THE STUDY OF A MULTIPERIPHERAL MODEL

WITH CONTINUED CROSS CHANNEL UNITARITY

by

Shirley Ann Jackson

S.B., Massachusetts Institute of Technology (1968)

SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF

PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

August, 1973

Signature of Author .

Department of Physics, August 23, 1973

Certified by . . . . .

Thesis Supervisor

Accepted by . . . . .

Chairman, Departmental Committee
on Graduate Students

TABLE OF CONTENTS

THE STUDY OF A MULTIPERIPHERAL MODEL

WITH CONTINUED CROSS CHANNEL UNITARITY

Shirley Ann Jackson

Submitted to the Department of Physics on August 23, 1973
in partial fulfillment for the requirements for the degree
of Doctor of Philosophy.

ABSTRACT

We derive a multiperipheral integral equation with
continued crossed channel unitarity for the absorptive part
for pion nucleon scattering from a Bethe-Salpeter equation
describing composite particle scattering. At high energy the
equation becomes homogeneous and we solve it via a variational
principle to obtain the behavior of the output trajectory
which falls with energy. We indicate the effect of turning
on and off the coupling constant in the strength function
resulting from the kernel of the equation. We indicate how
one may now use the composite particle amplitude resulting
from this equation to construct production and three body
amplitudes which may be used to investigate Toller angle
dependence and single particle inclusive distribution,
respectively.

Thesis supervisor: James E. Young

Title: Professor of Physics

ACKNOWLEDGEMENTS

I. INTRODUCTION

In recent years there has been much interest in ways of looking at multiparticle production phenomena and in models which attempt to set forth the basic dynamical mechanisms which give rise to such production. With regard to the first consideration, many physicists have come to study what are known as inclusive reactions. They study processes like $a + b \longrightarrow c_1 + c_2 + \ldots + c_k + X$, where X denotes an unknown system of particles. Experimentally such a process is realized by having a detection apparatus measure the momenta and types of the particles c_1 to c_k , i.e., the distribution of one (or a few) final particle(s) is analyzed, everything else being summed over. The main pieces of information are single particle spectra, e.g. the spectrum of the π^+ produced in p-p collisions. Next are correlations among secondaries or the analysis of the discrepancies between two particle inclusive distributions and the product of the relevant single particle inclusive distributions.

Mueller[99] introduced an approach to discussing inclusive data. He showed that the single particle inclusive distribution may be regarded as a certain discontinuity in a three-to-three (six point) amplitude for forward scattering in analogy with the connection between the total cross section and the discontinuity in total energy of a four point function for forward scattering.

If we wish to study a model for a single particle inclusive distribution for a particular final state particle and

compare the predictions of this model with experiment, we have two choices. First, we can try to directly calculate the single particle invariant function within the context of some model (e.g. the multiperipheral model for two-to-two scattering, which will be mentioned later), or we can construct a model for a three-to-three scattering process, and using Mueller's "optical" theorem, construct the relevant single particle distribution.

This thesis will be concerned with the latter approach. In particular, we construct what we call a multiperipheral equation with continued cross channel unitarity which describes the scattering of a composite particle with an "elementary" particle. In this case the scattering problem is $\pi N \to N \pi$ with the N being a composite of a pion and nucleon. This equation will yield for us a high energy composite-particle scattering amplitude. If now there is a mechanism for the formation of the composite in the initial state and the decay of the composite in the final state, then we can construct from the composite-particle scattering amplitude an amplitude describing three particles scattering into three particles. We have such a mechanism in the form of vertex functions (form factors) for the decay and formation of a composite particle from two "elementary" particles.

We can use our composite-amplitude scattering amplitude to investigate the following things. First, for the two-to-two problem, we extract the leading Regge singularity via a

variational principle. Then, we can do a Fourier-Bessel
analysis of the two-body amplitude to study the small
momentum transfer behavior of the model. Second, we attach
a vertex function for the decay of the final state compos-
ite-particle to construct a scattering amplitude describing
two particles scattering into three particles. Finally,
by attaching a vertex function for the formation of the
initial state composite particle, we construct an amplitude
describing three body scattering. This amplitude can be
used to calculate, via Mueller's Optical Theorem, a parti-
cular single particle inclusive distribution. We are
investigating whether a tripple Regge limit exists for this
problem.

The organization of the thesis is as follows: In
Chapter II we discuss single particle inclusive distributions
and the use of Mueller's Optical Theorem to obtain a single
particle distribution from a three-to-three scattering
amplitude. We also define the triple Regge limit. In Chap-
ter III we review the multiperipheral model. In Chapter IV
we discuss final state correlations and the definition of
the Toller angle. In Chapter V we discuss Fourier-Bessel
analysis of scattering amplitudes. In Chapter VI we discuss
the derivation of equations describing composite-particle
scattering as was done by Freedman, Lovelace, and Namyslowski[33]
and Aaron, Amado and Young[100], as well as the construction
of two-to-three and three-to-three scattering amplitudes from
the composite-particle amplitude via the attachment of

suitable vertex functions. Chapters VII, VIII, and IX are the heart of the thesis. Chapter VII deals with the formulation of our multiperipheral equation via a continuation of the Freedman, Lovelace, and Namyslowski and Aaron, Amado, and Young results to the cross channel, in particular it deals with what forms the Born term and the kernel take. Chapter VIII involves the actual solution of the homogeneous multiperipheral equation, including the extraction of Regge behavior and a bound on the Regge trajectory from a variational calculation. Finally in Chapter IX we indicate how to perform the investigations mentioned earlier, i.e. the Fourier-Bessel analysis of the two-to-two amplitude, the existence of final state correlations in the two-to-three amplitude and its Toller angle dependence, and the construction of the single particle inclusive distribution from the three-to-three amplitude and the triple Regge limit.

II. INCLUSIVE REACTIONS

A. Definitions

An inclusive reaction is a process of the form

$a + b \rightarrow c_1 + c_2 + \ldots + c_k + X$ where some of the reaction pro-
ducts are unobserved. This unknown system of particles is de-
noted by X. One measures the distribution of the particles
$c_1 \ldots c_k$ and sums over everything else.[99] X is characterized
by the missing mass squared, $M^2 = (p_a + p_b - p_{c_1} - \ldots - p_{c_k})^2$.
Graphically what one studies is:
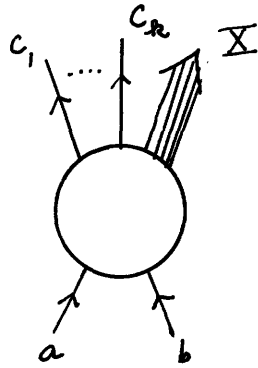
figure (2-1)

An inclusive reaction is different from an exclusive re-
action in that an exclusive measures all of the reaction pro-
ducts; namely, it is a process of the form $a + b \rightarrow c_1 + \ldots + c_k + \ldots + c_n$
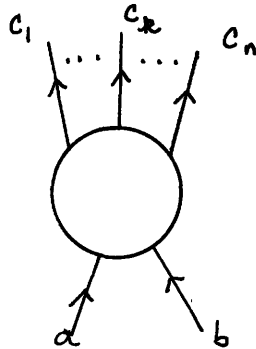
figure (2-2)

Inclusive processes can be constructed from exclusive processes

by simply summing over the quantum numbers of the final state particles one is not interested in or which are unobserved.

However, given the many-particle reactions resulting from experiments at present machine energies, it is a bit too ambitious to try to be completely exclusive. What this means is that given a reaction in which there are more than three final state particles, there are too many kinematical variables describing the process to handle comfortably. Therefore, one resorts to studying an essentially lower multiplicity reaction by considering one- and two-particle inclusive reactions. Our focus will be on single particle inclusive reactions.

B. Definition of a Single Particle Inclusive Distribution and Its Relation to the Mueller Optical Theorem

A single particle inclusive distribution is one describing the process: $a + b \rightarrow c + X$, where X denotes the unobserved system of particles, characterized by the missing mass squared, $M^2 = (p_a + p_b - p_c)^2$.
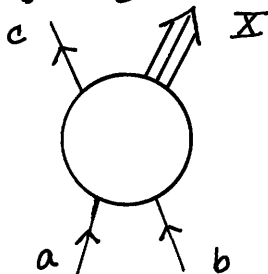
figure (2-3)

What is measured is the quantity: $\dfrac{d\sigma}{d\underset{\sim}{p_c}} = \dfrac{d^3\sigma}{d^3\underset{\sim}{p_a}}$

This is a one particle distribution or spectrum. It is easily seen that $\dfrac{d^3\sigma}{d^3\underset{\sim}{p_c}}$ is not Lorentz invariant. Instead we

may define the invariant distribution function

$$2E \frac{d^3\sigma}{d^3\underset{\sim}{p_c}} = \frac{d^3\sigma}{d^3\underset{\sim}{p_c}/2E} = f(\underset{\sim}{p_c}, s) = f(s, p_c^{\parallel}, p_c^2{}_\perp) \, . \quad (2-1)$$

Let us define some variables by examining the kinematics in the center of mass (C.M.) system. There the initial four-momenta are given by

$$P_a = (E_a, \underset{\sim}{p}) \qquad P_b = (E_b, -\underset{\sim}{p}) \qquad (2-2)$$

with $\underset{\sim}{p}$ chosen to be along the z-direction,

$$\underset{\sim}{p} = (0, 0, z) \, . \qquad (2-3)$$

The total center of mass energy squared is $s = W^2$, where

$$W^2 = \sqrt{\underset{\sim}{p}^2 + m_a^2} + \sqrt{\underset{\sim}{p}^2 + m_b^2} \qquad \text{and}$$

$$|\underset{\sim}{p}|^2 = \frac{[s - (m_a - m_b)^2][s - (m_a + m_b)^2]}{4s} = \frac{\lambda(s, m_a^2, m_b^2)}{4s} \quad (2-4)$$

with $\quad \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$.

We also have:

$$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} \qquad E_b = \frac{s + m_b^2 - m_a^2}{2\sqrt{s}} \qquad (2-5)$$

The magnitude of the momentum fixes s , for fixed masses. We make our choice of frame unique by requiring that P_c have no y-component, i.e.

$$P_c = (E, P_{cx}, 0, P_{cz}) \quad , \quad P_c^2 = m^2 \qquad (2\text{-}6)$$

The momentum transfer is

$$t = (P_a - P_c)^2 = m_a^2 + \mu^2 - 2EE_a + 2 \underset{\sim}{P_a} \cdot \underset{\sim}{P_c} \qquad (2\text{-}7)$$

In the physical region for the process indicated t is negative. The dot product is

$$\underset{\sim}{P_a} \cdot \underset{\sim}{P_c} = |\underset{\sim}{P}||\underset{\sim}{P_c}| \cos\theta = |\underset{\sim}{P}| P_c''$$

$$P_c'' = |\underset{\sim}{P_c}| \cos\theta \qquad (2\text{-}8)$$

The missing mass is given by

$$M^2 = (P_a + P_b - P_c)^2 = s + m^2 - 2EW \qquad (2\text{-}9)$$

Often it is more convenient to express the invariant function $d\sigma / d\underset{\sim}{P_c} / 2E$ in terms of t and M^2. Given that $dt/d\cos\theta = +2|\underset{\sim}{P_c}||\underset{\sim}{P}|$ and $dM^2/d|\underset{\sim}{P_c}|^2 = -2W/2E$, we have

$$2E\frac{d\sigma}{d\underset{\sim}{P_c}} = 2E\frac{d\sigma}{|\underset{\sim}{P_c}|^2 d|\underset{\sim}{P_c}| \, d\cos\theta \, d\phi} = \frac{4E}{(2\pi)} \frac{d\sigma}{|\underset{\sim}{P_c}| \, d|\underset{\sim}{P_c}|^2 d\cos\theta}$$

$$= (2) \frac{4W|\underset{\sim}{P}|}{(2\pi)} \frac{d\sigma}{dt \, dM^2} \qquad (2\text{-}10)$$

So

$$\frac{d\sigma}{dt \, dM^2} = \frac{1}{2} (2\pi) \frac{1}{4W|\underset{\sim}{P}|} F(s, P_c'', \underset{\sim}{P_{c\perp}}^2)$$

$$= \frac{1}{\lambda^{\frac{1}{2}}(s, m_a^2, m_b^2)} F(s, P_c'', \underset{\sim}{P_{c\perp}}^2) \qquad (2\text{-}11)$$

Mueller's Optical Theorem[99] says that:

$$F\left(s, p_c^{\parallel}, \underset{\sim}{p}_{c\perp}^2\right) = \frac{1}{(s, m_a^2, m_b^2)} \ disc_{M^2} \left\{ \right. \qquad \left. \right\}$$ (2-12)

That is:

$$F \ = \ \sum_X \left| \right. \left. \right|^2 \ = \ \sum_X $$

$$= \ \frac{1}{\lambda^{\frac{1}{2}}(s, m_a^2, m_b^2)} \ disc_{M^2} \left\{ \right. \qquad \left. \right\}$$

figure (2-4)

So:

$$\frac{d\sigma}{dt \, dM^2} \ = \ \frac{\pi}{2} \ \frac{1}{\lambda^{\frac{1}{2}}(s, m_a^2, m_b^2)} \ disc_{M^2} \left\{ \right. \qquad \left. \right\}$$ (2-13)

C.   The Triple Regge Limit

The triple Regge limit is defined as the limiting beha-
vior of the single particle inclusive distribution for t

fixed, s/M^2 large and M^2 large. From a Regge picture one expects the distribution to behave like:

$$\frac{1}{s^2} \ \gamma(t) \ \left(\frac{s}{M^2}\right)^{2\,\alpha(t)} (M^2)^{\alpha_V} \qquad\qquad (2\text{-}14)$$

The factor $(M^2)^{\alpha_V}$ arises from the leading vacuum exchange in the $b\bar{b}$ channel, the relevant large energy being M^2. The factor $\left(\frac{s}{M^2}\right)^{2\,\alpha(t)}$ comes from the traditional two body Regge limit. The factor $\gamma(t)$ will be explained presently.

We can arrive at the above result and perhaps gain some intuition by considering the following sequence of limits, in each instance assuming Regge behavior in the appropriate channel. First we consider the limit s/M^2 large, t fixed, M^2 fixed. If M^2 is fixed, then our single particle inclusive process is essentially like a two body scattering process:
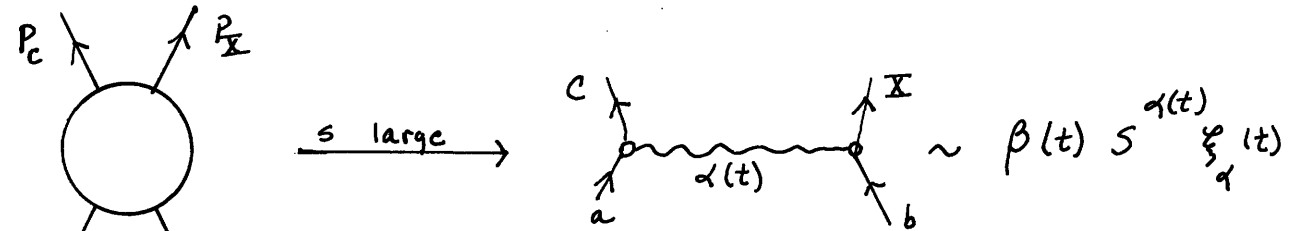
figure (2-5)

In our case we have:
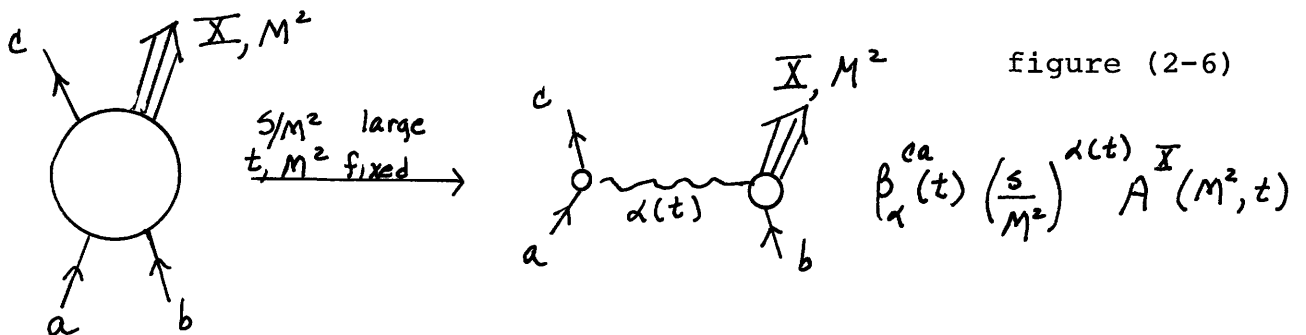
figure (2-6)

$$\frac{d\sigma}{dt\,dm^2} = \sum_X \left| \begin{array}{c} \end{array} \right|^2 = \frac{1}{s^2} \left| \beta_\alpha^{ca}(t) \right|^2 \left(\frac{s}{M^2} \right)^{2\alpha(t)} \sum_X \left| A^X(m^2,t) \right|^2 \tag{2-15}$$

Now $A^X(m^2,t)$ describes the process $\alpha(t) + b \longrightarrow X$
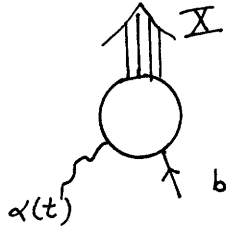
figure (2-7)

Therefore

$$\left| \beta_\alpha^{ca}(t) \right|^2 \left(\frac{s}{M^2} \right)^{2\alpha(t)} \sum_X \left| A^X(m^2,t) \right|^2$$

$$= \sum_X \quad = Disc_{M^2} \tag{2-16}$$

figure (2-8)

If now M^2 becomes large, with s remaining large and

fixed, this becomes

$$Disc_{M^2}$$
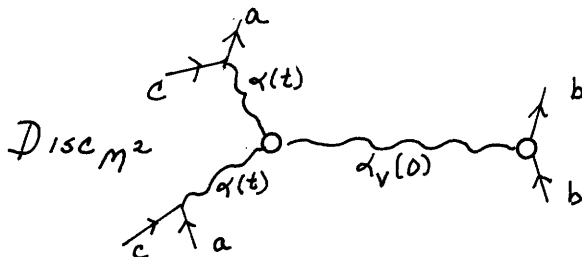
figure (2-9)

This assumes Regge behavior in the variable M^2 , in

which case the graph behaves like:

$$\gamma(t) \left(\frac{s}{M^2}\right)^{\alpha(t)} \left(M^2\right)^{\alpha_V(0)} \qquad (2\text{-}17)$$

$\gamma(t)$ contains the Regge couplings at the $b\bar{b}$ vertex and at the Reggeon-Reggeon-Vacuum trajectory vertex, and any signature factors.

This means finally that $\dfrac{d\sigma}{dt\, dM^2}$ behaves as:

$$\frac{d\sigma}{dt\, dM^2} \sim \frac{1}{s^2} \gamma(t) \left(\frac{s}{M^2}\right)^{2\alpha(t)} \left(M^2\right)^{\alpha_V(0)} \qquad (2\text{-}18)$$

The identification of α_V with the leading vacuum exchange in the $b\bar{b}$ channel is now clear (for M^2 being the relevant energy variable). The leading pole in helicity, i.e. the difference of $a\bar{c}$ and $\bar{a}c$ helicities, analytically continued and viewed from the $b\bar{b}$ channel, is given by the leading t-channel Regge singularities.[101]

De Tar et al[101] discovered these identifications by using an O(2,1) expansion for the connected Moller amplitude $\mathcal{M}$ for the process a + b + c $\longrightarrow$ a + b + c , where again

$$\frac{d^2\sigma}{dt\, dM^2} = \frac{const}{s^2} \, disc_{M^2} \, \mathcal{M} \qquad (2\text{-}19)$$

with $disc_{M^2} \mathcal{M}$ being the appropriate discontinuity.
They write $\mathcal{M}$ as

$$\mathcal{M}(t,s,M^2) = \int d\mu \, d\ell \sum_{\rho=1,2} e^{i\mu\lambda} \, d^{\ell}_{i\mu,\rho}(\xi) \, A^{\ell}_{i\mu,\rho}(t) \qquad (2\text{-}20)$$

where $i\mu$ is the analytic continuation of m (the diffe-
rence of the helicities of states $a\bar{c}$ and $\bar{a}c$ viewed from
the $b\bar{b}$ channel).[ ] The variable l is essentially the
analytic continuation of the total angular momentum for the
$b\bar{b}$ channel. The relation between the variables ζ and
η and s , t , M^2 are as follows, in the indicated
limits: for t fixed, s/M^2 large and M^2 large--the triple
Regge limit, $s/M^2 = \text{const} \cdot e^{|\eta|}$ and $M^2 = \text{const} \cdot e^{|\zeta|}$.

If we assume that the high ζ behavior is given by the lead-
ing Regge singularity $\alpha_V(o)$ coupling to the $b\bar{b}$ channel,
then

$$\sum_\rho d^l_{i\mu,\rho}(\zeta) \longrightarrow e^{l/\zeta l} \quad , \quad \zeta \longrightarrow \infty$$

The high η behavior is given by the leading helicity
in $i\mu$ at $n(t)$, then $e^{n(t)\eta} \sim e^{|\eta| 2\alpha(t)}$ as $\eta \rightarrow \pm\infty$.
So indeed

$$\frac{d\sigma}{dt\, dM^2} = \frac{1}{s^2}\, disc_{M^2}\, \mathcal{M} \sim \frac{1}{s^2} (M^2)^{\alpha_V(o)} \left(\frac{s}{M^2}\right)^{2\alpha(t)} \gamma(t) \quad (2\text{-}21)$$

as given before.

It is the existence of the triple Regge limit which we
investigate by computing a three-to-three scattering amplitude
taking the appropriate discontinuity and going to the limit
in the variables t , s/M^2, and M^2 .

III. REVIEW OF THE MULTIPERIPHERAL MODEL

A. Basic Assumptions and Mechanisms

The multiperipheral model, as originally formulated, [102]
is based on a mechanism for multiparticle production which
is a generalization to higher energy of the peripheral model.
The basic idea of the peripheral model is that one-pion ex-
change amplitudes dominate high energy processes (see fig.3-1).
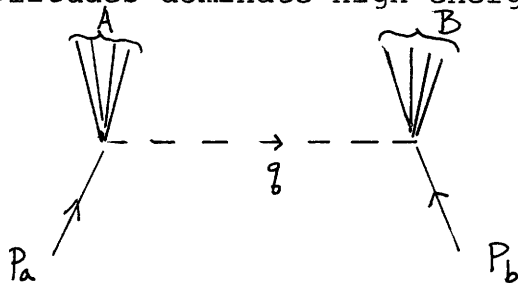
figure (3-1)

These amplitudes may be written

$$ T_{ab}(P_a, P_b; k_i) = \frac{T_{a\pi}(P_a, q, k_{a_i}) \, T_{b\pi}(P_b, q, k_{b_i})}{q^2 - \mu^2} \qquad (3\text{-}1) $$

where $T_{a\pi}(T_{b\pi})$ is the amplitude describing the
reaction between the $A(B)$ particle and the virtual pion,
giving system $A(B)$ as a final state, with k_{a_i} indica-
ting the momentum of the i-th particle in the a system. A
further assumption is that the dependence of the amplitudes
$T_{a\pi}(T_{b\pi})$ on the mass of the virtual pion is negligible
due to the small range of the interaction.

Multiperipheralism comes about as follows. One assumes
that if the center of mass energies of each of the subsystems

A and B are sufficiently large, then the one-pion exchange

may be extracted from the amplitudes $T_{a\pi}$ and $T_{b\pi}$.

Let us define the variables

$$S = (P_a + P_b)^2 = (k_a + k_b)^2 \qquad k_a = \sum_i k_{a_i} \quad k_b = \sum_i k_{b_i}$$

$$S_a = (P_a - q)^2 = k_a^2$$

$$S_b = (P_b + q)^2 = k_b^2 \qquad (3-2)$$

$$\sqrt{S} \gg \sqrt{S_a} + \sqrt{S_b}$$

The peripheralism of the amplitude $T_{a\pi}$, say, requires,

from energy conservation, that $\sqrt{S_a} \gg \sqrt{S_1} + \sqrt{S_2}$ where $\sqrt{S_1}$

and $\sqrt{S_2}$ are the energies in the centers of mass of the two

subprocesses into which the system A is split. Therefore any

subdivision of an amplitude degrades the energy of each group

of particles in the final state. If we repeat this procedure

for all high-energy amplitudes, we shall obtain a chain of

low energy amplitudes linked by virtual pions (see fig 3- 2).

figure (3-2)
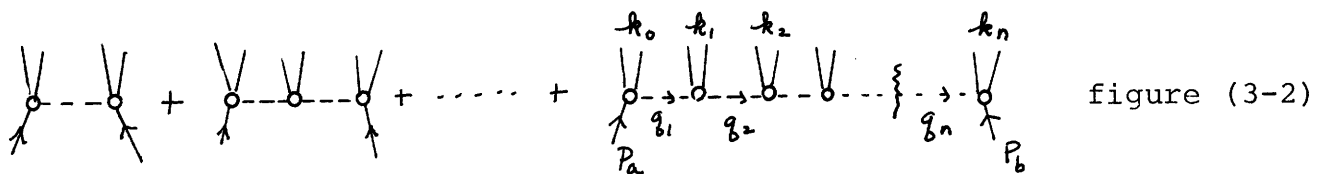
The amplitude for the n-th order peripheral graph is

$$T(P_a, P_b; k_i) = \frac{T^R(P_a, q_1; k_{0_i}) \, T^R(q_1, q_2; k_{1_i}) \cdots T^R(q_n, P_b; k_{n_i})}{(q_1^2 - \mu^2)(q_2^2 - \mu^2) \cdots (q_n^2 - \mu^2)} \qquad (3-3)$$

where the T^R are low energy amplitudes taken for the

$$q_i^2 = \mu^2 \qquad .$$

These results may be used to calculate the total cross
section for the scattering of particles a and b, and to cal-
culate the elastic non-forward amplitude. We consider the
process $p_a + p_b \rightarrow p'_a + p'_b$ for spinless particles and
compute the absorptive part of the elastic scattering ampli-
tude by using unitarity and the multiperipheral model for the
process $p_a + p_b \rightarrow$ n systems of particles. We call the in-
variant amplitude describing the elastic scattering process
$M(s,t)$ and its absorptive part $A(s,t)$. The total cross
section is given by

$$\sigma_{TOT} = \frac{1}{\lambda^{\frac{1}{2}}(s, P_a^2, P_b^2)} \, A(s,0) = \sum_n \sigma_n \qquad (3\text{-}4)$$

where $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ and σ_n
is the cross section for a + b $\rightarrow$ n systems of particles,
which is given by

$$\sigma_n(s) = \frac{1}{2g\sqrt{s}} \left(\frac{1}{(8\pi^4)}\right)^n \int \cdots \int \frac{d^4 q_1 \cdots d^4 q_n \, ds_0 \cdots ds_n}{(q_1^2 - \mu^2)^2 \cdots (q_n^2 - \mu^2)^2}$$

$$\times \, \delta(-k_0^2 - s_0) \, 2|k_0|\sqrt{s_0} \, \sigma^R(s_0) \cdots \delta(-k_n^2 - s_n) \, 2|k_n|\sqrt{s_n} \, \sigma^R(s_n) \qquad (3\text{-}5)$$

$$= \frac{1}{2g\sqrt{s}} \, A_n(s) \qquad (3\text{-}6)$$

where $\sigma^n(s_0) = \frac{1}{2|k_0|\sqrt{s_0}} \, A^R(s_0)$ $\qquad (3\text{-}7)$

and $A^R(s_0)$ is the absorptive part of the amplitude $T^R(s_0)$
given in equation (3-3),i.e. $A^R(s_0) = |T^R(s_0)|^2$. The
graphical representation of this is given in fig.3-3 .

figure (3-3)

$$S = (P_a + P_b)^2$$
$$t = 0$$
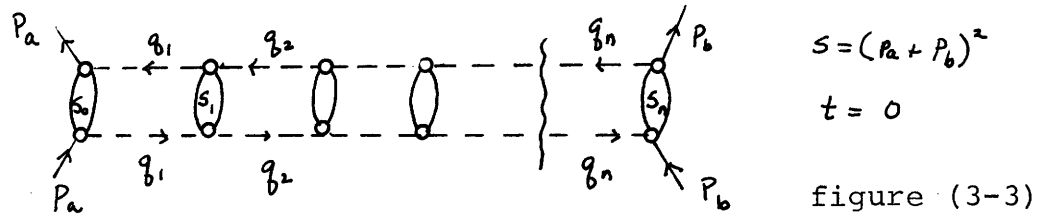
$A_n(s)$ is the absorptive forward amplitude with an n-system intermediate state. The contribution to the absorptive part of the non-forward amplitude from an intermediate state with n systems of particles formed via the multiperipheral mechanism is (fig.3-4)

$$A_n(s,t) = \frac{1}{(8\pi^4)^n} \int \cdots \int \frac{d^4 q_1 \cdots d^4 q_n \, ds_0 \cdots ds_n}{(q_1^2 - \mu^2)[(q_1 - \Delta)^2 - \mu^2] \cdots (q_n^2 - \mu^2)[(q_n - \Delta)^2 - \mu^2]}$$

$$\times \, \delta(k_0^2 - s_0) A^R(s_0, t) \, \delta(k_1^2 - s_1) A^R(s_1, t) \cdots \delta(k_n^2 - s_n) A^R(s_n, t) \qquad (3\text{-}8)$$

where $\quad \Delta = P_a - P_a' \quad ; \quad \Delta^2 = t \quad .$
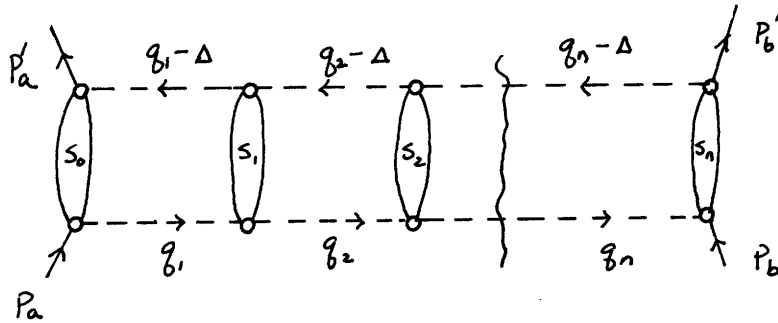
figure (3-4)

B. Forward Absorptive Amplitude

Going to equation (35), we can write for the n-th peripheral contribution to the absorptive forward amplitude

$$A_n(P_a, P_b) = \int ds_0 \, \frac{A^R(s_0)}{8\pi^4} \int d^4 p' \, \frac{\delta[(p_a - p')^2 - s_0] A_{n-1}(p', P_b)}{(p'^2 - \mu^2)^2} \qquad (3\text{-}9)$$

that is, $A_n(P_a, P_b)$ satisfies a recurrence relation. If we sum over all peripheral contributions we obtain an integral equation for the forward absorptive amplitude.

$$A(P_a, P_b) = \sum_n A_n(P_a, P_b)$$

$$A(P_a, P_b) = \int ds_0 \frac{A^R(s_0)}{8\pi^4} \int d^4p' \frac{\delta[(P_a - p')^2 - s_0] A(p', P_b)}{[p'^2 - \mu^2]^2} \tag{3-10}$$

Converting to invariant variables by using the Jacobian

$$Q(s, u; s', u'; s_0) = \int d^4p' \, \delta[(P_a - p')^2 - s_0] \, \delta[(p' + P_0)^2 - s'] \, \delta(p'^2 + u') \tag{3-11}$$

we obtain

$$A(s, u) = \int ds_0 \frac{A^R(s_0)}{8\pi^4} \int ds' du' \frac{Q(s, u; s', u'; s_0)}{(u' + \mu^2)^2} A(s', u') \tag{3-12}$$

where

$$Q(s, u, s', u', s_0) = \frac{\pi}{2(\mathcal{S}^2 + 4\mu^2 u)^{\frac{1}{2}}}$$

$$\times \theta(u\mathcal{S}'^2 + u'\mathcal{S}^2 - \mu^2 \mathcal{S}_0^2 - \mathcal{S}\mathcal{S}'\mathcal{S}_0 - 4\mu^2 u u')\theta(\sqrt{s} - \sqrt{s'} - \sqrt{s_0}) \tag{3-13}$$

with

$$\mathcal{S} = s + u - \mu^2$$
$$\mathcal{S}' = s' + u' - \mu^2 \tag{3-14}$$
$$\mathcal{S}_0 = s_0 + u + u'$$

For s large, the integral equation becomes

$$A(s, u) = \frac{1}{16\pi^3 s} \int ds_0 \, A^R(s_0) \int_{s'_{min}}^{s - 2\sqrt{s s_0}} ds' \int_{\frac{s'}{s}(u + \frac{s_0}{1 - \frac{s'}{s}})}^{s - s'} du \frac{A(s', u')}{(u' + \mu^2)^2} \tag{3-15}$$

Given that the region s' small and u' large does not contri-
bute to the asymptotic behavior (i.e. large values of s' and
small values of u' dominate), we can write for the integral
equation

$$A(s,u) = \frac{1}{16\pi^3} \int ds_0 \, A^R(s_0) \int_0^s \frac{ds'}{s} \int_{\frac{s'}{s}\left(u+\frac{s_0}{1-\frac{s'}{s}}\right)}^\infty du' \, \frac{A(s',u')}{(u'+\mu^2)^2} \qquad (3\text{-}16)$$

We easily see that the kernel is invariant under the transfor-
mation $s \longrightarrow cs$ and $s' \longrightarrow cs'$. This implies that the
solution of the equation may be written in the form

$$A(s,u) = s^\alpha \, \phi_\alpha(u) \qquad (3\text{-}17)$$

where $\left(x = \frac{s'}{s}\right) \phi_\alpha(u)$ must satisfy the equation

$$\phi_\alpha(u) = \frac{1}{16\pi^3} \int ds_0 \, A^R(s_0) \int_0^1 x^\alpha \, dx \int_{x\left(u+\frac{s_0}{1-x}\right)}^\infty du' \, \frac{\phi_\alpha(u')}{(u'+\mu^2)^2} \qquad (3\text{-}18)$$

This is a homogeneous Fredholm equation which has solutions
only for certain values of α , i.e. this is an eigenvalue
problem which determines α as a function of $A^R(s_0)$:
$$\alpha = \alpha\left(A^R(s_0)\right) \qquad .$$

If $\alpha \sim -1$, the lowest values of x will dominate and
therefore we may set the lower value of the u' integration to
zero and $\phi_\alpha(u)$ is u independent. We obtain

$$\alpha = -1 + \lambda \quad, \qquad (3\text{-}19)$$

where
$$\lambda = \frac{1}{16\pi^3\mu^2} \int ds_0 \, A^R(s_0) \qquad (3\text{-}20)$$

The condition $\alpha \sim -1$ implies λ is small which is known as the weak coupling approximation.

C. Non-Forward Absorptive Amplitude

If we go back to figure 3-4 and equation (3-8) for the n-th peripheral contribution to the non-forward absorptive amplitude and define

$$P = \frac{P_b + P_b'}{2} \qquad\qquad Q = \frac{P_a + P_a'}{2}$$

$$\Delta = \frac{P_a - P_a'}{2} = \frac{P_b' - P_b}{2} \quad , \qquad \Delta^2 = t \qquad\qquad (3\text{-}21)$$

$$P_a^2 = -u_1 \quad , \qquad\qquad P_a'^2 = -u_2$$

$$(Q + P) = (P_a + P_b)^2 = (P_a' + P_b')^2 = s$$

we obtain the recurrence relation

$$A_n(Q,\Delta,P) = \frac{1}{8\pi^4} \int d^4Q' \; \frac{A^R(Q',\Delta,Q) \; A_{n-1}(Q',\Delta,P)}{[(Q'+\Delta)^2 - \mu^2][(Q'-\Delta)^2 - \mu^2]} \qquad\qquad (3\text{-}22)$$

and the integral equation

$$A(Q,\Delta,P) = A^R(Q,\Delta,P) + \frac{1}{8\pi^4} \int d^4Q' \; \frac{A^R(Q',\Delta,Q) \; A(Q',\Delta,P)}{[(Q'+\Delta)^2 - \mu^2][(Q'-\Delta)^2 - \mu^2]} \qquad\qquad (3\text{-}23)$$

If we express this in terms of invariants and neglect the u_1' , u_2' dependence of A^R , we get

$$A(s, u_1, u_2, t) = A^R(s,t) + \frac{1}{8\pi^4} \int ds' \, du_1' \, du_2' \, ds_0 \; A^R(s_0,t)$$

$$\times \; \frac{K_t(s,u_1,u_2 ; s',u_1',u_2' ; s_0) \, A(s',u_1',u_2',t)}{(u_1' + \mu^2)(u_2' + \mu^2)} \qquad\qquad (3\text{-}24)$$

where $\quad K_t\,(s, u_1, u_2 ; s', u_1', u_2', s_0)$

$$= \int d^4 Q' \; \delta[(Q-Q')^2 - S_0] \; \delta[(Q'+\Delta)^2 + u_1'] \; \delta[(Q'-\Delta)^2 + u_2'] \; \delta[(Q'+P)^2 - s'] \quad (3\text{-}25)$$

In this problem t is a fixed parameter. At high energy K
becomes

$$K = \frac{1}{s}\, Q_t(u_1, u_2, u_1', u_2', x) = \frac{1}{ss'}\, T(\zeta, \zeta_1, \zeta_2) \qquad (3\text{-}26)$$

where

$$\zeta = -t\,(1-x) \qquad\qquad x = \frac{s'}{s}$$

$$\zeta_1 = u_1' - u_1 x - \frac{s_0 x}{1-x}$$

$$\zeta_2 = u_2' - u_2 x - \frac{s_0 x}{1-x} \qquad\qquad (3\text{-}27)$$

$$T(\zeta, \zeta_1, \zeta_2) = -\lambda(\zeta, \zeta_1, \zeta_2)$$

Assuming $A^R(s,t)$ decreases sufficiently rapidly with energy,
we obtain the integral equation

$$A(s, u_1, u_2, t) = \frac{1}{64\pi^4} \int ds_0\, A^R(s_0, t) \int \frac{ds'}{s}\, du_1'\, du_2'$$

$$\times\; \frac{Q_t\,(u_1, u_2, u_1', u_2', x)\, A(s', u_1', u_2', t)}{(u_1' + \mu^2)(u_2' + \mu^2)} \qquad (3\text{-}28)$$

Again the kernel is invariant under the transformation $s \to cs$
and $s' \to c s'$, implying that $A(s, u_1, u_2, t)$ may be written

$$A(s, u_1, u_2, t) = c(t)\, s^{\alpha(t)}\, \phi(u_1, u_2, t) \qquad (3\text{-}29)$$

where $\quad \phi(u_1, u_2, t)$ satisfies the integral equation

$$\phi(u_1, u_2, t) = \int \frac{A^R(s_0, t)}{8\pi^3} \, ds_0 \int_0^1 dx \; x^{\alpha(t)} \int du_1' \, du_2' \qquad (3\text{-}30)$$

$$\times \; \frac{Q_t(u_1, u_2, u_1', u_2', x) \; \phi(u_1', u_2', t)}{(u_1' + \mu^2)(u_2' + \mu^2)}$$

This is a two-dimensional homogeneous Fredholm equation which issatisfied only for certain values of α as a function of the fixed parameter t . To lowest order the relation between α and t is given by[103]

$$\int \frac{A^R(s_0, t)}{16\pi^3} \, ds_0 \int_0^1 dx \; x^{\alpha(t)} \int_0^1 \frac{dy}{(1-x)\mu^2 + s_0 x(1-x) - t(1-x)y(1-y)} = 1 \qquad (3\text{-}31)$$

If $\phi(u_1, u_2, t)$ is normalized by $\phi(-\mu^2, -\mu^2, t)$, then

$$A(s, t) = C(t) \, s^{\alpha(t)} \qquad (3\text{-}32)$$

We begin with an assumption about n-particle production via a multiperipheral mechanism. We can then sum all of the multiperipheral contributions by means of an integral equation for the off-mass shell absorptive amplitude $A(s, u)$ (for forward scattering) or $A(s, u_1, u_2, t)$ (for non-forward scattering). The kernel depends on a low energy amplitude A^R . In the limit of s large, the dilatation invariance of the kernel allows us to write

$$A(s, u) = s^{\alpha} \, \phi_{\alpha}(u)$$

where $\alpha = \alpha(A^R(s_0))$ or

$$A(s, u_1, u_2, t) = s^{\alpha(t)} \, \phi(u_1, u_2, t) \qquad (3\text{-}33)$$

where $\qquad \alpha(t) = \alpha(A^R(s_0, t))$

The problem we are then left with is the solution of a homogeneous integral equation for $\phi(u)$ or $\phi(u_1, u_2, t)$, which determines both α and ϕ. The corresponding on-mass-shell absorptive part then appears to be Regge behaved at high energy.

The assumptions of one-pion exchange dominance and negligible dependence of the low energy amplitudes on off shell masses neglect the possibilities of the exchange of higher mass states with spins greater than zero since the model is completely consistent only when the low energy interaction is in an s-wave state.[102] We will have something to say about this in our version of the multiperipheral model.

IV. FINAL STATE CORRELATIONS

Whenever we study a scattering process in which three or more particles are created, we have to consider the possibility of correlations between the final state particles. This is manifest in the degree of factorizability of the relevant amplitude or distribution describing the scattering. For an exclusive process, say, one with three reaction products, the factorizability question is answered by the Toller angle dependence, which will be discussed below. For an inclusive process, e.g. a two-particle inclusive process, factorizability is determined by the correlation function which measures the discrepancy between the two-particle inclusive distribution and the product of the two relevant single particle distributions.

A. Toller Angle

Bali, Chew, and Pignotti[22] studied multi-particle production via Toller variables. If we consider n-particle production, the diagram describing such a scattering process is below
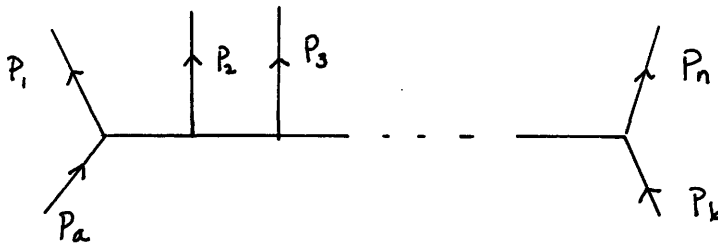
figure (4-1)

Following Bali et al we define three sets of variables:

N-1 t variables, N-1 ζ variables and N-2 ω variables.

The t variables are such that $t_{ij} = Q_{ij}^2 < 0$. The ζ_{ij} are conjugate variables to the t_{ij} , with $i\,\zeta_{ij}$ being the analytic continuation of the angle in the rest system of Q_{ij} between the direction of $\vec{p_i}$ and $\vec{p_j}$; the ζ_{ij} are real and range from 0 to ∞ independently of the other variables. The ζ_{ij} are linearly related to the subenergies s_{ij} . The ω_j , usually referred to as Toller angles, are understood in the following way: go into the rest frame of p_j where the spatial components of the two adjacent momentum transfers point in the same direction, call it z . If we consider a rigid rotation about the z-axis of all momenta to the left of the vertex j and an independent rigid rotation of all momenta to the right of the vertex j, the difference between these two rotation angles is ω_j , ranging from 0 to 2π . In other words, the angle ω_j describes a rigid rotation about the z-axis of the left hand side of the diagram with respect to the right hand side of the diagram. Assuming the multi-Regge hypothesis, we find for the amplitudes describing the 2 $\rightarrow$ n process:

$$M \sim f_1(t_{12})\, f_2(t_{12}, \omega_2, t_{23}) \cdots f_n(t_{n-1,n})$$
$$\times \left(\cosh \zeta_{12}\right)^{\alpha_{12}(t_{12})} \left(\cosh \zeta_{23}\right)^{\alpha_{23}(t_{23})} \cdots \qquad (4\text{-}1)$$

The vertex functions f_1 and f_n describe the coupling of two physical particles to a Regge trajectory, while the internal vertex functions f_j describe the coupling of two Regge trajectories to a physical particle. The f_j also contain the

Toller angle dependence which determines the factorizability of the amplitude.

If we consider a process with three reaction products (fig. 4-2)
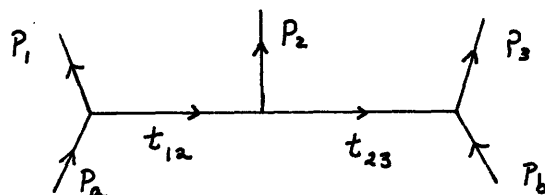
figure (4-2)

and write everything in terms of invariants, we find for the amplitude describing this process:

$$f \sim f_1(t_{12}) \; s_{12}^{\alpha(t_{12})} \; f_2(t_{12}, \omega_2, t_{23}) \; s_{23}^{\alpha(t_{23})} \; f_3(t_{23}) \qquad (4\text{-}2)$$

for s, s_{12}, s_{23} large and t_{12}, t_{23} small and fixed. It is this process which we wish to study by taking our composite-particle scattering amplitude, allowing the final state composite to decay and going to the limit s, s_{12}, s_{23} large; t_{12}, t_{23} fixed, to extract the Toller angle dependence in our model.

B. Correlations in Inclusive Reactions

If we consider the inclusive reaction

$$a + b \longrightarrow c_1 + \ldots + c_k + X$$

and define the associated invariant distribution function f_k by

$$\frac{d\sigma}{d^3 p_1 \cdots d^3 p_k} = \frac{f_k(p_1 \cdots p_k; s)}{2E_1 \cdots 2E_k} \qquad (4\text{-}3)$$

we find that, in general, a k-particle distribution cannot be obtained from distributions involving fewer particles due to correlations between particles. We may analyze this by defining correlation functions $c_k(p_1, \cdots, p_k)$ which vanish when there are no correlations. We define

$$f_1(p_1, s) = c_1(p_1)$$

$$f_2(p_1, p_2, s) = c_1(p_1, p_2) + c_1(p_1) c_2(p_2) \tag{4-4}$$

$$f_3(p_1, p_2, p_3, s) = c_3(p_1, p_2, p_3) + c_2(p_1, p_2) c_1(p_3)$$

$$+ c_2(p_1, p_3) c_1(p_2) + c_2(p_2, p_3) c_1(p_1)$$

$$+ c_1(p_1) c_1(p_2) c_1(p_3)$$

If there are no correlations $c_k(p_1, \cdots, p_k) = 0$ $k \geqslant 2$ and

$$f_k(p_1, \cdots, p_k, s) = f_1(p_1, s) \cdots f_1(p_k, s) \tag{4-5}$$

If the $c_k \neq 0$, the existence of correlations is implied.

Kinematic correlations are always present in particle reactions due to four-momentum conservation. Their importance varies from one part of phase space to the other. If in , say, a two-particle inclusive reaction the momenta of the two produced particles are small compared with their maximum values, kinematic correlations are unimportant since one is far from the phase space boundary. There is no general way of separating dynamical and kinematic correlations.

V. FOURIER-BESSEL ANALYSIS OF SCATTERING AMPLITUDES

The general characteristics, in most reactions, of the scattering of particles at high energies is that scattering angles are small and that angular distributions take the form of diffraction patterns. This suggests the use of an eikonal approximation. Blankenbecler and Goldberger[110] investigated a generalization to the eikonal approximation. Their result was that the high energy scattering amplitude was given as a Bessel transform with a $J_0(2bp \sin \theta/2)$ kernel, b being the impact parameter and p and θ the momentum and scattering angle in the C.M. Their representation satisfies unitarity automatically at high energy. They derived their result on the basis of small angle scattering. Predazzi[71] demonstrated that the Blankenbecler-Goldberger representation is valid for all energies and throughout the entire angular domain (θ : o to π). They express the spectral function of the representation in terms of the partial waves and give its inversion. We will now outline this for spinless particles.

We assume the existence of the usualpartial wave expansion for the scattering amplitude:

$$F(p,\theta) = \frac{1}{2ip} \sum_{l=0}^{\infty} (2l+1) f_{lp} P_l(\cos\theta) \qquad (5\text{-}1)$$

where

$$\frac{d\sigma}{d\Omega} = \left| F(p,\theta) \right|^2$$

$$f_\ell(p) = ip \int_{-1}^{1} d(\cos\theta) \, F(p,\theta) \, P_\ell(\cos\theta) = e^{2i\delta_\ell(p)} - 1$$

$$(5-2)$$

Partial wave unitarity is:

$$Im \, f_\ell(p)/2i = \left| f_\ell(p)/2i \right|^2 + \frac{1}{4}\left(1 - e^{-4 Im \, \delta_\ell(p)}\right)$$

$$(5-3)$$

Given that $(\quad)$

$$P_\ell(\cos\theta) = {_2}F_1\left(1+\ell, -\ell; 1; \sin^2\theta/2\right)$$

$$= (-1)^\ell \, {_2}F_1\left(1+\ell, -\ell; 1; \cos^2\theta/2\right)$$

$$(5-4)$$

$$(0 < \theta < \pi)$$

where $\ {_2}F_1\ $ is the Gauss hypergeometric function and $(\quad)$

$$\int_0^\infty dx \, x^{-\lambda} \, J_\nu(xy) \, J_\mu(xa) = y^\nu a^{-\lambda} \, \Gamma\left[\frac{\mu+\nu-\lambda+1}{2}\right]$$

$$\times \, {_2}F_1\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; \frac{y^2}{a^2}\right)$$

$$(a > y > 0 \, ; \quad Re(\mu+\nu)+1 > Re\,\lambda > -1)$$

$$(5-5)$$

If $\ \nu = 0\ $, $\ \mu = 2\ell+1\ $, $\ a = 1\ $, $\ y = \sin\theta/2\ $, then

$$P_\ell(\cos\theta) = \int_0^\infty J_0(x \sin\theta/2) \, J_{2\ell+1}(x) \, dx$$

$$(5-6)$$

and

$$F(p,\theta) = -i \int_0^\infty J_0(2bp \sin\theta/2) \, f(b,p) \, db \qquad (0 < \theta < \pi)$$

$$(5-7)$$

where

$$f(b,p) = \sum_{\ell=0}^{\infty} (2\ell+1)\, f_\ell(p)\, J_{2\ell+1}(2bp) \qquad (5\text{-}8)$$

Equation (5-7) is the equivalent of the Blankenbecler-Gold-berger representation for the scattering amplitude valid for $0 < \theta < \pi$, for all energies, it being a consequence of the existence of the usual partial wave expansion; equation (5-8) gives the spectral function in terms of the partial wave amplitudes. Predazzi[7] showed the domain of convergence of the representation in the $\cos\theta$ plane to be an ellipse with foci at ± 1, which therefore includes the physical region. This does not coincide with the Lehman ellipse in which the original partial wave expansion was known to be convergent.

If we write $2p\sin\theta/2 = (-t)^{\frac{1}{2}}$ (t = momentum transfer), we can express the scattering amplitude as a Bessel transform of zero order with $(-t)^{\frac{1}{2}}$ being the transform variable. We put

$$F(p,\theta) = F(p,t) \qquad (5\text{-}9)$$

then

$$f(b,p) = b \int_0^{\infty} d(-t)^{\frac{1}{2}}\, J_0(b\sqrt{-t})\, G(p,t) \qquad (5\text{-}10)$$

where $\qquad G(p,t) = i\,(-t)^{\frac{1}{2}}\, F(p,t)$.

$$(5\text{-}11)$$

We used the inversion

$$f(x) = \int_0^\infty dy \; g(y) \; J_0(xy) \; (xy)^{\frac{1}{2}}$$

$$g(y) = \int_0^\infty dx \; f(x) \; J_0(xy) \; (xy)^{\frac{1}{2}}$$

(5-12)

For $\theta \to 0$, equation (5-1) takes on the eikonal form

$$F(p,\theta) = -ip \int_0^\infty b \, db \; f(b,p) \; J_0(bp\theta)$$

(5-13)

For equation (5-8) to be valid $f(b,p)$ must have a finite cir-
cle of analyticity about the origin in the bp plane.
The partial wave amplitudes may be expressed in terms of the
spectral function thus:[25]

$$\int_{-1}^1 J_0(2bp \sin \theta/2) \; P_\ell(\cos\theta) \; d(\cos\theta) = \frac{2}{bp} \; J_{2\ell+1}(2bp)$$

$$f_\ell(p) = 2 \int_0^\infty f(b,p) \; J_{2\ell+1}(2bp) \; \frac{db}{b}$$

(5-14)

Matsumato and Tsujimura[18] formulate the Fourier-Bessel re-
presentation for helicity amplitudes describing the scattering
of particles with arbitrary spin. Impact parameter amplitudes
are expressed in terms of partial wave amplitudes. The par-
tial wave expansion of the s-channel helicity amplitude $A_{\lambda\mu}(s,t)$
for the scattering a + b $\longrightarrow$ c + d is:

$$A_{\lambda\mu}(s,t) = \sum_{j} (2j+1) \, d^{j}_{\lambda\mu}(\cos\theta) \, a^{j}_{\lambda\mu}(s) \qquad (5\text{-}15)$$

where $\lambda = \lambda_a - \lambda_b$; $\mu = \lambda_c - \lambda_d$, s and t are the usual Mandelstam variables and θ is the scattering angle. The rotation matrix $d^{j}_{\lambda\mu}$ is: [28]

$$d^{j}_{\lambda\mu}(\cos\theta) = d^{j}_{\lambda\mu}(z) = \left\{ \frac{\Gamma(j+\lambda+1)\,\Gamma(j-\mu+1)}{\Gamma(j-\lambda+1)\,\Gamma(j+\mu+1)} \right\}^{\frac{1}{2}}$$

$$\times \left(\frac{1+z}{2}\right)^{\frac{\lambda+\mu}{2}} \left(\frac{1-z}{2}\right)^{\frac{\lambda-\mu}{2}} \frac{F(\lambda-j,\lambda+j+1;1+\lambda-\mu;\frac{1-z}{2})}{\Gamma(1+\lambda-\mu)}$$

$$(5\text{-}16)$$

for $\lambda+\mu \geqslant 0$; $\lambda-\mu \geqslant 0$, F is the Gauss hypergeometric function. The impact parameter representation is introduced as:

$$A_{\lambda\mu}(s,t) = \left(\frac{1+z}{2}\right)^{\frac{\lambda+\mu}{2}} \int_0^\infty b\,db \, J_{\lambda-\mu}(b\sqrt{-t}) \, a_{\lambda\mu}(s,b)$$

$$0 \leq \sqrt{-t} \leq 2p \qquad (5\text{-}17)$$

(p : magnitude of center-of-mass momentum), derived on the basis of

$$A_{\lambda\mu}(s,t) \sim \left(\frac{1+z}{2}\right)^{\frac{\lambda+\mu}{2}} , \qquad \text{for } z \sim -1$$

$$(5\text{-}18)$$

The function $a_{\lambda\mu}(s,b)$ is the spectral function of the representation.

The spectral function is expressed in terms of the partial wave amplitudes as follows:

$$a_{\lambda\mu}(s,b) = \sum_{j} (2j+1) \, C_{\lambda\mu}^{j}(p) \, Z_{\lambda\mu}^{j}(2pb) \, a_{\lambda\mu}^{j}(s)$$

$$(5\text{-}19)$$

where

$$C_{\lambda\mu}^{j}(p) = 2p^{\lambda+\mu+1} \left\{ \frac{\Gamma(j-\lambda+1)\,\Gamma(j-\mu+1)}{\Gamma(j+\lambda+1)\,\Gamma(j+\mu+1)} \right\}^{\frac{1}{2}}$$

$$(5\text{-}20)$$

$$Z_{\lambda\mu}^{j}(2pb) = b^{\lambda+\mu-1} \, J_{2j+1}(2pb)$$

$$+ (-1)^{j-\lambda} \sum_{m=0}^{\lambda+\mu-1} \frac{1}{m!} (j+\lambda+1-m)_m$$

$$\times (j-\lambda+1)_m \, J_{2\lambda-m-1}(2pb)\left(\frac{2}{2pb}\right)$$

$$(5\text{-}21)$$

$$(\alpha)_m = \Gamma(\alpha+m)/\Gamma(\alpha)$$

$$(5\text{-}22)$$

Equation (5-19) is obtained from equation (5-15) by use of: (28)

$$\left(\frac{1+z}{2}\right)^{\frac{\lambda+\mu}{2}} C_{\lambda\mu}^{j}(p) \int_{0}^{\infty} b\,db \; J_{\lambda-\mu}(b\sqrt{-t}) \, Z_{\lambda\mu}^{j}(2pb)$$

$$(5\text{-}23)$$

$$= \begin{cases} d_{\lambda\mu}^{j}(z) & 0 < \sqrt{-t} < 2q \\ 0 & \sqrt{-t} > 2q \end{cases}$$

which itself derives from

$$\int_{0}^{\infty} x\,dx \; J_{\lambda-\mu}(xy) \, Z_{\lambda\mu}^{j}(xa)$$

$$(5\text{-}24)$$

$$= \begin{cases} \dfrac{2^{\lambda+\mu} \, y^{\lambda-\mu} \, \Gamma(j+\lambda+1)}{b^{2\lambda+1} \, \Gamma(j-\lambda+1)\,\Gamma(1+\lambda-\mu)} \, F\!\left(j+\lambda+1,\,-j+\lambda\,;\,1+\lambda-\mu\,;\,\dfrac{y^2}{a^2}\right) & b > a \\ \\ 0 & b < a \end{cases}$$

For pion-nucleon (πN) scattering which is spin 0 –

spin 1/2 scattering the representation becomes

$$A_{++}(s,t) = \left(\frac{1+z}{2}\right)^{\frac{1}{2}} \int_0^\infty b\,db\, J_0\left(b\sqrt{-t}\right) a_{++}(s,b) \qquad (5\text{-}25)$$

$$A_{+-}(s,t) = \int_0^\infty b\,db\, J_1\left(b\sqrt{-t}\right) a_{+-}(s,b) \qquad (5\text{-}26)$$

where

$$a_{++}(s,b) = \sum 4p^2\left(J_{2j+1}(2pb) + (-1)^{j-\frac{1}{2}} J_0(2pb)\right) a_{++}^j(s) \qquad (5\text{-}27)$$

$$a_{+-}(s,b) = \sum (2j+1)\frac{2p}{b} J_{2j+1}(2pb)\, a_{+-}^j(s) \qquad (5\text{-}28)$$

$$a_{\lambda\mu}(s,b) = \int_0^{2p} \sqrt{-t}\, d\sqrt{-t}\, \left(\frac{1+z}{2}\right)^{-\frac{\lambda+\mu}{2}} A_{\lambda\mu}(s,t)\, J_{\lambda-\mu}\left(b\sqrt{-t}\right) \qquad (5\text{-}29)$$

The importance of this representation derives from the following. The representation satisfies unitarity, the analyticity properties appear easy to impose and finally, significant for our purposes, the diffraction character of the scattering amplitude at high energy is a function of the behavior of the spectral function a (s,b) for small impact parameter, b. This may be seen with an example. If one assumes the behavior $\frac{d\sigma}{dt} = e^{at}$ for the differential cross section and correspondingly that the scattering amplitude $A(s,t) \sim e^{at/2}$, one finds a(s,b) $\sim \frac{e^{-b^2/2a}}{p}$. So the behavior of a(s,b) as a function of b determines the extent to which the differential cross section may be written in the usual diffractive way, e^{at} .

VI.   COMPOSITE-PARTICLE SCATTERING AND THREE BODY UNITARITY

Beginning with the Bethe-Salpeter equation for the three particle Green's function, Freedman, Lovelace and Namyslowski[31] derived a set of equations describing composite-particle sys-tem by assuming that in the interaction of three particles, any two-particle subsystem is dominated by bound states and resonances. Aaron, Amado, and Young[101] in a similar fashion derived a set of linear relativistic three-body equations for the scattering of a particle from a bound state or correlated pair of others. Both Freedman, Lovelace and Namyslowski and Aaron, Amado and Young combined the isobar idea with two- and three-body unitarity as suggested by Blankenbecler and Sugar.[14] The resultant equations obey two- and three-body unitarity exactly in the interaction channel. The basic mechanisms for this lie in the potential term (which is chosen to have a par-ticular three-particle cut) and in the composite particle pro-pagator (which is chosen to have the appropriate two-particle and three-particle cuts). Let us now discuss this in more detail.

A.   Two and Three-Body Unitarity---Structure of the Potential

and the Propagator

The composite-particle scattering equations have the form (fig. (6-1))

figure (6-1)

$$T(s) = B(s) + T(s) \, \hat{c}(s) \, B(s) \qquad (6-1)$$
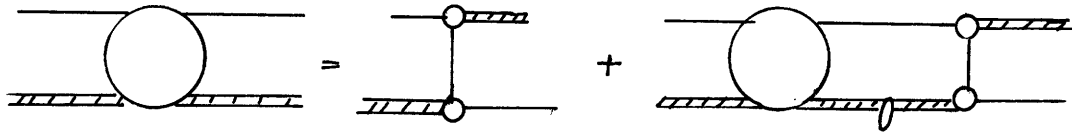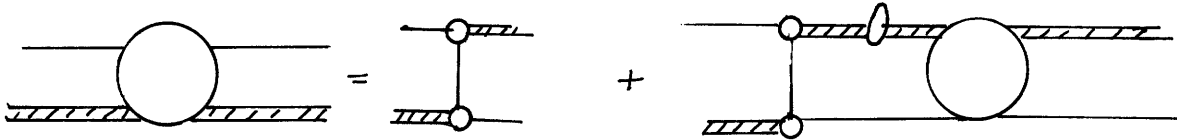
$$T(s) = B(s) + B(s) \, \hat{c}(s) \, T(s) \qquad (6-2)$$

where B(s) is the particle exchange graph, $\hat{c}(s)$ is the composite-particle propagator and T(s) is the amplitude describing the composite-particle scattering. The variable s is the square of the center-of-mass energy in the reaction channel. We can write

$$B(s^-) = B(s^-) - B(s^+) + T(s^+) - B(s^+) \, \hat{c}(s^+) \, T(s^+) \qquad (6-3)$$

$$B(s^+) = B(s^+) - B(s^-) + T(s^-) - B(s^-) \, \hat{c}(s^-) \, T(s^-) \qquad (6-4)$$

Using equations (6-1) and (6-4) we may write

$$T(s^-) = B(s^-) + \left[B(s^-) - B(s^+) \right] \hat{c}(s^-) \, T(s^-) \qquad (6-5)$$

$$+ T(s^+) \, \hat{c}(s^-) \, T(s^-) - T(s^+) \, \hat{c}(s^+) \, B(s^+) \, \hat{c}(s^-) \, T(s^-)$$

From equations (6-2) and (6-3) we may write

42.

$$T(s^-) = B(s^-) + \left[B(s^-) - B(s^+) \right] \tau(s^-) T(s^-) \tag{6-6}$$

$$+ T(s^+) \tau(s^-) T(s^-) - T(s^+) \tau(s^+) B(s^+) \tau(s^-) T(s^-)$$

We then find
$$T(s^+) - T(s^-) = T(s^+) \left\{ \tau(s^+) - \tau(s^-) \right\} T(s^-) \tag{6-7}$$

$$+ \left\{ I + T(s^+) \tau(s^+) \right\} \left\{ B(s^+) - B(s^-) \right\} \left\{ I + \tau(s^-) T(s^-) \right\}$$

Now terms like

$$\left[B(s^+) - B(s^-) \right] + \left[B(s^+) - B(s^-) \right] \tau(s^-) T(s^-)$$

$$+ T(s^+) \tau(s^+) \left[B(s^+) - B(s^-) \right]$$

correspond to cutting external lines. Since the external

lines are on the mass shell, they vanish. Therefore we obtain

$$T(s^+) - T(s^-) = T(s^+) \left[\tau(s^+) - \tau(s^-) \right] T(s^-) \tag{6-8}$$

$$+ T(s^+) \tau(s^+) \left[B(s^+) - B(s^-) \right] \tau(s^-) T(s^-)$$

The composite-particle scattering equations (with momentum

labels as in fig. (6-2)) are
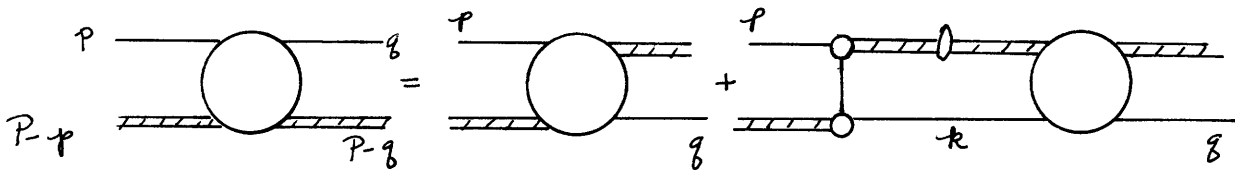
figure (6-2)

$$\langle p \mid T(s) \mid q \rangle = \langle p \mid B(s) \mid q \rangle + \frac{1}{(2\pi)^4} \int d^4k \langle p \mid B(s) \mid k \rangle$$

$$\times \tau(\sigma_k) \langle k \mid T(s) \mid q \rangle \tag{6-9}$$

with

$$\sigma_k = (P - k)^2$$

The discontinuity of T satisfies the relation (fig. (6-3)):
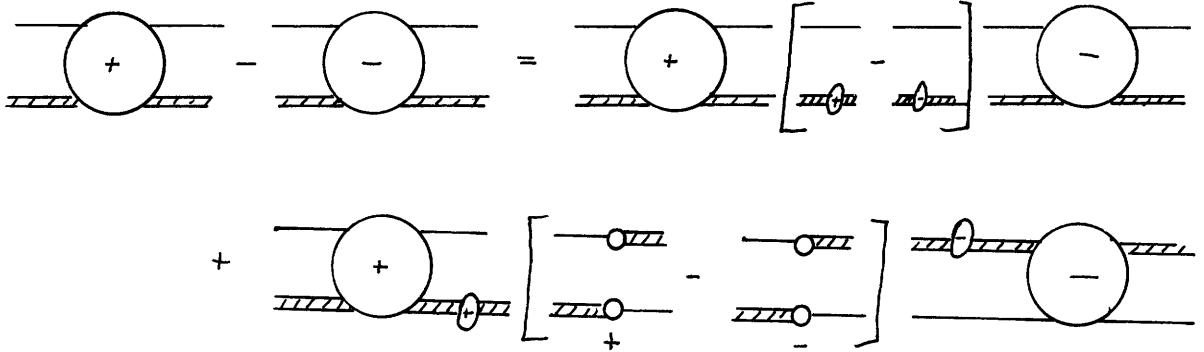
$$\langle p | T(s^+) | q \rangle - \langle p | T(s^-) | q \rangle$$

$$= (\tfrac{1}{2\pi})^4 \int d^4 k \, \langle p | T(s^+) | k \rangle \left[\hat{\imath}(\sigma_k^+) - \hat{\imath}(\sigma_k^-) \right] \langle k | T(s^-) | q \rangle$$

$$+ \tfrac{1}{(2\pi)^4} \int d^4 k \, d^4 k' \, \langle p | T(s^+) | k \rangle \, \hat{\imath}(\sigma_k^+) \left[\langle k | B(s^+) | k' \rangle - \langle k | B(s^-) | k' \rangle \right]$$

$$\times \, \hat{\imath}(\sigma_{k'}^-) \langle k' | T(s^-) | q \rangle \qquad . \quad (6\text{-}10)$$

We want expressions for the discontinuities in $\hat{\imath}(s)$ and $B(s)$ such that two-and three-body unitarity is satisfied. Unitarity says

$$T_{fL} - T_{fL}^+ = i \sum_n d\Omega_n \, T_{fn} \, T_{ni}^+ = i \sum_n d\Omega_n \, T_{fn}^+ \, T_{ni}$$

$$(6\text{-}11)$$

where

$$d\Omega_n = (2\pi)^4 \, \delta^{(4)} \left(P_f - \sum_{i=1}^n q_i \right) \prod_{i=1}^n \left(\frac{d^4 q_i}{(2\pi)^4} \, 2\pi \, \delta^+ (q_i^2 - m_i^2) \right)$$

$$(6\text{-}12)$$

is n-body phase space. Therefore from two-body and three-body

44.

unitarity we have (fig. (6-4))

figure (6-4)

$$\langle p | T(s^+) | q \rangle - \langle p | T(s^-) | q \rangle$$
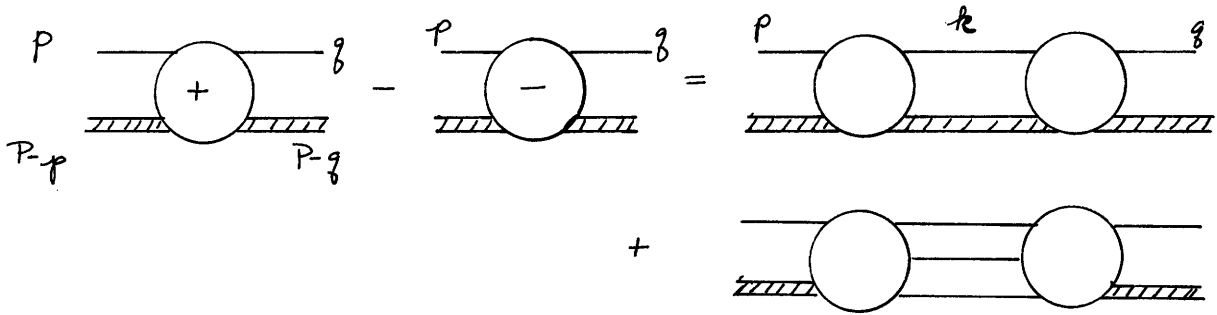
$$= i \frac{1}{(2\pi)^4} \int d^4k \; \delta^+(\sigma_k - \mu^2) \delta^+(k^2 - m^2) \langle p | T(s^+) | k \rangle \langle k | T(s^-) | q \rangle (2\pi)^2$$

$$+ i \frac{1}{(2\pi)^8} \int d^4k_1 \, d^4k_2 \, d^4k_3 \; \delta(P - k_1 - k_2 - k_3) (2\pi)^3 \delta^+(k_1^2 - m^2)$$

$$\times \delta^+(k_2^2 - m^2) \; \delta^+(k_3^2 - m^2) \langle p | T(s^+) | k_1 \, k_2 \, k_3 \rangle \langle k_1 \, k_2 \, k_3 | T(s^-) | q \rangle$$

(6-13)

The first term represents the situation where the composite
propagates as a stable particle in the intermediate state
(elastic bound state scattering); the second term represents
the situation where the composite breaks up in the intermediate
state and the same or a different composite is formed in the
final state. The term $\langle p | T(s) | k_1 \, k_2 \, k_3 \rangle$ is the
production (two-body $\longrightarrow$ three-body) amplitude. It has the
form[100] (fig. (6-5))
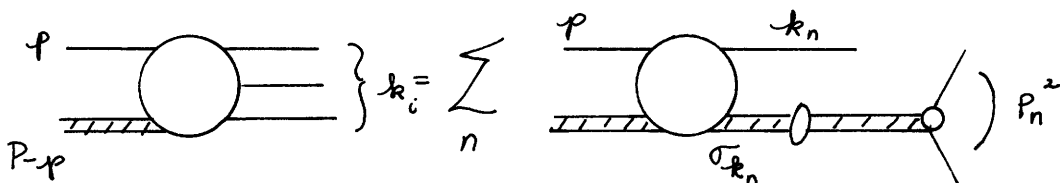
figure (6-5)

$$\langle p | T(s) | k_1, k_2, k_3 \rangle = \frac{1}{\sqrt{3!}} \sum_{n=1}^{3} \langle p | T(s) | k_n \rangle \, s(\sigma_{k_n}) \, \nu(-k_n^2)$$

$$(6-14)$$

where $p_1 = (k_2 - k_3)^2$, etc., ν is the vertex for the dissociation of the vertex, and s is a propagator which is related to t in a way to be demonstrated below. With the definition for $\langle p | T(s) | k_1, k_2, k_3 \rangle$ given in equation (6-14), equation (6-13) becomes

$$\langle p | T(s^+) | q \rangle - \langle p | T(s^-) | q \rangle$$

$$= \frac{i}{(2\pi)^2} \int d^4k \, \delta^+(\sigma_k - \mu^2) \, \delta^+(k^2 - m^2) \langle p | T(s^+) | k \rangle \langle k | T(s^-) | q \rangle$$

$$+ \frac{i}{(2\pi)^5} \int d^4k_1 d^4k_2 \, d^4k_3 \, \delta^+(P - k_1 - k_2 - k_3) \delta^+(k_1^2 - m^2) \, \delta^+(k_2^2 - m^2) \delta^+(k_3^2 - m^2)$$

$$\times \frac{1}{3!} \sum_{n,m=1}^{3} \langle p | T(s^+) | k_n \rangle s(\sigma_{k_n}^+) \nu(p_n^2) \, \nu(p_m^2) s(\sigma_{k_m}^-) \langle k_m | T(s^-) | q \rangle$$

$$(6-15)$$

The first term in equation (6-15) obviously contributes to the discontinuity of t . In the second term, m=n contributes to the discontinuity of t , it is the direct term corresponding to cutting the propagator bubble (fig. (66)).

"direct term"

figure (6-6)
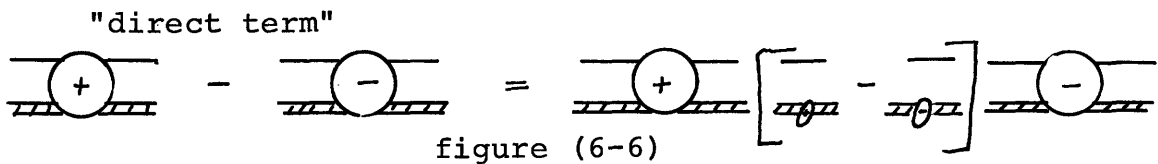
The terms m≠n correspond to the exchange of a particle between

bound states and will contribute to the discontinuity of $\mathcal{B}(s)$.

For $m \neq n$, the second term in equation (6-15) may be written:

$$\frac{i}{(2\pi)^5} \int d^4k \, d^4k' \, d^4k'' \, \delta^4(P-k-k'-k'') \, \delta^+(k^2-m^2) \, \delta^+(k'^2-m^2) \, \delta^+(k''^2-m^2)$$

$$\times \frac{1}{3!} \, i \, \langle p | T(s^+) | k \rangle \, S(\sigma_k^+) \, \mathcal{N}[(k''-k')^2] \, \mathcal{N}[(-k''-k)^2]$$

$$\times \, S(\sigma_{k'}^-) \, \langle k' | T(s) | q \rangle \tag{6-16}$$

$$= \frac{i}{(2\pi)^5} \int d^4k \, d^4k' \, \delta^+(k^2-m^2) \, \delta^+(k'^2-m^2) \, \delta^+[(P-k-k')^2-m^2]$$

$$\times \langle p | T(s^+) | k \rangle \, S(\sigma_k^+) \, \mathcal{N}[(P-k-2k')^2] \, \mathcal{N}[(P-2k-k')^2]$$

$$\times \, S(\sigma_{k'}^-) \, \langle k' | T(s) | q \rangle \tag{6-17}$$

Comparing equation (6-17) with the second term of equation (6-10), we obtain the relation

$$\hat{t}(\sigma_k^+) \left[\langle k | B(s^+) | k' \rangle - \langle k | B(s) | k' \rangle \right] \hat{t}(\sigma_{k'}^-)$$

$$= i \, \mathcal{N}[(P-k-2k')^2] \, S(\sigma_k^+) \, (2\pi)^3 \, \delta^+(k^2-m^2) \, \delta^+(k'^2-m^2) \, \delta^+[(P-k-k')^2-m^2]$$

$$\times \, S(\sigma_{k'}^-) \, \mathcal{N}[(P-2k-k')^2] \tag{6-18}$$

For m=n , writing $\rho_{12} = \frac{1}{2}(-k_1, -k_2)$,the second term in equation (6-15) becomes

$$\frac{i}{(2\pi)^5} \int d^4k \quad \frac{\delta^+(k^2-m^2)}{2(2\pi)^3} \langle p|T(s^+)|k\rangle \, S(\sigma_k^+) \, S(\sigma_k^-) \langle k|T(s^-)|g\rangle$$

$$\times \int d^4p_{12} \, (2\pi)^2 \, \mathcal{N}^2(p_{12}^2) \, \delta^+(k_1^2-m^2) \, \delta^+(k_2^2-m^2)$$

$$(6-19)$$

So from equations (6-19), (6-15), and (6-10) we obtain

$$\mathcal{T}(\sigma_k^+) - \mathcal{T}(\sigma_k^-) = i(2\pi)^2 \, \delta^+(k^2-m^2) \, \delta^+(\sigma_k - \mu^2)$$

$$+ \frac{i\,\delta^+(k^2-m^2)}{2(2\pi)^3} \, S(\sigma_k^+) \, S(\sigma_k^-) \int d^4p_{12} \, \mathcal{N}^2(p_{12}^2) \, \delta^+(p_1^2-m^2)$$

$$\times \, \delta^+(p_2^2-m^2)$$

$$(6-20)$$

Equations (6-19 and (6-20) suggest the identification

$$\mathcal{T}(\sigma_k) = (2\pi) \, \delta^+(k^2-m^2) \, S(\sigma_k)$$
$$(6-21)$$

Therefore we finally obtain

$$\langle k|B(s^+)|k'\rangle - \langle k|B(s^-)|k'\rangle$$

$$= i\,\mathcal{N}[(P-k-2k')^2] \, 2\pi \, \delta^+[(P-k-k')^2 - m^2] \, \mathcal{N}[(P-2k-k')^2]$$

$$(6-22)$$

with the constraints $k^2 = m^2$, $k'^2 = m^2$ from the δ function which has been factored off. We also get

$$S(\sigma_k^+) - S(\sigma_k^-) = 2\pi i \; \delta^+(\sigma_k - \mu^2)$$

$$+ \; \frac{S(\sigma_k^+)\,S(\sigma_k^-)}{2\,(2\pi)^4} \; i \int d^4 p_{12} \; v^2 (4 p_{12}^2)(2\pi)^2 \delta^+(p_1^2 - m^2)\,\delta^+(p_2^2 - m^2)$$

(6-23)

The potential term B is obtained from a dispersion relation in

s, assuming no cut contribution from the vertex functions.

We find in the three-body center-of-mass:

$$\langle k|\; B(s)\;|k'\rangle = \frac{v[(P - k - 2k')^2]\;\{\omega_k + \omega_{k'} + \omega_{k+k'}\}\;v[(P - 2k - k')^2]}{\omega_{k+k'}\left[(\omega_k + \omega_{k'} + \omega_{k+k'})^2 - s\right]}$$

(6-24)

where $\omega_k = (k^2 + m^2)^{\frac{1}{2}}$, it is energy dependent and com-

plex above the three-particle threshhold. The composite-par-

ticle propagator is found with greater difficulty and its cal-

culation as well as the details of the calculation of B(s) is

done in Appendix . What is important about the

composite-particle scattering equations is that they are li-

near, relativistically covariant equations which have the form

of two-body equations, but which obey two-body and three-body

unitarity exactly.

B. Form of Two-Body to Three-Body and Three-Body to Three-

Body Amplitudes

The production of two-body $\longrightarrow$ three-body amplitude

is as given in equation (6-14, namely

$$\langle p| T(s)|k_1 k_2 k_3 \rangle = \frac{1}{\sqrt{3!}} \sum_{n=1}^{3} \langle p| T(s)|k_n \rangle \, S(\sigma_{k_n}) \, \mathcal{N}(p_n^2)$$

$$(6\text{-}14)$$

This also suggests the form of the three-body to three-body amplitude, which is

$$\langle q_1 q_2 q_3 | T(s)|k_1 k_2 k_3 \rangle = \frac{1}{3!} \sum_{m,n=1}^{3} \mathcal{N}(p_m^2) \, S(\sigma_{q_m})$$

$$\times \langle q_m | T(s)|k_n \rangle \, S(\sigma_{k_n}) \, \mathcal{N}(p_n^2)$$

$$(6\text{-}25)$$

This means that if we compute the composite-particle amplitude from the theory, then given the composite-particle propagator $S(\sigma_k)$ (which we can calculate) and the vertex function $\mathcal{N}$, we can calculate two-body to three-body amplitudes and three-body to three-body amplitudes by a quadrature, i.e. simply by attaching the relevant composite-particle propagators and vertex functions.

VII. MULTIPERIPHERAL MODEL FOR COMPOSITE-PARTICLE SCATTERING

A. Composite-Particle Bethe-Salpeter Equation For Pion-Nucleon

Scattering in the t-Channel

Freedman, Lovelace, and Namyslowski[81] (FLN) and Aaron, Amado, and Young[100] (AAY) wrote down the following equation for composite-particle scattering:

$$T(s) = B(s) + B(s)\, \mathcal{T}(s)\, T(s) \qquad (7\text{-}1)$$

which graphically is (fig. (7-1)):

figure (7-1)
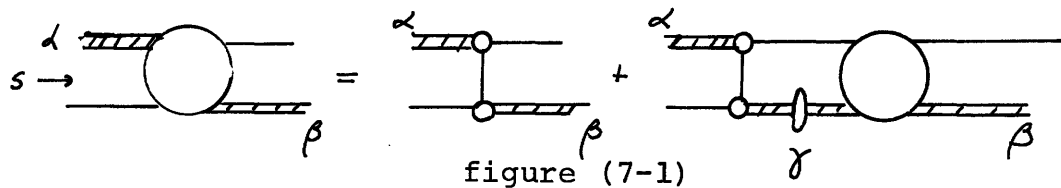
where the Born term is a one particle exchange term (a particle being exchanged between the two composites α and β).

We specialize to pion-nucleon (πN) scattering. We ask: what is the most general form for t channel scattering (where now t corresponds to s above) which gives us pion-nucleon out, and is such that when we look in the cross channel, we also get elastic pion-nucleon scattering? The equation we write down is (graphically):
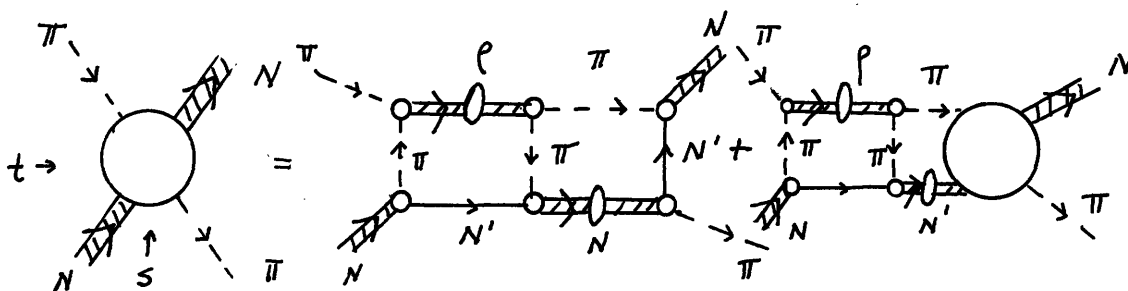
figure (7-2)

Our equation has the form:

$$T(t) = B(t) + K(t) \mathcal{L}(t) T(t) \qquad (7\text{-}2)$$

The form of our equation is obviously different than the form written down by FLN and AAY, but it is the minimal form which expresses the dynamics we feel to be important, namely we want

$\pi N \rightarrow N \pi$ in both the t and s channels. It also gives the effects of "doorway" states in the t channel as discussed in nuclear physics by Feshbach et al.[29] We believe these states to be part of the background and as such may contribute to singularities in angular momentum in t .

If we analytically continue this equation from the t channel to the s channel we obtain:
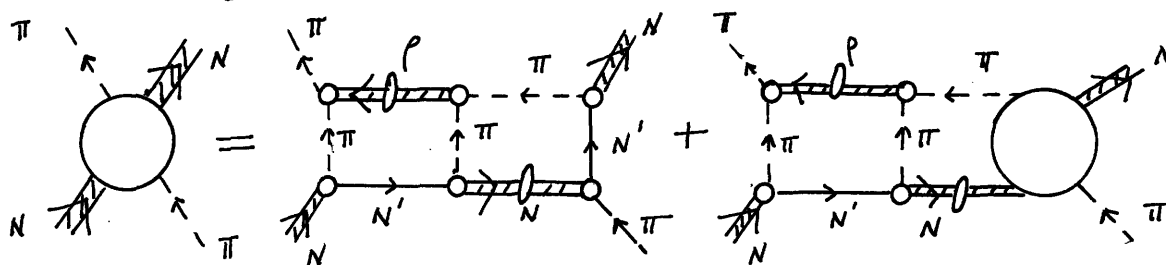
figure (7-3)

Now we suppose that in Born term the two pions (π) in the intermediate state are correlated, i.e., the rho meson (ρ) which is exchanged generates enough of a force to correlate the two pions in the form of a resonance, so that we then have

for the Born term:
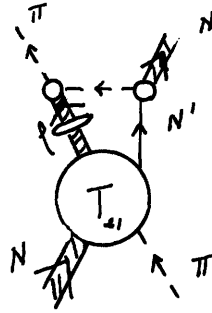
figure (7-4)

Making an assumption about resonance formation in fig.(7-4)

we have to lowest order for the Born term
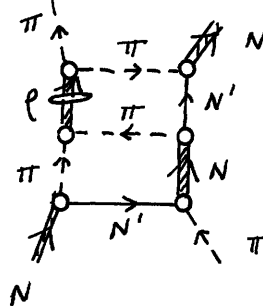
figure (7-5)

Our equation now becomes
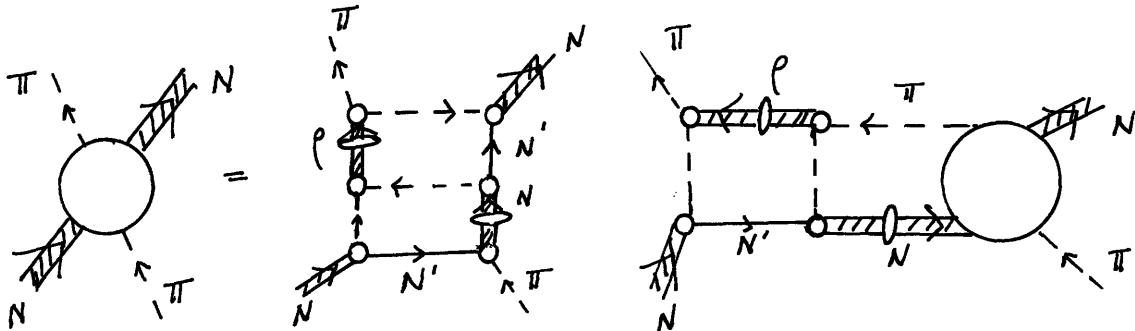
figure (7-6)

Now both our inhomogeneous term and kernel have the

intermediate state resonance structure. What we are saying

with this is that the scattering must always take place via

these "doorway" states. From nuclear physics[29] we expect

intermediate state resonances whenever there exist doorway

states whose coupling to the incident channel in the energy

domain being scrutinized is strong and whose coupling to other channels

is not. Even though we will go to high energy and these as-
sumptions may not seem valid for the inhomogeneous term, we
nevertheless retain this structure to include possible corre-
lations in S in the intermediate state. The possibility of
the exchanges in the kernel are a function of the momentum
transfer variables appropriate to the particles being exchanged.
In the multiperipheral model of Amati, Fubini, and Stang-
hellini, these momentum transfers are small. However, the
exchanges of the composites indicated are necessary given
the structure of the model. The correlations in t in the
kernel should be reflected in the angular momentum singularity
governing the high energy behavior in S .

Finally, to obtain a "multiperipheral" equation, following
the analytic continuation from the t channel to the s chan-
nel, we must take the S discontinuity of the terms in the
equation, i.e. the inhomogeneous term and the kernel.

What is important about this equation is that we begin
with a model which obeys exact two-body and three-body unita-
rity in t, since our original equation was derived within
the framework of FLN and AAY for t channel scattering.
Our resultant equation is such that the inhomogeneous term
has two-body and three-body cuts with strengths given by the
FLN and AAY theory since it is constructed as a thrice itera-
ted form of the single particle exchange graph in that theory.
The kernel is more subtle. It is constructed within FLN and
AAY theory in the t channel and then analytically continued

to the s channel. This is described below and worked out
in detail in the Appendix. This is the essence of our label-
ling our model a multiperipheral model with continued cross
channel unitarity, together with a plausibility assumption
based on work by Bertocchi, Fubini, and Tonin,[104] and
Abarbanel and Goldberger[105] demonstrating the equivalence
of the homogeneous multiperipheral integral equation to a
Wick rotated Bethe-Salpeter equation for bound state
scattering.

B. Analytic Continuation of Kernel from t to s-channel

We begin by considering the term in Figure (7-7) in
the t channel.
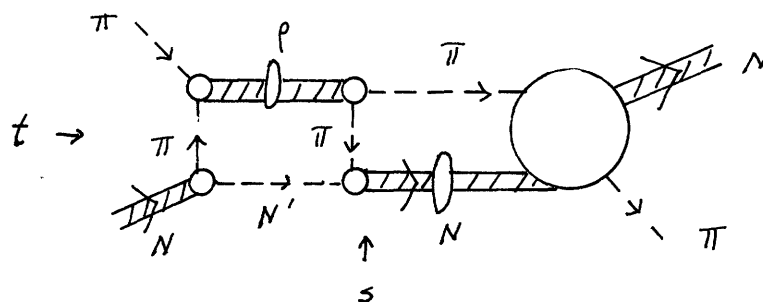
figure (7-7)

First we want to compute the absorptive part of the rho (ρ)
box after in it analytically continued from the t to the
s channel. Then we attach it to the absorptive part of the
amplitude itself as indicated to obtain the kernel term in
our integral equation. If we consider the rho box (fig. 7-8)
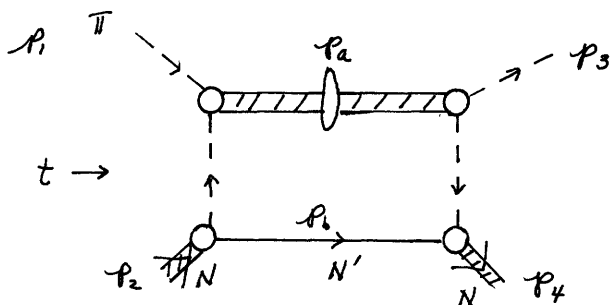
figure (7-8)

in the t channel with the momenta as indicated, it has spin
factors (for nucleon spins).

$$\bar{u}\,(p_4)\,\gamma_5\,\frac{p\!\!\!/_b + M'}{D(p_b^2)}\,\gamma_5\,u(p_2) \qquad\qquad (7\text{-}3)$$

$$= \bar{u}(p_4)\; -\;\frac{p\!\!\!/_b + M'}{D(p_b^2)}\;u(p_2) \qquad\qquad (7\text{-}4)$$

Using the property that the Dirac Gamma matrix γ_5 anticommutes with all other Dirac Gamma matrices. The quantity standing between the spinors is an invariant. Since what we wish to focus on are the invariant amplitudes A and B (in particular A) where the invariant pion-nucleon amplitude F is

$$F = -A + \frac{p\!\!\!/_1 + p\!\!\!/_2}{2}\,B \qquad\qquad (7\text{-}5)$$

If we take the trace of the expression between the spinors in equation (7-4) we obtain

$$\frac{M'}{D(p_b^2)}$$

(7-6)

This represents for us the contribution of the rho box to the invariant amplitude A. We will study the high energy behavior of A emerging from this, bearing in mind that we may determine an expression for the contribution to the B amplitude by multiplying by an appropriate variable and again taking the trace. We also have the rho meson spin to take account of. This will be treated below. The contribution of

the rho box in the t channel is then:

$$\left(\frac{1}{2\pi}\right)^3 \int d^4q \; \frac{\omega_{p+q} + \omega_p + E_q}{\omega_{p+q}\left[(\omega_{p+q} + \omega_p + E_q)^2 - t\right]} \; M' \; \frac{P_\mu\left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right]P_\nu'}{\left[\sigma_q - m_\rho^2 + iF(\sigma_q)\right]} \; \delta(q^2 - M'^2)$$

$$\times \; \frac{\omega_{p'+q} + \omega_{p'} + E_q}{\omega_{p'+q}\left[(\omega_{p'+q} + \omega_{p'} + E_q)^2 - t\right]}$$

(7-7)

where

$$\underset{\sim}{p} = \underset{\sim}{p_1} = -\underset{\sim}{p_2} \qquad \qquad \text{in } t \text{ center of mass}$$

$$\underset{\sim}{q} = \underset{\sim}{p_b} = -\underset{\sim}{p_a} \qquad \qquad (7\text{-}8)$$

$$\underset{\sim}{p'} = \underset{\sim}{p_3} = -\underset{\sim}{p_4}$$

We now continue this expression to the s channel (figure 7-9)
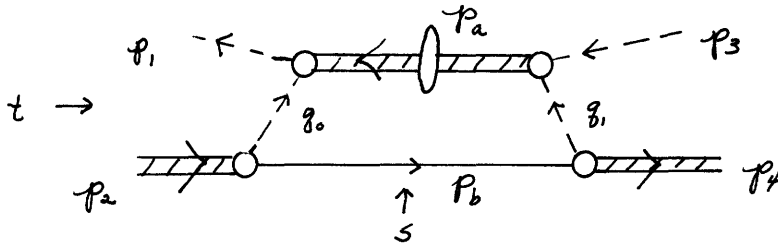
figure (7-9)

First we note that the following

$$\frac{\omega_{p+q} + \omega_p + E_q}{\omega_{p+q} \left[(\omega_{p+q} + \omega_p + E_q)^2 - t\right]} = \int_{t_{Th}}^{\infty} dt' \; \frac{\delta\left[(P' - p - q)^2 - \mu^2\right]}{t' - t}$$

(7-9)

i.e. the one particle exchange term is obtained from a dispersion integral in t of the discontinuity of the one particle exchange graph as given in equation (6-22). When we continue to the s-channel center of mass, the delta function becomes $\delta\left(q_o^2 - \mu^2\right)$ and we have

$$\int_{t_{Th}}^{\infty} dt' \; \frac{\delta\left[(P' - p - q)^2 - \mu^2\right]}{t' - t} \longrightarrow \delta(q_o^2 - \mu^2) \int_{t_{min}}^{t_{max}} \frac{dt'}{t' - t}$$

(7-10)

Similarly

$$\frac{\omega_{p'+q} + \omega_{p'} + E_q}{\omega_{p'+q} \left[(\omega_{p'+q} + \omega_{p'} + E_q)^2 - t\right]} = \int_{t_{Th}}^{\infty} dt'' \; \frac{\delta\left[(P' - p' - q)^2 - \mu^2\right]}{t'' - t}$$

(7-11)

continues to

$$\delta\left(q_1^2 - \mu^2\right) \int_{t_{min}}^{t_{max}} \frac{dt''}{t'' - t}$$

(7-12)

The justification for taking the delta functions outside the integrals is that, when continued, the delta functions are

non-vanishing only for the arguments of the delta function

giving the correct mass shell conditions for the intermediate

state pions. When we perform the t' and t'' integrals with

upper and lower limits 0 and $-\left[s - (2M^2 + m^2)\right]$, we obtain

for s large:

$$\left[\log \frac{(s)}{(-t)}\right]^2 \tag{7-13}$$

We take s large because we are defining an asymptotic kernel.

Let us define the variables

$$Q = \frac{p_1 + p_1'}{2} \qquad Q' = \frac{p_a + p_b}{2} \qquad N = \frac{p_2 + p_2'}{2}$$

$$\Delta = \frac{p_1 - p_1'}{2} \quad , \quad etc. \tag{7-14}$$

Then we have for the rho box

$$\frac{\log (s)/(-t)}{(2\pi)^3} \int d^4 Q' M' \frac{\delta(q_0^2 - \mu^2) \delta(q_1^2 - \mu^2)}{D(p_a^2) \; D(p_b^2)} P'_{1\mu} \left(q^{\mu\nu} - \frac{p_a^\mu p_a^\nu}{p_a^2}\right) P'_{2\nu} \tag{7-15}$$

where $\dfrac{1}{D(p_b^2)}$ is the analytic continuation of $\delta(q^2 - M'^2)$

from the t channel $c.m.$ to the s $c.m.$. Consider the rho

meson spin factors:

$$P'_{1\mu} \left(g^{\mu\nu} - \frac{P_a^\mu P_a^\nu}{P_a^2} \right) P'_{2\nu} \qquad (7\text{-}16)$$

$$= (P'_1 \cdot P'_2) - \frac{(P'_1 \cdot P_a)(P'_2 \cdot P_a)}{P_a^2} \qquad (7\text{-}17)$$

Now:

$$(P'_1 - P_a)^2 = q_0^2 = \mu^2$$

from mass shell delta function

$$\mu^2 + P_a^2 - 2P'_1 \cdot P_a = \mu^2 \qquad (7\text{-}18)$$

$$P'_1 \cdot P_a = \tfrac{1}{2} P_a^2$$

$$(P_2 + P_a)^2 = q_1^2 = \mu^2$$

$$\mu^2 + P_a^2 + 2P'_2 \cdot P_a = \mu^2 \qquad (7\text{-}19)$$

$$2P'_2 \cdot P_a = -P_a^2$$

$$P'_2 \cdot P_a = -\frac{P_a^2}{2}$$

$$(P'_1 + P'_2)^2 = s$$

$$\mu^2 + \mu^2 + 2P'_1 \cdot P'_2 = s$$

$$P'_1 \cdot P'_2 = \tfrac{1}{2} \left\{ s - 2\mu^2 \right\} \qquad (7\text{-}20)$$

The factor P_a^2 is a momentum transfer and is assumed small relative to the total center of mass energy squared, $(P'_1 + P'_2)^2$. So we find

$$P'_{1\mu}\left(g^{\mu\nu} - \frac{p_a^\mu p_a^\nu}{p_a^2}\right)P'_{2\nu} = \frac{1}{2}\left\{(p' + p_2')^2 - 2\mu^2\right\} + \frac{p_a^2}{4}$$

$$\sim \frac{(p_1' + p_2')^2}{2} = \frac{s}{2}$$

(7-21)

$\overline{D}'(p_b^2)$ is the analytic continuation of $\delta(g^2 - M'^2)$ from the t-channel c.m. to the s-channel c.m. Therefore it becomes

$$\delta(p_b^2 - M'^2) \longrightarrow \delta(p_b^2 + u_b) \qquad \text{where} \qquad u_b = -M'^2$$

(7-22)

where now M'^2 is negative and can assume a range of values which we sum over with a suitable weight function. We choose to sum over the masses with a weight function $\rho(u_b)$ which satisfies $\int \rho(u_b)\,du_b = 1$. The form for $\rho(u_b)$ is:

$$\rho(u_b) = \frac{\gamma^2}{(\gamma^2 + u_b)^2}$$

(7-23)

γ^2 being a cut-off parameter.

Therefore we finally get

$$\frac{1}{(2\pi)^3}\int du_b \frac{\gamma^2}{(\gamma^2 + u_b)^2} M' \frac{s}{2}\left(\log\left(\frac{s}{-t}\right)\right)^2$$

$$\times \int d^4 Q \frac{\delta(q_0^2 - \mu^2)\,\delta(q_1^2 - \mu^2)\,\delta(p_b^2 + u_b)}{D(p_a^2)}$$

(7-24)

Going to invariant variables we find this expression equal
to

$$(7\text{-}25)$$

$$\left(\log\left(\tfrac{S}{-t}\right)\right)^2 \frac{S}{2} \int du_b \frac{\gamma^2}{(u_b+\gamma^2)^2} M' \int du_a \frac{1}{u_a + m_\rho^2 + i\, g^2\left(\frac{4\mu^2+u_a}{u_a}\right)^{1/2}} \frac{\theta(J)}{\sqrt{J}}$$

where $\qquad J = s^2 H$

$$J = \int d^4 Q' \; \delta\left[(Q-Q')^2 - \mu^2\right] \delta\left[(Q'+\Delta)^2 + u_b\right]$$

$$\times \delta\left[(Q'-\Delta)^2 + u_a\right] \delta\left[(Q'+N)^2 - \mu^2\right]$$

$$(7\text{-}26)$$

and the expression for the rho (ρ) propagator is taken
from Appendix I .

So we get

$$\frac{\left(\log\frac{S}{-t}\right)^2}{2} \int du_b \frac{\gamma^2}{(u_b+\gamma^2)^2} M' \int du_a \frac{1}{u_a + m_\rho^2 + i\, g^2\left(\frac{4\mu^2+u_a}{u_a}\right)^{1/2}} \frac{\theta(H)}{\sqrt{H}}$$

$$(7\text{-}27)$$

Taking the discontinuity of this with respect to S , we
find

$$= \frac{\pi \log\left(\frac{S}{-t}\right)}{2} \int du_b \frac{\gamma^2}{(u_b+\gamma^2)^2} M'M \int du_a \frac{1}{u_a + m_\rho^2 + i\, g^2\left(\frac{4\mu^2+u_a}{u_a}\right)^{1/2}} \frac{\theta(H)}{\sqrt{H}}$$

$$(7\text{-}28)$$

The Jacobian is:

$$H = \left[- \left(\frac{u_b - u_a}{2} - \frac{u_1 - u_2}{2} \chi \right)^2 \right.$$

(7-29)

$$\left. - t(1-\chi) \left(\frac{(u_b + u_a) - (u_1 + u_2)\chi}{2} - \frac{s_0 \chi}{1-\chi} + \frac{t}{4}(1-\chi) \right) \right]$$

where $\qquad u_1 = - M^2 \qquad , \qquad u_2 = -\mu^2 \qquad , \qquad \chi = \frac{\mu^2}{s} \qquad$ (8-30)

This may be rewritten:

$$H(\zeta, \zeta_1, \zeta_2) = \frac{1}{4} \left(-\zeta^2 - \zeta_1^2 - \zeta_2^2 + 2\zeta\zeta_1 + 2\zeta\zeta_2 + 2\zeta_1\zeta_2 \right)$$

with

(7-31)

$$\zeta = - t(1-\chi)$$

$$\zeta_1 = u_b = u_1 \chi - \frac{\mu^2 \chi}{1-\chi}$$

(7-32)

$$\zeta_2 = u_a - u_2 \chi - \frac{\mu^2 \chi}{1-\chi}$$

For large s

$$\chi = \frac{\mu^2}{s} \longrightarrow 0$$

(7-33)

and

$$\zeta = -t \qquad\qquad -t > 0 \qquad\qquad \zeta > 0$$

$$\zeta_1 = u_b \qquad\qquad u_b > 0 \qquad\qquad \zeta_1 > 0$$

$$\zeta_2 = u_a \qquad\qquad u_a > 0 \qquad\qquad \zeta_2 > 0 \qquad (7\text{-}34)$$

Let us consider H as a function of ζ_1 :

(7-35)

$$H = \frac{1}{4} \left(-\zeta_1^2 + 2(\zeta + \zeta_2)\zeta_1 - \zeta^2 - \zeta_2^2 + 2\zeta\zeta_2 \right)$$

If $\zeta_1 = \chi$, then it can be written

$$H(x) = -\alpha x^2 + 2\beta x - \alpha' \qquad \text{(a parabola)} \qquad (7\text{-}36)$$

where $\qquad \alpha = 1 \quad, \quad \beta = (\zeta + \zeta_2) > 0 \quad, \quad \alpha' = (\zeta - \zeta_2)^2 > 0$

$$H'(x) = 0 = -2\alpha x + 2\beta$$

So the parabola is centered at $\quad x = \beta/\alpha = (\zeta + \zeta_2) > 0$

and $H > 0$ at this point as can be seen if we substitute in this value of x.

$$H''(x) = -2\alpha < 0$$

Therefore the parabola is concave down. The roots of the equation (7-36) are given by

$$x = \beta \mp \sqrt{\beta^2 - (\zeta - \zeta_2)^2}$$

$$\zeta_1^- = x_- = (\zeta + \zeta_2) - 2\sqrt{\zeta \zeta_2} = \left(\sqrt{\zeta} - \sqrt{\zeta_2}\right)^2 > 0$$

$$(7\text{-}37)$$

$$\zeta_1^+ = x_+ = (\zeta + \zeta_2) + 2\sqrt{\zeta \zeta_2} = \left(\sqrt{\zeta} + \sqrt{\zeta_2}\right)^2 > 0$$

$$(7\text{-}38)$$

The θ function in equation (7-25) says we must integrate in a region where H is positive. Given that H as a function of ζ is concave down, we must integrate in the shaded region in figure (7-10) but such that we're in the allowed range for the variable u_b $(0 \to \infty)$
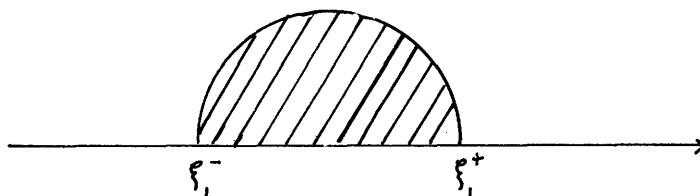
figure (7-10)

Since both roots are positive, we integrate from $\zeta_1^- \rightarrow \zeta_1^+$. Thus we have

$$\frac{\pi}{2} M M' \log S_0 \int d u_a \quad \frac{1}{u_a + m_\rho^2 + i \, g^2 \left(\frac{4 \mu^2 + u_a}{u_a}\right)^{\frac{1}{2}}}$$

$$\times \int_{\zeta_1^-}^{\zeta_1^+} d \zeta_1 \frac{\gamma^2}{(\zeta_1 + \gamma^2)^2} \quad \frac{1}{\left[-(\zeta_1 - \zeta_1^+)(\zeta_1 - \zeta_1^-)\right]^{1/2}}$$

$$(7\text{-}39)$$

$$= M \frac{\pi}{2} M' \log S \int d u_a \quad \frac{1}{u_a + m_\rho^2 + i \, g^2 \left(\frac{4 \mu^2 + u_a}{u_a}\right)^{1/2}}$$

$$\times \int_{\zeta_1^-}^{\zeta_1^+} d \zeta_1 \frac{\gamma^2}{(\zeta_1 + \gamma^2)^2} \quad \frac{1}{\left[(\zeta_1^+ - \zeta_1)(\zeta_1 - \zeta_1^-)\right]^{1/2}}$$

$$(7\text{-}40)$$

Consider the last integral and write:

$$\zeta_1 - \zeta_1^+ = \zeta_1 - \beta - \Delta = \chi - \Delta$$

$$(7\text{-}41)$$

$$\zeta_1 - \zeta_1^- = \zeta_1 - \beta + \Delta = \chi + \Delta$$

Then,
$$\zeta_1 = \chi + \beta$$
$$(7\text{-}42)$$

and
$$\left(\zeta_1 + \gamma^2 \right) = \left(x + \beta + \gamma^2 \right)^2 = \left(x + \eta \right)^2$$

(7-43)

So we have:

$$\int_{\zeta_1^-}^{\zeta_1^+} d\zeta_1 \frac{\gamma^2}{(\zeta_1 + \gamma^2)^2} \frac{1}{\left[(\zeta_1^+ - \zeta_1)(\zeta_1 - \zeta_1^-) \right]^{1/2}} = \int_{-\Delta}^{\Delta} dx \frac{\gamma^2}{(x + \eta)^2} \frac{1}{\left[(\Delta - x)(x + \Delta) \right]^{1/2}}$$

(7-44)

Let $\quad y = \frac{x}{\Delta} \quad , \quad x = \Delta y \; .$

We then obtain

$$\int_{-\Delta}^{\Delta} \frac{dx \; \gamma^2}{\Delta (x + \eta)^2 \left[(1-y)(1+y) \right]^{1/2}} = \gamma^2 \int_{-1}^{1} \frac{dy}{(\Delta y + \eta)^2 \left[(1-y)(1+y) \right]^{1/2}}$$

(7-45)

$$= \frac{\gamma^2}{\Delta^2} \int_{-1}^{1} \frac{dy}{\left(y + \frac{\eta}{\Delta} \right)^2 \left[(1-y)(1+y) \right]^{1/2}} = \frac{\gamma^2}{\Delta^2} \int_{-1}^{1} \frac{dy}{(y + \lambda)^2 \left[(1-y)(1+y) \right]^{1/2}}$$

(7-46)

$$= \frac{\gamma^2}{\eta^2} \int_{-1}^{1} \frac{dy}{\left(1 + \frac{y}{\lambda} \right)^2 \left[(1-y)(1+y) \right]^{1/2}}$$

(7-47)

$$\approx \frac{\gamma^2}{\eta^2} \int_{-1}^{1} \frac{1 - \frac{2y}{\lambda} \; dy}{\left[(1-y)(1+y) \right]^{1/2}}$$

(7-48)

if $\frac{y}{\lambda}$ small, for the integration region.

This can be so arranged with the parameter, γ^2.

Only the term with the constant numerator survives.

$$\int_{-1}^{1} \frac{d\gamma}{[(1-\gamma)(1+\gamma)]^{1/2}} = 2\int_{0}^{1} \frac{d\gamma}{[(1-\gamma)(1+\gamma)]^{1/2}}$$

(7-49)

$$= 2\int_{0}^{1} \frac{d\gamma}{\sqrt{(1-\gamma)(1-\rho^2\gamma)}} \qquad \rho^2 = -1 \;,\; \rho = i$$

(7-50)

$$= \frac{2}{\rho} Q_0\left[\frac{1}{2}\frac{(\rho^2+1)}{\rho}\right] = \pi$$

(7-51)

So our original integral (7-28) now becomes

$$M\frac{\pi}{2}M'\log S \int d u_a \frac{1}{u_a + m_\rho^2 + i g^2\left(\frac{4\mu^2+u_a}{u_a}\right)^{1/2}} \frac{\pi\gamma^2}{(-t+u_a+\gamma^2)^2}$$

(7-52)

Write $(-t+u_a+\gamma^2) = (u_a+b)^2$, so we have

$$M\frac{\pi^2\gamma^2}{2}M'\log S \int_{0}^{\infty} d u_a \frac{1}{(u_a+b)^2\left[u_a+m_\rho^2+i g^2\left(\frac{4\mu^2+u_a}{u_a}\right)^{1/2}\right]}$$

Now

(7-53)

$$\frac{1}{u_a + m_\rho^2 + i\, q^2 \left(\frac{4\mu^2 + u_a}{u_a}\right)^{1/2}} = \frac{u_a + m_\rho^2}{\left(u_a + m_\rho^2\right)^2 + q^4 \left(\frac{4\mu^2 + u_a}{u_a}\right)}$$

$$+ \frac{i\, q^2 \left(\frac{4\mu^2 + u_a}{u_a}\right)^{1/2}}{\left(u_a + m_\rho^2\right)^2 + q^4 \left(\frac{4\mu^2 + u_a}{u_a}\right)}$$

$$(7\text{-}54)$$

The coupling constant q^2 measures the probability of a rho
being formed from or dissociating into two pions. In this
sense it measures the coupling of the ρN channel to the πN
channel, because it gives the breakup and dissociation of
the rho meson which must occur if these channels are to be
coupled. A similar comment could be made about the coupling
constant in the composite nucleon propagator $(A1\text{-}27)$, however
we treat the nucleon without this coupling because of the
form of the equations, i.e. in adjoining this rho box to
the absorptive part in our equation, the absorptive part in
the integral equation must have a stable incoming nucleon
leg to correspond to the stable nucleon leg in the full
absorptive amplitude. This is what yields for us an integral
equation. We don't keep the imaginary part since we want a positive
definite absoprtive part.

C. Weak Coupling

To treat the weak coupling limit we set $q^2 = 0$.
Therefore we must integrate

$$\int_0^\infty du_a \; \frac{1}{(u_a + b)^2 (u_a + m_\rho^2)} = \int_0^\infty dx \; \frac{1}{(x+b)^2 (x + a)}$$

$$= \frac{1}{a^2} \int_0^\infty \frac{dy}{\left(y + \frac{b}{a}\right)^2 (1+y)} \qquad \begin{array}{l} a = m_\rho^2 \qquad b = \gamma^2 - t \\[2mm] y = \frac{x}{a} \end{array}$$

$$= \frac{1}{a^2} \; B(2,1) \; {}_2F_1\left(2, 2 \; ; \; 3 \; ; \; 1 - \frac{b}{a}\right)$$

$$= \frac{1}{2} \; \frac{1}{(\gamma^2 - t)^2} \qquad \text{using} \qquad F(2, 2, 3, z) = \frac{1}{z^2} \qquad (7 \cdot 55)$$

Therefore the box becomes

$$M \frac{\pi^2 \gamma^2 M'}{4} \log s \; \frac{1}{(\gamma^2 - t)^2} \qquad (7 \cdot 56)$$

Amati, Fubini, Stanghellini[102]; Bertocchi, Fubini, Tonin[103]; and Abarbanel, Saxton and Treiman[105] all noted the equivalence between the asymptotic diffraction multiperipheral equation and the Bethe-Salpeter equation (properly analytically continued). In our model the equivalence between the composite particle Bethe-Salpeter equation for t channel scattering (figure 7-2) and our multiperipheral absorptive part equation for s channel scattering (figure 7-3) is explicit because we have explicitly done the analytic continuation from the t channel to the s channel.

D. Strong Coupling Kernel (Box)

We now calculate the contribution of the rho (ρ) box for strong coupling. Consider the real part. For the real part we must compute

$$\int_0^\infty \frac{d u_a \; u_a (u_a + m_\rho^2)}{(u_a + b)^2 \left[u_a (u_a + m_\rho^2)^2 + g^4 (4\mu^2 + u_a) \right]} \qquad (7\text{-}57)$$

$$= \int_0^\infty \frac{dx \quad x (x+a)}{(x+b)^2 \left[x(x+a)^2 + d(x+c) \right]} \qquad (7\text{-}58)$$

where

$$a = m_\rho^2 \quad , \quad b = \gamma^2 - t \quad , \quad c = 4\mu^2$$

If c=a, since u is far from threshhold mass, we have for equation (7-58)

$$\int_0^\infty \frac{dx \quad x}{(x+b)^2 \left[x(x+a) + d \right]} \qquad (7\text{-}59)$$

$$= \int_0^\infty \frac{x \, f(x)}{(x+b)^2} \, dx = \int_0^\infty \frac{f(x)}{(x+b)} - b \int_0^\infty \frac{f(x)}{(x+b)^2} \qquad (7\text{-}60)$$

We get overall

$$\int_0^\infty \frac{x \, dx}{(x+b)^2 \left[(x+a) + d\right]}$$

$$= \frac{1}{2\sqrt{a^2 - 4d}} \left\{ \frac{1}{(z_1 - b)} \log\left(\frac{x_1}{b}\right) - \frac{1}{(x_2 - b)} \log\left(\frac{x_2}{b}\right) \right\}$$

$$- \frac{1}{2} \left\{ \frac{b(a - 2b)}{\left[b(b-a) + d\right]^2} \right\} \tag{7-61}$$

where

$$a = m_\rho^2 \quad , \quad b = \gamma^2 t \quad , \quad c = 4\mu^2 = a \quad , \quad d = q^4$$

$$(x_1 - b) = \left(\frac{a}{2} - b\right) + \frac{\sqrt{a^2 - 4d}}{2} \tag{7-62}$$

$$(x_2 - b) = \left(\frac{a}{2} - b\right) - \frac{\sqrt{a^2 - 4d}}{2}$$

If d is large,

$$x_1 - b \longrightarrow i q^2 \qquad\qquad x_2 - b \longrightarrow -i q^2$$

$$x_2 \longrightarrow -i q^2 \qquad\qquad x_1 \longrightarrow i q^2 \tag{7-63}$$

If q^2 is large, the first term is

$$\frac{1}{4 i q^2} \left\{ \frac{1}{i q^2} \log\left(\frac{i q^2}{b}\right) + \frac{1}{i q^2} \log\left(\frac{-i q^2}{b}\right) \right\} \tag{7-64}$$

$$= \frac{1}{4iq^2} \left\{ \frac{1}{iq^2} \log\left(\frac{q^4}{b^2}\right) \right\} \rightarrow -\frac{1}{4q^4} \log\left(\frac{q^4}{b^2}\right) \qquad (7\text{-}65)$$

the second term is

$$-\frac{1}{2} \left\{ \frac{b(a-2b)}{q^8} \right\} = \frac{const.}{q^8} \qquad (7\text{-}66)$$

Overall we get

$$\frac{const.}{q^8} - \frac{1}{4q^4} \log\left(\frac{q^4}{b^2}\right) \qquad (7\text{-}67)$$

$$const. = \frac{(\gamma^2-t)(m_\rho^2 - 2(\gamma^2-t))}{} \qquad (7\text{-}68)$$

Then finally we have

$$\frac{(\gamma^2-t)(m_\rho^2) - 2(\gamma^2-t)^2}{q^8} - \frac{1}{4q^4} = const. \frac{1}{q^4} \qquad (7\text{-}69)$$

VIII. SOLUTION TO THE HOMOGENEOUS EQUATION

A. Analytic Continuation of the Kernel and Homogeneous Equation

Because the Born term dies out at high energy (see Appendix III), our multiperipheral model with continued cross-channel unitarity then becomes a homogeneous integral equation problem (figure (8-1)).
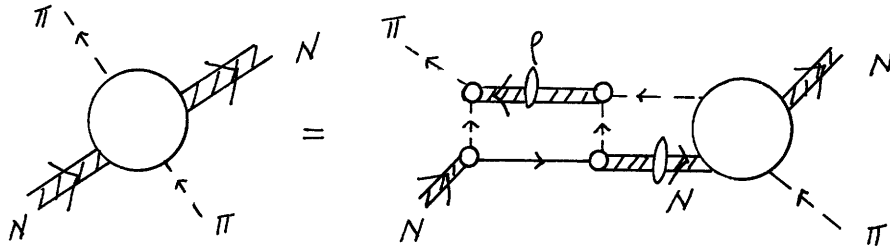
figure (8-1)

To proceed, we must express the equation in invariant variables and compute the kernel.

We have the problem of calculating figure (8-2) in invariant variables.
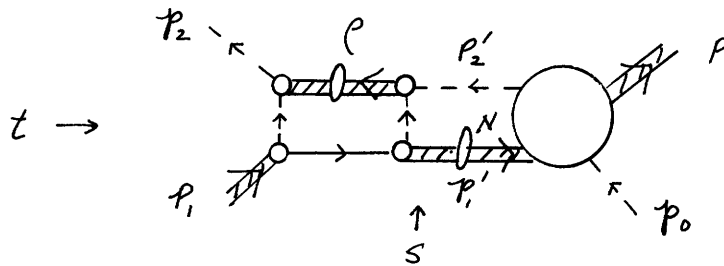
figure (8-2)

If we define

$$P = \frac{p_1 + p_2}{2} \qquad Q' = \frac{p_1' + p_2'}{2}$$

$$N = \frac{p_0 + p}{2} \qquad \Delta = \frac{p_1 - p_2}{2} \qquad , \; etc. \qquad (8\text{-}1)$$

We have for the graph in figure (8-2)

$$\frac{1}{(2\pi)^3} \int d^4 Q' \int du_2' \frac{K^2 M}{(u_2' + K^2)^2} \frac{A_\rho (p_1, p_1', \Delta)}{D(p_1'^2)} \delta(p_2'^2 + u_2')$$

$$(8\text{-}2)$$

where the $\int du_2' \frac{K^2}{(u_2' + K^2)^2}$ integral again comes from the continu-
ation of the pion mass shell delta function from the t to
the s channel and picking a weight function which itself
integrates to unity, and the factor M from again taking a
trace of the absorptive off mass shell amplitude before the
continuation is made. $D^{-1}(p_1'^2)$ is the nucleon propagator.
If we convert the integral to invariant variables using
the variables defined in equation (8-1) and the Jacobian

$$\int d^4 Q' \, \delta\left[(P - Q')^2 - S_0 \right] \, \delta\left[(Q' + \Delta)^2 + u_1' \right] \, \delta\left[(Q' - \Delta)^2 + u_2' \right]$$

$$\times \, \delta\left[(Q' + N)^2 - S' \right] = \frac{\theta(H)}{S\sqrt{H}} \qquad (8\text{-}3)$$

$$H = \left[-\left\{ \frac{(u_1' - u_2')}{2} - \frac{(u_1 - u_2)}{2} x \right\}^2 - t(1 - x) \right.$$

$$\left. \cdot \left\{ \frac{u_1' + u_2' - (u_1 + u_2) x}{2} - \frac{S_0 x}{1 - x} + \frac{t}{4}(1 - x) \right\} \right]$$

$$(8\text{-}4)$$

we find for the graph in figure (8-2)

$$\frac{1}{(2\pi)^3} \int ds_0 \; A_\rho(s_0,t) \; \int_0^1 dx \int_{u_1' min}^\infty du_1' \int_{u_2' min}^\infty du_2' \; \frac{K^2}{(u_2' + K^2)^2}$$

$$\times \; \frac{A(sx,t,u_1',u_2')}{D(u_1)} \; \frac{\theta(H)}{\sqrt{H}} \tag{8-5}$$

Our homogeneous integral equation then is

$$A(s,t,-M_1^2-\mu^2) = \frac{1}{(2\pi)^3} \int ds_0 \; A_\rho(s_0,t) \int_0^1 dx \int_{u_1' min}^\infty du_1'$$

$$\times \int_{u_2' min}^\infty du_2' \; \frac{K^2}{(u_2'+K^2)^2} \; \frac{A(sx,t,u_1',u_2')}{D(u_1')} \; \frac{\theta(H)}{\sqrt{H}}$$

$$\tag{8-6}$$

B. Extraction of Regge Behavior

Due to the dilatation invariance of the kernel under the transformation $s \to cs$, $s' \to cs$, we may write

$$A(s, t, -M_1^2 - \mu^2) = s^{\alpha(t)} \varphi(t, -M_1^2, -\mu^2) \qquad ; \quad \alpha(t) = \ell$$

The integral equation becomes

$$\varphi(t, -M_1^2 - \mu^2) = \frac{1}{(2\pi)^3} \int ds_0 \, A(s_0, t) \int_0^1 dx \, x^\ell \int_{u'_{1,min}}^\infty d u'_1$$

$$\times \int_{u'_{2\,min}}^\infty d u'_2 \, K^2 \, \frac{\varphi(t, u'_1, u'_2)}{D(u'_1)(u'_2 + K^2)^2} \, \frac{\theta(H)}{\sqrt{H}}$$

Again the Jacobian may be written

$$H = \frac{1}{4} \left(-\zeta^2 - \zeta_1^2 - \zeta_2^2 + 2\zeta_1 \zeta_2 + 2\zeta \zeta_1 + 2\zeta \zeta_2 \right) \tag{8-7}$$

where

$$\zeta = -t(1-x) \qquad\qquad x = s'/s$$

$$\zeta_1 = u'_1 - u_1 x - \frac{s_0 x}{1-x} \qquad\qquad u_1 = -M^2 \tag{8-8}$$

$$\zeta_2 = u'_2 - u_2 x - \frac{s_0 x}{1-x} \qquad\qquad u_2 = -\mu^2$$

or

$$H = \frac{x^2}{4}\left[-(u_1-u_2)^2 - 2t(u_1+u_2) - t^2\right]$$

$$-\frac{2x}{4}\left[-(u_1'-u_2')(u_1-u_2) - t(u_1+u_2) - t(u_1'+u_2') - 2s_0 t - t^2\right]$$

$$+\frac{1}{4}\left[-(u_1'-u_2')^2 - 2t(u_1'+u_2') - t^2\right] \qquad (8\text{-}9)$$

$$= -\alpha x^2 + 2\beta x - \alpha' \qquad (8\text{-}10)$$

Now

$$\alpha = +\left\{\left[t + (u_1+u_2)\right]^2 + (u_1-u_2)^2 - (u_1+u_2)^2\right\}$$

$$= +\left\{\left[t + u_1 + u_2\right]^2 - 4u_1 u_2\right\}$$

$$= +\left\{\left[t - (M^2+\mu^2)\right]^2 - 4M^2\mu^2\right\}$$

$$= +\lambda(t, M^2, \mu^2) = +\left[t - (M-\mu)^2\right]\left[t - (M+\mu)^2\right] \qquad (8\text{-}11)$$

The coefficient of $x^2 < 0$ since $\alpha > 0$ so the curve
H(x) is concave down. Therefore the region of x integration
is within the shaded region indicated in figure (8-3).
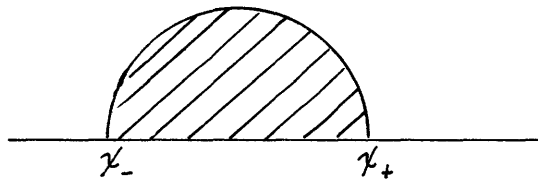
figure 8-3

If we define:

$$a = u_1' - u_2'$$

$$b = u_1 - u_2$$

$$c = u_1' + u_2' \qquad\qquad (8\text{-}12)$$

$$d = u_1 + u_2$$

then

$$H(x) = \frac{1}{4}\left\{-\left[2td + t^2 + b^2\right]x^2 + 2x\left[ct + dt + ab + 2S_0 t + t^2\right]\right.$$

$$\left. - \left[2tc + t^2 + a^2\right]\right\} \qquad\qquad (8\text{-}13)$$

$$= -\alpha x^2 + 2\beta x - \alpha' \qquad\qquad (8\text{-}14)$$

We have β always less than $\dfrac{\alpha + \alpha'}{2}$, i.e.

$$ct + dt + ab + 2S_0 t + t^2 < dt + ct + t^2 + \frac{a^2 + b^2}{2}$$

$$(8\text{-}15)$$

or

$$2S_0 t + ab < \frac{a^2 + b^2}{2}$$

$$(8\text{-}16)$$

$$2S_0 t < (a - b)^2$$

This is certainly true since $t < 0$ and $S_0 > 0$, and the

term on the right hand side of equation (8-16) is positive.

The roots $x_{\pm}$ of H(x) are given by

$$x_{\pm} = \frac{\beta \pm \sqrt{\beta^2 - \alpha\alpha'}}{\alpha}$$

The location of these with respect to the interval zero to one is what sets the x limits. A compilation of the possibilities is shown below.

Location of Roots		Limits of x-integral
(i) $\quad x_- > +1$ $\quad x_+ < 0 \qquad x_{\pm}$		$\int_0^1 dx$
(ii) $\quad 0 < x_- < 1 \quad , \quad x_+ > 1$		$\int_{x_-}^1 dx$
(iii) $\quad 0 < x_- \quad , \quad x_+ < 1$		$\int_{x_-}^{x_+} dx$
(iv) $\quad x_- < 0 \quad , \quad x_+ < 1$		$\int_0^{x_+} dx$

The limits on the x integration which are compatible with the requirements $\alpha > 0$ and $\beta < \frac{\alpha + \alpha'}{2}$ are those given in (iv). This means that $\alpha' < 0$. We have then

$$\alpha' = \left\{ (u_1' - u_2')^2 + (-t)\left[-t - 2(u_1' + u_2') \right] \right\} < 0$$

(8-17)

We define the variables

$$u_1' + u_2' = 2z$$

$$u_1' - u_2' = \rho \qquad\qquad (8\text{-}17)'$$

and thus replace the integrals over u_1' and u_2' by ones over z and ρ .

As before the assumption is that the difference term $(u_1' - u_2')^2 = \rho$ is small and it has no effect on $\beta < \frac{\alpha + \alpha'}{2}$, so we ignore its dependence in the terms of our integral equation and integrate it over a finite range of the order of the difference of the nucleon and pion masses squared. Then equation (8-17) yields the condition

$$- t - 2(u_1' + u_2') < 0$$

or

$$-t - 4z < 0 \qquad \text{where} \qquad u_1' + u_2' = 2z$$

$$z > -t/4$$

$$z_{min} = -t/4 = \bar{z}$$

C. Homogeneous Equation For Momentum Transfer Dependence

Our integral equation now is:

$$\phi(t, u_1, u_2) = \frac{1}{(2\pi)^3} \int ds_0 \, A(s_0, t) \int du_1'$$

$$\times \int du_2' \int_0^{\chi_+} \frac{d\chi \, \chi^{\alpha(t)}}{\sqrt{\alpha'} \sqrt{\left(1 - \frac{\chi}{\chi_+}\right)\left(1 - \frac{\chi}{\chi_-}\right)}} \, \frac{\phi(t, u_1', u_2') K^2}{D(u_1')(u_2' + K^2)^2}$$

(8-18)

Consider the integral

$$\int_0^{\chi_+} \frac{\chi^{\alpha(t)} \, d\chi}{\sqrt{\alpha'\left(1 - \frac{\chi}{\chi_-}\right)\left(1 - \frac{\chi}{\chi_+}\right)}}$$

(8-19)

where

$$\chi_+ \chi_- = \frac{\alpha'}{\alpha}$$

(8-20)

This equals

$$\frac{1}{\sqrt{\alpha'}} \int_0^{\chi_+} \frac{\chi^{\ell}}{\sqrt{\left(1 - \frac{\chi}{\chi_-}\right)\left(1 - \frac{\chi}{\chi_+}\right)}} \, d\chi$$

(8-21)

Let

$$y = \frac{\chi}{\chi_+} \quad , \quad \frac{\chi}{\chi_-} = \frac{\chi_+ \chi}{\chi_- \chi_+} = \rho^2 y$$

(8-22)

So the integral becomes

$$\frac{x_+^{\ell+1}}{\sqrt{\alpha'}} \int_0^1 \frac{y^\ell}{\sqrt{[(1-y)(1-\rho^2 y)]}} \, dy$$

$$= \frac{1}{\sqrt{\alpha'}} \frac{x_+^{\ell+1}}{\rho^{\ell+1}} Q_\ell \left[\frac{1}{2} \left(\frac{\rho^2+1}{\rho} \right) \right] \qquad (\quad) \qquad (8\text{-}23)$$

The details of this integration are given below. Consider

$$\int_0^1 \frac{y^\ell \, dy}{\sqrt{(1-y)(1-\rho^2 y)}} \qquad (8\text{-}24)$$

Now

$$\int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} \, dt$$

$$= \int_0^1 y^{b-1} (1-y)^{c-b-1} (1-yz)^{-a} \, dy$$

$$= \frac{\Gamma(b) \, \Gamma(c-b)}{\Gamma(c)} F(a,b,c;z) \qquad (36) \qquad (8\text{-}25)$$

This is Euler's formula for hypergeometric functions. Let
$z = \rho^2$ In our case,

$$\ell = b-1 \qquad b = \ell+1 \qquad a = \tfrac{1}{2}$$

$$-\tfrac{1}{2} = c-b-1 = c-\ell-2 \qquad (8\text{-}26)$$

$$c = \ell + 3/2 \qquad c-b = \tfrac{1}{2}$$

Therefore

$$\int_0^1 \frac{y^\ell \, dy}{\sqrt{(1-y)(1-\rho^2 y)}} = \frac{\Gamma(\ell+1) \; \Gamma(\frac{1}{2})}{\Gamma(\ell+\frac{3}{2})} F(\frac{1}{2}, \ell+1, \ell+\frac{3}{2}, \rho^2)$$

$$(8\text{-}27)$$

Now we have

$$Q_\ell(z) = \frac{1}{2^{\ell+1}} \; \pi^{\frac{1}{2}} \frac{\Gamma(\ell+1)}{\Gamma(\ell+\frac{3}{2})} \frac{1}{z^{\ell+1}} \; F\left(\frac{1}{2}\ell+1, \frac{1}{2}(\ell+1), \ell+\frac{3}{2}, z^{-2}\right)$$

$$(8\text{-}28)$$

and

$$Q_\ell(z) = \frac{1}{2^{\ell+1}} \; \pi^{\frac{1}{2}} \frac{\Gamma(\ell+1)}{\Gamma(\ell+\frac{3}{2})} \frac{1}{z^{\ell+1}} F\left(\frac{1}{2}(\ell+1), \frac{1}{2}\ell+1, \ell+\frac{3}{2}, z^{-2}\right)$$

$$(8\text{-}29)$$

This implies that

$$F\left(\frac{1}{2}\ell+1, \frac{1}{2}(\ell+1), \ell+\frac{3}{2}, z^{-2}\right) = F\left(\frac{1}{2}(\ell+1), \frac{1}{2}\ell+1, \ell+\frac{3}{2}, z^{-2}\right)$$

or

$$F\left(\frac{1}{2}, \ell+1, \ell+\frac{3}{2}, \rho^2\right) = F\left(\ell+1, \frac{1}{2}, \ell+\frac{3}{2}, \rho^2\right)$$

Thus we have

$$F\left(\ell+1, \frac{1}{2}, \ell+\frac{3}{2}, z\right) = (1+z)^{-\ell-1} F\left[\frac{1}{2}(\ell+1), \frac{1}{2}\ell+1, \ell+\frac{3}{2}, \frac{4z}{(1+z)^2}\right]$$

$$(8\text{-}30)$$

Let $\quad \dfrac{1}{x^2} = \dfrac{4z}{(1+z)^2} \qquad ; \qquad z = \rho^2 \qquad\qquad (8\text{-}31)$

So,

$$F(\ell+1, \tfrac{1}{2}, \ell+\tfrac{3}{2}, \rho^2) = \frac{1}{(1+\rho^2)^{\ell+1}} F\left[\tfrac{1}{2}(\ell+1), \tfrac{1}{2}\ell+1, \ell+\tfrac{3}{2}, \frac{4\rho^2}{(1+\rho^2)}\right]$$

$$= \frac{1}{(1+\rho^2)^{\ell+1}} F\left[\tfrac{1}{2}(\ell+1), \tfrac{1}{2}\ell+1, \ell+\tfrac{3}{2}, x^{-2}\right] \qquad (8\text{-}32)$$

$$= \frac{2^{\ell+1}}{\pi^{1/2}} \frac{\Gamma(\ell+\tfrac{3}{2})}{\Gamma(\ell+1)} x^{\ell+1} \frac{1}{(1+\rho^2)^{\ell+1}} Q_\ell(x \qquad (8\text{-}33)$$

where
$$x = \frac{1+\rho^2}{2\rho}$$

Therefore

$$\frac{\Gamma(\ell+1)\,\Gamma(\tfrac{1}{2})}{\Gamma(\ell+\tfrac{3}{2})} F\left(\tfrac{1}{2}, \ell+1, \ell+\tfrac{3}{2}, \rho^2\right)$$

$$= 2^{\ell+1} \frac{x^{\ell+1}}{(1+z)^{\ell+1}} Q_\ell\left[\frac{1+z}{2\sqrt{z}}\right]$$

$$= 2^{\ell+1} \frac{(1+\rho^2)^{\ell+1}}{(2\rho)^{\ell+1}} \frac{1}{(1+\rho^2)^{\ell+1}} Q_\ell\left[\tfrac{1}{2}\left(\frac{1+\rho^2}{\rho}\right)\right] \qquad (8\text{-}34)$$

We now calculate the argument of the Legendre function.

$$\frac{\rho^2 + 1}{\rho} = \frac{x_- + x_+}{x_-}\sqrt{\frac{x_-}{x_+}} = \frac{x_- + x_+}{\sqrt{x_+ x_-}}$$

$$= \frac{2\beta}{\alpha}\sqrt{\frac{\alpha}{\alpha'}} = \frac{2\beta}{\sqrt{\alpha\alpha'}} \qquad ; \qquad (8\text{-}35)$$

$$\frac{1}{2}\frac{\rho^2 + 1}{\rho} = \frac{\beta}{\sqrt{\alpha\alpha'}} \qquad (8\text{-}36)$$

Thus

$$\frac{x_+^{\ell+1}}{\rho^{\ell+1}} = \left[x_+ \left(\frac{x_-}{x_+}\right)^{1/2} \right]^{\ell+1} = \left(\sqrt{x_+ x_-}\right)^{\ell+1} = \left(\sqrt{\frac{\alpha'}{\alpha}}\right)^{\ell+1}$$

$$(8\text{-}37)$$

and

$$\frac{1}{\sqrt{\alpha'}}\frac{x_+^{\ell+1}}{\rho^{\ell+1}} = \frac{1}{\sqrt{\alpha'}}\left(\sqrt{\frac{\alpha'}{\alpha}}\right)^{\ell+1} \qquad (8\text{-}38)$$

Since β contains S_0 which we assume to be much larger than any momentum transfers in the problem, we have

$$Q_\ell\left(\frac{1}{2}\frac{\rho^2+1}{\rho}\right) = Q_\ell\left(\frac{\beta}{\sqrt{\alpha\alpha'}}\right) \longrightarrow \beta^{-\ell-1}\left(\sqrt{\alpha\alpha'}\right)^{\ell+1}$$

$$(8\text{-}39)$$

and

$$\frac{1}{\sqrt{\alpha'}}\frac{x_+^{\ell+1}}{\rho^{\ell+1}}Q_\ell\left(\frac{1}{2}\frac{\rho^2+1}{\rho}\right) = \frac{1}{\sqrt{\alpha'}}\left(\sqrt{\frac{\alpha'}{\alpha}}\right)^{\ell+1}\left(\sqrt{\alpha\alpha'}\right)^{\ell+1}\beta^{-\ell-1}$$

$$= \frac{1}{\sqrt{\alpha'}}\left(\alpha'\right)^{\ell+1}\beta^{-\ell-1} = \frac{1}{\sqrt{\alpha'}}\left(\alpha'\right)^{\ell+1}\frac{S_0^{-\ell-1}}{(-2t)^{\ell+1}} \qquad (8\text{-}40)$$

That is,

$$\beta = -2ts_0 \quad ; \quad \beta^{-\ell-1} = \frac{s_0^{-\ell-1}}{(-2t)^{\ell+1}} \tag{8-41}$$

Given that $A(s_0, t)$ behaves for large s_0 like $\log\left(\frac{s_0}{-t}\right) g(t)$, the s_0 integral becomes

$$\int \log\left(\frac{s_0}{-t}\right) s_0^{-\ell-1} ds_0 = \int ds_0 \log\left(\frac{s_0}{-t}\right) \frac{1}{s_0^{\ell+1}} \tag{8-42}$$

$$= \frac{-t}{(-t)^{\ell+1}} \int d\left(\frac{s_0}{-t}\right) \log\left(\frac{s_0}{-t}\right) \left(\frac{-t}{s_0}\right)^{\ell+1}$$

$$= \frac{1}{(-t)^{\ell}} \int dy_0 \, \log y_0 \, y_0^{-\ell-1}$$

$$= \frac{\Gamma(2)}{\ell^2 (-t)^{\ell}} \tag{8-43}$$

we then have:

$$\frac{(\alpha')^{\ell+1}}{\sqrt{\alpha'}} \frac{1}{(-2t)^{\ell+1}} \frac{1}{(-t)^{\ell}} = \left(\frac{\alpha'}{-2t}\right)^{\ell+1} \frac{1}{(-t)^{\ell}\sqrt{\alpha'}}$$

Now $\tag{8-44}$

$$\alpha' = \left\{ (u_1' - u_2')^2 + (-t)\left[-t - 2(u_1' + u_2')\right] \right\} \tag{8-45}$$

$$\approx (-t)\left[-t - 2(u_1' + u_2')\right] = (-2t)\left[-\frac{t}{2} - (u_1' + u_2')\right] .$$

So

$$\left(\frac{\alpha'}{-2t}\right)^{\ell+1} = \left(\frac{(-t)\left[-t - 2(u_1' + u_2')\right]}{-2t}\right)^{\ell+1} = \left[-\frac{t}{2} - (u_1' + u_2')\right]^{\ell+1}$$

(8-46)

and

$$\left(\frac{\alpha'}{-2t}\right)^{\ell+1} \frac{1}{(-t)^\ell} \frac{1}{\sqrt{\alpha'}} = \frac{1}{(-t)^\ell \sqrt{-2t} \left[-\frac{t}{2} - (u_1' + u_2')\right]^{1/2}} \left[-\frac{t}{2} - (u_1' + u_2')\right]^{\ell+1}$$

$$= \frac{1}{\sqrt{2} \, (-t)^{\ell+1/2}} \left[-\frac{t}{2} - (u_1' + u_2')\right]^{\ell+1/2}$$

$$= \frac{1}{\sqrt{2}} \left\{ \frac{\left[-\frac{t}{2} - (u_1' + u_2')\right]}{(-t)} \right\}^{\ell+1/2}$$

(8-47)

Our integral equation now is:

$$\phi(t, z) = \frac{a \, g(t)}{\ell^2} \int d\rho \int dz_0 \left\{ \frac{-\frac{t}{2} - 2z_0}{-t} \right\}^{\ell+1/2}$$

$$\times \frac{\phi(t, z_0)}{(z_0 + M^2)(z_0 + K^2)^2}$$

(8-48)

where all constants are absorbed in the factor a and we've redefined the integration variables as set forth in equation (8-17')', i.e. ρ in the above equation is $\rho = u_1' - u_2'$.

D. Solution to Homogeneous Equation

Our homogeneous   absorptive part equa tion is:

$$\phi(t,z) = \frac{a\, y(t)}{\ell^2} \int d\rho \int dz_0 \left\{ \frac{-t/2 - 2z_0}{t} \right\}^{\ell+1/2} \frac{\phi(t,z_0)}{(z_0+M^2)(z_0+K^2)^2}$$

(8-48)

Since z is essentially equal to M^2 we symmetrize our

kernel in the following way: we replace $2z_0$ by (z_0+z)

and set $K^2 = M^2 = z$ under the integral. The equation

now is

$$\phi(t,z) = \frac{a\, y(t)}{\ell^2} \int dz_0 \left\{ \frac{-t/2 - z - z_0}{t} \right\}^{\ell+1/2} \frac{\phi(t,z_0)}{(z+z_0)^3}$$

(8-49)

As it stands, this equation is very difficult to solve.
Since the kernel is essentially the partial wave projection
of the potential for this problem, we consider the effect
on the output trajectory of taking two forms for the
kernel, i.e. its form for $\ell + 1/2 = 1$ and $\ell + 1/2 = 0$.
This seems reasonable in that we expect ℓ to be somewhere
between -1 and 1, but probably not at either extreme. The
$\ell = -1$ would be equivalent to what emerges in most weak

coupling schemes with elementary exchanges. The $l=1$ value would saturate the Froissant bound.

E. Variational Calculation

If we have a homogeneous Fredholm equation of the second kind

$$\varphi(z) = \lambda \int_a^b K(z, z_0) \, \phi(z_0) \, dz_0 \qquad (8\text{-}50)$$

with a real positive definite symmetric kernel, there are non-zero solutions for a sequence of eigenvalues λ_n, with solutions $\phi_n(z)$ or eigenfunctions. They satisfy the variational principle that

$$\delta \left\{ \frac{\int \bar{\phi}(z) \, \phi(z) \, dz}{\int \bar{\phi}(z) \, K(z, z_0) \, \phi(z_0) \, dz_0 \, dz} \right\} = 0$$

$$(8\text{-}51)$$

The stationary values for this expression are the eigenvalues. A symmetric kernel is such that

$$K(z, z_0) = K(z_0, z) \qquad . \qquad (8\text{-}52)$$

Positive definiteness means

$$\int f(z)\, dz \int K(z, z_o)\, f(z_o)\, dz_o > 0 \qquad \text{for a function}$$

f finite in the range of integration appropriate for the equation using K.

Our kernel is

$$K(z, z_o) = \frac{-t/_2 - z_o}{(e + z_o)^3} = \frac{\mu - z_o}{(z + z_o)^3} \qquad (8\text{-}53)$$

We symmetrize it by writing:

$$\widetilde{K}(z, z_o) = \frac{\mu - z - z_o}{(z + z_o)^3} \qquad (8\text{-}54)$$

The positive definiteness follows from our choice of f(x) (see below).

This essentially results in a redefinition of μ and does not change the basic character of the kernel as a function of z_o (which is being integrated over in the integral equation).

We then consider

$$\lambda = min. \left\{ \frac{\int \bar{\phi}(z)\, \phi(z)\, dz}{\iint \bar{\phi}(z)\, K(z, z_o)\, \phi(z_o)\, dz_o\, dz} = min\; \frac{I_1}{I_2} \right.$$

$$(8\text{-}55)$$

to determine λ , hence ℓ since ℓ, as seen in

equation $(8\text{-}48)$ is contained in λ .

　　We pick a trial function

$$\phi = e^{-\alpha z}$$

and vary with respect to α . The lower limits on the

integrals are both $= \bar{z} = -t/2$. When we calculate I_1 and I_2

we find

$$I_1 = \frac{1}{2\alpha} e^{-2\alpha \bar{z}} \tag{8-56}$$

$$I_2 = \frac{1}{2} e^{-2\alpha \bar{z}} \left\{ -\frac{t}{4\bar{z}} + 2 - \frac{\alpha t}{2} \right\} + \frac{1}{2} E_i(-2\alpha \bar{z}) \left\{ 2 - \alpha t + 2\alpha z - \alpha^2 t z \right\} \tag{8-57}$$

$$\frac{I_1}{I_2} = \frac{1}{\left\{ -\frac{\alpha t}{4 z} + 2\alpha - \frac{\alpha^2 t}{2} \right\} + e^{2\alpha z} E_i(-2\alpha z) \left\{ 2\alpha - \alpha^2 t + 2\alpha^2 z - \alpha^3 t z \right\}} \tag{8-58}$$

$$= \frac{N}{D}$$

Thus we have

$$\delta\left(\frac{N}{D}\right) = \frac{\delta N}{D} - \frac{N \delta D}{D^2} = 0$$

$$= \frac{D \delta N - N \delta D}{D^2} = 0$$

$$\frac{\delta N}{D} = 0$$

so we are left with

$$N \delta D = 0 \qquad\qquad \delta D = 0$$

But

$$\delta D = \frac{\partial D}{\partial \alpha} \delta \alpha \qquad\qquad \text{for arbitrary variation} \quad \delta \alpha .$$

So our condition is $\dfrac{\partial D}{\partial \alpha} = 0$.

This yields the following transcendental equation which we solve numerically.

$$\left\{ -\frac{t}{4\bar{z}} - \alpha t + 4\alpha \bar{z} - 2 \right\} + e^{2\alpha\bar{z}} \, E_i \, (-2\alpha\bar{z}) \left\{ -2\alpha^3 t \bar{z}^2 + 4\alpha\bar{z} \right.$$

$$\left. + 8\alpha^2 \bar{z}^2 - 3\alpha^2 t \bar{z} - 2 \right\} = 0$$

We call $2\alpha\bar{z} = y$ and remembering $\bar{z} = -t/2$, our   (8-59)

equation to solve is

$$e^{y} E_i(-y) \left[\frac{y^3}{2} + \frac{7}{2} y^2 + 2y - 2 \right] = \left[3/2 - 3y \right]$$

(8-60)

$$e^{y} E_i(-y) A = RHS$$

(8-61)

In Table I we plot the left hand side of this equation versus the right hand side for various values of y. The curves represented by the two sides of the equation intersect at $y \approx 0.6$ or $\alpha \approx \frac{0.6}{[-t]}$

We substitute this value for α into our eigenvalue equation (7-55) and find

$$\lambda = \frac{c [-t]}{}$$

and

$$\ell^2 = const. \frac{f(t, g^2)}{(-t)}$$

$$\ell = \frac{h(t, g^2)}{(-t)^{1/2}}$$

This says for finite (negative t) ℓ has a finite value which falls off as we go to larger and larger negative t values. If g^2 is large enough, the factor $h(t, g^2)$ (see Chapter VII-D) falls off with a power of (g^2). Therefore, the height of the trajectory for a given value of t can be lowered by increasing g^2.

IX. CONSTRUCTING AMPLITUDES

A. The Two Body Amplitude from a Dispersion Relation

Following the discussion in BFT[103], we obtain the real part of our amplitude from the imaginary part $A(s,t)$ by using a fixed t dispersion relation. If the asymptotic form of $A(s,t)$ is given by $\phi(t) S^{\alpha(t)}$, write for $A(s,t)$ a dispersion relation with m subtractions, n being the minimum integer greater than $\alpha(t)$.

$$D(s,t) = \frac{S^m}{\pi} P \int_{(M+m)^2}^{\infty} \frac{A(s',t)}{S'^m(s'-s)} \, ds' \quad + \quad \text{polynomial in } S.$$

$$(9-1)$$

The maximum power of s in the polynomial is $m-1$. We find asymptotic behavior of D(s,t) from that of A(s,t) with the following approximations. 1) We argue that terms whose asymptotic behavior is smaller than the asymptotic form of A(s,t) cannot contribute to asymptotic form of D(s,t). 2) We extend integration range from 0 to ∞, with the contribution of the integration between 0 and $(M+m)^2$ negligible due to the fact that the integral contains no infrared divergence (as $s' \to 0$). 3) Since the asymptotic behavior of integrals is $S^{\alpha(t)}$, we neglect subtraction

polynomials (max. power $m-1$).   Therefore

$$D(s,t) = s^m \frac{\phi(t)}{\pi} P \int_0^\infty \frac{ds'}{s'^{m-\alpha}(s'-s)}$$

$$= -\frac{\phi(t)}{\pi} s^{\alpha(t)} P \int_0^\infty \frac{dx \; x^{\alpha-m}}{1-x} = -\phi(t) \, s^{\alpha(t)} \cot \pi \alpha(t)$$

$$(9-2)$$

So our full amplitude is

$$M(s,t) = D(s,t) + A(s,t)$$

$$= \phi(t) \, s^{\alpha(t)} \left[1 - \cot \pi \alpha(t) \right]$$

$$= \phi(t) \, s^{\alpha(t)} \left[\frac{\sin \pi \alpha(t) - \cos \pi \alpha(t)}{\sin \pi \alpha(t)} \right]$$

$$(9-3)$$

This obviously will not yield the correct signature for
the trajectory since we did not take proper account of the
cross channel effects or of the full spin treatment since
our output trajectory is only for the A amplitude in the
general form $F = A + \not{\phi} + B$ for pion nucleon scattering.
But the dispersion relation is done merely to indicate the
method.

B. Constructing Production and Three Body Amplitudes

   To construct a production amplitude from our cal-
culated absorptive part, we disperse the absorptive part
in s (as discussed above) to construct the full composite
particle scattering amplitude. We then perform a quadra-
ture and attach the unstable nucleon propagator and the
vertex function for the decay of the final state nucleon.
We have as shown in figure (9-1)

$$\langle k | T(s) | k' \rangle \quad S_N(\sigma_{k'}) \, \Lambda(p^2) \qquad (9\text{-}4)_a$$

figure (9-1)
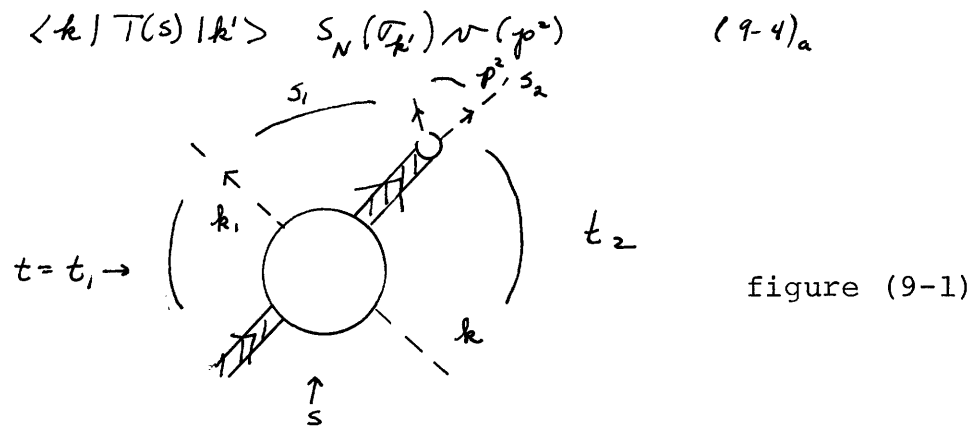
where $\langle k | T(s) k' \rangle$ is the composite particle
amplitude we have constructed and has the form $\beta(t) \, s^{\alpha(t)}$.
The unstable nucleon propagator as given in Appendix
is

$$S_N^{-1} = D(\sigma_k) = \frac{\sigma_k - M^2}{2M i} \left[1 + (\sigma_k - M^2) \frac{M^2 \gamma^2}{2\pi^2} \int_0^\infty \frac{dk \, \rho(k)}{(\sigma_k - x)(M^2 - x)^2} \right]$$

$$(9\text{-}4)_b$$

And the vertex function for composite particle dissociation

as given in Appendix II   is the vertex function calculated

from the Omnes equation, i.e. it is

$$(9-5)$$

$$\mathcal{N}(p^2) = \mathcal{N}(s_2) = \frac{6c \, \beta_2 \, (E-m)(W+m)}{\left[8\pi W \, (n - \alpha(s_2)) + (E-m)(W+m)\lambda(s_2)\beta_\alpha\right]}$$

This is then an explicit expression for the production

amplitude. If we wanted to consider the Toller angle

dependence wé would have to rewrite (9-4)$_a$ in terms of the

appropriate invariants s_1, s_2, t_1, t_2 and take the limit

$\dfrac{s_1 s_2}{s}$ fixed, s_1, s_2, s large and t_1, t_2 fixed. We

have next done this but the point is that once we have the

composite particle amplitude, we may (with some labor)

construct the production (two body $\rightarrow$ three body) amplitude.

Similarly to construct a complete three body amplitude

we would perform the quadrature indicated in figure

$$\mathcal{N}(p_1^2) \; S_N(\sigma_k) \; \langle k | T(s) | k' \rangle \; S_N(\sigma_{k'}) \; \mathcal{N}(p_2^2) \qquad (9-6)$$
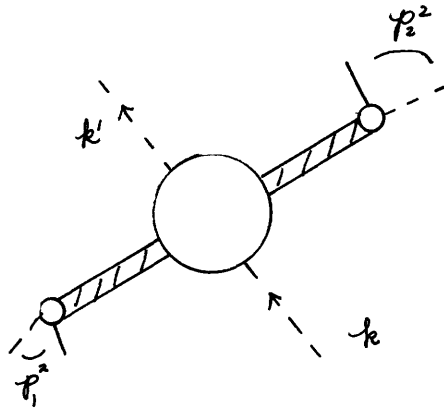
figure (9-2)

We now have a three body amplitude whose analytic properties

we know (because we know everything going into the quad-

rature) which we can use to study the existence of a triple

Regge limit after using Mueller's Optical Theorem discussed

in Chapter II. We can use in inclusive sum rules to study

the triple Regge vertex. Work in this direction is now

being done.

LHS	RHS	$3y$	$2y$	y	$\frac{1}{2}y^2$	$\frac{1}{2}y^3$	e^y	$E_i(-y)$	A
0.3	.636	1.2	.8	.4	.56	.032	1.49	-0.70	-0.61
0	.582	1.5	1.0	.5	.875	.062	1.65	-0.56	-0.63
-0.3	-0.45	1.8	1.2	.6	1.26	.108	1.82	-0.45	+0.52
-0.6	-0.95	2.1	1.4	.7	1.72	.172	2.01	-0.37	1.29

TABLE I

X.   CONCLUSIONS

We began with a Bethe-Salpeter equation describing composite particle scattering in the t channel. By analytically continuing this equation to the s channel and taking the s-discontinuity, we obtain a multiperipheral model for composite particle scattering. The essential differences between our multiperipheral model with continued cross channel unitarity and the regular multiperipheral models are the extra momentum transfer dependence due to the use of a weight function in continuing the unstable particle propagators from the t channel to the s channel, and the effect of the mass renormalization implied in keeping the principal value part (real part) of the rho box in the kernel of our equation. This equation at high energy becomes a homogeneous equation for the momentum transfer dependence.

We find that if we approximate the ℓ dependence of the kernel and symmetrize it, then we have an eigenvalue problem which we can solve by a variational technique. We do this and find that we get an output Regge trajectory which is real and falls off with increasing negative t , if equal mass (since $t < 0$ is region of validity of the calculation). We interpret this as follows. We assume that for large negative t , we are far away from any resonance (or bound state) region in the problem. For potential models,

which this is, we expect only a finite number of resonances
to be formed, i.e. the potential can bind only a finite
number of partial waves and therefore the trajectory
describing the resonance behavior must eventually turn over
and go to zero. We may contrast this with the Veneziano
model which has linear trajectories, hence continual
resonance formation, which suggests a bootstrap mechanism,
i.e. every time one potential exhausts itself, there is
another to continue to bind particles into bound states or
resonances. Therefore, it is probably unrealistic to
expect linear trajectories from multiperipheral or any
potential models. The interesting feature about this is
that trajectories must turn over if the model is unitary.
This, perhaps, is why the Veneziano model (as originally
proposed) is not unitary.

  With regard to our retaining the principal value
part of the rho box, two things must be said:
we want an absorptive part which is a positive definite
quantity, which it would not be if we kept the imaginary
part of the rho box. Further, since the imaginary part
comes from a threshhold factor at μ^2 (μ^2 = pion mass
squared), when we are far below threshhold factor to dis-
appear or have no effect except for a possible renormali-
zation of the mass of the originally unstable particle,
which we feel is provided for as the principal value part.

Also the mass renormalization of the rho meson is obviously
related to its lifetime and as such is related to the coupling
of the $\rho N'$ intermediate state to other channels, in this
instance πN . If we take the basic interaction given by the
Bethe-Salpeter equation for $t > 0$ and continue it to $t < 0$
there still exists the possibility of scattering through the
$\rho N'$ channel virtually although we are far away from the
2π threshhold. In this sense our principal value part gives
partly the probability of the transition occurring when we
are off the energy shell for that occurring. What we find
is that if we retain the mass renormalization in the form
of the coupling constant of the ρ to two pions (see
Chapter $VIII$) we find that as $q^2 \rightarrow \infty$, our amplitude falls
off, which is saying that when we're far from threshhold we
don't expect the coupling to be strong and the model forces
us to this by having the amplitude fall off rapidly as q^2
gets large. Stated another way, continued cross channel
unitarity keeps us from violating the Froissart bound.

Our treatment of the model has several shortcomings.
We do not treat isospin correctly except in the calculation
of the vertex functions. We don't treat spin fully except
in the calculation of the Born term which does not enter
our final equation. When we perform the dispersion
relation in s to form the full amplitude, we don't take
proper account of crossing. However, the virtue of this
work is that it does demonstrate the possibility of continu-

ing a Bethe-Salpeter equation to obtain a multiperipheral

model. It shows the possible utility of variational

techniques in tackling problems of this kind, for all

values of the coupling constant including not approximat-

ing the kernel. Finally, and importantly it shows that

minimal unitarization in t keeps us from violating the

Froissart bound.

What is left to be done in this model is the

investigation of the Toller angle dependence in production

amplitudes, investigation of the triple Regge limit in the

three to three amplitudes and an investigation of the analy-

tic properties in order to use this amplitude in inclusive

sum rules.

APPENDIX I: PROPAGATOR CALCULATIONS

Beginning with the expression in equation (6-23) for the
unstable particle propagator, $S(\sigma)$, Aaron, Amado and Young
derive the following equation for the inverse propagator

$$S(\sigma) = - \frac{1}{D(\sigma)} \qquad \text{(A1-1)}$$

$$D(\sigma) = (\sigma - \mu^2)\left[1 + \frac{(\sigma - \mu^2)}{(2\pi)^3} \int \frac{d^3k \, \mathcal{N}^2}{\omega_k (\sigma - \bar{\sigma})(\bar{\sigma} - \mu^2)^2} \right]$$

$$\text{(A1-2)}$$

$$\bar{\sigma} = 4(\underset{\sim}{k}^2 + m^2) = 4\omega_k^2$$

for a composite, of mass μ^2 , formed from equal mass spinless
particles of mass m^2. (rho meson (ρ) formed from two pions
in this case, and

$$D(\sigma) = \frac{\sigma - M^2}{2Mi}\left[1 + (\sigma - M^2)\frac{M^2\gamma^2}{2\pi^2}\int_0^\infty \frac{dk \, \rho(k)}{(\sigma - x)(M^2 - x)^2} \right]$$

where $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (A1-3)

$$\rho(k) = \frac{k^4 \mathcal{N}^2 \sqrt{x}}{E_k \omega_k (E_k + M)} \qquad \text{and} \qquad \begin{aligned} x &= (E_k + \omega_k)^2 \\ \mathcal{N}^2 &= \gamma_{NN'\pi}^2 \end{aligned} \qquad \text{(A1-4)}$$

for a composite nucleon formed from a pion and a nucleon.
These expressions yield the unstable particle propagators in
the bubble approximations.

A. Rho Meson Propagator

We have

(A1-5)

$$D(\sigma) = (\sigma - m_\rho^2) \left[1 + \frac{(\sigma - m_\rho^2)}{2(2\pi)^3} \int \frac{d^3k \; \gamma_{\rho\pi\pi}^2}{\omega_k (\sigma - \bar{\sigma})(\bar{\sigma} - \mu^2)^2} \right]$$

where we have replaced the $\rho\pi\pi$ vertex function with the $\rho\pi\pi$ coupling constant. Let us concentrate on the integral

$$\int \frac{d^3k \; \gamma_{\rho\pi\pi}^2}{\omega_k (\sigma - \bar{\sigma})(\bar{\sigma} - m_\rho^2)^2} \tag{A1-6}$$

$$= \int \frac{k^2 \, d|k| \, d\Omega \; \gamma_{\rho\pi\pi}^2}{(\sigma - 4\omega^2) \, \omega \, (m_\rho^2 - 4\omega^2)^2} \tag{A1-7}$$

Now $\omega^2 = k^2 + \mu^2$, μ^2 = pion mass

$$2\omega \, d\omega = 2k \, dk$$

$$d|k| = \frac{\omega \, d\omega}{|k|}$$

$$k = (\omega^2 - \mu^2)^{1/2} \tag{A1-8}$$

The integral becomes

$$\gamma_{\rho\pi\pi}^2 \; \frac{4\pi}{64} \int_\mu^\infty \frac{d\omega \, (\omega^2 - \mu^2)^{1/2}}{(a^2 - \omega^2)\left(\frac{m_\rho^2}{4} - \omega^2\right)^2} \tag{A1-9}$$

where

$$a^2 = \frac{\sigma}{4} \geqslant \mu^2$$

$$b^2 = \frac{m_\rho^2}{4} \geqslant \frac{\mu^2}{4} \tag{A1-10}$$

The integrand is an even function of ω and it has branch points at $\omega = \pm \mu$. Therefore we can perform the integral by integrating around the contour given in figure (A1-1).
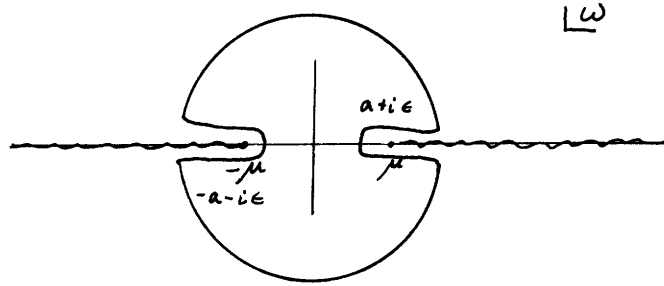
figure (Al-1)

a has a small positive imaginary part since it is above threshhold. We then pick up residue contributions at $\omega = \pm a$. At $\omega = a$, we get

$$\frac{(a^2 - \mu^2)^{1/2}}{(2a)(b^2 - a^2)^2} \qquad (Al-11)$$

At $\omega = -a$, we get

$$\frac{(a^2 - \mu^2)^{1/2}}{(2a)(b^2 - a^2)^2} \qquad (Al-12)$$

The net residue contribution then is

$$\frac{2(a^2 - \mu^2)^{1/2}}{(2a)(b^2 - a^2)^2} = \frac{\left(\frac{\sigma}{4} - \mu^2\right)^{\frac{1}{2}}}{\frac{\sigma}{2}^{1/2}\frac{(\sigma - m_\rho^2)^2}{16}} = \frac{16(\sigma - 4\mu^2)^{1/2}}{\sigma^{1/2}(\sigma - m_\rho^2)^2} \qquad (Al-13)$$

The net result we get for the integral is

$$\frac{2\pi i}{4}\sum Res. = 8\pi i \; \frac{(\sigma - 4\mu^2)^{1/2}}{\sigma^{1/2}(\sigma - m_\rho^2)^2} \qquad (Al-14)$$

Finally we obtain

$$D(\sigma) = \sigma - m_\rho^2 + \frac{i}{32\pi^2}\frac{(\sigma - 4\mu^2)^{\frac{1}{2}}}{\sigma^{1/2}(\sigma - m_\rho^2)^2}(\sigma - m_\rho^2)^2 \qquad (Al-15)$$

$$D(\sigma) = \sigma - m_\rho^2 + \frac{i}{32\pi^2}\left(\frac{\sigma - 4\mu^2}{\sigma}\right)^{1/2} \qquad \text{(A1-16)}$$

B. Nucleon Propagator

We have

$$D(\sigma) = \frac{\sigma - M^2}{2Mi}\left[1 + (\sigma - M^2)\frac{M^2\gamma^2}{2\pi^2}\int_0^\infty \frac{dk\, \rho(k)}{(\sigma - \chi)(M^2 - \chi)^2}\right] \qquad \text{(A1-17)}$$

Consider the integral

$$\int_0^\infty \frac{dk\,(E_k + \omega_k)\,k^4}{E_k\,\omega_k\,(E_k + M)\left[\sigma - (E_k + \omega_k)^2\right]\left[M^2 - (E_k + \omega_k)^2\right]^2} \qquad \text{(A1-18)}$$

We look at where the denominator of the integrand can vanish over the integration region. Since $k > 0$, none of the terms except $\left[\sigma - (E_k + \omega_k)^2\right]$ can vanish. Let us locate the zeroes of this function, i.e. the values of k where

$$\sigma - (E_k + \omega_k)^2 = 0 \qquad \text{(A1-19)}$$

They are

$$k = \pm\frac{\lambda^{1/2}(\sigma, \mu^2, M^2)}{2\sqrt{\sigma}} = \pm k_0 \qquad \text{(A1-20)}$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$.

$$\text{(A1-21)}$$

So we have for the nucleon propagator

$$\int_0^\infty \frac{dk \quad k^4 \, (E_k + \omega_k)}{E_k \, \omega_k \, (E_k + M) \left[M^2 - (E_k + \omega_k)^2 \right]^2 (k_0^2 - k^2)} \tag{Al-22}$$

We let

$$x = k^2 = \frac{\lambda(\sigma, \mu^2, M^2)}{4\sigma} \tag{Al-23}$$

The integral becomes

$$\frac{1}{2} \int_0^\infty \frac{dx \quad x^{3/2}}{E_x \, \omega_x \, (E_x + M) \left[M^2 - (E_x + \omega_x)^2 \right]^2 (a - x)} \tag{Al-24}$$

We can evaluate this using the contour given below (figure (A2)).
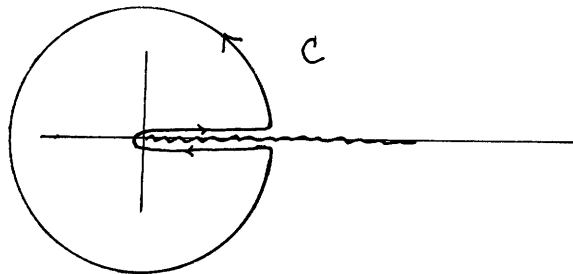
figure (Al-2)

and we have

$$\oint_C = 2 \int_0^\infty \qquad \text{or} \qquad \frac{1}{2} \int_0^\infty = \frac{1}{4} \oint_C$$

So we evaluate

$$\frac{1}{4} \oint_C \frac{dx \quad x^{3/2}}{E_x \, \omega_x \, (E_x + M) \left[M^2 - (E_x + \omega_x)^2 \right]^2 (a - x)} \tag{Al-25}$$

We pick up a residue contribution at x=a since a has a small positive imaginary part so it is above the cut. The result is

$$\frac{2\pi i \; a^{3/2}}{E_a W_a (E_a + M) \left[M^2 - (E_a + W_a)^2 \right]^2} \qquad\qquad \text{(A1-26)}$$

$$= \left[\frac{\lambda(\sigma, \mu^2, M^2)}{4\sigma} \right]^{3/2} \cdot \frac{2\pi i}{E_{k_0} W_{k_0} (E_{k_0} + M) \left[M^2 - \sigma \right]^2} \qquad \text{(A1-27)}$$

with k_0 given in equation (A1-20). Therefore, we obtain for the inverse nucleon propagator

$$D(\sigma) = \frac{\sigma - M^2}{2Mi} \left[1 + (\sigma - M^2) \frac{M^2 \gamma^2}{2\pi^2} \frac{2\pi i}{4} \left[\frac{\lambda(\sigma, \mu^2, M^2)}{4\sigma} \right]^{3/2} \frac{1}{E_{k_0} \omega_{k_0} (E_{k_0} + M)} \right.$$

$$\text{(A1-28)}$$

$$D(\sigma) = \frac{1}{2Mi} \left\{ \sigma - M^2 + \frac{i M^2 \gamma^2}{4\pi} \left[\frac{\lambda(\sigma, \mu^2, M^2)}{4\sigma} \right]^{3/2} \frac{1}{E_{k_0} \omega_{k_0} (E_{k_0} + M)} \right\}$$

$$\text{(A1-29)}$$

For the bare nucleon propagator (for nucleon formed in direct channel) we would have

$$\frac{2Mi}{K^2 - M^2}$$

where K is the nucleon bound state four momentum and M the bound state mass. Our prescription then is put in $2Mi$ for every nucleon line and divide by the relevant propagator factor. We retain this presciption for the unstable particle propagator even after we have continued the propagator to the cross channel. This gives us the factor $2Mi$ in equation (AI-29) for the nucleon propagator which will remain when we make our cross channel continuation. So when we calculate the box and attach it to the absorptive part in our integral equation, we pick up a product of factors $(2Mi)(2M'i) = -4M'M$

APPENDIX II: FORM FACTOR CALCULATION

A.   Unitarity Corrected Partial Wave Amplitudes

We wish to construct vertex functions (form factors)
for the decay of a composite-particle using unitarity correc-
ted partial wave amplitudes and the Omnes equation. This
calculation is based on work by Gheka and Visinescu[34].

Elastic unitarity for partial wave amplitudes reads

$$\text{Im } t_\ell(s) = \frac{k}{\sqrt{s}} \mid t_\ell(s) \mid^2 = \frac{\sqrt{s}}{k} \sin^2 \delta_\ell$$

(A2-1)

where δ_ℓ is the partial wave phase shift and is real in
the elastic region. This implies that $t_\ell(s)$ may be
written

$$t_\ell(s) = \frac{\sqrt{s}}{k} e^{i \delta_\ell} \sin \delta_\ell \qquad \text{(A2-2)}$$

or

$$\text{Im } \frac{1}{t_\ell(s)} = -\frac{k}{\sqrt{s}} \qquad \text{(A2-3)}$$

We define a function $h(s) = \frac{-ik}{\sqrt{s}} + h_1(s)$, where $h_1(s)$
is chosen to cancel the spurious $\sqrt{s}$ singularity and to
introduce no others. Therefore we write

$$h_1(s) = \frac{-k}{\sqrt{s}} f(s)$$

(A2-4)

where $f(s)$ has no singularity at s=0 and is an analytic function of s. We now express $t_\ell(s)$ in an N/D form, i.e.

$$t_\ell(s) = \frac{N(s)}{D(s)} \qquad \text{(A2-5)}$$

where $N(s)$ has no right hand cut, i.e. no unitarity cut. We now write $D(s) = 1 + W(s)N(s)$ and see if $W(s)$ can be chosen such that $t_\ell(s)$ is unitarity in the elastic region. We have

$$\frac{1}{t_\ell(s)} = \frac{1 + W(s)N(s)}{N(s)} \qquad \text{(A2-6)}$$

So, $$Im\, \frac{1}{t_\ell(s)} = Im\, W(s) = -\frac{k}{\sqrt{s}}$$

This implies that $W(s)$ is of the form

$$W(s) = -\frac{ik}{\sqrt{s}} + h_1(s) = h(s) \qquad \text{(A2-7)}$$

The function $h_1(s)$ has the form

$$h_1(s) = \frac{2}{\pi} \frac{k}{\sqrt{s}} \ln\left[\frac{\sqrt{s} + \alpha}{\beta}\right] \qquad \text{(A2-8)}$$

with α and β suitably chosen so that $h_1(s)$ cancels the spurious $\sqrt{s}$ singularity in $W(s)$. For equal mass scattering, say pion-pion ($\pi - \pi$) scattering, $h(s)$ is

$$h(s) = -\frac{ik}{\sqrt{s}} + \frac{2}{\pi} \frac{k}{\sqrt{s}} \ln\left[\frac{\sqrt{s} + 2k}{2\mu}\right] \qquad .$$

$$\text{(A2-9)}$$
That $h(s)$ cancels the $\sqrt{s}$ singularity is seen as follows:

$$s = 4(k^2 + \mu^2) \quad , \qquad \mu^2 = \text{pion mass}$$

$$k^2 = \frac{s - 4\mu^2}{4} \qquad \text{(A2-10)}$$

If s=0, this implies that $k^2 = -\mu^2$, so we have

$$h(0) = \frac{-ik}{(0)} + \frac{2}{\pi} \frac{k}{(0)} \ln[i]$$

Now $\ln[i] = \ln[e^{i\frac{\pi}{2}}] = i\frac{\pi}{2}$, so

$$h(0) = \frac{-ik}{(0)} + \frac{2}{\pi} \frac{k}{(0)} i\frac{\pi}{2} = \frac{-ik}{(0)} + \frac{ik}{(0)} = 0$$

Therefore, there is no singularity in $\sqrt{s}$ at s=0.

For unequal mass scattering:

$$|\underset{\sim}{k}| = \frac{\lambda(s, M^2, m^2)}{2\sqrt{s}} = \frac{[s-(M+m)^2][s-(M-m)^2]}{2\sqrt{s}}$$

$$(A2-11)$$

where M^2 , m^2 are the masses of the particles in question.

and $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$. When s=0

$$\lambda \rightarrow M^4 + m^4 - 2m^2 M^2 = (M^2 - m^2)^2$$

This implies that:

$$[s-(M+m)^2] \xrightarrow{s=0} -(M+m)^2$$

and

$$\ln\left\{\frac{i(M+m)}{(M+m)}\right\} = \ln i = \ln e^{i\frac{\pi}{2}} = i\frac{\pi}{2} \quad .$$

So

$$\ln\left\{\frac{\sqrt{s} + [s-(M+m)^2]^{\frac{1}{2}}}{(M+m)}\right\} \xrightarrow{s=0} i\frac{\pi}{2}$$

For unequal mass scattering then:

$$(A2-12)$$

$$h_1(s) = \frac{2}{\pi} \frac{k}{\sqrt{s}} \ln \left\{ \frac{\sqrt{s} + [s-(M+m)^2]^{\frac{1}{2}}}{(M+m)} \right\}$$

and

(A2-13)

$$h(s) = \frac{-ik}{\sqrt{s}} + h_1(s)$$

(A2-14)

This procedure then gives us a partial wave amplitude which obeys elastic unitarity of the form

$$T_\ell^I(s) = \frac{A_\ell^I(s)}{1 + h(s) A_\ell^I(s)}$$

(A2-15)

with $h(s)$ given in equation $(A2-14)$ for equal mass and unequal mass scattering respectively.

B. Omnes Equation

The Omnes equation which is an integral equation for the form factor embodying elastic unitarity is$(\quad)$

$$F(s) = 1 + \frac{s}{\pi} \int_{s_0}^{\infty} \frac{F(s') f^*(s') \rho(s') ds'}{s'(s'-s)}$$

(A2-16)

wherein $f(s)$ represents our unitarity corrected amplitude (in the elastic region) and $\rho(s)$ is a suitable weight function. This equation supposes that $F(s)$ satisfies a once subtracted dispersion relation given diagramming by
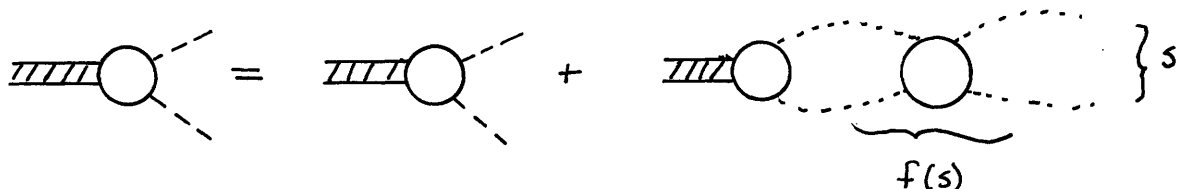
figure (A2-1)

which says that it decreases fairly fast with increasing distance from the two-particle threshold. The solution to this equation is

$$F(s) = \frac{f(s)}{f(s_o)} \qquad (A2\text{-}17)$$

as can be easily seen by substituting this form into equation (A2-16). The function $f(s)$ is

$$f(s) = \frac{f_I^{J\pm}(s)}{1 + \hbar(s)\, f_I^{J\pm}(s)} \qquad (A2\text{-}18)$$

which is the unitarity corrected particle wave amplitude, with $f_I^{J\pm}(s)$ depending on which composite-particle is dealt with and what its isospin is. This yields a form factor for the decay of a composite state which takes account of two-body intermediate states in a dispersion relation for the form factor.

C. Veneziano Model For Pion--Nucleon (πN) Scattering

In order to solve the Omnes equation, we need an expression for the kernel $f(s)$ of the integral equation. Since the composite whose dissociation we wish to describe with a form factor computed in this way is a nucleon composite made up of a nucleon and pion, we choose to construct the kernel $f(s)$ from a Veneziano model for pion-nucleon (πN) scattering. We do this because, if we wanted information about all possible resonances which could be formed with the pion and nucleon quantum numbers, the Veneziano model would provide it for us (within the narrow width approximation) since it can be expres-

sed as a sum over resonances in the relevant channel.

The Veneziano model we use is essentially the one derived by K. Igi[42].

This is given by

$$
\bar{A} = \beta_1 \left[\frac{\Gamma(1-\alpha(t))\,\Gamma(\tfrac{3}{2}-\alpha_{N_\alpha}(s))}{\Gamma(\tfrac{3}{2}-\alpha(t)-\alpha_{N_\alpha}(s))} - \frac{\Gamma(1-\alpha(t))\,\Gamma(\tfrac{3}{2}-\alpha_{N_\alpha}(u))}{\Gamma(\tfrac{3}{2}-\alpha(t)-\alpha_{N_\alpha}(u))} \right.
$$

$$
+ \frac{\Gamma(1-\alpha(t))\,\Gamma(\tfrac{3}{2}-\alpha_{\Delta_\delta}(s))}{\Gamma(\tfrac{3}{2}-\alpha(t)-\alpha_{\Delta_\delta}(s))} - \frac{\Gamma(1-\alpha(t))\,\Gamma(\tfrac{3}{2}-\alpha_{\Delta_\delta}(u))}{\Gamma(\tfrac{3}{2}-\alpha(t)-\alpha_{\Delta_\delta}(u))}
$$

$$
\left. + \frac{\Gamma(\tfrac{3}{2}-\alpha_{N_\alpha}(s))\,\Gamma(\tfrac{3}{2}-\alpha_{\Delta_\delta}(u))}{\Gamma(2-\alpha_{N_\alpha}(s)-\alpha_{\Delta_\delta}(u))} - \frac{\Gamma(\tfrac{3}{2}-\alpha_{\Delta_\delta}(s))\,\Gamma(\tfrac{3}{2}-\alpha_{N_\alpha}(u))}{\Gamma(2-\alpha_{\Delta_\delta}(s)-\alpha_{N_\alpha}(u))} \right]
$$

(A2-19)

$$
\bar{B} = \beta_2 \left[\frac{\Gamma(1-\alpha(t))\,\Gamma(\tfrac{1}{2}-\alpha_{N_\alpha}(s))}{\Gamma(\tfrac{3}{2}-\alpha(t)-\alpha_{N_\alpha}(s))} + \frac{\Gamma(1-\alpha(t))\,\Gamma(\tfrac{1}{2}-\alpha_{N_\alpha}(u))}{\Gamma(\tfrac{3}{2}-\alpha(t)-\alpha_{N_\alpha}(u))} + \frac{\Gamma(\tfrac{1}{2}-\alpha_{N_\alpha}(s))\,\Gamma(\tfrac{1}{2}-\alpha_{N_\alpha}(u))}{\Gamma(1-\alpha_{N_\alpha}(s)-\alpha_{N_\alpha}(u))} \right]
$$

$$
+ \beta_3 \left[\frac{\Gamma(1-\alpha(t))\,\Gamma(\tfrac{1}{2}-\alpha_{\Delta_\delta}(s))}{\Gamma(\tfrac{3}{2}-\alpha(t)-\alpha_{\Delta_\delta}(s))} + \frac{\Gamma(1-\alpha(t))\,\Gamma(\tfrac{1}{2}-\alpha_{\Delta_\delta}(u))}{\Gamma(\tfrac{3}{2}-\alpha(t)-\alpha_{\Delta_\delta}(u))} - \frac{\Gamma(\tfrac{1}{2}-\alpha_{\Delta_\delta}(s))\,\Gamma(\tfrac{1}{2}-\alpha_{\Delta_\delta}(u))}{\Gamma(1-\alpha_{\Delta_\delta}(s)-\alpha_{\Delta_\delta}(u))} \right]
$$

(A2-20)

$$A^{+} = \beta_{4}\left[\frac{\Gamma(1-\alpha(t))\,\Gamma(\frac{3}{2}-\alpha_{N_{\alpha}}(s))}{\Gamma(\frac{3}{2}-\alpha(t)-\alpha_{N_{\alpha}}(s))} + \frac{\Gamma(1-\alpha(t))\,\Gamma(\frac{3}{2}-\gamma_{N_{\alpha}}(u))}{\Gamma(\frac{3}{2}-\alpha(t)-\alpha_{N_{\alpha}}(u))} + \frac{\Gamma(\frac{3}{2}-\alpha_{N_{\alpha}}(s))\,\Gamma(\frac{3}{2}-\alpha_{N_{\alpha}}(u))}{\Gamma(2-\alpha_{N_{\alpha}}(s)-\alpha_{N_{\alpha}}(u))}\right]$$

$$+\ \beta_{5}\left[\frac{\Gamma(1-\alpha(t))\,\Gamma(\frac{3}{2}-\alpha_{\Delta_{\delta}}(s))}{\Gamma(\frac{3}{2}-\alpha(t)-\alpha_{\Delta_{\delta}}(s))} + \frac{\Gamma(1-\alpha(t))\,\Gamma(\frac{3}{2}-\alpha_{\Delta_{\delta}}(u))}{\Gamma(\frac{3}{2}-\alpha(t)-\alpha_{\Delta_{\delta}}(u))} - \frac{\Gamma(\frac{3}{2}-\alpha_{\Delta_{\delta}}(s))\,\Gamma(\frac{3}{2}-\alpha_{\Delta_{\delta}}(u))}{\Gamma(2-\alpha_{\Delta_{\delta}}(s)-\alpha_{\Delta_{\delta}}(u))}\right]$$

$$(A2\text{-}21)$$

$$B^{+} = \beta_{6}\left[\frac{\Gamma(1-\alpha(t))\,\Gamma(\frac{1}{2}-\alpha_{N_{\alpha}}(s))}{\Gamma(\frac{3}{2}-\alpha(t)-\alpha_{N_{\alpha}}(s))} - \frac{\Gamma(1-\alpha(t))\,\Gamma(\frac{1}{2}-\alpha_{N_{\alpha}}(u))}{\Gamma(\frac{3}{2}-\alpha(t)-\alpha_{N_{\alpha}}(u))}\right.$$

$$+\ \frac{\Gamma(1-\alpha(t))\,\Gamma(\frac{1}{2}-\alpha_{\Delta_{\delta}}(s))}{\Gamma(\frac{3}{2}-\alpha(t)-\alpha_{\Delta_{\delta}}(s))} - \frac{\Gamma(1-\alpha(t))\,\Gamma(\frac{1}{2}-\alpha_{\Delta_{\delta}}(u))}{\Gamma(\frac{3}{2}-\alpha(t)-\alpha_{\Delta_{\delta}}(u))}$$

$$\left.+\ \frac{\Gamma(\frac{1}{2}-\alpha_{N_{\alpha}}(s))\,\Gamma(\frac{1}{2}-\alpha_{\Delta_{\delta}}(u))}{\Gamma(1-\alpha_{N_{\alpha}}(s)-\alpha_{\Delta_{\delta}}(u))} - \frac{\Gamma(\frac{1}{2}-\alpha_{\Delta_{\delta}}(s))\,\Gamma(\frac{1}{2}-\alpha_{N_{\alpha}}(u))}{\Gamma(1-\alpha_{\Delta_{\delta}}(s)-\alpha_{N_{\alpha}}(u))}\right]$$

$$(A2\text{-}22)$$

where $A^{(\pm)}$ and $B^{(\pm)}$ are the usual invariant amplitudes for πN scattering. The trajectories are assumed linear. The trajectories have the following properties:

N_{α} : positive parity , $I = \frac{1}{2}$, even signature

$N_{\Delta_{\delta}}$: positive parity , $I = \frac{3}{2}$, odd signature .

Let us consider some general features of πN scattering. If we look at the general invariant amplitude which has the form

$$F_{\beta\alpha}(s,t,u) = \delta_{\beta\alpha}\, F^{(+)}_{(s,t,u)} + \tfrac{1}{2}\,[\tau_\beta, \tau_\alpha]\, F^{(-)}_{(s,t,u)}$$

(A2-23)

where τ_α , τ_β are isospin operators and

$$F(s,t,u) = -A(s,t,u) + \frac{\slashed{q} + \slashed{q}'}{2} B(s,t,u)$$

(A2-24)

with q and q' representing the initial and final momenta of the pions in the problem. It can be shown( ) that the one particle terms are only associated with the $B^{(\pm)}$ terms. The $F^{(\pm)}$ amplitudes are associated with states of definite isospin in the following way:

$$F_{I=\frac{1}{2}} = F^{(+)} + 2F^{(-)}$$
$$F_{I=\frac{3}{2}} = F^{(+)} - F^{(-)}$$

(A2-25)

If we were to express the amplitudes as matrix elements between helicity states $|JM\lambda_1,\lambda_2\rangle$ and perform the usual partial wave expansion, we would have:( )

$$F_{++} = \sum_J (2J+1)\, f^J_{++}\, d^J_{\frac{1}{2}\frac{1}{2}}(\theta)$$

(A2-26)

$$F_{+-} = \sum_J (2J+1)\, f^J_{+-}\, d^J_{-\frac{1}{2}\frac{1}{2}}(\theta)$$

(A2-27)

where

$$F_{++} = \langle \tfrac{1}{2}\, 0 \,|\, F \,|\, \tfrac{1}{2}\, 0 \rangle$$

and

$$F_{+-} = \langle -\tfrac{1}{2} 0 \mid F \mid \tfrac{1}{2} 0 \rangle$$

where $\tfrac{1}{2}$ and 0 label the helicities of the particles in the initial and final states. We have similar expressions for the partial wave amplitudes f_{++}^{J} and f_{+-}^{J}. If we were to form states of definite parity, we would find[56]:

$$f^{J+} = f_{++}^{J} - f_{--}^{J}$$
$$f^{J-} = f_{++}^{J} + f_{--}^{J}$$

(A2-28)

The f^{J+} are matrix elements of the partial wave amplitudes between states of definite parity, i.e.

$$f^{J+} = {}_{+}\langle \tfrac{1}{2} 0 \mid f^{J} \mid \tfrac{1}{2} 0 \rangle_{+}$$

$$= \langle \tfrac{1}{2} 0 \mid f^{J} \mid \tfrac{1}{2} 0 \rangle - \langle -\tfrac{1}{2} 0 \mid A^{J} \mid \tfrac{1}{2} 0 \rangle$$

(A2-29)

$$f^{J-} = {}_{-}\langle \tfrac{1}{2} 0 \mid f^{J} \mid \tfrac{1}{2} 0 \rangle_{-}$$

$$= \langle \tfrac{1}{2} 0 \mid f^{J} \mid \tfrac{1}{2} 0 \rangle + \langle -\tfrac{1}{2} 0 \mid A^{J} \mid \tfrac{1}{2} 0 \rangle$$

(A2-30)

with

$$\mid \tfrac{1}{2}, 0 \rangle_{\pm} = \frac{1}{\sqrt{2}} \left[\mid \tfrac{1}{2}, 0 \rangle \mp \mid -\tfrac{1}{2}, 0 \rangle \right]$$

being states of definite parity:

$$P \mid \tfrac{1}{2}, 0 \rangle_{\pm} = \pm (-1)^{J-\tfrac{1}{2}} \mid \tfrac{1}{2}, 0 \rangle_{\pm}$$

(A2-31)

since $\quad P \mid \tfrac{1}{2}, 0 \rangle = (-1)(-1)^{J-\tfrac{1}{2}} \mid -\tfrac{1}{2}, 0 \rangle \quad$ which derives from ( )

$$P \mid J M \lambda_{1} \lambda_{2} \rangle = \eta_{1} \eta_{2} (-1)^{J-\sigma_{1}-\sigma_{2}} \mid J M -\lambda_{1}, -\lambda_{2} \rangle$$

(A2-32)

where η_1 , η_2 are the intrinsic parities of the parti-
cles in question and σ_1 , σ_2 their spins.

The amplitudes F_{++} and F_{+-} may be expressed in terms
of the usual spin non-flip and spin flip amplitudes F_1 and
F_2 ( ).

$$F_{++} = \left(f_1 + f_2\right) \cos \theta/2$$
$$F_{+-} = \left(f_1 - f_2\right) \sin \theta/2 \tag{A2-33}$$

(θ = scattering angle)

and ( ) :

$$F_1 = \frac{E+m}{8\pi W} \left[A + (W-m)B\right] \tag{A2-34}$$

$$F_2 = \frac{E-m}{8\pi W} \left[-A + (W+m)B\right] \tag{A2-35}$$

W is the square root of the center-of-mass energy and E is
the energy of the nucleon in the center-of-mass and m is
its mass. What we finally obtain for the definite parity
partial wave amplitudes is

$$f^{J\pm} = \frac{1}{2}\int_{-1}^{1} dz \left[P_{J\pm\frac{1}{2}}(z) \, F_1 + P_{J\mp\frac{1}{2}}(z) \, F_2\right] \tag{A2-36}$$

$$(z = \cos\theta)$$

or

$$f_I^{J\pm} = \frac{1}{8\pi W} \frac{1}{2}\int_{-1}^{1} dz \left\{(E+m)\left[A^I + (W-m)B^I\right] P_{J\pm\frac{1}{2}}(z)\right.$$

$$\left. + (E-m)\left[-A^I + (W+m)B^I\right]P_{J\mp\frac{1}{2}}(z)\right\}$$

$$\tag{A2-37}$$

Returning to the Veneziano model, even parity trajectories

for $J = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \cdots$ occur in f^{J-} since the parity

then is $(-)(-1)^{J-\frac{1}{2}} = (-1)^{J+\frac{1}{2}} = (-1)^{2, 4, 6 \cdots} = +1$. Even

parity trajectories for $J = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \cdots$ occur in f^{J+}

since the parity is $(+)(-1)^{J-\frac{1}{2}} = (-1)^{0, 2, 4, \cdots} = +1$. Odd

parity trajectories for $J = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \cdots$ occur in f^{J+}

since the parity is $(+)(-1)^{J-\frac{1}{2}} = (-1)^{1, 3, 5, \cdots} = -1$. Odd

parity trajectories for $J = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \cdots$ occur in f^{J-}

since the parity is $(-)(-1)^{J-\frac{1}{2}} = (-1)^{J+\frac{1}{2}} = (-1)^{1, 3, 5, \cdots} = -1$.

A general Venezion amplitude $V_c^{ab}(\alpha(x), \alpha(y))$ may

be written: $(\quad)$

$$V_c^{ab}(\alpha(x), \alpha(y)) = \frac{\Gamma(a - \alpha(x)) \; \Gamma(b - \alpha(y))}{\Gamma[(c - \alpha(x) - \alpha(y))]} \qquad (A2-38)$$

$$= \sum_{n=a} \frac{(-1)^{n-a}}{(n-a)!} \; r_{n+b-c}(c - n - \alpha(y)) \frac{1}{n - \alpha(x)}$$

$$= \sum_{n=b} \frac{(-1)^{n-b}}{(n-b)!} \; r_{n+a-c}(c - n - \alpha(x)) \frac{1}{n - \alpha(y)}$$

$$(A2-39)$$

where $r_n(x)$ is the Pochhammer polynomial: [106]

$$r_n(x) = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1) \cdots (x+n-1)$$

$$r_n(-x) = (-1)^n \; r_n(x - n + 1)$$

$$(A2-40)$$

This means that the Veneziano amplitude may be expressed as a

sum over poles in either of the channels of interest. The

residue of $V_c^{ab}(\alpha(x), \alpha(y))$ at $\alpha(x) = n$ is

$$\frac{(-1)^{n-a}}{(n-a)!} \; r_{n+b-c}(c - n - \alpha(y)) \qquad (A2-41)$$

Similarly the residue of $V_c^{a\,b}\left(\alpha(x),\alpha(y)\right)$ at $\alpha(y)=n$ is

$$\frac{(-1)^{n-b}}{(n-b)!}\ r_{n+a-c}\left(c-n-\alpha(x)\right) \tag{A2-42}$$

The Pochhammer polynomial may also be written[106]

$$r_n(x)=\sum_{k=0}^{n}\rho_{nk}\,x^{n-k} \tag{A2-43}$$

where

$$\rho_{n0}=1$$
$$\rho_{nn}=0\quad,\quad n>0 \tag{A2-44}$$
$$\rho_{nk}=\sum_{i_1<i_2<\cdots<i_k=1}^{n-1}i_1\,i_2\cdots i_k\quad,\quad k\neq0,n$$
$$\rho_{nk}=\rho_{n-1,k}+(n-1)\,\rho_{n-1,k-1}$$

Finally we need

$$\int_{-1}^{1}dz\ P_\ell(z)\,z^j=\begin{cases}\dfrac{2^{\ell+1}\,j!\,(\frac{1}{2}j+\frac{1}{2}\ell)!}{(\frac{1}{2}j-\frac{1}{2}\ell)!\,(j+\ell+1)!}&,\ j\geqslant\ell,\ j-\ell\ \text{even}\\[4mm]0&\end{cases} \tag{A2-45}$$

because the residue of a particular partial wave amplitude at a pole is the partial wave projection of the relevant Pochhammer polynomial. The residue of $f_I^{J\pm}(s)$ at $\alpha(x)=n$ is called $f_I^\pm(n,J)$, i.e.

$$f_I^{J\pm}(x)\sim\frac{f_I^\pm(n,J)}{n-\alpha(x)}\quad,\quad \alpha(x)\longrightarrow n \tag{A2-46}$$

Now the Veneziano model for πN scattering given in equation ($A2-2$) has the feature of having the correct signatures for the Regge trajectories indicated. We will illustrate this in detail below for the nucleon trajectory, $\alpha_{N_\alpha}(s)$ and the trajectory $\alpha_{\Delta_s}(s)$. Also by eliminating the isospin doublet of the nucleon, the $\Delta(1236)$ and the $J^P = \frac{5}{2}^+$ particle on the N_α trajectory we can obtain certain relations between the coefficients of the various terms. To completely determine the coefficients we would need to work out further relations and also fit the model to data. However, we're only interested in the general features of the model and what form it gives for the vertex function for the composite nucleon. For further details the reader is referred to K. Igi.$^{(\ \ )}$

C.1. Positive Signature for the $\alpha_{N_\alpha}(s)$ Trajectory.

If $\alpha_{N_\alpha}(s)$ is positive signature trajectory, then for the particles on the trajectory $J - \frac{1}{2}$ = even. Any contribution from particles with $J - \frac{1}{2}$ = odd must vanish, i.e. the residue at the pole at $\alpha_{N_\alpha}(s) = J$ such that $J - \frac{1}{2}$ is odd must vanish. Let's check vanishing for $\alpha_{N_\alpha}(s) = \frac{3}{2}$, i.e. $J - \frac{1}{2}$ = odd, calculate residues at poles, look at partial wave of relevance. Since $\alpha_{N_\alpha}(s)$ has even parity we look at f^{J-} for $\alpha_{N_\alpha}(s) = \frac{3}{2}$ and for $I = \frac{1}{2}$.

$$f^-\left(\tfrac{3}{2}, \tfrac{3}{2}\right) = \tfrac{1}{2} \int_{-1}^{1} dz \left(P_1 \, \text{Res} \, f_1 + P_2 \, \text{Res} \, f_2 \right)$$

$$f_1 = \frac{E+m}{8\pi W}\left[A + (W-m)B\right]$$

$$f_2 = \frac{E-m}{8\pi W}\left[-A + (W+m)B\right]$$

$$f_{I=\frac{1}{2}} \quad f^+ + 2f^-$$

Res A^- :

$\alpha_{N_\alpha}(s) = \frac{3}{2}$

$$\beta_1\left[\frac{(-1)^0}{0!}\, r_{\frac{3}{2}+1-\frac{3}{2}}^{(\frac{3}{2}-\frac{3}{2}-\alpha(t)} + \frac{(-1)^0}{0!}\, r_{\frac{3}{2}+\frac{3}{2}-2}^{(2-\frac{3}{2}-\alpha_{\Delta_\delta}(u))}\right]$$

$$= \beta_1\left[-\alpha(t) + \frac{1}{2} - \alpha_{\Delta_\delta}(u)\right]$$

$$= \beta_1\left[-\gamma - bt + \frac{1}{2} - \delta - bu\right]$$

$$= \beta_1\left[-\gamma-\delta + \frac{1}{2} - b(c_1 + 2q_\beta^2 z) - b(c_2 - 2q_\beta^2 z)\right]$$

$$= \beta_1\left[\frac{1}{2}-\gamma-\delta - b(c_1 + c_2) - 2q_\beta^2 b z + 2q_\beta^2 b z\right]$$

$$= \beta_1\left[const.\right] = constant$$

Res A^+ : $\alpha_{N_\alpha}(s) = \frac{3}{2}$

$$\beta_4\left[\frac{(-1)^0}{0!}\, r_{\frac{3}{2}+1-\frac{3}{2}}^{(\frac{3}{2}-\frac{3}{2}-\alpha(t))} + \frac{(-1)^0}{0!}\, r_{\frac{3}{2}+\frac{3}{2}-2}^{(2-\frac{3}{2}-\alpha_{N_\alpha}(u))}\right]$$

$$= \beta_4\left[const.\right] = constant$$

$$\text{Res. } B^- : \quad \beta_2 \left[\frac{(-1)^1}{1!} \, r_{3/2 + 1 - 3/2} \, (3/2 - 3/2 - \alpha(t)) + \frac{(-1)^1}{1!} \, r_{3/2 + 1/2 - 1} \, (1 - 3/2 - \alpha_{N_\alpha}(u)) \right]$$

$$\alpha_{N_\alpha}(s) = 3/2$$

$$= \beta_2 \left[const. \right] = constant$$

$$\text{Res. } B^+ : \quad \beta_6 \left[\frac{(-1)^1}{1!} \, r_{3/2 + 1 - 3/2} \, (3/2 - 3/2 - \alpha(t)) + \frac{(-1)^1}{1!} \, r_{3/2 + \frac{1}{2} - 1} \, (1 - 3/2 - \alpha_{\Delta_\delta}(u)) \right]$$

$$\alpha_{N_\alpha}(s) = 3/2$$

$$= \beta_6 \left[const. \right] = constant$$

The residues of $A^+ + 2A^-$ and $B^+ + 2B^-$ at $\alpha_{N_\alpha}(s) = 3/2$ both equal constants.

$$f^-(3/2, 3/2) = \frac{1}{2} \int_{-1}^{1} dz \, (P_1 \, const. + P_2 \, const.) = 0$$

This implies that $\alpha_{N_\alpha}(s)$ has even signature.

C.2.   Negative Signature for the Δ_δ Trajectory

If $\alpha_{\Delta_\delta}(s)$ is negative signature trajectory, then for particles on the trajectory $J - \frac{1}{2}$ = odd. Contribution from particles with $J - \frac{1}{2}$ = even must vanish. Check vanishing for $\alpha_{\Delta_\delta}(s) = \frac{1}{2}$, i.e. $J - \frac{1}{2} = 0$ = even.

Since $\alpha_{\Delta_\delta}(s)$ has even parity, we look at f^{J+} for $I = \frac{3}{2}$ and we also look at $A^+ - A^-$ and $B^+ - B^-$.

$$f^+(\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{2} \int_{-1}^{1} dz \, (P_1 \, Res \, f_1 \; + \; P_0 \, Res \, f_2)$$

We need only look at B terms

$$Res \, B^+ : \atop \alpha_{\Delta_\delta}(s) = \frac{1}{2} \quad \beta_6 \left[\frac{(-1)^0}{0!} r_{\frac{1}{2} + 1 - \frac{3}{2}}(\tfrac{3}{2} - \tfrac{1}{2} - \alpha(t)) - \frac{(-1)^0}{0!} r_{\frac{1}{2} + \frac{1}{2} - 1}(1 - \tfrac{1}{2} - \alpha_{\Delta_\delta}(u)) \right]$$

$$= \beta_6 \left[1 - 1 \right] = 0$$

$$Res \, B^- : \atop \alpha_{\Delta_\delta}(s) = \frac{1}{2} \quad \beta_3 \left[\frac{(-1)^0}{0!} r_{\frac{1}{2} + 1 - \frac{3}{2}}(\tfrac{3}{2} - \tfrac{1}{2} - \alpha(t)) - \frac{(-1)^0}{0!} r_{\frac{1}{2} + \frac{1}{2} - 1}(1 - \tfrac{1}{2} - \alpha_{\Delta_\delta}(u)) \right]$$

$$= \beta_3 \left[1 - 1 \right] = 0$$

$$f^+(\tfrac{1}{2}, \tfrac{1}{2}) = 0 \qquad \text{which implies that } \alpha_{\Delta_\delta}(s)$$

has odd signature.

C.3. Elimination of Isospin Doublet of Nucleon

Consider vanishing of $f^+_{I=3/2}(\frac{1}{2}, \frac{1}{2})$ for no iso-spin doublet of nucleon. We only consider B terms, in particular $B^+ - B^-$ and evaluate the residue at $\alpha_{N_\alpha}(s) = \frac{1}{2}$.

$$f^+_{I=3/2}(\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{2} \int_{-1}^{1} dz\, (P_1 \operatorname{Res} f_1 + P_0 \operatorname{Res} f_2)$$

$$= \tfrac{1}{8\pi W} \tfrac{1}{2} \int_{-1}^{1} dz\, \{P_1 (E+m)(W-m)(B^+ - B^-) + P_0(E-m)(W+m)(B^+-B^-)\}$$

$$\operatorname{Res} B^+ :\quad \beta_6 \left[\frac{(-1)^0}{0!} \Gamma_{\frac{1}{2}+1}(\tfrac{3}{2}-\tfrac{1}{2}-\alpha(t)) + \frac{(-1)^0}{0!} \Gamma_{\frac{1}{2}+\frac{1}{2}-1}(1-\tfrac{1}{2}-\alpha_{\Delta_\delta}(u)) \right]$$
$$\alpha_{N_\alpha}(s) = \tfrac{1}{2}$$

$$= \beta_6 [1+1] = 2\beta_6$$

$$\operatorname{Res} B^- :\quad \beta_2 \left[\frac{(-1)^0}{0!} \Gamma_{\frac{1}{2}+1}(\tfrac{3}{2}-\tfrac{1}{2}-\alpha(t)) + \frac{(-1)^0}{0!} \Gamma_{\frac{1}{2}+\frac{1}{2}-1}(1-\tfrac{1}{2}-\alpha_{N_\alpha}(u)) \right]$$
$$\alpha_{N_\alpha}(s) = \tfrac{1}{2}$$

$$= \beta_2 [1+1] = 2\beta_2$$

$$B^+ - B^- = 2(\beta_6 - \beta_2)$$

Contribution from the P_1 term is equal to zero. Contribution from the P_0 term yields $\beta_6 = \beta_2$.

C.4. Elimination of Isospin Doublet of Δ (1236)

Elimination of isospin doublet of Δ (1236), i.e.

$$f_{I=1/2}^{-}(3/2,\, 3/2) \qquad \text{must equal zero.}$$

$$f_1^{I=3/2} = \frac{E+m}{8\pi W}\left[(A^+ + 2A^-) + (W-m)(B^+ + 2B^-)\right]$$

$$f_2^{I=3/2} = \frac{E-m}{8\pi W}\left[-(A^+ + 2A^-) + (W+m)(B^+ + 2B^-)\right]$$

$$\text{Res } A^+: \quad \beta_5\left[\frac{(-1)^0}{0!}\, r_{3/2+1-3/2}\left(\tfrac{3}{2}-\tfrac{3}{2}-\alpha(t)\right) - \frac{(-1)^0}{0!}\, r_{3/2+3/2-2}\left(2-\tfrac{3}{2}-\alpha_{\Delta_s}(u)\right)\right]$$

$$\alpha_{\Delta_s}(s)=3/2$$

$$= \beta_5\left[-\alpha(t) + \alpha_{\Delta_s}(u) - \tfrac{1}{2}\right]$$

$$= \beta_5\left[\underbrace{\delta - \gamma - \tfrac{1}{2} + b(C_2-C_1)}_{K_1} - 4q^2 b z\right]$$

$$= \beta_5\left[K_1 - 4q^2 b z\right]$$

$$\text{Res } A^-: \quad \beta_1\left[\frac{(-1)^0}{0!}\, r_{3/2+1-3/2}\left(\tfrac{3}{2}-\tfrac{3}{2}-\alpha(t)\right) - \frac{(-1)^0}{0!}\, r_{3/2+3/2-2}\left(2-\tfrac{3}{2}-\alpha_{N_\alpha}(u)\right)\right]$$

$$\alpha_{\Delta_s}(s)=3/2$$

$$= \beta_1\left[-\alpha(t) + \alpha_{N_\alpha}(u) - 1/2\right]$$

$$= \beta_1\left[\underbrace{\alpha - \gamma - \tfrac{1}{2} + b(c_2-c_1)}_{K_2} - 4q^2 b z\right]$$

$$= \beta_1\left[K_2 - 4q^2 b z\right]$$

$$\text{Res} \quad A^+ + 2A^- : \quad K_1 \beta_5 + 2K_2 \beta_1 - 4q^2 bz \left[\beta_5 + 2\beta_1 \right]$$
$$\alpha_{\Delta_\delta}(s) = 3/2$$

$$\text{Res} \quad B^+ \quad : \quad \beta_6 \left[\frac{(-1)'}{1!} r_{3/2 + 1 - 3/2} \left(\tfrac{3}{2} - 3/2 - \alpha(t) \right) - \frac{(-1)'}{1!} r_{3/2 + 1/2 - 1} \left(1 - 3/2 - \alpha_N(u) \right) \right]$$
$$\alpha_{\Delta_\delta}(s) = 3/2$$

$$= \beta_6 \left[K_3 + 4q^2 b z \right]$$

$$\text{Res} \quad B^- \quad : \quad \beta_3 \left[\frac{(-1)'}{1!} r_{3/2 + 1 - 3/2} \left(\tfrac{3}{2} - 3/2 - \alpha(t) \right) - \frac{(-1)'}{1!} r_{3/2 + 1/2 - 1} \left(1 - 3/2 - \alpha_\Delta(u) \right) \right]$$

$$= \beta_3 \left[K_4 + 4q^2 b z \right]$$

$$\text{Res} \quad B^+ + 2B^- : \quad \beta_6 K_3 + 2\beta_3 K_4 + 4q^2 b z \left[\beta_6 + 2\beta_3 \right]$$
$$\alpha_{\Delta_\delta}(s) = 3/2$$

$$f^-_{I + 1/2}(3/2, 3/2) = \frac{1}{8\pi N} \; 1/2 \int_{-1}^{1} dz \left\{ P_1 \; \text{Res} f_1 \Big|_{\alpha_{\Delta_\delta}(s) = 3/2} + P_2 \; \text{Res} f_2 \Big|_{\alpha_{\Delta_\delta}(s) = 3/2} \right\}$$

In P_1 term, the constants integrate to zero.

$$1/2 \int_{-1}^{1} dz \; z \; P_1(z) = 1/2 \int_{-1}^{1} z^2 \, dz = \frac{1}{2} \left. \frac{z^3}{3} \right]_{-1}^{1} = 1/3$$

In the P_2 term, the constants integrate to zero and the term proportional to z integrates to zero.

So we have:

$$\frac{1}{8\pi W} \; \frac{1}{2} \int_{-1}^{1} dz \left\{ P_1 \left[(E+m)(W-m)(B^+ + 2B^-) + (E+m)(A^+ + 2A^-) \right] \right\} = 0$$

$$\frac{E+m}{8\pi W} \; \frac{4q^2 b}{3} \left[-(\beta_5 + 2\beta_1) + (W-m)(\beta_6 + 2\beta_3) \right] = 0$$

which implies that either

$$(\beta_5 + 2\beta_1) = (W-m)(\beta_6 + 2\beta_3)$$

or simply $\beta_5 = -2\beta_1$, $\beta_6 = -2\beta_3$.

C.5. Elimination of Isospin Doublet of Particle With
$J^P = \frac{5}{2}^+$ on the N_α Trajectory

Elimination of isospin doublet of particle with $J^P = \frac{5}{2}^+$

on N_α trajectory, we look at $\mathcal{R}es\, f^{J+}$ at $\alpha(s) = \frac{5}{2}$, i.e.

$$f^+_{I=3/2}\left(\tfrac{5}{2}, \tfrac{5}{2}\right) = 0 = \tfrac{1}{2}\int_{-1}^{1} dz\, \left(P_3\, f_1^{I=3/2} + P_2\, f_2^{I=3/2}\right)$$

This involves $\mathcal{R}es\,(A^+ - A^-)$ as well as $\mathcal{R}es\,(B^+ - B^-)$

$$\mathcal{R}es\, A^+ : \quad \beta_4\left[\frac{(-1)^1}{1!}\, r_{5/2+1-3/2}^{\left(\frac{3}{2} - \frac{5}{2} - \alpha(t)\right)}\right.$$
$$\left. + \frac{(-1)^1}{1!}\, r_{5/2+3/2-2}^{\left(2 - \frac{5}{2} - \alpha_{N_\alpha}(u)\right)}\right]$$

$$= -\beta_4\left[r_2(-1-\alpha(t)) + r_1\left(-\tfrac{1}{2} - \alpha_{N_\alpha}(u)\right)\right]$$

$$\mathcal{R}es\, A^- : \quad -\beta_1\left[r_2(-1-\alpha(t)) + r_1\left(-\tfrac{1}{2} - \alpha_{\Delta_\delta}(u)\right)\right]$$

When we write $\alpha(t)$, $\alpha_{N_\alpha}(u)$, $\alpha_{\Delta_\delta}(u)$ in terms of z plus constant terms, the constant terms integrate out. We are left with an integral resulting from the P_3 term which implies that

$$const\,(\beta_4 - \beta_1) = 0 \quad \text{or} \quad \beta_4 = \beta_1$$

So we have finally the relations

1. $\quad \beta_6 = \beta_2$

2. $\quad (\beta_5 + 2\beta_1) = (W-m)(\beta_6 + 2\beta_3)$

3. $\quad \beta_4 = \beta_1$

D. Form Factor Calculation For $N \to N'\pi$

We will treat the problem where both N and N' have spin $\frac{1}{2}$. Then the nucleon N is itself on the N_α trajectory. So we have to calculate $f_I^{J^P} = f_{I=\frac{1}{2}}^{\frac{1}{2}+}$. For $I = \frac{1}{2}$, the terms of interest are $A^+ + 2A^-$ and $B^+ + 2B^-$ and

$$F_1^{I=\frac{1}{2}} = \frac{1}{8\pi W}(E+m)\left[(A^+ + 2A^-) + (W-m)(B^+ + 2B^-)\right]$$

$$F_2^{I=\frac{1}{2}} = \frac{1}{8\pi W}(E-m)\left[-(A^+ + 2A^-) + (W+m)(B^+ + 2B^-)\right]$$

If $\alpha_{N_\alpha}(s) = \frac{1}{2}$, we only consider the $B^{(\pm)}$ terms which contain the nucleon pole

$$f^{\frac{1}{2}+} = \frac{1}{2}\int_{-1}^{1} dz\,(P_1 F_1 + P_0 F_2) \qquad \text{(A2-47)}$$

Consider B^+ near $\alpha_{N_\alpha}(s) = \frac{1}{2}$, then

$$B^+(s,t,u) = \frac{2\beta_b}{n - \alpha(s)} \qquad , \qquad n = 1/2$$
$$\alpha_{N_\alpha}(s) \approx 1/2$$

from evaluating the Pochhammer polynomial numerators (which in this case equal one) for the terms in $B^+(s,t,u)$ contributing to the nucleon pole. Similarly we obtain

$$B^-(s,t,u) = \frac{2\beta_2}{n - \alpha(s)} \qquad , \qquad n = 1/2$$
$$\alpha_{N_\alpha}(s) \approx 1/2$$

But $\quad \beta_b = \beta_2 \quad$. Therefore, we obtain

$$(B^+ + 2B^-) = \frac{6\beta_2}{n - \alpha(s)} \qquad , \qquad n = \tfrac{1}{2}$$

The term $B^+ + 2B^-$ and hence F_1 and F_2 are constants with respect to z. Therefore only the second term in the integral in equation (A2-47) contributes. Thus we have

$$f_{I=\frac{1}{2}}^{\frac{1}{2}+}(s) = F_2 = \frac{6\beta_2}{n - \alpha(s)} \frac{(E-m)(W+m)}{8\pi W}$$

The unitarized pion nucleon partial wave amplitude is

$$f_{\pi N}(s) = \frac{f_{I=1/2}^{\frac{1}{2}+}(s)}{\left[1 + h(s) f_{I=1/2}^{\frac{1}{2}+}(s) \right]}$$

$$f_{\pi N}(s_0) = \text{a number} = C$$

Therefore

$$F(s) = \frac{f_{\pi N}(s)}{f_{\pi N}(s_0)} = \frac{6 c \beta_2 (E-m)(W+m)}{\left[8\pi W (n - \sigma(s)) + (E-m)(W+m) h(s) \beta_\alpha \right\}}$$

This has a behavior given roughly in the figure (A2-1) below

m_N = nucleon mass
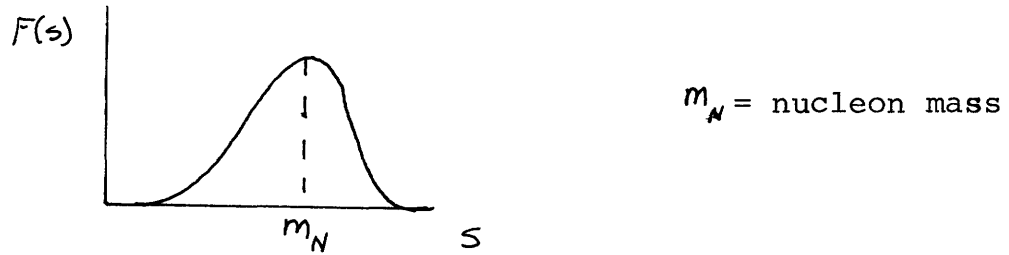
The unitarization weight is

$$h(s) = \frac{-ik}{\sqrt{s}} + \frac{2}{\pi} \frac{k}{\sqrt{s}} \ln \left\{ \frac{\sqrt{s} + \left[s - (M+m)^2 \right]^{\frac{1}{2}}}{(M+m)^2} \right\}$$

APPENDIX III: CONSTRUCTION AND HIGH ENERGY BEHAVIOR OF

INHOMOGENEOUS TERM

The inhomogeneous term in our integral equation is given
by figure (A3-1).

figure (A3-1)

which can be written

$$\int d^4_p \, d^4_q \, \langle k | B_{NN}(s) | p \rangle \, \left\{ \tau_N(\sigma_p^+) - \tau_N(\sigma_p^-) \right\} \langle p | B_{N\rho}(s) | q \rangle$$

$$\times \; \tau(\sigma_q) \, \langle q | B_{\rho N}(s) | k' \rangle$$

$$+ \int d^4_p \, d^4_q \, \langle k | B_{NN}(s) | p \rangle \, \tau_N(\sigma_p) \, \left\{ \langle p | B_{N\rho}(s^+) | q \rangle \right.$$

$$\left. - \langle p | B_{N\rho}(s^-) | q \rangle \right\} \, \tau_\rho(\sigma_q) \, \langle q | B_{\rho N}(s) | k' \rangle$$

$$+ \int d^4_p \, d^4_q \, \langle k | B_{NN}(s) | p \rangle \, \tau_N(\sigma_p) \, \langle p | B_{N\rho}(s) | q \rangle \left\{ \tau_\rho(\sigma_q^+) - \tau_\rho(\sigma_q^-) \right\}$$

$$\times \; \langle q | B_{\rho N}(s) | k' \rangle$$

(A3-1)

We now concentrate on the term with the discontinuity of the
one —particle exchange graph $B_{N\rho}(s)$. We do this because
we believe, since this term only contributes to three-body

phase space, it will give the largest contribution since three-body phase space is larger than two-body phase space.

In performing the intermediate state integrations, we take account of the fact that we want the inhomogeneous term in the high energy limit. Therefore, the states of interest to us are found near the phase space boundary, i.e., they are found within a shell in phase space of radius $q = |\vec{q}| \sim \sqrt{s}/2$, where the width of the shell is $dq = \frac{p}{E} dE \rightarrow dE$ at high energy. If $dE \sim \frac{1}{E}$, the width of the shell goes like $\frac{c}{\sqrt{s}}$. Such a behavior could come out of a flat rapidity distribution as in the multiperipheral model. Otherwise, we can say that we want the phase space interval of relevance to shrink as we go to high energy.

Now to actually carry out the integrations indicated we do the following: we expand the matrix elements (in the center-of-mass) in equation ($A3-1$) in terms of spherical harmonics to integrate out the angular dependence. We are then left with integrals over the magnitudes of the momenta of orbital angular momentum matrix elements. These can be rewritten as matrix elements between states of total angular momentum J, and we finally end up with an equation relating the matrix element of the inhomogeneous term between states of definite total angular momentum to an integral over magnitudes of momenta of matrix elements of one particle exchange graphs between states of definite total angular momentum.

Let us concretize this by considering the rho (ρ) meson box term, i.e.

$$\langle \underset{\sim}{p}, \nu | \ Box_\rho \ | \underset{\sim}{k}', r' \rangle = \sum_{t,i} \frac{1}{(2\pi)^3} \int \frac{d^3 \underset{\sim}{q}}{E_q D(\sigma_q)} \langle \underset{\sim}{p}, \nu | B_{N\rho}(s) | \underset{\sim}{q}, t, i \rangle$$

$$\times \ \langle \underset{\sim}{q}, t, i | \ B_{\rho N}(s) | \underset{\sim}{k}', r' \rangle \qquad (A3\text{-}2)$$

This is given in figure (A3-2).
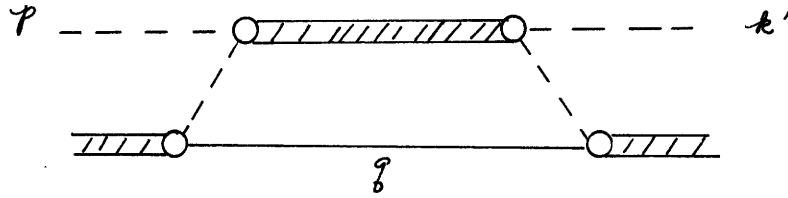
fig. (A3-2)

The labels ν , r' , t , i label the spin projections of the nucleons and the rho meson in the diagram. We may write

$$\langle \underset{\sim}{p}, \nu | \ Box_\rho | \underset{\sim}{k}', r' \rangle = \sum_{\substack{\ell \ell' \\ m m'}} \langle p, \ell, m, \nu | \ Box_\rho \ | k', \ell', m', r' \rangle$$

$$\times \ Y^*_{\ell m}(\hat{p}) \ Y_{\ell' m'}(\hat{k}') \qquad (A3\text{-}3)$$

$$\langle \underset{\sim}{p}, \nu | B_{N\rho}(s) | \underset{\sim}{q}, t, i \rangle = \sum_{\substack{\ell \ell' \\ m m'}} \langle p, \ell, m, \nu | B_{N\rho}(s) | q, \ell'', m'', t, i \rangle$$

$$\times \ Y^*_{\ell m}(\hat{p}) \ Y_{\ell'' m''}(\hat{q}) \qquad (A3\text{-}4)$$

$$\langle \underset{\sim}{q}, t, i | \ B_{\rho N}(s) | \underset{\sim}{k}', r' \rangle = \sum_{\substack{\ell''' \ell' \\ m''' m'}} \langle q, \ell''', m''', t, i | B_{\rho N}(s) | k', \ell', m', r' \rangle$$

$$\times \ Y^*_{\ell''' m'''}(\hat{q}) \ Y_{\ell' m'}(\hat{k}')$$

$$(A3\text{-}5)$$

Now

$$\int d\Omega_{\hat{q}} \; Y^*_{\ell''' m'''}(\hat{q}) \; Y_{\ell'' m''}(\hat{q}) = \delta_{\ell'' \ell'''} \delta_{m'' m'''}$$

Performing the sums over ℓ''', m''', we obtain

$$(A3-6)$$

$$\sum_{\substack{\ell \ell' \\ m m'}} \langle p, \ell, m, \mathcal{A} | \text{Box}_\rho | k', \ell', m', r' \rangle Y^*_{\ell m}(\hat{p}) Y_{\ell' m'}(\hat{k}')$$

$$= \frac{1}{(2\pi)^3} \sum_{\substack{\ell \ell' \ell'' \\ m m' m''}} \int \frac{q^2 dq}{E_q D(\sigma_q)} \langle p, \ell, m, \mathcal{A} | B_{N\rho}^{(s)} | q, \ell'', m'', t, i \rangle$$

$$\times \langle q, \ell'', m'', t, i | B_{\rho N}^{(s)} | k', \ell', m', r' \rangle Y^*_{\ell, m}(\hat{p}) Y_{\ell' m'}(\hat{k}')$$

$$(A3-7)$$

Call $\ell'' = \ell_1$, $m'' = m_1$, then we get

$$\langle p, \ell, m, \mathcal{A} | \text{Box}_\rho | k', \ell', m', r' \rangle$$

$$= \frac{1}{(2\pi)^3} \sum_{\substack{\ell_1, m_1 \\ t, i}} \int \frac{q^2 dq}{E_q D(\sigma_q)} \langle p, \ell, m, \mathcal{A} | B_{N\rho}^{(s)} | q, \ell_1, m_1, t, i \rangle$$

$$\times \langle q, \ell_1, m_1, t, i | B_{\rho N}^{(s)} | k', \ell', m', r' \rangle$$

$$(A3-8)$$

If we now use Clebsch-Gordon and Racah coefficients to project states of total angular momentum, J, we obtain:

$$\langle p, \ell, J, M_J \mid Box_\rho \mid k', \ell', J'', M_J'' \rangle$$

$$= \frac{1}{(2\pi)^3} \sum_{J' M_J'} \sum_{\ell_1, j} \int \frac{q^2 dq}{E_q D(\sigma_q)} \langle p, \ell, J, M_J \mid B_{N\rho}(s) \mid q, \ell_1, j, J', M_J' \rangle$$

$$\times \langle q, \ell_1, j, J', M_J' \mid B_{\rho N}(s) \mid k', \ell, J'', M_J'' \rangle \qquad (A3\text{-}9)$$

If we consider the entire absorptive term (write three-body cut) we obtain

$$\langle k, \ell, J, M_J \mid A \mid k', \ell', J', M_J' \rangle$$

$$= \frac{1}{(2\pi)^3} \sum_{\substack{J_1, J_2 \\ M_1, M_2}} \sum_{\substack{\ell_1, \ell_2 \\ j}} \int \frac{p^2 dp}{\omega_p D(\sigma_p)} \langle k, \ell, J, M_J \mid B_{NN}(s) \mid p, \ell_1, J_1, M_1 \rangle$$

$$\times \langle p, \ell_1, J_1, M_1 \mid B_{N\rho}(s) \mid q, \ell_2, j, J_2, M_2 \rangle$$

$$\qquad\qquad\qquad\qquad\qquad\qquad (A3\text{-}10)$$

$$\times \langle q, \ell_2, j, J_2, M_2 \mid B_{\rho N}(s) \mid k', \ell', J', M_J' \rangle$$

The states of total angular momentum J are formed as follows: for a pion-nucleon state, we have only the orbital angular momentum and the spin of the nucleon. The state is labelled by $\mid \ell m r \rangle$ with ℓ, m being the orbital angular momentum of the system and its magnetic projection, r being the projection of the nucleon spin. Therefore, we have

$$\mid J M_J \rangle = \sum_{m r} \mid \ell m r \rangle \langle \ell m r \mid J M_J \rangle$$

$$|J\, M_J\rangle = \sum_{m,r} |\ell m r\rangle\; C_{m\;\;r\;\;M_J}^{\ell\;\;\frac{1}{2}\;\;J} \tag{A3-11}$$

For a rho meson-nucleon state, we first combine the spins of the nucleon and rho meson and then couple this resultant angular momentum to the orbital angular momentum to form total angular momentum, J. Thus, we have

$$|J\, M_J\rangle = \sum_{\substack{i,t \\ m_\lambda,\, m_j}} |\ell_\lambda m_\lambda t\, i\rangle\; C_{i\;\;t\;\;m_j}^{1\;\;\frac{1}{2}\;\;j}\; C_{m_\lambda\;\;m_j\;\;M_J}^{\ell_\lambda\;\;j\;\;J} \tag{A3-12}$$

We then express the matrix element in equation $(A3\text{-}10)$ in terms of helicity states, do a helicity expansion to recapture the inhomogeneous term for high energy and continue it to the Breit frame (defined below).

Our original expression for the inhomogeneous term was

$$\langle \underset{\sim}{k},r\,|\,A(s)\,|\,\underset{\sim}{k'},r'\rangle = \int \frac{d^3p}{2\omega_p}\; \langle \underset{\sim}{k},r\,|\,B_{NN}(s)\,|\,\underset{\sim}{p},\lambda\rangle\; S_N(\sigma_p)$$

$$\times\; \langle \underset{\sim}{p},s\,|\,B_{\alpha x\rho}(s)\,|\,\underset{\sim}{k'},r'\rangle \tag{A3-13}$$

remembering that $\quad \mathcal{L}(\sigma_k) = \int(k^2 - m^2)\, S(\sigma_k)$

$$= \int \frac{d^3p}{2\omega_p}\; \frac{d^3q}{2E_q}\; \chi_s^{+}\left\{ E(\underset{\sim}{k},\underset{\sim}{p},s) + i\,\sigma\cdot\underset{\sim}{k}\times\underset{\sim}{p}\; F(\underset{\sim}{k},\underset{\sim}{p},s)\right\} \chi_r$$

$$\times\; S_N(\sigma_p)\, N_k N_p\; \chi_t^{+}\left\{ \left(\frac{\sigma\cdot\underset{\sim}{p}}{E_p+M} + \frac{\sigma\cdot\underset{\sim}{q}}{E_q+M'}\right)\tilde{B}_{N\rho}(\underset{\sim}{p},\underset{\sim}{q},s)\right\} \chi_s\; S_\rho(\sigma_q)\, N_p N_q$$

$$\times\; \chi_{r'}^{+}\left\{ \left(\frac{\sigma\cdot\underset{\sim}{q}}{E_q+M'} + \frac{\sigma\cdot\underset{\sim}{k'}}{E_{k'}+M}\right) B_{\rho N}(\underset{\sim}{q},\underset{\sim}{k'},s)\right\} \chi_t\; N_q N_{k'} \tag{A3-14}$$

where

$$E(\underline{k}, \underline{p}, s) = \frac{2Mi\,(E_{k+p} + \omega_k + \omega_p)}{E_{k+p}\left[(E_{k+p} + \omega_k + \omega_p)^2 - s\right]}\left[A(z)A_1(z)\,k^2 + B(z)B_1(z)\,p^2\right.$$

$$(A3\text{-}15)$$

$$\left. + \Big(A(z)B_1(z) + B(z)A_1(z)\Big)\,\underline{k}\cdot\underline{p}\right]$$

$$F(\underline{k}, \underline{p}, s) = \frac{2Mi\,(E_{k+p} + \omega_k + \omega_p)}{E_{k+p}\left[(E_{k+p} + \omega_k + \omega_p)^2 - s\right]}\left[A(z)B_1(z) - B(z)A_1(z)\right]$$

$$(A3\text{-}16)$$

$$\tilde{B}_{N\rho}(\underline{p}, \underline{q}, s) = \epsilon^\mu P_\mu\,\gamma_{NN'\pi}\,\gamma_{\rho\pi\pi}\,\delta\left[(P - p - q)^2 - \mu^2\right]$$

$$= \text{Disc}_s\, B_{N\rho}(\underline{p}, \underline{q}, s) \qquad (A3\text{-}17)$$

$$B_{\rho N}(\underline{q}, \underline{k}, s) = k'^\mu\,\epsilon_\mu\,\gamma_{NN'\pi}\,\gamma_{\rho\pi\pi}\,\frac{(\omega_{q+k'} + E_q + \omega_{k'})}{\omega_{q+k'}\left[(\omega_{q+k'} + E_q + \omega_{k'})^2 - s\right]}$$

$$(A3\text{-}18)$$

$$A(z) = \left[\frac{1}{E_{k+p} + M'} - \frac{1}{E_k + M}\right]$$

$$B(z) = \frac{1}{E_{k+p} + M'} \qquad\qquad (A3\text{-}19)$$

$$A_1(z) = \frac{1}{E_{k+p} + M'}$$

$$B_1(z) = \left[\frac{1}{E_{k+p} + M'} - \frac{1}{E_p + M}\right]$$

The non-spin flip nucleon exchange term becomes:

$$E(\underline{k},\underline{p},s)$$

$$= -\frac{1}{2M}\left\{ \frac{(A+Dz)(E_{k+p}+\omega_k+\omega_p)}{(E_{k+p}+\omega_k+\omega_p)^2 - S} - \frac{(Kz+L)(E_{k+p}+\omega_k+\omega_p)}{(E_{k+p})[(E_{k+p}+\omega_k+\omega_p)^2 - S]} \right\}$$

(A3-20)

where

$$A = 1$$

$$D = \frac{k\,p}{(E_k+M)(E_p+M)}$$

(A3-21)

$$K = p\,k\left\{ \frac{1}{E_k+M} + \frac{1}{E_p+M} - \frac{M'}{(E_k+M)(E_p+M)} \right\}$$

$$L = \left\{ \frac{k^2}{E_k+M} + \frac{p^2}{E_p+M} + M' \right\}$$

The spin flip nucleon exchange term becomes:

$$F(\underline{k},\underline{p},s)$$

$$= -\frac{\{E_{k+p}+C\}}{2M(E_k+M)(E_p+M)} \cdot \frac{(E_{k+p}+\omega_k+\omega_p)}{E_{k+p}[(E_{k+p}+\omega_k+\omega_p)^2 - S]}$$

(A3-22)

where

$$c = M' - E_k - E_p$$

For the pion exchange term which is cut, we have

$$\tilde{B}_{N\rho}(\underline{p},\underline{q},s) = \epsilon^\mu P_\mu \gamma_{NN'\pi}\gamma_{\rho\pi\pi}\delta[(P-p-q)^2 - \mu^2]$$

$$= \epsilon^\mu P_\mu \gamma_{NN'\pi}\gamma_{\rho\pi\pi}\frac{1}{2pq}\delta\left[\frac{(\omega_p+\omega_q-\sqrt{s})^2 - p^2 - q^2 - \mu^2}{2pq} - z\right]$$

(A3-23)

where $\quad p \cdot q = pq\tilde{z}$

and ( )

$$\epsilon^\mu P_\mu = \epsilon^0(q,\lambda) P_0 - \epsilon^K(q,\lambda) P_K$$

$$= \epsilon^0(q,\lambda) P_0 - (-1)^\nu \epsilon^\nu(q,\lambda) P_{-\nu}$$

$$= \frac{q^\lambda \omega_p}{m_p} - (-1)^\lambda P_{-\lambda} - \frac{(q\cdot p) q^\lambda}{m_p(m_p+\omega_q)}$$

$$= q^\lambda \left(\frac{\omega_p}{m_p} - \frac{qp\cos\theta}{m_p(m_p+\omega_q)} \right)(-1)^\lambda P_{-\lambda} = (A+B\cos\theta)q^\lambda - P_{-\lambda}(-1)^\lambda$$

$$= (A+B\cos\theta)\, q^\lambda - p^\lambda \qquad\qquad (A3-24)$$

$$B = \frac{-qp}{m_p(m_p+\omega_q)} \qquad ; \qquad A = \frac{\omega_p}{m_p}$$

For $\quad B_{pN}(q, k', s) \quad$ we have

$$k'^\mu \epsilon_\mu = (A' + B'\cos\theta)\, q_\lambda - k'_\lambda \qquad\qquad (A3-25)$$

$$B' = \frac{-q k'}{m_p(m_p+\omega_q)} \qquad ; \qquad A' = \frac{\omega_{k'}}{m_p} \qquad\qquad (A3-26)$$

For each of the pion exchange terms $\tilde{B}_{N\rho}$ and $B_{\rho N}$ we have
four parts. For $\tilde{B}_{N\rho}$ we have:

$$N_q N_p \chi_t^+ \frac{\sigma\cdot p}{E_p+M} \chi_s \tilde{\tilde{Z}}_{N\rho}(k, q, s)\, p^\lambda$$

$$\qquad\qquad (A3-27)$$

$$N_q N_p \chi_t^+ \frac{\sigma\cdot q}{E_q+M'} \chi_s \tilde{Z}_{N\rho}(k, q, s)\, p^\lambda$$

$$-N_g N_p \chi_t + \frac{\sigma \cdot p}{E_p + M} \chi_s \tilde{Z}_{Np}(p, q, s)(A + B\cos\theta)q^\lambda$$

$$-N_g N_p \chi_t + \frac{\sigma \cdot p}{E_q + M'} \chi_s \tilde{Z}_{Np}(p, q, s)(A + B\cos\theta)q^\lambda$$

where
$$\tilde{Z}_{Np}(p, q, s) = \delta\left[\frac{(\omega_p + \omega_g - \sqrt{s})^2 - (p^2 + q^2 + \mu^2)}{2pq} - z\right]_{(A3-29)}$$

Similarly for B_{pN} we have:

$$(A' + B'\cos\theta') q_\lambda Z_{pN}(q, k', s)\chi_{r'} + \left(\frac{\sigma \cdot q}{E_q + M'}\right)\chi_t N_g N_{k'}$$

$$(A' + B'\cos\theta') q_\lambda Z_{pN}(q, k', s)\chi_{r'} + \left(\frac{\sigma \cdot k'}{E_{k'} + M}\right)\chi_t N_g N_{k'} \qquad (A3-30)$$

$$-k'_\lambda Z_{pN}(q, k', s)\chi_{r'} + \left(\frac{\sigma \cdot q}{E_q + M'}\right)\chi_t N_g N_{k'}$$

$$-k'_\lambda Z_{pN}(q, k', s)\chi_{r'} + \left(\frac{\sigma \cdot k'}{E_{k'} + M}\right)\chi_t N_g N_{k'}$$

where
$$Z_{pN}(q, k', s) = \frac{\omega_{g+k'} + E_g + \omega_{k'}}{\omega_{g+k'}\left[(\omega_{g+k'} + E_g + \omega_{k'})^2 - s\right]} \qquad (A3-31)$$

If we call each of these terms $C(k_1, k_2, s)$ we can project out the orbital angular momentum L with component M by performing the integral

$$\int d\Omega_{\hat{k}_1} \, d\Omega_{\hat{k}_2} \; Y_{LM}(\hat{k}_1) \; Y^*_{L'M'}(\hat{k}_2) \; C(\underset{\sim}{k}_1, \underset{\sim}{k}_2, s)$$

to obtain

$$\langle k_1 \, L \, M \, r_1 | \; C \; | k_2 L' M' r_2 \rangle$$

We then construct states of definite total angular momentum J by multiplying the above by the appropriate Clebsch-Gordon coefficients and summing over magnetic projections as in equation $(A3\text{-}12)$. In carrying out this procedure, we write terms like Z_{PN} with no spin dependence as

$$Z(\underset{\sim}{k}_1, \underset{\sim}{k}_2, s) = \sum_{l,\sigma} Z_l \; Y^*_{l\sigma}(\hat{k}_1) \; Y_{l\sigma}(\hat{k}_2) \qquad (A3\text{-}32)$$

For the nucleon exchange term, we find

$$\langle k_1 \, L \, J \, M_J | \; E(\underset{\sim}{k}_1, \underset{\sim}{p}_1, s) | \; p_1 \, L' \, J' \, M'_J \rangle$$
$$= \delta_{JJ'} \, \delta_{LL'} \; E_L(\underset{\sim}{k}, \underset{\sim}{p}, s) \qquad (A3\text{-}33)$$

$$\langle k_1 , L, J, M_J | \; i \, \underset{\sim}{\sigma} \cdot \underset{\sim}{k} \times \underset{\sim}{p} \; F(\underset{\sim}{k}, \underset{\sim}{p}_1, s) | \; p, L', J', M'_J \rangle$$

$$= -6 \, k \, p \; \delta_{JJ'} \, \delta_{M_J M'_J} \, \delta_{LL'} \; W(LL \tfrac{1}{2} \tfrac{1}{2} ; 1J)$$

$$\times (-1)^{L + \frac{1}{2} - J} \left[\frac{L(L+1)}{6(2L+1)} \right]^{1/2} \left\{ F_{L+1} - F_{L-1} \right\} \quad (A3\text{-}34)$$

where $W(a \, b \, c \, d; e \, f)$ is a Racah coefficient.

For the product of the pion exchange terms we find

$$\langle p, L, J, M_J | \text{ OPE} | k', L', J', M_J' \rangle = \delta_{JJ'} \, \delta_{LL'} \, N_p \, N_q^2 \, N_{k'}$$

$$\times \; \frac{p^2 q \, k'}{(E_p + M)(E_{k'} + M)} \left\{ 3 B_{L+2}^{(1)} \; B_{L+1}^{(2)} \left[\frac{(L+1)(L+2)}{(2L+1)(2L+3)} \right] \right.$$

$$+ \; B_L^{(1)} B_{L+1}^{(2)} \left[\frac{3(L+1)^2}{(2L+1)(2L+3)} \right] \; + \; 3 B_L^{(1)} B_{L-1}^{(2)} \left[\frac{3L^2}{(2L+1)(2L-1)} \right]$$

$$+ \; 3 B_{L-2}^{(1)} \; B_{L-1}^{(2)} \left[\frac{L(L-1)}{(2L-1)(2L+1)} \right] \right\}$$

$$B^{(1)} \; B^{(2)} = \tilde{Z}_{Np} \, (A' + B' \cos \theta') \, Z_{pN}$$

$$\frac{p^2 q^2}{(E_p + M)(E_q + M')} \left[B_L^{(1)} B_L^{(2)} \; + \; 3 B_{L+2}^{(1)} B_L^{(2)} \frac{(L+1)(L+2)}{(2L+1)(2L+3)} \right.$$

$$+ 2 B_L^{(1)} B_L^{(2)} \frac{L(L+1)}{(2L-1)(2L+3)} \; + \; 3 B_{L-2}^{(1)} B_L^{(2)} \frac{L(L-1)}{(2L-1)(2L+1)} \right]$$

$$B^{(1)} B^{(2)} = \tilde{Z}_{Np} \, (A' + B' \cos \theta) \, Z_{pN}$$

$$\frac{p^2 q \, k'}{(E_p + M)(E_q + M')} \left\{ 3 B_{L+2}^{(1)} \; B_{L+1}^{(2)} \left[\frac{(L+1)(L+2)}{(2L+1)(2L+3)} \right] \right.$$

$$+ B_L^{(1)} B_{L+1}^{(2)} \left[\frac{3(L+1)^2}{(2L+1)(2L+3)} \right] + B_L^{(1)} B_{L-1}^{(2)} \left[\frac{3L^2}{(2L-1)(2L+1)} \right]$$

$$+ 3 B_{L-2}^{(1)} B_{L-1}^{(2)} \left[\frac{L(L-1)}{(2L+1)(2L-1)} \right] \Big\}$$

$$B^{(1)} B^{(2)} = \widetilde{Z}_{Np} \left(- z_{pN} \right)$$

$$\frac{p^2 k'^2}{(E_p + M)(E_{k'} + M)} \left\{ B_L^{(1)} B_L^{(2)} \left[1 + \frac{2L(L+1)}{(2L-1)(2L+3)} \right] \right.$$

$$+ 3 B_{L+2}^{(1)} B_{L+2}^{(2)} \left[\frac{(L+1)(L+2)}{(2L+1)(2L+3)} \right] + 3 B_{L-2}^{(1)} B_{L-2}^{(2)} \left[\frac{L(L-1)}{(2L-1)(2L+1)} \right] \Big\}$$

$$B^{(1)} B^{(2)} = \widetilde{Z}_{Np} \left(- z_{pN} \right)$$

$$\frac{p q^3}{(E_q + M')^2} \left\{ B_{L+1}^{(1)} B_L^{(2)} \left[\frac{3(L+1)(L+2) + L(L+1) + (L+1) + (L+1)(2L+3)}{(2L+1)(2L+3)} \right] \right.$$

$$+ B_{L-1}^{(1)} B_L^{(2)} \left[\frac{L}{(2L+1)} + \frac{L(L+1)}{(2L+1)(2L-1)} + \frac{3L(L-1)}{(2L+1)(2L-1)} \right] \Big\}$$

$$B^{(1)} B^{(2)} = \widetilde{Z}_{Np} \left(A' + B' \cos\theta' \right) z_{pN}$$

$$\frac{3pq^2 k'}{(E_q+M')(E_k+M)} \left\{ B_{L-1}^{(1)} B_{L-1}^{(2)} \frac{L}{(2L+1)} + B_{L+1}^{(1)} B_{L+1}^{(2)} \frac{L+1}{(2L+1)} \right\}$$

$$B^{(1)} B^{(2)} = \widetilde{Z}_{N\rho} (A' + B' \cos\theta) Z_{\rho N}$$

$$\frac{3pq^2 k'}{(E_q+M')^2} \left\{ B_{L-1}^{(1)} B_{L-1}^{(2)} \frac{L}{(2L+1)} + B_{L+1}^{(1)} B_{L+1}^{(2)} \frac{L+1}{(2L+1)} \right\}$$

$$B^{(1)} B^{(2)} = \widetilde{Z}_{N\rho} (-Z_{\rho N})$$

$$\frac{pq k'^2}{(E_q+M')(E_k+M)} \left\{ 3 B_{L+1}^{(1)} B_{L+2}^{(2)} \frac{(L+1)(L+2)}{(2L+1)(2L+3)} + B_{L+1}^{(1)} B_L^{(2)} \left[\frac{L+1}{(2L+1)} + \frac{L(L+1)}{(2L+1)(2L+3)} \right] \right.$$

$$\left. + B_{L-1}^{(1)} B_L^{(2)} \left[\frac{L}{(2L+1)} + \frac{L(L+1)}{(2L+1)(2L-1)} \right] + 3 B_{L-1}^{(1)} B_{L-2}^{(2)} \frac{L(L-1)}{(2L-1)(2L+1)} \right\}$$

$$B^{(1)} B^{(2)} = \widetilde{Z}_{N\rho} (-Z_{\rho N})$$

$$\frac{pq^3}{(E_p+M)(E_q+M')} \left\{ 3 B_{L+1}^{(1)} B_L^{(2)} \frac{(L+1)(L+2)}{(2L+1)(2L+3)} + B_{L+1}^{(1)} B_L^{(2)} \left[\frac{(L+1)}{(2L+1)} + \frac{L(L+1)}{(2L+1)(2L+3)} \right] \right.$$

$$\left. + B_{L-1}^{(1)} B_L^{(2)} \left[\frac{L}{(2L+1)} + \frac{L(L+1)}{(2L+1)(2L-1)} \right] + 3 B_{L-1}^{(1)} B_L^{(2)} \frac{L(L-1)}{(2L+1)(2L-1)} \right\}$$

$$B^{(1)} B^{(2)} = -(A + B \cos\theta) \widetilde{Z}_{N\rho} (A' + B' \cos\theta) Z_{\rho N}$$

$$\frac{3p\hat{q}^2 k'}{(E_p+M)(E_{k'}+M)} \left\{ B_{L-1}^{(1)} B_{L-1}^{(2)} \frac{L}{(2L+1)} + B_{L+1}^{(1)} B_{L+1}^{(2)} \frac{L+1}{(2L+1)} \right\}$$

$$B^{(1)} B^{(2)} = -(A + B\cos\theta)\, \tilde{Z}_{N\rho}\, (A' + B'\cos\theta)\, Z_{\rho N}$$

$$\frac{3p\hat{q}^2 k'}{(E_p+M)(E_g+M')} \left\{ B_{L-1}^{(1)} B_{L-1}^{(2)} \frac{L}{(2L+1)} + B_{L+1}^{(1)} B_{L+1}^{(2)} \frac{L+1}{(2L+1)} \right\}$$

$$B^{(1)} B^{(2)} = -(A + B\cos\theta)\, \tilde{Z}_{N\rho}\, (-Z_{\rho N})$$

$$\frac{p\hat{q}\,k'^2}{(E_p+M)(E_{k'}+M)} \left\{ 3 B_{L+1}^{(1)} B_{L+2}^{(2)} \frac{(L+1)(L+2)}{(2L+1)(2L+3)} + B_{L+1}^{(1)} B_{L}^{(2)} \left[\frac{L+1}{(2L+1)} + \frac{L(L+1)}{(2L+1)(2L+3)} \right] \right.$$

$$\left. + B_{L-1}^{(1)} B_{L}^{(2)} \left[\frac{L}{(2L+1)} + \frac{L(L+1)}{(2L+1)(2L-1)} \right] + 3 B_{L-1}^{(1)} B_{L-2}^{(2)} \frac{L(L-1)}{(2L-1)(2L+1)} \right\}$$

$$B^{(1)} B^{(2)} = -(A + B\cos\theta)\, \tilde{Z}_{N\rho}\, (-Z_{\rho N})$$

$$\frac{3\hat{q}^4}{(E_g+M')^2}\, B_L^{(1)} B_L^{(2)}$$

$$B^{(1)} B^{(2)} = -(A + B\cos\theta)\, \tilde{Z}_{N\rho}\, (A' + B'\cos\theta)\, Z_{\rho N}$$

$$\frac{\hat{q}^3 k'}{(E_g+M')(E_{k'}+M)} \left\{ 3 B_L^{(1)} B_{L+1}^{(2)} \left[\frac{(L+1)(L+2)}{(2L+1)(2L+3)} \right] + B_L^{(1)} B_{L+1}^{(2)} \left[\frac{(L+1)}{(2L+1)} + \frac{L(L+1)}{(2L+1)(2L+3)} \right] \right.$$

$$+ B_L^{(1)} B_{L-1}^{(2)} \left[\frac{L}{(2L+1)} + \frac{L(L+1)}{(2L+1)(2L-1)} \right] + 3 B_L^{(1)} B_{L-1}^{(2)} \left[\frac{L(L-1)}{(2L+1)(2L-1)} \right] \Big\}$$

$$B^{(1)} B^{(2)} = -(A + B\cos\theta)\, \widetilde{Z}_{N\rho}\, (A' + B'\cos\theta)\, Z_{\rho N}$$

$$\frac{q^3 k'}{(E_q + M')^2} \Big\{ 3 B_L^{(1)} B_{L+1}^{(2)} \left[\frac{(L+1)(L+2)}{(2L+1)(2L+3)} \right] + B_L^{(1)} B_{L+1}^{(2)} \left[\frac{(L+1)}{(2L+1)} + \frac{L(L+1)}{(2L+1)(2L+3)} \right]$$

$$+ B_L^{(1)} B_{L-2}^{(2)} \left[\frac{L}{(2L+1)} + \frac{L(L+1)}{(2L+1)(2L-1)} \right] + 3 B_L^{(1)} B_{L-2}^{(2)} \left[\frac{L(L-1)}{(2L+1)(2L-1)} \right] \Big\}$$

$$B^{(1)} B^{(2)} = -(A + B\cos\theta)\, \widetilde{Z}_{N\rho}\, (-Z_{\rho N})$$

$$\frac{q^2 k'^2}{(E_q + M')(E_{k'} + M)} \Big\{ B_L^{(1)} B_L^{(2)} + 3 B_L^{(1)} B_{L+2}^{(2)} \frac{(L+1)(L+2)}{(2L+1)(2L+3)}$$

$$+ 2 B_L^{(1)} B_L^{(2)} \frac{L(L+1)}{(2L-1)(2L+3)} + 3 B_L^{(1)} B_{L-2}^{(2)} \frac{L(L-1)}{(2L-1)(2L+1)} \Big\}$$

$$B^{(1)} B^{(2)} = -(A + B\cos\theta)\, \widetilde{Z}_{N\rho}\, (-Z_{\rho N}) \qquad (A3\text{-}35)$$

Partial Wave Projection of the Inhomogeneous Term

In performing the partial wave projection of the spin independent part of the exchange terms, we use the method of partial fractions, i.e. we consider a term of the form:

$$\frac{E_{k+p} + \omega_k + \omega_p}{E_{k+p}\left[(E_{k+p} + \omega_k + \omega_p)^2 - s\right]} \qquad (A3 \cdot 35)'$$

$$\omega_k = \sqrt{k^2 + m^2} \quad , \quad E_{k+p} = \sqrt{(k+p)^2 + M^2}$$

We can re-express the denominator in terms of partial fractions:

$$\frac{1}{E_{k+p}\left[(E_{k+p} + \omega_k + \omega_p)^2 - s\right]} = \frac{\alpha}{E_{k+p}} + \frac{\beta}{E_{k+p} + \omega_k + \omega_p - \sqrt{s}} + \frac{\gamma}{E_{k+p} + \omega_k + \omega_p + \sqrt{s}}$$

$$= \alpha \, a + \beta \, \mathcal{B} + \delta \, \mathcal{C}$$

where

$$\alpha = \frac{1}{(\omega_k + \omega_p)^2 - s}$$

$$\beta = -\frac{1}{2\sqrt{s}\,(\omega_k + \omega_p - \sqrt{s})}$$

$$\delta = \frac{1}{2\sqrt{s}\,(\omega_k + \omega_p + \sqrt{s})}$$

If we take the partial wave projection of the term in $(A3 \cdot 35)'$

$$\frac{1}{2}\int_{-1}^{1} dz \; P_\ell(z) \left\{ \frac{E_{k+p} + \omega_k + \omega_p}{E_{k+p}\left[(E_{k+p} + \omega_k + \omega_p)^2 - s\right]} \right\} = \alpha \, a_\ell + \beta \, \mathcal{B}_\ell + \delta \, C_\ell$$

$$(A3 - 36)$$

we obtain:

$$\alpha \, a_\ell = \alpha \left[(\omega_k + \omega_p) \frac{a^{1/2}}{b} \sum_{n=0}^{\ell} \frac{f^{(n)}(0)}{n!} (-1)^n Q_\ell\left(-\frac{1}{\beta}\right) + \delta_{\ell 0} \right] \qquad (A3\text{-}37)$$

where we've written

$$E_{k+p} = \left(\underset{\sim}{k}^2 + \underset{\sim}{p}^2 + 2\underset{\sim}{k}\cdot\underset{\sim}{p} + M^2\right)^{1/2} = (a + bz)^{1/2}$$

$$\underset{\sim}{k}^2 = |\underset{\sim}{k}|^2 = k^2$$

$$a = k^2 + p^2 + M^2 \quad ; \qquad b = \underset{\sim}{k}\,\underset{\sim}{p} = |\underset{\sim}{k}||\underset{\sim}{p}|$$

and

$$(a + bz)^{1/2} = a^{1/2}\left(1 + \frac{b}{a}z\right)^{1/2} = a^{1/2}\left(1 + \gamma z\right)^{1/2}$$

$$= a^{1/2} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \gamma^n z^n$$

$$\beta \, B_\ell = -\beta \left[\frac{a}{b} + \chi\right] Q_\ell(\chi) - \sqrt{s}\,\frac{a^{1/2}}{b} \sum_{n=0}^{\ell} \frac{f^{(n)}(0)}{n!} \gamma^n \chi^n Q_\ell(\chi)$$

$$\qquad\qquad (A3\text{-}38)$$

$$- \beta(\omega_k + \omega_p)(\omega_k + \omega_p - \sqrt{s})\left(-\frac{1}{b}\right) Q_\ell(\chi)$$

where

$$\chi = \frac{(\omega_k + \omega_p - \sqrt{s})^2 - (k^2 + p^2 + M^2)}{2kp} \qquad\qquad (A3\text{-}39)$$

$$\delta \, C = -\delta\left[\frac{a}{b} + \chi'\right] Q_\ell(\chi') + \sqrt{s}\,\frac{a^{1/2}}{b} \sum_{n=0}^{\ell} \frac{f^{(n)}(0)}{n!} \gamma^n \chi'^n Q_\ell(\chi')$$

$$- \delta(\omega_k + \omega_p)(\omega_k + \omega_p + \sqrt{s})\left(-\frac{1}{b}\right) Q_\ell(\chi')$$

$$\qquad\qquad (A3\text{-}40)$$

where

$$\chi' = \frac{(\omega_k + \omega_p + \sqrt{s})^2 - (k^2 + p^2 + M^2)}{2kp} \qquad (A3\text{-}41)$$

If we take s large and integrate near the phase space boundary then both k and p become large. In fact, for k and p large, $\omega_k + \omega_p \sim \sqrt{s}$ and $(\omega_p + \omega_k - \sqrt{s}) = 0$, except for a small correction factor (given below). Therefore,

$$\frac{\alpha}{E_{k+p}} + \frac{\beta}{E_{k+p} + (\omega_k + \omega_p - \sqrt{s})} \sim \frac{\alpha}{E_{k+p}} + \frac{\beta}{E_{k+p}} = \frac{\alpha + \beta}{E_{k+p}}$$

$$\alpha = \frac{1}{(\omega_k + \omega_p)^2 - s} = \frac{1}{(\omega_k + \omega_p - \sqrt{s})(\omega_k + \omega_p + \sqrt{s})}$$

Since ω_k and ω_p represent center-of-mass energies for the particles with momenta k and p (refer to figure $A3\text{-}1$).

$$\omega_k = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}$$

$$\omega_p = \frac{s + m_4^2 - m_3^2}{2\sqrt{s}}$$

$$\omega_k + \omega_p = \sqrt{s} + \frac{\mathcal{M}^2}{\sqrt{s}} \quad , \quad \mathcal{M}^2 = \frac{m_1^2 - m_3^2 + m_4^2 - m_3^2}{2}$$

$$\omega_k + \omega_p - \sqrt{s} = \frac{\mathcal{M}^2}{\sqrt{s}}$$

$$\omega_k + \omega_p + \sqrt{s} = 2\sqrt{s} + \frac{\mathcal{M}^2}{\sqrt{s}} \sim 2\sqrt{s}$$

Therefore, $\qquad (\omega_k + \omega_p - \sqrt{s})(\omega_k + \omega_p + \sqrt{s}) \sim 2\mathcal{M}^2$

α becomes

$$\alpha \sim \frac{1}{2m^2}$$

$$\beta = -\frac{1}{2\sqrt{s}} \frac{1}{(\omega_k + \omega_p - \sqrt{s})} \approx -\frac{1}{2\sqrt{s}} \frac{1}{\frac{m^2}{\sqrt{s}}} = -\frac{1}{2m^2}$$

We see that, in the limit of large s, $\alpha + \beta = 0$. This implies we only need to keep the δ term in equation (A3-40). In the denominator of δ we have:

$$\omega_k + \omega_p + \sqrt{s} \sim 2\sqrt{s}$$

So
$$\delta \sim \frac{1}{2\sqrt{s} \, 2\sqrt{s}} = \frac{1}{4s}$$

Effectively, we're left with considering

$$\frac{1}{4s} \frac{E_{k+p} + \omega_k + \omega_p}{E_{k+p} + \omega_k + \omega_p + \sqrt{s}}$$

Consider δC :

$$\delta C = \left\{ -\delta \frac{a}{b} Q_\ell(x') - \delta x' Q_\ell(x') \right\}$$

$$+ \delta \sqrt{s} \frac{a^{1/2}}{b} \sum_{n=0}^{\ell} \frac{f^{(n)}(0)}{n!} \beta^n (x')^n Q_\ell(x')$$

$$+ \delta (\omega_k + \omega_p)(\omega_k + \omega_p + \sqrt{s}) \frac{1}{b} Q_\ell(x')$$

Now

$$a^{1/2} = (k^2 + p^2 + \mu^2)^{1/2} \approx \left(\frac{s}{4} + \frac{s}{4}\right)^{1/2} = \frac{\sqrt{s}}{2}$$

$$b = 2kp \sim \frac{s}{2}$$

This implies that

$$\frac{\sqrt{s}\, a^{1/2}}{b} = \frac{1}{\sqrt{2}} \frac{s}{\frac{s}{2}} \sim \sqrt{2}$$

also $\quad (\omega_k + \omega_p) \sim \sqrt{s}$, so that $\quad (\omega_k + \omega_p + \sqrt{s}) \sim 2\sqrt{s}$

$$\frac{a}{b} + x' = \frac{a}{b} + \frac{c-a}{b} = \frac{c}{b} \sim \frac{4s}{s/2} \sim 8 = z_0$$

$$\frac{a}{b} \sim \frac{s/2}{s/2} \sim 1$$

The argument of the Q_ℓ , using the definition ($A3.41$) for x'

becomes $\quad x' \sim \dfrac{4s - s/2}{s/2} \sim \dfrac{8s - s}{s} \sim 7 = x_0$

We obtain

$$-\delta z_0\, Q_\ell(x_0) + \delta\sqrt{2} \sum_{n=0}^{\ell} \frac{f^{(n)}(0)}{n!} \beta_0^{n} (x_0)^{n} Q_\ell(x_0) + \delta\, 4\, Q_\ell(x_0)$$

Finally we get:

$$-4\delta\, Q_\ell(x_0) + \delta\sqrt{2} \sum_{n=0}^{\ell} \frac{f^{(n)}(0)}{n!} \beta_0^{n} (x_0)^{n} Q_\ell(x_0)$$

$$\delta \rightarrow \frac{1}{4S}$$

we obtain

$$-\frac{1}{S} Q_\ell(x_0) + \frac{\sqrt{2}}{4S} \sum_{n=0}^{\ell} \frac{f^{(n)}(0)}{n!} \beta_0^n x_0^n Q_\ell(x_0)$$

$$= \frac{A}{S} Q_\ell(x_0) = \frac{C_3}{S} \quad \text{for} \quad B_{\rho N}^{\cup}$$

Let us look at nucleon exchange term for non-flip.

$$\frac{(A + D z)[E_{k+p} + \omega_k + \omega_p]}{(E_{k+p} + \omega_k + \omega_p)^2 - S} - \frac{(Kz + L)(E_{k+p} + \omega_k + \omega_p)}{E_{k+p}[(E_{k+p} + \omega_k + \omega_p)^2 - S]}$$

$$= (A + Dz)(E_{k+p} + \omega_k + \omega_p) \left\{ \frac{\alpha_1}{E_{k+p} + \omega_k + \omega_p - \sqrt{S}} + \frac{\beta_1}{E_{k+p} + \omega_k + \omega_p + \sqrt{S}} \right\}$$

$$- (Kz + L)(E_{k+p} + \omega_k + \omega_p) \left\{ \frac{\alpha}{E_{k+p}} + \frac{\beta}{E_{k+p} + \omega_k + \omega_p - \sqrt{S}} + \frac{\gamma}{E_{k+p} + \omega_k + \omega_p - \sqrt{S}} \right\}$$

In the first expression the α_1 term dominates, it looks like $\frac{\alpha_1}{E_{k+p}}$. In this second expression, the α and β terms cancel for large S and integration near the phase space boundary, so we obtain:

$$\alpha_1 A \left\{ (\omega_k + \omega_p) \frac{a^{1/2}}{b} \sum_{n=0}^{\ell} \frac{f^{(n)}(0)}{n!} (-1)^n Q_\ell\left(-\frac{1}{\beta}\right) + \delta_{\ell 0} \right\}$$

$$+ \alpha_1 D \left\{ (\omega_k + \omega_p) \frac{a^{1/2}}{b} \sum_{n=0}^{\ell} \frac{f^{(n)}(0)}{n!} (-1)^n \left(-\frac{1}{\beta}\right) Q_\ell\left(-\frac{1}{\beta}\right) + \frac{\delta_{\ell 1}}{2\ell+1} \right\}$$

and $\qquad \alpha_1 = \dfrac{1}{2\sqrt{5}}$

we finally get

$$\alpha_1 A \left\{ \sqrt{2} \sum_{n=0}^{\ell} \frac{f^{(n)}(0)}{n!} (-1)^n Q_\ell(\alpha_0) + \delta_{\ell 0} \right\} \qquad \ell \geqslant 0$$

$$+ \alpha_1 D \left\{ \sqrt{2} \sum_{n=0}^{\ell-1} \frac{f^{(n)}(0)}{n!} (-1)^n \alpha_0 \, Q_\ell(\alpha_0) + \frac{\delta_{\ell 1}}{2\ell+1} \right\} \qquad \ell \geqslant 1$$

$$\longrightarrow \quad \frac{1}{2\sqrt{5}} \, C_1$$

The nucleon exchange term for non-spin flip

B_{NN} becomes $\qquad i \dfrac{N_k \, N_p \, C_1}{2\sqrt{5}}$.

Consider the partial wave projection of the pion exchange term which is cut:

$$B_{N\rho} : \qquad \frac{\pi i}{2pq} \, P_L(x)$$

$$x = \frac{(\omega_p + \omega_q - \sqrt{5})^2 - (p^2 + q^2 + \mu^2)}{2pq} \sim \frac{-5/2 - \mu^2}{5/2}$$

x goes to $-1-\dfrac{2\mu^2}{s^+} \sim -1-\dfrac{2\mu^2}{s}+i\varepsilon \longrightarrow -1+i\varepsilon = \tilde{x}$

$$B_{N\rho} \longrightarrow \frac{\pi i}{2pq} P_L(\hat{x})$$

The partial wave projection of $(A+Bz)B_{N\beta}$ is

$$(A+Bx)\frac{\pi i}{2pq}P_L(x) \ .$$ Therefore

for high s

$$(A+B\tilde{x})\frac{\pi i}{2pq}P_L(\hat{x})$$ becomes $$(A-B)\frac{\pi i}{2pq}P_L(\hat{x})$$

Since

$$A=\frac{\omega_p}{m_\rho} \qquad B=\frac{-qp}{m_\rho(m_\rho+\omega_q)} \longrightarrow -\frac{p}{m_\rho}$$

Now $A-B \sim \dfrac{2p}{m_\rho}$. Therefore

$$\frac{2p}{m_\rho}\frac{\pi i}{2pq}P_L(\hat{x}) \longrightarrow \frac{\pi i}{m_\rho q}P_L(\hat{x}) = \frac{\pi i\, G}{q} \text{ for } (A+Bz)B_{N\rho}$$

We must multiply this by $N_p N_q$ to obtain the full term.
For the uncut pion exchange term, the partial wave projection
by similar analysis can be shown to be

$$B_{\rho N}(A'+B'z') \longrightarrow \frac{A'C_3}{s} + \frac{B'C_3'}{s}$$

$$A'=\frac{\omega_{k'}}{m_\rho} \qquad B'=\frac{-k'p}{m_\rho(m_\rho+\omega_q)} \longrightarrow -\frac{k'}{m_\rho}$$

$$\frac{k'C_3}{s} - \frac{k'C_3'}{s} \longrightarrow \frac{k'F}{s} \text{ times } N_{q\,k'}$$

Looking at the inhomogeneous term (figure   )

We consider

$$\int \frac{q^2 \, dq}{2E_q} \left[disc \; B_{N\rho} (p, q, s) \right]^J S(\sigma_q) \; B_{\rho N}^J (q, k', s)$$

Since $(A + B z)$ and $(A' + B' z)$ will introduce

extra factors of momentum, then the terms involving both

$(A + B z)$ and $(A' + B' z)$ should be least convergent.

There exist four such terms:

$$+(A + B \cos\theta) \; z_{N\rho} \; (A' + B' \cos\theta) \; z_{\rho N}$$

$$-(A + B \cos\theta) \; z_{N\rho} \; (A' + B' \cos\theta) \; z_{\rho N}$$

$$-(A + B \cos\theta) \; z_{N\rho} \; (A' + B' \cos\theta) \; z_{\rho N}$$

$$-(A + B \cos\theta) \; z_{N\rho} \; (A' + B' \cos\theta) \; z_{\rho N}$$

which have different L projections.

So for the integral above, with $S(\sigma_k) \longrightarrow \frac{1}{s}$, we obtain

$$-\frac{1}{2s} \int \oint dq \; N_p \, N_q \; \frac{\pi i}{\not q} G \; q^2 \; \frac{k' F}{s} \, N_q \, N_{k'}$$

$$= -\frac{N_p \, N_{k'}}{2 s^2} \int dq \; \pi i \, G \, q^2 \; \frac{q}{2M'} \, F \, k'$$

$$= \frac{-N_p \, N_{k'} \; k' \, F \, \pi i \, G}{4 M' s^2} \int_{\frac{\sqrt{s}}{2}(1 - \frac{2c}{s})}^{\frac{\sqrt{s}}{2}} q^3 \, dq$$

$$= \frac{c \, s}{8} \left\{ \frac{-N_p \, N_{k'} \; k' \, F \, \pi i \, G}{4 M' s^2} \right\}$$

we obtain

$$-\frac{N_p \, N_{k'} \; k' \, F \, \pi i \, G \, c}{32 \, M' \, s} \quad = \quad Box_p^{\;J}$$

Now consider

$$\int \frac{p^2 \, dp}{2\omega_p} \; B_{NN}^{\;J} \, (k, p, s) \; S(\sigma_p) \; Box_p^{\;J} (p, k', s)$$

$$= \frac{1}{2s} \int p \, dp \; \frac{i \, N_k \, N_p \, C}{2\sqrt{s}} \left(\frac{-N_p \, N_{k'} \; k' \, F \, \pi i \, G \, c}{32 \, M' \, s} \right)$$

$$= \frac{N_k \, N_{k'} \; k' \, \pi \, c^2 \, F \, G}{128 \, M' \, s^2 \sqrt{s}} \int p \, dp \; N_p^2$$

Integrating near the phase space boundary

$$= \frac{N_k \, N_{k'} \, k' \; \pi \, c^2 \, FG}{256 \, M' M \quad s^2 \sqrt{s}} \int_{\frac{\sqrt{s}}{2}\left(1 - \frac{2c}{s}\right)}^{\frac{\sqrt{s}}{2}} p^2 \, dp$$

we get

$$= \frac{N_k \, N_{k'} \, k' \; \pi \, c^3 \, FG}{(256) \, 4 \, M' M \; s^2}$$

This term goes as $\frac{1}{s}$ for s large (i.e. k , k' large, $N_k \, N_{k'} \longrightarrow \sqrt{s}$, $k' \to \sqrt{s}$). Therefore it dies out at high energy.

A similar analysis shows that the spin flip term dies out even faster at high energy.

Since we merely have to convert this to a helicity partial wave amplitude (using Clebsch-Gordon Coefficients) and form the partial wave series to recapture the full contribution to the inhomogeneous term. We have discovered that the high energy behavior of the contribution to the inhomogeneous absorptive part coming from cutting the three internal lines indicated in figure (A3-1) is such that it dies out at very large s. When we repeat this procedure for the terms where the unstable particle propagators are cut, the contribution is even more convergent. What this means is that if we go to the high energy limit in our continued multiperipheral equation, the equation will reduce to a homogeneous integral equation.

164.

REFERENCES

1. H. D. I. Abarbanel, Phys. Rev. D **3**, 2227 (1971).

2. H. D. I. Abarbanel, preprint (1972), NAL TH-Y-77.

3. H. D. I. Abarbanel, G. F. Chew, M. L. Goldberger and L. M. Saunders, Ann. Phys. **73**, 156 (1972).

4. E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963).

5. M. Ademollo, G. Veneziano and S. Weinberg, Phys. Rev. Lett. **22**, 83 (1969).

6. S. L. Adler, Phys. Rev. **137**, B1022 (1965).

7. A. Alessandrini and R. L. Omnes, Phys. Rev. **139**, B167 (1965).

8. R. Aviv, R. L. Sugar and R. Blankenbecler, preprint.

9. V. Balakrishnan, preprint (1972), TIFR/TH/72-8.

10. V. Barger and R. J. N. Phillips, Phys. Rev. **187**, 2210 (1969).

11. G. Barton, _Dispersion Techniques in Field Theory_, (W. A. Benjamin, Inc., New York), 1965.

12. W. W. Bell, _Special Functions for Scientists and Engineers_, (D. Van Nostrand Co., Ltd., London), 1968.

13. J. D. Bjorken and T. T. Wu, Phys. Rev. **130**, 2566 (1963).

14. R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).

15. E. Byckling and K. Kajantie, _Particle Kinematics_, (John Wiley and Sons, New York), 1973.

16. C.-F. Chan, preprint (1972), LBL-1038.

17. S.-J. Chang, D. Gordon, F. E. Low and S. B. Treiman, Phys. Rev. **D4**, 3055 (1971).

18. G. F. Chew, Phys. Rev. **140**, 1427 (1965).

19. G. F. Chew and C. DeTar, Phys. Rev. **180**, 1577 (1969).

20. G. F. Chew, R. Karplus, S. Gasiorowicz and F. Zachariasen, Phys. Rev. 110, 265 (1958).

21. G. F. Chew and F. E. Low, Phys. Rev. 101, 1570 (1956).

22. G. F. Chew and A. Pignotti, Phys. Rev. 176, 2112 (1968).

23. G. F. Chew and A. Pignotti, Phys. Rev. Lett. 20, 1078 (1968).

24. M. Creutz, F. E. Paige and L.-L. Wang, preprint.

25. R. E. Cutkosky, Phys. Rev. 96, 1135 (1954).

26. Dwight, Tables of Integrals and Other Mathematical Data, (MacMillan & Co., New York), 1961, p.144-167.

27. Erdelyi, Magnus, Oberhettinger, Tricomi and Bateman, Manuscript Project, "Higher Transcendental Functions", Volume I.

28. Erdelyi, Magnus, Oberhettinger, Tricomi and Bateman, Manuscript Project, "Tables of Integral Transforms", Volumes I and II.

29. H. Feshbach, A. Kerman and R. H. Lemmer, Ann. Phys. 41, 230 (1967).

30. J. Finkelstein and K. Kajantie, Nuovo Cim. 56A, 661 (1968).

31. D. Z. Freedman, C. Lovelace and J. M. Namyslowski, Nuovo Cim. 43A, 2850 (1966).

32. S. Fubini and G. Veneziano, preprint (1970), MIT CTP-128.

33. M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).

34. Gr. Ghika and M. Visinescu, Lett. Nuovo Cim. 3, 9 (1970).

35. J. S. Goldstein, Phys. Rev. 91, 1516 (1953).

36. I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals Series and Products, (Academic Press, New York), 1965.

37. R. Haag, Phys. Rev. $\underline{112}$, 669 (1958).

38. I. G. Halliday and L. M. Saunders, Phys. Rev.D $\underline{2}$, 2969 (1970).

39. J. H. Hetherington and L. H. Schick, Phys. Rev. $\underline{137}$, B935 (1965).

40.   Hildebrand, Methods of Applied Mathematics, (Prentice-Hall, Englewood Cliffs, N.J.), 1965.

41. D. Horn, preprint (1971), Ref. TH 1387-CERN.

42. K. Igi, Phys. Rev. Lett. $\underline{28B}$ (1968).

43. M. Jacob, preprint (1971), Ref. TH 1340-CERN.

44. C. E. Jones, F. E. Low and J. E. Young, preprint (1970).

45. C. E. Jones, F. E. Low and J. E. Young, preprint (1972).

46. E. Jahnke and F. Emde, Tables of Functions, (Dover Publications, New York), 1943.

47. B. W. Lee and R. F. Sawyer, Phys. Rev. $\underline{127}$, 2266 (1962).

48. H. Lee, preprint (1972).

49. H. Lehmann, Nuovo Cim. $\underline{10}$, 3541 (1958).

50. C. Lovelace, Strong Interaction and High Energy Physics, Ed. Moorhouse, British Universities Summer School (1963), p.437.

51. C. Lovelace, Phys. Rev. $\underline{135}$, B1225 (1964).

52. C. Lovelace, Phys. Lett. $\underline{28B}$, 264 (1968).

53. C. Lovelace, Nucl. Phys. $\underline{B12}$, 253 (1969).

54. F. E. Low, preprint (1969), BNL 50162 (T-527).

55. H. Margenau and G. M. Murphy, Mathematics of Physics and Chemistry, (D. Van Nostrand & Co., Inc., Princeton), 1962.

56. A. Martin and T. Spearman, Elementary Particle Theory, (John Wiley and Sons, Inc., New York), 1970.

57. I. O. Moen and W. J. Zakrzewski, preprint (1972), DAMTP 72/45.

58. I. O. Moen and W. J. Zakrzewski, preprint (1973), DAMTP 73/5.

59. I. O. Moen and W. J. Zakrzewski, preprint (1973), DAMTP 73/6.

60. P. Morse and H. Feshbach, Methods of Theoretical Physics, Volumes I and II, (McGraw-Hill, New York), 1953.

61. A. H. Mueller and I. J. Muzinich, Ann. Phys. $\underline{57}$, 500 (1970).

62. N. Nakanishi, Prog. Theor. Phys. Suppl. No. 43, 1 (1969).

63. Y. Nambu, Prog. Theor. Phys. $\underline{5}$, 614 (1950).

64. J. M. Namyslowski, preprint (1966), Stanford ITP-237.

65. R. G. Newton, The Complex j Plane, (W. A. Benjamin, Inc., New York), 1964.

66. R. G. Newton, Scattering Theory of Waves and Particles, (McGraw-Hill, New York), 1966.

67. K. L. Nielsen, College Mathematics, (Barnes & Noble, Inc., New York), 1958.

68. R. Omnes and M. Froissan, Mandelstam Theory and Regge Poles, (W. A. Benjamin, Inc., New York), 1963.

69. S. Pinsky and W. I. Weisberger, preprint (1969).

70. J. C. Polkinghorne, J. Math. Phys. $\underline{5}$, 431 (1964).

71. E. Predazzi, Ann. Phys. $\underline{36}$, 250 (1966).

72. E. E. Salpeter and H. A. Bethe, Phys. Rev. $\underline{84}$, 1232 (1951).

73. A. I. Sanda, preprint, NAL TH-Y-25.

74. C. Schwartz and C. Zamach, Phys. Rev. $\underline{141}$, 1454 (1966).

75. D. Silverman and C.I Tan, Phys. Rev. D $\underline{3}$, 991 (1971).

76. V. Singh, Phys. Rev. $\underline{129}$, 1889 (1963).

77. D. Sivers and J. Yellin, Ann. Phys. **55**, 107 (1969).

78. E. J. Squires, Complex Angular Momentum and Particle Physics, (W. A. Benjamin, Inc., New York), 1964.

79. J. D. Stack, Phys. Rev. **164**, 1904 (1967).

80. H. P. Stapp, preprint (1965).

81. L. Stodolsky, Phys. Rev. D **1**, 2683 (1970).

82. R. L. Sugar and R. Blankenbecler, Phys. Rev. **183**, 1387 (1969).

83. C. I Tan, preprint (1971).

84. C. I Tan, preprint (1971).

85. G. B. Thomas, Calculus and Analytic Geometry, (A. W. Publishing Co., Inc., Reading, Mass.), 1962.

86. P. D. Ting, Phys. Rev. D **2**, 2982 (1970).

87. D. M. Tow, Phys. Rev. D **2**, 154 (1970).

88. T. L. Trueman and T. Yao, Phys. Rev. **132**, 2741 (1963).

89. E. P. Tryon, Phys. Rev. D **4**, 1221 (1971).

90. G. Veneziano, Nuovo Cim. **57B**, 190 (1968).

91. G. Veneziano, Proceedings of the International School of Subnuclear Physics, 1970.

92. G. Veneziano, preprint (1971), MIT CTP-200.

93. F. Wagner, preprint (1969), Ref. TH 1012-CERN.

94. L. L. C. Wang, Phys. Rev. **142**, 1187 (1966).

95. G. C. Wick, Phys. Rev. **96**, 1124 (1954).

96. Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).

97. J. E. Young, preprint, Inclusive Reactions-A Survey, (1971).

98. F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).

99. A. H. Mueller, Phys. Rev. D2, 2963 (1970).

100. R. Aaron, R. Amado, J. Young, Phys. Rev. **174**, 2022 (1968).

101. C. DeTar, C.E. Jones, F.E. Low, J.Weis, J. Young, C. I Tan, Phys. Rev. Lett. **26**, 675 (1971).

102. D. Amati, A. Stanghellini, S. Fubini, N.C. 26, 896 (1962).

103. L. Bertocchi, S. Fubini, M. Tonin, N.C. 25, 626 (1962).

104. M. L. Goldberger, (Princeton preprint).

105. A. Abarbanel, M. L. Goldberger, S. Trueman, Phys. Rev. Lett. (1969).

106. F. E. Paige, M.I.T. Ph.D Thesis (1970).

107. C. Lovelace and D. Masson, N.C. 26, 472 (1962).

108. J. Bjorken and T. T. Wu, Phys. Rev. 13D, 2566 (1963).

109. J. E. Young, Riv. Del N.C., 2, 88 (1972).

110. R. Blankenbecler and M. Goldberger, Phys. Rev. 126, 766 (1962).

111. M. Baker and R. Blankenbecler, Phys. Rev. 128, 415 (1962).

112. V. Matsumoto and T. Tsujimura, (preprint).

113. H. A. Jahn and J. Hope, Phys. Rev. 93, 318 (1954).

114. M. Goldberger, M. Grisaru, S. MacDowell and D. Wong, Phys. Rev. 120, 2250 (1960).

BIOGRAPHICAL NOTE

Shirley Jackson was born and raised in Washington, D. C. and attended public schools there. She did her undergraduate work at the Massachusetts Institute of Technology, receiving a S.B. in Physics in June, 1968. She has been supported during her graduate career by a National Science Foundation Traineeship, a Ford Foundation Advanced Study Fellowship and a Martin Marietta Corporation Graduate Fellowship. Her hobbies include tennis, hiking, cycling, photography and reading.