Aspects of Quantum Theory in 1+1 and Slightly More Dimensions

by

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Abstract

We consider four problems in (1+1)-dimensional physics. Each of these problems had important connections to the physical behavior of (3+1)-dimensional systems. First, we consider problem of fermions interacting with multiple bosonic solitons. We describe a new approximation scheme for determining the fermion energy spectrum and apply it to (1+1)-dimensional two-component fermions coupled to scalar field solitons. Second, we study (1+1)-dimensional behavior in particles falling toward a Schwarzchild black hole. Using a non-covariant choice for the momentum cutoff, we examine the photon self-energy integral. We find evidence that the photons acquire an effective mass with a nonzero imaginary part, so that the photons may decay. Third, we consider cold fermions trapped in a high aspect ratio potential, which confines the particles to move in only one direction. The purely (1+1)-dimensional aspects of this problem have been extensively studied. We examine the corrections that arise because of the underlying (3+1)-dimensional character of the situation and determine the zero-temperature shifts in the (1+1)-dimensional energy spectrum. Fourth, we present a toy model, which is related, by analogy to the problem of electron-inhabited bubbles in liquid helium. An analysis of the 1-dimensional model suggests that the recent suggestion that the electron bubbles may split in two is incorrect.

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Chapter 1

Introduction

1.1 The relationships between physics in 1+1 and 3+1 dimensions

Although the physical world in (3+1)-dimensional, it is possible to learn a great deal by studying lower-dimensional theories. Theories with fewer dimensions are useful not only as theoretical laboratories, but also because they model (3+1)-dimensional physics when the relevant degrees of freedom are confined to a lower-dimensional subspace for kinematic or dynamical reasons. We shall concentrate the on physics of systems with only a single spatial dimension, considering four separate problems that relate to such systems. However, each of our (1+1)-dimensional problem is closely related to important (3+1)-dimensional questions. These relationships between (3+1)- and (1+1)-dimensional physics take many forms. The lower-dimensional problem may be physically embedded in a more realistic physical system, but two of the dimensions can be somehow eliminated from the problem; or the (1+1)-dimensional situation might allow a greatly simplified development and analysis of an approximation scheme that is equally valid in higher dimensions; or the relationship may be merely through an analogy, via which the essential components of a complex system with many degrees of freedom may be brought into clearer focus. All three of these situations will occur here.

Many (1+1)-dimensional systems have the advantage of simplicity. These systems may have well-known and accessible exact solutions, and even when they do not, it is often easier to implement useful approximation schemes in such systems of low dimensionality. Therefore, if it is possible to connect the physics of a (3+1)-dimensional problem with related phenomena in 1+1 dimensions, a great deal may be learned about real physics through an analysis of a simpler, dimensionally-reduced system.

1.2 Fermion masses in the presence of solitons

In Chapter 2, we consider the problem of fermions interacting with bosonic solitons [2]. Fundamental problems of this nature arise in the study of non-perturbative effects in both 3+1 and 1+1 dimensional quantum field theories. Some exact solitary wave solutions are known for the boson field equations in each case. However, the lower-dimensional solutions are much simpler.

The (1+1)-dimensional solitons also have substantially simpler interactions with fermions. The coupling of fermionic particles to bosonic field configurations with non-zero topological indices gives rise to zero-energy fermion modes, and the $\mathcal{O}(\hbar)$ wavefunctions for these modes can be simply expressed in 1+1 dimensions. [The (3+1)-dimensional zero-mode wavefunctions are generally much more complicated.] We develop an approximation scheme, based upon these exact solutions to the Dirac equation, for studying the fermionic energy spectrum in the presence of multiple-soliton field configurations.

We illustrate the method in (1+1)-dimensional ϕ^4-Yukawa theory, where we may perform many calculations exactly. However, the method is equally applicable in higher dimensions. We illustrate this with a simple (3+1)-dimensional example, with isospinor fermions interacting with a multiple-monopole potential. This particular problem is solvable, because the energies of the zero-modes are not displaced from zero. For more general problems in 3+1 dimensions, the discrete energy spectrum will change in the presence of several solitons, and our method will still be useful, although it will probably only be possible to determine the energies numerically.

1.3 Dimensional reduction near black holes

In Chapter 3, we examine a form of dimensional reduction that may occur near Schwarzchild black holes [3]. Classically, a particle falling toward the event horizon of such a black hole has its motion strongly focused into the (1+1)-dimensional t-r subspace. We suggest that a much more profound reduction in dimensionality may also occur. If the momentum cutoffs for quantum-mechanical loop integrations are chosen in a certain natural manner (although in a manner that breaks general covariance), the system may become approximately (1+1)-dimensional. The dimension d of the loop integral is reduced from 3+1 to $(1+\epsilon)+1$, and it is possible to estimate the value of ϵ for modified theories of gravity, in terms of the length scale at which General Relativity breaks down.

We look at the effects of this situation. First we consider the purely (1+1)-dimensional case, where the photon self-energy gives rise to a Schwinger mass term. Then we look at the $\mathcal{O}(\epsilon)$ effects. At this order, the mass acquires an imaginary term, which corresponds to photon decay. We calculate and interpret this term, making use of an analogy between the dimensional reduction that we are studying and the dimensional reduction that occurs in the classical electrodynamics of waveguides.

In Appendix A, we present a dimension-independent calculation of the photon self-energy. This calculation, for arbitrary dimension $2 \leq d < 4$ is an important building block for the photon decay calculation.

1.4 Degenerate fermions trapped in one dimension

In Chapter 4, we consider a system of degenerate fermions. We connect two active fields of research, fermion trapping and fermions in 1+1 dimensions, by considering fermions that are trapped in an axially symmetric potential with a very high aspect ratio, so that the particles may move freely along only one axis.

The theory of (1+1)-dimensional fermions has been the subject of extensive work by condensed matter physicists. The most important result of this work was the

discovery that fermionic quasi-particles do not exist in this system. Instead, the fundamental excitations are bosonic phonons formed by the coherent excitation of fermion-hole pairs. This continues to be the case even in the presence of certain arbitrarily strong interactions. This known as "Luttinger liquid" behavior.

We analyze the corrections to the (1+1)-dimensional theory that arise in the trapping scenario outlined above. We find the energy shifts of the Luttinger liquid phonons with second order perturbation theory. The problem involves the evaluation of several difficult sums and products of matrix elements, and outline three complementary ways in which the final result may be obtained. Furthermore, by comparing these complementary results, we can make a connection between the physical Fermi momentum k_F and the bandwidth of the Luttinger liquid.

Two of the calculations in this chapter are performed in appendices. In Appendix B, we establish a simple result for the matrix elements of coherent state operators between harmonic oscillator number states. The result plays an important role in our evaluation of fermion matrix elements between differing Luttinger liquid states. In Appendix C, we evaluate an integral that arises in our analysis and interpret the result in terms of an effective momentum cutoff for the bosonized theory. However, we ultimately shall find that this cutoff is never physically relevant.

1.5 Electrons bubbles in liquid helium

In Chapter 5, we present a model relevant to the study of electron-inhabited bubbles in liquid helium [4]. Recently, it has been suggested that this system may exhibit a previously-unanticipated form of particle fractionalization. However, since liquid helium is a complex condensed matter system, with many strongly coupled degrees of freedom, it is difficult to analyze without making many approximations.

To get around this problem, we present a toy model, which captures the essential physics of the electron bubble problem. The model has only one spatial dimension and a very small number of particles. The model is readily amenable to an analytical solution, which uses only the Born-Oppenheimer and WKB approximations, and the

applicability of these approximations is clearly evident. Our analysis of this system may then be carried over, by analogy, to the liquid helium problem, and in the latter context, our results contradict the assertion that the electrons in the fluid are becoming fractionalized.

Chapter 2

Fermion Energies in a Multi-Soliton Background

2.1 Boson fields and fermions in 1+1 dimensions

The (1+1)-dimensional Dirac equation, when coupled to a scalar field, has the interesting property that with the scalar field in a single-soliton state, the fermionic spectrum can include a single zero-energy mode, so that the electric charge of the system is half-integral [26]. However, while the one-soliton case is exactly solvable, the problem of fermions in the presence of multiple interacting solitons is difficult to analyze exactly. Graham and Jaffe [19, 20] have developed a numerical method for calculating the fermionic energies in a multi-soliton background, and Goldhaber, Litvintsev, and van Nieuwenhuizen [17] used a direct expansion of the the Dirac equation in powers of the energy to approximate the same quantity analytically. We shall examine the two-soliton case using an another alternative method, deriving an analytic approximation for the fermionic energies. Our approximation method can also be applied to problems in 3+1 dimensions, where numerical calculations are difficult.

The free Dirac equation in 1+1 dimensions is $i\gamma^\mu\partial_\mu\psi + m\psi = 0$, where a suitable choice for the Dirac matrices is $\alpha = \sigma_2$ and $\beta = \sigma_1$. If we use this representation,

then the Hamiltonian $H = \alpha p + \beta m$ becomes

$$H = -i\sigma_2\partial_x + \sigma_1 m = \begin{bmatrix} 0 & -\partial_x + m \\ \partial_x + m & 0 \end{bmatrix}. \tag{2.1}$$

We shall replace the fermion mass m with a scalar potential $U(x)$, which is time-independent (or varies very slowly in time), so that energy is conserved.

We are interested in the case where the potential $U(x)$ that appears in the Dirac equation is derived from a scalar field ϕ. We shall take the Lagrange density for this field to be

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\partial_x\phi)^2 - V(\phi). \tag{2.2}$$

We shall concentrate on the "Mexican hat" ϕ^4 potential,

$$V(\phi) = \frac{\lambda^2}{4}\left(\phi^2 - \frac{a^2}{\lambda^2}\right)^2, \tag{2.3}$$

although similar results hold for for the sine-Gordon potential [42, 48].

The ϕ^4 potential has two degenerate vacua, at $\phi = \pm\frac{a}{\lambda}$. There are also solutions that interpolate between the two vacua (i.e. solitons). For a stationary soliton centered at $x = x_0$, the solution is

$$\phi_{S,x_0}(x) = \frac{a}{\lambda}\tanh\left[\frac{\mu}{2}(x - x_0)\right], \tag{2.4}$$

where $\mu^2 \equiv 2a^2$ is the mass squared of the light quanta of the scalar field. As $x \to \pm\infty$, $\phi_{S,x_0}(x) \to \pm\frac{a}{\lambda}$. There is also an antisoliton solution $\phi_{A,x_0}(x) = -\phi_{S,x_0}(x)$, which goes the other direction, from $+\frac{a}{\lambda}$ at $x = -\infty$ to $-\frac{a}{\lambda}$ at $+\infty$. Finally, there are solutions that resemble spatially alternating solitons and antisolitons. However, treating these solutions as collections of individual solitons and antisolitons is a good approximation only when the solitons and antisolitons are widely separated, so that they disturb one-another only minimally. In general, these solitons must be moving relative to one-another, but we shall assume that this motion is very slow, so that $\phi(x)$ depends very weakly on time. In three spatial dimensions, there do exist exact

multi-soliton (multi-monopole) static solutions to bosonic field equations [51].

We may introduce massless fermions coupled to this scalar field by a Yukawa interaction, $U(x) = g\phi(x)$. When ϕ is in a vacuum state, this generates a fermion mass $m = g\frac{a}{\lambda}$. When ϕ has the soliton profile ϕ_{S,x_0}, there is known to be a zero-energy bound state localized around the zero of the soliton [26, 48, 12], with wavefunction

$$\psi_+ = C\left\{\cosh\left[\frac{\mu}{2}(x-x_0)\right]\right\}^{-2m/\mu}\begin{bmatrix}1\\0\end{bmatrix}. \tag{2.5}$$

C is a real normalization constant. For the antisoliton ϕ_{A,x_0} the corresponding zero mode wavefunction is

$$\psi_- = C\left\{\cosh\left[\frac{\mu}{2}(x-x_0)\right]\right\}^{-2m/\mu}\begin{bmatrix}0\\1\end{bmatrix}. \tag{2.6}$$

2.2 Fermion energies in a two-solition background

We shall investigate how the energies of the fermion bound states are affected in the two-soliton case. We restrict our attention to situations for which $\frac{\mu}{m} > 2$, when there exists only a single fermionic bound state on each of the isolated soliton and antisoliton. We shall consider an antisoliton centered at x_1 and a soliton centered at x_2, where $\mu(x_2 - x_1) \gg 1$. We shall approximate ϕ as a sum $\phi = \phi_{A,x_1} + \phi_{S,x_2} + \frac{a}{\lambda}$ and use ψ_+ and ψ_- to represent the bound state wavefunctions (2.5) and (2.6) centered at x_2 and x_1, respectively.

When the solitons are far apart, we may expect the potential due to one soliton to be a small perturbation of the Dirac equation for the bound state localized around the other soliton. In this spirit, we shall calculate the energy expectation values for the symmetrized states $\Psi_\pm \equiv \frac{1}{\sqrt{2}}(\psi_+ \pm \psi_-)$. For $H\psi_+$, we have

$$\begin{aligned} H\psi_+ &= [-i\sigma_2\partial_x + \sigma_1 g\phi_{S,x_2}]\psi_+ + \sigma_1 g\left[\phi_{A,x_1} + \frac{a}{\lambda}\right]\psi_+ \\ &= 0 + \sigma_1[g\phi_{A,x_1} + m]\psi_+. \end{aligned} \tag{2.7}$$

2.2.1 Energies for light fermions

We shall first evaluate the energy expectation values $E_\pm \equiv \int_{-\infty}^{\infty} dx\, \Psi_\pm^\dagger H \Psi_\pm$ under the approximations $\frac{\mu}{m} \gg 1$ and $\mu(x_2 - x_1) \gg 1$, where elementary expressions can be obtained explicitly. Then we shall present an exact analysis for general $\frac{\mu}{m} > 2$, which reduces to the previous when $\frac{\mu}{m} \gg 1$.

To find $E_\pm$, we need to integrate quantities such as $(\psi_+^\dagger \sigma_1 \psi_-)\phi_{A,x_1}$. The characteristic size of ϕ_{A,x_1} is $\frac{1}{\mu}$, and the characteristic decay length of $\psi_\pm$ is $\frac{1}{m}$. If $\frac{\mu}{m} \gg 1$, then the fermion wavefunctions decay very little over the width of the antisoliton, and it is a good approximation to replace ϕ_{A,x_1} by the step function $-\frac{a}{\lambda}\mathrm{sgn}(x - x_1)$ in the integral. Approximating ϕ_{A,x_1} with this step function gives

$$H\psi_+ = \sigma_1 m[1 - \mathrm{sgn}(x - x_1)]\psi_+. \tag{2.8}$$

The corresponding expression for $H\psi_-$ is

$$H\psi_- = \sigma_1 m[1 + \mathrm{sgn}(x - x_2)]\psi_-. \tag{2.9}$$

Combining (2.8) and (2.9), we get

$$H\Psi_\pm = \frac{1}{\sqrt{2}}\sigma_1 m\{[1 - \mathrm{sgn}(x - x_1)]\psi_+ \pm [1 + \mathrm{sgn}(x - x_2)]\psi_-\}. \tag{2.10}$$

Since $\psi_+^\dagger \sigma_1 \psi_+ = \psi_-^\dagger \sigma_1 \psi_- = 0$, only the cross terms in $\Psi_\pm^\dagger H \Psi_\pm$ are nonzero. Using the relations

$$\psi_+^\dagger \sigma_1 \psi_- = \psi_-^\dagger \sigma_1 \psi_+ = C^2 \left\{\cosh\left[\frac{\mu}{2}(x - x_1)\right]\cosh\left[\frac{\mu}{2}(x - x_2)\right]\right\}^{-2m/\mu}, \tag{2.11}$$

the expression for $E_\pm$ is

$$\begin{aligned} E_\pm &= \pm m\frac{C^2}{2}\int_{-\infty}^{\infty} dx \left\{\cosh\left[\frac{\mu}{2}\left(x - y + \frac{\Delta x}{2}\right)\right]\cosh\left[\frac{\mu}{2}\left(x - y - \frac{\Delta x}{2}\right)\right]\right\}^{-2m/\mu} \\ &\quad \times \left[2 - \mathrm{sgn}\left(x - y + \frac{\Delta x}{2}\right) + \mathrm{sgn}\left(x - y - \frac{\Delta x}{2}\right)\right], \end{aligned} \tag{2.12}$$

where $y \equiv \frac{x_2+x_1}{2}$, and $\Delta x \equiv x_2 - x_1$ is the separation between the solitons. Shifting he integration $x \to (x-y)$ gives

$$\begin{aligned} E_\pm &= \pm m\frac{C^2}{2}\int_{-\infty}^{\infty} dx \left\{\cosh\left[\frac{\mu}{2}\left(x+\frac{\Delta x}{2}\right)\right]\cosh\left[\frac{\mu}{2}\left(x-\frac{\Delta x}{2}\right)\right]\right\}^{-2m/\mu} \\ &\quad \times\left[2-\operatorname{sgn}\left(x+\frac{\Delta x}{2}\right)+\operatorname{sgn}\left(x-\frac{\Delta x}{2}\right)\right], \end{aligned} \tag{2.13}$$

which depends only on Δx. Thus we see that the zero-energy eigenstate of the one-soliton background bifurcates into two states, symmetrically displaced above and below $E=0$. To proceed further with our estimate of $E_\pm$, we must approximate the wavefunctions, using

$$\cosh\left[\frac{\mu}{2}(x-x_0)\right] \approx \frac{1}{2}\left[\theta(x-x_0)e^{\frac{\mu}{2}(x-x_0)} + \theta(x_0-x)e^{-\frac{\mu}{2}(x-x_0)}\right]. \tag{2.14}$$

The approximation (2.14) is natural in this context, since the approximate wavefunctions produced by (2.14) are the exact wavefunctions corresponding to perfectly rectangular solitons. Using (2.14), $E_\pm$ reduces to an integral over exponentials and step functions, which may be evaluated exactly, yielding,

$$E_\pm \approx \pm(16)^{m/\mu}C^2 e^{-m\Delta x}. \tag{2.15}$$

It only remains to calculate C. Using the same approximation (2.14) for cosh and demanding that $\int \psi_+^\dagger \psi_+ \, dx = 1$ gives

$$C^2 \approx (16)^{-m/\mu} m, \tag{2.16}$$

so that

$$E_\pm \approx \pm m e^{-m\Delta x}. \tag{2.17}$$

An interaction energy of this nature has been previously suggested [26].

We are now in a position to show that the $\Psi_\pm$ are the correct approximate states for which to calculate the energy. If we instead used some other linear combination

$(b\psi_+ + c\psi_-)$, with $|b|^2 + |c|^2 = 1$, the expectation value of the energy would be $E' = 2(\mathrm{Re}\{bc^*\})E_+$. E' takes its minimum value (corresponding to the best approximation to the lowest-energy wavefunction) when $c = -b$. The best approximation to the higher-energy wavefunction must be orthogonal to our expression for the lower-lying state, and so it has $c = b$. So the best linear combinations of ψ_+ and ψ_- are $\frac{1}{\sqrt{2}}(\psi_+ \pm \psi_-)$.

2.2.2 Interactions with fermions of more general mass

To obtain the result (2.17), we needed to make the approximation $\frac{\mu}{m} \gg 1$. We now present an exact evaluation, which shows that the above is the leading term for large $\frac{\mu}{m}$.

If we insert the exact expressions (2.4), (2.5), and (2.6) into (2.7) (and the corresponding expression for $H\psi_-$), we get

$$\begin{aligned} H\Psi_\pm &= \frac{1}{\sqrt{2}}\sigma_1 m\left\{1 - \tanh\left[\frac{\mu}{2}\left(x - y + \frac{\Delta x}{2}\right)\right]\right\}\psi_+ \\ &\quad \pm\frac{1}{\sqrt{2}}\sigma_1 m\left\{1 + \tanh\left[\frac{\mu}{2}\left(x - y - \frac{\Delta x}{2}\right)\right]\right\}\psi_-. \end{aligned} \tag{2.18}$$

Multiplying this by $\Psi_\pm^\dagger$ from the left and integrating over the shifted variable $(x - y)$ gives

$$\begin{aligned} E_\pm &= \pm m\frac{C^2}{2}\int_{-\infty}^{\infty} dx\left\{\mathrm{sech}\left[\frac{\mu}{2}\left(x + \frac{\Delta x}{2}\right)\right]\mathrm{sech}\left[\frac{\mu}{2}\left(x - \frac{\Delta x}{2}\right)\right]\right\}^{2m/\mu} \\ &\quad \times\left\{2 - \tanh\left[\frac{\mu}{2}\left(x + \frac{\Delta x}{2}\right)\right] + \tanh\left[\frac{\mu}{2}\left(x - \frac{\Delta x}{2}\right)\right]\right\}. \end{aligned} \tag{2.19}$$

When the integral is performed, the two tanh terms contribute equally, giving

$$\begin{aligned} E_\pm &= \pm mC^2\int_{-\infty}^{\infty} dx\left\{\mathrm{sech}\left[\frac{\mu}{2}\left(x + \frac{\Delta x}{2}\right)\right]\mathrm{sech}\left[\frac{\mu}{2}\left(x - \frac{\Delta x}{2}\right)\right]\right\}^{2m/\mu} \\ &\quad \times\left\{1 + \tanh\left[\frac{\mu}{2}\left(x - \frac{\Delta x}{2}\right)\right]\right\}. \end{aligned} \tag{2.20}$$

We may simplify (2.20) using hyperbolic identities, in particular

$$\operatorname{sech}(a+b) = \frac{\operatorname{sech} a \operatorname{sech} b}{1 + \tanh a \tanh b}. \tag{2.21}$$

Letting $a = \frac{\mu}{2}(x - \frac{\Delta x}{2})$ and $b = \frac{\mu}{2}\Delta x$ in (2.21) gives

$$E_{\pm} = \pm mC^2 \operatorname{sech}^n\left(\frac{m}{n}\Delta x\right) \int_{-\infty}^{\infty} dx \frac{\operatorname{sech}^{2n}\left[\frac{m}{n}\left(x - \frac{\Delta x}{2}\right)\right]\left\{1 + \tanh\left[\frac{m}{n}\left(x - \frac{\Delta x}{2}\right)\right]\right\}}{\left\{1 + \alpha \tanh\left[\frac{m}{n}\left(x - \frac{\Delta x}{2}\right)\right]\right\}^n}, \tag{2.22}$$

where we have introduced the dimensionless parameters $n \equiv \frac{2m}{\mu}$ and $\alpha \equiv \tanh(\frac{m}{n}\Delta x)$. Making the substitution $u = \tanh[\frac{m}{n}(x - \frac{\Delta x}{2})]$ in (2.22) simplifies this integral substantially, giving

$$E_{\pm} = \pm mC^2 \operatorname{sech}^n\left(\frac{m}{n}\Delta x\right) \int_{-1}^{1} du \frac{n}{m} \frac{(1+u)^n (1-u)^{n-1}}{(1+\alpha u)^n}. \tag{2.23}$$

To get a final expression for the energy, we must again calculate C. The normalization gives

$$\begin{aligned} \frac{1}{C^2} &= \int_{-\infty}^{\infty} dx \operatorname{sech}^{2n}\left[\frac{m}{n}(x - x_0)\right] \\ &= \int_{-1}^{1} du \frac{n}{m}(1-u^2)^{n-1} \\ &= \frac{n}{m}\sqrt{\pi}\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}, \end{aligned} \tag{2.24}$$

which agrees with (2.16) as $n \to 0$.

The remaining integral in (2.23) may also be found analytically. The result involves the hypergeometric function $F(a_1, a_2; b_1; z)$. In terms of this F, the integral is

$$\int_{-1}^{1} du \frac{(1+u)^n (1-u)^{n-1}}{(1+\alpha u)^n} = \sqrt{\pi}\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}(1+\alpha)^{-n} F\left(n, n; 1+2n; \frac{2\alpha}{1+\alpha}\right). \tag{2.25}$$

Fortunately, this simplifies significantly when combined with (2.24), so that the final

formula for the fermion energies is

$$E_{\pm} = \pm m \operatorname{sech}^n\left(\frac{m}{n}\Delta x\right)(1+\alpha)^{-n} F\left(n, n; 1+2n; \frac{2\alpha}{1+\alpha}\right). \tag{2.26}$$

Since the solitons are far apart, $\alpha \approx 1 - 2e^{-\frac{2m}{n}\Delta x} \equiv 1 - \epsilon$ is close to one, and it is natural to expand (2.25) around $\alpha = 1$. If we expand the integral in (2.25) to $\mathcal{O}(\epsilon)$, we find,

$$\int_{-1}^{1} du \frac{(1+u)^n (1-u)^{n-1}}{(1+\alpha u)^n} = \int_{-1}^{1} du\,(1-u)^{n-1} + \epsilon\left[n\int_{-1}^{1} du \frac{u(1-u)^{n-1}}{(1+u)}\right]. \tag{2.27}$$

The $\mathcal{O}(\epsilon^0)$ term in (2.27) has value $\frac{2^n}{n}$. However, the $\mathcal{O}(\epsilon)$ term is divergent, because $F(n, n; 1+2n; z)$ is not analytic at $z = 1$. Instead, we have

$$\int_{-1}^{1} du \frac{(1+u)^n (1-u)^{n-1}}{(1+\alpha u)^n} = \frac{2^n}{n} + n2^{n-1}(\epsilon \ln \epsilon) + \mathcal{O}(\epsilon). \tag{2.28}$$

Using (2.28) and $\epsilon = 2e^{-\frac{2m}{n}\Delta x}$, the $\mathcal{O}(\epsilon \ln \epsilon)$ expression for $E_{\pm}$ is

$$E_{\pm} \approx \pm m\left[\frac{2^n}{\sqrt{\pi}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]\operatorname{sech}^n\left(\frac{m}{n}\Delta x\right)\left[1 - n\,(2m\Delta x)\,e^{-\frac{2m}{n}\Delta x}\right]. \tag{2.29}$$

As $n \to 0$, this agrees with the earlier result (2.17), through terms of $\mathcal{O}(n)$.

The expression (2.26) gives the energies for the fermion bound states that lie closest to zero energy. When $n < 1$, the one-soliton system has only the single bound state (2.5) [12], so the soliton-antisoliton system we are considering has two bound states, with energies given by (2.26). The fermion energies depend on Δx, so (2.26) generates a contribution to the effective potential between the soliton and the antisoliton. The character of this potential term depends on which fermionic states are occupied. In the vacuum, the state Ψ_- is occupied, while Ψ_+ is empty. E_- is negative and for $\alpha \approx 1$ becomes more negative as Δx decreases, creating an attractive interaction between the soliton-antisoliton pair. Since this interaction exists in the absence of any fermions or antifermions, it must be generated by virtual particle effects. If either a single fermion or a single antifermion is present (i.e. both

states are either filled or empty), the total energy vanishes, and these states do not contribute to the effective potential. Evidently, the presence of a single particle causes a cancellation of the virtual particle corrections mentioned above. In the presence of both a fermion and an antifermion (Ψ_+ occupied and Ψ_- empty), the potential term is repulsive. This presents an interesting picture, with the fermions introducing novel interactions between the soliton-antisoliton pair.

If the positions of the soliton and antisoliton are reversed, so that $x_1 > x_2$, a calculation of the energies proceeds along the same lines. We could extend (2.26) by replacing Δx by $|\Delta x|$ (both in the sech term and in α) and multiplying by $\mathrm{sgn}(x_2 - x_1)$. The energies obtained would be valid whenever $\mu|\Delta x| \gg 1$. However, there are additional subtleties that arise when we allow Δx to change signs. This involves the solitons passing through one-another. The ϕ^4 solitons do not pass through one-another to emerge undistorted [8]; a sine-Gordon soliton and antisoliton will pass through each other without distortion, but when the solitons are in precisely the same place, the field takes a vacuum value everywhere, so there are no bound states at that instant [42]. In either case, it is impossible to identify which bound state for $x_2 > x_1$ corresponds to which bound state for $x_1 > x_2$. However, (2.26) does have the attractive feature that the energies $E_\pm$ approach the energies of stationary continuum solutions as $\Delta x \to 0$ (even though (2.26) is not valid for small values of the separation).

2.3 Extensions to 3+1 dimensions

The simplest (3+1)-dimensional soliton with a known zero-energy fermion mode is the 't Hooft-Polyakov monopole [26, 47, 41] coupled to isospinor fermions (in the fundamental representation). The monopole is described by a scalar field $\Phi_M = \Phi_M^a T^a$ and a vector field $A_M^0 = 0$, $A_M^i = A_M^{ia} T^a$, where T^a is the 2×2 isospin generator.

Using the representation

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix}, \ \beta = i \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \tag{2.30}$$

for the Dirac matrices, the associated zero mode is

$$\psi_M = \begin{bmatrix} \chi_+ \\ 0 \end{bmatrix}, \ \chi_+^{\nu n} = f(|\mathbf{r} - \mathbf{r}_M|)(s_+^\nu s_-^n - s_-^\nu s_+^n), \tag{2.31}$$

where ν is the spinor index, n is the isospin index, $\mathbf{r}_M$ is the position of the monopole, and f is a known function of the boson fields. The $s_\pm^{\nu,n}$ are orthonormal spinor, isospinor basis vectors. The corresponding zero mode for an antisoliton with boson fields Φ_A, $A_A^0 = 0$, and A_A^i is

$$\psi_A = \begin{bmatrix} 0 \\ \chi_- \end{bmatrix}, \ \chi_-^{\nu n} = f(|\mathbf{r} - \mathbf{r}_A|)(s_+^\nu s_-^n - s_-^\nu s_+^n). \tag{2.32}$$

To apply our method to this system, we consider the approximate monopole-antimonopole configuration $\Phi = \Phi_M + \Phi_A + \Phi_0$ (where Φ_0 is the vacuum value of the field as $r \to \infty$), $A^\mu = A_M^\mu + A_A^\mu$ and calculate the expectation value of the energy for the state

$$\Psi_\pm \equiv \frac{1}{\sqrt{2}}(\psi_M + \psi_A) = \frac{1}{\sqrt{2}} \begin{bmatrix} \chi_+ \\ \chi_- \end{bmatrix} \tag{2.33}$$

in this background. This energy is

$$\begin{aligned} E_\pm &= \frac{1}{2}\int d^3r \qquad\qquad (2.34) \\ &\times \ [\chi_+^\dagger \ \chi_-^\dagger] \begin{bmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{p} - \boldsymbol{\sigma}\cdot\mathbf{A}^a T^a - iG\Phi^a T^a \\ \boldsymbol{\sigma}\cdot\mathbf{p} - \boldsymbol{\sigma}\cdot\mathbf{A}^a T^a + iG\Phi^a T^a & 0 \end{bmatrix} \begin{bmatrix} \chi_+ \\ \chi_- \end{bmatrix}, \end{aligned}$$

where G is the strength of the Yukawa coupling. Since the states $\chi_\pm$ are singlets of spin plus isospin, the terms with $\mathbf{p}$ and Φ vanish immediately. The vector field term gives a contribution proportional to $\delta_i^a A^{ia}$, which vanishes for the monopole-antimonopole

profile, because A^{ia} is traceless in (i, a), since it involves ϵ^{iaj}. So in this approximation, there are two degenerate zero-energy fermion modes for the monopole-antimonopole system.

The isovector fermion coupled to the same boson field does not obviously produce zero-energy modes, and exact solutions of the Dirac equation in the presence of even a single monopole are difficult to obtain. However, our method would be applicable to this and other problems even if the single-soliton fermion zero-mode wavefunctions were only known approximately or numerically. The interaction energies could be found by numerically integrating the cross terms in the expectation value of the Hamiltonian. For this reason, our method has much broader usefulness that other methods which have been used to solve the purely (1+1)-dimensional problem.

Chapter 3

(1+1)-Dimensional Behavior and Photon Decay Near the Schwarzschild Horizon

In a recent paper [9], the question is raised whether classical General Relativity can exist as an emergent phenomenon—as the low-energy limit of an underlying quantum system. In this view, the singularity at the Schwarzschild event horizon represents a failure of the effective description owing to the divergence of a characteristic coherence length. We shall introduce an additional element to this model: a natural change in the dimensionality of virtual particle loop integrals of the quantum system near the horizon. This change will have important implications. In particular, it may cause photons to decay when they near the event horizon of a black hole, as suggested in [9].

3.1 (1+1)-dimensional behavior in classical black hole systems

In the Schwarzschild metric, the line element is

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (3.1)$$

(The velocity of light is set to one.) Classically, this metric concentrates the motion in the radial direction near $r = r_S$. This can be seen by examining the spatial line element

$$dl^2 = \left(1 - \frac{r_S}{r}\right)^{-1} \left[dr^2 + \left(1 - \frac{r_S}{r}\right) r^2 d\Omega^2\right]. \tag{3.2}$$

For r near r_S, the angular variables are suppressed, and motion is confined to the two-dimensional t-r subspace.

More specifically, consider the geodesic equation, $\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0$, where $\dot{x}^\mu$ is the derivative of the position with respect to an affine parameter.

The temporal and angular equations,

$$\ddot{t} + \frac{r_S}{r(r - r_S)}\dot{t}\dot{r} = 0, \tag{3.3}$$

$$\ddot{\theta} + 2\frac{\dot{r}}{r}\dot{\theta} - \frac{1}{2}\sin 2\theta\,\dot{\phi}^2 = 0, \tag{3.4}$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0, \tag{3.5}$$

can be readily integrated to give

$$\dot{t}\frac{r - r_S}{r} = \tau, \tag{3.6}$$

$$r^2 \sin^2\theta\,\dot{\phi} = m, \tag{3.7}$$

$$(r^2\dot{\theta})^2 + \frac{m^2}{\sin^2\theta} = l^2. \tag{3.8}$$

τ, m, and l^2 are integration constants. Near the Schwarzschild horizon we may allow all particles to be massless and take the geodesics to be null. Then the constants will diverge, but we retain their ratios, which remain finite. From the θ equation, it is clear that for $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$, $\ddot{\theta} = 0$ also, so the motion remains in the equatorial plane, with $m^2 = l^2$.

Using the other three equations, the radial equation

$$\ddot{r} + \left(1 - \frac{r_S}{r}\right)\frac{r_S}{2r^2}\dot{t}^2 + \left(1 - \frac{r_S}{r}\right)^{-1}\frac{r_S}{2r^2}\dot{r}^2 + (r_S - r)(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) = 0, \tag{3.9}$$

becomes

$$\ddot{r} + \frac{\tau^2 r_S}{2r(r-r_S)} - \frac{r_S}{2r(r-r_S)}\dot{r}^2 - \frac{l^2(r-r_S)}{r^4} = 0. \tag{3.10}$$

In this form, the equation may be integrated to give

$$\dot{r} = \pm\tau\sqrt{1-\left(1-\frac{r_S}{r}\right)\frac{\lambda^2}{r^2}}, \tag{3.11}$$

where $\lambda = l/\tau$, and a possible integration constant is set to zero for null geodesics.

Dividing the ϕ and r equations (3.7) and (3.11), by the t equation (3.6), gives at $\theta = \frac{\pi}{2}$, $m = l$,

$$\frac{d\phi}{dt} = \left(1-\frac{r_S}{r}\right)\frac{\lambda}{r^2}, \tag{3.12}$$

$$\frac{dr}{dt} = \pm\left(1-\frac{r_S}{r}\right)\sqrt{1-\left(1-\frac{r_S}{r}\right)\frac{\lambda^2}{r^2}}. \tag{3.13}$$

Consider a photon trajectory just outside the event horizon, at radius $r = r_S + \Delta$. For $\Delta \ll r_S$, the time-evolution equations (3.12) and (3.13) become

$$\frac{d\Delta}{dt} = \pm\frac{\Delta}{r_S}\sqrt{1-\frac{\lambda^2}{r_S^3}\Delta}, \tag{3.14}$$

$$\frac{d\phi}{dt} = \frac{\lambda}{r_S^3}\Delta. \tag{3.15}$$

The equations, with the lower sign in (3.14), may be integrated, yielding

$$\Delta = \frac{r_S^3/\lambda^2}{\cosh^2\frac{t}{2r_S}} \tag{3.16}$$

$$\phi = \frac{2r_S}{\lambda}\tanh\frac{t}{2r_S}. \tag{3.17}$$

Equations (3.16) and (3.17) show that if $\frac{\lambda}{r_S}$ is sufficiently small ($\ll \pi$), then the radial motion will be much more rapid than the angular motion. This is the regime we are interested in, where the concentration of motion in the radial direction reduces some aspects of the system to 1+1 dimensions.

3.2 Dimensional reduction in quantum loop calculations

We shall consider a much more profound reduction in the dimensionality of the system. Taking local coordinates r, x, and y, the spatial line element is

$$dl^2 = -g_{rr}dr^2 + dx^2 + dy^2. \tag{3.18}$$

In the model [9], the singularity at $r = r_S$ represents a real physical effect, not merely a coordinate artifact, so these are very natural linear coordinates. If we suppose that p^r, p^x, and p^y are cut off at the same scale in a loop integral, $\mathbf{p}^2 \equiv -p_i p^i$ will be dominated by $-g_{rr}(p^r)^2$. (This supposition strongly breaks general covariance, of course.) This is the kind of situation we would like to analyze. However, a sharp momentum cutoff is not gauge invariant; to study the photon self-energy, we shall translate this idea into gauge-invariant language, using dimensional regularization.

In the renormalization prescription described above, one of the spatial dimensions provides the dominant contribution to $\mathbf{p}^2$. In the language of dimensional regularization, this can be seen as a reduction in the effective dimensionality d of the momentum integral to $d < 4$.

We shall find the effective dimensionality by examining the volume element of this system, because the momentum cutoff in a given direction and the volume contribution of that direction are closely related. To see this, consider for the moment a theory that is regulated by a lattice at short distances. The volume of a region counts the number of lattice points in that region. The lattice spacing a_i in a given direction governs the density of lattice points along that axis, so a length L in the x_i-direction contributes an amount $\alpha\frac{L}{a_i}$ to the volume, where α is a scaling constant independent of direction. The lattice spacing also corresponds directly to the momentum cutoff in that direction, $p_i^{max} = \frac{\pi}{a_i}$. So the dependence of the volume on a given dimension and the momentum cutoff in that direction are intimately linked.

So we look for an expression for the effective dimensionality (to be used in dimen-

sional regularization) in terms of the volume element. In Schwarzschild space-time, the spatial volume element is

$$dV = \sqrt{g}\, dr\, d\theta\, d\phi. \tag{3.19}$$

Here, $g = -g_{rr} r^4 \sin^2\theta$ is the determinant of the spatial metric $-g_{ij}$. The classical geodesic problem suggests that the radial direction should always contribute one effective dimension, while the angular directions may contribute less than one. We wish to determine the effective dimensionality by an integral over an effective volume. To a achieve a reasonable result, the effective volume is defined by rescaling a factor of $\sqrt{-g_{rr}}$ from each direction. So the new volume element reads

$$dV' = \frac{r^2 \sin\theta}{-g_{rr}}\, dr\, d\theta\, d\phi = g_{tt} dV_E, \tag{3.20}$$

where dV_E is a Euclidean volume element.

From dV', we need to find an expression for the number of effective dimensions. This expression should have several properties. The dimension corresponding to a volume element $dx_1\, dx_2 \cdots dx_n$ should be n (with dV_E corresponding to 3 dimensions). So the dimension function should be additive where the volume element is multiplicative; this is the fundamental property of a logarithm. So a natural choice for the effective spatial dimension d_s is

$$d_s = \frac{\ln \int_0^\Lambda dV'}{\ln \Lambda}. \tag{3.21}$$

Evaluating this gives us

$$d_s = \frac{\ln \int_0^\Lambda dV_E}{\ln \Lambda} + 2\frac{\ln \sqrt{g_{tt}}}{\ln \Lambda} = 3 + 2\frac{\ln \sqrt{g_{tt}}}{\ln \Lambda}. \tag{3.22}$$

Equation (3.22) has some problems. The most striking one is that Λ appears to be a dimensional quantity, which would make $\frac{\ln \sqrt{g_{tt}}}{\ln \Lambda}$ ambiguous. The obvious solution is that the coordinates must be nondimensionalized, to make Λ dimensionless. How-

ever, it is not at all obvious how to nondimensionalize the coordinates. Fortunately, we do not need to deal with that question directly. Regardless of the coordinates' dimensions, dV_E will always contribute three dimensions to d_s. To analyze the last term in (3.22), we introduce the natural condition that the angular factors in the volume element can not contribute any less than zero dimensions each. That is,

$$\frac{\ln \sqrt{g_{tt}}}{\ln \Lambda} \geq -1. \tag{3.23}$$

Since $g_{tt} < 1$ and the logarithm is strictly increasing, this condition may be rewritten as

$$\Lambda^{-1} \leq \sqrt{g_{tt}}. \tag{3.24}$$

In classical General Relativity, $g_{tt}(r_S) = 0$, but in the model [9], g_{tt} drops to a nonzero minimum value

$$g_{tt}^{min} \sim 1 - \frac{r_S}{r_S + \delta} \approx \frac{\delta}{r_S}, \tag{3.25}$$

where δ is a small length that characterizes how the classical Schwarzschild singularity is cut off by the underlying quantum system. We expect δ to be related to the Planck length. Thus we estimate $\Lambda \sim \sqrt{r_S/\delta}$. Close to the event horizon, at $r = r_S + \Delta$, where $\delta \ll \Delta \ll r_S$ and $g_{tt} \sim \Delta/r_S$, the effective dimension of the system is

$$d_s = 1 + 2\left(1 - \frac{\ln \Delta/r_S}{\ln \delta/r_S}\right) = 1 + 2\left(\frac{\ln \delta/\Delta}{\ln \delta/r_S}\right). \tag{3.26}$$

So we will look at the problem of Quantum Electrodynamics in $(1+\epsilon)+1$ dimensions, where

$$\epsilon = 2\left(\frac{\ln \delta/\Delta}{\ln \delta/r_S}\right). \tag{3.27}$$

3.3 Photon mass generation and decay

We shall only consider the contribution to the photon self-energy from massless particles. This should be a good approximation near the event horizon. Photons coming in from spatial infinity are highly blueshifted at $r = r_S + \Delta$, so the momentum scale

in the photon propagator is large compared to any invariant momentum scale (such as the electron mass). The same argument may also be phrased in different terms. The energy of a comoving electron of mass m_e is $\sqrt{g_{tt}}\, m_e$, so near the event horizon, the apparent electron mass becomes small. So it is reasonable to consider massless particles.

The one-loop photon self-energy due to a single species of charged massless fermions is

$$i\Pi^{\mu\nu}(q) = 2\,\mathrm{tr}\,\mathrm{I}\,(q^2\eta^{\mu\nu} - q^\mu q^\nu)\int_0^1 dx \int_k \frac{x(1-x)}{[k^2 + q^2x(1-x)]^2}. \tag{3.28}$$

Here, I is the unit matrix in spinor space, and the the Minkowski metric is denoted by $\eta^{\mu\nu}$, to avoid confusion with the GR metric $g^{\mu\nu}$. Since $d \to 4$ as $r \to \infty$, the Dirac matrices should be four-dimensional. Equation (3.28), derived in Appendix A, gives, upon a d-dimensional k-integration,

$$i\Pi^{\mu\nu}(q) = -i\,\mathrm{tr}\,\mathrm{I}\,(q^2\eta^{\mu\nu} - q^\mu q^\nu)\frac{e^2}{(-q^2)^{2-d/2}}\frac{1}{(4\sqrt{\pi})^{d-1}}\frac{(1-d/2)\pi}{\sin\frac{\pi d}{2}}\frac{1}{\Gamma(\frac{d}{2}+\frac{1}{2})}. \tag{3.29}$$

There is a subtlety in the use of (3.29). In dimensional regularization, it is usual to reduce all aspects of the problem to d dimensions. In our case, only the loop integral is d-dimensional. There are still four Dirac matrices, and the photon remains a four-component field. However, so long as the external photon momentum q lies in the d-dimensional subspace, equation (3.29) remains valid, and the metric $\eta^{\mu\nu}$ is d-dimensional.

The $d = 4$ and $d = 2$ cases of equation (3.29) are well understood. Since we have $d = 2 + \epsilon$, we expand around $d = 2$. Evaluating equation (3.29) with this value of d, we get

$$i\Pi^{\mu\nu}(q) = -i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\frac{2e^2}{\pi}\frac{1}{(-q^2)^{1-\epsilon/2}}\left\{\frac{\pi^{1/2-\epsilon/2}}{2^{1+2\epsilon}}\frac{\epsilon\pi/2}{\sin\frac{\epsilon\pi}{2}}\frac{1}{\Gamma(\frac{3}{2}+\frac{\epsilon}{2})}\right\}. \tag{3.30}$$

The bracketed term in (3.30) is unity at $\epsilon = 0$. It is purely real, so will only contribute higher-order corrections to the real and imaginary parts of the self-energy.

At $d = 2$, we get Schwinger's well-known result that the photon becomes mas-

sive [44]. The self-energy is

$$i\Pi^{\mu\nu}(q)|_{d=2} = i\frac{2e^2}{\pi}\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right). \tag{3.31}$$

(This differs by a factor of two from the usual result, because here we have used four-dimensional Dirac matrices.) The residue of the pole at $q^2 = 0$ gives the photon mass $m_\gamma^2 = \frac{2e^2}{\pi}$.

For $d = 2 + \epsilon$, the result is only slightly different. Instead of having $\frac{1}{-q^2}$, we have $\frac{1}{(-q^2)^{1-\epsilon/2}}$, which we expand about $\epsilon = 0$ to get

$$\frac{1}{(-q^2)^{1-\epsilon/2}} \approx \frac{1}{-q^2}\left[1 + \frac{\epsilon}{2}\ln(-q^2)\right]. \tag{3.32}$$

As in equation (3.31), there is a pole at $q^2 = 0$. However, the residue is shifted by the second term in (3.32). To zeroth order in ϵ, the pole in the propagator is shifted to $\frac{2e^2}{\pi}$. To first order, the pole is further shifted to the value of $[1 + \frac{\epsilon}{2}\ln(-q^2)]$ evaluated at the new pole location. This shifts the pole to

$$\begin{aligned} m_\gamma^2 &= \frac{2e^2}{\pi}\left[1 + \frac{\epsilon}{2}\ln\left(-\frac{2e^2}{\pi}\right)\right] \\ &= \frac{2e^2}{\pi}\left[1 + \frac{\epsilon}{2}\ln(-1) + \frac{\epsilon}{2}\ln\left(\frac{2e^2}{\pi}\right)\right] \\ &\approx \frac{2e^2}{\pi}\left(1 - i\frac{\pi\epsilon}{2}\right). \end{aligned} \tag{3.33}$$

Equation (3.33) is correct to lowest order in ϵ in both its real and imaginary parts. The sign of the imaginary part has been chosen so that photons decay rather than appear.

We must now turn to the question of how to interpret equation (3.33). By expanding around $d = 2$, we have introduced a number of two-dimensional conventions. The e^2 appearing in (3.33) is the two-dimensional value of the electromagnetic coupling. In two dimensions, e has mass dimension one, so $\frac{2e^2}{\pi}$ is indeed a mass squared. We must relate the e in (3.33) (which we will henceforth refer to as e_2) to the four-dimensional electron charge e_4.

We may relate the two charges by comparing the actions in two and four dimensions. In four dimensions, the electromagnetic Lagrangian density is

$$\mathcal{L}_4 = -\frac{1}{4e_4^2} F_{\mu\nu} F^{\mu\nu}. \tag{3.34}$$

Then the action is

$$S_4 = \int dt\, r^2 dr\, d\Omega\, \mathcal{L}_4. \tag{3.35}$$

The action S_2 derived from the two-dimensional Lagrangian $\mathcal{L}_2$ should the same as S_4, up to a constant factor C. So we have

$$\int dt\, dr \mathcal{L}_2 = -C \int dt\, r^2 dr\, d\Omega \frac{1}{4e_4^2} F_{\mu\nu} F^{\mu\nu}. \tag{3.36}$$

We must perform the angular integrals on the right-hand side of (3.36) to determine $\mathcal{L}_2$. This means doing an integral over the submanifold orthogonal to the t-r subspace. This orthogonal submanifold is a sphere, and the integral over it will depend upon the radius at which the integral is evaluated. We are interested in in radii $r \approx r_S$ (which is the only region where the integration over angles is justified). If $F^{\mu\nu}$ is spherically symmetric, the angular integral gives 4π, and we can set $r = r_S$ to get

$$\int dt\, dr\, \mathcal{L}_2 = -4\pi r_S^2 C \int dt\, dr\, \frac{1}{4e_4^2} F_{\mu\nu} F^{\mu\nu}. \tag{3.37}$$

We can now read off the value of $\mathcal{L}_2$,

$$\mathcal{L}_2 = -\frac{1}{4} \frac{4\pi r_S^2 C}{e_4^2} F_{\mu\nu} F^{\mu\nu}, \tag{3.38}$$

so that the two-dimensional charge is

$$e_2^2 = \frac{1}{4\pi r_S^2 C} e_4^2. \tag{3.39}$$

The constant C includes differences in how the field operators are normalized in two and four dimensions. So although the precise numerical relation between e_2 and e_4

has not been determined, the dependence of e_2 on r_S is unambiguous.

In deriving equation (3.39), we assumed that the field configuration was spherically symmetric. We can also evaluate the angular integral for more general field configurations, although this adds additional ambiguities. If the field $F^{\mu\nu}$ is in an $l > 0$, $m = 0$ multipole state, the integral becomes

$$\int dt\, dr\, \mathcal{L}_2 = -C \int dt\, r^2 dr\, d\Omega\, P_l(\cos\theta)^2 \frac{1}{4e_4^2} F_{\mu\nu}F^{\mu\nu}. \tag{3.40}$$

(The additional angular fields coming from derivatives of $P_l(\cos\theta)$ are suppressed by $\frac{1}{r_S}$ and have been dropped.) Since the maximum value of $P_l(\cos\theta)$ is $P_l(1) = 1$, the $F^{\mu\nu}$ appearing in equation (3.40) is the maximum value of the field over all angles. It is consistent with our earlier identification of the two- and four-dimensional fields to identify this $F^{\mu\nu}$ with the field appearing in $\mathcal{L}_2$ (although other identifications could also be consistent). Evaluating the integral then gives us

$$e_2^2 = \frac{2l+1}{4\pi r_S^2 C} e_4^2. \tag{3.41}$$

A similar calculation may be done for $m \neq 0$, but the result (with these conventions) depends on m explicitly, not merely on l. Despite this problem, (3.41) remains a good candidate for an m-independent multipole field mass.

3.4 Interpretation of the mass term: similarities to classical electrodynamics

We must also address the question of how to interpret the imaginary part of (3.33). To help with the interpretation, we shall use an analogy to a much simpler dimensional reduction problem—the electromagnetic field in a rectangular waveguide [5]. This simple problem in classical electrodynamics has many similarities to the QED problem under consideration.

Consider a rectangular waveguide with metal walls. The waveguide has dimen-

sions a in the x-direction and b in the y-direction. (We will presume that a and b are comparable in magnitude.) The waves propagate freely in the z-direction. The boundary conditions on this system restrict the wavevector of the electromagnetic field in the interior to be

$$\mathbf{k} = \frac{\pi n_x}{a}\hat{\mathbf{x}} + \frac{\pi n_y}{b}\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}. \tag{3.42}$$

The numbers n_x and n_y are positive integers; at least one of n_x and n_y must be nonvanishing for fields to exist. The frequency $\omega = |\mathbf{k}|$ satisfies

$$\omega^2 = k_z^2 + \pi^2\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}\right). \tag{3.43}$$

Since propagation only occurs along the z-axis, it is natural to look at this system in the t-z subspace, where the wavevector is simply k_z. Then (3.43) looks like the energy-momentum relation for a relativistic particle of mass $m_{wg}^2 = \pi^2(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2})$.

So in $1+1$ dimensions, a photon in a waveguide acquires an effective mass. The scale of this mass is a^{-1}, where a is the characteristic size of the system in the neglected dimensions. This is the same scaling we found previously. In equation (3.33), the scale of the photon mass was r_S^{-1}, and r_S is the length scale of the event horizon in the angular directions. According to (3.41), the black hole system actually has a whole hierarchy of photon masses. The waveguide also exhibits this property; different n_x and n_y values give different values of m_{wg}^2. (These results are similar to those found in Kaluza-Klein theories, although the higher modes are not strongly suppressed here.)

The waveguide system also exhibits another important property—decay. Through interactions in the x- and y-dimensions, a photon can disappear from the interior of the waveguide. This can occur in a variety of ways, depending on the regime. We mention the two simplest regimes and discuss the interpretation of decays in these regimes. At low frequencies, $\omega \ll \nu$, where ν is the collision frequency for electrons in the metal walls, the magnetic field drives surface currents which dissipate energy through resistive heating. This leads to a low-frequency energy loss

$$\Gamma \equiv \frac{\langle P\rangle}{\langle U\rangle} \propto \frac{1}{\sqrt{a^3\sigma}}, \tag{3.44}$$

where σ is the conductivity of the metal walls. The behavior is different at higher frequencies, $\nu \ll \omega < \omega_p$, where ω_p is the plasma frequency, related to the electron density n_e by $\omega_p^2 = \frac{n_e e^2}{m_e}$. In this regime, the electromagnetic field is exponentially damped in the walls, but photons can tunnel through the walls and escape from the waveguide. However, the decay rate does not have a simple dependence on a and ω_p.

The decay rate in the waveguide depends strongly on the regime, but in each regime, the decay rate depends primarily on the length a and some other length parameter. In the regimes outlined above, the length parameters are provided by σ and ω_p. In the black hole model, the imaginary part of m_γ^2 is governed by ϵ, which depends on the inverse mass scale r_S, as well as on the lengths δ and Δ.

The two decay regimes outlined above possess very different decay processes. The first regime is dissipative. Through interactions, photons in the waveguide decay into something else—thermal excitations of the metal walls. In the second regime, the photon does not actually decay. Instead, it escapes from the waveguide and the corresponding $1+1$-dimensional subspace. Despite their differences, these processes would each contribute an imaginary part to m_{wg}^2.

Analogously, the imaginary term in equation (3.33) could have two different origins. The photons could be decaying into other particles, as is suggested in [9]. Alternatively, the effective decay rate in (3.33) could correspond to photons being scattered out of the $1+1$ dimensional t-r subspace into states with large angular momenta. Our simple treatment does not allow us to distinguish between the two. However, either process would be novel—an effect caused by the finite minimum of g_{tt} and controlled in magnitude by δ.

Although it is not directly relevant to the problem at hand, it is worth mentioning one further suggestive aspect of the waveguide analogy. In $2+1$ dimensions, the photon self-energy (3.29) is still singular at $q^2 = 0$, but that singularity is weaker than a simple pole in q^2 [28]. This introduces ambiguity as to whether or not the photon has mass in $2+1$ dimensions. The waveguide analogy of $2+1$-dimensional QED is the propagation of the electromagnetic field between two parallel plates of separation a in the x-direction. However, in this case, the wavevector in the x-direction, $\frac{\pi n_x}{a}$, is

allowed to vanish. So a photon in this system may or may not behave as if it were massive.

Chapter 4

Perturbative Energy Shifts of the Luttinger Liquid Due to the Presence of Additional Dimensions

4.1 The Luttinger model

A low-temperature system of fermions in one dimension has the known property of bosonization [43, 45, 15]. The coherent fermion-hole excitations of the system behave like bosons, with a spectrum that remains exactly solvable even in the presence of certain arbitrarily strong interactions. This phenomenon is described by the Luttinger and Tomonaga-Luttinger models.

The main physical applications of the Luttinger model occur in the context of the quantum Hall effect [52, 53]. However, there is another situation in which we expect Luttinger-type behavior to arise quite naturally. If a large number of degenerate fermions are trapped in a potential well with a very high aspect ratio, so that the particles are effectively constrained to move along a single axis, we would expect one-dimensional behavior to dominate. We shall consider a system of this type. Since the trapping potential cannot be purely one-dimensional, the second and third dimensions will affect the energy levels of the system. We shall calculate these energy shifts, by

treating the effects of the small transverse dimensions as perturbations around a purely one-dimensional model. Since the field of fermion trapping is advancing very rapidly [21, 13], such calculations should be highly relevant in the analysis of future research.

After a brief summary of the one-dimensional Luttinger model, we shall introduce a physically motivated interaction Hamiltonian in Section 4.2. We then calculate the perturbative energy shift due to this interaction in two different ways. In Section 4.3, we use bosonized forms for the fermion operators. The results of the bosonized calculation depend upon the cutoff of the model. We may determine the physically meaningful value of the cutoff by comparing the bosonized result with the result of our second calculation (Section 4.4), which uses an entirely different, simplified model of the one-dimensional Fermi gas. Finally, in Section 4.5, we extend our results to systems with additional interactions and discuss the significance of our findings. Our main results are expressions (4.48) and (4.54), which describe the energy shifts of the bosonic modes.

The phenomenon of bosonization is expressed in the Luttinger model, in which the noninteracting (spinless) fermionic Hamiltonian is approximated by

$$H_0 = v_F \sum_k \left[(k - k_F) a_k^\dagger a_k + (-k - k_F) b_k^\dagger b_k \right], \tag{4.1}$$

where a (b) operators correspond to right-(left-)moving fermions and v_F is the Fermi velocity. (We set $\hbar = 1$.) In the Luttinger model, the further approximation is made that this Hamiltonian is valid for all momenta k, but that states corresponding to a_k and b_{-k} for $k < 0$ are filled [37, 40]. The introduction of fictional negative-energy states, which lie far from the Fermi surface, should have little effect on the physics at low temperatures.

With these approximations, the Hamiltonian may be expressed in the form

$$H_0 = \frac{2\pi v_F}{L} \sum_{q>0} [\rho_+(q)\rho_+(-q) + \rho_-(-q)\rho_-(q)], \tag{4.2}$$

where the $\rho_\pm$ operators

$$\begin{aligned} \rho_+(q) &= \sum_k a^\dagger_{k+q} a_k \\ \rho_-(q) &= \sum_k b^\dagger_{k+q} b_k \end{aligned} \tag{4.3}$$

obey the Bose commutation relations $[\rho_+(-q), \rho_+(q')] = [\rho_-(q), \rho_-(-q')] = \delta_{qq'} \frac{qL}{2\pi}$ and $[\rho_+(q), \rho_-(q')] = 0$. The bosonic quanta created and annihilated by the $\rho_\pm$ operators have energies $\omega_q = v_F q$ (taking $q > 0$). It is also possible to express the fermion field operators in terms of the bosonic $\rho_\pm$ operators. We shall make use of this representation in Section 4.3.

The Tomonaga-Luttinger model is formed by adding scattering effects [50], via the interaction Hamiltonian

$$H_1 = \frac{1}{2L} \sum_q \{2g_2(q)\rho_+(q)\rho_-(-q) + g_4(q)[\rho_+(q)\rho_+(-q) + \rho_-(q)\rho_-(-q)]\}. \tag{4.4}$$

The g_2 and g_4 are the Fourier transforms of the interaction potentials. This model is exactly solvable, since $H_0 + H_1$ remains bilinear in the $\rho_\pm$. We will return to this interaction later, but first we shall consider interactions between the Luttinger liquid bosons and fermions in excited states of the trap.

4.2 Interaction Hamiltonians

The Luttinger model should be a good description of a system of fermions in a narrow, axially symmetric trap, in which no radially or angularly excited states are occupied. We shall add to this system additional four-fermion interaction terms. These new interactions will be similar in structure to H_1 (when H_1 is expanded in terms of fermion operators). These interactions must conserve linear momentum along and angular momentum around the axis of the trap, as well as fermion number. One such

interaction term is

$$H' = \sum_{k,k',q} C_{k,k',q} c^\dagger_{k-q} d_k (a^\dagger_{k'+q} a_{k'} + b^\dagger_{k'+q} b_{k'}). \tag{4.5}$$

The operators c_k and d_k correspond to arbitrary fermion states, subject to the condition that at least one of them corresponds to an excited state of the trap (i.e. is not an a_k or a b_k). This ensures that we shall be studying new effects. The effective coupling $C_{k,k',q}$ will depend upon the radial and angular quantum numbers corresponding to c_k and d_k (in part because $C_{k,k',q}$ includes integrals over the radial and angular eigenmodes of the trap). Angular momentum conservation demands that c_k and d_k annihilate states with the same angular momentum.

We are only interested in interactions that will renormalize the bosonic energies. The simplest way for bosonized operators to arise from from an interaction Hamiltonian such as H' is for $C_{k,k',q}$ to be effectively independent of k and k'. Then the $a^\dagger_{k'+q}$ and $a_{k'}$ operators combine to form a boson operator, as in (4.3), and the $b^\dagger_{k'+q}$ and $b_{k'}$ combine similarly. This leads to an interaction of the form

$$H_2 = \frac{1}{2L} \sum_{q \neq 0} g(q) \left(\sum_k c^\dagger_{k-q} d_k \right) [\rho_+(q) + \rho_-(q)] + \text{h.c.}. \tag{4.6}$$

We shall calculate the energy shifts of the low-lying bosonic states to second order in H_2. (The first-order shifts obviously vanish.) If both c_{k-q} and d_k correspond to excited states of the trap, then the operator H_2 preserves the number of excited fermions. Hence, H_2 has zero matrix element between low-lying (Luttinger liquid) states and states with excited fermions. So we only get nonvanishing contributions from terms where one of c_k and d_k corresponds to an excited fermion state, while the other corresponds to a low-lying mode. Moreover, the excited mode must have angular momentum $\ell = 0$ about the axis of the trap.

Let us fix $q > 0$. We shall calculate the energy shift for the $\rho_+(q)$ quanta. In order for these quanta to be well-behaved Luttinger liquid phonons, we must have $\omega_q \ll \epsilon_F \equiv \frac{1}{2m} k_F^2$, or $q \ll k_F$. We may write the interactions affecting the $\rho_+(q)$

quanta as

$$H_{2,+q} = \frac{1}{L} g(q) \left\{ \left[\sum_k (c^\dagger_{k-q} d_k + d^\dagger_{k-q} c_k) \right] \rho_+(q) + \left[\sum_k (c^\dagger_{k+q} d_k + d^\dagger_{k+q} c_k) \right] \rho_+(-q) \right\}, \tag{4.7}$$

where d_k annihilates an excited fermion and c_k annihilates a low-lying fermion. [If c_k annihilates a right-moving fermion, additional terms are actually be required in order make $H_{2,+q}$ hermitian. However, these terms do not involve ρ_+, so their matrix elements are smaller by a factor of $\mathcal{O}(L^{-1/2})$ and may be neglected.] We shall denote by Δ the energy gap for the $\ell = 0$ states corresponding to the operators d_k.

4.3 Energy shift calculated with bosonized operators

We shall calculate the perturbative energy shift arising from (4.7) in two different models. In this section, we use the bosonized field operators of the Luttinger model to perform the calculation. It is possible to obtain a closed-form expression for the energy shift in two different situations—when the temperature vanishes or when the energy gap is very large. In these situations, we find that the leading contributions to the the infinite sum over intermediate states may be resummed. This resummation allows us to find a simple expression for the energy shift.

We shall restrict our attention to the case in which c_k is a_k (rather than b_k). In this case, the matrix element which we must calculate is $\langle \{m_{+p}\}; K | H_{2,+q} | \{n_{+p}\} \rangle$. The state $|\{n_{+p}\}\rangle$ is a number state of the Luttinger model; the $\{n_{+p}\}$ are the occupation numbers of the ρ_+ quanta. The state $|\{m_{+p}\}; K\rangle$ has differing occupation numbers $\{m_{+p}\}$, as well as an excited state fermion with momentum K.

The matrix element is simply

$$\begin{aligned} \langle \{m_{+p}\}; K | H_{2,q} | \{n_{+p}\} \rangle &= \frac{1}{L} g(q) \langle \{m_{+p}\}; K | \sum_k \left[d^\dagger_{k-q} a_k \rho_+(q) \right. \\ &\quad \left. + d^\dagger_{k+q} a_k \rho_+(-q) \right] | \{n_{+p}\} \rangle \end{aligned}$$

$$= \frac{1}{L} g(q) \sum_k \delta_{K,k-q} \sqrt{\left(\frac{qL}{2\pi}\right)(n_{+q}+1)} \langle \{m_{+p}\}|a_k|\{n'_{+p}\}\rangle$$
$$+ \frac{1}{L} g(q) \sum_k \delta_{K,k+q} \sqrt{\left(\frac{qL}{2\pi}\right) n_{+q}} \langle \{m_{+p}\}|a_k|\{n''_{+p}\}\rangle, \quad (4.8)$$

where the occupation numbers $\{n'_{+p}\}$ and $\{n''_{+p}\}$ are the same as the $\{n_{+p}\}$, except that $n'_{+q} = n_{+q} + 1$ and $n''_{+q} = n_{+q} - 1$. As $L \to \infty$, $\{n'_{+p}\}$ and $\{n''_{+p}\}$ differ from $\{n_{+p}\}$ only on a set of measure zero, so we may replace the matrix elements $\langle \{m_{+p}\}|a_k|\{n'_{+p}\}\rangle$ and $\langle \{m_{+p}\}|a_k|\{n''_{+p}\}\rangle$ appearing in (4.8) by $\langle \{m'_{+p}\}|a_k|\{n_{+p}\}\rangle$ and $\langle \{m''_{+p}\}|a_k|\{n_{+p}\}\rangle$, where $m'_{+q} = m_{+q} - 1$ and $m''_{+q} = m_{+q} + 1$. We shall eventually be summing over all possible sets of occupation numbers $\{m_{+p}\}$ and shall only be interested in cases where the energy difference between $|\{m_{+p}\}\rangle$ and $|\{m'_{+p}\}\rangle$ or $|\{m''_{+p}\}\rangle$ is negligible, so we may ultimately replace both matrix elements in (4.8) with $\langle \{m_{+p}\}|a_k|\{n_{+p}\}\rangle$. So we see that the crucial quantity to calculate is $\langle \{m_{+p}\}|a_k|\{n_{+p}\}\rangle$, for an arbitrary set of occupation numbers $\{m_{+p}\}$. We shall perform this calculation by taking a Fourier transform,

$$\langle \{m_{+p}\}|a_k|\{n_{+p}\}\rangle = \frac{1}{\sqrt{L}} \int_0^L dx\, e^{-ikx} \langle \{m_{+p}\}|\psi_+(x)|\{n_{+p}\}\rangle, \quad (4.9)$$

so we must now turn our attention to the field operator $\psi_\pm$.

The bosonized form of the fermion field operator is [39, 35, 25, 22, 24]

$$\psi_\pm(x) = \frac{1}{\sqrt{2\pi\alpha}} U_\pm \exp[\pm i k_F x \mp i\phi(x) + i\theta(x)]. \quad (4.10)$$

In the pure Luttinger model, this is only an operator identity when we take the limit $\alpha \to 0$ [49]. However, we shall interpret α as a finite cutoff, so that $v_F \alpha^{-1}$ is the bandwidth of the Luttinger liquid [45, 35, 34, 31, 10]. The finiteness of α corresponds to a deviation of the energy spectrum from the Luttinger form far from the Fermi surface. We shall discuss which values of α are physically meaningful in Sections 4.4 and 4.5. The $U_\pm$ operators decrease the total particle number on their corresponding branches by unity; these operators are necessary because the bosonic fields conserve the total fermion number.

The bosonic quantities ϕ (the boson field) and θ (the integral of the momentum conjugate to ϕ) are given by

$$-i\phi = -\frac{\pi}{L}\sum_{p\neq 0}\frac{1}{p}e^{-\alpha|p|/2}e^{-ipx}[\rho_+(p)+\rho_-(p)] + i\pi\frac{N_+ + N_-}{L}x \tag{4.11}$$

$$i\theta = -\frac{\pi}{L}\sum_{p\neq 0}\frac{1}{p}e^{-\alpha|p|/2}e^{-ipx}[\rho_+(p)-\rho_-(p)] + i\pi\frac{N_+ - N_-}{L}x, \tag{4.12}$$

where N_+ and N_- are the numbers of right- and left-moving fermions, respectively. So the argument of the exponential in $\psi_\pm$ is

$$\pm ik_F x \mp i\phi + i\theta = \pm ik_F x \mp \frac{2\pi}{L}\sum_{p\neq 0}\frac{1}{p}e^{-\alpha|p|/2}e^{-ipx}\rho_\pm(p) \pm \frac{2\pi i N_\pm}{L}x. \tag{4.13}$$

Using $k_F = \frac{2\pi N_\pm}{L}$, this becomes,

$$\pm ik_F x \mp i\phi + i\theta = \pm 2ik_F x \mp \frac{2\pi}{L}\sum_{p>0}\frac{1}{p}e^{-\alpha p/2}\left[e^{-ipx}\rho_\pm(p) - e^{ipx}\rho_\pm(-p)\right]. \tag{4.14}$$

Since $[\rho_\pm(p), \rho_\pm(p')] = 0$ if $p \neq \pm p'$, the elements of the sum in (4.14) all commute with one-another. This allows us to write $\psi_\pm$ in the product form

$$\psi_\pm(x) = \frac{1}{\sqrt{2\pi\alpha}}U_\pm e^{\pm 2ik_F x}\prod_{p>0}\exp\left\{\pm\left(\frac{2\pi}{pL}\right)e^{-\alpha p/2}\left[e^{ipx}\rho_\pm(-p) - e^{-ipx}\rho_\pm(p)\right]\right\}. \tag{4.15}$$

So the study of the matrix elements of $\psi_\pm$ between different Luttinger liquid states leads naturally to the study of harmonic oscillator matrix elements of the form $\langle m|e^{\lambda A^\dagger - \lambda^* A}|n\rangle$, where $A^\dagger$ and A are raising and lowering operators. In our case, since $\sqrt{2\pi/(pL)}\rho_\pm$ is a canonically normalized ladder operator,

$$\lambda = \mp\sqrt{\frac{2\pi}{pL}}e^{\mp ipx}e^{-\alpha p/2}. \tag{4.16}$$

It is important to note that λ is $\mathcal{O}(L^{-1/2})$.

In terms of $l \equiv m - n$, the general formula for the matrix element in question is

$$\begin{aligned}\langle m|e^{\lambda A^\dagger - \lambda^* A}|n\rangle &= \frac{1}{|l|!}\left(\frac{m!}{n!}\right)^{\frac{1}{2}\mathrm{sgn}(l)} \lambda^{(|l|+l)/2}(-\lambda^*)^{(|l|-l)/2} \\ &\quad \times \left\{e^{|\lambda|^2/2} F\left[\max(m,n)+1; |l|+1; -|\lambda|^2\right]\right\}, \qquad (4.17)\end{aligned}$$

where $F(a;b;z)$ is a confluent hypergeometric function. This formula is derived in Appendix B.

The formulas above pertain to both ψ_+ and ψ_-, but we shall henceforth restrict our attention to ψ_+ only. We shall now proceed to calculate the energy shift for the case in which the temperature is effectively zero when compared to the transverse energy scale of the trap. The fact that the temperature vanishes allows us to make some substantially simplifications to the mathematics. We shall show explicitly that these simplifications are not valid (and would give rise to unphysical results) when the "transverse temperature" is finite.

It may initially appear that the bracketed factors in (4.17) may be neglected, since both $e^{|\lambda|^2/2}$ and $F[\max(m,n)+1; |l|+1; -|\lambda|^2]$ are power series in $|\lambda|^2$, and $|\lambda|^2$ is $\mathcal{O}(L^{-1})$. Each power series is dominated by its first term in the large L limit, and for both series, that term is unity. However, these terms may not actually be neglected in the $L \to \infty$ limit, because there are an infinite number of such terms, corresponding to all the allowed momenta. Since the density of states in momentum space is proportional to L, the infinite product does not approach unity.

Let us denote the infinite product in question by $\mathcal{P}$; that is, $\mathcal{P}$ is defined to be

$$\mathcal{P} = \prod_{p>0} \exp\left[\left(\frac{\pi}{pL}\right)e^{-\alpha p}\right] F\left[\max\left(m_{+p}, n_{+p}\right)+1; |m_{+p}-n_{+p}|+1; -\left(\frac{2\pi}{pL}\right)e^{-\alpha p}\right]. \qquad (4.18)$$

When we calculate the energy shift, we sum up terms consisting of a matrix element squared divided by an energy difference. If the characteristic energy differences are $\mathcal{O}(L)$, then the sum of the matrix elements squared must also be $\mathcal{O}(L)$ if we are to obtain a nonvanishing contribution. This is only possible if the occupation numbers of a macroscopic number of states states are $\mathcal{O}(L)$. This is a natural situation at

finite temperature; however, at $T = 0$, the occupation numbers must be small, so that the excitation energy of the system is not extensive.

So at $T = 0$ we only get a nonvanishing contribution when the difference in energies between $|\{m_{+p}\}\rangle$ and $|\{n_{+p}\}\rangle$ grows more slowly than L. That is, almost all of the m_{+p} must be equal to the n_{+p}. We may then separate out those values of p at which m_{+p} and n_{+p} differ and get

$$\begin{aligned}\mathcal{P} &= \prod_{p,l_{+p}\neq 0}\left\{\exp\left[\left(\frac{\pi}{pL}\right)e^{-\alpha p}\right]F\left[\max\left(m_{+p},n_{+p}\right)+1;|l_{+p}|+1;-\left(\frac{2\pi}{pL}\right)e^{-\alpha p}\right]\right\} \\ &\times\prod_{p,l_{+p}=0}\left\{\exp\left[\left(\frac{\pi}{pL}\right)e^{-\alpha p}\right]F\left[n_{+p}+1;1;-\left(\frac{2\pi}{pL}\right)e^{-\alpha p}\right]\right\}. \qquad (4.19)\end{aligned}$$

The number of terms in the first product must grow more slowly than L. Since each of these terms has the form $1+\mathcal{O}(L^{-1})$ as $L\to\infty$, the entire product is one in that limit.

The second product in (4.19) is more complicated. Using the Kummer transformation formula $F(a;b;z) = e^z F(b-a;b;-z)$ and the fact that $F(-n;1;z)$ is just the Laguerre polynomial $L_n(z)$ [1], we may rewrite $\mathcal{P}$ as

$$\mathcal{P} = \prod_{p,l_{+p}=0}\left\{\exp\left[-\left(\frac{\pi}{pL}\right)e^{-\alpha p}\right]L_{n_{+p}}\left[\left(\frac{2\pi}{pL}\right)e^{-\alpha p}\right]\right\} \qquad (4.20)$$

We may evaluate this explicitly at $T = 0$, where $n_{+p} = 0$ for almost all p. The product of the Laguerre polynomials is one, and all that remain are the exponentials. Each momentum p represented in the product has the form $p(r) = 2\pi r/L$, for some positive integer r, so we have

$$\mathcal{P}(T=0) = \exp\left[-\sum_{r,l_{+p(r)}=0}\frac{e^{-2\pi\alpha r/L}}{2r}\right]. \qquad (4.21)$$

Since $l_{+p} = 0$ for almost all p, we may extend the sum to include all positive integers r. Then the sum is just the Taylor series for $\frac{1}{2}\log\left(1-e^{-2\pi\alpha/L}\right)$, so, to leading order

in L^{-1}, we have

$$\mathcal{P}(T=0) = \sqrt{\frac{2\pi\alpha}{L}}. \tag{4.22}$$

The factor of $\sqrt{\frac{\alpha}{L}}$ will be necessary in order to make our final result finite.

If we were to assume that the only terms which contributed had $m_{+p} = n_{+p}$ for almost all p even when $T > 0$, we would obtain the expression

$$\mathcal{P} = \sqrt{\frac{2\pi\alpha}{L}} \left\{ \prod_{p, l_{+p}=0} L_{n_{+p}} \left[\left(\frac{2\pi}{pL} \right) e^{-\alpha p} \right] \right\}. \tag{4.23}$$

We shall use this expression to demonstrate explicitly that our assumption is inconsistent unless T vanishes.

We may now turn our attention to remaining factors in the matrix elements—those arising from the unbracketed terms in (4.17). We shall divide the matrix elements of this operator into groups. Each matrix element $\langle \{m_{+p}\} | \psi_+ | \{n_{+p}\} \rangle$ is labeled by an integer $j \geq 0$, where j is the number of quanta by which $\{m_{+p}\}$ and $\{n_{+p}\}$ differ. That is,

$$j = \sum_{p>0} |m_{+p} - n_{+p}|. \tag{4.24}$$

According to our earlier argument, at $T = 0$, j grows must grow more slowly than L, and we shall assume that this is also the case for finite T (although we shall eventually show that this assumption is inconsistent).

Each of the j quanta by which $\{m_{+p}\}$ and $\{n_{+p}\}$ differ corresponds to a momentum $p_i > 0$, $i \in 1, 2, \ldots, j$. Since j is smaller than $\mathcal{O}(L)$, in the continuum limit all the p_i are distinct, except on a set of measure zero. So we shall only need to use (4.17) in the case where $|l| \leq 1$. Keeping this in mind, we may write the matrix elements of ψ_+ as

$$\langle \{m_{+p}\} | \psi_+(x) | \{n_{+p}\} \rangle = \frac{\mathcal{P}}{\sqrt{2\pi\alpha}} e^{2ik_F x} \prod_{i=1}^{j} \begin{cases} -\sqrt{\frac{2\pi}{p_i L}} e^{-ip_i x} e^{-\alpha p_i/2} \sqrt{n_{+p_i} + 1} \\ \sqrt{\frac{2\pi}{p_i L}} e^{ip_i x} e^{-\alpha p_i/2} \sqrt{n_{+p_i}} \end{cases}. \tag{4.25}$$

The upper term applies if $m_{+p_i} > n_{+p_i}$ (i.e. if $m_{+p_i} = n_{+p_i} + 1$), and the lower term otherwise. We note that this matrix element is $\mathcal{O}(L^{-j/2})$. Such a matrix element

can only give rise to a nonvanishing energy shift if it is multiplied by an appropriate positive power of L.

We may find the corresponding matrix element of a_k by taking the Fourier transform (4.9). We have

$$\begin{aligned}\langle\{m_{+p}\}|a_k|\{n_{+p}\}\rangle &= \frac{\mathcal{P}}{\sqrt{2\pi\alpha}}\left(\frac{2\pi}{L}\right)^{j/2}\frac{1}{\sqrt{L}}\int_0^L dx\left(\prod_{i=1}^{j}\frac{1}{\sqrt{p_i}}\begin{cases}-\sqrt{n_{+p_i}+1}\\ \sqrt{n_{+p_i}}\end{cases}\right)\\ &\times \exp\left(-ikx+2ik_Fx+\sum_{i=1}^{j}\mp ip_ix\right)\exp\left(-\frac{1}{2}\alpha\sum_{i=1}^{j}p_i\right). \quad (4.26)\end{aligned}$$

For each i, we have two possibilities, depending upon the relative magnitudes of m_{+p_i} and n_{+p_i}. In each case, the upper (lower) sign in the exponential corresponds to the upper (lower) square root. The integral over x just gives $L\delta_{k,2k_F+\sum_{i=1}^{j}\mp p_i}$. The matrix element squared is then

$$\begin{aligned}|\langle\{m_{+p}\}|a_k|\{n_{+p}\}\rangle|^2 &= \frac{\mathcal{P}^2}{2\pi\alpha}\left(\frac{2\pi}{L}\right)^{j}L\delta_{k,2k_F+\sum_{i=1}^{j}\mp p_i} \quad (4.27)\\ &\times\exp\left(-\alpha\sum_{i=1}^{j}p_i\right)\prod_{i=1}^{j}\frac{1}{p_i}\begin{cases}n_{+p_i}+1\\ n_{+p_i}\end{cases}.\end{aligned}$$

To find the perturbative energy shift due to terms of this form, we must divide the matrix element squared by the energy defect and integrate over all possible momenta. The phase-space factors arising from the integrations will cancel out the j factors of $2\pi/L$ in (4.27). The energy defect is given by

$$E_{\{n'_{+p}\};k\mp q}-E_{\{n_{+p}\}}=\Delta+\frac{1}{2m}\left[(k\mp q)^2-k_F^2\right]\pm\omega_q+\sum_{i=1}^{j}\pm\omega_{p_i}. \quad (4.28)$$

$\Delta+\frac{1}{2m}(k\mp q)^2$ is the energy of the excited fermion; $\frac{1}{2m}k_F^2$ is the energy lost by the Luttinger liquid ground state with the removal of one fermion; and $\pm\omega_q+\sum_{i=1}^{j}\pm\omega_{p_i}$ is the energy change due to the phonons gained or lost in the intermediate state. There are 2^j possible choices of signs in the sum over the quanta created or annihilated by a_k.

The momentum range over which we must integrate extends from $e^{-\gamma}p_{\min} \equiv e^{-\gamma}\frac{2\pi}{L}$ to $+\infty$. The Euler-Mascheroni constant γ enters when we change from a sum to an integral, through the formula $\lim_{m\to+\infty}\left(\sum_{r=1}^{m}\frac{1}{r} - \int_1^m \frac{dr}{r}\right) = \gamma$. We may then absorb this γ-dependence into the limits of integration. When we integrate all the p_i over this range, we must also divide by $j!$. This factor arises from our overcounting of intermediate states—permuting the p_i does not give rise to a new state.

So the energy shift consists of terms of the form

$$\begin{aligned}\Delta E_j &= -\frac{\mathcal{P}^2}{2\pi\alpha}\frac{q}{2\pi L}|g(q)|^2\left(\frac{2\pi}{L}\right)^j \sum_k \sum_{s=\pm1}\sum_{t=1}^{2^j}\frac{1}{j!}\left(\frac{L}{2\pi}\right)^j \\ &\times \int_{\frac{p_{\min}}{e^\gamma}}^{+\infty}\left(\prod_{i=1}^{j} dp_i \frac{e^{-\alpha p_i}}{p_i}\left\{\begin{array}{c} n_{+p_i}+1 \\ n_{+p_i}\end{array}\right\}\right) \\ &\times \frac{L\delta_{k,2k_F+\sum_{i=1}^{j}\mp p_i}\left(n_{+q}+\frac{s+1}{2}\right)}{\Delta+\frac{1}{2m}\left[(k-sq)^2-k_F^2\right]+s\omega_q+\sum_{i=1}^{j}\pm\omega_{p_i}}\end{aligned} \tag{4.29}$$

The sum over t runs over the 2^j choices of sign (and of $n_{+p_i}+1$ or n_{+p_i}). The integrand in (4.29) behaves as p_i^{-1} for small values of p_i and as p_i^{-3} for large values of p_i, so the integral is dominated by the region where all the p_i are small.

Near $p_i = p_{\min}$, p_i and ω_{p_i} are each $\mathcal{O}(L^{-1})$, so we may neglect $\sum_{i=1}^{j}\mp p_i$ and $\sum_{i=1}^{j}\pm\omega_{p_i}$ unless j is $\mathcal{O}(L)$ or greater. Since we have assumed that j is smaller than $\mathcal{O}(L)$, we may drop $\sum_{i=1}^{j}\mp p_i$ and $\sum_{i=1}^{j}\pm\omega_{p_i}$. We shall not drop the $\exp\left(-\alpha\sum_{i=1}^{j}p_i\right)$ term; this factor is needed to provide a cutoff for the momentum integral.

By dropping the two sums, we simplify the integral substantially. The expression becomes

$$\Delta E_j = -\frac{\mathcal{P}^2}{4\pi^2\alpha}\sum_{s=\pm1}\sum_{t=1}^{2^j}\frac{|g(q)|^2 q\left(n_{+q}+\frac{s+1}{2}\right)\frac{1}{j!}\prod_{i=1}^{j}\int_{p_{\min}/e^\gamma}^{+\infty}dp_i\frac{e^{-\alpha p_i}}{p_i}\left\{\begin{array}{c} n_{+p_i}+1 \\ n_{+p_i}\end{array}\right\}}{\Delta+\frac{1}{2m}\left[(2k_F-sq)^2-k_F^2\right]+s\omega_q}. \tag{4.30}$$

We may now perform the sum over t, to get

$$\Delta E_j = -\frac{\mathcal{P}^2}{4\pi^2\alpha}\sum_{s=\pm 1}\frac{|g(q)|^2 q\left(n_{+q}+\frac{s+1}{2}\right)\frac{1}{j!}\left[\int_{p_{\min}/e^\gamma}^{+\infty} dp_i\,\frac{e^{-\alpha p_i}}{p_i}\,(2n_{+p_i}+1)\right]^j}{\Delta+\frac{1}{2m}\left[(2k_F-sq)^2-k_F^2\right]+s\omega_q}. \quad (4.31)$$

Furthermore, we may easily sum up all the terms with different values of j, getting the exponential function

$$\Delta E_+ = -\frac{\mathcal{P}^2}{4\pi^2\alpha}\sum_{s=\pm 1}\frac{|g(q)|^2 q\left(n_{+q}+\frac{s+1}{2}\right)\exp\left[\int_{p_{\min}/e^\gamma}^{+\infty} dp_i\,\frac{e^{-\alpha p_i}}{p_i}\,(2n_{+p_i}+1)\right]}{\Delta+\frac{1}{2m}\left[(2k_F-sq)^2-k_F^2\right]+s\omega_q}. \quad (4.32)$$

(The subscript $+$ in ΔE_+ indicates that this energy shift only includes the terms arising from ψ_+—i.e. the terms in which c_k is a_k.) It now remains to evaluate the integral and add the contributions from the terms for which c_k is b_k.

It is impossible to evaluate the integral for a general set of n_{+p_i}. However, we may simplify our expression for the energy shift substantially by separating the two terms in the integral and transforming the integral back into an infinite sum. We have

$$\exp\left[\int_{\frac{p_{\min}}{e^\gamma}}^{+\infty} dp_i\,\frac{e^{-\alpha p_i}}{p_i}\,(2n_{+p_i}+1)\right] = \exp\left[2\sum_{p_i>0}\left(\frac{2\pi}{p_i L}\right)e^{-\alpha p_i}n_{+p_i}\right] \quad (4.33)$$

$$\times\exp\left[\sum_{p_i>0}\left(\frac{2\pi}{p_i L}\right)e^{-\alpha p_i}\right]. \quad (4.34)$$

The second factor on the right-hand side of (4.33) contains the same integral that we encountered when we evaluated $\mathcal{P}(T=0)$; in fact, this term is exactly $[\mathcal{P}(T=0)]^{-2}$. So we define a new infinite product $\mathcal{D}$ as

$$\begin{aligned}\mathcal{D} &= \mathcal{P}^2\exp\left[\int_{\frac{p_{\min}}{e^\gamma}}^{+\infty} dp_i\,\frac{e^{-\alpha p_i}}{p_i}\,(2n_{+p_i}+1)\right]\\ &= \prod_{p_i>0}\left\{\exp\left[\left(\frac{2\pi}{p_i L}\right)e^{-\alpha p_i}n_{+p_i}\right]L_{n_{+p_i}}\left[\left(\frac{2\pi}{p_i L}\right)e^{-\alpha p_i}\right]\right\}^2. \quad (4.35)\end{aligned}$$

$\mathcal{D}$ is an overall factor multiplying our expression for ΔE_+. It is clear that $\mathcal{D}=1$ when $T=0$ (since almost all the n_{+p} are then zero). For $T>0$, $\mathcal{D}$ contains all the energy shift's temperature dependence, so it is natural to calculate its thermal

average. We shall find that this average diverges for all finite temperatures; this unphysical result signals that we can not assume that j is smaller than $\mathcal{O}(L)$ unless the temperature is strictly vanishing.

Denoting the inverse temperature by β, we find that the thermal average of $\mathcal{D}$ is

$$\langle \mathcal{D} \rangle = \sum_{\{n_{+p}\}} \left\{ \exp\left[-\beta v_F \sum_{p>0} (p n_{+p}) \right] \mathcal{D} \right\}. \tag{4.36}$$

The sum over the $\{n_{+p}\}$ runs over all possible sets of occupation numbers. This sum may be factorized into an infinite product of sums indexed by p. This allows us to calculate the thermal average separately for each value of the momentum, so $\langle \mathcal{D} \rangle$ becomes

$$\langle \mathcal{D} \rangle = \prod_{p>0} \sum_{n_{+p}=0}^{\infty} \left(\exp\left[-\beta v_F p n_{+p} \right] \left\{ \exp\left[\left(\frac{2\pi}{pL} \right) e^{-\alpha p} n_{+p} \right] L_{n_{+p}} \left[\left(\frac{2\pi}{pL} \right) e^{-\alpha p} \right] \right\}^2 \right). \tag{4.37}$$

The sum is now in the form of a generating function for the $[L_n(x)]^2$. That is,

$$\langle \mathcal{D} \rangle = \prod_{p>0} \sum_{n=0}^{\infty} [L_n(x)]^2 z^n, \tag{4.38}$$

with $z = \exp\left[\left(\frac{2\pi}{pL} \right) e^{-\alpha p} - \beta v_F p \right]$ and $x = \left(\frac{2\pi}{pL} \right) e^{-\alpha p}$.

The sum of this generating function is known [18]. In fact, when $|z| < 1$,

$$\sum_{n=0}^{\infty} [L_n(x)]^2 z^n = \frac{1}{1-z} \exp\left(-\frac{2xz}{1-z} \right) I_0\left(2x \frac{\sqrt{z}}{1-z} \right), \tag{4.39}$$

where I_0 is the modified Bessel function $I_0(u) = J_0(iu)$. The series diverges if $|z| > 1$. So $\langle \mathcal{D} \rangle$ converges only if $\exp\left[\left(\frac{2\pi}{pL} \right) e^{-\alpha p} - \beta v_F p \right] < 1$ for all p. Since $\left(\frac{2\pi}{pL} \right) e^{-\alpha p} - \beta v_F p$ is a strictly decreasing function of p, we need only consider $p = p_{\min}$. At this value of p, we may neglect $e^{-\alpha p}$, so the condition for convergence becomes $\beta > (v_F p_{\min})^{-1}$. As $L \to \infty$, this can not be satisfied at any finite temperature. So the thermal average of $\mathcal{D}$ is equal to unity at $T = 0$ (when $z = 0$ for all p) and diverges for $T > 0$. {For finite L and very large v_F, the condition that $\beta > (v_F p_{\min})^{-1}$ gives us a qunatitative

estimate of how small the termperature must be in order for our approximation to be valid. Current experiments with degenerate fermions [21, 13] typically achieve temperatures for which $\beta^{-1} \sim \epsilon_F$. In order for the $T = 0$ approximation to be valid, the temperature must be smaller by a factor of $\mathcal{O}(N)$, so this regime is far beyond the limits of current technique.}

The divergence of $\langle \mathcal{D} \rangle$ at nonzero T indicates the breakdown of our approximation. We may see this more explicitly by looking at the argument of the exponential in (4.32). The Taylor series $\sum_{j=0}^{\infty} \frac{x^j}{j!}$ for e^x is dominated by the terms for which j is $\mathcal{O}(x)$. At $T = 0$, the argument of the exponential appearing in (4.32) is $\int_{\frac{p_{\min}}{e^\gamma}}^{+\infty} dp_i \frac{e^{-\alpha p_i}}{p_i} = \log\left(\frac{L}{2\pi\alpha}\right)$, which is $\mathcal{O}(\log L)$. Therefore, the dominant contributions to the exponent come from values of j that are likewise $\mathcal{O}(\log L)$. However, when $T > 0$, the average number of quanta present in each of the lowest-lying modes is $\mathcal{O}(L)$ unless $\beta > (v_F p_{\min})^{-1}$. The argument of the exponential and the dominant values of j are then $\mathcal{O}(L \log L)$. This contradicts our assumption that j is smaller than $\mathcal{O}(L)$.

We have shown that when the temperature is nonzero, we must receive a substantial contribution from terms for which j is $\mathcal{O}(L)$ or larger, so the methods we have used so far do not apply to this case. However, the $T = 0$ treatment does provide us with several insights that will prove useful when we analyze the $T > 0$ case. We shall return to that case later; however, there are several more points about the zero-temperature situation that we must consider first.

Thus far, we have assumed that α^{-1} is the correct cutoff for the momentum integral appearing in (4.32). However, there is also another cutoff we must consider. In order to obtain a physically meaningful result, we must determine which cutoff is the relevant one.

A real Luttinger liquid has a finite bandwidth $v_F \alpha^{-1}$, because the phonon description breaks down at large values of p_i. The Luttinger model only describes the low-energy excitations of the fermion system. When p_i becomes comparable to k_F, the energy $\frac{(k_F + p_i)^2}{2m} - \frac{k_F^2}{2m}$ of an excitation can no longer be approximated by $v_F p_i$ and is no longer much smaller than ϵ_F, so we expect the physical cutoff α^{-1} to be $\mathcal{O}(k_F)$.

The second cutoff arises from the energy denominator (4.28). When p_i is large

compared to $\sqrt{m\Delta}$, the integrand decays as p_i^{-3}. If there existed an intermediate range of p_i values, where $m\Delta/k_F \ll p_i \ll \sqrt{m\Delta}$, the integrand would decay as p_i^{-2} in this region. However, since we must have $\Delta > \frac{1}{2m}k_F^2$ in order to have an energy gap at all, such a region can not exist. So the rapid p_i^{-3} decay at large values of p_i will provide a cutoff for the integral which we expect to be $\mathcal{O}(\sqrt{m\Delta})$.

We must determine which of these two cutoff scales is smaller, since the smaller one determines the behavior of the energy shift. When $\Delta \gg \epsilon_F$, the first cutoff, near the Fermi momentum, is smaller. However, when the energy gap is comparable to the Fermi energy, $\sqrt{m\Delta}$ is itself $\mathcal{O}(k_F)$. Without more precise values for the cutoffs, it is not clear which one matters in this situation.

We shall now estimate the cutoff $p'_{\max}$ arising from the p_i^{-3} behavior of (4.28). To do this, we shall again make use of the fact that the integral over the p_i is dominated by the region where the p_i are small. The value of $p'_{\max}$ is determined by the behavior of the integrand when not all the p_i are small; at least one of the p_i must be comparable to $\sqrt{m\Delta}$. However, the dominant contribution under such conditions comes from the region where only a single p_i is large compared to $p_{\min}$. If more than one p_i is large, the contribution is much smaller. This means that we may determine $p'_{\max}$ by analyzing the case in which all but one of the p_i are negligible.

More precisely, we approximate $\delta_{k,2k_F+\sum_{i=1}^{j}\mp p_i}$ by $\delta_{k,2k_F-p_i}$ and $\sum_{i=1}^{j}\pm\omega_{p_i}$ by ω_{p_i}. The choice of signs corresponds to the fact that the virtual phonon at momentum p_i must be created, not annihilated, since almost all the phonon modes are empty of quanta when $T = 0$. One of the p_i integrations is now nontrivial. We shall perform this integration explicitly and extract the cutoff from it. Our prescription for doing this is that

$$\int_{\frac{p_{\min}}{e^\gamma}}^{+\infty} \frac{dp_i}{p_i} \frac{1}{\Delta + \frac{1}{2m}[(2k_F - p_i)^2 - k_F^2] + \omega_{p_i}} \approx \frac{1}{\Delta + 3\epsilon_F} \log\left(\frac{p'_{\max}}{p_{\min}}\right). \tag{4.40}$$

The integral in (4.40) is elementary but complicated. The calculation is worked

out in Appendix C. The result (C.5) is

$$p'_{\max} \approx e^{\gamma}\sqrt{2m(\Delta+3\epsilon_F)}\exp\left[\sqrt{\frac{\epsilon_F}{\Delta+2\epsilon_F}}\left(\tan^{-1}\sqrt{\frac{\epsilon_F}{\Delta+2\epsilon_F}}+\frac{\pi}{2}\right)\right]. \tag{4.41}$$

Although this estimate is rather inexact, it shows the general character of the cutoff. $p'_{\max}$ has its minimum value at $\Delta \approx 4.79\epsilon_F$, where $p'_{\max} \approx 10.45k_F$. However, we have not yet determined whether this is ever the relevant cutoff.

We shall defer the final resolution of this question until Section 4.4. In the meantime, we shall make the replacement

$$\int_{\frac{p_{\min}}{e^{\gamma}}}^{+\infty} dp_i \frac{e^{-\alpha p_i}}{p_i} \to \log\left(\frac{Lp_{\max}}{2\pi}\right), \tag{4.42}$$

where in all cases, $p_{\max}$ is $\mathcal{O}(k_F)$. Using the prescription (4.42) to evaluate the integral and inserting our expression for $\mathcal{P}(T=0)$, we get

$$\Delta E_+ \approx -\frac{1}{4\pi^2}\sum_{s=\pm 1}\frac{|g(q)|^2 q p_{\max}\left(n_{+q}+\frac{s+1}{2}\right)}{\Delta+\frac{1}{2m}\left[(2k_F-sq)^2-k_F^2\right]+s\omega_q}. \tag{4.43}$$

Since $q \ll k_F$, we may neglect all the q-dependent terms in the denominator of (4.43). We may then trivially perform the sum over s. Thus far, we have restricted our attention to the case for which c_k corresponds to a right-moving fermion. The terms in which c_k corresponds to a left-moving particle also contribute. When the q-dependent terms in the denominator are neglected, the two branches contribute equally, so we have

$$\Delta E \approx -\frac{|g(q)|^2 q p_{\max}}{\pi^2(\Delta+3\epsilon_F)}\left(n_{+q}+\frac{1}{2}\right). \tag{4.44}$$

(4.44) is the energy shift for a single bosonic mode, with momentum $+q$. To obtain the total shift for the system, we must add in the similar renormalization of the left-moving mode with momentum $-q$ and integrate over q. Since the density of momentum states is proportional to L, the total shift is extensive, as it must be.

We shall now turn our attention to the $T > 0$ case. The crucial simplification that allowed us to evaluate the $T = 0$ energy shift in closed form was the fact that the

energy defects were effectively independent of the intermediate states. A remarkable cancellation then occurred between the prefactor $\mathcal{P}$ and the exponential that arose when we summed over all all intermediate states. A similar cancellation will occur here.

When $T > 0$, we may not necessarily neglect $\sum_{i=1}^{j} \mp p_i$ and $\sum_{i=1}^{j} \pm\omega_{p_i}$, because j is not generally small. The presence of these two sums gives rise to a nontrivial energy denominator. However, if Δ is formally very large, so that any other energy may be neglected compared to it, the energy defect again becomes independent of the p_i. As in the previous case, the temperature is negligible when compared with the transverse energy scale; that is, the "transverse temperature" is effectively vanishing. With this simplification, it will be possible to obtain an analytical expression for the energy shift.

If we again assemble the various terms that compose the energy shift (now assuming that Δ is very large), we have

$$\Delta E_+ = -\sum_{s=\pm 1} \frac{1}{2\pi\alpha} \frac{|g(q)|^2 q \left(n_{+q} + \frac{s+1}{2}\right)}{2\pi} \sum_{\{m_{+p}\}} \frac{\left|\langle\{m_{+p}\}| \prod_{p>0} e^{\lambda_{+p} A_{+p}^\dagger - \lambda_{+p}^* A_{+p}} |\{n_{+p}\}\rangle\right|^2}{\Delta}, \tag{4.45}$$

where $\lambda_{+p} = -\sqrt{2\pi/(pL)} e^{-\alpha p/2}$, and $A_{+p}^\dagger$ and A_{+p} are the canonical ladder operators corresponding to the $\rho_+(\pm p)$ quanta.

Because the energy defect in (4.45) is independent of the $\{m_{+p}\}$, we may rewrite the energy shift as

$$\begin{aligned} \Delta E_+ \;=\; & -\sum_{s=\pm 1} \frac{|g(q)|^2 q \left(n_{+q} + \frac{s+1}{2}\right)}{4\pi^2 \alpha \Delta} \langle\{n_{+p}\}| \left\{ \prod_{p>0} \left(e^{\lambda_{+p} A_{+p}^\dagger - \lambda_{+p}^* A_{+p}} \right)^\dagger \right. \\ & \left. \times \left[\sum_{\{m_{+p}\}} |\{m_{+p}\}\rangle\langle\{m_{+p}\}| \right] \prod_{p>0} \left(e^{\lambda_{+p} A_{+p}^\dagger - \lambda_{+p}^* A_{+p}} \right) \right\} |\{n_{+p}\}\rangle. \end{aligned} \tag{4.46}$$

The operator in brackets is just the identity, so we may drop it. Furthermore, since

$\lambda_{+p}A^\dagger_{+p} - \lambda^*_{+p}A_{+p}$ is antihermitian,

$$\prod_{p>0}\left(e^{\lambda_{+p}A^\dagger_{+p}-\lambda^*_{+p}A_{+p}}\right)^\dagger = \left[\prod_{p>0}\left(e^{\lambda_{+p}A^\dagger_{+p}-\lambda^*_{+p}A_{+p}}\right)\right]^{-1}, \tag{4.47}$$

and the entire matrix element expression is unity.

This shows that, when Δ is very large, we recover a similar expression to the one we obtained in a different extreme limit, that of zero temperature. The total energy shift is

$$\Delta E = -\frac{|g(q)|^2 q}{\pi^2 \alpha \Delta}\left(n_{+q} + \frac{1}{2}\right). \tag{4.48}$$

This expression is actually identical to the large Δ limit of (4.44). In that limit, the Fermi energy may be neglected compared to Δ, and the correct cutoff $p_{\max}$ is just α^{-1}, because $p'_{\max}$ grows as $\mathcal{O}(\sqrt{m\Delta})$. However, we still do not know to what actual value α corresponds. We shall now turn our attention to a different model, which will allow us to answer that question.

4.4 Energy shift for a more physical model

In this section, we present an alternative model—one which is useful when $T = 0$ and Δ is very large. Using this model, we shall again calculate the energy shift. From our calculation we may extract the physically relevant value of α^{-1}, the cutoff arising from the breakdown of the phonon description. Moreover, we shall find in Section 4.5 that our calculation requires only slight modification in the presence of strong interactions between the low-lying fermion states.

The Luttinger model does not accurately represent the fermions lying deep within the Fermi sea, so it is not necessary to use a bosonized form for the matrix elements of c_k when $k \not\approx k_F$. Since a basic assumption of the Luttinger liquid theory is that the presence or absence of additional fermions far from the Fermi surface should have minimal effect on the low-energy excitations, we shall simply assume that the action of c_k ($k \not\approx k_F$) on a Luttinger liquid state is to create a hole in the Fermi sea, without

affecting the phonon configuration in any way. The only exception to this prescription is that when the fermion operators are combined in the form (4.3), as they are in H_2, we shall used the bosonized expressions.

Our approximation will require that $\Delta \gg \epsilon_F$. This condition ensures that the energy shift is not dominated by the terms with the smallest energy defects; instead, the defects are dominated by Δ and depend only weakly on the energy of the fermion annihilated by c_k. So the entire Fermi sea will contribute to the energy shift, and the region near the Fermi surface (where our approximation for c_k is invalid) will make a comparatively small contribution. The condition that $T = 0$ ensures that the occupied states are precisely those with $|k| < k_F$.

Now we must identify the states between which we shall be calculating matrix elements. As before, we shall have the state $|\{n_{+p}\}\rangle$ on the right. We shall find the energy shift of this state due to interactions with the states $|\{n_{+p}\}; k \mp q; k\rangle$. These states have bosonic occupation numbers $\{n'_{+p}\}$ which are the same as $\{n_{+p}\}$, except that $n'_{+q} = n_{+q} \pm 1$. These states also have an excited fermion with momentum $k \mp q$ and a hole in the Fermi sea with momentum $-k$.

The matrix elements we shall need are then

$$\langle\{n'_{+p}\}; k - sq; k|\rho_+(sq)d^\dagger_{k-sq}c_k|\{n_{+p}\}\rangle = \sqrt{\frac{qL}{2\pi}\left(n_{+q} + \frac{s+1}{2}\right)}\theta(k_F - |k|), \quad (4.49)$$

for $s \pm 1$. The step function ensures that the matrix elements vanish when $|k| > k_F$. The energy defects are

$$E_{\{n'_{+p}\};k-sq;k} - E_{\{n_{+p}\}} = \Delta + \frac{1}{2m}\left[(k - sq)^2 - k^2\right] + s\omega_q, \quad (4.50)$$

so the energy shift is

$$\begin{aligned}\Delta E &= -\frac{1}{L^2}|g(q)|^2 \sum_{k=-k_F}^{k_F}\sum_{s=\pm1} \frac{|\langle\{n'_{+p}\}; k - sq; k|\rho_+(sq)d^\dagger_{k-sq}c_k|\{n_{+p}\}\rangle|^2}{\Delta + \frac{1}{2m}\left[(k - sq)^2 - k^2\right] + s\omega_q} \\ &= -\frac{1}{L^2}|g(q)|^2\left(\frac{qL}{2\pi}\right)\sum_{k=-k_F}^{k_F}\sum_{s=\pm1} \frac{n_{+q} + \frac{s+1}{2}}{\Delta + \frac{1}{2m}\left[(k - sq)^2 - k^2\right] + s\omega_q}. \quad (4.51)\end{aligned}$$

Converting the sum over k into an integral, we have

$$\begin{aligned}
\Delta E &= -\frac{1}{L^2}|g(q)|^2\left(\frac{qL}{2\pi}\right)\sum_{s=\pm 1}\left(n_{+q}+\frac{s+1}{2}\right)\left(\frac{L}{2\pi}\right)\int_{-k_F}^{k_F}dk\,\frac{1}{\left(\Delta+\frac{q^2}{2m}+s\omega_q\right)-s\frac{q}{m}k} \\
&= -\frac{|g(q)|^2 q}{4\pi^2}\sum_{s=\pm 1}\left(n_{+q}+\frac{s+1}{2}\right)\left(\frac{sm}{q}\right)\log\left[\frac{\left(\Delta+\frac{q^2}{2m}+s\omega_q\right)+s\frac{q}{m}k_F}{\left(\Delta+\frac{q^2}{2m}+s\omega_q\right)-s\frac{q}{m}k_F}\right] \\
&= -\frac{|g(q)|^2 m}{4\pi^2}\sum_{s=\pm 1}\left(n_{+q}+\frac{s+1}{2}\right)\log\left[1+\frac{2qk_F/m}{\left(\Delta+\frac{q^2}{2m}+s\omega_q\right)-\frac{q}{m}k_F}\right]. \qquad (4.52)
\end{aligned}$$

We have already required that Δ be large compared with the Fermi energy. Since $q \ll k_F$, we may ignore all other energies compared to Δ and approximate the logarithm in (4.52) by its leading term. This gives us

$$\Delta E = -\frac{|g(q)|^2 q k_F}{\pi^2 \Delta}\left(n_{+q}+\frac{1}{2}\right). \qquad (4.53)$$

As in Section 4.3, we must sum over all values of q to get an extensive energy shift.

We now compare the result (4.53) with (4.44). The models leading to these expressions are both valid when $T = 0$ and the energy gap is large, so the two expressions must agree under these circumstances. In fact, (4.44) and (4.53) agree in their region of common validity exactly if $p_{\max} = k_F$. Since k_F is substantially smaller than any possible value of $p'_{\max}$, we must conclude that the cutoff α^{-1} is equal the Fermi momentum when $T = 0$. We then see that $\alpha^{-1} \ll p'_{\max}$ for all values of Δ, so the general expression for the $T = 0$ energy shift is

$$\Delta E = -\frac{|g(q)|^2 q k_F}{\pi^2(\Delta + 3\epsilon_F)}\left(n_{+q}+\frac{1}{2}\right). \qquad (4.54)$$

This expression should be valid for all values of $\Delta > \epsilon_F$. We shall discuss the relationship between α and the temperature in Section 4.5.

4.5 Extensions and discussion

The model outlined in Section 4.4 may be applied to systems with more general interactions. In particular, our techniques may be applied to the interacting Tomonaga-Luttinger model. The Tomonaga-Luttinger Hamiltonian $H_0 + H_1$ is diagonalized by the Bogoliubov-transformed operators [7]

$$\rho'_{\pm}(q) = \frac{1}{2}\left(X + X^{-1}\right)\rho_{\pm}(q) + \frac{1}{2}\left(X - X^{-1}\right)\rho_{\mp}(q), \tag{4.55}$$

where

$$X = \left(\frac{v_F + \frac{g_4(q)}{2\pi} + \frac{g_2(q)}{2\pi}}{v_F + \frac{g_4(q)}{2\pi} - \frac{g_2(q)}{2\pi}}\right)^{1/4}. \tag{4.56}$$

The $\rho'_{\pm}$ satisfy the same commutation relations as the $\rho_{\pm}$. To determine the effects of the additional interaction H_2, we only need relate $\rho_+(q) + \rho_-(q)$ to $\rho'_+(q) + \rho'_-(q)$. From (4.55), it is obvious that

$$\rho_+(q) + \rho_-(q) = \left(\frac{v_F + \frac{g_4(q)}{2\pi} - \frac{g_2(q)}{2\pi}}{v_F + \frac{g_4(q)}{2\pi} + \frac{g_2(q)}{2\pi}}\right)^{1/4} [\rho'_+(q) + \rho'_-(q)], \tag{4.57}$$

so the expression (4.53) for ΔE needs only two modifications. We must use the Tomonaga-Luttinger frequency $\omega'_q = q[(v_F + \frac{g_4}{2\pi})^2 - (\frac{g_2}{2\pi})^2]^{1/2}$ and multiply by the magnitude squared of the factor given in (4.57). However, ω'_q must still be negligible compared to Δ, so we simply have

$$\Delta E = -\frac{|g(q)|^2 q k_F}{\pi^2 \Delta}\left|\frac{v_F + \frac{g_4(q)}{2\pi} - \frac{g_2(q)}{2\pi}}{v_F + \frac{g_4(q)}{2\pi} + \frac{g_2(q)}{2\pi}}\right|^{1/2}\left(n_{+q} + \frac{1}{2}\right). \tag{4.58}$$

It is slightly more difficult to perform this calculation using the bosonized operators. The mixing of the the right-moving and left-moving phonons in $\psi_{\pm}$ adds to the complexity of the calculations [45, 24, 16, 14]. Performing the calculation, we find that the matrix elements factorize into ρ'_+ terms and ρ'_- terms. The cancellations that appeared in Section 4.3 at $T = 0$ now occur separately for the two types of terms. The end result is the same as (4.58), provided we set $p_{\max} = \alpha^{-1} = k_F$ and interpret

Δ as representing $\Delta + 3\epsilon_F$ when Δ is $\mathcal{O}(\epsilon_F)$.

Our main results are (4.48) and (4.54), the expressions for the energy shift. It is interesting to note that (4.54) remains regular—without a pole or a branch point—at $\Delta = \epsilon_F$. Although the ground state changes qualitatively at this point, as fermions "spill over" out of the Luttinger liquid into the radially excited state, the phonon energies remain finite. This occurs because the energy defect (4.28) does not vanish for small q, since the excited fermions have large momenta $k \approx 2k_F$. This momentum may be interpreted as arising from a chiral anomaly or as the reciprocal lattice vector of a one-dimensional crystal; in either case, the extra momentum comes from the bulk motion of the Luttinger liquid. The fact that (4.54) is regular at $\Delta = \epsilon_F$ suggests that the phonons may remain metastable even when $\Delta < \epsilon_F$. This would be an interesting "softening" of the expected phase transition in this region. However, in order to understand this situation properly, other interactions than (4.7) must also be considered.

Since the cutoff parameter α is related to the renormalization group flow for the Luttinger liquid [10], the identification of k_F with α^{-1} has interesting implications. In the Luttinger model, α is effectively a free parameter; its value needs to be specified on the basis of physical considerations external to the model. Our results at $T = 0$ suggest that $v_F k_F = 2\epsilon_F$ is the maximum physically meaningful bandwidth for the system. This order of magnitude for the cutoff has a clear physical basis, as discussed in Section 4.3. Since the Luttinger model (with spin) is known to be equivalent to several other condensed matter models [10, 6, 32, 33, 36, 23] and field theory models [25, 11], the specification of a particular value of α may have interesting implications for the related cutoffs of these analogous systems.

Although we have identified $\alpha^{-1} = k_F$ as the correct physical cutoff at $T = 0$, our results do not tell us anything about the renormalization group flow for nonzero temperatures. Moreover, although (4.48) is valid for finite temperatures, it is only meaningful in an extremely singluar limit; all the temperature-dependence of the expression has been removed by making Δ very large. The temperature-independent expression (4.48) is consistent with a nontrivial T-dependence of the energy shift,

becuase ΔE is actually formally vanishing in this limit, since it is proportional to the ratio $\left[|g(q)|^2\alpha^{-1}q\right]/\Delta$. So the questions of the finite-temperature energy shift and the nature of the renormalization group flow for $T > 0$ remain open.

We have used several complementary techniques to evaluate the energy shift. The bosonized method used in Section 4.3 is valid in a wider range of situations; however, we needed the calculations of Section 4.4 to determine the unknown constant $p_{\max}$ appearing the bosonized result. We found a frequency shift that is always $\mathcal{O}[|g(q)|^2 q k_F/\Delta]$, even for nonzero temperatures or in the presence of strong Tomonaga-Luttinger-type interactions. When experimenters are able to build fermionic traps with sufficiently high aspect ratios to observe Luttinger liquid behavior, formulas of this type should be readily testable.

Chapter 5

One-Dimensional Toy Model for a Proposed Form of Electron "Splitting" in Liquid Helium

5.1 The problem and our toy model

Recently, Maris [38] has suggested that the splitting of an electron-inhabited bubble in liquid helium may result in a division of the electron into separate pieces. Each piece would behave like only a fraction of the original electron. This phenomenon would be entirely different from the any of the accepted forms of fermion fractionization [26, 46, 30]. Maris' interpretation has been criticized [27] as corresponding to an ordinary quantum superposition phenomenon and not indicating any fractionization of the electron. We shall discuss this criticism further.

Maris considers an electron trapped in bubble in liquid helium. The repulsive interactions between the single electron and the electron clouds of the helium atoms support the bubble and keep the electron confined in a deep potential well. The electron is initially in the ground (1s) state of the well, before being excited into the 1p state. Maris considers the time evolution of the coupled electron-bubble system after the electron is excited.

Maris treats the electron quantum mechanically and the bubble classically. The electron's 1p wave function is not spherically symmetric, so it exerts different pressures at different points on the bubble. Maris treats this pressure gradient classically and studies its effect on the bubble's shape. He concludes that the bubble will deform and eventually split into two separate bubbles, each containing half of the electron's wave function.

We contend that the classical analysis does not capture the correct time evolution of the bubble's shape. When the entire system is treated quantum-mechanically, the electron's wave function becomes entangled with the wave function of the bubble. The end result is not two bubbles, each containing half of the electron, but rather a system in an entangled state—a superposition of two bubbles in different positions. If the position of the electron is measured, the system will collapse into a state with only a single bubble, surrounding the location where the electron is found.

To shed light on this problem, we shall consider a toy model which is similar to the liquid helium system in many important ways. Our model is one-dimensional and contains only three interacting particles, so we may learn a great deal about the behavior of the system using simple, analytical techniques. We shall make particular use of the Born-Oppenheimer approximation.

The system contains two heavy objects of equal mass M (the "atoms") and a lighter object of mass m (the "electron"), moving on a segment of length L with periodic boundary conditions. The atoms and electron interact through repulsive potentials. If we let x_1, x_2, and x_e be the positions of the atoms and the electron and set $\hbar = 1$, the Hamiltonian is

$$H = K + V \tag{5.1}$$

$$K = -\frac{1}{2M}\frac{\partial^2}{\partial x_1^2} - \frac{1}{2M}\frac{\partial^2}{\partial x_2^2} - \frac{1}{2m}\frac{\partial^2}{\partial x_e^2} \tag{5.2}$$

$$V = V_0\left\{\cos\left[\frac{2\pi}{L}(x_1 - x_2)\right] + 2\epsilon\cos\left[\frac{2\pi}{L}(x_e - x_1)\right] + 2\epsilon\cos\left[\frac{2\pi}{L}(x_e - x_2)\right]\right\} \tag{5.3}$$

We shall insist the the potentials be strongly confining; V_0 must be much larger than the kinetic energy scale $\frac{1}{mL^2}$. We shall also require that $\epsilon < 1$, but we shall assume

that it is $\mathcal{O}(1)$.

It is convenient to separate out the center of mass motion and define new coordinates,

$$X = \frac{Mx_1 + Mx_2 + mx_e}{2M+m} \tag{5.4}$$

$$y = x_2 - x_1 \tag{5.5}$$

$$x = x_e - \frac{x_1 + x_2}{2}. \tag{5.6}$$

In these coordinates, we have

$$K = -\frac{1}{2(2M+m)}\frac{\partial^2}{\partial X^2} - \frac{1}{M}\frac{\partial^2}{\partial y^2} - \frac{1}{2\mu}\frac{\partial^2}{\partial x^2} \tag{5.7}$$

$$\begin{aligned} V &= V_0\left\{\cos\left(\frac{2\pi}{L}y\right) + 2\epsilon\cos\left[\frac{2\pi}{L}\left(x-\frac{y}{2}\right)\right] + 2\epsilon\left[\frac{2\pi}{L}\left(x-\frac{y}{2}\right)\right]\right\} \\ &= V_0\left[\cos\left(\frac{2\pi}{L}y\right) + 4\epsilon\cos\left(\frac{\pi}{L}y\right)\cos\left(\frac{2\pi}{L}x\right)\right], \end{aligned} \tag{5.8}$$

where $\mu \equiv \frac{2mM}{2M+m}$ is a reduced mass for the system.

Writing the full wave function as

$$\Psi(x_1, x_2, x_e) = e^{iPX}\psi(x, y), \tag{5.9}$$

the Hamiltonian governing the relative motion wave function ψ is

$$H_{\text{rel}} = -\frac{1}{M}\frac{\partial^2}{\partial y^2} - \frac{1}{2\mu}\frac{\partial^2}{\partial x^2} + V_0\left[\cos\left(\frac{2\pi}{L}y\right) + 4\epsilon\cos\left(\frac{\pi}{L}y\right)\cos\left(\frac{2\pi}{L}x\right)\right]. \tag{5.10}$$

Since Ψ is periodic in x_1, x_2, and x_e, ψ must have the "helical" boundary conditions

$$\psi(x+L, y) = \psi(x, y) \tag{5.11}$$

$$\psi(x, y+L) = \psi(x+L/2, y). \tag{5.12}$$

Although these boundary conditions mix x and y, the entire physical region is contained within the bounds $0 \le x < L$, $0 \le y < L$; so (5.11) and (5.12) do not cause

any additional mixing of the x and y dynamics.

5.2 Solution of the model by the Born-Oppenheimer method

We shall analyze the behavior of ψ using the Born-Oppenheimer approximation. This approximation is justified by the existence of two widely separated mass scales, m and M. Since $M \gg m \approx \mu$, the atoms move very slowly compared to electron. We may consider y to be an adiabatically varying parameter and solve the Schrödinger equation governing the x motion. The Hamiltonian for the electron's relative motion is

$$H_{\mathrm{rel}\,e} = -\frac{1}{2\mu}\frac{d^2}{dx^2} + 4\epsilon V_0 \cos\left(\frac{\pi}{L}y\right)\cos\left(\frac{2\pi}{L}x\right). \tag{5.13}$$

We shall distinguish two different parameter regimes. When $\epsilon V_0 \left|\cos\left(\frac{\pi}{L}y\right)\right|$ is large compared to $\frac{1}{\mu L^2}$, the potential $V_{\mathrm{rel}\,e}$ in (5.13) is strongly confining. However, when $\frac{1}{\mu L^2} \gg \epsilon V_0 \left|\cos\left(\frac{\pi}{L}y\right)\right|$, the electron is nearly free. We must analyze these two cases separately.

We shall assume for now that y lies in the region $0 \le y \le L/2$. The minimum value of $\cos\left(\frac{\pi}{L}y\right)$ occurs when $y = L/2$. In the vicinity of this point, $4\epsilon V_0 \cos\left(\frac{\pi}{L}y\right) \approx 4\epsilon V_0\left(\frac{\pi}{L}\Delta y\right)$, where $\Delta y \equiv L/2 - y$. So the transition between the two regimes occurs when $\Delta y \sim \frac{1}{V_0 m L}$. Because $\frac{1}{V_0 m L^2}$ is small, $\frac{\Delta y}{L} \ll 1$ in both the weakly confined and transition regions. The potential is strongly confining for all values of y, except for in a comparatively small region around $\Delta y = 0$.

Let us first consider the strongly confined regime. The potential $V_{\mathrm{rel}\,e}$ has its minimum at $x = L/2$. Near $x = L/2$, $V_{\mathrm{rel}\,e}$ has the form

$$V_{\mathrm{rel}\,e}(x) \approx 4\epsilon V_0 \cos\left(\frac{\pi}{L}y\right)\left[-1 + \frac{1}{2}\left(\frac{2\pi}{L}\Delta x\right)^2\right], \tag{5.14}$$

for $\Delta x \equiv x - L/2$. So $H_{\mathrm{rel}\,e}$ becomes

$$H_{\mathrm{rel}\,e} \approx -\frac{1}{2\mu}\frac{d^2}{dx^2} - 4\epsilon V_0 \cos\left(\frac{\pi}{L}y\right) + \frac{1}{2}\mu\left[\frac{4\pi}{L}\sqrt{\frac{\epsilon V_0 \cos\left(\frac{\pi}{L}y\right)}{\mu}}\right]^2 (\Delta x)^2. \tag{5.15}$$

This has solutions that are approximately given by harmonic oscillator wave functions $\phi_n(\Delta x)$ centered at $\Delta x = 0$. The corresponding energies are

$$E_{\mathrm{rel}\,e,n} \approx -4\epsilon V_0 \cos\left(\frac{\pi}{L}y\right) + \left(n+\frac{1}{2}\right)\left(\frac{4\pi}{L}\right)\sqrt{\frac{\epsilon V_0 \cos\left(\frac{\pi}{L}y\right)}{\mu}}. \tag{5.16}$$

The second term in (5.16) is smaller than the first by a factor of $\mathcal{O}\left(\frac{1}{\sqrt{V_0 m L^2}}\right)$, provided $n+\frac{1}{2}$ is $\mathcal{O}(1)$.

Now we consider the $\frac{1}{\mu L^2} \gg \epsilon V_0 \left|\cos\left(\frac{\pi}{L}y\right)\right|$ regime. In this case, the electron is nearly free. The electron states are plane waves subject to the boundary condition (5.11), and the energies are

$$E_{\mathrm{rel}\,e,n} \approx \frac{2\pi^2}{\mu L^2} n^2. \tag{5.17}$$

The ground state corresponds to $n = 0$, and the first electronically excited state is the $n = 1$ state that is an odd function of Δx.

If we extend the energy eigenvalues to cover all possible values of y (not just $0 \le y \le L/2$) and restrict our attention to the two lowest-lying electronic states, we get

$$E_{\mathrm{rel}\,e,n} \approx \begin{cases} -4\epsilon V_0 \left|\cos\left(\frac{\pi}{L}y\right)\right| + \left(n+\frac{1}{2}\right)\left(\frac{4\pi}{L}\right)\sqrt{\frac{\epsilon V_0 \left|\cos\left(\frac{\pi}{L}y\right)\right|}{\mu}} \\ \frac{2\pi^2}{\mu L^2} n^2 \end{cases} \tag{5.18}$$

The upper expression for $E_{\mathrm{rel}\,e,n}$ is valid whenever $|y - L/2| \gg \frac{1}{\epsilon V_0 \mu L}$ (taking $0 \le y < L$), while the lower is correct if $|y - L/2| \ll \frac{1}{\epsilon V_0 \mu L}$. If we include only terms which are zeroth-order in $\frac{1}{V_0 m L^2}$, we have

$$E_{\mathrm{rel}\,e,n} \approx -4\epsilon V_0 \left|\cos\left(\frac{\pi}{L}y\right)\right|. \tag{5.19}$$

We obtain this simplified form because the two expressions for $E_{\mathrm{rel}\,e,n}$ agree near

$y = L/2$ in this approximation (i.e. they both vanish). However, we must still keep in mind that this form for $E_{\mathrm{rel}\,e,n}$ is not quantitatively accurate near $y = L/2$.

This yields an effective potential for the relative motion of the two atoms, given by

$$\begin{aligned} V_{\mathrm{eff}}(y) &= V_0 \left[\cos\left(\frac{2\pi}{L}y\right) - 4\epsilon \left|\cos\left(\frac{\pi}{L}y\right)\right|\right] \\ &= -\left(1+2\epsilon^2\right) V_0 + 2V_0 \left[\left|\cos\left(\frac{\pi}{L}y\right)\right| - \epsilon\right]^2 . \end{aligned} \tag{5.20}$$

[$V_{\mathrm{eff}}(y)$ has a cusp at $y = L/2$, but we have already noted that we expect $E_{\mathrm{rel}\,e,n}$ to have a slightly different form in the region around this point. This effect should smooth out V_{eff} in the vicinity of this local maximum.] If $\epsilon < 1$, this effective potential has symmetry-breaking minima at $y_0 \equiv \frac{L}{\pi}\cos^{-1}\epsilon$ and $L - y_0$ (or, equivalently, at y_0 and $-y_0$). At these points, $V_{\mathrm{eff}}(y) = -(1+2\epsilon^2)V_0$. If $\epsilon \geq 1$, the electron-atom repulsion overpowers the atom-atom repulsion, and the minimum energy configuration has $y = 0$; this is possible because the repulsive cosine potential lacks a hard core region. It was for this reason that we required that $\epsilon < 1$. We need ϵ to be $\mathcal{O}(1)$ so that our formula for V'_{eff} is valid at $y = y_0$. That is, y_0 must lie in the regime where the electron is tightly confined; if $\epsilon < \frac{1}{\sqrt{V_0 \mu L^2}}$, then y_0 will lie too close to $y = L/2$.

The two minima of y correspond to different arrangements of the system. Since the system is translationally invariant, we may choose to fix $x_1 = 0$, so that y becomes the position of atom 2. If $y = y_0$, the minimum of the the electron potential $V_{\mathrm{rel}\,e}$ lies at $x = L/2$. So the electron will be localized near $x_e = L/2 + y_0/2$. Since $0 < y_0 < L/2$, the order of the particles as we move in the positive direction is atom 1, atom 2, electron; because the system is periodic, this is equivalent to atom 2, electron, atom 1. If $y = L - y_0$, $V_{\mathrm{rel}\,e}$ localizes x near $x = 0$, so the electron wave function is peaked at $x_e = L/2 - y_0/2$. The order of the particles becomes atom 1, electron, atom 2. In each case, the electron pushes the two atoms apart. The distance from the atom on the electron's left to the one on the electron's right (measured through the electron) is $L - y_0 > L/2$.

Since $V_{\mathrm{eff}}(y)$ is a even function of y, the exact eigenstates of this potential must be

states of definite parity. The parity eigenstates may be constructed as superpositions

$$\psi(x,y) \approx \frac{1}{\sqrt{2}}[\Phi(y-y_0)\phi_n(x-L/2) \pm \Phi(L-y_0-y)\phi_n(x)], \tag{5.21}$$

where $\Phi(y-y_0)$ is an approximate eigenstate of the Hamiltonian $H'_{\rm eff} = -\frac{1}{M^2}\frac{d^2}{dy^2} + V'_{\rm eff}$, localized around $y = y_0$. These superpositions are not, strictly speaking, Born-Oppeneheimer states, but they should be good approximations to the energy eigenstates. The specific form (5.21) is dependent upon parity invariance, but the general superposition structure is not. For a general (asymmetric) potential with two local minima, the energy eigenstates are superpositions of states localized around those two minima. If $|+\rangle$ is a state located at one minimum and $|-\rangle$ is located at the other, then the eigenstates are $\frac{1}{\sqrt{2}}(v_\pm|+\rangle \pm v_\mp|-\rangle)$; the $v_\pm$ are related to the matrix elements of the Hamiltonian by

$$v_\pm = \sqrt{1 \pm \frac{\delta}{\sqrt{\delta^2 + 4|\langle +|H|-\rangle|^2}}}, \tag{5.22}$$

where $\delta \equiv \langle +|H|+\rangle - \langle -|H|-\rangle$ is the difference in energies between the two minima. δ is assumed to be small compared to $\langle +|H|+\rangle$ and comparable in magnitude to $\langle +|H|-\rangle$.

The form (5.21) for ψ holds in the region $0 \le y \le L$, $0 \le x \le L$; it may be extended to other values of x and y using the boundary conditions (5.11–5.12). There are no low-lying energy eigenstates for which the mean atomic separation is $L/2$ that do not have this superposition form, because $y = L/2$ is a local maximum of $V'_{\rm eff}$.

The two superposition states in (5.21) are not degenerate. There will be some mixing between the two states, shifting the energy of each. We shall now show that this energy shift is very small, so that the expressions given in (5.21) are very good approximations to the exact wave functions.

In the WKB approximation, the energy difference between the two states is $\Delta E = \frac{\omega T}{\pi}$ [29]; ω is the frequency of classical oscillations about the minima, and T is the

tunneling amplitude $T = \exp\left[-\int \sqrt{M(V_{\text{eff}} - E)}\, dx\right]$, where the integration extends over the barrier region. The small oscillation frequency ω is just

$$\begin{aligned} \omega &= \sqrt{\left(\frac{2}{M}\right) \frac{d^2}{dy^2} V_{\text{eff}}(y)\Big|_{y=y_0}} \\ &= \left(\frac{2\pi}{L}\right) \sqrt{\frac{2V_0(1-\epsilon^2)}{M}}. \end{aligned} \tag{5.23}$$

Because of the model's periodic boundary conditions, T is actually composed of two terms, T_1 and T_2. These correspond to the amplitude for the particle to tunnel to the right (from y_0 to $L - y_0$) and for it to tunnel to the left (from y_0 to $-y_0$), respectively. If we neglect the quantum fluctuations around the minima, then T is given by

$$\begin{aligned} T &= T_1 + T_2 \\ T &= \exp\left\{-\int_{y_0}^{L-y_0} dy \sqrt{M[V_{\text{eff}}(y) - V_{\text{eff}}(y_0)]}\right\} \\ &\quad + \exp\left\{-\int_{-y_0}^{y_0} dy \sqrt{M[V_{\text{eff}}(y) - V_{\text{eff}}(y_0)]}\right\} \\ &= \exp\left[-\frac{2L\sqrt{2MV_0}}{\pi} \int_{\cos^{-1}\epsilon}^{\frac{\pi}{2}} du\, (\epsilon - \cos u)\right] \\ &\quad + \exp\left[-\frac{2L\sqrt{2MV_0}}{\pi} \int_0^{\cos^{-1}\epsilon} du\, (\cos u - \epsilon)\right] \\ &= \exp\left[-\frac{2L\sqrt{2MV_0}}{\pi} \left(\epsilon \sin^{-1}\epsilon + \sqrt{1-\epsilon^2} - 1\right)\right] \\ &\quad + \exp\left[-\frac{2L\sqrt{2MV_0}}{\pi} \left(\sqrt{1-\epsilon^2} - \epsilon \cos^{-1}\epsilon\right)\right]. \end{aligned} \tag{5.24}$$

We conclude that the energy difference between the two states, $\Delta E = \frac{\omega(T_1+T_2)}{\pi}$ is an exponentially small function of the large parameter $\sqrt{V_0 M L^2}$, so the mixing between the two superposition states is minimal.

5.3 The relationship between our model and the electron "splitting" problem

We may now relate our toy model to the liquid helium problem. In each system, the electron repels the atoms surrounding it. The electron-filled bubble in the liquid helium system corresponds to the "bubble" the electron in the toy model creates by forcing apart the atoms on its left and its right.

The states corresponding to (5.21) in the liquid helium system are superpositions of different position states of the bubble. A single bubble exists at either of two locations; the two spatially separated bubbles do not coexist simultaneously. A classical treatment of the bubble fails to account for these states. The analogue in the toy model of treating the bubble as a purely classical object is the assumption that the wave function must be an unentangled product of a function of x and a function of y. Specifically, the two-bubble, split-electron state from [38] corresponds to the toy model state

$$\psi_{\text{class}}(x,y) = \frac{1}{2}[\Phi(y-y_0)+\Phi(L-y_0-y)][\phi_0(x)-\phi_0(x-L/2)], \tag{5.25}$$

which is clearly not an eigenstate of the energy.

The analogue of $\psi_{\text{class}}(x,y)$ for the helium system may be written schematically as

$$\psi^{\text{He}}_{\text{class}}(\mathbf{X}_b,\mathbf{X}_e) = \frac{1}{2}[\psi^b_1(\mathbf{X}_b)+\psi^b_2(\mathbf{X}_b)][\psi^e_1(\mathbf{X}_e)-\psi^e_2(\mathbf{X}_e)]. \tag{5.26}$$

$\mathbf{X}_b$ and $\mathbf{X}_e$ are the coordinates of the bubble and electron, respectively, while ψ^b_i and ψ^e_i are appropriate wave functions, localized at two different positions indexed by i. It is easy to see that this wave function is not an energy eigenstate. The $\psi^b_1(\mathbf{X}_b)\psi^e_2(\mathbf{X}_e)$ and $\psi^b_2(\mathbf{X}_b)\psi^e_1(\mathbf{X}_e)$ terms correspond to the electron and the bubble being in different locations. These configurations are unstable; the empty bubble will rapidly collapse, and a new bubble will form around the electron. This process will occur even while the two bubbles are still splitting apart; as the bubble splits, the bubble and electron wave functions will become entangled. For this reason, it is incorrect to treat the

bubble classically.

We also note that, if (5.26) were a stationary state, it would be possible to send faster-than-light signals between the two bubbles. We would begin by separating the two bubbles by a large distance. Then we measure whether or not the electron is present within one bubble. If we find the electron in that bubble, we know that the electron is not present in the other bubble, and the other bubble consequently will collapse. Similarly, if we do not find the electron in the first bubble, it must be in the second bubble. The second bubble with then expand, because it now contains an entire electron. In either case, our measurement has affected the size of the second bubble in a measurable way, so someone observing the second bubble would immediately know that we had performed the measurement[1]. Of course, this paradox does not arise for the superposition wave functions

$$\psi^{\mathrm{He}}(\mathbf{X}_b, \mathbf{X}_e) = \frac{1}{\sqrt{2}}[\psi_1^b(\mathbf{X}_b)\psi_1^e(\mathbf{X}_e) \pm \psi_2^b(\mathbf{X}_b)\psi_2^e(\mathbf{X}_e)]; \tag{5.27}$$

a measurement of the electron's position reveals a full-sized bubble at the location where the electron is found and no bubble at all in the other location.

Finally, we point out that the superposition states displayed in (5.21) are exactly the sort of states discussed in [27]. If we attempt to measure the fermion number between atom 1 on the left and atom 2 on the right, we find that the expectation value is fractional—with value $\frac{1}{2}$. However, the presence of the two nearly degenerate superposition states leads to a large dispersion in this localized fermion number. An analogous phenomenon occurs for the liquid helium wave function (5.27). In each case, the large dispersion indicates that the fractionization is a characteristic of the expectation values only—not of the eigenvalues.

[1]We would like to thank S. Glashow for pointing out the acausal nature of this situation

Appendix A

Photon Self-Energy Integral

To derive (3.28) in a dimension-independent manner, we begin with the integral

$$i\Pi^{\mu\nu}(q) = -e^2 \int_k \mathrm{tr} \frac{\gamma^\mu (\not{k} + \not{q}) \gamma^\nu \not{k}}{(k+q)^2 k^2}. \tag{A.1}$$

Introducing a Feynman parameter x, setting $M^2 = -x(1-x)q^2$, shifting the integration variable $k \to k - xq$, and dropping all odd-k terms leaves

$$i\Pi^{\mu\nu} = -e^2 \int_0^1 dx \int_k \mathrm{tr} \frac{\gamma^\mu \not{k} \gamma^\nu \not{k} - x(1-x)\gamma^\mu \not{q} \gamma^\nu \not{q}}{(k^2 - M^2)^2}. \tag{A.2}$$

Evaluating the traces for D-dimensional Dirac matrices and a d-dimensional integration over k gives

$$\begin{aligned} \mathrm{tr}(\gamma^\mu \not{k} \gamma^\nu \not{k}) &= \mathrm{tr}(-\gamma^\mu \gamma^\nu k^2 + 2\gamma^\mu \not{k} k^\nu) \\ &= \mathrm{tr}\left(-\gamma^\mu \gamma^\nu k^2 + \frac{2}{d}\gamma^\mu \gamma^\nu k^2\right) \\ &= \left(\frac{2}{d} - 1\right) D \eta^{\mu\nu} k^2 \qquad \text{(A.3)} \\ \mathrm{tr}(\gamma^\mu \not{q} \gamma^\nu \not{q}) &= -D\eta^{\mu\nu} q^2 + 2Dq^\mu q^\nu \qquad \text{(A.4)} \end{aligned}$$

The integrand with the k^2 from (A.3) may be transformed to resemble the rest of

the integrand according to

$$\int_k \frac{k^2}{(k^2-M^2)^2}\left(\frac{2}{d}-1\right) = \int_k \frac{1}{k^2-M^2}\left(\frac{2}{d}-1\right) + \int_k \frac{M^2}{(k^2-M^2)^2}\left(\frac{2}{d}-1\right). \quad \text{(A.5)}$$

The first term of (A.5) may be Wick-rotated to Euclidean space and evaluated using integration by parts, to give

$$\begin{aligned}
\int_k \frac{1}{k^2-M^2}\left(\frac{2}{d}-1\right) &= i\int \frac{d\Omega_d k^{d-1}dk}{-k^2-M^2}\left(\frac{2-d}{d}\right) \\
&= \frac{i}{d}\int \frac{d\Omega_d dk\, k^2}{k^2+M^2}(d-2)k^{d-3} \\
&= \frac{-i}{d}\int d\Omega_d dk\, k^{d-2}\frac{d}{dk}\frac{k^2}{k^2+M^2} \\
&= -\frac{2}{d}\int_k \frac{M^2}{(k^2+M^2)^2}. \qquad \text{(A.6)}
\end{aligned}$$

Adding this to the other term in (A.5) gives

$$\int_k \frac{k^2}{(k^2-M^2)^2}\left(\frac{2}{d}-1\right) = -\int_k \frac{M^2}{(k^2-M^2)^2}. \quad \text{(A.7)}$$

Combining (A.7) with the integrals having numerator (A.4) gives the self-energy result, equation (3.28).

Appendix B

Harmonic Oscillator Matrix Elements

To obtain the general formula (4.17), we first apply the Baker-Campbell-Hausdorff formula to the operator $e^{\lambda A^\dagger - \lambda^* A}$. Since $[-\lambda^* A, \lambda A^\dagger] = -|\lambda|^2$ commutes with both A and $A^\dagger$, we have

$$e^{\lambda A^\dagger - \lambda^* A} = e^{-\lambda^* A} e^{\lambda A^\dagger} e^{|\lambda|^2/2}. \tag{B.1}$$

This reduces the problem to the determination of $\langle n|e^{-\lambda^* A} e^{\lambda A^\dagger}|m\rangle$.

There are three cases: $m = n$, $m > n$, and $m < n$. The first case is the simplest. If $m = n$, each factor of A from $e^{-\lambda^* A}$ must be paired with a factor of $A^\dagger$ from $e^{\lambda A^\dagger}$; otherwise, the term does not contribute. So we get

$$\begin{aligned}
\langle m|e^{-\lambda^* A} e^{\lambda A^\dagger}|m\rangle &= \langle m|1 + (-\lambda^* A)\left(\lambda A^\dagger\right) + \frac{1}{(2!)^2}(-\lambda^* A)^2\left(\lambda A^\dagger\right)^2 + \cdots |m\rangle \\
&= \sum_{i=0}^{\infty} (-1)^i |\lambda|^{2i} \frac{1}{(i!)^2} \frac{(m+i)!}{m!} \\
&\equiv F\left(m+1; 1; -|\lambda|^2\right).
\end{aligned} \tag{B.2}$$

For the case of $m > n$, there must be l more factors of $A^\dagger$ than factors of A for a term to contribute. This immediately leads to the series

$$\langle m|e^{-\lambda^* A} e^{\lambda A^\dagger}|n\rangle = \langle m|\frac{1}{l!}\left(\lambda A^\dagger\right)^l + \frac{1}{(l+1)!}(-\lambda^* A)\left(\lambda A^\dagger\right)^{l+1}$$

$$
\begin{aligned}
&+\frac{1}{2!}\frac{1}{(l+2)!}\left(-\lambda^* A\right)^2\left(\lambda A^\dagger\right)^{l+2}+\cdots|n\rangle \\
&= \lambda^l\frac{1}{l!}\sum_{i=0}^{\infty}\frac{1}{i!}\frac{l!}{(l+i)!}(-1)^i|\lambda|^{2i}\frac{(m+i)!}{m!}\sqrt{\frac{m!}{n!}} \\
&= \lambda^l\frac{1}{l!}\sqrt{\frac{m!}{n!}}F\left(m+1;l+1;-|\lambda|^2\right). \qquad \text{(B.3)}
\end{aligned}
$$

In the third case, l is negative. The calculation proceeds along essentially the same lines as for the $l > 0$ case. There must be $|l|$ more factors of A than $A^\dagger$ for a term to be nonzero, so we get

$$
\begin{aligned}
\langle m|e^{-\lambda^* A}e^{\lambda A^\dagger}|n\rangle &= \langle m|\frac{1}{|l|!}\left(-\lambda^* A\right)^{|l|}+\frac{1}{(|l|+1)!}\left(-\lambda^* A\right)^{|l|+1}\left(\lambda A^\dagger\right) \\
&\quad +\frac{1}{2!}\frac{1}{(|l|+2)!}\left(-\lambda^* A\right)^{|l|+2}\left(\lambda A^\dagger\right)^2+\cdots|n\rangle \\
&= (-\lambda^*)^{|l|}\frac{1}{|l|!}\sum_{i=0}^{\infty}\frac{1}{i!}\frac{|l|!}{(|l|+i)!}(-1)^i|\lambda|^{2i}\frac{(n+i)!}{n!}\sqrt{\frac{n!}{m!}} \\
&= (-\lambda^*)^{|l|}\frac{1}{|l|!}\sqrt{\frac{n!}{m!}}F\left(n+1;|l|+1;-|\lambda|^2\right). \qquad \text{(B.4)}
\end{aligned}
$$

Combining equations (B.1), (B.2), (B.3), and (B.4), we get (4.17).

Appendix C

Estimation of the Cutoff $p'_{\max}$ for the Luttinger Liquid Energy Shift Integral

We shall now calculate the cutoff $p'_{\max}$ that arises from the behavior of the energy defect (4.28). We begin with the integration formula

$$\int_{\frac{p_{\min}}{e^\gamma}}^{+\infty} \frac{dp_i}{ap_i + bp_i^2 + cp_i^3} = \frac{1}{a}\log\left(\frac{e^\gamma}{p_{\min}}\sqrt{\frac{a + be^{-\gamma}p_{\min} + ce^{-2\gamma}p_{\min}^2}{c}}\right) \tag{C.1}$$
$$+\frac{b}{2a\sqrt{-b^2+4ac}}\left[i\log\left(\frac{2ic}{\sqrt{-b^2+4ac}}\right) - i\log\left(\frac{-2ic}{\sqrt{-b^2+4ac}}\right)\right.$$
$$\left.+i\log\left(\frac{\sqrt{-b^2+4ac} - ib - 2ice^{-\gamma}p_{\min}}{\sqrt{-b^2+4ac}}\right) - i\log\left(\frac{\sqrt{-b^2+4ac} + ib + 2ice^{-\gamma}p_{\min}}{\sqrt{-b^2+4ac}}\right)\right].$$

In our case, we have $a = \Delta + 3\epsilon_F$, $b = -\frac{k_F}{m}$, and $c = \frac{1}{2m}$. We shall define x to be the frequently-appearing ratio $x \equiv -\frac{b}{\sqrt{-b^2+4ac}} = \sqrt{\frac{\epsilon_F}{\Delta+2\epsilon_F}}$; this is a measure of the size of the energy gap.

To simplify the expression (C.1), we first note that since $p_{\min}$ is small compared to any other momentum scale, we may approximate $a + be^{-\gamma}p_{\min} + ce^{-2\gamma}p_{\min}^2 \approx a$ and $ib + 2ice^{-\gamma}p_{\min} \approx ib$. This reduces the first term on the right-hand side to $\frac{1}{a}\log\left(\frac{e^\gamma}{p_{\min}}\sqrt{\frac{a}{c}}\right)$.

We now turn our attention to the complex terms. Since these terms appear in conjugate pairs, we need only calculate their real parts. Each term has an overall factor of i times a logarithm, so we shall only need the logarithms' imaginary parts. Because $\frac{2ic}{\sqrt{-b^2+4ac}}$ is purely imaginary, we may simplify the second and third terms on the right-hand side of (C.1) to

$$\begin{aligned} -\frac{x}{2a}\left[i\log\left(\frac{2ic}{\sqrt{-b^2+4ac}}\right) - i\log\left(\frac{-2ic}{\sqrt{-b^2+4ac}}\right)\right] &= -\frac{x}{2a}\left[i\log i - i\log(-i)\right] \\ &= -\frac{x}{2a}\left[i\left(i\frac{\pi}{2}\right) - i\left(-i\frac{\pi}{2}\right)\right] \\ &= \frac{\pi x}{2a}. \end{aligned} \tag{C.2}$$

We may evaluate the last two terms of (C.1) similarly, getting

$$\begin{aligned} -\frac{x}{2a}\left[\log(1+ix) - i\log(1-ix)\right] &= -\frac{x}{2a}i\left[i\tan^{-1}x - i\tan^{-1}(-x)\right] \\ &= \frac{x}{a}\tan^{-1}x. \end{aligned} \tag{C.3}$$

So the entire expression becomes

$$\int_{\frac{p_{\min}}{e^\gamma}}^{+\infty} \frac{dp_i}{ap_i + bp_i^2 + cp_i^3} = \frac{1}{a}\log\left(\frac{e^\gamma L\sqrt{2m\Delta + 3k_F^2}}{2\pi}\right) + \frac{1}{a}\left[x\left(\tan^{-1}x + \frac{\pi}{2}\right)\right]. \tag{C.4}$$

We may absorb the second term on the right-hand side of (C.4) into the logarithm by exponentiating it. We then identify the cutoff as

$$p'_{\max} \approx e^\gamma\sqrt{2m\Delta + 3k_F^2}\exp\left[x\left(\tan^{-1}x + \frac{\pi}{2}\right)\right]. \tag{C.5}$$

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