Lines on Fano Hypersurfaces

by

Roya Beheshti Zavareh

B.Sc. in Mathematics, Sharif University, Tehran, Iran, 1999

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Author . . . .
Department of Mathematics
May 13, 2003

Certified by . . . . May 14 2003
A. Johan de Jong
Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
Chairman, Pavel Etingof
Department Committee on Graduate Students

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Abstract

In this thesis, the Hilbert scheme of lines on smooth hypersurfaces is studied. The main result is that the Hilbert scheme of lines on any smooth Fano hypersurface of degree $d \leq 6$ in $\mathbb{P}^n_k$ has the expected dimension $2n - d - 3$, if k is an algebraically closed field of characteristic zero.

Thesis Supervisor: A. Johan de Jong
Title: Professor of Mathematics

To my parents, Fatemeh Ganjizadeh and Heshmatollah Beheshti

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Contents

Chapter 1

Introduction

Let k be an algebraically closed field of characteristic zero. For a subvariety X of $\mathbb{P}^n_k$, consider the Hilbert scheme of lines on X. It is a subscheme of the Grassmannian of lines in $\mathbb{P}^n_k$ and is called the Fano variety of lines on X. We denote it by $\mathcal{F}(X)$. For general hypersurfaces, these schemes have been studied classically, but not much is known when X is not general. In this thesis, some properties of $\mathcal{F}(X)$ are investigated when X is an arbitrary smooth Fano hypersurface.

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree n. When $\mathcal{F}(X)$ is smooth and has the expected dimension, $n-3$, it is known that $\mathcal{F}(X)$ is not uniruled. In Chapter 2, we generalize this to the following theorem.

Theorem A. *Let k be an algebraically closed field of characteristic zero and $X \subset \mathbb{P}^n_k$ any smooth hypersurface of degree n. Let Y be an irreducible component of $\mathcal{F}(X)$ such that the lines corresponding to its closed points sweep out a divisor in X. Then Y is not uniruled.*

In Chapter 4, we study the dimension of the Hilbert scheme of lines on Fano hypersurfaces. It is easy to see that for a general hypersurface X of degree d in $\mathbb{P}^n$, $\mathcal{F}(X)$ is integral and has the expected dimension $2n-d-3$. When X is not general, $\mathcal{F}(X)$ is not necessarily irreducible or reduced. However, it is conjectured that it has always the right dimension if $d \leq n$.

Conjecture B. *Let k be an algebraically closed field of characteristic p and $X \subset \mathbb{P}^n_k$ any smooth hypersurface of degree d, $d \leq n$. If $p = 0$ or $p > d$, then $\dim \mathcal{F}(X) = 2n-d-3$.*

We prove the following theorem.

Theorem C. *Conjecture B holds for $d \leq 6$ if k has characteristic zero.*

When $\mathrm{char}(k) = p > 3$ and d is large, the above conjecture is not true. It is easy to see that the family of lines contained in a Fermat hypersurface of degree $p+1$ in $\mathbb{P}^n, \sum_{i=0}^n X_i^{p+1} = 0$, has dimension at least $2n-6$ which is larger than the expected dimension of $\mathcal{F}(X)$: Fix a point $P = (p_0; \ldots; p_n)$ in X. A point $Q = (q_0; \ldots; q_n)$ is on a line $l \subset X$ which passes through P if and only if

$$\sum_{0 \le i \le n} (p_i + tq_i)^{p+1} = 0 \qquad t \in k,$$

or equivalently

$$\sum q_i^{p+1} = 0, \quad \sum p_i^p q_i = 0, \quad \sum p_i q_i^p = 0.$$

The intersection of these hypersurfaces has dimension at least $n-3$ and hence the dimension of lines passing through each point of X is at least $n-4$. Therefore $\dim \mathcal{F}(X) \ge 2n-6$. Also, It can be shown that if $\mathrm{char}(k) = 0$, then the Fermat hypersurface of degree $m \ge n$ contains a $(2n-3)$-parameter family of lines (see [2], Exercise 2.5). Hence the conditions in Conjecture B are necessary.

Chapter 3 is devoted to proving a proposition on the singularities of second fundamental forms of a hypersurface and also proving a theorem of Landsberg in details. These are both used in the proof of Theorem C.

In Chapter 5, another method is used to prove Theorem C when $d = 5$. We show that

Theorem D. *The Abel-Jacobi map of a smooth quintic threefold is non-zero if it contains a 1-parameter family of lines.*

It is shown then that such a threefold cannot be a general hyperplane section of a smooth quintic fourfold.

Conventions

1. All schemes are considered over a fixed algebraically closed field of characteristic zero unless otherwise stated.

2. For any projective variety $X \subset \mathbb{P}^n$, $\mathcal{F}(X)$ always denotes the Fano variety of lines on X.

3. For a scheme X and a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X, we denote by $\mathcal{F}/\mathrm{tor}$ the sheaf obtained from dividing out $\mathcal{F}$ by its torsion.

Chapter 2

Rational curves on the Fano variety of lines on hypersurfaces

A variety Y of dimension m is called uniruled if there exists a variety Z of dimension $m-1$ and a dominant rational map $\mathbb{P}^1 \times Z \dashrightarrow Y$. Since our base field is algebraically closed, this implies that there is a rational curve through every point of Y.

Theorem 2.1. *Let X be a smooth hypersurface of degree $d \geq n-1$ in $\mathbb{P}^n$ and $\mathcal{Y}$ an irreducible component of $\mathcal{F}(X)$. If the lines corresponding to the closed points of $\mathcal{Y}$ sweep out a divisor in X, then $\mathcal{Y}$ is not uniruled.*

Remark 2.2. If X is any smooth hypersurface of degree at least n in $\mathbb{P}^n$, then the lines in X cannot cover X (see Lemma 4.5). So the lines corresponding to the points of $\mathcal{Y}$ cover a subvariety of codimension at least 1. If X is a general hypersurface of degree n in $\mathbb{P}^n$, then $\mathcal{F}(X)$ is irreducible and the normal bundle of a general line l in X is isomorphic to $\mathcal{O}_l^{n-3} \oplus \mathcal{O}_l(-1)$ (see [6], 4.4). Hence the lines in X cover a divisor and from the theorem above, we can conclude that $\mathcal{F}(X)$ is not uniruled. If X is a smooth hypersurface of degree $n-1$, then X is covered by lines and hence there are many such $\mathcal{Y}$'s.

Proof of Theorem 2.1. Assume on the contrary that $\mathcal{Y}$ is covered by rational curves. Without loss of generality, we can assume that the base field is uncountable. We define Σ_e to be the ruled surface $\mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ over $\mathbb{P}^1$.

Let C be a rational curve on $\mathcal{F}(X)$ and $S \subset C \times X$ the family of lines parametrized by C. Let $\nu : \mathbb{P}^1 \longrightarrow C$ a dominant map. If we pullback S under ν, we get a ruled surface over

$\mathbb{P}^1$ which is isomorphic to Σ_{e_C} for a non-negative integer e_C and a map $f_C : \Sigma_{e_C} \longrightarrow X$.

$$\begin{array}{ccc} \Sigma_{e_C} & \simeq \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e_C)) & \xrightarrow{f_C} X \\ \big\downarrow \pi & & \\ \mathbb{P}^1 & & \end{array}$$

Let $\mathcal{M}or(\Sigma_e, X)$ be the scheme parameterizing the morphisms from Σ_e to X. For every e, the scheme $\mathcal{M}or(\Sigma_e, X)$ has countably many components. Since our base field is uncountable and the lines corresponding to the points of $\mathcal{Y}$ cover a subvariety of dimension $n-2$ of X, there is a rational curve C in $\mathcal{Y}$ and an irreducible component of $\mathcal{M}or(\Sigma_{e_C}, X)$ denoted by $\mathcal{Z}$, such that $\mathcal{Z}$ contains f_C and the image of the map

$$\begin{array}{rcl} \Gamma : \mathcal{Z} \times \Sigma_{e_C} & \longrightarrow & X \\ (g, p) & \longmapsto & g(p) \end{array}$$

is $(n-2)$-dimensional. Fix one such curve C and let $e = e_c$, $\Sigma = \Sigma_{e_C}$ and $f = f_C : \Sigma \longrightarrow X$.

There is an injective map from the tangent sheaf of Σ to the pullback of the tangent sheaf of X. Denote the quotient by $\mathcal{F}$

$$0 \longrightarrow T_\Sigma \longrightarrow f^*T_X \longrightarrow \mathcal{F} \longrightarrow 0. \tag{2.1}$$

To get a contradiction, we compute $h^0(\Sigma, (\bigwedge^{n-4} \mathcal{F})/\mathrm{tor})$ in two different ways. First, we use the fact that the deformations of f cover a divisor in X to show that $h^0(\Sigma, (\bigwedge^{n-4} \mathcal{F})/\mathrm{tor})$ is positive and then, we prove that it is zero, using some computations with exact sequences of powers of tangent sheaves.

Step 1: $h^0(\Sigma, (\bigwedge^{n-4} \mathcal{F})/\mathrm{tor}) > 0$. Let p be a general point of Σ and consider the maps

$$\alpha : H^0(\Sigma, f^*T_X) \longrightarrow (f^*T_X)_p \simeq T_{X,f(p)},$$

and

$$\beta : T_{\Sigma,p} \longrightarrow T_{X,f(p)}.$$

Lemma 2.3. *For a general point p of Σ the image of the map*

$$H^0(\Sigma, f^*T_X) \oplus T_{\Sigma,p} \xrightarrow{\alpha\oplus\beta} T_{X,f(p)},$$

is at least $(n-2)$-dimensional.

Proof. Since the image of Γ is $(n-2)$-dimensional, the same is true for the image of the map induced on the Zariski tangent spaces at a general point (f,p). The Zariski tangent space to $\mathcal{Z}$ at f is a subspace of the tangent space to $\mathcal{M}or(\Sigma, X)$ at f which is naturally isomorphic to $H^0(\Sigma, f^*T_X)$ and $\alpha \oplus \beta$ is the induced map on tangent spaces. This proves the lemma. □

Consider the following commutative diagram.

$$\begin{array}{ccc} H^0(\Sigma, T_\Sigma) & \longrightarrow & H^0(\Sigma, f^*T_X) \\ \downarrow & & \downarrow \alpha \\ T_{\Sigma,p} & \xrightarrow{\beta} & T_{X,f(p)} \end{array}$$

For a general point p of Σ, the map $H^0(\Sigma, T_\Sigma) \longrightarrow T_{\Sigma,p}$ is surjective. Therefore, for such a point, $\beta(T_{\Sigma,p}) \subset \alpha(H^0(\Sigma, f^*T_X))$. So by the last lemma, the image of α is $(n-2)$-dimensional.

Now, look at the sequence (2.1). We have shown that the image of α is $(n-2)$-dimensional. This implies that the global sections of f^*T_X generate an $(n-2)$-dimensional subspace at a general point of Σ. Therefore global sections of $\mathcal{F}$ generate an $(n-4)$-dimensional subspace at a general point of Σ and hence $h^0(\Sigma, (\bigwedge^{n-4}\mathcal{F})/\mathrm{tor}) > 0$.

Step 2: $h^0(\Sigma, (\bigwedge^{n-4}\mathcal{F})/\mathrm{tor}) = 0$. By the next lemma, there is an injective map

$$0 \longrightarrow (\bigwedge^{n-4}\mathcal{F})/\mathrm{tor} \otimes \bigwedge^2 T_\Sigma \longrightarrow \bigwedge^{n-2} f^*T_X. \tag{2.2}$$

Lemma 2.4. *Let X be an integral scheme. Consider an exact sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \mathcal{N} \xrightarrow{\phi} \mathcal{M} \xrightarrow{\psi} \mathcal{F} \longrightarrow 0,$$

where $\mathcal{N}$ is locally free of rank r and $\mathcal{M}$ is locally free of rank m. For every positive integer s, there is an injective map from $\bigwedge^r \mathcal{N} \otimes \bigwedge^s \mathcal{F}/\mathrm{tor}$ to $\bigwedge^{r+s} \mathcal{M}$.

Proof. Let x be a point of X and $e_1, e_2, \ldots, e_r$ be a basis for $\mathcal{N}_x$. We define a map γ_x : $\bigwedge^r \mathcal{N}_x \otimes \bigwedge^s \mathcal{F}_x/\mathrm{tor} \longrightarrow \bigwedge^{r+s} \mathcal{M}_x$ as follows. For $n \in \mathcal{N}_x$, let $\tilde{n} \in \mathcal{M}_x$ denotes an arbitrary lifting of n. Every element of $\bigwedge^r \mathcal{N}_x$ can be written as sum of elements of form $a(e_1 \wedge \cdots \wedge e_r)$. Define γ_x to be

$$\gamma_x((ae_1 \wedge \cdots \wedge e_r) \otimes (n_1 \wedge \cdots \wedge n_s)) = a(\phi(e_1) \wedge \cdots \wedge \phi(e_r) \wedge \tilde{n}_1 \wedge \cdots \wedge \tilde{n}_s)$$

It is easy to see that the map does not depend on the chosen liftings, and we can glue the γ_x's to get a map as desired. To show that this map injective, observe that it is injective at the generic point of X, hence its kernel is a torsion sheaf. Since $\bigwedge^r \mathcal{N} \otimes \bigwedge^s \mathcal{F}/\mathrm{tor}$ is torsion-free, it cannot have a torsion subsheaf. $\square$

Twist the map in (2.2) with the canonical sheaf of Σ, ω_Σ. We get another injective map

$$0 \longrightarrow (\bigwedge^{n-4} \mathcal{F})/\mathrm{tor} \longrightarrow \bigwedge^{n-2} f^*T_X \otimes \omega_\Sigma.$$

Now we compute $h^0(\Sigma, \bigwedge^{n-2} f^*T_X \otimes \omega_\Sigma)$ and show that it is zero. This will conclude the proof of Theorem 2.1. Pullback the sequence of tangent sheaves under f

$$0 \longrightarrow f^*T_X \longrightarrow f^*T_{\mathbb{P}^n} \longrightarrow f^*\mathcal{O}_X(d) \longrightarrow 0. \tag{2.3}$$

By considering the $(n-1)$-th exterior power of the above sequence and twisting it with ω_Σ, we get another exact sequence

$$0 \longrightarrow \bigwedge^{n-1} f^*T_X \otimes f^*\mathcal{O}(-d) \otimes \omega_\Sigma \longrightarrow \bigwedge^{n-1} f^*T_{\mathbb{P}^n} \otimes f^*\mathcal{O}(-d) \otimes \omega_\Sigma \longrightarrow \bigwedge^{n-2} f^*T_X \otimes \omega_\Sigma \longrightarrow 0.$$

Applying the long exact sequence of cohomology, we see that it is enough to prove two things:

(1) $h^0(\Sigma, \bigwedge^{n-1} f^*T_{\mathbb{P}^n} \otimes f^*\mathcal{O}_X(-d) \otimes \omega_\Sigma) = 0$.

(2) $h^1(\Sigma, \bigwedge^{n-1} f^*T_X \otimes f^*\mathcal{O}_X(-d) \otimes \omega_\Sigma) = 0$.

To compute the dimension of these cohomology groups, we use another exact sequence which is the pullback of the Euler sequence for $\mathbb{P}^n$

$$0 \longrightarrow f^*\mathcal{O}_X \longrightarrow f^*\mathcal{O}_X(1)^{\oplus n+1} \longrightarrow f^*T_{\mathbb{P}^n} \longrightarrow 0. \tag{2.4}$$

From (2.3) and (2.4), we get

$$\bigwedge^{n-1} f^*T_X \otimes \mathcal{O}_X(-d) \otimes \omega_\Sigma \simeq \bigwedge^{n} f^*T_{\mathbb{P}^n} \otimes \mathcal{O}_X(-2d) \otimes \omega_\Sigma$$
$$\simeq f^*\mathcal{O}_X(n+1-2d) \otimes \omega_\Sigma$$

Hence we have $h^1(\Sigma, \bigwedge^{n-1} f^*T_X \otimes f^*\mathcal{O}_X(-d) \otimes \omega_\Sigma) = h^1(\Sigma, f^*\mathcal{O}_X(n+1-2d) \otimes \omega_\Sigma) = h^1(\Sigma, f^*\mathcal{O}_X(2d-n-1))$, by Serre duality.

Let F be the class of a fiber of π and C be the class of the section with $C^2 = -e$. Recall that the Picard group of Σ is the free abelian group generated by F and C and the intersection products are given by

$$C^2 = -e, \quad F^2 = 0, \quad C \cdot F = 1.$$

Let $f^*\mathcal{O}_X(1) = aC + bF$. The image of a line of ruling of Σ under f is a line and hence we get

$$1 = f^*\mathcal{O}_X(1) \cdot F = (aC + bF) \cdot F = a.$$

Also $f^*\mathcal{O}_X(1) \cdot C \geq 0$. Hence we get

$$-e + b = (C + bF) \cdot C \geq 0.$$

Now, we have

$$f^*\mathcal{O}_X(2d-n-1) = (2d-n-1)C + b(2d-n-1)F,$$

and

$$\begin{aligned} h^1(\Sigma, f^*\mathcal{O}_X(2d-n-1)) &= h^1(\mathbb{P}^1, \pi_* f^*\mathcal{O}_X(2d-n-1)) \\ &= h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((2d-n-1)b) \otimes \pi_*((2d-n-1)C)) \\ &= h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((2d-n-1)b) \otimes (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-(2d-n-1)e))) \\ &= 0. \end{aligned}$$

This proves (2). To see that (1) is true, observe that the sequence (2.4) leads to another short exact sequence

$$0 \longrightarrow \bigwedge^{n-1} f^*T_{\mathbb{P}^n}\otimes f^*\mathcal{O}_X(-d)\otimes\omega_\Sigma \longrightarrow f^*\mathcal{O}_X(n-d)^{\oplus n+1}\otimes\omega_\Sigma \longrightarrow f^*\mathcal{O}_X(n+1-d)\otimes\omega_\Sigma \longrightarrow 0,$$

and since $n \leq d$, $h^0(\Sigma, f^*\mathcal{O}_X(n-d)\otimes\omega_\Sigma) = 0$. This completes the proof of Theorem 2.1. □

Chapter 3

Fundamental forms of hypersurfaces

Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d given by a homogeneous polynomial F. Fix a point $p \in X$. For $1 \le k \le d$, let Y_p^k be the degree k hypersurface in $\mathbb{P}^n$ given by the homogeneous polynomial

$$\sum_{\substack{(i_1,\dots i_k) \\ 0\le i_1,\dots,i_k \le n}} \frac{\partial^k F}{\partial x_{i_1}\dots\partial x_{i_k}}(p)\, x_{i_1}\dots x_{i_k} \tag{3.1}$$

Notice that Y_p^1 is just the embedded tangent space to X at p. The restriction of (3.1) to the tangent space $T_{X,p}$ gives a k-form which is called the k-th fundamental form of X at p.

If $p = (1; 0; \dots; 0)$, then we can write F as

$$F = x_0^{d-1}F_1 + x_0^{d-2}F_2 + \dots + F_d, \tag{3.2}$$

where F_i is a homogeneous polynomial of degree i in $x_1, \dots, x_n$. In this case, the intersection of Y_p^i and the hyperplane $x_0 = 0$ is the hypersurface defined by F_i. Also, the intersection of the hypersurfaces $Y_p^1, Y_p^2, \dots, Y_p^k$ is the same as the intersection of the hypersurfaces defined by $F_1, F_2, \dots F_k$.

Lemma 3.1. *For a smooth point p of X and $0 \le k \le d = \deg(X)$,*

(a) The point p is in Y_p^k.

(b) If a point q is in the intersection of $Y_p^1, Y_p^2, \dots, Y_p^k$, then the line passing through p

and q lies in this intersection.

(c) A point q is in the intersection of $Y_p^1, Y_p^2, \ldots, Y_p^d$ if and only if the line passing through q and p lies in X.

Proof. We can assume $p = (1; 0; \ldots; 0)$ and use the above argument. □

Definition 3.2. *We say that a line l passing through p has contact to order k with X at p if l is contained in the intersection of $Y_p^1, \ldots Y_p^k$.*

So a line l has contact to order 1 at p if it lies in the tangent hyperplane at p, and has contact to order d if it lies in X. Again, if $p = (1; 0; \ldots; 0)$ and we write F as (3.2), then a line has contact to order k if it lies in the intersection of $F_1, \ldots, F_k$.

By Lemma 3.1, the union of all lines having contact to order k at p form a cone with vertex p. Notice that by definition, this cone is the intersection of k hypersurfaces in $\mathbb{P}^n$ and hence its expected dimension is $n - k$. The next theorem says that generally the expected dimension is obtained unless the whole cone lies in X.

Theorem 3.3 (Landsberg). *Let $X \subset \mathbb{P}^n$ be a hypersurface and p a general point of X. Denote the union of all lines in $\mathbb{P}^n$ having contact to order k to X at p by Σ^k. If an irreducible component Σ_0^k of Σ^k has dimension greater than $n - k$, then Σ_0^k lies in X.*

In section (3.1), we prove the above theorem. The proof is taken from [8] but the notation might be slightly different. In section (3.2), we prove a proposition on the singularities of Y_p^2 for a general $p \in X$.

3.1 Intersection of the fundamental forms.

For a given integer n, let $GL_{n+1}(k)$ be the space of invertible matrices of size $n+1$ over k. Let $\phi : GL_{n+1}(k) \to \mathbb{P}^n$ be the map given by $\phi([V_0, \ldots, V_n]) = V_0$, where by $[V_0, \ldots V_n]$ we mean the matrix having $V_0, \ldots, V_n$ as its columns. Let Ω be the matrix of global one forms on $GL_{n+1}(k)$ given by

$$\Omega_f = f^{-1}\mathrm{d}f \quad \text{for} \quad f \in GL_{n+1}(k).$$

Denote the entries of Ω by $\omega_{ij}, 0 \leq i, j \leq n$. We have

$$\mathrm{d}x_{ij}(f) = \sum_{0 \leq k \leq n} f_{ik} \cdot \omega_{kj}(f) \quad \text{for} \quad f = (f_{ij})_{0 \leq i,j \leq n} \in GL_{n+1}(k). \tag{3.3}$$

Notation. In what follows, $f = (f_{ij})_{0\leq i,j\leq n}$ is always an element of $GL_{n+1}(k)$. We denote the columns of f by $V_0, \dots, V_n$. Hence the i-th entry of V_j is f_{ij}.

Let X be an integral hypersurface of degree d in $\mathbb{P}^n$ given by the homogeneous polynomial F. For given integers $0 \leq i_1, i_2, \dots, i_m \leq n$, we define regular functions $r^m_{i_1,\dots,i_m}$ on $GL_{n+1}(k)$ as follows:

$$r^m_{i_1,\dots,i_m}(f) = \frac{\partial^m F}{\partial V_{i_1}\dots\partial V_{i_m}}(V_0), \quad f = [V_0, \dots, V_n],$$

More precisely,

$$r^m_{i_1,\dots,i_m}(f) = \sum_{\substack{(j_1,\dots,j_m)\\ 0\leq j_1,\dots,j_m\leq n}} f_{j_1,i_1}\cdots f_{j_m,i_m}\frac{\partial^m F}{\partial x_{j_1}\dots\partial x_{j_m}}(V_0). \tag{3.4}$$

Since the ω_{ij}'s form a basis for the space of 1–forms at every point of $GL_{n+1}(k)$, we can express the derivative of the functions $r_{i_1,\dots,i_m}$ as linear combinations of ω_{ij}'s. The next lemma says what the coefficients of the combination are when $i_1 = \cdots = i_m$.

Lemma 3.4. *For $0 \leq j \leq n$,*

$$\mathrm{d}r^m_{i,i,\dots,i} = \sum_{0\leq t\leq n} r^{m+1}_{i,\dots,i,t}\,\omega_{t0} + m\sum_{0\leq t\leq n} r^m_{i,\dots,i,t}\,\omega_{ti}.$$

Proof. By partial derivation, we have

$$\mathrm{d}r^m_{i,\dots,i} = \sum_{(j_1,\dots,j_m)}\sum_{1\leq l\leq m}(f_{j_1,i}\cdots\hat{f_{j_l,i}}\cdots f_{j_m,i}\frac{\partial^m F}{\partial x_{j_1}\dots\partial x_{j_m}}(V_0))\,\mathrm{d}x_{j_l,i}$$
$$+\sum_{(j_1,\dots,j_m)} f_{j_1,i}\cdots f_{j_m,i}\,\mathrm{d}(\frac{\partial^m F}{\partial x_{j_1}\dots\partial x_{j_m}}(V_0)).$$

Hence the assertion follows from the equality

$$\mathrm{d}(\frac{\partial^m F}{\partial x_{j_1}\dots\partial x_{j_m}}(V_0)) = \sum_{0\leq t\leq n}\frac{\partial^{m+1}F}{\partial x_{j_1}\dots\partial x_{j_m}\partial x_t}(V_0)\,\mathrm{d}x_{t0},$$

and equation (3.3). □

Before going on, we need another easy lemma.

Lemma 3.5. *For $0 \leq i_1, \dots, i_m \leq n$, we have $r^{m+1}_{i_1,\dots,i_m,0} = (d-m)r^m_{i_1,\dots i_m}$, where $d = \deg(X)$.*

Proof. By definition we have

$$\begin{aligned} r^{m+1}_{i_1,\dots,i_m,0} &= \sum_{(j_1,\dots,j_m)} f_{j_1,i_1}\cdots f_{j_m,i_m}(\sum_t f_{t0}\frac{\partial}{\partial x_t}\frac{\partial^m F}{\partial x_{j_1}\dots\partial x_{j_m}}(V_0)) \\ &= (d-m)\sum_{(j_1,\dots,j_m)} f_{j_1,i_1}\cdots f_{j_m,i_m}\frac{\partial^m F}{\partial x_{j_1}\dots\partial x_{j_m}}(V_0) \\ &= (d-m)r^m_{i_1,\dots,i_m}. \end{aligned}$$

□

Let $\mathcal{F}_X \subset GL_{n+1}(k)$ be the set of all matrices with columns $[V_0,\dots,V_n]$ such that V_0 is a smooth point of the hypersurface X and $V_1,\dots V_{n-1}$ are points in the embedded tangent plane to X at V_0, namely the hyperplane defined by the linear polynomial

$$\sum_t (\frac{\partial F}{\partial x_t}(V_0))\; x_t.$$

The scheme $\mathcal{F}_X$ is a smooth locally closed subvariety of GL_{n+1} and there is a map $H^0(GL_{n+1},\Omega^1_{GL_{n+1}}) \longrightarrow H^0(\mathcal{F}_X,\Omega^1_{\mathcal{F}_X})$. Denote by ω'_{ij} the image of ω_{ij} under this map.

Lemma 3.6. *At every point of $\mathcal{F}_X$,*

(a) $\omega'_{n0}=0$.

(b) $\omega'_{00},\omega'_{10},\dots,\omega'_{n-1,0}$ are independent 1-forms.

Proof. (a) We have

$$r_0 = \sum_t f_{t0}\frac{\partial F}{\partial x_t}(V_0) = dF(V_0) = 0.$$

Hence we get $\mathrm{d}r_0 = 0$. By Lemma 3.4,

$$0 = \mathrm{d}r_0 = \sum_{0\le t\le n} r_{0t}\,\omega'_{t0} + \sum_{0\le t\le n} r_t\,\omega'_{t0}.$$

Since $V_1,\dots V_{n-1}$ are in the tangent plane to X at V_0, $r_1=\cdots=r_{n-1}=0$, and by Lemma 3.5, we have

$$r_{0t}=r_t=0 \quad \text{for} \quad 0\le t\le n-1,$$

therefore

$$r_n\,\omega'_{n0}=0.$$

Because all the matrices in $\mathcal{F}_X$ are invertible, V_n is not in the tangent hyperplane at V_0 and therefore $r_n \neq 0$. Hence we get $\omega'_{n0} = 0$.

To prove part (b), observe that the map $\phi : GL_{n+1}(k) \to \mathbb{P}^n$ factors through the map $\phi' : GL_{n+1}(k) \to \mathbb{A}^{n+1}$ and $\mathrm{d}x_{t0}$ is the pullback of $\mathrm{d}x_t$ under ϕ'. Let $X' \subset \mathbb{A}^{n+1}$ be the affine cone over X. We know that $\mathrm{d}x_0, \ldots, \mathrm{d}x_n$ form a space of dimension n at every smooth point of X'. Since the map $\phi'|_{\mathcal{F}_X} \longrightarrow X_{smooth}$ is smooth (the fibers are linear) and surjective, the pullback of these 1-forms, make an n-dimensional space at every point of $\mathcal{F}_X$. On the other hand, f is invertible and $f \cdot \Omega = \mathrm{d}f$. Therefore $\omega'_{00}, \omega'_{10}, \ldots, \omega'_{n-1,0}, \omega'_{n0}$ form an n-dimensional space at every point and by part (a), we know that $w'_{n0} = 0$. So the first n-forms are linearly independent. This proves part (b). □

For a point $p \in X$, let Σ_p^k be the intersection of hypersurfaces $Y_p^1, Y_p^2, \ldots, Y_p^k$. By our definition, the underlying space of Σ_p^k is the union of all lines in X passing through p with contact to order k with X.

Lemma 3.7. *For $f = [V_0, \ldots, V_n] \in \mathcal{F}_X$, the followings hold.*

(a) $V_i \in \Sigma_{V_0}^k$ if and only if $r_i^1(f) = r_{i,i}^2(f) = \cdots = r_{i,i,\ldots,i}^k(f) = 0$.

(b) Assume that $V_1 \in \Sigma_{V_0}^k$. Then V_i is in the Zariski tangent space of $\Sigma_{V_0}^k$ at point V_1 if and only if

$$r_i^1(f) = r_{1,i}^2(f) = \cdots = r_{1,1,\ldots,1,i}^k(f) = 0.$$

Proof. Part (a) is trivial by equations (3.1) and (3.4). We prove part (b) in case $k = 2$. The proof in the general case is similar.

The point V_i is in the Zariski tangent space of $\Sigma_{V_0}^2$ at V_1 if and only if it is in the Zariski tangent spaces of the hypersurfaces $Y_{V_0}^1$ and $Y_{V_0}^2$ at point V_1. The equations of these tangent spaces are given by

$$\sum_t \left(\frac{\partial F}{\partial x_t}(V_0)\right) x_t = 0$$

and

$$\sum_t \left(\frac{\partial}{\partial x_t} \Big(\sum_{m,j} \frac{\partial^2 F}{\partial x_m \partial x_j}(V_0) x_m \cdot X_j \Big)(V_1) \right) x_t$$

$$= 2 \sum_{t,m} \frac{\partial^2 F}{\partial x_t \partial x_m}(V_0) f_{m1} x_t.$$

If we evaluate these two linear polynomials at the point V_i, we get $r_i(f)$ and $r_{1,i}(f)$. □

Proof of Theorem 3.3. Let $\dim \Sigma_p^k = s > n - k$ for a general point of X. Define $\mathcal{G}_X \subset \mathcal{F}_X$ be the set of all invertible matrices $f = [V_0, \ldots, V_n]$ such that V_0 is a general point of X, V_1 is a general point of an irreducible s-dimensional component of $\Sigma_{V_0}^k$, and $V_2, \ldots V_s$ are in the Zariski tangent space of $\Sigma_{V_0}^k$ at V_1. Notice that V_0 is also in this tangent space and hence $V_0, V_1, \ldots, V_s$ span the Zariski tangent space of $\Sigma_{V_0}^k$ at V_1. The scheme $\mathcal{G}_X$ is smooth and there is a map $H^0(GL_{n+1}, \Omega^1_{GL_{n+1}}) \longrightarrow H^0(\mathcal{G}_X, \Omega^1_{\mathcal{G}_X})$. Denote the image of ω_{ij} by ν_{ij}. By the last lemma we have

$$r^i_{1,1,\ldots,1} = 0 \quad \text{for} \quad 1 \leq i \leq k \tag{3.5}$$

and

$$r^i_{1,\ldots,1,t} = 0 \quad \text{for} \quad 1 \leq i \leq k,\ 2 \leq t \leq s. \tag{3.6}$$

By looking at the derivative of (3.5) for $1 \leq i \leq k-1$, we get

$$\begin{aligned} 0 &= \mathrm{d}r^i_{1,1\ldots,1} \\ &= \sum_{0 \leq t \leq n} r^{i+1}_{1,\ldots,1,t}\, \nu_{t0} + i \sum_{0 \leq t \leq n} r^i_{1,\ldots,1,t}\, \nu_{t1} \\ &= \sum_{s+1 \leq t \leq n} r^{i+1}_{1,\ldots,1,t}\, \nu_{t0} + i \sum_{s+1 \leq t \leq n} r^i_{1,\ldots,1,t}\, \nu_{t1}, \end{aligned}$$

where the equality in the second line comes from Lemma 3.4 and the equality in the third line comes from equations (3.5) and (3.6), hence

$$\sum_{s+1 \leq t \leq n} r^{i+1}_{1,\ldots,1,t}\, \nu_{t0} = -i \sum_{s+1 \leq t \leq n} r^i_{1,\ldots,1,t}\, \nu_{t1} \qquad 1 \leq i \leq k-1.$$

Putting the coefficients of these $k-1$ equations in matrices, we get two $k-1$ by $n-s$ matrices A and B such that

$$A \cdot \begin{bmatrix} \nu_{s+1,0} \\ \cdot \\ \cdot \\ \cdot \\ \nu_{n0} \end{bmatrix} = B \cdot \begin{bmatrix} \nu_{s+1,1} \\ \cdot \\ \cdot \\ \cdot \\ \nu_{n1} \end{bmatrix}.$$

The matrix B is $(k-1)$ by $(n-s)$, and by our assumption on the dimension of $\Sigma^k_{V_0}$, $k-1 \geq n-s$. The kernel of this matrix is empty, since otherwise by Lemma 3.7, there would be another direction in the tangent space of $\Sigma^k_{V_0}$ at V_1, which is not possible by our assumption that $V_0, \ldots, V_s$ span the tangent space of $\Sigma^k_{V_0}$ at V_1. Therefore there is a matrix C such that $C \cdot B = I$. This implies that $\nu_{s+1,1}, \ldots, \nu_{n,1}$ are linear combinations of $\nu_{s+1,0}, \ldots, \nu_{n0}$. By taking derivatives of $r^k_{1,\ldots,1} = 0$, we get

$$0 = \sum_{0 \leq t \leq n} r^{k+1}_{1,\ldots,1,t}\, \nu_{t0} + k \sum_{s+1 \leq t \leq n} r^k_{1,\ldots,1,t}\, \nu_{t1}.$$

By Lemma 4.5, $\nu_{n0} = 0$ and $\nu_{00}, \ldots, \nu_{n-1,0}$ are linearly independent.n [Notice that in Lemma 4.5, we proved the independence of these forms on $\mathcal{F}_X$. The same proof works here, by generic smoothness and the fact that we chose V_0 and V_1 to be general.] We get

$$r^{k+1}_{1,\ldots,1,1} = r^{k+1}_{1,\ldots,1,2} = \cdots = r^{k+1}_{1,\ldots,1,s} = 0$$

on $\mathcal{G}_X$. This implies that V_1 is in $\Sigma^{k+1}_{V_0}$ and the tangent space of Σ^{k+1} at V_1 is the same as the tangent space of $\Sigma^k_{V_0}$ at V_1. By continuing this argument, we get the desired result.

□

For later use, we reformulate the above theorem in the following corollary.

Corollary 3.8. *If X is a hypersurface in $\mathbb{P}^n$ such that the dimension of the family of lines passing through its general point is k, then the lines passing through the general point of X form a cone of degree at most $(n-k)!$ and it is contained in the proper intersection of $n-k$ polynomials of degree $1, 2, \ldots, n-k$ in $\mathbb{P}^n$.*

3.2 Singularities of the second fundamental form.

Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d given by the homogeneous polynomial F. Fix a point p in X and as before let Y^k_p be the hypersurface defined by equation (3.1). Denote by Z^k_p the intersection of Y^k_p and the embedded tangent plane to X at p, Y^1_p. It is a degree k hypersurface in $\mathbb{P}^{n-1}$.

Proposition 3.9. *For a general point p of X, the singular points of Z^2_p are contained in Z^k_p and are singular points of Z^k_p for $2 \leq k \leq d = \deg X$.*

To prove the proposition, we need a lemma. We skip the proof of the lemma since it is similar to the previous ones.

Lemma 3.10. *For a point $f = [V_0, \dots, V_n] \in \mathcal{F}_X$, V_j is a singular point of $Z^k_{V_0}$ if and only if*

$$r^k_{j,\dots,j,t}(f) = 0 \quad \textit{for} \quad 0 \le t \le n-1.$$

Proof of the proposition. Since Z^2_p is a quadric, the singular points of Z^2_p form a linear subvariety. By ([5], 2.6), this linear subvariety is contained in X and it is a fiber of the Gauss map. Let s be the dimension of the singular locus of Z^2_p.

We restrict our functions to those matrices $f = [V_0, \dots, V_n]$ in $\mathcal{F}_X$ such that $V_1, \dots V_s$ are singular points of $Z^2_{V_0}$. Since $V_1, V_2, \dots, V_s$ are in X, by Lemma 3.10,

$$r^t_{j,\dots,j} = 0 \quad 1 \le j \le s, 1 \le t \le d,$$

therefore,

$$0 = \mathrm{d}r_j = \sum_{0 \le t \le n-1} r_{jt}\, \omega_{t0} + \sum_{0 \le t \le n} r_t\, \omega_{tj} \qquad j = 1, \dots, s.$$

We know that $r_1 = r_2 = \cdots = r_{n-1} = 0$ since $V_1, \dots, V_{n-1}$ are in the tangent space of X at V_0. Also V_n is not in the tangent hyperplane because the matrix is invertible and hence $r_n \neq 0$. By the last lemma, $r_{j0} = \cdots = r_{j,n-1} = 0$, hence $\omega_{nj} = 0$. By Lemma 3.4,

$$\begin{aligned} 0 &= \mathrm{d}r_{jj} \\ &= \sum_{t=0}^{n} r^3_{j,j,t}\, \omega_{t0} + 2 \sum_{t=0}^{n} r^2_{j,t}\, \omega_{tj} \\ &= \sum_{t=0}^{n-1} r^3_{j,j,t}\, \omega_{t0}. \end{aligned}$$

The forms $\omega_{00}, \dots \omega_{n-1,0}$ are linearly independent. Hence

$$r_{j,j,0} = \cdots = r_{j,j,n-1} = 0.$$

This implies that V_j is a singular point of Y^3_p. By repeating this argument, we see that $V_1, \dots, V_s$ are singular point of all Y^k_p's. □

Chapter 4

Dimension of the Fano variety of lines on hypersurfaces

Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d given by the homogeneous polynomial F and $\mathcal{F}(X)$ the Fano variety of lines of X. Let $G(1,n)$ denotes the Grassmannian which parametrizes lines in $\mathbb{P}^n$. Since X is a hypersurface, it is easy to describe $\mathcal{F}(X)$ as a subscheme of $G(1,n)$. On $G(1,n)$ we have a universal rank 2 subbundle of $\mathcal{O}_{G(1,n)}^{n+1}$ such that its restriction to any point $[l] \in G(1,n)$ is identified with the rank 2 subspace of $k^{\oplus n+1}$ whose projective space is $l \subset \mathbb{P}^n$. If we denote by S this universal subbundle, then F gives rise to a section of $\mathrm{Sym}^d(S^\vee)$, and the (scheme theoretic) zero locus of this section is exactly $\mathcal{F}(X)$. Therefore the ideal sheaf of $\mathcal{F}(X)$ is locally generated by $d+1$ elements and if the corresponding global section of $\mathrm{Sym}^d(S^\vee)$ is regular, then the dimension of $\mathcal{F}(X)$ is $\dim G(1,n) - (d+1) = 2n-d-3$. We refer to the number $2n-d-3$ as the expected dimension of $\mathcal{F}(X)$.

Lemma 4.1. *For every hypersurface X of degree d in $\mathbb{P}^n$, $\dim \mathcal{F}(X) \geq 2n-d-3$. For a general X, $\mathcal{F}(X)$ has dimension $2n-d-3$ if $d \leq 2n-3$ and is empty otherwise.*

Proof. Let $G(1,n)$ be the Grassmannian of lines in $\mathbb{P}^n$ and H be the projective space parameterizing hypersurfaces of degree d in $\mathbb{P}^n$. Let $I \subset H \times G(1,n)$ be the incidence variety and p_1 and p_2 the projections to H and $G(1,n)$ respectively. For any line $l \subset \mathbb{P}^n$ there is a surjective map

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(l, \mathcal{O}(d)).$$

The kernel of this map corresponds to the hypersurfaces of degree d which contains l, so the inverse image of l under p_2 is a linear subvariety of codimension $d+1$ in H. Hence I is irreducible and smooth of codimension $d+1$.

If $d > 2n-3$, the above argument shows that $\dim I < \dim H$ and the fiber of p_1 is empty for a general hypersurface of degree d.

Now assume $d \leq 2n-3$. To prove the assertion, we need to show that p_1 is surjective. Since I is smooth, it is enough to show that the the map induced on Zariski tangent spaces is surjective for at least one point of I. Fix a line $l \in G(1,n)$ and assume it is given by $x_2 = x_3 = \cdots = x_n = 0$. The kernel of the map on the Zariski tangent spaces induced by p_1 at $(X,l) \in I$ is the tangent space to $p_1^{-1}(X) \simeq \mathcal{F}(X)$ at l, which is isomorphic to $H^0(l, N_{l/X})$. So it is enough to show that there exists a hypersurface X such that it contains l and $h^0(l, N_{l/X}) = \dim H - \dim I = 2n-d-3$. Any hypersurface X which contains l is given by an equation of the form $x_2 f_2 + x_3 f_3 + \cdots + x_n f_n = 0$. For such a hypersurface, we have an exact sequence

$$0 \longrightarrow N_{l/X} \longrightarrow \mathcal{O}_l(1)^{n-1} \xrightarrow{(f_2,\ldots,f_n)} \mathcal{O}_l(d) \longrightarrow 0.$$

It is easy to see that we can find $f_2, \ldots, f_n$ such that the map $H^0(l, \mathcal{O}_l(1)^{n-1}) \longrightarrow H^0(l, \mathcal{O}_l(d))$ is surjective and hence $h^0(l, N_{l/X}) = h^0(l, \mathcal{O}_l(1)^{n-1}) - h^0(l, \mathcal{O}_l(d)) = 2n-d-3$. □

As was mentioned before, when X is general, it can be shown that $\mathcal{F}(X)$ is smooth and hence for a general line $l \subset X$

$$N_{l/X} \simeq \begin{cases} \mathcal{O}_l^{d-1} \oplus \mathcal{O}_l(1)^{n-1-d} & \text{if } d \leq n-1, \\ \mathcal{O}_l^{2n-3-d} \oplus \mathcal{O}_l(-1)^{d-n+1} & \text{if } d \geq n-1. \end{cases}$$

When X is not general, this might fail. If X is the Fermat hypersurface of degree 4 in $\mathbb{P}^4$, then $\mathcal{F}(X)$ has 40 curves each with multiplicity 2 and the normal bundle of each line l is $\mathcal{O}_l(1) \oplus \mathcal{O}_l(-2)$ (see [2], Exercise 2.5).

In this chapter we will prove Theorem C.

Theorem 4.2 (Theorem C). *If $X \subset \mathbb{P}^n$ is any smooth Fano hypersurface of degree $d \leq 6$, then $\mathcal{F}(X)$ has the expected dimension $2n-d-3$.*

Reduction to the case of $n = d$. Before giving the proof of the theorem, we show that to prove Conjecture B and hence Theorem 4.2, it is enough to prove it for $d = n$.

Lemma 4.3. *If Conjecture B holds for $d = n$, then it holds for $d \leq n$.*

Proof. We assume that the conjecture holds for (d, m), $d \leq m+1$ and show that it holds for $(d, m+1)$. Let X be a smooth hypersurface of degree d in $\mathbb{P}^{m+1}$ and X' a general hyperplane section of X. Since X' is a smooth hypersurface of degree d in $\mathbb{P}^m$, by the induction hypothesis, $\dim \mathcal{F}(X') = 2m - d - 3$. Let $\mathcal{Y}$ be an irreducible component of $\mathcal{F}(X)$. By the following lemma, either a codimension two subvariety of $\mathcal{Y}$ lies in $\mathcal{F}(X')$ or all the lines corresponding to the points of $\mathcal{Y}$ pass through the same point x. In the latter case, all these lines are contained in the intersection of X and the tangent hyperplane at x. Hence $\dim \mathcal{Y} \leq m-1$ and the equality holds if and only if X is a hyperplane, therefore

$$\dim \mathcal{Y} \leq m - 2 \leq 2(m+1) - d - 3.$$

□

Lemma 4.4. *Let $\mathcal{Y}$ be an irreducible subvariety of $G(1,n)$ that intersects the family of lines in a given hyperplane in codimension greater than 2. Then all the lines corresponding to the points of $\mathcal{Y}$ pass through the same point.*

Proof. It is enough to show that every two lines of $\mathcal{Y}$ intersect. For a closed point $[l] \in \mathcal{Y}$ let $\Lambda_l \subset (\mathbb{P}^n)^*$ be the set of hyperplanes which contain l. It is a linear subvariety of dimension $n-2$. To show that l and l' intersect is equivalent to show that $\dim(\Lambda_l \cap \Lambda_{l'}) = n-3$. Let $\dim \mathcal{Y} = s$. By our assumption, every hyperplane which contains l, contains a family of dimension at least $s-1$ of lines in $\mathcal{Y}$. Let $I \subset \mathcal{Y} \times \Lambda_l$ be the incidence variety and p_1 and p_2 be the projections from I to $\mathcal{Y}$ and Λ_l. The dimension of any fiber of p_2 is at least $s-1$. Thus $\dim I \geq s - 1 + \dim \Lambda_l = s + n - 3$, and the dimension of any fiber of p_1 is at least $s + n - 3 - \dim \mathcal{Y} = n - 3$.

□

4.1 Proof of Theorem C for $d \leq 5$.

Lemma 4.5. *A smooth hypersurface of degree d in $\mathbb{P}^n$ is not covered by lines if $d \geq n$.*

Proof. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface which is covered by lines. Let $I \subset X \times \mathcal{F}(X)$ be the incidence variety and (q, l) be a general point of I. Consider the following commutative diagram:

$$\begin{array}{ccc} T_{I,(q,l)} & \xrightarrow{\alpha_2} & T_{\mathcal{F}(X),l} \simeq H^0(l, N_{l/X}) \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \phi} \\ T_{X,q} & \xrightarrow{\psi} & N_{l/X}|_q \end{array}$$

where α_1 and α_2 are the maps on the Zariski tangent spaces induced by the projection maps from I to X and $\mathcal{F}(X)$ and ψ is the map induced by the natural map from the tangent sheaf of X to the normal sheaf of l in X.

Let $N_{l/X} \simeq \mathcal{O}_l(a_1) \oplus \cdots \oplus \mathcal{O}_l(a_{n-2})$ be the decomposition of $N_{l/X}$ into line bundles. Since X is covered by lines, the dimension of the image of α_1 is $n-1$ and the dimension of the image of $\psi \circ \alpha_1$ is $n-2$ since the kernel of ψ is $T_{l,q}$, so $\dim \phi(H^0(l, N_{l/X})) = n-2$. This implies that each a_i is non-negative. On the other hand, by the following exact sequence, $a_1 + \cdots + a_{n-2} = n-1-d$.

$$0 \longrightarrow N_{l/X} \longrightarrow N_{l/\mathbb{P}^n} \longrightarrow \mathcal{O}_l(d) \longrightarrow 0$$

Thus we get $n - 1 - d \geq 0$.

□

Now we are ready to prove Theorem 4.2 for $d \leq 5$.

$n = d = 3$. The above lemma shows that X is not covered by lines and therefore there are only finitely many lines on X.

$n = d = 4$. By the above lemma, the union of all lines in X form a 2-dimensional subvariety. Any 2-dimensional subvariety of $\mathbb{P}^4$ which contains a 2-parameter family of lines is a linear subvariety. This shows that $\dim \mathcal{F}(X) = 1$ since X is smooth and cannot contain a $\mathbb{P}^2$ by the Lefschetz hyperplane theorem.

$n = d = 5$. Assume on the contrary that $\dim \mathcal{F}(X) \geq 3$ and let $\mathcal{Y}$ be an irreducible subvariety of $\mathcal{F}(X)$ whose dimension is 3. The lines corresponding to the closed points of $\mathcal{Y}$ cannot cover X by Lemma 4.5. Hence they sweep out a 3-dimensional subvariety X' of X. Consider a general projection of X' to $\mathbb{P}^4$. We get a hypersurface in $\mathbb{P}^4$ with a 1-parameter family of lines passing through its general point. By Corollary 3.8, the degree of the cone of lines passing through a general point of X' is at most 2. So the same is true for X'. Namely, the cone of lines passing through its general point has degree at most two and hence it is a cone over a rational curve. This implies that $\mathcal{Y}$ is covered by rational curves which is not possible by Theorem 2.1.

4.2 Proof of Theorem C for $d = 6$.

Outline of proof. Without loss of generality, we can assume that our base field is uncountable and by Lemma 4.3, we can assume $n = d = 6$. Let X be a smooth hypersurface of degree 6 in $\mathbb{P}^6$ such that $\dim \mathcal{F}(X) > 2n - d - 3 = 3$. For this to happen, there should exists a codimension one subvariety of X, denoted by X', with a 1-parameter family of lines passing through its general point. Let Σ be the cone of lines passing through a general point of X'. We use Theorem 3.3 to show that $\deg \Sigma \leq 6$ and that Σ is a component of the proper intersection of two hyperplanes, a quadric and a cubic in $\mathbb{P}^6$. Let C be a hyperplane section of Σ which does not pass through its vertex. We use the results of Chapter 2 and 3 to show that C is a non-rational curve of degree at most 6 which lies on a non-singular quadric surface in $\mathbb{P}^3$ (Step 1). We prove then that in each case deformations of a cone over such a curve in X cannot cover a codimension one subvariety of X (Step 2).

Step 1.

Assume on the contrary that $\dim \mathcal{F}(X) > 2n-d-3 = 3$. Let $\mathcal{Y}$ be an irreducible subvariety of $\mathcal{F}(X)$ such that $\dim \mathcal{Y} = 4$. Let $I \subset X \times \mathcal{Y}$ be the incidence variety and π_X and $\pi_{\mathcal{Y}}$ be the projections from I to X and $\mathcal{Y}$. Since $\mathcal{Y}$ is irreducible and the fibers of $\pi_{\mathcal{Y}}$ are lines, I is irreducible of dimension 5. Let $r = \dim \pi_X(I)$. Note that by Lemma 4.5, X is not covered by lines and hence $r \leq 4$. Also $r \geq 4$ for if $r = 3$, then the dimension of the fiber of a general point in $\pi_X(I)$ is 2-dimensional and hence $\pi_X(I)$ is a linear subvariety of dimension 3 in X. This is not possible since X is smooth. The rest is to argue that $r \neq 4$.

Convention. From now on, we assume that $X' = \pi_X(I)$, $r = \dim X' = 4$ and p is a general point of X'.

The family of lines in $\mathcal{Y} \subset \mathcal{F}(X)$ passing through p is one dimensional. Since the base field is uncountable, there is an irreducible component Σ of this family with the following property:

($*$) For a general point q of X', there is an irreducible cone of lines in X' passing through q with the same Hilbert polynomial as Σ.

Now we show that $\deg \Sigma \leq 6$. Let $\pi : X' \to \mathbb{P}$ be a general linear projection of X' into a 5-dimensional linear subvariety of $\mathbb{P}^6$ and Z the image of X' under π. Let $(x_0, x_1, \ldots, x_5)$ be a system of homogenous coordinates on $\mathbb{P}$ and H the equation of Z with respect to these coordinates. We can assume that $\pi(p) = (1; 0; \ldots; 0)$ and write H as

$$H = X_0^{r-1} H_1 + X_0^{r-2} H_2 + \cdots + H_r,$$

where r is the degree of H and H_i is a homogeneous polynomial of degree i in $X_1, \ldots X_5$. Let $Z^i \subset \mathbb{P}$ be the hypersurface defined by H_i. Since π is generically injective, the dimension of the family of lines in Z passing through $\pi(p)$ is one, and so by Corollary 3.8, $\dim(Z^1 \cap Z^2 \cap Z^3) = 2$. Since $\pi(\Sigma)$ is a component of this intersection, $\deg(\pi(\Sigma)) \leq 6$ and hence

$$\deg \Sigma = \deg \pi(\Sigma) \leq 6.$$

Moreover, if Γ denotes the embedded tangent plane to X at p, the above projection gives an isomorphism from Γ to $\mathbb{P}$. Therefore Σ sits in the proper intersection of three hypersurfaces of degrees 1,2 and 3 in Γ. Let C be a hyperplane section of Σ which does not pass through p. The above argument shows that C is a curve which lies in a $\mathbb{P}^3$ and is a component of the proper intersection of a quadric Q and a cubic T in $\mathbb{P}^3$.

Lemma 4.6. *With the above notation,*
(a) C is not rational.
(b) The quadric Q is irreducible and non-singular.

Proof. The first part follows from Theorem 2.1. If Q is reducible, then since C is in the intersection of Q and a cubic, it is either a line, a conic or a singular cubic in $\mathbb{P}^2$, and hence rational. This is not possible by part (a). If Q is irreducible and singular at a point q, then by Proposition 3.9, q is contained in T and it is a singular point of T. We can assume that $q = (1; 0; 0; 0)$ and Q is the zero locus of the polynomial $X_1^2 = X_2 X_3$. Hence T is given by a polynomial of the form $X_0G_1 + G_2$, where G_1 and G_2 are homogeneous polynomials of degree 2 and 3 in X_1, X_2, X_3. Therefore, the irreducible components of the intersection of Q and T are rational curves. In particular C is rational. Again, this is not possible by part (a).

□

Step 2.

Recall that X' is the union of all lines in X corresponding to the closed points of $\mathcal{Y}$ and p is a general point of X'. We assumed that there is a two dimensional cone of lines, denoted by Σ, passing through p.

Lemma 4.7. *The global sections of $\bigwedge^2 N_{\Sigma/X}$ generate a non-zero subspace at a general point of Σ.*

Proof. The proof is similar to the the proof of Lemma 4.5. Let $\mathcal{G}$ be the Hilbert scheme parameterizing closed subschemes of X with the same Hilbert polynomial as Σ. Let $I \subset X \times \mathcal{G}$ be the incidence variety.

Consider the following commutative diagram:

$$\begin{array}{ccc} T_{I,(q,\Sigma)} & \xrightarrow{\alpha_2} & T_{\mathcal{G},\Sigma} \simeq H^0(\Sigma, N_{\Sigma/X}) \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \phi} \\ T_{X,q} & \xrightarrow{\psi} & N_{\Sigma/X}|_q \end{array}$$

where q is a general point of Σ. Since $\dim X' = 4$, the image of α_1 is 4-dimensional and hence the image of $\psi \circ \alpha_2$ is 2-dimensional. The diagram is commutative, so the image of ϕ is 2-dimensional. □

By step 1, we know what Σ could possibly be. Hence we can try to compute $h^0(\Sigma, \bigwedge^2 N_{\Sigma/X})$ directly and get a contradiction using the above lemma.

Lemma 4.8. *If X is a smooth hypersurface of degree d in $\mathbb{P}^n$ and Y is a subvariety of X, then there is an exact sequence of normal sheaves on Y*

$$0 \longrightarrow N_{Y/X} \longrightarrow N_{Y/\mathbb{P}^n} \longrightarrow \mathcal{O}_Y(d) \longrightarrow 0.$$

Proof. It is enough to show that the sequence of conormal sheaves

$$\mathcal{I}_{X/\mathbb{P}^n} \otimes \mathcal{O}_Y \longrightarrow \mathcal{I}_{Y/P^n} \otimes \mathcal{O}_Y \longrightarrow \mathcal{I}_{Y/X} \otimes \mathcal{O}_Y \longrightarrow 0$$

is exact on the left and splits locally. Therefore, we can assume that X is a smooth affine hypersurface in $\mathbb{A}^n$ given by a polynomial F and Y is a subvariety of X. Let I and J be the ideals defining X and Y in $\mathbb{A}^n$. We want to show that the following sequence is exact on the left

$$I/IJ \xrightarrow{\phi} J/J^2 \xrightarrow{\psi} (J/I)/(J/I)^2 \longrightarrow 0. \tag{4.1}$$

If $\phi(\overline{FK}) = 0$, then

$$FK = G_1 H_1 + \cdots + G_m H_m$$

for some polynomials $G_1, \ldots, G_m, H_1, \ldots, H_m \in J$. Hence $\dfrac{\partial(FK)}{\partial x_i}(p) = 0$ for $1 \leq i \leq n$ and $p \in Y$. Since X is non-singular, $K(p) = 0$. Therefore $K \in J$ and $FK \in IJ$.

Next, we show that the above sequence splits locally. Since X is smooth, Ω^1_X is locally free and there is a locally splitting sequence of $\mathcal{O}_X$ modules

$$0 \longrightarrow I/I^2 \xrightarrow{d} \Omega^1_{\mathbb{A}^n} \otimes \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

Therefore locally there exists a map $s : \Omega^1_{\mathbb{A}^n} \otimes \mathcal{O}_X \longrightarrow I/I^2$ such that $s \circ d = \mathrm{id}_{I/I^2}$. Now, it is easy to see that the composition of the maps

$$J/J^2 \xrightarrow{d} \Omega^1_{\mathbb{A}^n} \otimes \mathcal{O}_Y \xrightarrow{s \otimes \mathrm{id}} I/I^2 \otimes \mathcal{O}_Y$$

splits sequence (4.1).

□

Using the above lemma, we get the following exact sequence of $\mathcal{O}_\Sigma$-modules

$$0 \longrightarrow N_{\Sigma/X} \longrightarrow N_{\Sigma/\mathbb{P}^6} \longrightarrow \mathcal{O}_\Sigma(6) \longrightarrow 0.$$

Since this sequence splits locally, we get two exact sequences

$$0 \longrightarrow (\bigwedge^3 N_{\Sigma/X})(-6) \longrightarrow (\bigwedge^3 N_{\Sigma/\mathbb{P}^6})(-6) \longrightarrow \bigwedge^2 N_{\Sigma/X} \longrightarrow 0, \tag{4.2}$$

and

$$0 \longrightarrow (\bigwedge^4 N_{\Sigma/X})(-12) \longrightarrow (\bigwedge^4 N_{\Sigma/\mathbb{P}^6})(-12) \longrightarrow (\bigwedge^3 N_{\Sigma/X})(-6) \longrightarrow 0. \qquad (4.3)$$

By step 1, $\deg \Sigma \leq 6$ and it is a cone over a non-rational curve. Hence $3 \leq \deg \Sigma \leq 6$. We analyze different cases separately. In each case, first, we get some information on $H^0(\Sigma, \bigwedge^2 N_{\Sigma/X})$ by applying the long exact sequence of cohomology to sequence (4.2) and then, we use Lemma 4.7 to get a contradiction. Note that if Σ is a complete intersection, then $N_{\Sigma/\mathbb{P}^6}$ is locally free of rank 4 and $\bigwedge^4 N_{\Sigma/X}(-12) = 0$. Therefore from sequence (4.3) we get $\bigwedge^3 N_{\Sigma/X}(-6) \simeq \bigwedge^4 N_{\Sigma/\mathbb{P}^6}(-12)$.

Case 1: $\deg \Sigma = 3$.

By Lemma 4.6, Q is irreducible and smooth. The curve C is a divisor of type (1, 2) on Q and therefore it is rational. This is not possible by Lemma 4.6.

Case 2: $\deg \Sigma = 4$.

The curve C is either of type $(1,3)$ or $(2,2)$ as a divisor on Q. The former cannot happen because a curve of type $(1,3)$ on a quadric is rational. In the latter case, C is a complete intersection of two hyperplanes and two quadric in $\mathbb{P}^6$ and

$$N_{\Sigma/\mathbb{P}^6} \simeq \mathcal{O}_\Sigma(2)^{\oplus 2} \oplus \mathcal{O}_\Sigma(1)^{\oplus 2}.$$

Therefore we get $h^1(\Sigma, \bigwedge^3 N_{\Sigma/X}(-6)) = h^1(\Sigma, \mathcal{O}_\Sigma(6)) = 0$ and $h^0(\Sigma, \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6)) = h^0(\Sigma, \mathcal{O}_\Sigma(-1)^{\oplus 2} \oplus \mathcal{O}_\Sigma(-2)^{\oplus 2}) = 0$. This implies that $h^0(\Sigma, \bigwedge^2 N_{\Sigma/X}) = 0$ which is not possible by Lemma 4.7.

Case 3: $\deg \Sigma = 5$.

In this case, C is of type $(2,3)$ and possibly singular. Let U be the complement of the vertex of Σ and $\pi : U \to C$ be the projection map. Since the sequences (4.2) and (4.3) split locally, we can divide the sheaves in these sequences by their torsions. So we get a short exact sequence

$$0 \longrightarrow \bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor} \longrightarrow \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6)/\mathrm{tor} \longrightarrow \bigwedge^2 N_{\Sigma/X}/\mathrm{tor} \longrightarrow 0. \qquad (4.4)$$

and an isomorphism

$$\bigwedge^4 N_{\Sigma/\mathbb{P}^6}(-12)/\mathrm{tor} \simeq \bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor}. \tag{4.5}$$

Now, we try to do the same as in the previous case. Namely, we try to compute $h^0(\Sigma, \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6)/\mathrm{tor})$ and $h^1(\Sigma, \bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor})$. In this case, we show that $h^0(U, \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6)|_U) = 0$ and there is a sheaf of $\mathcal{O}_\Sigma$-modules $\mathcal{G}$ and a map

$$\phi : \bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor} \longrightarrow \mathcal{G},$$

such that ϕ is an isomorphism on U and $h^1(\Sigma, \mathcal{G}) = 0$.

Let's first see why this implies that $\deg \Sigma \neq 5$. Since the sequence (4.4) splits locally, we can extend it to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor} & \to & \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6)/\mathrm{tor} & \to & \bigwedge^2 N_{\Sigma/X}/\mathrm{tor} & \to & 0 \\ & & \downarrow \phi & & \downarrow \psi & & \| & & \\ 0 & \to & \mathcal{G} & \to & \mathcal{G}' & \to & \bigwedge^2 N_{\Sigma/X}/\mathrm{tor} & \to & 0, \end{array}$$

where ψ is also an isomorphism on U. Since $h^1(\Sigma, \mathcal{G}) = 0$, any global section of $\bigwedge^2 N_{\Sigma/X}/\mathrm{tor}$ is the image of a global section of $\mathcal{G}'$ and hence zero because $H^0(U, \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6)|_U) = 0$ and ψ is an isomorphism on U. This contradicts Lemma 4.7.

Computing $H^0(\Sigma, \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6)/\mathrm{tor})$.

We show that $h^0(\Sigma, \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6)/\mathrm{tor}) = h^0(U, \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6)|_U) = 0$. Since Σ sits in a $\mathbb{P}^4$, there is a locally splitting exact sequence of normal sheaves

$$0 \longrightarrow N_{\Sigma/\mathbb{P}^4} \longrightarrow N_{\Sigma/\mathbb{P}^6} \longrightarrow \mathcal{O}_\Sigma(1)^{\oplus 2} \longrightarrow 0. \tag{4.6}$$

This gives us another exact sequence

$$0 \longrightarrow (\bigwedge^2 N_{\Sigma/\mathbb{P}^4}(1))^{\oplus 2} \longrightarrow \bigwedge^3 N_{\Sigma/\mathbb{P}^6} \longrightarrow N_{\Sigma/\mathbb{P}^4}(2) \longrightarrow 0.$$

Twist the above sequence with $\mathcal{O}_\Sigma(-6)$ and restrict the new sequence to U,

$$0 \longrightarrow (\bigwedge^2 N_{\Sigma/\mathbb{P}^4}(-5))^{\oplus 2}|_U \longrightarrow \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6)|_U \longrightarrow N_{\Sigma/\mathbb{P}^4}(-4)|_U \longrightarrow 0.$$

Recall that π is the projection from U to C. We can extend π to a map from the complement of the vertex of Σ in $\mathbb{P}^4$ to $\mathbb{P}^3$. This map is flat and hence $N_{\Sigma/\mathbb{P}^4}|_U \simeq \pi^* N_{C/\mathbb{P}^3}$. On the other hand, by Lemma 4.8, there is an exact sequence

$$0 \longrightarrow N_{C/Q} \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow \mathcal{O}_C(2) \longrightarrow 0. \tag{4.7}$$

Since $N_{C/Q}$ is a line bundle, $N_{C/\mathbb{P}^3}$ is locally free of rank 2. Also, we have $\deg N_{C/Q} = C^2 = 12$, therefore

$$\deg \bigwedge^2 N_{C/\mathbb{P}^3}(-5) = \deg N_{C/Q}(-3) = -3.$$

Hence none of $\bigwedge^2 N_{C/\mathbb{P}^3}(-5)$ and $N_{C/\mathbb{P}^3}(-4)$ has global sections. The result follows.

Computing $H^1(\Sigma, \bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor})$.

We prove that there exists a sheaf $\mathcal{G}$ with the desired property. Namely, there exists a map $\phi : \bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor} \to \mathcal{G}$, such that ϕ is an isomorphism on U and $h^1(\Sigma, \mathcal{G}) = 0$. First, notice that by (4.5) and (4.6)

$$\bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor} \simeq \bigwedge^4 N_{\Sigma/\mathbb{P}^6}(-12)/\mathrm{tor} \simeq \bigwedge^2 N_{\Sigma/\mathbb{P}^4}(-10)/\mathrm{tor}.$$

Lemma 4.9. *Let C be a curve of type $(2,3)$ on a smooth quadric Q in $\mathbb{P}^3$. There is an exact sequence of $\mathcal{O}_{\mathbb{P}^3}$-modules*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4)^{\oplus 2} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow i_*(\mathcal{O}_C) \longrightarrow 0.$$

Proof. Let P be the polynomial defining the quadric. We have

$$H^0(\mathbb{P}^3, \mathcal{I}_C(3)) \geq H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) - H^0(C, \mathcal{O}_C(3)) = 6.$$

So there are two degree 3 polynomials G_1 and G_2 such that C is the intersection of Q and the hypersurfaces defined by G_1 and G_2. Define the map $\phi : \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^3}$

to be given by polynomials (G_1, G_2, P).

Let T_1 and T_2 be the hypersurfaces defined by G_1 and G_2 respectively. The intersection of T_1 and Q is the union of C and a line which we can assume is given by $X_0 = X_1 = 0$. Hence there exist polynomials $L_i, R_i, i = 0, 1$ such that

$$\begin{aligned} X_0 G_2 &= L_0 G_1 + R_0 P, \\ X_1 G_2 &= L_1 G_1 + R_1 P. \end{aligned} \tag{4.8}$$

Define $\psi : \mathcal{O}_{\mathbb{P}^3}(-4)^{\oplus 2} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-2)$ to be given by the pair $((L_0, -X_0, R_0), (L_1, -X_1, R_1))$. It is easy to see that these generate all the relations between G_1, G_2 and P and that ψ is injective.

□

The sequence in the previous lemma gives a similar exact sequence for Σ in $\mathbb{P}^4$. By restricting these exact sequences to C and Σ and dualizing them, we get the following exact sequences for C and Σ

$$0 \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow \mathcal{O}_C(3)^{\oplus 2} \oplus \mathcal{O}_C(2) \longrightarrow \mathcal{O}_C(4)^{\oplus 2},$$
$$0 \longrightarrow N_{\Sigma/\mathbb{P}^4} \longrightarrow \mathcal{O}_\Sigma(3)^{\oplus 2} \oplus \mathcal{O}_\Sigma(2) \longrightarrow \mathcal{O}_\Sigma(4)^{\oplus 2}.$$

Hence, there is an injective morphism

$$0 \longrightarrow \bigwedge^2 N_{\Sigma/\mathbb{P}^4}/\mathrm{tor} \xrightarrow{\phi} \mathcal{O}_\Sigma(5)^{\oplus 2} \oplus \mathcal{O}_\Sigma(6).$$

Denote the quotient of ϕ by $\mathcal{F}$. There is an exact sequence

$$0 \longrightarrow \bigwedge^2 N_{\Sigma/\mathbb{P}^4}(-10)/\mathrm{tor} \xrightarrow{\phi} \mathcal{O}_\Sigma(-5)^{\oplus 2} \oplus \mathcal{O}_\Sigma(-4) \xrightarrow{\eta} \mathcal{F}(-10) \longrightarrow 0. \tag{4.9}$$

Claim:

a) $h^0(U, \mathcal{F}(-10)|_U) = 0$.

b) $h^1(\Sigma, \mathcal{O}_\Sigma(-5)^{\oplus 2} \oplus \mathcal{O}_\Sigma(-4)) = 0$.

Proof of the Claim: In a similar way as the proof of Lemma 4.9, we get the following exact sequence

$$0 \longrightarrow N_{C/Q} \longrightarrow \mathcal{O}_C(3)^{\oplus 2} \longrightarrow \mathcal{O}_C(4)^{\oplus 2}.$$

On the other hand, from sequence (4.7), we get $\bigwedge^2 N_{C/\mathbb{P}^3} \simeq N_{C/Q}(2)$. If we twist the above exact sequence with $\mathcal{O}_C(2)$, we get another short exact sequence

$$0 \longrightarrow \bigwedge^2 N_{C/\mathbb{P}^3} \xrightarrow{\phi'} \mathcal{O}_C(5)^{\oplus 2} \xrightarrow{\eta'} \mathcal{O}_C(6)^{\oplus 2}.$$

Denote the images of η' by $\mathcal{F}'$. Since $H^0(C, \mathcal{F}'(-10))$ is a subspace of $H^0(C, \mathcal{O}_C(-4)^{\oplus 2})$, $\mathcal{F}'(-10)$ does not have any global section and the same it true for its pullback under π. To conclude that $\mathcal{F}(-10)|_U$ does not have any global section, we observe that there is an exact sequence

$$0 \longrightarrow \pi^*\mathcal{F}'(-10) \longrightarrow \mathcal{F}(-10)|_U \longrightarrow \mathcal{O}_U(-4) \longrightarrow 0.$$

This finishes the proof of the first part of the claim. To prove the second part of the claim, we use the following lemma.

Lemma 4.10. *Let $\Sigma \subset \mathbb{P}^4$ be a cone over a curve C of type $(2,3)$ on a smooth quadric Q in $\mathbb{P}^3$. Then $H^1(\Sigma, \mathcal{O}_\Sigma(m)) = 0$ for every m.*

Proof. Since $\mathcal{I}_{C/Q} \simeq \mathcal{O}_Q(-2,-3)$, $H^1(Q, \mathcal{I}_C(m)) = 0$ for every m ([6], III, Ex. 5.6). This implies that the map $H^0(Q, \mathcal{O}_Q(m)) \longrightarrow H^0(C, \mathcal{O}_C(m))$ is surjective for every m and hence the map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m)) \longrightarrow H^0(C, \mathcal{O}_C(m))$$

is surjective for every m. The above map factors through the map $H^0(\Sigma, \mathcal{O}_\Sigma(m)) \to H^0(C, \mathcal{O}_C(m))$, so this map is also surjective. Now, look at following exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma(m-1) \longrightarrow \mathcal{O}_\Sigma(m) \longrightarrow i_*(\mathcal{O}_C(m)) \longrightarrow 0.$$

The map $H^1(\Sigma, \mathcal{O}_\Sigma(m-1)) \to H^1(\Sigma, \mathcal{O}_\Sigma(m))$ is injective for every m, and for large enough m, $H^1(\Sigma, \mathcal{O}_\Sigma(m)) = 0$. The assertion follows. □

This finishes the proof of the claim. Now, in sequence (4.9), let $\mathcal{E}$ be the subsheaf of $\mathcal{F}(-10)$ generated by the global sections of $\mathcal{F}(-10)$. It is supported at the vertex of Σ. Let $\mathcal{G}$ be the inverse image of $\mathcal{E}$ under η. It is a subsheaf of $\mathcal{O}_\Sigma(-5)^{\oplus 2} \oplus \mathcal{O}_\Sigma(-4)$, which makes the following sequence exact

$$0 \longrightarrow \bigwedge^2 N_{\Sigma/\mathbb{P}^4}(-10)/\mathrm{tor} \xrightarrow{\phi} \mathcal{G}' \longrightarrow \mathcal{E} \longrightarrow 0.$$

By the above claim, it is evident that $H^1(\Sigma, \mathcal{G}) = 0$ and ϕ is an isomorphism on U.

Case 4: $\deg \Sigma = 6$.

In this case, Σ is a complete intersection of two hyperplanes, a quadric and a cubic in $\mathbb{P}^6$, and

$$N_{\Sigma/\mathbb{P}^6} = \mathcal{O}_\Sigma(1)^{\oplus 2} \oplus \mathcal{O}_\Sigma(2) \oplus \mathcal{O}_\Sigma(3).$$

Hence, we have $h^1(\Sigma, \bigwedge^4 N_{\Sigma/\mathbb{P}^6}(-12)) = h^1(\Sigma, \mathcal{O}_\Sigma(-5)) = 0$ and also $h^0(\Sigma, \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6)) = h^0(\Sigma, \mathcal{O}_\Sigma^{\oplus 2} \oplus \mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(-2)) = 2$. Therefore we have

$$h^0(\Sigma, \bigwedge^2 N_{\Sigma/X}) = 2.$$

Lemma 4.11. *For every non zero $\alpha \in H^0(\Sigma, N_{\Sigma/X})$, there exists $\beta \in H^0(\Sigma, N_{\Sigma/X})$ such that $\alpha \wedge \beta \neq 0$.*

Proof. It is enough to show that there exist α and β such that $\alpha \wedge \beta \neq 0$. This is clear by Lemma 4.7. □

Now, we show that every non-zero global section of $N_{\Sigma/X}$ has finitely many zeros. Let $\alpha \in H^0(\Sigma, N_{\Sigma/X})$ be a non-zero global section. By last lemma, there exists $\beta \in H^0(\Sigma, N_{\Sigma/X})$ such that $\alpha \wedge \beta \neq 0$. Recall that we have an exact sequence:

$$0 \longrightarrow \bigwedge^3 N_{\Sigma/X}(-6) \xrightarrow{\phi} \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6) \longrightarrow \bigwedge^2 N_{\Sigma/X} \longrightarrow 0.$$

We showed that $h^1(\Sigma, \bigwedge^3 N_{\Sigma/X}(-6)) = 0$. Thus, $\alpha \wedge \beta$ is the image of some r in $H^0(\Sigma, \bigwedge^3 N_{\Sigma/\mathbb{P}^6}(-6)) = H^0(\Sigma, \mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(-2))$. Without loss of generality we can assume $r = (1, 0, 0, 0)$. Let L_1, L_2, G, K be the polynomials defining the hyperplanes, the quadric and the cubic in $\mathbb{P}^6$ whose intersection is Σ, and F the polynomial defining X. There are homogeneous polynomials H_1, H_2, R, P such that

$$F = H_1 L_1 + H_2 L_2 + RG + PK.$$

The map ϕ is given by (H_1, H_2, R, P), thus if $\alpha(q) = 0$ then $H_2(q) = R(q) = P(q) = 0$. The set $\{q \in \Sigma | H_2(q) = R(q) = P(q) = 0\}$ is zero-dimensional, since otherwise there will be a point in the intersection of this subvariety and the hypersurface $H_1 = 0$ and this point will be a singular point of X. Therefore α has finitely many zeros.

Since $h^0(\Sigma, N_{\Sigma/X}) \geq 4$ and $h^0(\Sigma, \bigwedge^2 N_{\Sigma/X}) = 2$, there are independent global sections α and β such that $\alpha \wedge \beta = 0$. Let C' be a hyperplane section of Σ which does not contain any zero of α. We have $(\alpha \wedge \beta)|_{C'} = 0$ and α is no where zero. Hence there exist a constant r such that $\alpha|_{C'} = r\beta|_{C'}$. Therefore $\alpha - r\beta$ is a global section of $H^0(\Sigma, N_{\Sigma/X})$ having a one dimensional set of zeros and so by the previous argument it is zero. It contradicts our assumption that α and β are linearly independent.

Chapter 5

Quintic threefolds with many lines

In this chapter, we use Hodge theory to give another proof of Theorem 4.2 for $d = 5$ assuming that our base field is $\mathbb{C}$. By Lemma 4.3, it is enough to consider only the case $n = d = 5$. So let X be a smooth quintic fourfold. We want to show that $\dim \mathcal{F}(X) = 2$. The method of proof is to take a general pencil of hyperplane sections $X_{t, t \in \mathbb{P}^1}$ of X, and show that the Abel-Jacobi map for X_t is zero. This, as we will see, implies that X_t cannot contain a 1-parameter family of lines.

Abel-Jacobi maps. We recall some well-known facts on Abel-Jacobi maps. Let Y be a smooth threefold and $J(Y)$ the intermediate Jacobian of Y which is the complex torus defined as

$$J(Y) = H^{3,0}(Y)^\vee \oplus H^{2,1}(Y)^\vee / H_3(Y),$$

where $H_3(Y) = H_3(Y, \mathbb{Z})/(\text{torsion})$ is embedded in $H^{3,0}(Y)^\vee \oplus H^{2,1}(Y)^\vee$ via integration along cycles. The Abel-Jacobi map is a map from the group of algebraic 1-cycles on Y which are homologous to zero into $J(Y)$ and is defined as follows. An algebraic 1-cycle η, which is homologically equivalent to zero, is the boundary of a topological 3-cycle V. The image of η under Abel-Jacobi map sends a 3-form ω to $\int_V \omega$.

Let C be an integral curve parameterizing a family $Z_{c, c \in C}$ of algebraic 1-cycles on Y, and fix a base point $c_0 \in C$. There is map from C to the group of algebraic 1-cycles homologous to zero on Y sending c to $Z_{c_0} - Z_c$. Hence we get a map

$$\alpha : C \longrightarrow J(Y).$$

This map is called the Abel-Jacobi map associated to C.

The derivative of the Abel-Jacobi map. Let C be as above and assume it is non-singular. In this case, it can be shown that α is a holomorphic map between complex manifolds and it is possible to compute the derivative of this map at a point corresponding to a smooth curve on Y.

The derivative of the Abel-Jacobi map at c is a map

$$\mathrm{d}\alpha : T_{C,c} \longrightarrow T_{J(Y),0} \simeq H^{3,0}(Y)^\vee \oplus H^{2,1}(Y)^\vee$$

which is the composition of the Kodaira-Spencer map

$$\beta : T_{C,c} \longrightarrow H^0(Z_c, N_{Z_c/Y})$$

and a map

$$\gamma : H^0(Z_c, N_{Z_c/Y}) \longrightarrow H^{3,0}(Y)^\vee \oplus H^{2,1}(Y)^\vee \simeq H^0(X, \Omega_Y^3)^\vee \oplus H^1(Y, \Omega_Y^2)^\vee.$$

In ([3], p.28-29) it is proved that if Z_c is a smooth curve on Y, then the transpose of γ

$$\gamma^\vee : H^0(Y, \Omega_Y^3) \oplus H^1(Y, \Omega_Y^2) \longrightarrow H^0(Z_c, N_{Z_c/Y})^\vee \simeq H^1(Z_c, N_{Z_c/Y}^\vee \otimes \Omega_{Z_c}^1)$$

is given as follows. It is zero on the first summand and the composition of two maps on the second summand

$$H^1(Y, \Omega_Y^2) \longrightarrow H^1(Z_c, \Omega_Y^2 \otimes \mathcal{O}_{Z_c}) \longrightarrow H^1(Z_c, N_{Z_c/Y}^\vee \otimes \Omega_{Z_c}^1),$$

where the second map is derived from the exact sequence

$$0 \longrightarrow \bigwedge^2 N_{Z_c/Y}^\vee \longrightarrow \Omega_Y^2 \otimes \mathcal{O}_{Z_c} \longrightarrow \Omega_{Z_c}^1 \otimes N_{Z_c/Y}^\vee \longrightarrow 0. \tag{5.1}$$

Theorem 5.1. *If Y is a smooth quintic threefold which contains a 1-parameter family of lines, then the Abel-Jacobi map of Y is non-zero.*

Proof. Let C be a curve parameterizing a 1-dimensional family of lines on Y. By passing to the normalization of C, we may assume it is non-singular. Fix a point $c \in C$ corresponding to a line $l \subset Y$. To prove that the Abel-Jacobi map $\alpha : C \to J(Y)$ is non-zero, we show that its derivative at c is non-zero. By the above argument, it is enough to show that the composition of the maps

$$H^1(Y, \Omega_Y^2) \xrightarrow{\phi} H^1(l, \Omega_Y^2|_l) \xrightarrow{\psi} H^1(l, \Omega_l^1 \otimes N_{l/Y}^\vee) \tag{5.2}$$

is non-zero. There is an exact sequence of normal bundles on l

$$0 \longrightarrow N_{l/Y} \longrightarrow N_{l/\mathbb{P}^4} \simeq \mathcal{O}_l(1)^{\oplus 3} \longrightarrow N_{Y/\mathbb{P}^4} \otimes \mathcal{O}_l \simeq \mathcal{O}_l(5) \longrightarrow 0.$$

Therefore, we have $\bigwedge^2 N_{l/Y} \simeq \mathcal{O}_l(-2)$ and hence $H^1(l, \bigwedge^2 N_{l/Y}^\vee) = 0$. Thus by applying the long exact sequence of cohomology to sequence (5.1), we conclude that ψ is an isomorphism.

Now, we show that ϕ is non-zero. Consider the following exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-5) \longrightarrow \Omega_{\mathbb{P}^4}|_Y \longrightarrow \Omega_Y \longrightarrow 0,$$

and its third exterior power

$$0 \longrightarrow \Omega_Y^2 \longrightarrow \Omega_{\mathbb{P}^4}^3|_Y \otimes \mathcal{O}_Y(5) \longrightarrow \Omega_Y^3 \otimes \mathcal{O}_Y(5) \simeq \mathcal{O}_Y(5) \longrightarrow 0. \tag{5.3}$$

By restricting sequence (5.3) to l, we get a commutative diagram

$$\begin{array}{ccc} H^0(Y, \mathcal{O}_Y(5)) & \xrightarrow{\eta} & H^1(Y, \Omega_Y^2) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \phi} \\ H^0(l, \mathcal{O}_l(5)) & \xrightarrow{\eta'} & H^1(l, \Omega_Y^2|_l) \end{array}$$

Since γ is surjective, we only need to show that the map η' is non-zero. To prove that η' is non-zero, restrict sequence (5.3) to l and observe that that $h^1(l, \Omega_{\mathbb{P}^4}^3(5)|_l) = 0$, so it is enough to show that $h^1(l, \Omega_Y^2|_l) > 0$. Let $N_{l/Y} \simeq \mathcal{O}_l(a_1) \oplus \mathcal{O}_l(a_2)$. We know that $a_1 + a_2 = -2$, and at least one of them is positive, since there is a 1-parameter family of lines on Y. Therefore, there are two possibilities for $N_{l/Y}$, it is isomorphic to either $O_l \oplus \mathcal{O}_l(-2)$ or $\mathcal{O}_l(1) \oplus \mathcal{O}_l(-3)$. In each case, $h^1(l, \Omega_l \otimes N_{l/Y}^\vee) > 0$. The assertion now follows from applying long exact sequence of cohomology to sequence (5.1).

□

Proof of Theorem 4.2 for $d = 5$. Let $X \subset \mathbb{P}^5$ be a smooth hypersurface of degree 5. Take a general pencil of hyperplane sections $X_{t,\, t\in\mathbb{P}^1}$ of X. We claim that there are at most finitely many lines on X_t for a general $t \in \mathbb{P}^1$. To prove the claim, we use the following theorem.

Theorem 5.2. *([4], 14.2) Let $X \subset \mathbb{P}^{2m+1}$ be a smooth hypersurface of degree $\geq 2 + (3/(m-1))$ and $X_{t,\, t\in\mathbb{P}^1}$ a general hyperplane section of X. Then for a general t, the map $H_{2m-1}(V, \mathbb{Z}) \longrightarrow H_{2m-1}(X_t, \mathbb{Z})$ is zero, where V is any codimension $m-1$ subvariety of X_t.*

Choose a general $t \in \mathbb{P}^1$. Let $Y = X_t$ and assume on the contrary that Y contains a 1-parameter family of lines. Let C be a curve parameterizing such a family and $\alpha : C \longrightarrow J(Y)$ the Abel-Jacobi map. The map α induces a map

$$\alpha_* : H_1(C, \mathbb{Z}) \longrightarrow H_1(J(Y), \mathbb{Z}) = H_3(Y, \mathbb{Z}).$$

The family $l_{c,\, c\in C}$ traces a surface V in Y and the image of α_* lies in the image of $H_3(V, \mathbb{Z}) \longrightarrow H_3(Y, \mathbb{Z})$. Therefore by Theorem 5.2, α_* is zero. To conclude that α is zero, observe that there is a factorization of α

$$C \longrightarrow \mathrm{Alb}(C) \longrightarrow J(C),$$

and hence the induced factorization of α_*

$$H_1(C, \mathbb{Z}) \xrightarrow{\simeq} H_1(\mathrm{Alb}(C), \mathbb{Z}) \xrightarrow{\alpha_*} H_1(J(C), \mathbb{Z}).$$

□

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