-ups," and to outside professionals and experts in design reviews or juries. The Escher's World program included all of these aspects of the design studio model in the work of the participants.

A day in the studio

Generally working days in the program began with a warm-up activity involving traditional materials: for example, one day began with participants cutting out a shape from construction paper and then figuring out a way to make the same shape at one-half size. After the warm-up activity, participants were introduced to some piece of the functionality of the software they were using for the program: on the day that began with cutting out shapes at different scales, for instance, participants were shown how to construct dilations in the computational medium. Participants spent the rest of the day (approximately 2 to 3 hours) on design activities using the concepts introduced.

Design work typically began with the discussion of some "master works" from prominent artists (because of the abstract geometric nature of some of their pieces, examples drew heavily from the work of Kandinsky, Klee, Escher, Lewitt, Mondrian, and Picasso). This discussion of master works was followed by the presentation of a "design challenge." Typically, participants were asked to make a design using the mathematical concept introduced in the warm-up activity. Participants worked on the design challenge in an initial design phase lasting 45 minutes to an hour, consulting periodically with program leaders or peers for technical help with the software or for more detailed desk crits on their emerging designs. These initial design explorations were followed by a pin-up, where each participant presented his or her work-in-progress and got technical, aesthetic, and mathematical comment and feedback from peers and program leaders.

During the pin-up, additional exemplars were shown, and the design challenge was revised to include additional criteria. For example, on the day when participants began working with dilations, the original design challenge was to make an interesting composition using multiple dilations of a one or more shapes using a single vanishing point. The challenge was later revised to include the idea of balanced composition: participants were asked to make two designs using the same set of dilated objects, one balanced composition and one unbalanced. Participants typically worked on the revised challenge for another hour or so, with desk crits continuing throughout. The day usually ended with a pin-up and discussion of participants' final products.

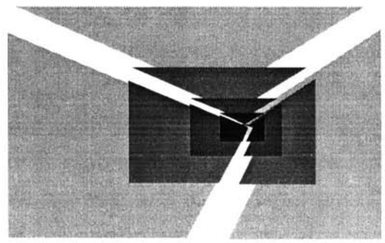


Figure 16: Participant work in Escher's World: a balanced composition using multiple dilations.

There were eight of these relatively structured days, mostly towards the beginning of the program (six in the first nine days) while participants got used to the software and to the workings of the design studio. At the end of the first week, participants were given the opportunity to work on a charette—a design term that refers to a relatively quick exploration of a complete design project—taking an entire day to produce a design that used the tools and techniques the participants had been working with that week. At the end of the second week, participants prepared a design for review by professors of mathematics and architecture who came to the studio for a day; for many participants, these designs became the basis for their final museum piece. Following the design review, emphasis shifted toward work on the final project, and participants spent most of the last two weeks designing pieces for the museum and then installing the museum exhibit.

Content

Over the course of the program, participants were introduced to mathematical ideas such as curvature, parallel and perpendicular lines, translation, rotation, dilation, reflection, and fractal recursion. Participants also worked with design principles such as simplest shapes, form, negative space, color, depth, and balance. The mathematical ideas were taken from the basic concepts of transformational geometry and were chosen because of their suitability for exploration with the particular software used in the project. The design ideas were taken from Rudolph Arnheim's work on design theory (Arnheim 1974) and chosen because of their suitability for exploration in conjunction with the mathematical topics of the program.[7]

Equipment and Software

Escher's World summer program used 12 Macintosh computers connected by a an ethernet network to a black-and-white laser printer, an ink-jet color printer, and a large

[7] Thanks are due to Joseph Press, a doctoral candidate in the MIT Department of Architecture and one of the program leaders, for suggesting and then incorporating Arnheim's framework into the program.

format (36" by 70") color plotter. At the beginning of the program, participants often worked in pairs or groups on a single computer, and often used different computers on different days. As the program progressed, participants increasingly worked on their own projects, each using the same machine from day to day. One of the computers functioned as a server, and all participant work from each day was archived on the server for later study. Another computer was connected to a 27" NTSC video monitor for use by participants or program leaders during pin-ups or other group discussions. All of the computers in the project were equipped with the Geometer's Sketchpad (Jackiw 1995), a commercial software program designed for use in geometry class rooms (see below in the results section for a more detailed discussion of the functionality of the Sketchpad program).

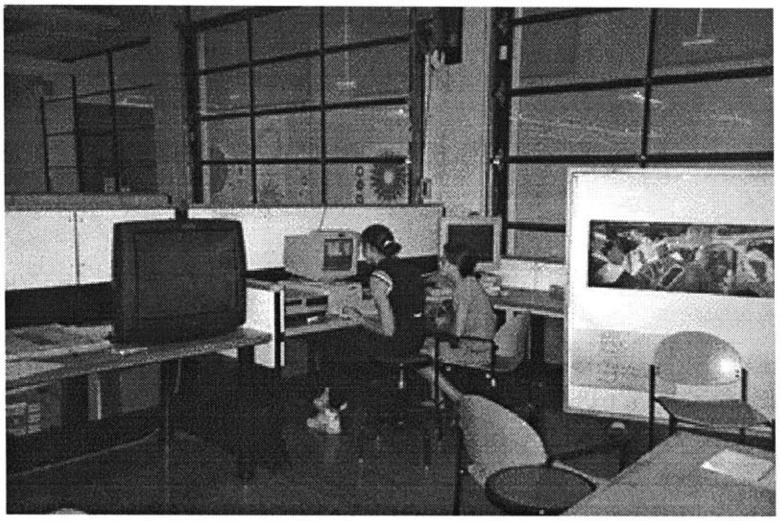


Figure 17: Participants working in the Escher's World studio at the MIT School of Architecture and Planning

Data Collection and Processing

Interviews

Data for the Escher's World summer program was collected in several ways. First, each participant came to MIT for a series of three interviews: one immediately before the program (pre-interview), one immediately after the program (post-interview), and a final interview three months after the conclusion of the program. In these interviews, participants were asked about their prior experiences and feelings towards art, mathematics, and learning in general (pre-interview), about their experiences in the program (post-interview), and about how these experiences changed their views of art, mathematics, and learning (post-interview and final interview). In each of the interviews, participants also described works of art by Escher and Matisse, constructed designs using a set of cut-out figures, and took a short test of 18 geometry questions taken from a variety of current and past geometry text books (see Moise and Downs 1971, Manfre, Moser et al. 1994, Rubenstein, Craine et al. 1995b, Rubenstein, Craine et al. 1995a, Serra 1997, Aichele, Hopfensberger et al. 1998).

What rigid motions will move triangle P
onto triangle Q in the drawing below?

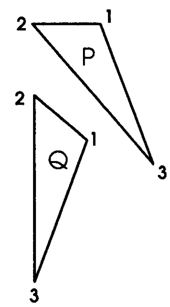


Figure 18: Sample geometry question from participant interviews

Design histories

Pre- post- and final interviews provided baseline data to see if the program helped participants learn concepts in mathematics and design. And, of course, these interviews also give some information about which parts of the program were effective in helping participants think about these issues in new ways. But the bulk of the data from the project is in audio recordings of the conversations participants had with the program leaders during their design work—and in the designs-in-process and final images participants produced throughout their work in the project. During the program, each of the program leaders wore a "tie-clip" microphone connected to a portable tape-recorder in his or her pocket. This allowed the program leaders to move about in the studio and have a record of their various interactions with participants.

Most of these interactions were, of course, "desk crits" where the program leader and a participant discussed the participant's design work in some detail. The nature of the desk crit is that it begins with a discussion of the designer's prior work—the design steps that led him or her to the desk crit discussion. This description of design steps is followed by a discussion of the design goals underlying the work, and then a conversation about problems, possible solutions, and potential courses of action. In an ideal desk crit, the critic even has a chance to probe the designer for the design "lessons" that he or she has learned and incorporated into the design. In other words, the desk crit provides a natural opportunity to gather information about and insight into a participant's design process and design thinking. The desk crits in the Escher's World program were, of course, not uniformly at this "ideal" level, but with some effort and prior training of the program leaders, the recorded desk crits did provide a means to conduct clinical interviews at key points in participants' design processes.

At the end of each day, each of the program leaders was responsible for providing an annotated telescopic transcription of his or her desk crits. This meant providing a detailed account of all of his or her desk crit conversations—usually a summary of the comments of critic and participant in each of the exchanges in the discussion, with verbatim excerpts for key points. These summaries were supplemented by images from the participant's designs, illustrating the critical moments in the participant's work.

At the conclusion of the program, these transcriptions were combined into a single, illustrated "design record" for each participant: that is, the three different (and often overlapping) accounts of a participant's design work from the transcriptions of the program leaders were combined into a single design history for each participant on each day. These design histories were clearly not comprehensive in the sense that they do not represent a complete recording of participants' design work. However, the design histories do give a detailed account of participants' work over an extended period of time. As such, they provide the major source of information for understanding how participants learned mathematical ideas during their design activities in the Escher's World program.

Data Analysis

Design episodes

One key feature of work in a "studio" setting is that students work on projects in a series of design cycles or episodes. In a typical design studio, these design cycles let participants revisit issues and explore them in depth over time (see Schon 1985, Shaffer 1996c). The nature of these cycles is the subject of continuing research and theorizing (see Schon 1985, Akin 1986, Shaffer 1996c, Simon 1996), but in general we can say that design episodes in the Escher's World project were characterized by (1) the recognition of some initial problem or sub-problem, condition, or issue ("I wanted to explore with rotation," "I just wanted to use circles," "I wanted something to go in and out"), (2) a series of steps or "design moves" taken to understand and or resolve the initial issue ("I put in hot colors," "I dilated it, and made two other shapes and moved them around," "I polygoned (sic) them"), and (3) some insight into the nature of the initial problem that suggests a conclusion, an end to the exploration, and/or a more interesting issue to pursue in a subsequent design episode ("I just made a bunch of triangles; it's not fun at all," "I see movement—it goes outwards and going on spearing off into different directions," "Oh cool... uhh... I'm getting dizzy... wait a minute—I've got to save this!").

For the purposes of coding and analysis, the design histories of each participant were divided into a series of design episodes, identifying a statement of the problem or issue being addressed (an initial condition), a series of design steps (an exploration), and a conclusion to the episode (an insight). In some of the design episodes recorded there were gaps in the record, either because participants were not consciously aware, for example, of the issue they were addressing in a particular design episode, or because a critic did not ask in detail about every step of the design process. Rather than interpolating or interpreting understanding or intent on the part of participants, the analysis presented here accepts that the design record from the desk crits is thorough but not necessarily exhaustive.

Action, reason, concept

The goal of this analysis is to examine how participants developed mathematical understanding. In order to make it possible to look for places where participants were gaining insights into the concepts of transformational geometry, the design episodes were further subdivided into descriptions of actions, descriptions of reasons, and descriptions of underlying principles. "Actions" refer to evidence about specific steps taken by participants during their design process: "I made… a totally different shape, and then I dilated it and stretched it around"; "I switched the shapes so they all looked the same"; "I drew a box in my mind around the perfect square that would fit around this, and then tried to fill in the whiteness." "Reasons" refer to comments that explain why a participant took a particular action: "the spacing was boring, [and made it look] too much like a face"; "I wanted the [viewer] to focus on one of [the circles], so instead of just seeing a square, they'll see all the things that make it up"; "I switched the shading to show how [this] shape was used to make the star." Finally, "concepts" refer to places in the desk crit where participants identify abstract principles or general rules to explain their design choices: "the mirror line creates symmetry, and symmetry creates equal parts on both sides, which makes it balanced"; "when you want something to be the center, you really shouldn't let anything go out of its field"; "if you need something repeated… you can almost always [define a] transformation for it."

Clearly, these categories of action, reason, and concept bear some resemblance to Piaget's stages of pre-operational, concrete operational, and formal-operational thought. Piaget's stages refer to the progressive coordination of actions and their increasingly abstract internal representation (see, for example, Piaget 1948). Similarly, the categories of action, reason, and concept refer to the ability to describe and explain actions in increasingly abstract language.

Coding

Each design episode was thus divided into a matrix for the purpose of coding, with columns representing the initial conditions, exploration, and insight of the episode (and the flow of the episode through time), and the rows representing the levels of abstraction (action, reason, concept) in each stage. Of course, for the reasons discussed above, in many of the design episodes one or more of the cells of the matrix were empty. This was particularly true early in the program when participants were less likely to describe their work in conceptual terms.

	Initial Conditions	*Exploration*	*Insights*
Actions			
Reasons			
Concepts			

Figure 19: Table showing the coding matrix for design episodes

Once the design episodes were thus categorized, all of the cells for all of the episodes were coded for the presence (or absence) of mathematical ideas, design principles, collaboration with program leaders, collaboration with peers, use of exemplars, and other categories relevant to the analysis. In addition, codes were added about episodes as a whole, including whether they were a continuation of an exploration begun in a previous design episode, whether the topic being explored was a new topic or one that had appeared in an earlier exploration, and so on. For obvious reasons, a significant portion of the analysis in the results section looked at design episodes where participants had conceptual insights about mathematical ideas ("dilation is the creation of apparent perspective"; "[a] mirror line creates symmetry, and symmetry creates equal parts on both sides, which makes [a design] balanced").

Unfortunately, resource constraints have made it impossible thus far to have multiple coders examine the data. Instead, to maintain consistency in the coding a single person (the author) completed all of the coding. Each design episode was examined a total of seven times in the process of coding for various categories, and the most significant categories and codes (such as codes for mathematics and design) were coded or checked twice for accuracy.

Qualitative and Quantitative Analyses

As is, hopefully, clear in the description above, the strength—and ultimately the validity—of this analysis and the results reported depend on the richness of the qualitative data collected about the design activities and mathematics learning of the 12 participants in the Escher's World summer program. At the same time, the coding method described above does make it possible to use statistical techniques to provide insight into underlying processes at work in the experiences of the participants. In particular, this analysis uses, at times, logistic regressions (described in more detail in the results section) and paired t-tests to explore patterns in various observations about participants' activities in the program. However, it should be noted that the small size and potential bias of the sample makes it impossible to generalize directly from these statistical results to implications for all learners. Rather, the purpose of the regressions is to support the underlying qualitative analysis as a description of the learning experience of these 12 participants in the Escher's World program.

Results

The results of the Escher's World summer program are presented here in two parts. First is a general overview of the participants' learning from the program, drawing mostly on mathematics and design test results from pre-, post-, and final interviews. Following this overview of the "effects" of the project is a more detailed analysis of the participants' design activity. This second analysis describes the cognitive and social processes through which participants developed their mathematical understanding during design activity.

General Results

The general results of the Escher's World program support the conclusion of earlier workshops in the Escher's World project: namely, that participants can learn geometry through design activities (see Shaffer 1997b). In particular, in tests of transformational geometry knowledge, participants' scores rose significantly between pre-and post- interviews (mean pre=9.5, mean post=12.25; $p<.01$). These gains were stable in final interviews three months later (mean final=12.0), the results of which were significantly different than the pre-interview scores ($p<.05$) but not statistically different from post-interview scores ($p>.49$; see Figure 20). These general results are supported by observations of mathematics learning from design episodes, where participants reported some kind of mathematical insight in 22% of design episodes (176/806), and all participants reported having conceptual mathematical insights (mean=4.7 conceptual insights about mathematics during the program).

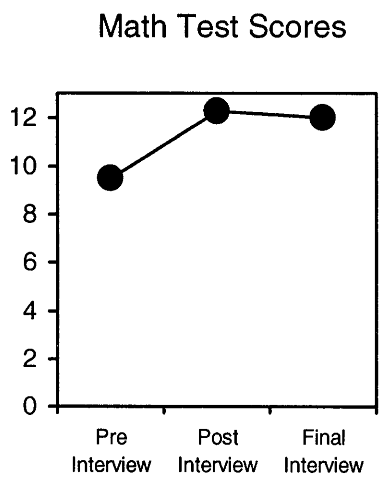


Figure 20: Scores on a test of transformational geometry knowledge rise significantly after the program ($p<.01$) and remain significantly higher three months later ($p<.05$)

A similar change can be observed in participants' performance on design tasks before and after the program. Participants were shown a copy of "Sky and Water I" by M.C. Escher (not reproduced here for copyright reasons) and asked to "describe what the image looks like for someone had not seen it." Even from a simple measure such as total number of words in participants' descriptions, it is clear that they have more to say after the program (mean response length pre = 82.3 words, post = 127.6 words; $p<.01$). Participants' responses also reflected a pattern observed in earlier work on the Escher's World project (Shaffer 1997b) where descriptions of images showed increasingly sophisticated use of analytic terms from mathematics and design after completing

Escher's World activities. In this case, for example, before the program one participant described the Escher image as: "a fish transforming into a bird. There's a fish at the bottom, and there's a bird at the top. And as it goes a little higher, it sort of transforms into the bird." After the program, the same participant described the image in mode detail and with more analytical language:

> A fish, then showing another fish, two fish above, and three fish, and going on. Except as the fish goes upward, it turns more into a weird design. And [that] creates—I don't know if a lot of people know what negative space is, but it creates a negative space or an atmosphere to show this seagull coming out.... At the top, the seagull is the most realistic, because here all you see is this black log with two wings, and as it goes up it gets more detailed and detailed. Same thing with the fish.

A change in participants' design ability can also be seen in designs they made with pre-cut construction paper during interviews. In post- and final interview images, participants made increasingly abstract forms, incorporating design principles from the program to create more compelling images than in their pre-interview designs with the same materials (see Figure 21).

Pre

Post

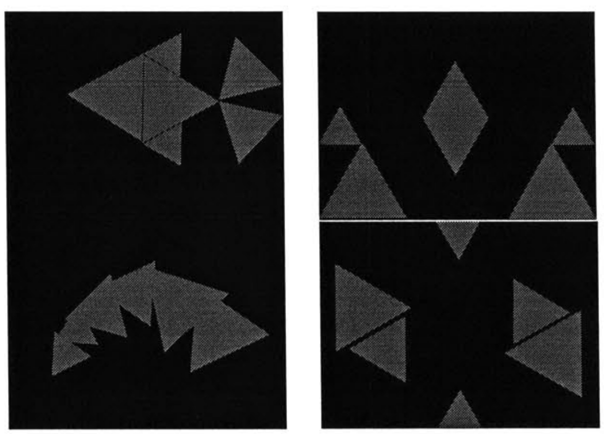


Figure 21: Two images sets of images from participant interviews. In each set, the image from the pre-interview is on top, and the image from the post-interview with the same participant is on the bottom. The later images use more abstract forms more dynamic compositions.

Participants also reported deeper and more profound—though more individual—transformations as a result of the Escher's World project. One participant, for example, who was particularly analytical and logical in his thinking, discovered that he did indeed have artistic talent. Both he and his teachers now see him as more creative in his work at school. Another participant discovered the joy of participating in group discussions; now she talks more in classes at school and is getting all A's. In her final interview, she proudly displayed the pager that her parents had bought her as a reward for good grades.

Still another participant regretted setting himself up as the "class clown" and overindulging in cookies during breaks. "Something I've learned," he said, "is that... I should have tried to control myself so I wouldn't be seen as an eating slob." In general, participants came away from Escher's World with a changed view of mathematics. As one participant put it: "I see that math is such a more broader, bigger area of a whole bunch of different things." And of course, a common sentiment was simply: "I wish school was like this."

The digital studio thus provided a powerful context for participants to make discoveries about themselves, about learning in general—and about the specific domains of mathematics and art. But if participants in the Escher's World summer program clearly learned about geometry and design through working on projects in a computational studio setting, the question still remains: what was it about participants' activity in the digital studio that helped them develop their understanding of the transformational geometry inherent in their design work?

Mathematical Reflection through Design

Design episodes

As described above in the Methods section, the principle unit of analysis of participants' design activity in the Escher's World program was the design episode. To provide a sense of scale: participants engaged in an average of 67 design episodes each over 13 days—between 5 and 6 episodes per day on an average day when the figures are adjusted for participants who missed one or more days of the program. The average length of a design episode was approximately 15 minutes, although some explorations were much shorter (as short as 2-3 minutes) and some were spread over more time, including breaks and other interruptions. Participants, on average, had a total of 11 "conceptual insights" during their design activity. That is, about once each day, on average, participants came to understand and be able to articulate an abstract principle about mathematics, design, or some other aspect of their design activity during their design work.

Kinds of mathematical understanding

There are, of course, different kinds of mathematical understanding, just as there are different kinds of understanding in general. Theorists have long made the distinction between declarative and procedural knowledge: "knowing that" something is true versus "knowing how" to do something (see, eg, Schon 1985, Dreyfus and Dreyfus 1986). Similarly, the distinction is often made between specific knowledge and generalized or abstract thinking, including discussions of how "abstract" and "general" ideas may be similar but not synonymous (Cobb 1986, Perkins and Salomon 1989, Boaler 1993, Gruber and Voneche 1995, Noss and Hoyles 1996). More recently, mathematics educators have begun to look more closely at the idea of abstraction itself, suggesting that abstract thinking is not disconnected from particular instances, but rather represents a web of connections between ideas (Wilensky 1991, Noss, Healy et al. 1996). In this view, abstract knowledge is characterized by a rich set of associations that make it broadly applicable.

Explicit knowledge of mathematical principles is thus not the only meaningful goal of mathematics learning. At the same time, it is clear that the ability to think reflectively is a critical skill for learners to develop, and an important part of mathematical understanding is the ability to form general principles and conclusions from specific observations. This kind of reflective thinking about mathematical activity can be seen in participants' design histories in design episodes where participants have "conceptual" insights about the mathematical ideas that underlie their design activity (see above in the Methods section for how design episodes were coded for conceptual mathematical insights). Understanding the nature of design episodes where participants show this kind of reflective mathematical insight is thus an important part of understanding mathematical learning in the digital design studio. That is, an important part of understanding how participants learned mathematical ideas in the Escher's World program is understanding how they came to understand and articulate ideas like: "an arc really does have a center," "dilation is the creation of apparent perspective," or "when you rotate something its actually in a circle—you can rotate it a lot of [ways] and it will always make a circle."

A model of reflective insight

One question that naturally arises in an environment like the Escher's World program is whether the process of developing reflective insight about mathematics is unique to each participant, whether there are several different paths or processes involved, or whether there are general processes at work common across all the learners. Obviously in any of these cases, there will be variations between individuals. But the question is whether these variations are so great as to make it meaningless to talk about a reflective process in general terms without referring to the specific circumstances of the person learning.

One way to look at this question is in quantitative terms. In particular, it is possible to look at features of design activity that make reflective mathematical insight more likely using a fixed-effects logistic regression model. Briefly, a logistic model computes the likelihood, expressed as an "odds ratio," that an outcome will occur (in this case, reflective mathematical insight) based on a given condition or set of conditions. The odds ratio gives the increased likelihood of the outcome given a particular condition. So, for example, if we were to compute a logistic model for whether it will snow based on the "season" of the year, we might find that "winter" has an odds ratio of 50—that is, it is 50 times more likely to snow in the winter than it is in other seasons. A logistic model can also determine whether a given condition is statistically significant in predicting the outcome, and how much of the observed variability (variance) of the outcome is explained by the model.

Of course the design episodes in the Escher's World program were not independent events. Rather, they were repeated observations of a small number of participants over time. A "fixed effects" logistic model takes this into account by controlling for the time-invariant effects of individual participants. If we assume that each participant came to the program with his or her own unique abilities, interests, strengths, and so on, a fixed effects model essentially removes this individual variation and models only the design activities that are associated with reflective mathematical insight *in general* for this group of learners. As it turns out, this kind of model for the Escher's World program (see Figure 22) can predict instances of conceptual mathematical insight with $R^2 > .70$. This suggests that there are some significant features in the development of reflective mathematical thinking that these individual learners had in common.

The remainder of the results section looks at these features in an attempt to explain how the participants in the Escher's World program learned mathematics through design. The analysis that follows looks at the role of collaboration and the role expressive intent in the development of conceptual thinking about mathematics. Following these discussions, the paper presents a short case study and a second analytical model that describe the development of collaboration and expressive intent in the design studio context.

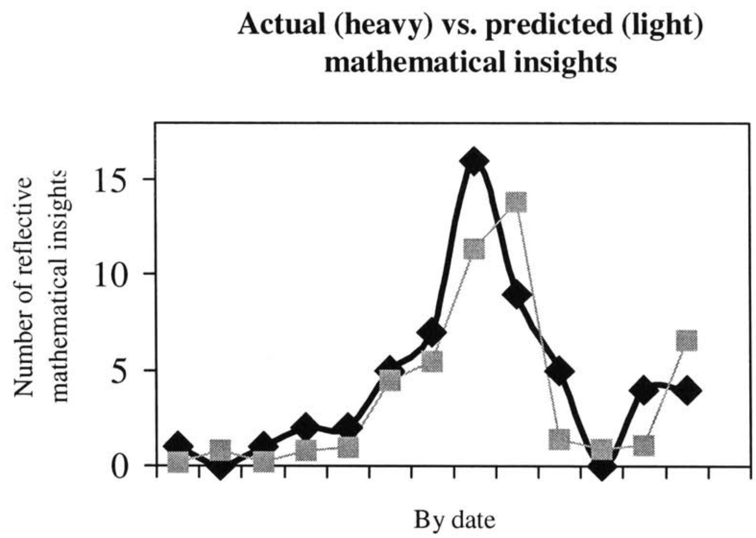


Figure 22: Graph shows actual (heavy line, diamonds) reflective insights about mathematics during design episodes by date in the program as well as reflective mathematical insights predicted (light line, squares) by a fixed-effects logistic regression model ($p<.001$, $R^2>.70$)

Collaboration

Theorists who write about the influence of collaboration in the learning process suggest an important aspect of design activity is that the creation of an artifact (real or virtual) makes cognitive activity public, and thus an object for discussion (see , eg Papert 1980, Bruner 1996, Bell 1997, Enyedy, Vahey et al. 1997). The production of artifacts provides opportunities to talk not only about the ideas they represent, but also about the processes involved in creating them (Papert 1980).

This aspect of design learning was clearly at work in the Escher's World program. We can see the in the data the importance of collaborative conversations in helping participants turn their design activity into reflective insights about mathematics. Participants were significantly more likely to have abstract mathematical insights if they had a conversation with someone else about abstract ideas while making sense of what happened during their design activity. In the logistic model of reflective mathematical insight, nearly 50% of the variability in whether or not participants express a mathematical insight is determined by whether or not they collaborate with program leaders or with peers while they are working through the implications of their design activity (see Figure 23).

For example, in one design episode, a participant observed that it was not easy to produce a compelling form in the negative space of his design. One of the program leaders asked: "Why is that?" This led to a discussion in which the participant explained that the constraint of mirror symmetry in the design made it impossible to create different design effects on opposite sides of the mirror line. While such a question may seem simple, and perhaps obvious to a more experienced mathematician or designer, it was apparently important in pushing the participant to articulate his understanding of mirror symmetry and its graphic consequences. Overall in the program, participants were 54 times more likely to have a reflective mathematical insight during conversations with a program leader ($p<.001$) and 30 times more likely to have such an insight in collaboration with a peer ($p<.001$) than with no collaboration at all.

Variance in Reflective Mathematical Insight Explained by
by Logistic Regression Model

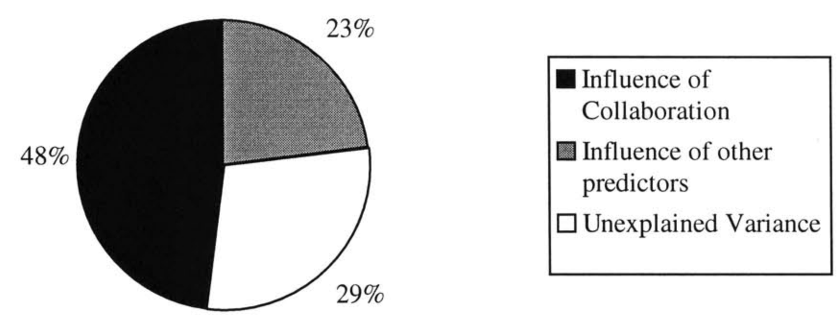


Figure 23: Pie chart showing importance of collaboration in reflective insight. Percentages shown are R^2 values from fixed-effect logistic regression models.

It is true, of course, that the data collection method used in this study—clinical interviews as a part of design desk crits—was likely to over-represent the effects of collaboration on participants' work in the studio. Indeed, collaborations with program leaders were involved in 80% of the design episodes where participants had reflective mathematical insights. But although the effects of collaboration may be overstated here due to bias in the data collection method, it is clear that collaboration did play a significant role in participants' mathematical insights.

In addition to showing that collaboration plays an important role in reflective insight, the data from the Escher's World summer program also suggest what kinds of collaborative strategies were more effective in helping participants develop and express conceptual understanding about mathematics. Collaborative interactions in the design episodes were characterized as Probing when the collaborator (program leader or peer) asked a question to elicit more information or further explanation from the participant doing the design work. Interactions were coded for Validation when the collaborator made a judgement (positive or negative) about the work being discussed. Similarly, interactions were coded for Suggestion or Explanation where collaborators offered suggestions of possible design steps or design ideas, or when collaborators offered explanations of ideas encountered in the design work or of function of the computational tool.

A separate logistic regression indicated the effects of these different collaborative strategies. All of these modes of collaboration made it more likely for the designer to explore the mathematical aspects of his or her design in an abstract or conceptual way ($p<.05$ for each of probing, validating, suggesting, and explaining). Probing or validating a participant's work were substantially more helpful than making suggestions or offering explanations in generating reflective mathematical insight—18 and 12 times more likely for the former, but only 6 and 4 times more likely for the latter.

Effects of various collaborative actions on reflective insight

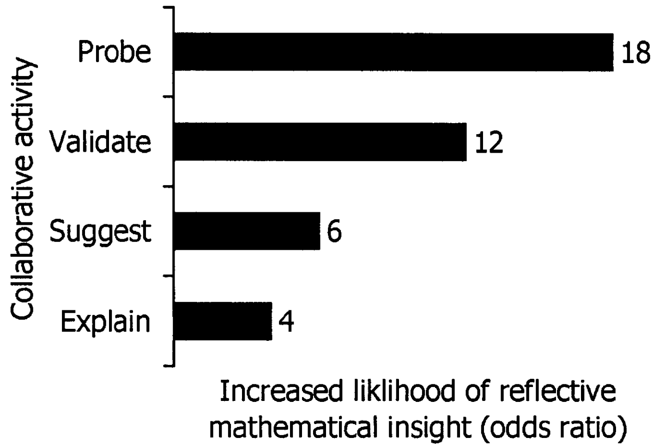


Figure 24: Graph shows the effect of various collaborative activities on the likelihood of reflective insight. Numbers indicate the odds ratio compared to no collaborative activity; thus, probing makes it 18 times more likely that the designer will have a reflective insight compared to no collaboration. Note that all of these colaborative styles increase the chances of reflective insight, but that probing and validation are more effective than suggesting and explaining.

These collaborative activities—particularly the collaborations with facilitators—took place primarily during desk crits. The data also suggests, however, that other aspects of the studio environment promoted constructive collaborative interactions. Pinups, for example, provided participants with opportunities to get feedback from their peers. In pinups, participants could present several versions of the same graphic concept for comment and comparison, learning in the process that "everyone has different opinion on how they like it," and that the final creative decision is necessarily up to the designer. As one participant succinctly put it: "It's hard to make something everyone likes."

Pinups also gave participants a chance to practice their skills at giving feedback to others. Vygotsky described the process of development as the progressive internalization of cognitive activities that individuals are first able to do only with the help of others (Vygotsky 1978). In the pinups, we see a similar process at work, where participants model their questions for peers on the questions they have heard program leaders ask, as when one participant turned to another and said: "I have a good David question: What made you decide how to do this? And: Why? That's a Joseph question!" Joseph and David were, of course, two of the program leaders, and in this pinup example, we see both cognitive and social scaffolding as one participant asks another difficult questions using the program leaders as models. This kind of "public" collaboration in pinups led to more direct collaboration in other settings. Participants were more likely to collaborate with their peers during design activities when referring to earlier interactions in a pinup ($p<.01$).

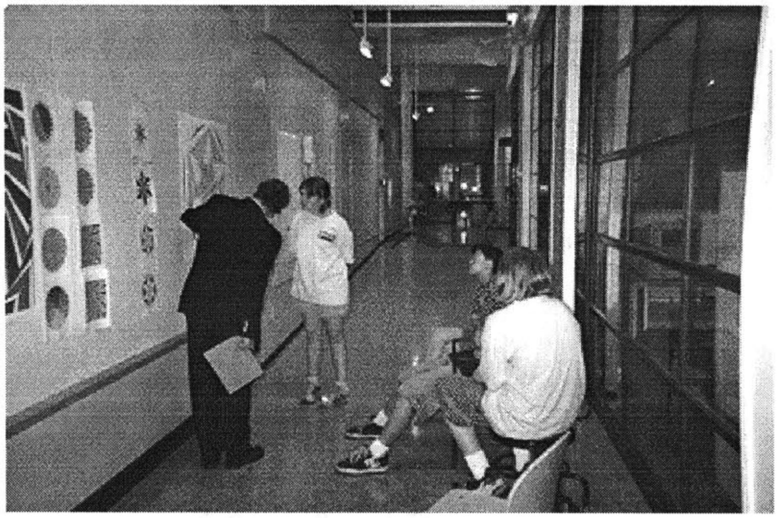


Figure 25: Participants in Escher's World discuss their work in a design review with the Dean of the MIT School of Architecture and Planning

The other major collaborative format of the design studio, the design review or jury, also appears to have had a significant effect on participants' learning in the Escher's World program. As shown in Figure 26, participants work with abstract mathematical ideas and participants collaboration overall rose dramatically in the days leading up to the main design review, where participants prepared designs for comment and criticism from a panel of designers and professional mathematicians. After the review, participants continued to work on their designs, but seemed to enter more of a "production" mode, marked by less intensity of effort and more attention to refining ideas already in place. Participants were six times more likely to work with abstract mathematical ideas in their design activity in the days before the pinup than in the first week of the program, and four times more likely to collaborate with a program leader about abstract ideas ($p<.05$ for both results).

Effect of Design Review on Work in the Studio

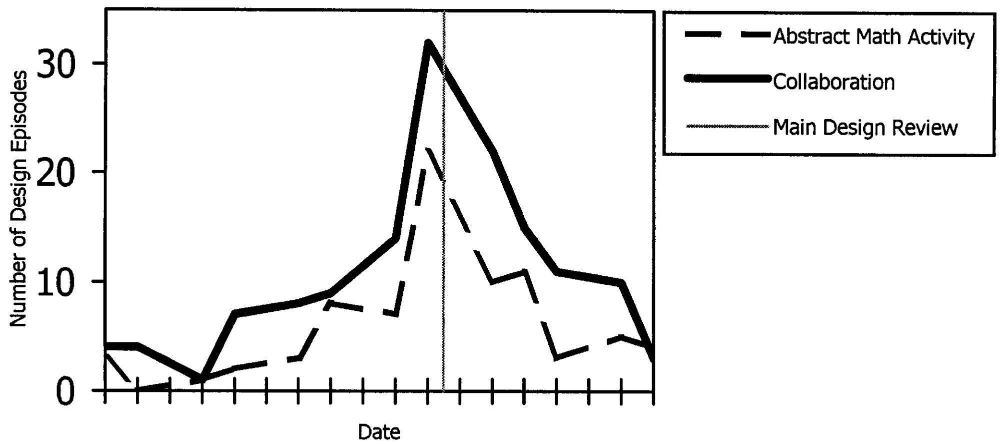


Figure 26: Graph shows how mathematical activity and collaboration rise in the days preceding the main design review

What we see in the data form the Escher's World program, in other words, is that collaboration played a significant role in helping participants articulate the mathematical principles underlying their designs. Moreover, some of the specific structures of the design studio model—including pinups, design reviews, and particularly desk crits—provided rich contexts in which this collaboration took place. In order to understand the development of reflective thinking in the context of the design studio, we thus need to know more about these collaborations, and in particular, we need to explain how deep collaborative conversations about abstract concepts come about in the studio context.

Conceptual thinking

Whether "conceptual thinking" exists as a skill separate from any particular domain and whether the ability to "think abstractly" can transfer from one context to another are matters of both great concern and considerable disagreement among theorists of learning and development. Piaget argued that there is such a thing as abstract thinking (or "formal operational thought"), and at least strongly implied that there are correlations in a person's ability to think abstractly across different domains: this is the now-famous idea of "stages" of cognitive development (see Gruber and Voneche 1995). More recent authors in the Piagetian tradition have suggested that the ability to think conceptually or abstractly does exist in individual skill areas, but that such ability develops more or less independently for different skills (Fischer 1980). An even stronger critique of the generalizability of abstract thinking comes from Lave and other theorists of the "situated cognition" school, who argue that skills are learned in specific contexts and that the concept of "transfer" is meaningless, or at best plays a minimal role in development (see Boaler 1993). Still other authors argue that abstract thinking exists, but is far less important in expert behavior than the rich body of specific examples that comes with experience in a domain (Dreyfus and Dreyfus 1986).

As Perkins and Salomon point out in their review of the debate on the generalizability of cognitive skills (Perkins and Salomon 1989), the reality of individual development probably lies somewhere in between the most extreme theoretical positions. Evidence collected over a number of studies suggests that "abstract thinking" as an independent skill does not account for all—or even most—of a person's proficiency in particular domains. However, there are also important aspects of learning that seem to use cognitive techniques that work across domains: heuristics, strategies, and general ways of thinking about new situations that make some people more effective than others at problem-solving. Perkins and Salomon argue that such general cognitive skills are multi-purpose tools, like human hands, that are used in similar—though not precisely the same—ways in different situations.

For the participants in the Escher's World program, "expressive intent" appears to have functioned as a general cognitive skill in this sense. As the program progressed, participants increasingly started their design explorations with a concept that they wanted their design to represent—they tried to, as one participant put it, "pick something that illustrates something else." These expressive goals were sometimes purely mathematical ("the idea was to show how dilation changed size of something without changing its location"), sometimes mostly about design considerations ("[I] like when one object makes up another object"), and sometimes combined both aesthetic and geometric concerns ("I wanted to express the negative space of circles... no, of rotation"). In some cases, participants had multiple expressive intents within a single design. So, for example, one participant created a striking black-and-white image (which eventually became his final project) exploring simultaneously how straight lines can form a circle, how rotating forms can create negative space, and how a line continues on forever (see Figure 27). These ideas, which the participant articulated repeatedly as the goals of his design work, emerged more clearly as the image progressed from draft to final product.

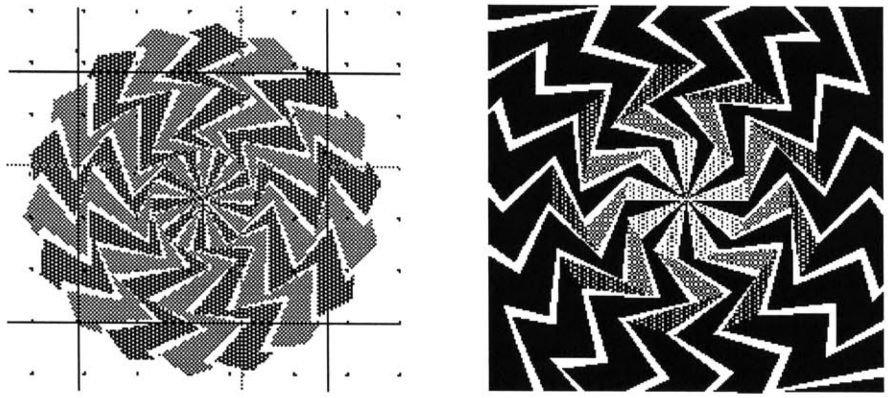


Figure 27: Participants in the Escher's World program began to express explicit design ideas in their work. In this case both the draft (left) and final (right) images were intended to show how straight lines can form a circle, how rotating forms can create negative space, and how a line continues on forever.

It is significant (in both the statistical and conceptual sense of the word) that starting—and more generally exploring—with this kind of explicit expressive intent made it more likely that participants would have reflective insights from their design activity. Design episodes with conceptual activity in the initial conditions or exploration phases were more likely to have conceptual thinking in the insight phase of the exploration ($p<.001$). As shown in Figure 28, this was true in the domains of both mathematics and design: expressive intent and reflective activity supported abstract insight.

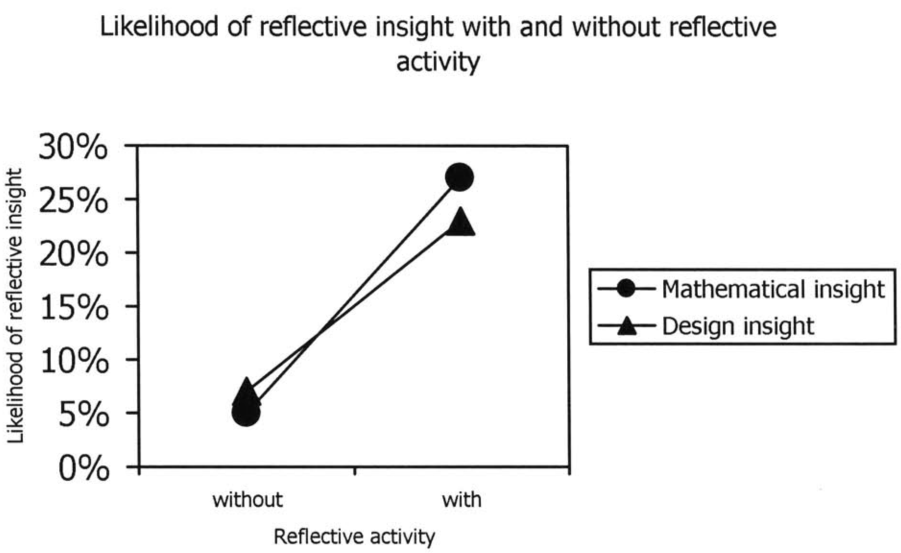


Figure 28: Reflective activity tends to promote reflective insight.

If we look more closely at the idea of "reflective activity," however, the story becomes more complex, and more interesting. Not surprisingly, starting an exploration with an abstract mathematical idea in mind made it more likely that a participant in Escher's World would come to an abstract mathematical insight ($p<.001$). Starting with an abstract design principle, on the other hand, made it significantly *less* likely that a participant would have an abstract insight about mathematics ($p<.05$). For the learners in Escher's World, the subject matter of design—ideas about negative space, movement, depth, form, and composition—did not appear to scaffold mathematical understanding directly.

What does seem to have been supporting development of mathematical learning, however, was the expressive nature of the design activities. Participants were more likely to start an exploration with a mathematical idea if they had worked earlier with a specific abstract expressive intent ($p<.05$), and participants were dramatically more likely to investigate an abstract mathematical idea in the context of an explicit expressive goal ($p<.001$). In the digital studio, in other words, mathematical ideas were explored in the context of expressive ideas for a design. Participants explored mathematical concepts by trying to "say" something with their designs, like the participant who asked: "Could I say something like: I like to use rotation to show negative space because the negative space changes so much?"

The expressive intent in participants' designs thus provided a framework for thinking about mathematical ideas and design principles. As shown in the next figure, if we look at the episodes where participants started with an initial concept for their design, 63% (41/65 episodes) involved an explicit statement of expressive intent. Of these, about half were about design (20/41) and half about mathematics (18/41). But math and design only overlapped 11% of the time (7/65 episodes). If design ideas were supporting participants' development of mathematical thinking, it was not through specific design principles, but rather through the expressive nature of design activity. "Abstract thinking" may not be a general, transferable skill from domain to domain. But it seems that for the participants in the Escher's World digital studio, "expressive intent" was an idea that helped develop abstract thinking across disciplines. Thinking about expression provided a connection between creative design activity and conceptual thinking about mathematics.

Initial Concepts for Design

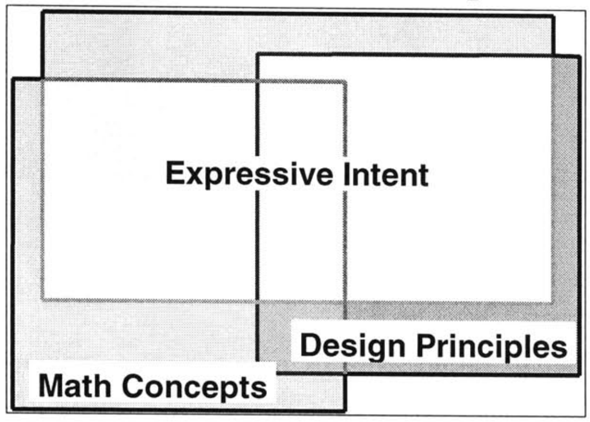


Figure 29: Diagram shows the overlap of mathematics concepts, design principles, and expressive intent in the initial conditions of participants' design episodes. The areas on the diagram are proportional to the total number of design episodes in each category. Thus, expressive intent shows a substantial overlap with math concepts and design principles, but mathematics and design are relatively separate from each other.

The development of reflective insight

In order to understand the development of reflective, abstract insight about mathematics by the participants in the Escher's World program, we thus have to explain (at least) two phenomena. The first is the process by which participants began to collaborate with others about their design work. The second is the way in which participants came to approach their design work with expressive intent, and especially with the explicit goal of representing an abstract idea in their images. In addressing these issues, this section of the analysis presents excerpts from one participant's design work over the course of the Escher's World program. The general features of this design story are then explored quantitatively to describe a general path to reflective thinking for the participants in Escher's World.

B – 's Story

The first day of the Escher's World program began with a design challenge: to make a square out of circles. Participants looked at several examples of solutions to this design problem (see Figure 30), and were also showed how to construct a circle using the software tool. Later, after a pinup where participants shared their solutions to the original challenge, the problem was revised. Participants spent the remainder of the day working on drawing straight-edged figure of any kind using curved lines (arcs or circles).

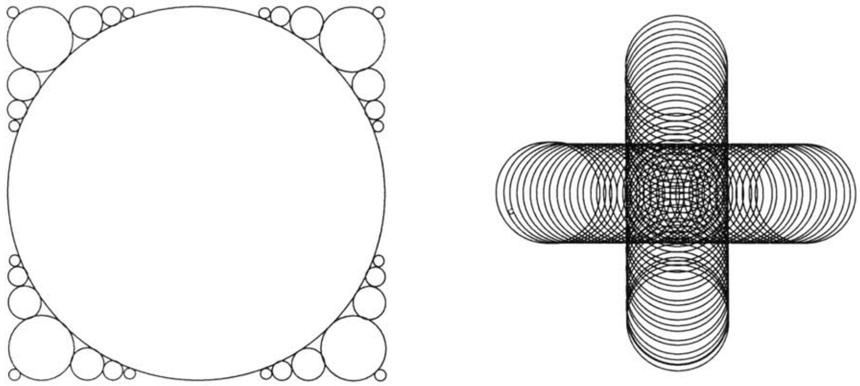


Figure 30: Two examples of drawing a square using circles that participants were shown during the Escher's World program.

B—'s response to these challenges was first to draw the outline of a square "by hand" and "by eye" (see Figure 31), and only after talking with program leaders and seeing the work of her peers did B— begin to use straight lines as "guides" for her drawing. B—'s work on these first design challenges shows progressively more sophisticated use of the idea of curvature and properties of arcs: in the second image in Figure 31, for example, she uses a single circle with large radius to form one "straight" side of a square; in the final image she uses arcs to "wrap around" the bottom vertices of the triangle. But at this early stage, B— showed little evidence of thinking explicitly about these issues. About the middle image she said only "I just wanted to use little circles," and when asked why she chose a triangle for the final design, she replied: "It was a simple shape. I'm just fiddling around."

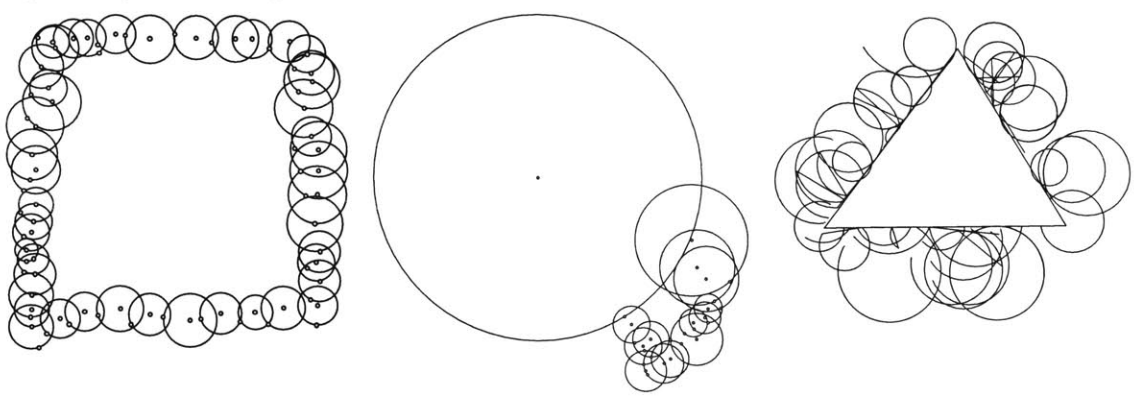


Figure 31: B—'s responses to the challenge of drawing straight edges with curves.

The next day participants were asked to make an image of a ball, and were given two rubber balls of different sizes and with different coefficients of restitution (bounciness) to use as models. The challenge was two-fold. First, the images of this round object were to be made using only straight lines. Second, and perhaps more important, the designs were supposed to convey an interpretation of some interesting aspect of the ball: its weight, the texture of its surface, the way it bounces, and so on. In response to this second challenge, we can already begin to see a change in B—'s work (see Figure 33). In particular, after making one drawing of a ball "bouncing off of a ceiling," B— found herself struggling to get the artistic effect she wanted. She called over a program leader and in the course of the desk crit explained: "I want the bottom lighter... not lighter, more open, not really there...." As B— and the critic explored the drawing, they talked about her expressive intent (what she wanted from the drawing) and also about the underlying construction of the image.

What they discovered was that by using a circle as a "guide" for the line segments that make up the ball, B— had inadvertently given herself a way to alter the artistic effect of the image. (The circle was "hidden" by the program, but its position was still determining the location of the line segments in the image). By "dragging" the center point of the circle, B— could spread out the segments on the lower edge of the ball. Armed with this discovery, she returned to the design with both a clear intent and a means to manipulate the image. B— spent the remainder of her working time adjusting and reconstructing the design so that, as she said of her final image, "when it goes out, it's sort of... motion" representing the movement of the bouncing ball.

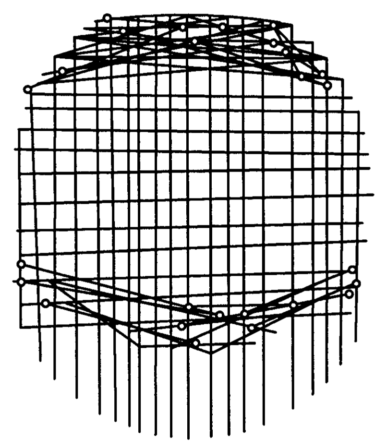


Figure 32: B—'s original image of a ball bouncing off of a ceiling.

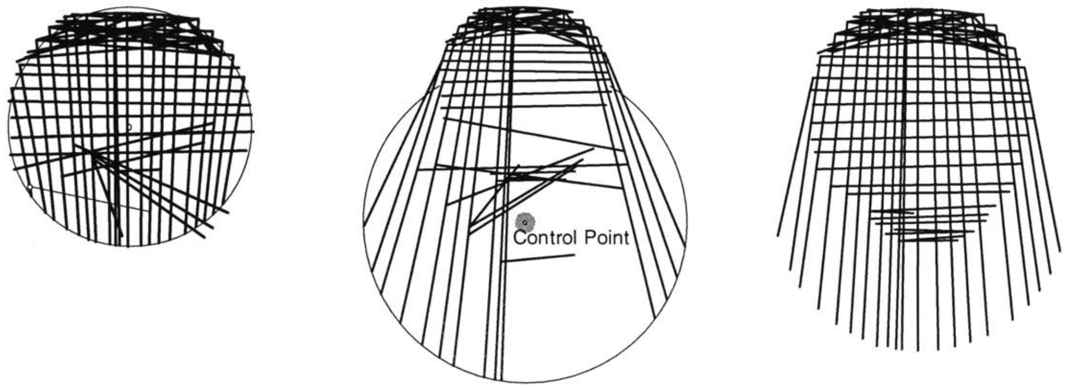


Figure 33: B— and the critic explored the underlying construction of the image, and then B— was able to use the tool to realize her expressive intent in a final design (right). The middle sketch is a reconstruction—the coloring and labeling of the center point were added, and several line segments were removed for clarity.

Moving ahead a week or so, B— was working on a design challenge involving rotation and discovered that she could make an "exploding" negative space in the center of the image (see Figure 34; the "vibrating" white space at the center of the rotation is what B— called "explosive"). This idea became the focus of a series of design explorations, a topic that B— returned to consistently in the following days, and ultimately the subject of her final project. What in particular about B—'s schooling or upbringing might have made explosions such a compelling theme for her is left to the reader's speculation, but whatever the motive, the problem of making this kind of visually active white space was clearly quite compelling for B—.

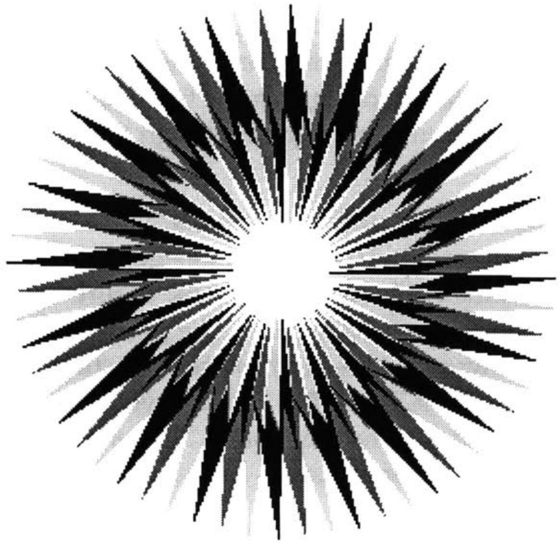


Figure 34: B— discovered that it was possible to create an "exploding" negative space (the vibrating what space) at the center of a rotated image

B—'s discovery of exploding negative space was accompanied by a realization about the design process. As she put it: "You should start with a few [shapes], play around with them until you find some nice shapes, and then expand [on that]." The question for B— was how to "expand" this design idea. After discussing the problem in a series of desk crits with one of the program leaders, B— began a series of carefully-conducted explorations into the workings of "exploding" spaces. She made a set of designs (see Figure 35)—some 10-20 in all—determining the factors that make a rotated shape look explosive: a "nice pointy shape," "enough" points, a dark color, and so on.

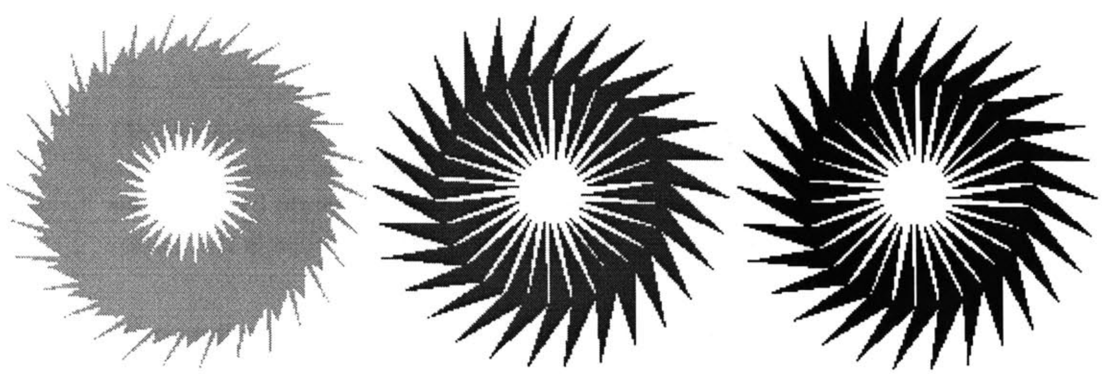


Figure 35: B— explored the factors that make an exploding negative space from a rotated shape

At this point, B— found herself stuck. She knew she wanted to make a design that explored the idea of exploding negative space, but was not sure how to proceed. In a desk crit, a program leader suggested that she think about making "rings of explosions," and perhaps use the same shape in each ring—as if a real object was exploding and sending off shards in all directions. B— liked this idea, but she ran into trouble trying to execute it: she could make an explosion in the inner ring, or in the outer ring, but not in both (see Figure 36). As she had done earlier in designing the bouncing ball, B— called over a program leader when she had trouble getting her design to behave as she expected (and wanted) using the software.

B—'s first explanation for the problem was that there was something wrong with the shape she had chosen. The program leader suggested that they change the color of one of the shapes on each ring of the exploding design (see Figure 36) so they could watch carefully as they changed the shape dynamically. The problem, it turned out, was not with the shape. The problem was that B— had rotated the shape by the same amount in both the inner and outer rings. "Oh," said B—, "you have to rotate the outside one more times--less degrees--[because] they're farther apart.... It's a bigger circle, [and] with a bigger circle you need to rotate it more times to keep it pointy."

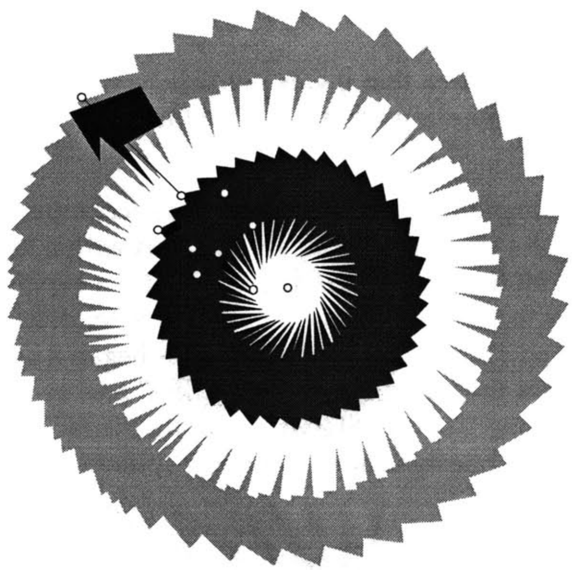


Figure 36: B— and the critic use this image to explore the nature of rotation. B— was frustrated that she could not make the same "explosion" in both rings, until she realized that she needed to rotate by a smaller angle in a large circle to keep the distance between objects the same

With this mathematical insight about the relationships among the size (radius) of a circle, the angle of rotation (subtended angle), and the distance between object and image (an arc or chord length), B— was able to complete her project. The final image (see Figure 37) is of a negative space explosion in the background sending shapes into the foreground flying off in all directions within a rectangular frame.

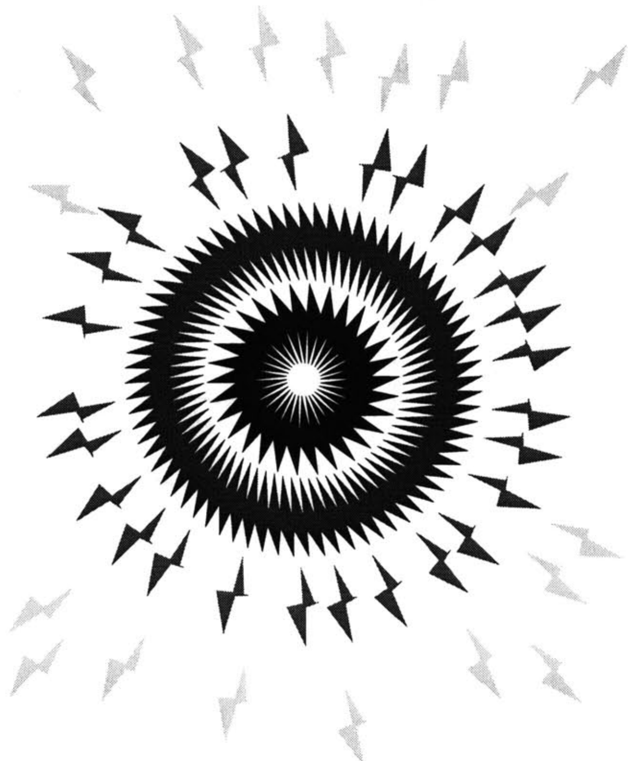


Figure 37: B—'s final image for the museum exhibit showing an explosion in the negative space of the image sending identical shapes off in all directions.

Patterns across design stories

One of the central issues in B—'s work in Escher's World is clearly her use of the software tool, and particularly the way in which the tool distinguishes between objects that are "drawn" and those that are "constructed." All of the computers in Escher's World were equipped with the Geometer's Sketchpad (Jackiw 1995), a commercial software program designed for use in geometry class rooms. Sketchpad is one of a (relatively small) number of "dynamic geometry" environments that are currently available. A dynamic geometry environment allows users to create basic geometric figures (circles, lines, polygons), and to change their attributes (location, size, orientation, color). Users can also define mathematical relationships between objects, including geometric transformations. So, for example, one could create a point and a polygon, and then create the image of the polygon rotated around the point by a given angle. When objects are moved in a dynamic geometry environment, mathematical relationships are preserved. The display is updated in real time as objects are "dragged" on the screen. In this way, users can explore the effects of mathematical relationships quickly and easily.

The difference between "drawing" and "constructing" a figure in Sketchpad depends on how the user has—or has not—used these mathematical relationship in creating his or her design. So, for example, Figure 38 shows two shapes made using the

Sketchpad program. The first (a1) looks like a square. But when its vertices are moved (a2, a3), it undergoes continuous deformation, revealing that it was constructed as an arbitrary quadrilateral and then moved to look like a square. The second shape (b1) was constructed as a square using perpendicular lines and a circle (see Figure 39 for construction details). As a result, when its vertices are moved (b2, b3), it changes size and orientation, but remains a square because of the mathematical relationships among its parts.

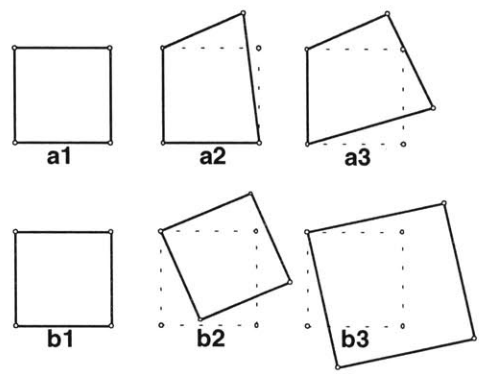


Figure 38: Constructing versus drawing in Sketchpad. Square a1 was drawn in Sketchpad: it looks like a square, but when vertices are moved (a2, a3), the square does not hold its shape. Square b1 was constructed using parellel and perpendicular lines and a circle (to make the sides the same length). When the vertices are moved (b2, b3), the figure can change size and orientation, but is mathematically constrained to remain a square.

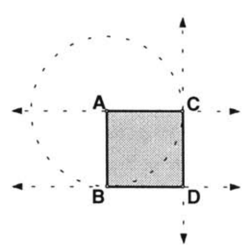


Figure 39: One method for constructing a square in Sketchpad is to make segment AB, then to construct lines perpendicular to AB through points A and B. The circle with center at A and radius of length AB gives point C. Constructing a line perpendicular to AC through point C gives point D, the final vertex of the square.

This distinction between constructing and drawing was an important issue for participants in Escher's World, and was the subject of a number of collaborative conversations, especially during the first week (see Figure 40). In the first week this focus on construction served as a bridge to thinking about expressive intent: design episodes focusing on expressive intent tended to follow conversations about construction ($p<.052$), just as we saw in B—'s work.

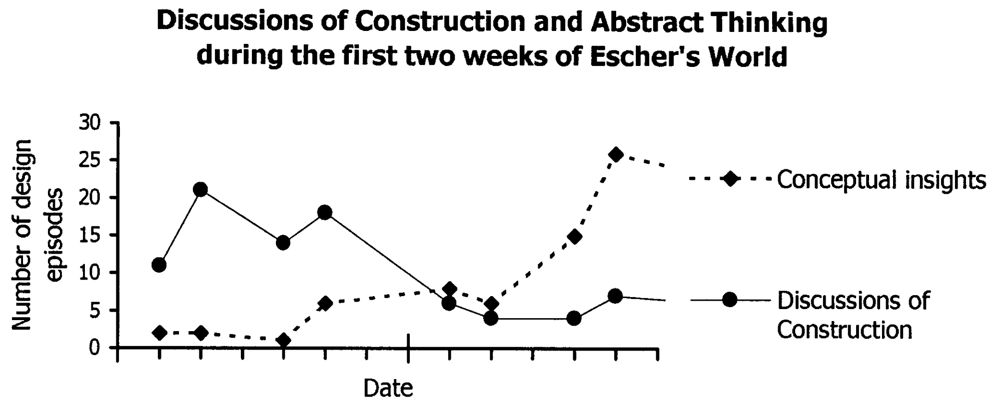


Figure 40: Graph shows how participants focused on issues of construction (versus drawing) early in the program, and then moved to other abstract thinking.

In B—'s design history, we saw that these conversations about construction occurred when B— was trying to achieve a particular design effect and was frustrated because of the way she had implemented her design idea in the software. The developmental progression seems to have been that based on prior knowledge about "mathematical objects" like circles and lines, B— came up with a plan for responding to a design challenge: a scheme combining prior knowledge with ideas seen in exemplars from peers or program leaders. This created an expectation as to how a design built in the software could, would, and should function. When this expectation did not turn out to be correct, B— turned to a program leader for help. The resulting discussion encouraged B— to think about issues of construction and expressive intent—and, as described above, ultimately helped her think about the underlying abstract mathematical ideas in her work.

One way to understand this process of expectation, surprise, and collaboration is in terms of the concept of variance. In Piagetian terms, an "object" is anything that is invarient under a set of transformations (Gruber and Voneche 1995). One of the earliest and most important cognitive steps that young children take is in developing the idea of object permanence: that people and things can continue to exist even when we can not see or sense them. Later, we create more abstract invarient "objects": things like squares, and gravity, and love. What makes these concepts "real" to us is the extent to which we see them as existing consistently through a range of transformations in the world around us. We understand these ideas to the extent that we know how they will behave in a given set of circumstances. In Piagetian terms, our understanding consists of sets of "operations": groups of objects and the actions we can predictably perform with them (Gruber and Voneche 1995; see also Noss and Hoyles 1996).

With or without the Piagetian vocabulary, it seems clear that understanding is deeply intertwined with expectation. In B—'s design history, we see how based on her knowledge of squares (and later of rotation), she expected that the software (and thus her design) would act in certain ways. But when her designs were drawn and not constructed, the objects broke down—sometimes quite literally. This breakdown of expectations—the creation of variance in what she thought were invarient "objects"—raised questions for B—, and she directed these questions at the software, at her own understanding, and ultimately at her peers and at the program leaders. These questions led to productive discussions, and then to explorations of the idea of constructing objects rather than making drawings. The idea of construction, in turn, helped B— think about her expressive intent, and eventually about the abstract mathematical ideas and general design principles in her work.

This developmental process—object and example leading to scheme, leading to variance, leading to collaboration, leading to construction, expressive intent, and finally to reflection (see Figure 41)—was not, however, evident in B—'s work only. In logistic regression analyses, mathematical "objects" such as squares, circles, and lines ($p<.01$), exemplars ($p<.01$), effects of the software ($p<.001$), and collaboration during explorations ($p<.001$) all had statistically significant effects on whether participants addressed the problem of "constructing" mathematical relationships in their designs. The pattern described above and shown in Figure 41 appears to have been a common route to conceptual competence and reflective thinking for participants in the Escher's World program. Collaboration, construction, and expressive intent played a significant role in helping these learners turn prior mathematical knowledge and design examples into abstract mathematical understanding.

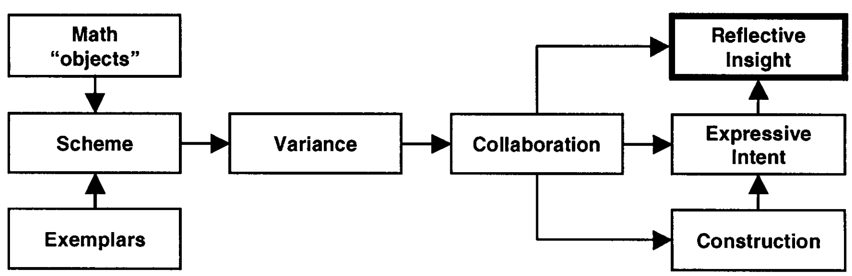


Figure 41: Schematic diagram of participant learning in the Escher's World digital studio using the Geometer's Sketchpad program.

Individual differences

The analysis presented above clearly focuses on similarities among the participants in the Escher's World program rather than on differences between them. But certainly there were plenty of differences among these twelve 12-to-14 year olds. One participant compared the program to his play-group, where he and his friends build models with Lego blocks. Another was well into the transformations of puberty, leaving notes to "the flirt" and helping to run a swim program in the afternoons after Escher's

World. One participant was remarkable in his ability to focus on his work, going through significantly more design episodes than any other participant but still managing to be popular and social. Another participant was called "the mad scientist" by one of the program leaders because of his intensity, incredible attention to detail, and his almost total inability to work constructively with others.

Often these differences among participants showed up in their work. The participant who often played the role of "class clown" liked to work with what he called "wacky shapes" in his designs. B— focused her work on "explosions." One of the participants, who appeared to be coming to terms with sex and sexuality, made a series of images, based on a poem she wrote, of dagger-like objects moving in and out of spherical shapes. Another participant who was enthralled by ideas of chaos and nihilism produced a dark and sinister-looking final project.

One of the interesting and important aspects of a digital studio is clearly the way that expressive activities make it possible for learners with a range of interests to engage with mathematical ideas on their own individual terms. This is certainly an idea that has been explored by other authors, including theorists such as Parker, Dewey, and Papert (Parker 1894/1969, Dewey 1915, Papert 1993), and as is suggested in the conclusion below, more work should be done to explore the role that such differences play in the studio setting.

This analysis, however, focuses on patterns of development across the participants in the Escher's World program. One of the surprising and interesting things about the results of the program is that despite the obvious differences in background, experience, maturity, and temperament of the participants, there are developmental patterns and processes that seem to be common elements of learning in a digital studio. These patterns shed light on the nature of learning in a studio setting, and on how collaboration, expressive activity, and digital media can help develop reflective thinking through design activity.

Discussion

The developmental progression described in the results section and shown schematically in Figure 41 above gives us a way of understanding the role that various aspects of the digital design studio play in mathematics learning. This section of the paper gives a summary, based on the results above, of the importance of design knowledge and expressive activity, studio pedagogy, and computational tools in creating a "digital studio" environment for mathematics learning.

Design knowledge, expressive activity, and mathematics learning

One of the working hypotheses of the Escher's World project was that in a studio setting, participants' design knowledge would scaffold their mathematical understanding. It seemed reasonable to expect that participants would use, for example, their understanding of a design principle like "balanced composition" to get deeper insights into related mathematical ideas like "mirror symmetry." There certainly were examples of participants talking about mathematical ideas in the context of design ("dilation is the creation of apparent perspective"), but as described above, in general working explicitly with design principles made participants *less* likely to have mathematical insights.

The influence of design principles and design knowledge on mathematics learning appears, then, to be either indirect or in some way unconscious. The "unconscious" effects of design knowledge—and, indeed, the role of tacit knowledge in general in the digital studio—are beyond the scope of the analysis presented here; hopefully they will be the subject of another study, based either on a subsequent analysis of this data or on another Escher's World-like experiment.

If specific design principles do not directly support explicit mathematical thinking, design activity does provide a more general context (and a model) for thinking conceptually about mathematics. The key bridge between design and mathematics is the concept of "expressive intent": the notion that images convey ideas—mathematical ideas, design ideas, or perhaps both. Part of thinking about design is thinking about expression and communication, and this, rather than specific graphic design concepts, helped participants think reflectively about the underlying mathematics of their work in the studio.

Expressive activity is at the heart of the design studio, and in some ways it is appropriate to think of the design studio as an environment designed to help people learn *to be* expressive *through being* expressive. The results from Escher's World suggest that expressive intent is a general cognitive resource or skill that can be used to support reflective thinking in a variety of domains—design and mathematics in the Escher's World project, but potentially in other domains as well. One might consider, for example, using the studio model to explore statistics through political or environmental issues, or outside of the domain of mathematics entirely, to learn about history through creative writing. The key concept in all of these cases is that expressive intent can provide a context for thinking about abstract ideas—and that the structure of the design studio provides a model for thinking and learning in an expressive context.

Pedagogical structures of the design studio and mathematics learning

The analysis of participant learning in the design studio presented above looked at several structural features of the design studio: particularly design reviews, pinups, and desk crits. Reviews, pinups, and crits are all forms of social interaction designed to support constructive collaboration between a learner and peers or experts (or both). As we saw in the results section above, collaboration during conceptual thinking played an important role in abstract insight: participants were more likely to think reflectively about mathematical ideas with the help of a peer or an expert. The desk crit—and more generally the organization of the design studio—provide the opportunities for such conversations to take place. The open space, loosely-structured schedule, independent projects, and high ratio of experts to learners make it possible for sustained, detailed collaborations to take place regularly. The structure of the desk crit, moreover, supports reflective thinking about emerging designs, about mathematics ideas, and about design principles. Key questions of the desk crit are: What are you trying to say (or show) here? Do you think this shows that? Why or why not? And as we have seen, these kinds of probing questions played a significant role in helping participants in Escher's World think reflectively about mathematics.

The public forums of pinups and design reviews also play a role in design learning. Design reviews motivate deeper exploration and provide a sense of audience for learners as they think about what ideas they are trying to express. Pinups provide continuing validation, perhaps of a more socially loaded—and therefore more compelling—form than feedback from a lone expert in a desk crit. Because of the nature of the data collected in this particular study, the analysis here has focused more on individual collaborations in desk crits than on the developmental role of these public presentations. This analysis clearly suggests, however, that the "pedagogical structures" of the design studio play a significant role in turning creative design activity into abstract mathematical insights—and that additional work should be done to study the role of reviews and pinups in design learning.

Digital design tools and mathematics learning

The design studio model provides structures for collaboration. The expressive nature of design activities provides scaffolding for abstract thinking. And the design tools used in the Escher's World program played a substantial role in making it possible to use these features of the design studio to build mathematical understanding. Certainly it is possible to learn about mathematics through design without computational tools: in separate projects Loeb and Willett have shown that students can learn mathematics through traditional design activities (Willett 1992, Loeb 1993), and based on the Escher's World project, an arts-education consultant has taught mathematics using art activities in the New York City public schools. The more significant question here is not whether computational tools are *required* to learn mathematics through expressive activities, but rather what role tools such as the Geometer's Sketchpad can play in helping to create expressive mathematics learning environments.

Theorists of new media and learning have argued that computational tools let learners create artifacts (real or virtual) which can become the center of a collaborative conversation (see, Papert 1980, Bruner 1996). This certainly was the case in Escher's World, where much of the work of the studio took place in conversations over designs constructed using the Sketchpad program. But the analysis above goes beyond this to suggest that some particular properties of the software made these conversations more likely, and more likely to lead to reflective thinking.

The learning process described above in the results section shows that participants began their designs with an expectation, based on prior knowledge and on examples of other work, of how particular mathematical "objects" would behave in the software. But a key feature of the Sketchpad program is the distinction between "drawn" and "constructed" figures: both may look the same on the screen, but "constructed" figures are produced by explicitly creating mathematical constraints on points, lines, and curves. When a design is moved dynamically, these constraints are preserved, and a constructed figure retains its underlying mathematical form. A figure that was merely draw by eye breaks down when points or lines are moved on the screen. This variance between expected and observed behavior led participants to collaborative conversations, and the resulting discussions and explorations of what it means to "construct" a design rather than merely "draw" helped participants think about the expressive intent of their design work. This, in turn, and in conjunction with further collaborations, led to mathematical insight.

Noss and Hoyles (Noss and Hoyles 1996) argue that one important feature of software for mathematics learning is the extent to which a tool demands that a learner describe his or her expressive intent explicitly in the computational medium. They refer to this property as "autoexpressive," and while the term may be cumbersome, the idea—that a tool reflects in its behavior the extent to which an idea has been represented explicitly—plays a significant role in transforming creative activity into abstract thinking. The results described above add to this idea in two ways: first, they point out the important role that collaboration plays in this "autoexpressive" process. Second, they describe the process (prior knowledge, exemplar, scheme, variance, collaboration, construction, and expressive intent) by which pedagogical structures of the design studio interact with digital media to foster reflective thinking.

There is, however, another aspect of the Geometer's Sketchpad that seems important in the context of mathematics learning in Escher's World. Sketchpad does not *require* explicit construction of objects to produce graphic designs. Rather, it *rewards* the use of mathematical ideas by making it possible to build more elaborate and more dramatic images. As participants used more mathematical concepts (and design principles) in their work, their designs became more compelling, such that by the end of four weeks, it was hard to believe that the designs from the first day were made by the same participants as those who completed final projects for display at the MIT Museum. And, of course, in some sense, they were not the same participants: they had been transformed by their collective, collaborative, expressive experiences in the studio. They were thinking reflectively about mathematics and design, and they were able to use their understanding of mathematics and design to create dramatic expressions in a powerful medium. It was not only the mathematical constraints of Sketchpad but also its artistic affordances that provided opportunities for reflective thinking about mathematics.

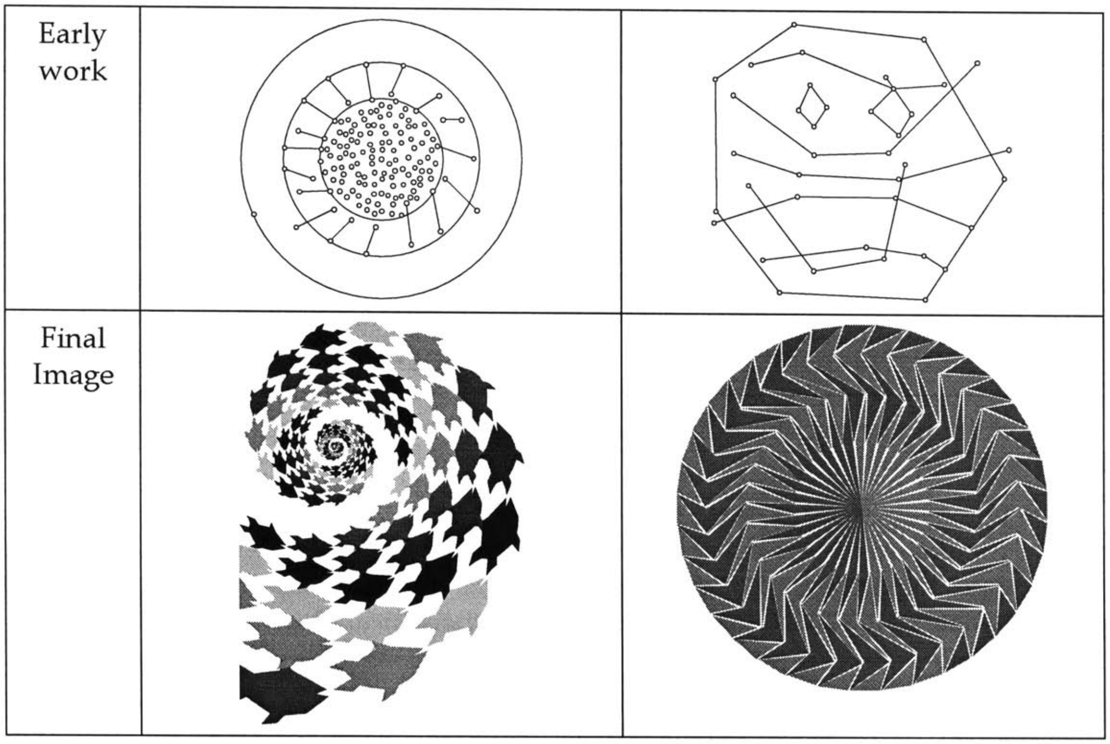


Figure 42: Two participants' final images compared to their drawings from early in the program

Conclusion

Previous work in the Escher's World project has suggested that students learned mathematics, and learned to like mathematics more, in a digital studio setting(Shaffer 1997b). This study focused on the cognitive processes by which such learning take place. The question was: what features of a digital studio support abstract mathematical thinking?

The analysis presented here suggests three features of the digital studio were particularly significant in supporting mathematics learning. The first was collaboration. The interactions of participants with peers and with program leaders played a significant role in turning design explorations into reflective mathematical thinking. The second significant feature of the studio learning environment was the expressive nature of the design activities. Specifically, the concept of "expressive intent" helped participants think about mathematical ideas in abstract ways. The idea that a design had to "say something" provided a language for thinking about that "something"—in this case geometric ideas—in conceptual terms. The third significant feature of the Escher's World studio was the nature of the tool being used. In particular, the distinction between "drawn" and "constructed" figures in the Geometer's Sketchpad pushed participants to think more deeply about mathematical ideas and mathematical objects. Questions about how to "construct" a desired effect led to collaborative conversations about the mechanics of the tool, about expressive intent, and ultimately about the underlying mathematical ideas.

There is clearly more work to be done in identifying the important features of the studio as a learning environment, and in understanding how these features relate to one another. For example, it would be useful to make a more detailed study of a single learner in a studio setting to shed more light on the development of tacit mathematical and design knowledge. Similarly, an important addition to this work would be a closer study of interactions in pinups and design reviews. These venues—particularly the pinups—have received little attention in the design literature, but appear to be an important junction of individual cognitive development and powerful social forces. Finally, the analysis presented above focuses primarily on the studio learning process across individuals. It would be useful to have further work exploring the role that differences between individuals play in the studio setting. How, for example, does the studio environment incorporate individual preferences, prior experiences, and strengths and weakness of learners?

While there is clearly further research to be done, the results obtained thus far do suggest a number of interesting and potentially important conclusions about mathematics learning: First, the Escher's World project as a whole suggests that expressive activities are a promising venue for thinking about deep mathematical ideas, and that the design studio is a useful model for structuring such activities. Second, the experiences of the participants in Escher's World show that computational media can play an important role in creating such new and compelling environments for mathematics learning. And finally, this research suggests that more work can and should be done to explore how the studio model works, and how it might be extended beyond a research setting to have a significant impact on mathematics learning and on the use of new media in education more generally.

Conclusion

Thus we come to the end of this collection of investigations, which up to now have loosely gone under the title of "Escher's World," after the Dutch artist M.C. Escher, whose work so richly integrated complex mathematics and compelling design. However, if these investigations have been successful, this is not the end of the story, but rather the beginning.

If we understand expression as a process by which people create external reflections of parts of their internal life, then computational media have a tremendous expressive potential—and not just in the field of mathematics. Computers make it easier for more people to "say" meaningful things about their world. Someone with little formal background in statistics can use spreadsheets and other data-analysis software to look for patterns in numerical data and thus understand meaningful aspects of his or her world. Someone with little artistic or photographic training can use a digital camera to capture images and then edit those images to record his or her understanding of the significance places, people, and events. Computation empowers concrete representations to solve complex problems (Turkle and Papert 1990, Kaput 1992), creates new representations (Resnick 1991, Resnick 1994b, Resnick 1994a, Resnick, Bruckman et al. 1996), and makes it easier to use existing representations (Kaput 1992, Doerr 1996) to create understanding and convey meaning.

As a growing literature on "situated" and "distributed" cognition suggests, expressive learning (indeed, any learning) does not take place in isolation (see Boaler 1993, Salomon 1993, Greeno 1997). We learn within contexts that are mediated by social, physical, intellectual, technological and intentional factors. That is, what and how we learn depends on who we are with, where we are, what we already know, what technology is available, and what we are trying to accomplish. New media can play an important role in making learning more expressive, but simply putting a computational object into a room is not enough to transform thinking—or education.

Creating new media for expressive learning is not just about deciding what to put in the box; it also depends on creating an appropriate learning environment in which the box will function effectively. Creating new media for expressive learning means creating a new *system* for learning, of which computational devices are an important component. The goal of this research has been to create such a learning system and to examine in detail how that system functions. While the specific domains of these investigations have been mathematics and design, the range of expressive power of computational media suggests that the theoretical ideas about expressive learning explored here could be profitably used in other disciplines. The "lessons" of Escher's World can be applied across the curriculum.

Significant Contributions to Knowledge

Since the writings of Francis Parker and John Dewey (Parker 1894/1969, Dewey 1915, Dewey 1938, Dewey 1958), educators have been excited by the possibilities of learning through design activities. At their best, design activities make it possible to learn about important ideas in a creative and compelling way. The introduction of computational media to education made this idea only more appealing, as thoughtful educators saw how computers could make it possible to explore even more areas of human understanding in an open-ended, design-based environment (Papert 1980, Turkle and Papert 1990, Harel and Papert 1991, Kafai and Harel 1991a, Kafai and Harel 1991b, Resnick and Ocko 1991, Papert 1993, Wilensky 1995, Kafai 1996, Resnick, Bruckman et al. 1996).

The body of work presented in this dissertation has explored the computer as an expressive medium for learning mathematics by looking at existing environments where expression and expressive activity are at the center of the learning process. The architectural design studio is an example of one such environment, where the physical and temporal structure of the learning space supports specific pedagogical elements such as extended projects, public presentations, and desk crits, all of which are designed to help students develop expressive architectural ideas as solutions to design problems.

The projects described in the preceding chapters adopt this framework, using the power of computational media as a bridge to bring mathematical development into the design learning system. In building this bridge, the projects attempted to answer three fundamental questions about expression, computation, and the design studio: (1) What is the relationship between expression and cognition? (2) What is the relationship between expression and computation? (3) What is the relationship between computation, expression, and the learning environment?

In answering these questions, the project went beyond previous work on learning-by-design by (1) looking at students' learning during an intensive exposure to the computational learning environment, (2) making specific links between expressive activity and particular learning outcomes, and (3) investigating the architectural design studio as a specific model of learning through design.

The studies above describe the nature of expression, its relationship to computational media, and its relationship to learning in general. More important, the success of these projects represents an important step in finding a practical and theoretical grounding for the successful use of computers to transform education. The point is not that all students should learn mathematics by making works of art. Rather, the idea is that computers can help us think about ways of encountering mathematics (and potentially other subjects) that are more "artistic" as we look at what it means to be expressive in a variety of domains.

Appendix

Postcards from Escher's World

During the month of July 1997, twelve young artists came to MIT as part of the Escher's World project to take a journey through the worlds of mathematics and art. Using computers as well as traditional media, these artists spent four weeks exploring the elements and operations of transformational geometry through advanced projects in graphic design. The "postcards" from this journey through Escher's world of geometry and design show how mathematical functions such as rotation, dilation, and translation can be used to create artistic effects such as movement, depth, and emergent forms in a static, two-dimensional medium. Like the work of the Dutch artist M.C. Escher (1898-1972) that inspires these pieces, the results are both pleasing to the eye and provocative for the mind.

The artists would like to thank David Williamson Shaffer, Joseph Press, Laura Bouwman, Don Stidsen, Kimberly Alexander Shilland, Seymour Papert, William Mitchell, Mitchel Resnick, James Kaput, Paul Goldenberg, Thomas Gallemore, Benjamin Kou, Jason Han, Mary Leen, Arthur Ganson, Diane McLaughlin, Sonja Laska, Florence Williams, and Carole Li for their help, advice, and support in making this exhibit possible.

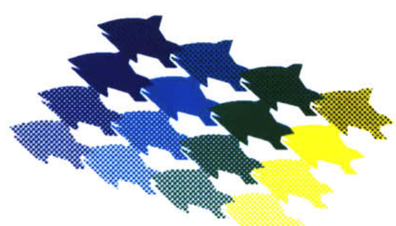
The following pages document the exhibit "Postcards from Escher's World."

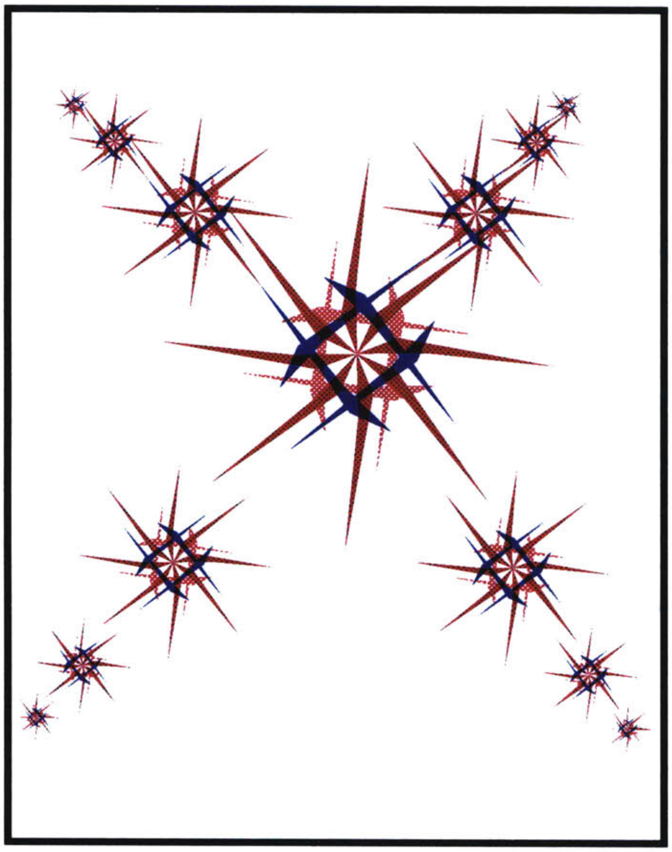
Dan Bender

Newton, Massachusetts

I am 13 years old and I go to Day Middle School. I was born in Boston on December 16, 1983. I enjoy drawing, playing sports, and watching late night TV. In Escher's World, I enjoyed learning new ideas about math and expressing them through my work.

This piece is entitled "A Fishy Situation." In this piece, I expressed many mathematical concepts that I learned in Escher's World. In order to create this, I spent several days designing a spiral made out of a fractal using fish. At first, I didn't dilate the fish that were in rows of four, so they were cramped up on the inside row of the spiral. Then I experimented with the amount of dilation and found one that fits. I traveled down many different roads during my explorations of this piece, but I ended up back where I had started with a few different colors and an extra row of fish.



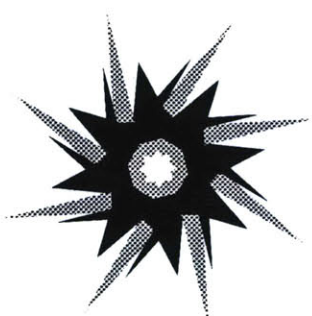


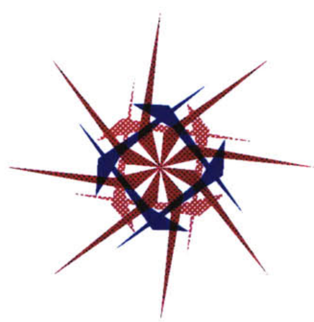
Gerald Bernard

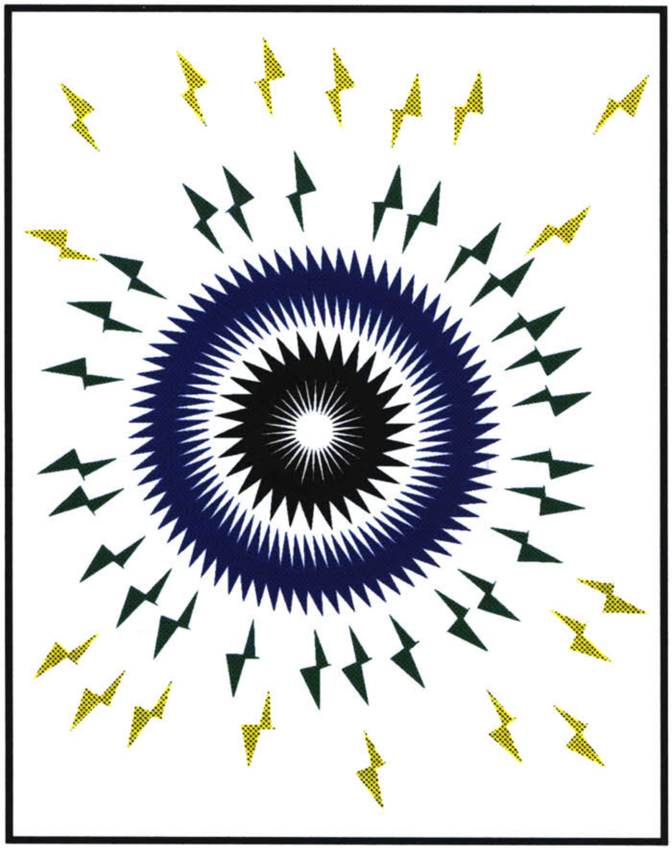
Milton, Massachusetts

I am 12 years old and I go to Pierce Middle School. I love playing basketball with my friends. I have a puppy, we have had him for about 8 months.

My pictures are about rotation, dilating, and translation. The first one in black and gray with one image turning one way and the other turning the other way. The second picture is similar, but in red and blue with longer and different lines. The third picture I saw too much space so I added another image that is bigger than both images. I did not like my third picture so I made a frame in blue. My final piece I dilated the two pieces four times in four spots.





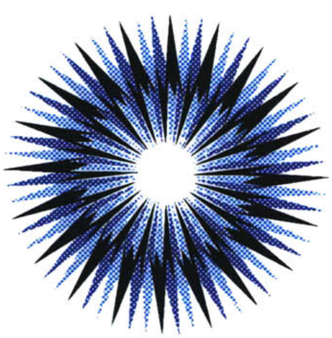


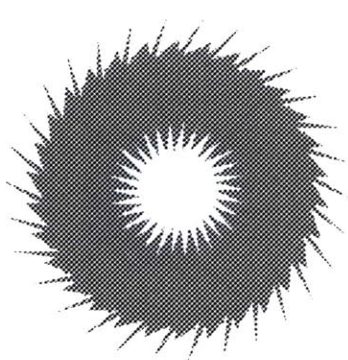
Hanna Kreiger-Benson

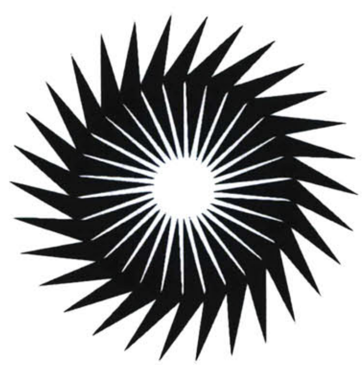
Belmont, Massachusetts

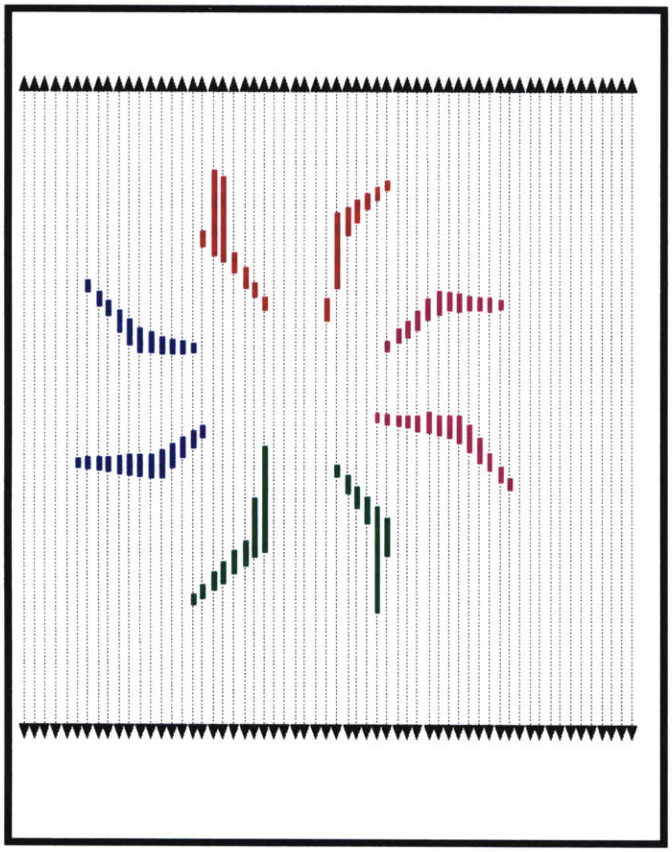
I am twelve years old and entering seventh grade, as a Homeschooler. I attended the Solomon Schechter day school until sixth grade. I am interested in art and math, and like Escher's World.

My picture grew out of the idea of exploding negative space. I discovered that by rotating a pointy shape, I could make the negative space seem to explode or move, almost pulsating or wiggling. I was stuck until David suggested to have the explosion not just a symmetrical circle, so I made rings of explosions, and had them look like they were going off the page.







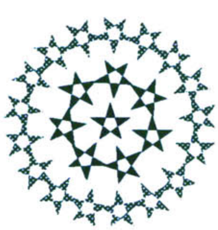


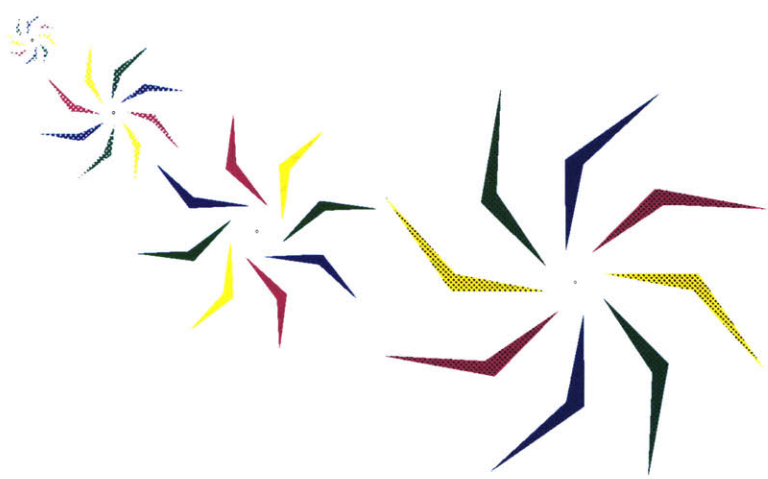
Ian Carlson

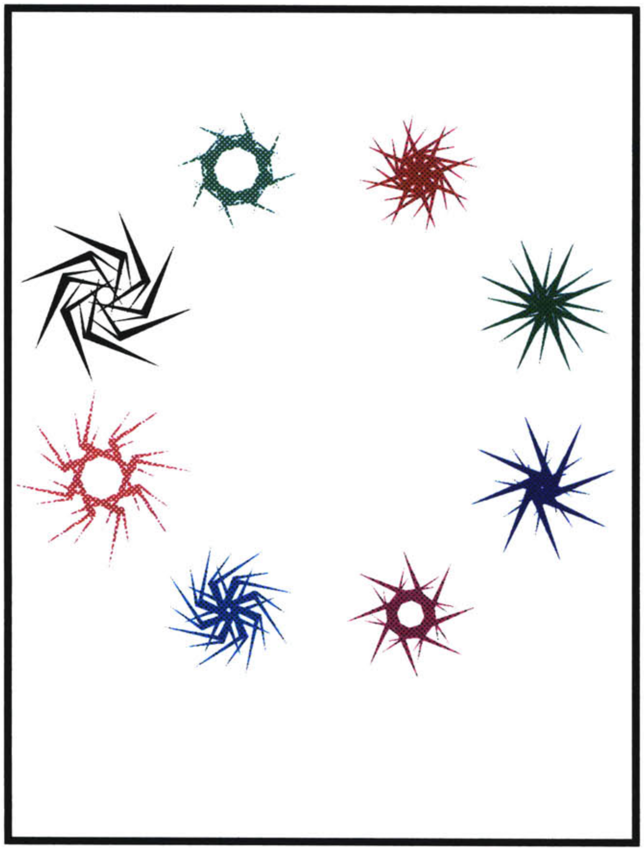
Cambridge, MA

I was born in 1985 in Boston, MA. I go to the Cambridgeport school in Cambridge and enjoy soccer, drawing and going to the movies.

My big idea is inspired from a piece by Victor Vasarely. It shows a circle made of lines on a base made of lines. Since the base lines were parallel and the circle lines perpendicular it made the circle stand out. I tried to do the exact same thing except I used color instead of line direction. The dotted lines help to make the color stand out. The basic concepts I used were rotation and dilation because each line has a duplicate next to it to make it look thicker. I took a rotated polygon and dragged it across a base of vertical lines. Then I was able to draw the colored lines over the polygon to create a spatial effect of reconstructing the polygon.





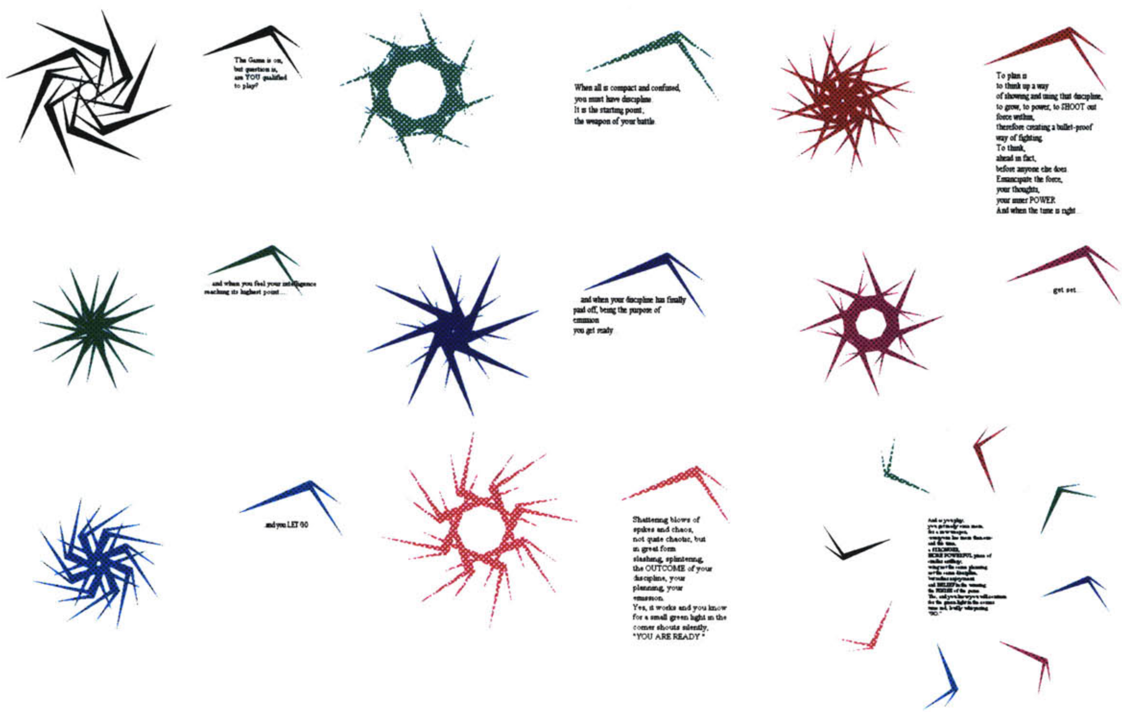


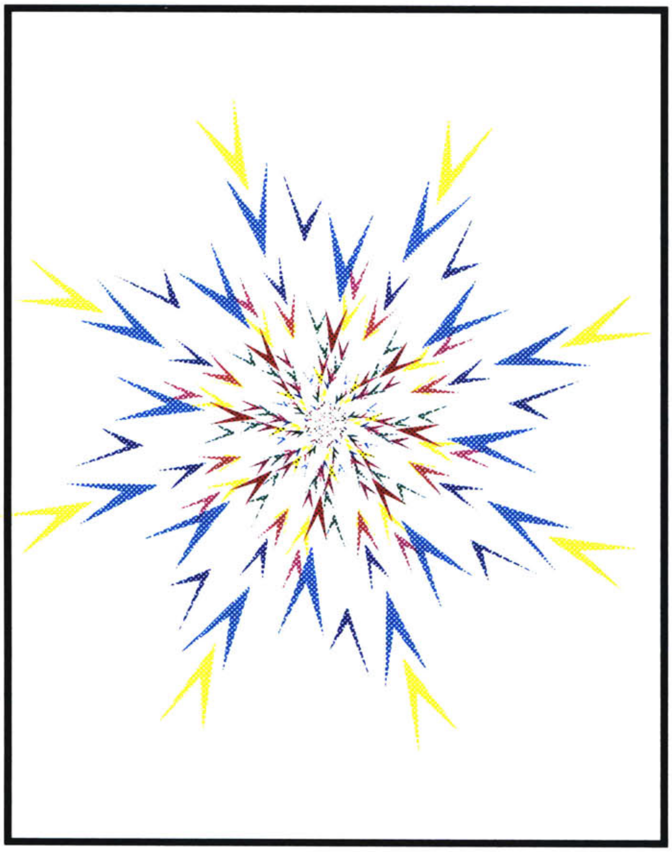
Melissa Li

Boston, Massachusetts

I was born in Hong Kong on June 17, 1983. When I was only a month old, I immigrated to America, settling in Boston. I am now 14 years old and am going to be a freshman at Boston Latin School. For most of my life, I enjoyed music, performing arts, and writing. A lot of my works at Escher's World connect mathematical ideas with views and thoughts on life.

This piece entitled "The Game," is a rotation sequence. Originally, it was supposed to be a packet full of separate rotations with text on each page. However, it was later expressed as one whole drawing. This piece explains the mathematical concept of rotation and how a polygon, when pulled in a straight line across the center, will appear opposite. The math is also combined with poetic text explaining the "Game of Life" and "Game of Battle" both at once.





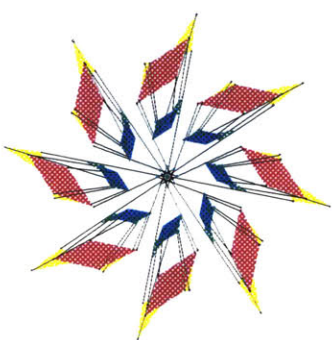
Nadia Fermin

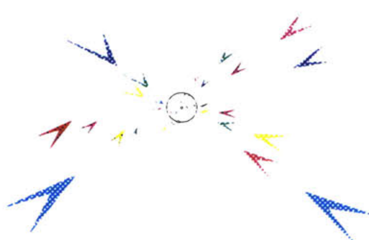
Newton, MA

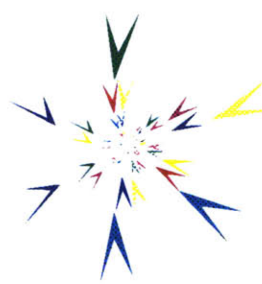
I was born in Boston. I go to Oak Hill Middle School in Newton. I have two sisters, (twins) their names are Loma and Melody. I love art and computers, so that is why I came to Escher's World.

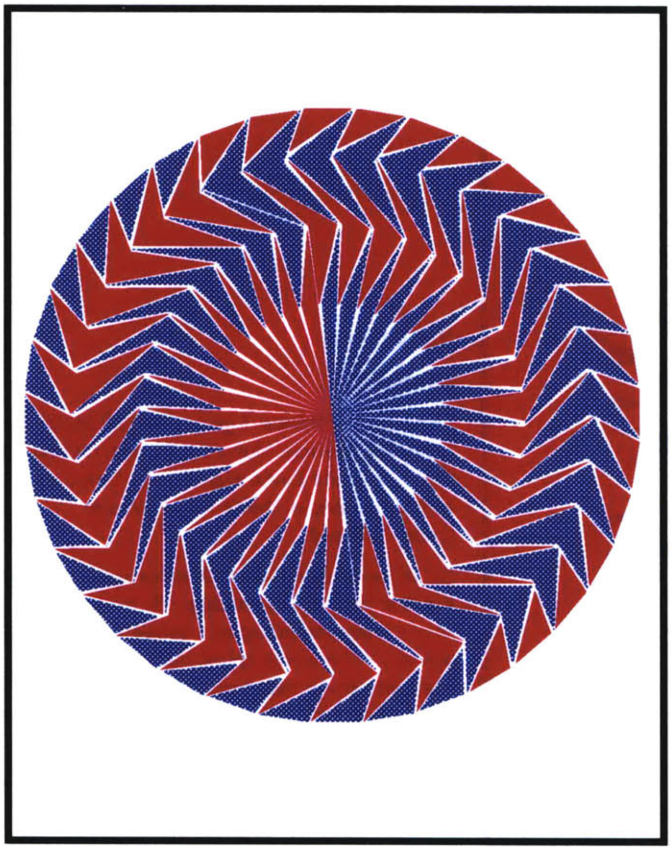
All of my art has to do with either spirals, using a mirror line, and/or rotating. My big picture shows arrows spiraling as it spirals it changes colors. I did not only spiral, but I rotated so I can even out the negative and positive spaces. The drawing that has three big shapes, I used a mirror line and moved it around a lot.

I guess the rest of the drawings have to do with rotating and fooling around with shapes.









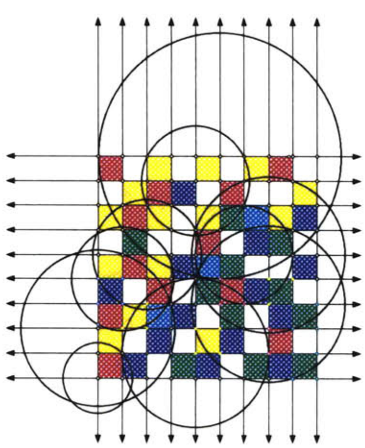
Stephan N. Brooks

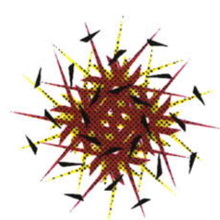
Arlington, MA

I was born in Santa Clara, CA. I am going to Arlington High School. I have one sister and one brother, twins. I like to fold origami, play video games and play soccer. My favorite kind of peanut is honey roasted.

My big picture is an improved version of my smaller images, but I think this image is brighter, more complex and powerful. It clearly shows the mountain and valley illusions created by the boomerang-like shapes. You may have noticed that this drawing has the same colors as the Korean flag. However, it is completely irrelevant.

The other pictures are based mostly on rotation and color.



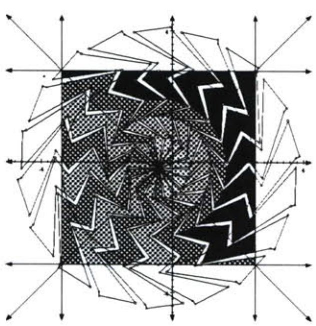


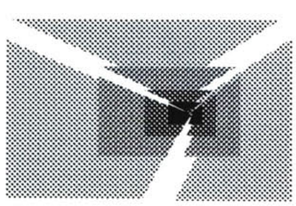
Theodore Kuttner

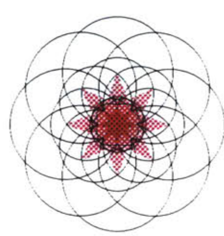
Boston, MA

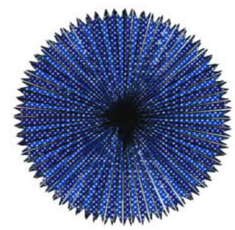
I was born on April 14th, 1984. I have lived in Boston all my life. This fall, I will be starting 8th grade at Boston Latin School. I am in the honors math program there and have taken an optional art course which I hope to continue this year.

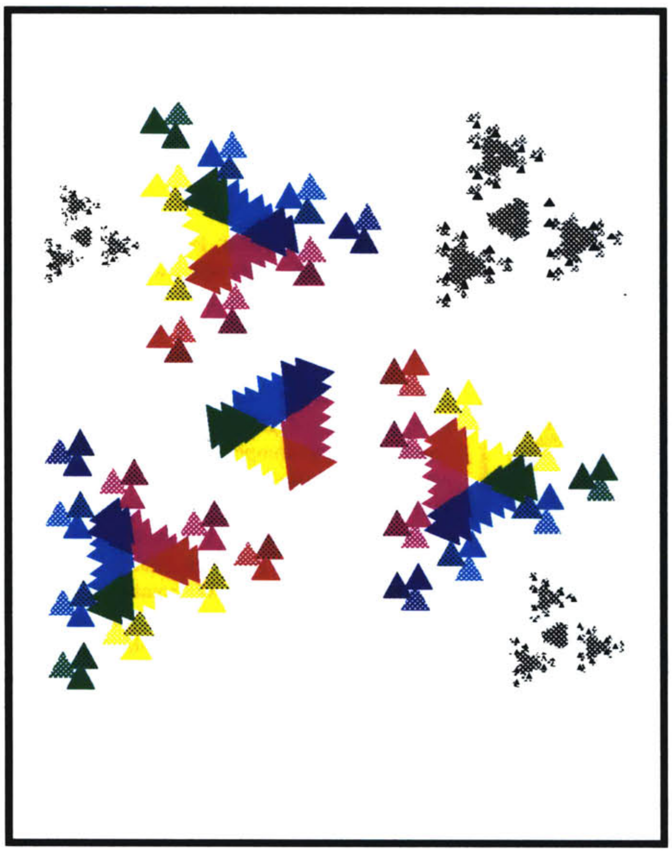
This piece is a metaphor of our lives. It depicts the chaos and confusion caused by the fierce competition and complications of the business world and ultimately leads to the decay and malfunction of the socio-psychological system. The picture began as one line rotated once and sealed off to form a polygon. That polygon was then set into an eighteen fold rotation with only a three degree difference between each one. The negative space evolved into a visible, material object on the page. At every angle in the original line, a full circle appeared from the rotations. So three objects exist on this page: the original polygons, the negative space, and the circles. All compete for visibility, thus creating chaos.









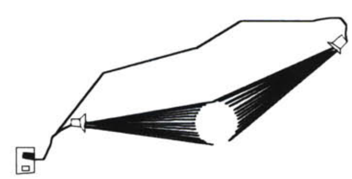


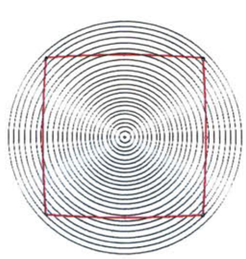
Ted Offner

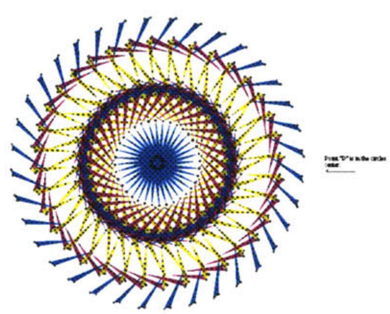
Belmont, MA

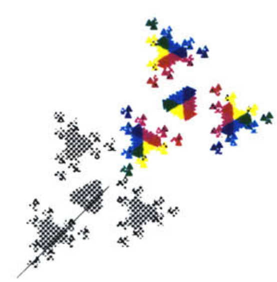
I am 13 and in the 8th grade at Chenery middle school in Belmont. I have two sisters, one 10 and one 3. I like going to Harvard Square, the movies, and writing computer programs with Q-basic that trick people into thinking that something is horribly wrong. During the winter I am on the Otis Ridge Ski Team. Also, I believe that UPS is taking over the world.

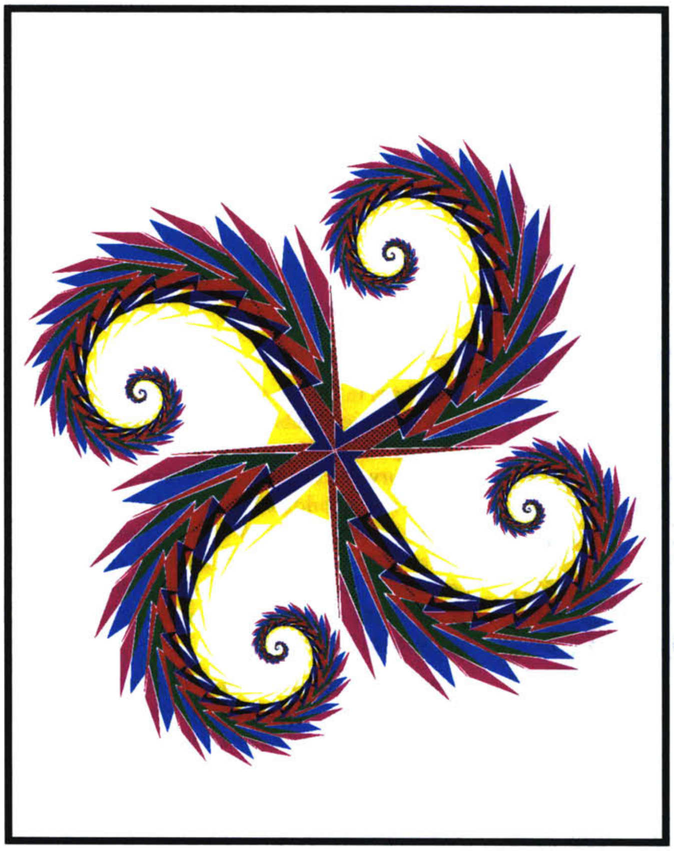
This picture started out with one triangle. After much rotating and coloring, I got the complete color section. Then I dilated it to smaller sizes and made the smaller dilations black. I placed the black figures in the empty space. The drawing you see below is really a bunch of circles with four slightly curved lines, but the circles make them seem straight, so the red "lines" seem to form a square.











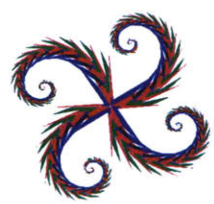
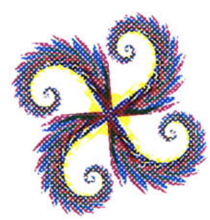
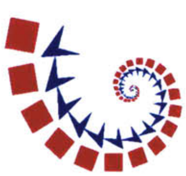
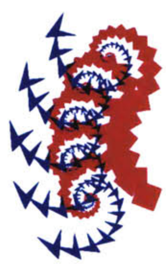
Timothy Ho

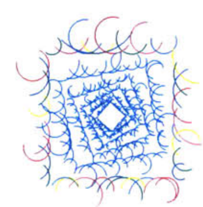
Dover, Mass.

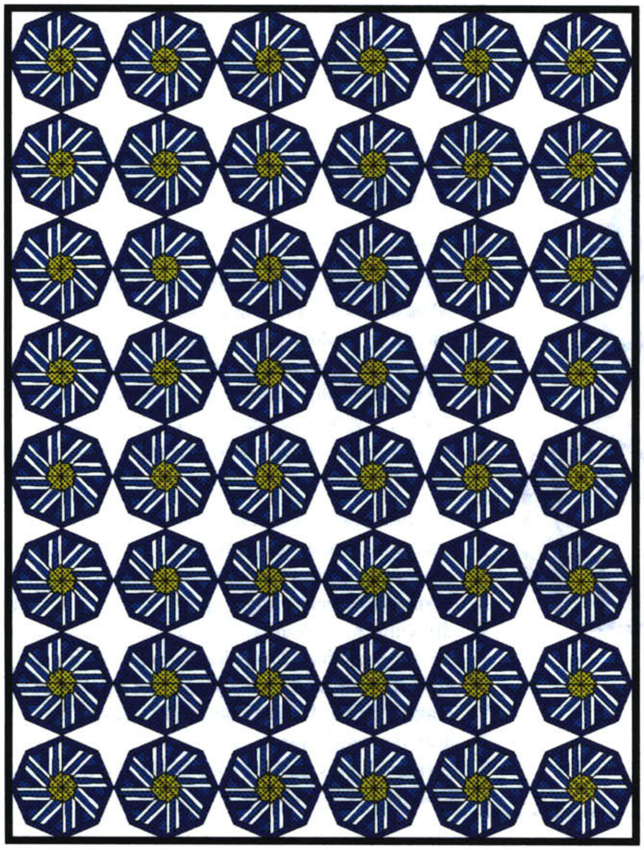
I was born on April 19, 1984 in Boston, Massachusetts. I live in Dover and go to Dover-Sherborn Middle School. I will be going to 8th grade this year and am 13.35 years old. I was really bored this summer and decided to join Escher's World because I like drawing and working with math. I also like drawing art on my own time.

My first drawing is colorful arcs creating a square, and then spiraling inward. My second drawing is a blue triangle and using its three corners to create a triple spiral. My third to sixth drawings are variations and process drawings of my final product. I also have a big drawing of colorful arcs making a square, and rotating it to make a circle. I finally decided to have a final drawing using spirals, and I thought spiraling certain objects made it look funky. So have fun looking at my drawing.





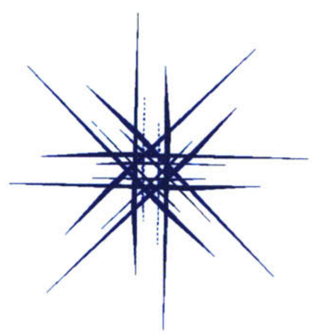


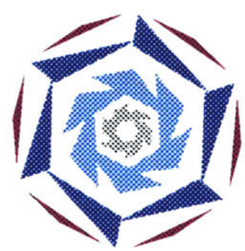
Ulrike Baigorria

Boston, Massachusetts

I was born in Boston on April 6, 1984 but I have been living in Cambridge all my life. I will be entering the eighth grade at the I.S.P. program at The Henry Wadsworth Longfellow Elementary School in Cambridge. I love to dance. I have a brother Matias who is 19 years old and is studying in Germany. I love watching and playing basketball. When I was in Escher's World, I tried to use all the methods we learned in at least one of the pictures I worked on.

How do you like my picture? Well, stick around and I'll tell you how I made it. I made it by taking one triangle and putting a pattern in it and then I rotated the triangle by 45 degrees until they made an octagon. After the octagon was made I translated it until it filled the page so it looked like what you see now. Bye bye...





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