Adiabatic Limit and Szegő Projections

by

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Abstract

Given a smooth closed manifold X and an adapted complex structure on T^*X near the zero section, the behavior of the Szegő projection S_ϵ on the co-sphere bundle $\mathbb{S}_\epsilon^*X$ as the radius ϵ decreases to 0 is investigated. It is shown that, provided $\dim X \geq 3$, the family S_ϵ, $0 < \epsilon < \epsilon_0$, can be understood up to $\epsilon = 0$ as an element of Ψ_{aH}^*, an algebra of operators on $\mathbb{S}^*X \times [0, \epsilon_0)$ constructed by combining the calculus of the Heisenberg pseudodifferential operators and the adiabatic limit formalism of Mazzeo and Melrose. The limit at $\epsilon = 0$ of an element $A \in \Psi_{aH}^*$ is a family A_x, $x \in X$, of translation invariant Heisenberg operators on $\mathbb{R}^n \times \mathbb{S}^{n-1}$. Most importantly, Ψ_{aH}^* provides the appropriate setting for proving that, for small ϵ, the push-forward map sending smooth CR functions on $\mathbb{S}_\epsilon^*X$ to their fiber average is an isomorphism onto $C^\infty(X)$, a question earlier studied by Hörmander, Lebeau, Boutet de Monvel and Guillemin, and Epstein and Melrose.

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Contents

Introduction

The cotangent bundle of a smooth closed manifold X carries, near the 0-section, complex structures for which $X \hookrightarrow T^*X$ is a totally real submanifold. Furthermore, Guillemin and Stenzel [9], and Lempert and Szőke [12] proved that, once a real-analytic structure and a real-analytic metric h on X are chosen, there exists a unique complex structure near X in T^*X adapted both to the canonical symplectic structure on T^*X and to the metric h (see §4.1). The co-ball bundles $\mathbb{B}^*_\epsilon X$ of small radius ϵ, $0 < \epsilon < \epsilon_0$, called Grauert tubes, are then strictly pseudoconvex. Moreover, the induced strictly pseudoconvex CR structure on their boundaries, the co-sphere bundles $\mathbb{S}^*_\epsilon X$, has the same underlying contact structure as the one induced by the symplectic structure on T^*X.

On a general CR manifold one has, in analogy with the $\overline{\partial}$-operator on complex manifolds, the $\overline{\partial}_b$-complex of Kohn and Rossi and its associated Laplacian, $\Box_b$. For embeddable strictly pseudoconvex CR manifolds, as for example the co-sphere bundles $\mathbb{S}^*_\epsilon X$ are, the kernel of $\overline{\partial}_b$ on functions is infinite dimensional and the Szegő projection S is the orthogonal projection on it. It turns out that both S and the parametrices for $\Box_b$ are pseudodifferential operators of type $(\frac{1}{2},\frac{1}{2})$, for which the main pseudodifferential techniques fail. This has led in turn to the construction of the Heisenberg algebra of pseudodifferential operators, Ψ^*_H, defined in the more general setting of contact manifolds. Such algebras have been constructed from different perspectives by Dynin [4], Beals and Greiner [1], Taylor [17] and Epstein and Melrose [6]. The Toeplitz subalgebra of operators of the form SAS with $A \in \Psi^*_H$ has been earlier extended in a different direction in the work of Boutet de Monvel and Guillemin [2].

We are interested in the behavior of the Szegő projections S_ϵ on $\mathbb{S}^*_\epsilon X$ as $\epsilon \searrow 0$ and in understanding its uniform limit in an appropriate sense (Theorem 4.3). We use this to prove that the push-forward map sending smooth CR functions on $\mathbb{S}^*_\epsilon X$, or equivalently holomorphic functions on the Grauert tube smooth up to the boundary, to their fiber average:

$$P : S_\epsilon C^\infty(\mathbb{S}^*_\epsilon X) \longrightarrow C^\infty(X), \quad Pu(x) = \int_{\mathbb{S}^*_{\epsilon,x}X} u(x,y)\, dy, \quad x \in X$$

is an isomorphism for ϵ small enough (Theorem 4.8). Boutet de Monvel and Guillemin [2] and Guillemin [8] proved that P is Fredholm in this setting and, earlier, Hörmander [10] and Lebeau [11] had showed P is an isomorphism for $X = \mathbb{R}^n$ and for $X = \mathbb{S}^n$ respectively. Epstein and Melrose [5] considered a closely related map, namely integration over the closed balls $\mathbb{B}^*_{\epsilon,x}X$ instead of the spheres $\mathbb{S}^*_{\epsilon,x}X$, and proved it is an isomorphism for ϵ small by methods similar to the ones here.

Interest in the map P arises from the correspondence it gives between pseudodifferential operators on X and Toeplitz operators on $\mathbb{S}^*X$ and its connection to index theory. Also, after extending P to distributions, one has $WF(f) = \operatorname{singsupp} u \subset \mathbb{S}^*X$ for pairs of distributions $f \in C^{-\infty}(X)$ and $u \in S_\epsilon C^{-\infty}(\mathbb{S}^*X)$ with $Pu = f$.

If $\mathbb{S}^*X = T^*X \setminus \mathbf{0}/\mathbb{R}_+$ denotes the canonical cosphere bundle with its canonical contact structure, then each S_ϵ is in $\Psi^0_H(\mathbb{S}^*X)$. This is a consequence of choosing an adapted complex structure on T^*X.

Consequently, we prefer to think of the family S_ϵ as a family of Szegő projections associated to a smooth family of CR structures on $\mathbb{S}^*X$. Geometrically, this might be seen as expanding each tube of radius ϵ to a fixed radius, or, equivalently, as keeping the metric on the fiber constant while blowing up the metric on the base and hence making the fibers fan out. This scenario suggests that, at $\epsilon = 0$, the problem becomes localized over the fiber, that is the geometry degenerates as $\epsilon \searrow 0$, at each $x \in X$, to a tube $T_xX \times \mathbb{S}^*_xX$. Similarly, the limit of S_ϵ at $\epsilon = 0$ can be understood as a family of Szegő projections S_x, $x \in X$, on $T_xX \times \mathbb{S}^*_xX \cong \mathbb{R}^n \times \mathbb{S}^{n-1}$, invariant under translations in the $\mathbb{R}^n$ variable.

The above process is called an adiabatic limit. To make it precise we combine the Heisenberg calculus with the adiabatic limit formalism of Mazzeo and Melrose [13] and Epstein and Melrose [5] into an algebra $\Psi^*_{aH}(\mathbb{S}^*X)$ of adiabatic Heisenberg operators on $\mathbb{S}^*X \times [0, \epsilon_0)$. The importance of the technique rests on the fact that now one can obtain information for $\epsilon > 0$ by understanding the limit $\epsilon = 0$, i.e. the model case $X = \mathbb{R}^n$. The construction is global in the fibers and thus it is also well suited for studying the push-forward map above.

We begin by recalling in Chapter 1 the basics of the Heisenberg calculus on a contact manifold and the interpretation of its principal symbol in the isotropic calculus on a symplectic vector bundle. We continue in Chapter 2 with the study of the model case $X = \mathbb{R}^n$, i.e. of the translation invariant Heisenberg operators on $\mathbb{R}^n \times \mathbb{S}^{n-1}$, or, more suggestively, on the co-sphere bundle $\mathbb{S}^*V = V \times \mathbb{S}V^*$ of a vector space V. We denote the space of these by $\Psi^*_{tiH}(V \times \mathbb{S}V^*)$. As pointed out earlier, this will be our limiting case. Explicit description of the Szegő projection, the invertibility of P and the invertibility of $\Box_b$ on $(0,1)$-forms (provided $n \geq 3$) are all readily obtained in this setting.

In Chapter 3 the algebra $\Psi^*_{aH}(\mathbb{S}^*X)$ of adiabatic Heisenberg operators on $\mathbb{S}^*X \times [0, \epsilon_0)$ is constructed. Operators $A \in \Psi^*_{aH}$ restrict for each $\epsilon > 0$ to a Heisenberg operator on $\mathbb{S}^*X$, $A_\epsilon \in \Psi^*_H(\mathbb{S}^*X)$, and at $\epsilon = 0$ to a family, called the normal operator, of translation invariant Heisenberg operators $N(A)_x \in \Psi^*_{tiH}(T_xX \times \mathbb{S}^*_xX)$, $x \in X$. A main property, and indeed the main reason for constructing $\Psi^*_{aH}(\mathbb{S}^*X)$, is that for an elliptic A, the invertibility of its normal operator implies the invertibility of A for small ϵ (Proposition 3.15).

In the last Chapter we return to the complex setting, interpret the CR structures on the boundaries of the Grauert tubes as an adiabatic family, and conclude, using the results of Chapter 3 and Chapter 2, the invertibility of the associated $\Box^{0,1}_b$ (for $n \geq 3$) in Ψ^*_{aH}. One of the main results (Theorem 4.3), the extension of the family S_ϵ to $\epsilon = 0$ as an element of

Ψ^*_{aH}, follows from this.

The fact that P is Fredholm for $\epsilon > 0$ follows by showing that SP^*PS is an elliptic Toeplitz operator on $\mathbb{S}^*X$ and similarly that PSP^* is an elliptic pseudodifferential operator on X. We extend the statement to the adiabatic setting and derive the invertibility of SP^*PS and PSP^* for ϵ small. Finally this proves that, for small ϵ, P is an isomorphism (Theorem 4.8).

Chapter 1

Preliminaries and notation

Following Epstein and Melrose [6], we recall the algebra $\Psi_H^*(M)$ of Heisenberg pseudodifferential operators on a contact manifold M. By analogy with the classical calculus of pseudodifferential operators where radial homogeneity on T^*X is used to define the space of symbols, parabolic homogeneity with respect to the contact direction $L \subset T^*M$ is used here to introduce a space of parabolic symbols and to define $\Psi_H^*(M)$. Such algebras of parabolic pseudodifferential operators had been previously constructed by Dynin [4], Beals and Greiner [1] and Taylor [17]. In [6] both the classical and the Heisenberg pseudodifferential operators are included into a larger algebra $\Psi_{eH}^*(M)$, the algebra of extended Heisenberg operators. Earlier Boutet de Monvel and Guillemin [2] considered the Toeplitz subalgebra and extended it to the more general setting of symplectic cones.

1.1 Heisenberg pseudodifferential operators

We start by recalling briefly the *radial compactification* $\overline{V}$ of a vector space V, the *quadratic compactification* ${}^q\overline{V}$ of V, and, if $L \subset V$ is a subspace, the *L-parabolic compactification* ${}^L\overline{V}$. They are all closed balls obtained by attaching smoothly to V a sphere at infinity.

For the radial and quadratic compactifications, the sphere at infinity $\mathbb{S}V = (V \setminus 0)/\mathbb{R}^+$ is defined by the radial $\mathbb{R}^+$-action $v \to \tau v$, $\tau \in \mathbb{R}^+$ and the smooth structure near the boundary is generated by smooth homogeneous functions on $V \setminus 0$ of degree 0 and -1 in the case of $\overline{V} = V \sqcup \mathbb{S}V$ and by those of degree 0 and -2 in the case of ${}^q\overline{V} = V \sqcup \mathbb{S}V$. If ρ and ρ_q are boundary defining functions for $\overline{V}$ and ${}^q\overline{V}$ respectively, then the corresponding spaces of one-step polyhomogeneous symbols of order m are:

$$S^m(V) = \rho^{-m}C^\infty(\overline{V}), \qquad S_q^m(V) = \rho_q^{-m/2}C^\infty({}^q\overline{V})$$

and we clearly have $S_q^m(V) \subsetneq S^m(V)$.

On the other hand, for ${}^L\overline{V}$ we use the parabolic $\mathbb{R}^+$-action on $L \times V/L$

$$(l, v) \to (\tau^2 l, \tau v), \tau \in \mathbb{R}^+$$

to define the sphere at infinity ${}^L\mathbb{S}V = (L \times V/L \setminus (0,0))/\mathbb{R}^+$ and set ${}^L\overline{V} = V \sqcup {}^L\mathbb{S}V$. Given a decomposition $V = L \oplus H$, the smooth homogeneous functions on $V \setminus 0$ of degree 0 and -1 under the parabolic $\mathbb{R}^+$-action on $V = L \oplus H$ generate the smooth structure on ${}^L\overline{V}$ near the boundary. Moreover the smooth structure is independent of the choice of H and so is the intersection of ${}^L\mathbb{S}V$ with the closure of H in ${}^L\overline{V}$, the equatorial sphere. The space of one-step polyhomogeneous L-parabolic symbols of order m is defined by

$$S_L^m(V) = \rho^{-m} C^\infty({}^L\overline{V})$$

for ρ a boundary defining function.

All these definitions can be extended invariantly to the case of vector bundles. We refer to [6] for details on this as well as for most of the material in this chapter.

Let now M^{2n+1} be a contact manifold, compact without boundary, assumed to be transversally oriented. Denote by

$$L \subset T^*M \quad \text{and} \quad H \subset TM \tag{1.1}$$

the (oriented) contact line bundle and its annihilator, the contact hyperplane bundle. A contact form α is a non-vanishing section of L and it satisfies the non-degeneracy condition $\alpha \wedge (d\alpha)^n \neq 0$; this is also equivalent with $d\alpha_p$ being a symplectic form on H_p at each $p \in M$.

The space of L-parabolic symbols on T^*M is defined as above by

$$S_L^m(T^*M) = \rho^{-m} C^\infty({}^L\overline{T^*M}) \tag{1.2}$$

ρ being a boundary defining function for ${}^L\overline{T^*M}$. Their inverse Fourier transforms will give, via the Riemann-Weyl fibration, the Schwartz kernels of the Heisenberg operators.

For a choice of a metric h on M the Riemann-Weyl fibration is the diffeomorphism

$$W : TM \supset U \to V \subset M \times M, \quad W(x,v) = (\exp_x(v/2), \exp_x(-v/2))$$

from a neighborhood of the 0-section to a neighborhood of the diagonal $\Delta = \mathrm{Diag}(M)$.

The fiber-wise Fourier transform is invariantly defined from rapidly decreasing half-densities on a vector bundle $V \to M$ to half-densities on the dual vector bundle:

$$\mathcal{F} : \dot{C}^\infty(V, \Omega^{\frac12}) \longrightarrow \dot{C}^\infty(V^*, \Omega^{\frac12}),$$
$$(\mathcal{F}\phi)(v^*) = \int_{V_p} e^{-iv^*(v)}\, \phi(v)\, |\omega_p^k/k!|^{1/2}, \qquad p \in M,\ v^* \in V_p^*$$

and extends to tempered distributions:

$$\mathcal{F} : C^{-\infty}(V, \Omega^{\frac12}) \longrightarrow C^{-\infty}(V^*, \Omega^{\frac12});$$

here ω_p is the canonical symplectic form on $V_p \times V_p^*$ and k the rank of V. In constructing Ψ_H^* we will take $V = TM$ and, as usual, use the half-density $|\omega^{2n+1}/(2n+1)!|^{1/2}$ defined by the canonical symplectic form ω on T^*M to trivialize $\Omega^{\frac{1}{2}}(T^*M)$.

For a compact manifold with corners N we use the notations

$$\dot{C}^\infty(N, E) \quad \text{and} \quad C^{-\infty}(N, E) = \left(\dot{C}^\infty(N, E^* \otimes \Omega)\right)'$$

for the space of smooth sections vanishing with all derivatives at ∂N and for the space of extendible distributional sections of E respectively. Taking N to be any of the compactifications of V discussed above gives the same space of functions on V and therefore no particular choice for a compactification is made in the definition of $\mathcal{F}$.

The space $\Psi_H^m(M; \Omega^{\frac{1}{2}}, \Omega^{\frac{1}{2}})$ of *Heisenberg operators of order* $m \in \mathbb{R}$ consists of operators

$$A : C^\infty(M, \Omega^{\frac{1}{2}}) \longrightarrow C^\infty(M, \Omega^{\frac{1}{2}})$$

with Schwartz kernels smooth outside the diagonal

$$A \in C^{-\infty}(M \times M, \Omega^{\frac{1}{2}}), \quad \text{singsupp } A \subset \Delta$$

and such that

$$\mathcal{F} W^*(\chi A) \in S_L^m(T^*M) = \rho^{-m} C^\infty({}^L\overline{T^*M}) \tag{1.3}$$

for some choice of a metric h in the definition of W and for any $\chi \in C_c^\infty(M \times M)$, supp $\chi \subset V$ and $\chi \equiv 1$ near Δ.

The *principal symbol* $\sigma_m(A) = [a] \in S_L^m / S_L^{m-1}$, where $a = \mathcal{F} W^*(\chi A)$, is independent of the choices made above and gives a short exact sequence:

$$\mathbf{0} \longrightarrow \Psi_H^{m-1} \hookrightarrow \Psi_H^m \xrightarrow{\sigma_m} S_L^m / S_L^{m-1} \longrightarrow \mathbf{0}. \tag{1.4}$$

We will give next a second interpretation to the space S_L^m / S_L^{m-1}. The product # on S_L^m / S_L^{m-1} for which σ_m is a homomorphism will be postponed to the next section, described there in the context of the isotropic calculus.

Since L is trivial, ${}^L\mathbb{S}^*M = \partial({}^L\overline{T^*M})$ splits globally in two hemisphere bundles

$${}^L\mathbb{S}^*M = {}^L\mathbb{S}_+^*M \cup {}^L\mathbb{S}_-^*M,$$

with their interiors identified with the hyperplane bundle

$$\begin{aligned} W = T^*M/L \otimes L^{-\frac{1}{2}} &\xrightarrow{\cong} \text{int}({}^L\mathbb{S}_\pm^*M), \\ w = v \otimes l^{-\frac{1}{2}} &\longrightarrow [v, l], \end{aligned} \tag{1.5}$$

where l in the expression of w is chosen to be in the positive and respectively negative direction of L. Given $l \in L^\pm$, the notation $l^{-\frac{1}{2}}$ is used for the element in $L^{-\frac{1}{2}}$ defined by $l \in C^\infty(L^* \setminus 0)$, $l^{-\frac{1}{2}}(l^*) = \pm |l^*(l)|^{-\frac{1}{2}}$.

The maps (1.5) extend as smooth diffeomorphism

$$ {}^q\overline{W}_\pm \longrightarrow {}^L\mathbb{S}^*_\pm M $$

from the quadratic compactification of W, ${}^q\overline{W}$, to the upper and lower hemisphere; the signs specify the identification (1.5) used.

Most importantly, the principal symbol $\sigma_m(A)$ splits in two components corresponding to the two hemispheres and we have the identification

$$ \begin{aligned} S^m_L(T^*M)/S^{m-1}_L(T^*M) &\longrightarrow \rho_q^{-\frac{m}{2}} C^\infty({}^q\overline{W}_+, L^{-\frac{m}{2}}) \oplus \rho_q^{-\frac{m}{2}} C^\infty({}^q\overline{W}_-, L^{-\frac{m}{2}}) \\ [a] &\longrightarrow (a_+, a_-), \quad a_+|_{\mathbb{S}W} \equiv a_-|_{\mathbb{S}W}, \end{aligned} \tag{1.6} $$

the equality of a_+ and a_- being in Taylor series at $\mathbb{S}W$. Above, $L \to {}^q\overline{W}$ is the lift of the contact line bundle $L \to M$ to the hyperplane bundle $W \to M$.

Heisenberg operators acting between sections of complex vector bundles E and F on M:

$$ A : C^\infty(M, E) \longrightarrow C^\infty(M, F) $$

identified as always with their Schwartz kernels:

$$ A \in C^{-\infty}(M \times M, F \boxtimes (E^* \otimes \Omega)) = C^{-\infty}(M \times M, \mathrm{Hom}(E \otimes \Omega^{-1}, F)) $$

are defined by:

$$ \Psi^m_H(M; E, F) = \Psi^m_H(M; \Omega^{\frac12}, \Omega^{\frac12}) \otimes_{C^\infty(M\times M)} C^\infty(M \times M, \mathrm{Hom}(E \otimes \Omega^{-\frac12}, F \otimes \Omega^{-\frac12})). $$

The vector bundles $\mathrm{Hom}(E, F)$ and $E \boxtimes F$ over $M \times M$ are those with fibers

$$ \mathrm{Hom}(E,F)_{p,p'} = \hom(E_{p'}, F_p), \quad (E \boxtimes F)_{p,p'} = E_p \otimes F_{p'}, \qquad (p, p') \in M \times M $$

and we have $\mathrm{Hom}(E, F) \cong F \boxtimes E^*$.

The corresponding range for the principal symbol map is the space of

$$ (a_+, a_-) \in \rho_q^{-\frac{m}{2}} C^\infty({}^q\overline{W}_+, L^{-\frac{m}{2}} \otimes \hom(E,F)) \oplus \rho_q^{-\frac{m}{2}} C^\infty({}^q\overline{W}_-, L^{-\frac{m}{2}} \otimes \hom(E,F)) \tag{1.7} $$

with $a_+|_{\mathbb{S}W} \equiv a_-|_{\mathbb{S}W}$. Again, as in (1.6), $L^{-\frac{m}{2}} \otimes \hom(E, F)$ is the lift from M to W.

1.2 Isotropic algebras

Let (W^{2n}, ω) be a symplectic vector space. The *isotropic* and the *quadratic isotropic algebras* on W are the spaces of symbols:

$$
\begin{aligned}
&\Psi^*_{\text{is}}(W) = \bigcup_{m\in\mathbb{R}} \Psi^m_{\text{is}}(W), \qquad \Psi^*_{\text{Is}}(W) = \bigcup_{m\in\mathbb{R}} \Psi^m_{\text{Is}}(W),\\
&\Psi^m_{\text{is}}(W) = \rho^{-m} C^\infty(\overline{W}), \qquad \Psi^m_{\text{Is}}(W) = \rho_q^{-m/2} C^\infty({}^q\overline{W})\\
&\Psi^m_{\text{Is}}(W) \subset \Psi^m_{\text{is}}(W)
\end{aligned} \tag{1.8}
$$

with the non-commutative product:

$$
a\#b\,(w) \;=\; \pi^{-2n} \int_{W\times W} e^{2i\omega(w_1,w_2)}\, a(w+w_1)\, b(w+w_2)\, |\omega^n_{w_1}\omega^n_{w_2}/(n!)^2|, \tag{1.9}
$$

$a, b \in \Psi^*_{\text{is}}(W)$. The smoothing elements in these algebras are the Schwartz functions on W, $\Psi^{-\infty}_{\text{is}}(W) = \Psi^{-\infty}_{\text{Is}}(W) = \dot{C}^\infty(\overline{W}) = \dot{C}^\infty({}^q\overline{W})$; they form an ideal in both Ψ^*_{is} and Ψ^*_{Is}.

It is sometimes useful to represent these algebras as algebras of operators on $\mathcal{S}(V)$ for $V \subset W$ a Lagrangian subspace. More precisely, the choice of a Lagrangian complement, U, gives a symplectic isomorphism:

$$
W = V \oplus U \longrightarrow V \oplus V^*, \qquad w = v + u \longrightarrow (v,\, \omega(\cdot, u)|_V),
$$

where on $V \oplus V^*$ the symplectic form $\omega_0((v_1, v_1^*), (v_2, v_2^*)) = v_2^* \cdot v_1 - v_1^* \cdot v_2$ is used. This identifies $\Psi^*_{\text{is}}(W) \cong \Psi^*_{\text{is}}(V \oplus V^*)$ and $\Psi^*_{\text{Is}}(W) \cong \Psi^*_{\text{Is}}(V \oplus V^*)$ which can now be understood as full symbols spaces for algebras of operators on V, $\Psi^*_{\text{iso}}(V)$ and $\Psi^*_{\text{Iso}}(V)$ respectively:

$$
\Psi^m_{\text{iso}}(V) \underset{q_W}{\overset{\sigma_W}{\rightleftarrows}} \Psi^m_{\text{is}}(W), \qquad \Psi^m_{\text{Iso}}(V) \underset{q_W}{\overset{\sigma_W}{\rightleftarrows}} \Psi^m_{\text{Is}}(W)
$$

The above correspondences are given by the Weyl symbol and quantization maps:

$$
\begin{aligned}
A(v, v') \;&=\; (2\pi)^{-n} \int_{V^*} e^{i(v-v')\cdot v^*} a(\frac{v+v'}{2}, v^*)\, dv^*,\\
a(v, v^*) \;&=\; \int_V e^{iv'\cdot v^*} A(v + v'/2, v - v'/2)\, dv'
\end{aligned}
$$

and the product $\#$ corresponds to the composition of operators, $q_W(a\#b) = q_W(a) \circ q_W(b)$. Even if no particular representation as above if fixed, we will still refer to the elements of $\Psi^*_{\text{is}}(W)$ and $\Psi^*_{\text{Is}}(W)$ as isotropic and quadratic isotropic operators respectively.

Let now $J \in \hom(W)$, $J^2 = -Id$, be a complex structure on W compatible with the symplectic structure, that is

$$
\omega(Ju, Jv) = \omega(u, v) \quad \text{and} \quad \omega(u, Ju) > 0, \quad \forall\, u, v \in W
$$

or equivalently

$$H(u,v) = \omega(u, Jv), \quad u, v \in W \quad \text{is a positive inner product on } W.$$

For any Lagrangian subspace $V \subset W$ this gives an orthogonal Lagrangian decomposition $V \oplus JV = W$; in fact we can choose linear orthogonal coordinates (x, ξ) on W and identify (W, ω) with $(\mathbb{R}^{2n}_{x,\xi}, dxd\xi)$, $J\partial_x = \partial_\xi$.

The harmonic oscillator, $H = \Delta_x + x^2$, is a second order isotropic operator on $\mathbb{R}^n$, $H \in \Psi^2_{\text{Iso}}(\mathbb{R}^n) \subset \Psi^2_{\text{iso}}(\mathbb{R}^n)$, with symbol $h(x,\xi) = |x|^2 + |\xi|^2 \in \Psi^2_{\text{Is}}(\mathbb{R}^{2n}) \subset \Psi^2_{\text{is}}(\mathbb{R}^{2n})$ defined in fact invariantly on W:

$$h \in \Psi^2_{\text{Is}}(W) \subset \Psi^2_{\text{is}}(W), \qquad h(u) = H(u,u) = \omega(u, Ju).$$

The eigenvalues of H are of the form $n + 2k$, $k \in \mathbb{N}$. The ground state, the eigenspace associated to the lowest eigenvalue, is one-dimensional and the projection on it is a smoothing isotropic operator on $\mathbb{R}^n$. Its symbol is defined again invariantly on W:

$$s = 2^n e^{-h} \in \Psi^{-\infty}_{\text{is}}(W).$$

and satisfies:

$$s\#s = s, \ (h-n)\#s = s\#(h-n) = 0.$$

We will refer to h and s as the *harmonic oscillator* and the *ground state projection.*

The isotropic algebras Ψ^*_{is} and Ψ^*_{Is} can be constructed in the more general setting of symplectic vector bundles $W \to M$ by defining the product (1.9) fiber-wise; however, since a global Lagrangian splitting might not exist, we do not have in general the representations as operators, Ψ^*_{iso} and Ψ^*_{Iso}. Vector bundle coefficients can also be added. A complex vector bundle E on M lifts to W to be trivial along the fibers; taking sections of (the lift of) E in (1.2) the spaces $\Psi^*_{\text{is}}(W; E)$ and $\Psi^*_{\text{Is}}(W; E)$ are defined and the product (1.9) is well-defined $\Psi^*_{\text{is}}(W; E) \# \Psi^*_{\text{is}}(W; F) = \Psi^*_{\text{is}}(W; E \otimes F)$ for $F \to M$ a second vector bundle.

We return now to the principal symbol map in the Heisenberg calculus, (1.6) and (1.7). First, the hyperplane bundle $W = T^*M/L \otimes L^{-\frac{1}{2}} \to M$ is canonically a symplectic vector bundle. To see this, recall that each section of L gives H in (1.1) a symplectic vector bundle structure and the same holds for the duals, L^{-1} and T^*M/L. Then on the fibers of W the form $\omega(v_1 \otimes l^{-1/2}, v_2 \otimes l^{-1/2}) = (l^{-1/2} \otimes l^{-1/2})(v_1, v_2)$, $l > 0$ gives a symplectic structure and so does $-\omega$; we write W_+ or W_- to indicate the symplectic structure used.

The symbol space in the short exact sequence (1.4) can now be identified, as in (1.6), with the set of

$$(a_+, a_-) \in \Psi^m_{\text{Is}}(W_+; L^{-\frac{m}{2}}) \oplus \Psi^m_{\text{Is}}(W_-; L^{-\frac{m}{2}}), \quad a_+|_{\mathbb{S}W} = a_-|_{\mathbb{S}W} \tag{1.10}$$

and most important, as shown in [6]:

$$\sigma_{m+m'}(AB)_\pm = \sigma_m(A)_\pm \# \sigma_{m'}(B)_\pm, \quad A \in \Psi^m_H, \ B \in \Psi^{m'}_H.$$

In the case of vector bundle coefficients, $\Psi^m_{\mathrm{Is}}(W_\pm, L^{-\frac{m}{2}} \otimes \hom(E,F))$ should be used in (1.10).

1.3 Hermite ideals and Toeplitz operators

Taking Heisenberg operators with symbols vanishing rapidly at the lower or upper hemisphere we get the *upper and lower Hermite ideals*:

$$\mathcal{I}^*_\pm(M) = \{A \in \Psi^*_H(M) : \mathcal{F}\,W^*(\chi A) \equiv 0 \text{ at } {}^L\mathbb{S}^*_\mp M\}.$$

Their principal symbol satisfies:

$$\sigma_m(A)_\pm \in \Psi^{-\infty}_{\mathrm{Is}}(W_\pm; L^{-\frac{m}{2}}),\ \ \sigma_m(A)_\mp = 0, \quad A \in \mathcal{I}^m_\pm(M).$$

Given a family of ground state projections $s \in \Psi^{-\infty}_{\mathrm{Is}}(W_+)$ associated to a compatible complex structure on W a *generalized Szegő projection* on M is an operator

$$S \in \mathcal{I}^0_+(M) \text{ with } \sigma_0(S) = s,\ S^2 = S$$

and self-adjoint for some trivialization of the density bundle. Such operators exist; in fact their space has countable many connected components and the relative index map of Epstein and Melrose [6], R-Ind$(S, S') = \mathrm{Ind}(S'S)$, gives a relative labeling of the components.

The *Toeplitz algebras* associated to a generalized Szegő projection S are given by:

$$\mathcal{T}_S(M) = \{A \in \mathcal{I}^*_+(M) : A = SA = AS\} = \{SBS \in \mathcal{I}^*_+(M) : B \in \Psi^*_H(M)\}$$

Chapter 2

The model case; translation invariant Heisenberg operators on $\mathbb{R}^n \times \mathbb{S}^{n-1}$

Consider the co-sphere bundle $\mathbb{S}^*V = V \times \mathbb{S}V^* = V \times (V^* \setminus \{\mathbf{0}\})/\mathbb{R}_+$ of a n dimensional real vector space V with its canonical contact structure and denote by L the oriented contact line bundle over $\mathbb{S}^*V$, $L \subset T^*(\mathbb{S}^*V)$; the orientation used here is such that the positive half-line is given by:

$$L^+_{x,y} = \{(\xi, 0) : [\xi] = y\} \subset V^* \times T^*_y(\mathbb{S}V^*), \quad (x, y) \in V \times \mathbb{S}V^*.$$

Note that this contact structure is translation invariant in the V-variables, or, in other words, L is a pull-back from $\mathbb{S}V^*$ to $V \times \mathbb{S}V^*$. In light of this, we will mostly regard L as a line bundle over the sphere $\mathbb{S}V^*$.

For simplicity, we will occasionally identify $\mathbb{S}^*V$ with $\mathbb{R}^n \times \mathbb{S}^{n-1}$ by introducing linear coordinates on V. Later in the section, our choice of CR structure on $\mathbb{S}^*V$ will be obtained via by embedding $\mathbb{S}^*V$ in $\mathbb{C}^n$ as the tube of radius 1 around $\mathbb{R}^n$ in the standard euclidean metric. However, only the contact structure is needed for the Heisenberg calculus and hence our coordinate free approach.

2.1 Definition and properties

Briefly, translation invariant Heisenberg operators on $\mathbb{S}^*V = V \times \mathbb{S}V^*$ are just Heisenberg operators (in the sense that their symbols are L-parabolic symbols on $T^*(\mathbb{S}^*V)$) that are invariant under translations in the V-variables and subject to an additional decay condition on their Schwartz kernels, needed for composition. The translation invariance property we require for our operators:

$$A : \mathcal{S}(V \times \mathbb{S}V^*) \longrightarrow \mathcal{S}(V \times \mathbb{S}V^*) \tag{2.1}$$

forces their Schwartz kernels to be of the form $K(x, y, x', y') = K_A(x - x', y, y')$, i.e. convolution in the V-variables, formally written as:

$$Au(x, y) = \int_{V \times \mathbb{S}V^*} K_A(x - x', y, y')\, u(x', y')\, dx' \nu_{y'},$$

with $\nu_{y'}$ a density on $\mathbb{S}V^*$. Consequently, their symbols $a(y, \xi, \eta)$, $y \in \mathbb{S}V^*$, $\xi \in V^*$, $\eta \in T^*_y(\mathbb{S}V^*)$ will be independent of the x-variable, $x \in V$.

Remark 2.1. By the above, we prefer to view the symbols as being defined on the vector bundle:

$$V^* \oplus T^*(\mathbb{S}V^*) \longrightarrow \mathbb{S}V^* \tag{2.2}$$

instead of the cotangent bundle $T^*(V \times \mathbb{S}V^*) \to V \times \mathbb{S}V^*$ that we would normally use. The contact line bundle L, regarded by translation invariance as a bundle over the sphere $\mathbb{S}V^*$ instead of the whole space $\mathbb{S}^*V = V \times \mathbb{S}V^*$, is a subbundle of (2.2).

More precisely, following the recipe in [6], a parabolic symbol of order m will be, as in (1.2), an element of:

$$S^m_L = \rho^{-m} C^\infty(\overline{^L V^* \oplus T^*(\mathbb{S}V^*)}), \tag{2.3}$$

where $\overline{^L V^* \oplus T^*(\mathbb{S}V^*)}$ is the fiber-wise L-parabolic compactification of the bundle (2.2).

Consider the fiber Fourier transform $\mathcal{F}$ between tempered distributions on $V \oplus T(\mathbb{S}V^*)$ and tempered distributions on $V^* \oplus T^*(\mathbb{S}V^*)$ and the Riemann-Weyl fibration

$$W : V \times U_1 \to V \times U_2, \quad W = (id, W_{\mathbb{S}})$$

where

$$W_{\mathbb{S}} : T(\mathbb{S}V^*) \supset U_1 \to U_2 \subset \mathbb{S}V^* \times \mathbb{S}V^*$$

is a Riemann-Weyl fibration for some metric on $\mathbb{S}V^*$.

Definition 2.2. *The space $\Psi^m_{tiH}(V \times \mathbb{S}V^*)$ of translation invariant Heisenberg operators of order m on $\mathbb{S}^*V = V \times \mathbb{S}V^*$ consists of the operators (2.1) with kernels*

$$A \in C^{-\infty}(V \times \mathbb{S}V^* \times \mathbb{S}V^*),$$

smooth outside the 'diagonal' $\{0\} \times \mathrm{Diag}(\mathbb{S}V^*)$, rapidly decreasing at infinity in V and near the diagonal satisfying :

$$\mathcal{F}W^*(\chi A) \in S^m_L \tag{2.4}$$

for any cut-off function χ with $\operatorname{supp}\chi \subset U_2$ and $\chi \equiv 1$ near $\mathrm{Diag}(\mathbb{S}V^*)$.

As in the compact case, Heisenberg operators between sections of vector bundles can be considered, $\Psi^m_{tiH}(V \times \mathbb{S}V^*; E, F)$. However, here the bundles $E, F \to V \times \mathbb{S}V^*$ are assumed to be pull-backs of bundles on the sphere $\mathbb{S}V^*$, also denoted E and F.

Define the *full symbol* $\sigma(A)$ and the *principal symbol* $\sigma_m(A)$ as the class of $\mathcal{F}W^*(\chi A)$ in $S^m_L/S^{-\infty}_L$ and respectively in S^m_L/S^{m-1}_L. The principal symbol is invariantly defined and gives a short exact sequence:

$$\mathbf{0} \longrightarrow \Psi^{m-1}_{tiH}(V \times \mathbb{S}V^*) \hookrightarrow \Psi^m_{tiH}(V \times \mathbb{S}V^*) \xrightarrow{\sigma_m} S^m_L/S^{m-1}_L \longrightarrow \mathbf{0}. \tag{2.5}$$

As usual, this has an interpretation in terms of the isotropic calculus on the symplectic vector bundle

$$W_\pm = (V^* \oplus T^*(\mathbb{S}V^*))/L \otimes L^{-\frac{1}{2}} \to \mathbb{S}V^*$$

the sign specifying, as in §1.2, the symplectic structure on the fibers of $W_\pm$.

We also have an algebra structure:

Proposition 2.3. *The space of translation invariant Heisenberg operators $\Psi^*_{tiH}(V \times \mathbb{S}V^*)$ is closed under composition*

$$\Psi^m_{tiH} \circ \Psi^{m'}_{tiH} \subset \Psi^{m+m'}_{tiH}$$

and the principal symbol map in (2.5) is a graded algebra homomorphism:

$$\sigma_{m+m'}(AB) = \sigma_m(A) \# \sigma_{m'}(B), \quad A \in \Psi^m_{tiH},\, B \in \Psi^{m'}_{tiH}.$$

Proof. Note that any $A \in \Psi^m_{tiH}(V \times \mathbb{S}V^*)$ can be decomposed as $A = A_0 + A_1$ with $A_1 \in \Psi^{-\infty}_{tiH} \equiv \mathcal{S}(V \times \mathbb{S}V^* \times \mathbb{S}V^*)$ and $A_0 \in \Psi^m_{tiH}$ with compact support, $A_0 \in C_0^{-\infty}(V \times \mathbb{S}V^* \times \mathbb{S}V^*)$. The proposition follows as in the compact case once we know that $\Psi^{-\infty}_{tiH}$ is an ideal in Ψ^*_{tiH}, which in turn follows as for classical pseudodifferential operators. □

Most of the definitions in the compact case can be repeated here without change. We mention the upper and lower *Hermite ideals*, denoted $\mathcal{I}^m_+(V \times \mathbb{S}V^*)$ and $\mathcal{I}^m_-(V \times \mathbb{S}V^*)$, defined as the space of operators in Ψ^m_{tiH} with full symbols vanishing to infinite order at the lower and upper hemisphere respectively. Also, given a choice of a compatible complex structure J on the fibers of $W \to \mathbb{S}V^*$, a field of harmonic oscillators are obtained. The *generalized Szegő projections* are then fixed as those self-adjoint projections $S \in \mathcal{I}^0_+(V \times \mathbb{S}V^*)$ having as principal symbol $s \in \Psi^{-\infty}_{\mathrm{Is}}(W_+)$, the projections onto the ground states of the above harmonic oscillators.

Given $A \in \Psi^m_{tiH}(V \times \mathbb{S}V^*)$ denote by $\widehat{A}(\xi, y, y')$ the Fourier transform of its kernel $A(x, y, y')$ in the x-variable and by $\widehat{A}(\xi)$, $\xi \in V^*$, the operator on $\mathbb{S}V^*$ having as Schwartz kernel $\widehat{A}(\xi, y, y')$, $y, y' \in \mathbb{S}V^*$. By analogy with the suspended algebra of Melrose, we will call $\widehat{A}(\cdot)$ the *indicial family* of A and have:

Lemma 2.4. *For an operator $A \in \Psi^m_{tiH}(V \times \mathbb{S}V^*)$, the indicial family*

$$\widehat{A} \in C^\infty(V^*, \Psi^m(\mathbb{S}V^*))$$

is a smooth family of pseudodifferential operators of order m on the sphere $\mathbb{S}V^*$. In partic-

ular

$$\widehat{A} \in C^\infty(V^*, \Psi^{-\infty}(\mathbb{S}V^*)) \qquad \text{for } A \in \mathcal{I}^*_\pm,$$
$$\widehat{R} \in \mathcal{S}(V^*, \Psi^{-\infty}(\mathbb{S}V^*)) \qquad \text{for } R \in \Psi^{-\infty}_{tiH}.$$

Proof. Follows directly from the symbol estimates for the symbol of A. □

Remark 2.5. A more precise description of the indicial families than provided by the previous lemma can be obtained. Such a description is not needed here, but it is useful in understanding the space of generalized Szegő projections, a questions that will be taken up elsewhere.

For an elliptic operator $A \in \Psi^m_{tiH}$, i.e. with principal symbol invertible as isotropic operators, the standard construction of a parametrix gives a remainder $R \in \Psi^{-\infty}_{tiH}$. Using the indicial family a little bit more can be achieved:

Lemma 2.6. *Given $A \in \Psi^m_{tiH}$ elliptic, there exist left and right parametrices $P, P' \in \Psi^{-m}_{tiH}$ such that the following is true for the remainders $R = Id - PA$ and $R' = Id - AP'$:*

$$\widehat{R}, \widehat{R'} \in C^\infty_0(V^*, \Psi^{-\infty}(\mathbb{S}V^*)).$$

Consequently, $\widehat{A}(\xi) \in \Psi^m(\mathbb{S}V^*)$ is invertible for $\xi \in V^*$ large.

Proof. Standard arguments yield a left-parametrix $B \in \Psi^{-m}_{tiH}$, $BA = I - E$ with remainder $E \in \Psi^{-\infty}_{tiH}$. Passing to indicial families, i.e. taking Fourier transform in the x-variables, gives:

$$\widehat{B}(\xi)\widehat{A}(\xi) = I - \widehat{E}(\xi), \quad \forall \xi \in V^*; \quad \widehat{E} \in \mathcal{S}(V^*, \Psi^{-\infty}(\mathbb{S}V^*));$$

the same notation, I, is being used for both the identity operator in Ψ^0_{tiH} and the identity in $\Psi^0(\mathbb{S}V^*)$.

For ξ large, $|\xi| > r'$, the operator norm of $\widehat{E}(\xi)$ acting on $L^2(\mathbb{S}V^*)$is small enough to make $I - \widehat{E}(\xi)$ invertible with inverse $I - \widehat{F}(\xi)$. The operators $\widehat{F}(\xi)$ are smoothing as follows from $\widehat{F}(\xi) = -\widehat{E}(\xi) - \widehat{E}(\xi)^2 + \widehat{E}(\xi)\widehat{F}(\xi)\widehat{E}(\xi)$ and, after extending smoothly to $|\xi|$ small, they form a family $\widehat{F} \in C^\infty(V^*, \Psi^{-\infty}(\mathbb{S}V^*))$. In fact $\widehat{F}(\xi)$ is rapidly decreasing as $|\xi| \to \infty$, i.e. $\widehat{F}$ is the indicial family of a smoothing operator $F \in \Psi^{-\infty}_{tiH}$. To see this choose a perturbation $E_1 \in \Psi^{-\infty}_{tiH}$ of E, having a compactly supported associated family $\widehat{E_1}(\cdot) \in \mathcal{S}(\mathbb{R}^n_\xi, \Psi^{-\infty}(\mathbb{S}^{n-1}))$, $\operatorname{supp} \widehat{E_1}(\cdot) \subset \{|\xi| < r''\}$, such that $I - E + E_1$ is invertible (this can be arranged by taking $\widehat{E_1}$ to equal $\widehat{E}$ on a large ball, an so making the L^2 norm of the kernel $E - E_1$ small). The inverse of $I - E + E_1$ is of the form $I - F_1$ with $F_1 \in \Psi^{-\infty}_{tiH}$. Note that $\widehat{F}(\xi) = \widehat{F_1}(\xi)$ for $|\xi| > \max\{r', r''\}$ and hence $F \in \Psi^{-\infty}_{tiH}$.

Taking now $P = (I - F)B$ and $R = I - (I - F)(I - E)$ proves the lemma. □

2.2 The $\overline{\partial}_b$ complex and the Szegő projection

Given a euclidean metric on V, one can embed $V \times \mathbb{S}V^* \subset T^*V$ via a contact diffeomorphism as the unit cosphere bundle in that metric. Normal coordinates on V identify then T^*V with $\mathbb{C}^n$

$$T^*V \ni (x,y) = y_i dx_i \longrightarrow z = x - iy \in \mathbb{C}^n$$

and we use this to fix a complex structure on T^*V.

In coordinates the $\overline{\partial}$-operator is given by:

$$\overline{\partial} = \sum_j \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})\,(dx_j + idy_j)$$

and its indicial family (though not defined previously in this context, the analogy is straightforward) by:

$$\widehat{\overline{\partial}}(\xi) = \sum_j \frac{i}{2}(\xi_j - \partial_{y_j})\,dy_j; \tag{2.6}$$

here the identification of the trivial form bundles on T^*V with their restriction at $x = 0$ was used, identifying $dx_j + idy_j$ with dy_j. By direct computation, it follows from (2.6) that:

$$e^{-\xi\cdot y}\,\widehat{\overline{\partial}}(\xi)\,e^{\xi\cdot y} \;=\; \widehat{\overline{\partial}}(0) \;=\; -\frac{i}{2}d_y, \tag{2.7}$$

where d_y is the exterior derivative on $\mathbb{R}^n$ in the y-variable.

The above complex structure gives rise to a strictly pseudoconvex CR structure on the co-sphere bundle $V \times \mathbb{S}V^* = \{(x,y) : |y| = 1\}$ underlying the contact structure introduced above. Note that the one-form $\alpha = i\overline{\partial}(|y|^2)|_{\mathbb{S}^*V} = y_i dx_i$ is a positive contact form on $\mathbb{S}^*V$.

The $\overline{\partial}_b$-complex of Kohn and Rossi, and the associated Laplacian, $\Box_b = \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b$, are Heisenberg operators of order 1 and 2 respectively. Of course, in our case, they are invariant under translations in the base variable. From (2.7), one gets for the indicial family of $\overline{\partial}_b$:

$$\widehat{\overline{\partial}}_b(\xi) \;=\; e^{\xi\cdot y}\,\widehat{\overline{\partial}}_b(0)\,e^{-\xi\cdot y} \;=\; -\frac{i}{2}e^{\xi\cdot y}\,d_{\mathbb{S}}\,e^{-\xi\cdot y}, \tag{2.8}$$

with $d_{\mathbb{S}}$ denoting the exterior derivative on $\mathbb{S}V^*$.

For $\Box_b$ on $(0,q)$-forms we have:

Lemma 2.7. *For $n \geq 3$, $\Box_b^{0,q}$ is invertible in Ψ^2_{tiH} for $0 < q < n-1$.*

Proof. The principal symbol of $\Box_b^{0,q}$ is $\sigma_2(\Box_b^{0,q})_\pm = -(h \mp (n-1-2q))$, where h is the field of harmonic oscillators on W. Since the eigenvalues of h are of the form $n-1+2k$ with $k \geq 0$, it follows that $\sigma_2(\Box_b^{0,q})_\pm$ is invertible in the isotropic algebra for $q \neq 0$ and $q \neq n-1$. Thus $\Box_b^{0,q}$ is elliptic and, by Lemma 2.6, a parametrix $P \in \Psi^{-2}_{tiH}$ can be constructed

$$\Box_b \circ P = P \circ \Box_b = Id - R, \qquad R \in \Psi^{-\infty}_{tiH}$$

such that the indicial family of R has compact support, i.e. $\widehat{R} \in C_0^\infty(\mathbb{R}^n, \Psi^{-\infty}(\mathbb{S}^{n-1}))$. This proves that $\widehat{\Box_b}(\xi)$ is invertible for ξ large.

From (2.7), one gets for the indicial family of $\overline{\partial}_b$:

$$\widehat{\overline{\partial}_b}(\xi) \;=\; e^{\xi\cdot y}\,\widehat{\overline{\partial}_b}(0)\,e^{-\xi\cdot y} \;=\; -\frac{i}{2}e^{\xi\cdot y}\,d_{\mathbb{S}}\,e^{-\xi\cdot y}, \tag{2.9}$$

with $d_{\mathbb{S}}$ denoting the exterior derivative on $\mathbb{S}V^*$. This implies that $\widehat{\Box_b}(\xi)$ is invertible everywhere. Denote by $\widehat{L} \in C^\infty(\mathbb{R}^n, \Psi^{-2}(\mathbb{S}^{n-1}))$ the inverse family. We have:

$$\widehat{L}(\xi) \;=\; \widehat{L}(\xi)\,(\widehat{\Box_b}(\xi)\widehat{P}(\xi) + \widehat{R}(\xi)) \;=\; \widehat{P}(\xi) + \widehat{L}(\xi)\widehat{R}(\xi).$$

which proves that $\widehat{L}$ is in fact the indicial family of an operator $L \in \Psi^{-2}_{tiH}$, the sum of P and a smoothing operator. □

The *Szegő projection* S, defined as the projection on $\ker\overline{\partial}_b = \ker\Box_b^{0,0}$, can be written as:

$$S = Id - \overline{\partial}_b^*\,(\Box_b^{0,1})^{-1}\,\overline{\partial}_b.$$

As follows from (2.9), the indicial family of S consists of rank 1 projections on the $\ker\widehat{\overline{\partial}_b}(\xi) = \text{span}\{e^{\xi\cdot y}\}$:

$$\widehat{S} \in C^\infty(V^*, \Psi^{-\infty}(\mathbb{S}V^*)), \qquad S \in \mathcal{I}^0_+$$
$$\widehat{S}(\xi, y, y') = \frac{e^{\xi\cdot y}}{||e^{\xi\cdot\cdot}||_{L^2(\mathbb{S}V^*)}}\,\frac{e^{\xi\cdot y'}}{||e^{\xi\cdot\cdot}||_{L^2(\mathbb{S}V^*)}}.$$

The Schwartz kernel of S follows from this or could have been computed directly by elementary means:

$$S(x - x', y, y') = (2\pi)^{-n}\int_{V^*} e^{i(x-iy-x'-iy')\cdot\xi}\,\frac{1}{||e^{\xi\cdot\cdot}||^2_{L^2(\mathbb{S}V^*)}}\,d\xi.$$

Chapter 3

Adiabatic Heisenberg Algebra

The adiabatic Heisenberg algebra $\Psi^*_{aH}(\mathbb{S}^*X)$ of operators on the co-sphere bundle $\mathbb{S}^*X$ of a compact manifold X is constructed here. Adiabatic calculi for a fibration have been studied before by Mazzeo and Melrose [13] and Epstein and Melrose [5]. The fact that we can combine this formalism with the Heisenberg calculus rests on $\mathbb{S}^*X \to X$ being a Legendre fibration. In fact this is the only such Legendre fibration with compact fibers, at least for dim $X \geq 3$.

3.1 Adiabatic structure for a Legendre fibration

Let X^n be a smooth compact manifold without boundary and

$$\phi : M^{2n-1} = \mathbb{S}^*X = (T^*X \setminus \{0\})/\mathbb{R}_+ \longrightarrow X$$

its canonical co-sphere bundle. Mazzeo and Melrose [13] defined the adiabatic tangent and cotangent bundle, ${}^aTM$ and ${}^aT^*M$, for a general fibration $\phi : M \to X$. We will briefly recall the construction here and show that in our case, $M = \mathbb{S}^*X$, the contact structure can be easily added to the construction.

Following [13], consider the space $\mathcal{V}_a$ of vector fields on $M \times [0, \epsilon_0)$ tangent both to the fibers $M \times \{\epsilon\}$ of the fibration $M \times [0, \epsilon_0) \to [0, \epsilon_0)$ and the fibers of $M \times \{0\} \to X$:

$$\mathcal{V}_a = \{V \in C^\infty(M \times [0, \epsilon_0), T(M \times [0, \epsilon_0))) : V\epsilon = 0,$$
$$V \textit{ tangent to } \mathbb{S}^*_xX \textit{ at } \epsilon = 0, \forall x \in X\}$$

In local coordinates $x_1, ..., x_n, y_1, ..., y_{n-1}$ on M that trivialize ϕ, x's being coordinates on X and y's coordinates on $\mathbb{S}^{n-1}$, these vector fields can be written as linear combination of $\epsilon\partial_{x_1}, ..., \epsilon\partial_{x_n}, \partial_{y_1}, ..., \partial_{y_{n-1}}$. As shown by Mazzeo and Melrose [13], $\mathcal{V}_a$ is the space of the C^∞ sections of a vector bundle of rank $2n - 1 (= \dim M)$ over $M \times [0, \epsilon_0)$:

$$\mathcal{V}_a = C^\infty(M \times [0, \epsilon_0), {}^aTM)$$

and

$$\epsilon\partial_{x_1}, ..., \epsilon\partial_{x_n}, \partial_{y_1}, ..., \partial_{y_{n-1}}$$

form a basis for ${}^aTM$.

The natural inclusion $\mathcal{V}_a \hookrightarrow C^\infty(M \times [0, \epsilon_0), \pi^*TM)$, where $\pi : M \times [0, \epsilon_0) \to M$ is the projection on the left factor, gives a natural map of bundles over $M \times [0, \epsilon_0)$ which is an isomorphism for $\epsilon > 0$:

$$\iota : {}^aTM \longrightarrow \pi^*TM \cong TM \times [0, \epsilon_0); \tag{3.1}$$

identifying ${}^aTM|_\epsilon$ (the restriction of ${}^aTM$ to $M \times \{\epsilon\}$) with TM, $\epsilon > 0$. At $\epsilon = 0$ the range of ι is the tangent bundle to the fibers of ϕ and its kernel $\epsilon\,\phi^*TX \subset {}^aTM|_0$ (the restriction to $\epsilon = 0$ of the pull-back of TX to $M \times [0, \epsilon_0)$, rescaled by ϵ; it is naturally isomorphic with ϕ^*TX; for more on rescaling vector bundles see [15]). With $TM_\phi \subset {}^aTM|_0$ denoting the natural subbundle of vectors 'tangent to the fiber' (i.e. of vectors $v \in {}^aT_{p,0}M$, $p \in M$ for which a vector field $V \in \mathcal{V}_a$ exists with $V_{p,0} = v$ and tangent to all the fibers $\mathbb{S}^*_{\phi(p)}X \times \{\epsilon\}$, $0 \le \epsilon < \epsilon_0$) we have at $\epsilon = 0$ the natural splitting of ${}^aTM$:

$$\begin{aligned} {}^aTM|_0 &= \epsilon\,\phi^*TX \oplus TM_\phi; \\ {}^aT_{x,y,0}M &= \epsilon\,T_xX \oplus T_y(\mathbb{S}^*_xX), \quad x \in X,\, y \in \mathbb{S}^*_xX. \end{aligned} \tag{3.2}$$

Taking the dual of (3.1) we get:

$$\iota^* : T^*M \times [0, \epsilon_0) \cong \pi^*T^*M \longrightarrow {}^aT^*M,$$

which is also an isomorphism for $\epsilon > 0$. Here ${}^aT^*M$ is the dual of ${}^aTM$; in local coordinates a generic point in ${}^aT^*M$ is $\xi_1\frac{dx_1}{\epsilon} + ... + \xi_n\frac{dx_n}{\epsilon} + \eta_1 dy_1 + ... + \eta_{n-1}dy_{n-1}$. Dually to (3.2) we have at $\epsilon = 0$:

$$\begin{aligned} {}^aT^*M|_0 &= \frac{1}{\epsilon}\phi^*T^*X \oplus T^*M_\phi, \\ {}^aT^*_{x,y,0}M &= \frac{1}{\epsilon}\,T^*_xX \oplus T^*_y(\mathbb{S}^*_xX), \quad x \in X,\, y \in \mathbb{S}^*_xX, \end{aligned} \tag{3.3}$$

where T^*M_ϕ is the dual of TM_ϕ, co-tangent vectors to the fibers.

The exterior powers ${}^a\Lambda^*M = \Lambda^*({}^aT^*M)$ also have splittings as above at $\epsilon = 0$. Their sections will be the adiabatic forms written in local coordinates as:

$$\alpha = a_{I,J}(x, y, \epsilon)\frac{dx_I}{\epsilon^{|I|}}dy_J$$

Also the differential on M, which, for $\epsilon > 0$, can be lifted by ι^* to act on sections of ${}^a\Lambda^*M$, extends to $\epsilon = 0$:

$$d : C^\infty(M \times [0, \epsilon_0), {}^a\Lambda^pM) \longrightarrow C^\infty\left(M \times [0, \epsilon_0), {}^a\Lambda^{p+1}M\right), \quad p = 0, \ldots, 2n-1, \tag{3.4}$$

in local coordinates taking the form:

$$d\alpha = (\epsilon\partial_{x_i} a_{I,J})(x,y,\epsilon)\,\frac{dx_i}{\epsilon}\frac{dx_I}{\epsilon^{|I|}}dy_J \;+\; (-1)^{|I|}(\partial_{y_j} a_{I,J})(x,y,\epsilon)\,\frac{dx_I}{\epsilon^{|I|}}dy_j dy_J.$$

Recall that the space of s-densities on a k-dimensional real vector space V is

$$\Omega^s V = \{w : \Lambda^k V \setminus \mathbf{0} \to \mathbb{C},\ w(t\gamma) = |t|^s w(\gamma)\,\forall\gamma \in \Lambda^k V \setminus 0,\ t \in \mathbb{R}\setminus 0\}, \quad s \in \mathbb{R},$$

and, in the case of a vector bundle $E \to Z$, the fiber-wise construction gives a line bundle $\Omega^s E \to Z$. The 1-density bundle Ω^1 will be shortly denoted by Ω.

On $M\times[0,\epsilon_0)$ there are a few possible choices of density bundles. In addition to the usual ones, $\Omega^s = \Omega^s(T(M\times[0,\epsilon_0)))$, we also have the adiabatic density bundles ${}^a\Omega^s = \Omega^s({}^aTM)$ and the ones in the ϵ variable, $\Omega^s_\epsilon = \Omega^s(T[0,\epsilon_0))$, given by the lift to $M\times[0,\epsilon_0)$ of the tangent bundle to $[0,\epsilon_0)$. Note that ${}^a\Omega\otimes\Omega_\epsilon \cong \epsilon^{-n}\Omega$ or equivalently:

$$ {}^a\Omega^{\frac{1}{2}} \otimes (\epsilon^n\, {}^a\Omega^{\frac{1}{2}} \otimes \Omega_\epsilon) \cong \Omega. \tag{3.5}$$

In local coordinates, $a(x,y,\epsilon)|\frac{dx}{\epsilon^n}dy|^{1/2}$ is a section of ${}^a\Omega^{\frac{1}{2}}$. We are going to use adiabatic densities in a more general context than defined here, namely on ${}^aT^*M$, ${}^aTM$, on their compactifications, and on the double space M^2_a defined in the next section. All these manifolds have ϵ as a defining function for the boundary or a face of the boundary and also all have a fibered boundary face. In general, adiabatic structures can be defined at such a fibered boundary face by following the construction above, starting with $\mathcal{V}_a$. It is beyond our objective to include a complete treatment of this, since we are interested only in adiabatic density bundles, which can always be trivialized or described as above in terms of other density bundles. For example, on ${}^aT^*M$, $a(x,y,\xi,\eta,\epsilon)|\frac{dx}{\epsilon^n}dy\,d\xi\,d\eta|^{1/2}$ is an adiabatic half-density associated to the fibration ${}^aT^*M \to X$ at $\epsilon = 0$.

To investigate the behavior of the contact structure as $\epsilon \to 0$ consider over $\mathbb{S}^*X\times[0,\epsilon_0)$ the trivial line bundle $L \subset {}^aT^*(\mathbb{S}^*X)$ with positive half-line given by:

$$L^+_{x,y,\epsilon} = \{\xi \in \frac{1}{\epsilon}T^*_xX \,:\, [\epsilon\,\xi] = y\} \subset {}^aT^*_{x,y,\epsilon}(\mathbb{S}^*X), \quad x\in X,\ y\in\mathbb{S}^*_xX, \epsilon \geq 0;$$

multiplication by ϵ is used above to identify $1/\epsilon\, T^*_xX$ with T^*_xX. In particular, at ϵ=0, using this and (3.3):

$$L^+_{x,y,0} \cong \{(\xi,0) \,:\, [\xi] = y\} \subset T^*_xX \oplus T^*_y(\mathbb{S}^*_xX). \tag{3.6}$$

The line bundle defined above will be the *adiabatic contact line* in the sense of the following lemma:

Lemma 3.1. *For $\epsilon \geq 0$, the restriction of L to $\mathbb{S}^*X\times\{\epsilon\}$ is identified through ι^* with the contact line bundle on $\mathbb{S}^*X$. On the other hand, at $\epsilon = 0$, for every $x \in X$ it restricts to a* *line bundle* $L^x \subset \frac{1}{\epsilon} T^*_x X \oplus T^*(\mathbb{S}^*_x X)$ *over* $\mathbb{S}^*_x X$ *identified via (3.6) with the contact structure on the model space* $T^*_x X \oplus \mathbb{S}^*_x X$ *of Chapter 2. Moreover, every contact form* α *on* $\mathbb{S}^* X$ *lifts to a non-vanishing section* $\tilde{\alpha} = \epsilon^{-1} \iota^* \pi^* \alpha$ *of* ${}^a T^* M$ *spanning* L.

Proof. The only thing to be checked is that $\tilde{\alpha}$ is well-defined and non-zero. For every $V \in \mathcal{V}_a$ we have $\iota^* \pi^* \alpha(V)|_{\epsilon=0} = \pi^* \alpha(\iota_* V)|_{\epsilon=0} = \alpha(V|_{\epsilon=0}) = 0$ since, by definition, $V|_{\epsilon=0}$ is tangent to the fibers of ϕ and so annihilated by the contact form. This proves that $\tilde{\alpha} = \epsilon^{-1} \iota^* \pi^*_L \alpha$ is a well-defined section of ${}^a T^* M$. To prove that is non-zero, test it on the Reeb vector field R: $\tilde{\alpha}(\epsilon R) = \alpha(R) = 1$. □

Remark 3.2. An adiabatic contact form $\tilde{\alpha}$ can be defined as a form

$$\tilde{\alpha} : M \times [0, \epsilon_0) \to {}^a T^* M \ \text{ s.t. } \ \tilde{\alpha} \wedge (d\tilde{\alpha})^{n-1} \neq 0$$

where d is the differential in (3.4).

In our case, that is the co-sphere bundle with its canonical contact structure, the above lemma provides such an example, namely $\tilde{\alpha} = \alpha/\epsilon$ for α a contact form on $\mathbb{S}^* X$. It can be easily seen in local coordinates why $\tilde{\alpha}$ is an adiabatic form and why it induces at $\epsilon = 0$ a contact structure on the model spaces $T_x X \times \mathbb{S}^*_x X$.

3.2 The double space

Note the existence of a Schwartz kernel theorem, i.e. a bijective correspondence between operators:

$$A : \dot{C}^\infty(M \times [0, \epsilon_0), {}^a\Omega^{\frac{1}{2}}) \longrightarrow C^{-\infty}(M \times [0, \epsilon_0), {}^a\Omega^{\frac{1}{2}}) \tag{3.7}$$

and their Schwartz kernels

$$K_A \in C^{-\infty}(M \times M \times [0, \epsilon_0), {}^a\Omega^{\frac{1}{2}}) \tag{3.8}$$

given by:

$$\begin{aligned} &\langle Au, v \rangle = \langle K_A, \pi^*_L v \cdot \pi^*_R u \rangle, \\ &\forall\ u \in \dot{C}^\infty(M \times [0, \epsilon_0), {}^a\Omega^{\frac{1}{2}}),\ v \in \dot{C}^\infty(M \times [0, \epsilon_0), {}^a\Omega^{\frac{1}{2}} \otimes \Omega_\epsilon), \end{aligned} \tag{3.9}$$

where $\pi_L, \pi_R : M \times M \times [0, \epsilon_0) \to M \times [0, \epsilon_0)$ are the projection off the second and respectively first factor in M. The adiabatic density bundle in (3.8) is associated to the product fibration $M \times M \to X \times X$. In local coordinates

$$K_A = K_A(x, y, x', y', \epsilon) |\frac{dx}{\epsilon^n} \frac{dx'}{\epsilon^n} dy\, dy'|^{\frac{1}{2}}.$$

The operators in $\Psi^*_{aH}(M)$ will be defined in §3.4 by selecting kernels as in (3.8) with a special type of singularity at the diagonal $\operatorname{Diag} M \times [0, \epsilon_0)$. In fact the precise description of the kernels will be given on a blown-up version of $M \times M \times [0, \epsilon_0)$, the double space M^2_a

defined next. By blow-up the space of functions vanishing to infinite order at the boundary does not change and hence neither the space of extendible distributions:

$$\beta_* : C^{-\infty}(M_a^2, {}^a\Omega^{\frac{1}{2}}) \xrightarrow{\cong} C^{-\infty}(M \times M \times [0, \epsilon_0), {}^a\Omega^{\frac{1}{2}}). \tag{3.10}$$

For the exact construction of the kernels of the operators in $\Psi^*_{aH}(\mathbb{S}^*X)$ as distributions on M_a^2 see Definition 3.7.

We refer to [14] and [15] for more on blow-ups and analysis on manifolds with corners.

Definition 3.3. *The double space M_a^2 is defined as the radial blow-up in $M \times M \times [0, \epsilon_0)$ of the fiber diagonal D at $\epsilon = 0$:*

$$\beta : M_a^2 = [M \times M \times [0, \epsilon_0) \,;\, D] \longrightarrow M \times M \times [0, \epsilon_0),$$
$$D = M \times_X M \times \{0\} = \{(p, p', 0); \phi(p) = \phi(p')\}.$$

Given the fibrations $\phi : M \longrightarrow X$, $\psi : N \longrightarrow X$, the notation $M \times_X N$ stands for the submanifold $\{(p, q) \in M \times N; \phi(p) = \psi(q)\}$ of $M \times N$.

The blown-up space M_a^2 is a manifold with corners of codimension 2. Its boundary has two faces (assuming M is connected): the closure of

$$\beta^{-1}(M \times M \times \{0\} \setminus D)$$

and the front face

$$\mathrm{ff}(M_a^2) = \overline{\beta^{-1}(D)}.$$

By construction, $\mathrm{ff}(M_a^2)$ is the inward-pointing hemisphere bundle of D in $M \times M \times [0, \epsilon_0)$. It is canonically isomorphic with the radial compactification of $N\,\mathrm{Diag}(X) \times_X M \times_X M$, the normal bundle of D as a submanifold of $M \times M$ (here $N\,\mathrm{Diag}(X)$ is the normal bundle to the diagonal in $X \times X$). Thus

$$\mathrm{int}(\mathrm{ff}(M_a^2)) \cong TX \times_X M \times_X M \tag{3.11}$$

and, most importantly, it is fibered over X:

$$\begin{array}{ccc} \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} & \text{------} & \mathrm{int}(\mathrm{ff}(M_a^2)) \\ & & \downarrow \tau \\ & & X \end{array} \tag{3.12}$$

$$\tau^{-1}(x) \cong T_xX \times \mathbb{S}^*_xX \times \mathbb{S}^*_xX, \quad x \in X.$$

Denote by Δ the diagonal $\mathrm{Diag}(M) \times (0, \epsilon_0) = \{(p, p, \epsilon) : p \in M, \epsilon > 0\} \subset M \times M \times [0, \epsilon_0)$ and by

$$\Delta_a = \overline{\beta^{-1}(\Delta)} \subset M_a^2$$

the closure of its lift to M_a^2. Δ_a intersects the boundary of M_a^2 transversally in the interior of the front face, in fact in the 0-section Diag(M) of $\Delta_0 = TX \times_X \mathrm{Diag}(M)$:

$$\Delta_a \cap \mathrm{ff}(M_a^2) = \mathrm{Diag}(M) \subset \Delta_0.$$

Let (x, y, x', y', ϵ) be local coordinates on $M \times M \times [0, \epsilon_0)$ near Δ, where (x, y) and (x', y') are the same coordinates on the left and the right copy of M. Then (X, X', y, y', ϵ) are local coordinates on M_a^2, near $\Delta_a \cup \Delta_0$, where:

$$\begin{aligned} x &= \frac{\epsilon}{2}X + X', \quad x' = -\frac{\epsilon}{2}X + X' \text{ or equivalently} \\ X &= \frac{x - x'}{\epsilon},\ X' = \frac{x + x'}{2} \text{ for } \epsilon > 0. \end{aligned} \tag{3.13}$$

In terms of these $\Delta_a = \{X = 0,\ y = y'\}$, $\Delta_0 = \{y = y',\ \epsilon = 0\}$ and the fibers of τ are given by $\tau^{-1}(x) = \{X' = x,\ \epsilon = 0\}$, $x \in X$.

In these local coordinates, $\phi(X, X', y, y', \epsilon)|dX \frac{dX'}{\epsilon^n} dy\, dy'|^{\frac{1}{2}}$ is an adiabatic half-density on M_a^2.

Remark 3.4. The structure of M_a^2 is essential in what follows, especially in understanding how does a family $A_\epsilon \in \Psi_H^m(M)$, $\epsilon > 0$, of Heisenberg operators glue at $\epsilon = 0$ with a family $A_x \in \Psi_{tiH}^m(T_xX \times \mathbb{S}_x^*X)$, $x \in X$, of translation invariant Heisenberg operators to form an adiabatic Heisenberg operator $A \in \Psi_{aH}^m(M)$. The slices $M \times M \times \{\epsilon\}$, $\epsilon > 0$, forming the interior of M_a^2, and the fibers $T_xX \times \mathbb{S}_x^*X \times \mathbb{S}_x^*X$ of the important part of its boundary, $\mathrm{ff}(M_a^2)$, are the spaces on which the kernels of the A_ϵ's and A_x's have been defined in Chapter 1 and Chapter 2 respectively. Moreover the singularities of the kernels occur only at Δ_a. We will have this in mind when defining the kernel of the adiabatic Heisenberg operator A in §3.4.

3.3 Riemann-Weyl fibration

Let h be a Riemannian metric on X and $g \in C^\infty(M; T^*M \otimes T^*M)$ a symmetric 2-cotensor on $M = \mathbb{S}^*X$, positive definite when restricted to the fibers of ϕ. For $\epsilon > 0$ sufficiently small (suppose for $\epsilon < \epsilon_0$)

$$g_\epsilon = g + \frac{\phi^* h}{\epsilon^2} \tag{3.14}$$

gives a family of metrics on M. This lifts and extends to a positive definite symmetric 2-cotensor $\tilde{g} \in C^\infty(M \times [0, \epsilon_0); {}^aT^*M \otimes {}^aT^*M)$, the natural choice of a metric in the adiabatic setting (in fact, such a family of metrics is the motivation for the construction of ${}^aTM$, [13]). Moreover, at $\epsilon = 0$, the splittings (3.2) and (3.3) are orthogonal with respect to $\tilde{g}$.

For each $\epsilon > 0$, let $w_\epsilon(v) = (\exp_{p,\epsilon}(v), \exp_{p,\epsilon}(-v))$, $v \in T_pM$ be the Riemann-Weyl fibration associated with the metric g_ϵ on M. It extends to:

$$w : {}^aTM \overset{\iota}{\hookrightarrow} TM \times [0, \epsilon_0) \supset U \longrightarrow V \subset M \times M \times [0, \epsilon_0)$$

having as range at $\epsilon = 0$ the fiber diagonal D. We claim that it lifts to the blow-up space M_a^2 to give a Riemann-Weyl fibration:

Lemma 3.5. *There exist open neighborhoods U of the 0-section in ${}^aTM$ and V of Δ_a in M_a^2 with $\Delta_0 \subset V \cap \partial M_a^2 \subset \mathrm{ff}(M_a^2)$ and a b-diffeomorphism:*

$$W : {}^aTM \supset U \longrightarrow V \subset M_a^2,$$

satisfying the following:
a) $\beta \circ W = w$; i.e. for $\epsilon > 0$ its restriction $W_\epsilon : {}^aTM|_\epsilon \supset U|_\epsilon \to V|_\epsilon \subset M \times M \times \{\epsilon\}$ is a Riemann-Weyl fibration on M (associated to g_ϵ);
b) at $\epsilon = 0$, W restricts to $W_0 : {}^aTM|_0 \supset U|_0 \to V|_0 \subset int(\mathrm{ff}(M_a^2))$; it further restricts, for each $x \in X$, to be:

$$W^x : \epsilon T_xX \oplus T(\mathbb{S}_x^*X) \supset \epsilon T_xX \oplus U^x \longrightarrow T_xX \oplus V^x \subset T_xX \oplus \mathbb{S}_x^*X \oplus \mathbb{S}_x^*X,$$
$$W^x = (\epsilon^{-1}Id, W_{\mathbb{S}_x^*}),$$

the Riemann-Weyl map in the model case,§2.1. Above the map $\epsilon^{-1}Id$ is the identification $\epsilon T_xX \to T_xX$ and $W_{\mathbb{S}_x^*} : T(\mathbb{S}_x^*) \supset U^x \to V^x \subset \mathbb{S}_x^*X \oplus \mathbb{S}_x^*X$ the Riemann-Weyl map on $\mathbb{S}_x^*X$ associated to $g|_{\mathbb{S}_x^*X}$.

Proof. We define W for $\epsilon > 0$ to be equal with w and show that it extends to be C^∞ up to $\epsilon = 0$. Clearly, conclusion a is then automatically satisfied.

First we find an expression for w. In fact, by the metric identification of the tangent and cotangent bundles, we consider w to be defined on ${}^aT^*M$. Let $|\cdot|$ be the length function on ${}^aT^*M$ fixed by the metric $\tilde{g}$. At a point $\gamma = \xi_i \frac{dx_i}{\epsilon} + \eta_i\, dy_i \in {}^aT^*M$, we have $l(\gamma) = \frac{1}{2}|\gamma|^2 = \frac{1}{2}[h^{ij}(x)\xi_i\xi_j + g^{ij}(x,y)\eta_i\eta_j + O(\epsilon)]$. The Hamiltonian vector field is given by

$$H_l = h^{ik}(x)\xi_i\,(\epsilon\partial_{x_k}) + g^{ik}(x,y)\eta_i\partial_{y_k} - \frac{1}{2}\partial_{y_k}g^{ij}(x,y)\,\eta_i\eta_j\partial_{\eta_k} + O(\epsilon),$$

with $O(\epsilon)$ representing an adiabatic vector field on ${}^aT^*M$ multiplied by ϵ. By an adiabatic vector field on ${}^aT^*M$ we mean a vector field on ${}^aT^*M$ which, at $\epsilon = 0$, is tangent to the fibers of ${}^aT^*M \to X$ (a base for these vector fields is given by $\epsilon\partial_{x_k}$, ∂_{y_k}, ∂_{ξ_k} and η_k).

The map w is given then by $w : (\bar{x}, \bar{y}, \bar{\xi}, \bar{\eta}, \epsilon) \to (x(1), y(1), x'(1), y'(1), \epsilon(1))$, where $(x, y, \xi, \eta, \epsilon)$ and $(x', y', \xi', \eta', \epsilon')$ are integral curves of H_l, i.e. solutions of:

$$\begin{aligned} &\frac{dx_k}{dt} = \epsilon\, h^{ik}(x)\xi_i + O(\epsilon^2) && \frac{d\xi_k}{dt} = O(\epsilon) \\ &\frac{dy_k}{dt} = g^{ik}(x,y)\eta_i + O(\epsilon) && \frac{d\eta_k}{dt} = -\frac{1}{2}\partial_{y_k}g^{ij}(x,y)\eta_i\eta_j + O(\epsilon) \\ &\frac{d\epsilon}{dt} = 0 \end{aligned} \tag{3.15}$$

with initial data, at $t = 0$, $(\bar{x}, \bar{y}, \bar{\xi}, \bar{\eta}, \epsilon)$ and $(\bar{x}, \bar{y}, -\bar{\xi}, -\bar{\eta}, \epsilon)$ respectively.

In the local coordinates (X', X, y, y', ϵ) introduced above on M_a^2 near $\Delta_a \cup \Delta_0$, the map W is given by $W : (\bar{x}, \bar{y}, \bar{\xi}, \bar{\eta}, \epsilon) \to (X'(1), X(1), y(1), y'(1), \epsilon)$ with $(X', X, y, y', \xi, \xi', \eta, \eta')$

a solution of the system of differential equations obtained from (3.15) by the change of coordinates: $x = \frac{\epsilon}{2}X + X'$, $x' = -\frac{\epsilon}{2}X + X'$. With the exception of

$$\begin{aligned} &\frac{dX'_k}{dt} = O(\epsilon), \\ &\frac{dX_k}{dt} = \frac{1}{2}[h^{ik}(\frac{\epsilon}{2}X + X')\xi_i - h^{ik}(-\frac{\epsilon}{2}X + X')\xi'_i] + O(\epsilon), \end{aligned} \tag{3.16}$$

the other equations of the system are basically unchanged (modulo the substitution of x and x' in terms of X and X'). The solution is also subject to the initial conditions $X'(0) = \bar{x}$, $X(0) = 0$, $y(0) = y'(0) = \bar{y}$, $\xi_k(0) = -\xi'_k(0) = \bar{\xi}_k$, $\eta_k(0) = -\eta'_k(0) = \bar{\eta}_k$.

At $\epsilon = 0$, note that $\xi_k(\cdot) = -\xi_k(\cdot) = \bar{\xi}_k$ and $X'(\cdot) = \bar{x}$. The equation (3.16) rewrites $\frac{dX_k}{dt} = h^{ik}(\bar{x})\bar{\xi}_i$ with solution $X_k(t) = [h^{ik}(\bar{x})\bar{\xi}_i]\,t$, $t \in \mathbb{R}$. Note that $\frac{T^*_{\bar{x}}X}{\epsilon} \ni \bar{\xi} \to X(1) \in T_{\bar{x}}X$ is, by the metric identification of the tangent and cotangent bundles, the map $\epsilon^{-1}Id$ in the statement of the lemma. The remaining equations of the system:

$$\begin{aligned} \frac{dy_k}{dt} &= g^{ik}(\bar{x}, y)\eta_i, & \frac{d\eta_k}{dt} &= -\frac{1}{2}\partial_{y_k} g^{ij}(\bar{x}, y)\eta_i\eta_j, \\ \frac{dy'_k}{dt} &= g^{ik}(\bar{x}, y')\eta'_i, & \frac{d\eta'_k}{dt} &= -\frac{1}{2}\partial_{y'_k} g^{ij}(\bar{x}, y')\eta'_i\eta'_j. \end{aligned}$$

solve to give the Riemann-Weyl map $W_{\mathbb{S}^*_{\bar{x}}} : (\bar{y}, \bar{\eta}) \to (y(1), y'(1))$ on $\mathbb{S}^*_{\bar{x}}X$, again the metric identification of $T(\mathbb{S}^*_{\bar{x}}X)$ and $T^*(\mathbb{S}^*_{\bar{x}}X)$ being employed. So at $\epsilon = 0$ the map W restricts to W_0 proving part b of the lemma.

We still have to check that W is a local diffeomorphism near the 0-section of ${}^aT^*M$, i.e. that its differential at $\bar{\xi} = 0$, $\bar{\eta} = 0$ is non-degenerate. By construction this is clear for $\epsilon > 0$. At $\epsilon = 0$, from the above equations (or from the form of W_0) it follows that the differential is:

$$\frac{D(X'(1), X(1), y(1), y'(1), \epsilon)}{D(\bar{x}, \bar{\xi}, \bar{y}, \bar{\eta}, \epsilon)}\Bigg|_{\bar{\xi}=0, \bar{\eta}=0, \epsilon=0} = \begin{pmatrix} I & \mathbf{0} & \mathbf{0} & * \\ \mathbf{0} & h(\bar{x}) & \mathbf{0} & * \\ \mathbf{0} & \mathbf{0} & D(W_{M_{\bar{x}}})|_{\bar{\eta}=0} & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \end{pmatrix};$$

this is non-degenerate since $h(\bar{x})$ is positive definite and

$$D(W_{\mathbb{S}^*_{\bar{x}}})|_{\bar{\eta}=0} = \begin{pmatrix} I & g(\bar{x}, \bar{y}) \\ I & -g(\bar{x}, \bar{y}) \end{pmatrix}$$

is the differential of the diffeomorphism $W^{\bar{x}}$. So W is a diffeomorphism from some open neighborhood U of the 0-section in ${}^aT^*M$ to an open neighborhood V of Δ_a. The results obtained above at $\epsilon = 0$ (i.e. the domain of definition of W_0) show that we can choose the neighborhoods U and V subject to the extra conditions in the statement of the lemma. □

3.4 Adiabatic Heisenberg operators and principal symbol

As already stated in §3.2, the operators A in (3.7) are identified with their Schwartz kernels $K_A \in C^{-\infty}(M \times M \times [0, \epsilon_0), {}^a\Omega^{\frac{1}{2}})$ or, equivalently, with $\tilde{K}_A \in C^{-\infty}(M_a^2, {}^a\Omega^{\frac{1}{2}})$, the lift of K_A to M_a^2. The identification is given by (3.9) or, after inserting $K_A = \beta_* \tilde{K}_A$, by:

$$(3.17) \qquad \langle Au, v\rangle = \langle \tilde{K}_A, \beta^*(\pi_L^* v \cdot \pi_R^* u)\rangle.$$

In view of this we will often use the same notation both for an operator A and its Schwartz kernel $A = \tilde{K}_A$.

To define Ψ_{aH}^*, we will follow the construction of Ψ_H^* in [6] and adapt it to the adiabatic setting. We will define a class of kernels $A \in C^{-\infty}(M_a^2, {}^a\Omega^{\frac{1}{2}})$, with singularities at $\Delta_a \subset M_a^2$ of the same type as Ψ_H^* uniformly up to $\epsilon = 0$. More precisely, we will use the adiabatic contact structure of §3.1 to define parabolic symbols on ${}^aT^*M$ and then use the inverse Fourier transform and the Riemann-Weyl fibration of §3.3 to characterize the Schwartz kernels $A \in C^{-\infty}(M_a^2, {}^a\Omega^{\frac{1}{2}})$ that the operators in Ψ_{aH}^* should have.

The adiabatic symbols will be defined on ${}^{La}\overline{T}^*M$, the L-parabolic compactification of the adiabatic cotangent bundle ${}^aT^*M \to M \times [0, \epsilon_0)$. It is a manifold with corners of codimension 2 and has two boundary faces, the restriction ${}^{La}\overline{T}^*M|_0$ of ${}^{La}\overline{T}^*M$ to $\epsilon = 0$ and the parabolic cosphere bundle ${}^{La}\mathbb{S}^*M$. With ρ denoting a defining function for the boundary face ${}^{La}\mathbb{S}^*M$ we have:

Definition 3.6. *For any $m \in \mathbb{R}$, the space of 1-step polyhomogeneous adiabatic L-parabolic symbols of order m on M is*

$$S_{La}^m({}^aT^*M) = \rho^{-m} C^\infty({}^{La}\overline{T}^*M),$$

also denoted shortly by S_{La}^m. Also set $S_{La}^{-\infty} = \bigcap_m S_{La}^m$.

Note that $S_{La}^{-\infty} = \dot{C}_{\mathbb{S}}^\infty({}^{La}\overline{T}^*M)$, the space of smooth functions vanishing to infinite order at ${}^{La}\mathbb{S}^*M$.

The fiber-wise Fourier transform on smooth densities, defined as in Chapter 1:

$$\mathcal{F} : \dot{C}^\infty({}^a\overline{T}M, {}^a\Omega^{\frac{1}{2}}) \longrightarrow \dot{C}^\infty({}^a\overline{T}^*M, {}^a\Omega^{\frac{1}{2}}),$$
$$(\mathcal{F}f)(\gamma) = \int_{v \in {}^aT_{(p,\epsilon)}M} e^{-i\gamma(v)} f(v)\, \lambda_{p,\epsilon}, \quad \gamma \in {}^aT^*_{(p,\epsilon)}M,$$

extends to act on distributions:

$$\mathcal{F} : C^{-\infty}({}^a\overline{T}M, {}^a\Omega^{\frac{1}{2}}) \longrightarrow C^{-\infty}({}^a\overline{T}^*M, {}^a\Omega^{\frac{1}{2}}).$$

Above $\lambda_{p,\epsilon}$ is the canonical half-density on ${}^aT_{p,\epsilon}M \oplus {}^aT^*_{p,\epsilon}M$. As always, if ω is the (adiabatic) symplectic form on ${}^aT^*M$, we use $|\omega^{2n-1}/(2n-1)!|^{\frac{1}{2}}$ to trivialize the half-density bundle on ${}^aT^*M$.

We are now ready for:

Definition 3.7. *An adiabatic Heisenberg operator of order $m \in \mathbb{R}$ on M is an operator A as in (3.7) whose Schwartz kernel $A = \tilde{K}_A$ satisfies:*

(3.18) $A \in C^{-\infty}(M_a^2, {}^a\Omega^{\frac{1}{2}})$ *with* $\operatorname{singsupp} A \subset \Delta_a$;

(3.19) $A \equiv 0$ *at* $\partial M_a^2 \setminus \operatorname{int}(\operatorname{ff}(M_a^2))$ *(i.e. vanishes to infinite order);*

(3.20) $\mathcal{F}\,W^*(\chi A) \in S^m_{La}$ *for some choice of a metric as in (3.14) (and hence of a Riemann-Weyl fibration W as in §3.3) and for every $\chi \in C_c^\infty(M_a^2)$, with $\operatorname{supp}\chi \cap \operatorname{ff}(M_a^2) \subset \overline{V}$, and $\chi \equiv 1$ near $\Delta_a \cup \Delta_0$.*

We denote by $\Psi^m_{aH}(M; {}^a\Omega^{\frac{1}{2}}, {}^a\Omega^{\frac{1}{2}}) \subset C^{-\infty}(M_a^2, {}^a\Omega^{\frac{1}{2}})$ the space of such operators.

For complex vector bundles E and F over $M \times [0, \epsilon_0)$, the space of the adiabatic Heisenberg operators from sections of E to sections of F is:

$$\Psi^m_{aH}(M; E, F) = \\ = \Psi^m_{aH}(M; {}^a\Omega^{\frac{1}{2}}, {}^a\Omega^{\frac{1}{2}}) \otimes_{C^\infty(M_a^2)} C^\infty\big(M_a^2, \beta^* \operatorname{Hom}(E \otimes {}^a\Omega^{-\frac{1}{2}}, F \otimes {}^a\Omega^{-\frac{1}{2}})\big).$$

The *principal symbol* of an operator A as above is defined as the class:

$$\sigma_m(A) = [\mathcal{F}\,W^*(\chi A)] \in S^m_{La}/S^{m-1}_{La}.$$

Remark 3.8. As follows directly from Definition 3.7 and the first part of Lemma 3.5, an adiabatic operator $A \in \Psi^*_{aH}(M)$ can be understood on $(0, \epsilon_0)$ as a smooth family of Heisenberg operators on $M = \mathbb{S}^*X$

$$A_\epsilon \in \Psi^*_{aH}(M), \quad 0 < \epsilon < \epsilon_0,$$

defined by restricting the kernel of A to the slices $M \times M \times \{\epsilon\}$, $\epsilon > 0$, of M_a^2. Of course, the symbol $a = \mathcal{F}\tilde{W}^*(\chi K_A)$ and the principal symbol of A restrict for each $\epsilon > 0$ to the ones of A_ϵ:

$$\sigma_m(A)|_{M \times \{\epsilon\}} = \sigma_m(A_\epsilon), \quad 0 < \epsilon < \epsilon_0.$$

The restriction of A to $\epsilon = 0$ will be described in the next section.

We have:

Proposition 3.9. *The principal symbol $\sigma_m(A)$ is independent of the choice of the metric (3.14) and the cut-off function χ in its definition. It gives a short exact sequence:*

(3.21) $$0 \to \Psi^{m-1}_{aH} \longrightarrow \Psi^m_{aH} \xrightarrow{\sigma_m} S^m_{La}/S^{m-1}_{La} \to 0.$$

Proof. Let a_1 and a_2 be $\mathcal{F}W^*(\chi A)$ for two different choices of metric and cut-off function. To prove the independence, we have to show that $q = \rho^m(a_1 - a_2) \in C^\infty({}^{La}\overline{T}^*M)$ is in fact in $\rho C^\infty({}^{La}\overline{T}^*M)$. For $\epsilon > 0$ this is, in view of the remark above, the independence for Heisenberg operators proved in [6]. Consequently $q \equiv 0$ on ${}^{La}\mathbb{S}^*M$ for $\epsilon > 0$ and, being smooth, it also vanishes at $\epsilon = 0$, i.e. $q \in \rho C^\infty({}^{La}\overline{T}^*M)$. The exactness of the sequence follows easily. □

The parabolic co-sphere bundle splits into the upper and lower hemisphere bundles

$$ {}^{La}\mathbb{S}^*M = {}^{La}\mathbb{S}^*_+M \cup {}^{La}\mathbb{S}^*_-M $$

diffeomorphic to the quadratic compactifications ${}^q\overline{W}_+$ and ${}^q\overline{W}_-$ of the the hyperplane bundle:

$$ W = {}^aT^*M/L \otimes L^{-\frac{1}{2}} \longrightarrow M \times [0, \epsilon_0). $$

Each non-zero section $\tilde{\alpha}$ of L, i.e an adiabatic contact form, gives a symplectic structure on the annihilator of L, the contact hyperplane $H \subset {}^aTM$. Thus, as is in the non-adiabatic case, W is canonically a symplectic vector bundle and hence, as in §1.2, isotropic algebras can be defined on its fibers.

Remark 3.10. The range of the principal symbol in (3.21) can be replaced, as in (1.10), with the set of

$$ (a_+, a_-) \in \Psi^m_{\mathrm{Is}}(W_+; L^{-\frac{m}{2}}) \oplus \Psi^m_{\mathrm{Is}}(W_-; L^{-\frac{m}{2}}), \quad \text{s.t.} \quad a_+|_{\mathbb{S}W} \cong a_-|_{\mathbb{S}W} \tag{3.22} $$

the equality of a_+ and a_- being in Taylor series at $\mathbb{S}W$. As in §1.2, in the case of vector bundle coefficients $\Psi^m_{\mathrm{Is}}(W_\pm, L^{-\frac{m}{2}} \otimes \hom(E, F))$ should be used above.

The lower and upper *Hermite ideals*, $\mathcal{I}^*_{a\pm}(M)$, *generalized Szegő projections* and the associated *Toeplitz algebras*, $\mathcal{T}^*_{aS}(M)$, can be defined as in the compact case of [6], recalled in Chapter 1.

Remark 3.11. Given an adapted complex structure on the fibers of W and hence a field of ground state projections $s \in \Psi^{-\infty}_{\mathrm{Is}}(W_+)$, it can be shown that generalized Szegő projections $S \in \mathcal{I}^*_{a+}(M)$ with $\sigma_0(S)_+ = s$ exist. Since this will not be needed later, we do not prove it here. However, such examples of generalized Szegő projections will be provided in the next chapter, Theorem 4.3.

3.5 Normal operator

For $A \in \Psi^*_{aH}(M; E, F)$ the normal operator is defined as the restriction of A to the front face:

$$ N(A) = A|_{\mathrm{ff}(M^2_a)}. $$

Recall from §3.2 that $\mathrm{ff}(M^2_a)$ is fibered over X, (3.11) and (3.12), with fibers:

$$ T_xX \times \mathbb{S}^*_xX \times \mathbb{S}^*_xX, \quad x \in X. $$

We will show that the restriction of the kernel of A to each of this fibers is the kernel of a translation invariant Heisenberg operator on $T_xX \times \mathbb{S}^*_xX$:

$$N(A)_x \in \Psi^*_{tiH}(T_xX \times \mathbb{S}^*_xX; E_{0x}, F_{0x}), \quad x \in X.$$

The bundles $E_0 \to M$ and $F_0 \to M$ are the restrictions of $E, F \to M \times [0, \epsilon_0)$ to $M \times \{0\}$. E_{0x} and F_{0x} are the restrictions to the fibers $\mathbb{S}^*_xX$, $x \in X$, of M, and, as in §2.1, they are the appropriate vector bundle coefficients of translation invariant Heisenberg operators on $T_xX \times \mathbb{S}^*_xX$. This restrictions should also be understood at the level of kernels, i.e. at $\mathrm{ff}(M^2_a)$.

To prove this we need to understand better the restriction to $\epsilon = 0$ of the symbols in Definition 3.6, or, for the beginning, the structure of the boundary face ${}^{La}\overline{T}^*M|_0$ on which the symbols are defined. This is of course the L-parabolic compactification of the vector bundle ${}^aT^*M|_0 \to M$ which splits according to (3.3). Further restriction of ${}^aT^*M|_0$ and ${}^{La}\overline{T}^*M|_0$ to the fibers $\mathbb{S}^*_xX$, $x \in X$, of $M = \mathbb{S}^*X$ gives:

$${}^aT^*M|_{x,0} = T^*_xX \oplus T^*(\mathbb{S}^*_xX) \longrightarrow \mathbb{S}^*_xX, \qquad x \in X$$

and

$${}^{La}\overline{T}^*M|_{x,0} = {}^{L^x}\overline{T^*_xX \oplus T^*(\mathbb{S}^*_xX)} \longrightarrow \mathbb{S}^*_xX, \qquad x \in X,$$

where L^x is the restriction of the adiabatic contact line L to $\mathbb{S}^*_xX$ at $\epsilon = 0$ as described in Lemma 3.1. By the same lemma, L^x is the translation invariant contact structure on $T^*_xX \oplus T^*(\mathbb{S}^*_xX)$.

We claim that

Proposition 3.12. *The normal operator N_a, defined as the restriction of the kernels to the front face $\mathrm{ff}(M^2_a)$, gives a short exact sequence:*

$$(3.23) \quad 0 \longrightarrow \epsilon\Psi^m_{aH}(M; E, F) \hookrightarrow \Psi^m_{aH}(M; E, F) \xrightarrow{N_a} \Psi^m_{tiH}(TX \times_X \mathbb{S}^*X; E|_0, F|_0) \longrightarrow 0.$$

Moreover, for each $x \in X$

$$\sigma_m(N(A)_x) = \sigma_m(A)|_{\mathbb{S}^*_xX \times \{0\}}.$$

Proof. Follows directly from the construction of $\Psi^m_{aH}(M)$ and $\Psi^m_{tiH}(TX \times_X \mathbb{S}^*X)$, i.e. all the data in Definition 3.20 and Definition 2.2 match. □

3.6 Mapping properties and composition

The operators $A \in \Psi^m_{aH}(M; {}^a\Omega^{\frac{1}{2}}, {}^a\Omega^{\frac{1}{2}}) \subset C^{-\infty}(M^2_a, {}^a\Omega^{\frac{1}{2}})$ have better mapping properties than initially suggested in (3.7). Of course, for $\epsilon > 0$ they inherit the mapping properties of the Heisenberg operators on M. Few words have to be said at $\epsilon = 0$.

Recall the two possible representations of its kernel, A and K_A. They are identified via

(3.10), identification written in local coordinates as:

$$A(X,X',y,y',\epsilon)|dX\frac{dX'}{\epsilon^n}dy\,dy'|^{\frac{1}{2}} \longleftrightarrow K_A(x,y,x',y',\epsilon)|\frac{dx}{\epsilon^n}\frac{dx'}{\epsilon^n}dy\,dy'|^{\frac{1}{2}}. \tag{3.24}$$

For example, the kernel of the identity operator is given in this two representations by

$$Id = \delta(X)\delta(y-y')|dX\frac{dX'}{\epsilon^n}dy\,dy'|^{\frac{1}{2}}$$

and

$$K_{Id} = \epsilon^n\delta(x-x')\delta(y-y')|\frac{dx}{\epsilon^n}\frac{dx'}{\epsilon^n}dy\,dy'|^{\frac{1}{2}}.$$

As follows from (3.9) and (3.17), the action of A on

$$u = u(x,y,\epsilon)|\frac{dx}{\epsilon^n}dy|^{\frac{1}{2}} \in \dot{C}^\infty(M\times[0,\epsilon_0),{}^a\Omega^{\frac{1}{2}})$$

can be expressed in terms of K_A as:

$$(Au)(x,y,\epsilon)|\frac{dx}{\epsilon^n}dy|^{\frac{1}{2}} = \int_M K_A(x,y,x',y',\epsilon)\;u(x',y',\epsilon)\,|\frac{dx'}{\epsilon^n}dy'||\frac{dx}{\epsilon^n}dy|^{\frac{1}{2}}. \tag{3.25}$$

and in terms of $A \in C^{-\infty}(M_a^2,{}^a\Omega^{\frac{1}{2}})$:

$$(Au)(x,y,\epsilon)|\frac{dx}{\epsilon^n}dy|^{\frac{1}{2}} = \int A(X,x-\frac{\epsilon}{2}X,y,y',\epsilon)\;u(x-\epsilon X,y',\epsilon)|dXdy'||\frac{dx}{\epsilon^n}dy|^{\frac{1}{2}}. \tag{3.26}$$

We have:

Proposition 3.13. *Each $A \in \Psi^m_{aH}(M;{}^a\Omega^{\frac{1}{2}},{}^a\Omega^{\frac{1}{2}})$ defines a continuous linear operator*

$$A : \dot{C}^\infty(M\times[0,\epsilon_0),{}^a\Omega^{\frac{1}{2}}) \longrightarrow \dot{C}^\infty(M\times[0,\epsilon_0),{}^a\Omega^{\frac{1}{2}}) \tag{3.27}$$

and by duality

$$A : C^{-\infty}(M\times[0,\epsilon_0),{}^a\Omega^{\frac{1}{2}}) \longrightarrow C^{-\infty}(M\times[0,\epsilon_0),{}^a\Omega^{\frac{1}{2}}).$$

Furthermore, $A : C^\infty(M\times[0,\epsilon_0),{}^a\Omega^{\frac{1}{2}}) \longrightarrow C^\infty(M\times[0,\epsilon_0),{}^a\Omega^{\frac{1}{2}})$.
For maximally residual *operators* $A \in \epsilon^\infty\Psi^{-\infty}_{aH}(M;{}^a\Omega^{\frac{1}{2}},{}^a\Omega^{\frac{1}{2}})$, *i.e. those with kernels* $A \in \dot{C}^\infty(M_a^2,{}^a\Omega^{\frac{1}{2}}) = \dot{C}^\infty(M\times M\times[0,\epsilon_0),{}^a\Omega^{\frac{1}{2}})$, *more is true:*

$$A : C^\infty([0,\epsilon_0),C^{-\infty}(M))\otimes C^\infty({}^a\Omega^{\frac{1}{2}}) \longrightarrow \dot{C}^\infty(M\times[0,\epsilon_0),{}^a\Omega^{\frac{1}{2}}). \tag{3.28}$$

Furthermore, this mapping property characterizes the maximally residual operators.

Proof. We start with the maximal residual ones. For these the statement is a slight variation of the well known mapping properties of smoothing operators on closed manifolds. Indeed, as a smooth family of smoothing operators on M, $A \in \epsilon^\infty \Psi_{aH}^{-\infty}$ maps into smooth densities. Additionally, the vanishing of A to infinite order at $\epsilon = 0$ proves (3.28). Reversing the argument, an operator having this mapping property is a smooth family of smoothing operators on M and thus it has a smooth kernel A on $M \times M \times [0, \epsilon_0)$. The extra vanishing requirement for its range implies that in fact A vanishes to infinite order at $\epsilon = 0$.

Returning to (3.27), Au is smooth for $\epsilon > 0$ since, as can be seen in (3.25), it is just the parametric version of the fact that the Heisenberg operators preserve the smoothness. To prove that Au is smooth up to $\epsilon = 0$ we use the representation (3.26), in which the kernel A is smooth outside $X = 0$ and, most importantly, rapidly decreasing as $|X| \to \infty$. Because of this rapid decrease the smoothness of the integral in X and y' follows as in the compact case, i.e. by integration by parts and symbol estimates. In fact, this proves that Au is smooth on $M \times [0, \epsilon_0)$ for u smooth, even without any rapid vanishing assumption on u. Of course, since any power of ϵ in u can be taken in front of the integral, this also proves (3.27). □

Vector bundle coefficients can be added without difficulty to the mapping properties above.

As follows from Proposition 3.13, adiabatic Heisenberg operators can be composed. We claim that the composite is also an adiabatic Heisenberg operator. Of course, for $\epsilon > 0$ this is just a parametric form of the composition in $\Psi_H^*(M)$. In fact, this is the case uniformly to $\epsilon = 0$.

First recall from [6] the composition formula for Heisenberg operators. The kernel of $A \in \Psi_H^m(M)$ is given in terms of its symbol $a \in S_L^m(T^*M)$ by (1.3), which in local coordinates near the diagonal is written as:

$$A(x, y, x', y') = (2\pi)^{-(2n-1)} \int e^{i\left((x-x')\cdot\xi + (y-y')\cdot\eta\right)} \, a\big(\frac{x+x'}{2}, \frac{y+y'}{2}, \xi, \eta\big) \, d\xi d\eta. \tag{3.29}$$

Choose now local coordinates $z = (x, y) = (t, u, y)$ on $M = \mathbb{S}^*X$ better adapted to the contact structure and to the fibration, i.e. in which a contact form is given by

$$\alpha = dt + \sum_{i=1}^{n-1} y_i \, du_i$$

The vector fields ∂_{y_i} and $\partial_{u_i} - y_i\partial_t$ span locally the contact field, whereas ∂_t is the Reeb vector field. Instead of the dual coordinates $(\xi, \eta) = (\tau, \mu, \eta)$ on the fibers of $T^*(\mathbb{S}^*X)$, we prefer to work with the symbols of the above vector fields as linear coordinates on the fibers. In these new coordinates on $T^*(\mathbb{S}^*X)$:

$$(z, \sigma) = (t, u, y, \tau, \zeta, \eta), \quad \zeta_i = \mu_i - y_i\tau, \tag{3.30}$$

the parabolic dilatation is given by $(\tau, \zeta, \eta) \to (\lambda^2\tau, \lambda\zeta, \lambda\eta)$, $\lambda > 0$.

Write $a^\#$ for a in the new coordinates, $a^\#(z, \sigma) = a(z, (\xi, \eta))$. The upper and lower part of the principal symbol of A, lying in the isotropic algebras on the symplectic hyperplane bundle W, are then given by:

$$\sigma_m(A)_\pm(z,\bar{\zeta},\bar{\eta}) = \lim_{\tau\to\infty} \tau^{-m/2}\, a^{\#}(z,\pm\tau,\tau^{1/2}\bar{\zeta},\tau^{1/2}\bar{\eta}) \tag{3.31}$$

where $(\bar{\zeta},\bar{\eta})$ are local coordinates on W.

In the new coordinates, (3.29) reads as:

$$\begin{gathered} A(z,z') = (2\pi)^{-(2n-1)} \int e^{i\phi(z,z',\sigma)}\, a^{\#}\big(\frac{z+z'}{2},\sigma\big)\, d\sigma, \\ \phi(z,z',\sigma) = (z-z')\cdot\sigma + (u-u')\cdot\frac{y+y'}{2}\tau \end{gathered} \tag{3.32}$$

and

$$a^{\#}(z,\sigma) = \int e^{-i\phi(z+w/2,z-w/2,\sigma)} A\big(z+\frac{w}{2}, z-\frac{w}{2}\big)\, dw. \tag{3.33}$$

It is shown in [6] that if A and B are two Heisenberg operators then so is the composite $C = A\circ B$. Its symbol can be computed using (3.32) and (3.33):

$$\begin{aligned} c^{\#}(z,\sigma) \;=\; \pi^{-(4n-2)} \int & e^{2i(z_1\sigma_2 - z_2\sigma_1 + (u_1y_2-u_2y_1)\tau)}\; a^{\#}(t+t_1-u_1(y+y_2),u+u_1,y+y_1,\sigma+\sigma_1) \\ & b^{\#}(t+t_2-u_2(y+y_1),u+u_2,y+y_2,\sigma+\sigma_2) dz_1 dz_2 d\sigma_1 d\sigma_2. \end{aligned} \tag{3.34}$$

This expression is slightly different from the one obtained in [6], but only because we use local coordinates in which the expression of the contact form in not symmetric. We will however do this computation, or rather a somewhat more complicated version of it, in §4.3.

Let now $A \in \Psi^m_{aH}(M; {}^a\Omega^{\frac12}, {}^a\Omega^{\frac12}) \subset C^{-\infty}(M^2_a, {}^a\Omega^{\frac12})$ be an adiabatic Heisenberg operator on M. As follows from Definition 3.7, the representations (3.24) of its kernel are written as:

$$K_A(x,y,x',y',\epsilon) = (2\pi)^{-(2n-1)} \int e^{i\left((x-x')\cdot\xi/\epsilon + (y-y')\cdot\eta\right)} a\big(\frac{x+x'}{2},\frac{y+y'}{2},\epsilon,\xi,\eta\big)\, d\xi d\eta \tag{3.35}$$

in local coordinates near $\mathrm{Diag}(M)$ for $\epsilon > 0$ and as

$$A(X,X',y,y',\epsilon) = (2\pi)^{-(2n-1)} \int e^{i\left(X\cdot\xi + (y-y')\cdot\eta\right)}\, a\big(X',\frac{y+y'}{2},\epsilon,\xi,\eta\big)\, d\xi d\eta \tag{3.36}$$

in local coordinates near $\Delta_a \cup \Delta_0$ for small ϵ. Above, a is a parabolic symbol on ${}^aT^*M$ and $a(x,y,\epsilon,\xi,\eta)$ is its local coordinate expression at a point $\gamma = \xi_i \frac{dx_i}{\epsilon} + \eta_i dy_i$ in ${}^aT^*M$.

Take on M local coordinates $z = (x,y) = (t,u,y)$ as above in which the adiabatic contact form is given by

$$\tilde{\alpha} \;=\; \alpha/\epsilon \;=\; \frac{dt}{\epsilon} + \sum_{i=1}^{n-1} y_i \frac{du_i}{\epsilon}.$$

As in (3.13), the induced coordinates on M_a^2 near $\Delta_a \cup \Delta_0$ are:

$$(T, T', U, U', y, y', \epsilon) \quad \text{with}$$
$$T = \frac{t-t'}{\epsilon},\ T' = \frac{t+t'}{2},\ U = \frac{u-u'}{\epsilon},\ U' = \frac{u+u'}{2}.$$

Also take (z, ϵ, σ) as coordinates on ${}^aT^*M$, where (z, σ) are given in (3.30).

By analogy with (3.32), writing (3.35) and (3.36) in the new coordinates gives

$$K_A(z, z', \epsilon) = (2\pi)^{-(2n-1)} \int e^{i\phi(z,z',\sigma,\epsilon)}\, a^{\#}\big(\frac{z+z'}{2}, \epsilon, \sigma\big)\, d\sigma, \tag{3.37}$$
$$\phi(z, z', \sigma, \epsilon) = \phi(z, z', \tilde{\sigma}) \quad \text{with} \quad \tilde{\sigma} = (\frac{\tau}{\epsilon}, \frac{\zeta}{\epsilon}, \eta)$$

and

$$A(T, T', U, U', y, y', \epsilon) = (2\pi)^{-(2n-1)} \int e^{i\phi'(Z,Z',\sigma)}\, a^{\#}\big(Z', \epsilon, \sigma\big)\, d\sigma, \tag{3.38}$$
$$Z = (T, U, y - y'), \quad Z' = (T', U', \frac{y+y'}{2}) \quad \text{and} \quad \phi'(Z, Z', \sigma) = Z \cdot \sigma + U \cdot \frac{y+y'}{2}\tau.$$

Taking Fourier transform in (3.38) and (3.37) gives $a^{\#}$ in terms of A:

$$a^{\#}(z, \epsilon, \sigma) = \int e^{-i\phi'(Z,z,\sigma)}\, A\big(T, t, U, u, y + \frac{v}{2}, y - \frac{v}{2}, \epsilon\big)\, dZ, \qquad Z = (T, U, v) \tag{3.39}$$

and of K_A:

$$a^{\#}(z, \epsilon, \sigma) = \int e^{-i\phi'(\tilde{Z},z,\tilde{\sigma})}\, K_A\big(z + \frac{\tilde{Z}}{2}, z - \frac{\tilde{Z}}{2}, \epsilon\big)\, dZ, \tag{3.40}$$
$$Z = (T, U, v), \quad \tilde{Z} = (\epsilon T, \epsilon U, v).$$

Finally, the adiabatic Heisenberg operators form an algebra:

Theorem 3.14. *Adiabatic Heisenberg pseudodifferential operators are closed under composition:*

$$\Psi_{aH}^{m} \circ \Psi_{aH}^{m'} \subset \Psi_{aH}^{m+m'}$$

and the principal symbol map in (3.21) and the normal operator in (3.23) are algebra morphisms:

$$\sigma_{m+m'}(A \circ B)_{\pm} = \sigma_m(A)_{\pm} \# \sigma_{m'}(B)_{\pm},$$
$$N(A \circ B) = N(B) \circ N(B).$$

Proof. For the composition, it suffices to consider kernels supported near $\Delta_a \cup \Delta_0$. As follows from (3.25):

$$K_{A\circ B}(x,y,x',y',\epsilon) = \int K_A(x,y,x'',y'',\epsilon)\, K_B(x'',y'',x',y',\epsilon)|\frac{dx''}{\epsilon^n}dy''|$$

and lifting to M_a^2:

$$(A\circ B)(X,X',y,y',\epsilon) = \int A(X-X'',X'+\frac{\epsilon}{2}X'',y,y'',\epsilon)\cdot \\ B(X'',X'-\frac{\epsilon}{2}(X-X''),y'',y',\epsilon)|dX''dy''|.$$

The same computation as in (3.34), i.e. rewriting the previous expressions using (3.37)-(3.40), shows that the kernel of $C = A\circ B$ can be represented in the form (3.37), or equivalently (3.38), with $c^{\#}$ given by:

$$c^{\#}(z,\epsilon,\sigma) = \pi^{-(4n-2)} \int e^{2i(z_1\sigma_2 - z_2\sigma_1 + (u_1y_2-u_2y_1)\tau)}\; a^{\#}(t+\epsilon t_1-\epsilon u_1(y+y_2),u+\epsilon u_1,y+\epsilon y_1,\sigma+\sigma_1) \\ b^{\#}(t+\epsilon t_2-\epsilon u_2(y+y_1),u+\epsilon u_2,y+\epsilon y_2,\sigma+\sigma_2)dz_1dz_2d\sigma_1d\sigma_2.$$

This expression is uniform in ϵ up to $\epsilon = 0$ and hence the argument in the non-adiabatic case can be implemented without change. See [6] and a version of it later in §4.3. Briefly, one needs to understand the behavior of $c^{\#}$ near the parabolic cosphere bundle, i.e. to understand the limit in (3.31). The stationary phase lemma is employed for computing the limit and it also gives the isotropic composition of the symbols as stated in the statement. For the composition of the normal operators, take $\epsilon = 0$ in the expression of $c^{\#}$ and keep the first coordinate, x, in $z = (x,y)$ fixed. The composition of normal operators is obtained. □

3.7 Ellipticity and invertibility

Proposition 3.15. *Let $A \in \Psi^m_{aH}(M)$ be* fully elliptic, *i.e. it has invertible principal symbol $\sigma_m(A)_\pm \in \Psi^m_{\mathrm{Is}}(W_\pm)$ in the isotropic algebra and invertible normal operator $N(A) \in \Psi^m_{tiH}(TX\times_X \mathbb{S}^*X)$. Then there exists a parametrix $B \in \Psi^{-m}_{aH}$ such that:*

$$A\circ B - Id,\; B\circ A - Id \in \epsilon^\infty \Psi^{-\infty}_{aH}(M).$$

Moreover, A restricts to be invertible in the adiabatic Heisenberg algebra on $M\times[0,\epsilon_1)$ for some $\epsilon_1 < \epsilon_0$.

Proof. The general parametrix construction can be implemented here, using the invertibility of both $\sigma_m(A)$ and $N(A)$, to reduce at each step the order of the error and increase its vanishing order in ϵ. More precisely, by Proposition 3.9, Remark 3.10 and Proposition 3.12, there exists an operator $B_0 \in \Psi^{-m}_{aH}(M)$ with

$$\sigma_{-m}(B_0)_\pm = \sigma_m(A)^{-1}_\pm \quad\text{and}\quad N(B_0)_x = N(A)^{-1}_x,\; \forall x\in X.$$

This gives an error
$$Id - A \circ B_0 = \epsilon R_1 \in \epsilon \Psi^{-1}_{aH}(M).$$
Assume now that $B_j \in \Psi^{-m-j}_{aH}(M)$, $j = 0, \dots, k-1$, have been constructed to further reduce the error to:
$$Id - A \circ (B_0 + \epsilon B_1 + \cdots \epsilon^{k-1} B_{k-1}) = \epsilon^k R_k \in \epsilon^k \Psi^{-k}_{aH}(M).$$
Again, by the invertibility of $\sigma_m(A)$ and $N(A)$, there exist $B_k \in \Psi^{-m-k}_{aH}(M)$ with
$$\sigma_{-m-k}(B_k)_\pm = \sigma_m(A)^{-1}_\pm \# \sigma_{-k}(R_k)_\pm \quad \text{and} \quad N(B_k)_x = N(A)^{-1}_x \circ N(R_k)_x,\ \forall x \in X.$$
This gives the next step in the induction and the asymptotic sum of $\epsilon^k B_k$, $k \in \mathbb{N}_0$, provides a right parametrix and, as usual, also a left parametrix.

For the second part, recall that the remainder $R = Id - A \circ B \in \epsilon^\infty \Psi^{-\infty}_{aH}(M)$ is a smooth function on $M \times M \times [0, \epsilon_0)$ (and also on M^2_a) vanishing to infinite order at $\epsilon = 0$. In particular, its norm can be made arbitrarily small by restricting to $[0, \epsilon_1)$ for small enough ϵ_1. Consequently, $Id - R$ is invertible, for example as an operator on $\dot{C}^\infty(M \times [0, \epsilon_1))$. Moreover, its inverse is of the form
$$(Id - R)^{-1} = Id - R'$$
with
$$R' = -R - R^2 + RR'R.$$
Proposition 3.13 implies that $RR'R$ is maximally residual and hence $R' \in \epsilon^\infty \Psi^{-\infty}_{aH}(M)$. Thus $Id - R'$ is in the adiabatic Heisenberg algebra (on $M \times [0, \epsilon_1)$) and $B' = B \circ (Id - R')$ is the required inverse of A. □

The proof above can be adapted to show that a similar result also holds for Toeplitz operators:

Proposition 3.16. *Let $S \in \mathcal{I}^0_{a+}(M)$ be a generalized Szegő projection and $SAS \in \mathcal{I}^m_{a+}(M)$ a* fully elliptic *Toeplitz operator, meaning that the principal symbol is invertible on the range of $s = \sigma_0(S)$ and normal operator is invertible on the range of $N(S)$. Then SAS is invertible as a Toeplitz operator on $M \times [0, \epsilon_1)$ for some $\epsilon_1 < \epsilon_0$.*

Chapter 4

Degeneration of the Szegő projection and the push-forward map

In defining the adiabatic Heisenberg operators in the earlier sections only the canonical contact structure on $\mathbb{S}^*X$ has been used. On the other hand, as will be recalled in the next section, if X is real-analytic and a real-analytic metric is given on X a complex structure can be introduced in the neighborhood of the zero section in T^*X such that the sphere bundles $\mathbb{S}^*_\epsilon X$ in this metric carry a strictly pseudoconvex Cauchy-Riemann structure. The underlying contact structures of these CR structures are all contactomorphic with the one on $\mathbb{S}^*X$.

The initial objective of the current chapter is to put together these notions, the adiabatic structure on $\mathbb{S}^*X \times [0, \epsilon_0)$ and the CR structures on $\mathbb{S}^*_\epsilon X$, $0 < \epsilon < \epsilon_0$. An adiabatic Szegő projection S is then constructed in Ψ^0_{aH} which for positive ϵ's restricts to the Szegő projection on $\mathbb{S}^*_\epsilon X$ and has as limit at $\epsilon = 0$ the model ones on $\mathbb{R}^n \times \mathbb{S}^{n-1}$. The existence of S allows us to use the adiabatic Heisenberg calculus to prove that the push-forward map is an isomorphism for ϵ small.

4.1 The Cauchy-Riemann structure

Suppose that a real-analytic structure refining the differentiable one has been chosen on X; this is always possible as follows from Nash [16]. By a result of Bruhat and Whitney [3], X, as a real-analytic manifold, can be embedded as a totally real submanifold in an n-dimensional complex manifold $\tilde{X}$, the complexification of X. In fact, Grauert [7] proves the existence of a smooth strictly plurisubharmonic function $\rho \geq 0$ in a neighborhood of X such that $X = \rho^{-1}(0)$; in particular the tubes $\rho^{-1}([0, \epsilon)$, called Grauert tubes, are strictly pseudoconvex. As shown by Guillemin-Stenzel [9] and Lempert-Szöke [12], this complex structure can now be transferred to a neighborhood of the zero section in T^*X such that the Grauert tubes are exactly the co-ball bundles associated to some given real-analytic metric on X.

More precisely, following [9], let $h : T^*X \to [0, \infty)$ denote the square of the length function, $h(\xi) = |\xi|^2$, for some real-analytic metric on X, also denoted by h. Also consider the involution $\sigma : T^*X \to T^*X$, $\sigma(\xi) = -\xi$. Then there exist a σ-invariant neighborhood U of X in T^*X and a unique complex structure on U such that

$$\sigma \text{ is anti-holomorphic,} \tag{4.1}$$

$$-\operatorname{Im}\overline{\partial}h = \beta = \sum \xi_i \, dx_i, \tag{4.2}$$

(this complex structure is in fact the conjugate of the one originally introduced in [9]). Additionally the inclusion $i : X \to U \subset T^*X$ is an isometric embedding, where on U the Kähler metric given by the Kähler form $\omega = d\beta = i\partial\overline{\partial}h$ is considered.

For $\epsilon > 0$ small enough, $0 < \epsilon < \epsilon_0$, the co-ball bundles

$$\mathbb{B}^*_\epsilon X = \{h(x,\xi) \leq \epsilon^2\} \subset U$$

are strictly pseudoconvex. The strictly pseudoconvex CR structure induced on their boundaries, $\mathbb{S}^*_\epsilon X$, is defined by the $n-1$ rank complex subbundle

$$T^{1,0}\mathbb{S}^*_\epsilon X = T^{1,0}\mathbb{B}^*_\epsilon X|_{\mathbb{S}^*_\epsilon X} \cap \mathbb{C} \otimes T\mathbb{S}^*_\epsilon X.$$

Set $T^{0,1}\mathbb{S}^*_\epsilon X = \overline{T^{1,0}\mathbb{S}^*_\epsilon X}$ and note that $T^{1,0}\mathbb{S}^*_\epsilon X \oplus T^{0,1}\mathbb{S}^*_\epsilon X = \mathbb{C} \otimes \ker \alpha_\epsilon$, where

$$\alpha_\epsilon = i\overline{\partial}g|_{\mathbb{S}^*_\epsilon X} = -i\partial g|_{\mathbb{S}^*_\epsilon X} = -\operatorname{Im}\overline{\partial}g|_{\mathbb{S}^*_\epsilon X} = \beta|_{\mathbb{S}^*_\epsilon X} \tag{4.3}$$

is a contact form on $\mathbb{S}^*_\epsilon X$ and $\ker \alpha_\epsilon \subset T(\mathbb{S}^*_\epsilon X)$ the contact hyperplane field. Also denote by J_ϵ the restriction to $\ker \alpha_\epsilon$ of the complex structure on $U \subset T^*X$ and by g_ϵ the restriction to $\mathbb{S}^*_\epsilon X$ of the metric on U.

As (4.3) shows, the main consequence of (4.2) is that the contact structure on $\mathbb{S}^*_\epsilon X$ as a strictly pseudoconvex CR manifold (given by α_ϵ) agrees with the one induced as a submanifold in T^*X (given by $\beta|_{\mathbb{S}^*_\epsilon X}$).

We will next transfer these CR structures on $\mathbb{S}^*_\epsilon X$, $0 < \epsilon < \epsilon_0$ to an adiabatic family of CR structures on $\mathbb{S}^*X = (T^*X \setminus \{\mathbf{0}\})/\mathbb{R}_+$. First note that the canonical projections

$$p_\epsilon : \mathbb{S}^*_\epsilon X \to \mathbb{S}^*X$$

are contact diffeomorphisms for each $\epsilon > 0$. Also note that the homogeneity of the length function $h : T^*X \to [0,\infty)$ and the definition of α_ϵ imply that the contact forms α_ϵ/ϵ are all mapped by p_ϵ's into the same contact form $\alpha := (p_\epsilon^{-1})^*(\alpha_\epsilon/\epsilon)$ on $\mathbb{S}^*X$. In fact, as shown in §3.1, $\alpha_\epsilon/\epsilon^2$ extends to an adiabatic contact form

$$\tilde{\alpha} : \mathbb{S}^*X \times [0,\epsilon_0) \to {}^aT(\mathbb{S}^*X), \quad \tilde{\alpha}_\epsilon = (p_\epsilon^{-1})^*(\alpha_\epsilon/\epsilon^2).$$

We can use now the identification

$$(p_\epsilon)_* : \ker \alpha_\epsilon \subset T(\mathbb{S}^*_\epsilon X) \longrightarrow \ker \tilde{\alpha}_\epsilon \subset {}^aT(\mathbb{S}^*X)|_\epsilon, \qquad 0 < \epsilon < \epsilon_0,$$

to transfer the complex structure J_ϵ on $\ker \alpha_\epsilon$ to a complex structure $\widetilde{J}_\epsilon$ on $\ker \tilde{\alpha}_\epsilon$ for each positive ϵ. Of course this just introduces a strictly pseudoconvex CR structure on each copy of $\mathbb{S}^*X$ in $\mathbb{S}^*X \times (0, \epsilon_0)$. Our objective is to show that it is well-behaved as $\epsilon \searrow 0$:

Proposition 4.1. *$\widetilde{J}$ extends to a complex structure on* $\ker\tilde{\alpha}$ *up to* $\epsilon = 0$. *Moreover, at* $\epsilon = 0$, *it induces for all* $x \in X$ *the CR structure on* $T_xX \times \mathbb{S}^*_xX$ *associated as in §2.2 to the metric* h_x *on* T_xX.

Proof. Denote by $\mathcal{V}'$ the set of vector fields V' in the interior of $\mathbb{B}^*_{\epsilon_0}X \subset T^*X$ tangent to each $\mathbb{S}^*_\epsilon X$, such that $V' = 0$ at the zero section in T^*X and $\beta(V') \equiv 0$. Note that

$$JV' \in \mathcal{V}', \quad \forall V' \in \mathcal{V}'.$$

They all lift to be smooth adiabatic vector fields on $\mathbb{S}^*X \times [0, \epsilon_0)$; this can be seen by identifying $\mathbb{S}^*X \times [0, \epsilon_0)$ with the blow-up of the zero section in T^*X and checking the statement in local (polar) coordinates. We use the same notation, $\mathcal{V}'$, for their lifts. They do not span all of $\mathcal{V}_a$, but they span over $C^\infty(\mathbb{S}^*X \times [0, \epsilon_0))$ the set $\mathcal{V}'_a \subset \mathcal{V}_a$ of those annihilated by $\tilde{\alpha}$, i.e. each $V \in \mathcal{V}'_a$ can be written as a finite sum:

$$V = \sum a_i V'_i, \quad V'_i \in \mathcal{V}', \quad a_i \in C^\infty(\mathbb{S}^*X \times [0, \epsilon_0)). \tag{4.4}$$

The complex structure J can now be extended from $\mathcal{V}'$ to $\mathcal{V}'_a$:

$$\widetilde{J}V = \sum a_i\, JV'_i, \quad V'_i \in \mathcal{V}'.$$

It is well defined since for $\epsilon > 0$ we recover the $\widetilde{J}$ introduced above; hence two representation (4.4) give the same $\widetilde{J}V$ for $\epsilon > 0$ and by continuity also at $\epsilon = 0$. Being linear over $C^\infty(\mathbb{S}^*X \times [0, \epsilon_0))$, $\widetilde{J}$ descends from $\mathcal{V}'_a$ to a smooth complex structure on the hyperplane bundle $\ker\tilde{\alpha} \subset {}^aT(\mathbb{S}^*X)$.

For the second part fix $\epsilon = 0$ and $x \in X$. A verification in normal coordinates at x proves the statement.

□

4.2 The adiabatic $\overline{\partial}_b$ complex and the Szegő projection

The definition of the $\overline{\partial}_b$-complex on CR manifolds works in the adiabatic context without change, i.e. the adiabatic contact form, $\tilde{\alpha}$, the adiabatic complex structure, $\tilde{J}$, and the adiabatic metric are used instead to construct the exterior powers ${}^a\Lambda^{0,q}(\mathbb{S}^*X)$, $0 \leq q \leq n-1$, to endow them with Hermitian metrics and represent them as subbundles of the complexified adiabatic exterior powers of Chapter 3, ${}^a\Lambda^q(\mathbb{S}^*X) \otimes \mathbb{C}$.

For $0 \leq q \leq n-2$, the $\overline{\partial}_b$ operator on $(0,q)$-forms is given by

$$\overline{\partial}_b : C^\infty(\mathbb{S}^*X \times [0, \epsilon_0), {}^a\Lambda^{0,q}) \longrightarrow C^\infty(\mathbb{S}^*X \times [0, \epsilon_0), {}^a\Lambda^{0,q+1}), \quad \overline{\partial}_b = \Pi_{q+1} \circ d,$$

where d is the differential in (3.4) and Π_q the orthogonal projection from ${}^a\Lambda^q(\mathbb{S}^*X) \otimes \mathbb{C}$ onto ${}^a\Lambda^{0,q}(\mathbb{S}^*X)$.

It is not hard to see that $\overline{\partial}_b$ is an adiabatic Heisenberg operator of order 1. At each $\epsilon > 0$, $\overline{\partial}_b$ restricts to $\overline{\partial}_b^\epsilon$, the operator associated to the CR structure on $\mathbb{S}_\epsilon^* X$. The normal operator $N(\overline{\partial}_b)_x$ is, for each $x \in X$, the translation invariant $\overline{\partial}_b$ on $T_x X \times \mathbb{S}_x^* X \cong \mathbb{R}^n \times \mathbb{S}^{n-1}$. Similar statements hold for its associated Laplacian, $\Box_b$.

Proposition 4.2. *For $n \geq 3$ and $0 < q < n-1$, $\Box_b^{0,q}$ is invertible as an element of the adiabatic Heisenberg algebra on $M \times [0, \epsilon_1)$ for some $\epsilon_1 < \epsilon_0$.*

Proof. Follows from Proposition 3.15 and the invertibility of the normal operator given in Lemma 2.7. □

This allows us to define the (adiabatic) *Szegő projection* on $M \times [0, \epsilon_1)$ by:

$$S = Id - \overline{\partial}_b^* \, (\Box_b^{0,1})^{-1} \, \overline{\partial}_b \in \Psi_{aH}^0(\mathbb{S}^* X).$$

Of course, as follows from the above formula, for each ϵ positive the restriction of S to $\mathbb{S}^* X \times \{\epsilon\}$:

$$S_\epsilon \in \Psi_H^0(\mathbb{S}^* X)$$

is the standard Szegő projection associated to the CR structure on $\mathbb{S}_\epsilon^* X$. Also its normal operator is a family

$$N(S)_x = S^x \in \Psi_{tiH}^0(T_x X \times \mathbb{S}_x^* X), \quad x \in X$$

of translation invariant Szegő projections as in §2.2. This concludes:

Theorem 4.3. *For $n \geq 3$, the family of Szegő projections S_ϵ on the the co-sphere bundles $\mathbb{S}_\epsilon^* X$, $\epsilon > 0$, extends to $\epsilon = 0$ as an element of $\Psi_{aH}^0(\mathbb{S}^* X)$.*

4.3 The push-forward map

Given a metric g on $\mathbb{S}^* X$ consider the push-forward:

$$P : C^\infty(\mathbb{S}^* X) \to C^\infty(X),$$
$$Pu(x) = \int_{\mathbb{S}_x^* X} u(x,y) \, |dg_x(y)|, \quad x \in X,$$

defined as fiber-wise average. The density $|dg_x|$ is defined by the restriction g_x of the metric to the fiber $\mathbb{S}_x^* X$. Also consider the pull-back:

$$P^* : C^\infty(X) \to C^\infty(\mathbb{S}^* X), \quad P^* f = f \circ \phi.$$

In this section we are interested in understanding the maps PSP^* and SP^*PS as elements of the pseudodifferential calculus on X and of the Heisenberg calculus on $\mathbb{S}^* X$ respectively. Using the tools developed in Boutet de Monvel and Guillemin [2], Guillemin [8] has shown that these operators are elliptic and concluded that P is Fredholm on the range of S. We are going to reprove this using the calculus of Epstein and Melrose [6] and extend it to the adiabatic setting in the next section. In fact, slightly more general, we will consider the maps PAP^* and AP^*PB for $A, B \in \mathcal{I}^0_+(\mathbb{S}^*X)$ elements of the Hermite ideal and identify their symbols in the isotropic calculus.

Since we want to avoid the theory of Fourier integral operators with complex phase, we will show first that PAP^* and AP^*PB have the right wave front relations and then, by localizing near points in $\mathbb{S}^*X$ in the positive contact direction, that their singularities are of the expected type.

We have for $A \in \mathcal{I}^0_+(\mathbb{S}^*X)$:

$$WF'(A) \subset L^+ \times_{\mathbb{S}^*X} L^+ \subset T^*(\mathbb{S}^*X) \times T^*(\mathbb{S}^*X)$$

and for P and P^*:

$$\begin{aligned} WF'(P) &= \{(\gamma, \phi^*\gamma) : \gamma \in T^*X \setminus 0\} = \{(x,\xi), (x,y,\xi,0)\} \subset T^*X \times T^*(\mathbb{S}^*X), \\ WF'(P^*) &= \{(\phi^*\gamma, \gamma) : \gamma \in T^*X \setminus 0\} = \{((x,y,\xi,0),(x,\xi))\} \subset T^*(\mathbb{S}^*X) \times T^*X. \end{aligned}$$

Denoting by $C \subset T^*X \times T^*(\mathbb{S}^*X)$ and $C^* \subset T^*(\mathbb{S}^*X) \times T^*X$ the graphs of the identity map:

$$Id : T^*X \setminus 0 \to L^+, \quad (x,\xi) \to (x,[\xi],\xi,0)$$

we get by composing the above relations:

$$\begin{aligned} WF'(PA) &\subset C \subset T^*X \times T^*(\mathbb{S}^*X), \\ WF'(AP^*) &\subset C^* \subset T^*(\mathbb{S}^*X) \times T^*X \end{aligned} \tag{4.5}$$

and finally:

$$\begin{aligned} WF'(PAP^*) &\subset \mathrm{Diag}(T^*X \setminus 0) \subset T^*X \times T^*X, \\ WF'(AP^*PB) &\subset \mathrm{Diag}(L) \subset T^*(\mathbb{S}^*X) \times T^*(\mathbb{S}^*X). \end{aligned}$$

Proposition 4.4. *If $A \in \mathcal{I}^0_{H,+}(\mathbb{S}^*X)$ then $PAP^* \in \Psi^{-\frac{n-1}{2}}(X)$ and*

$$\sigma_{-\frac{n-1}{2}}(PAP^*) = \int_V \sigma_0(A)|_V \, \mu \in C^\infty(\mathbb{S}^*X, L^{\frac{n-1}{2}})$$

where $V \to \mathbb{S}^*X$ is the Lagrangian subbundle of $T^*(\mathbb{S}^*X)/L \otimes L^{-\frac{1}{2}}$ given by:

$$V_{x,y} = T^*_x X / L_{x,y} \otimes L^{-\frac{1}{2}}_{x,y}, \quad (x,y) \in \mathbb{S}^*X$$

and $\mu_{x,y} \in C^\infty(V_{x,y}, \Omega \otimes L^{\frac{n-1}{2}}_{x,y})$ is the lift to $V_{x,y}$ of the density $|dg_x(y)|$ on $T_y(\mathbb{S}^*_x X)$.

Before proving the proposition, the statements above need to be understood canonically. First note that $T^*_x X / L_{x,y} \otimes L^{-1}_{x,y}$ is canonically isomorphic with $T_y(\mathbb{S}^*_x X)$; in general, for an oriented line in a vector space $L \subset W$ and $y \in \mathbb{S}W$ the class of an element in L, the isomorphism is given by:

$$W/L \otimes L^{-1} \ni [w] \otimes l^{-1} \to v \in T_y(\mathbb{S}W)$$
$$v(f) := \partial_s f([l + sw])|_{s=0}, \text{ for any } f \in C^\infty(\mathbb{S}^*X).$$

Thus $dh_x(y)$ lifts to a density $\mu'_{x,y}$ on $T^*_x X/L_{x,y} \otimes L^{-1}_{x,y}$ which in turn can be seen as the density $\mu_{x,y}$ on $V_{x,y} = T^*_x X/L_{x,y} \otimes L^{-\frac{1}{2}}_{x,y}$ with coefficients in $L^{\frac{n-1}{2}}_{x,y}$):

$$\mu_{x,y}(v_1 \wedge \ldots \wedge v_{n-1})(l^*) := \mu'_{x,y}(l^*(v_1) \wedge \ldots \wedge l^*(v_{n-1})), \quad \forall l^* \in L^* \setminus 0.$$

We still have to check that sections of $L^{-m} \to \mathbb{S}^*X$ are naturally the principal symbols of pseudodifferential operators of order m on X. If s is a section of L^{-m} define on T^*X:

$$a(\gamma) := s([\gamma])(l^*) \cdot |\gamma(l^*)|^m, \quad \gamma \in L_{[\gamma]}\backslash 0 \subset T^*X, \ l^* \in L^*_{[\gamma]} \setminus 0.$$

Note that a is well-defined by the homogeneity of s and that it is homogeneous of order m.

Proof of Proposition 4.4. Formally for the beginning, the kernel of PAP^* can be written as

$$PAP^*(x,x')|dx'| = \int_{\mathbb{S}^*_x X} \int_{\mathbb{S}^*_{x'} X} A(x,y,x',y')\,|dx'\,dy'||dg_x(y)|. \tag{4.6}$$

The wave front relations (4.5) show that the only non-rapidly decreasing contribution of A to the full symbol c of PAP^* at a point $x \in X$ in the direction $\xi \in T^*_x X$ is given by the singularities of A near the point (x,y,x,y) with $y = [\xi]$. In other words, for computing $c(x,\xi)$ we can localize A near the point (x,y,x,y), $y = [\xi]$, in (4.6). Using local coordinates on $\mathbb{S}^*X$ adapted to the contact structure, as in §3.6, and expressing A in terms of its symbol a we get:

$$PAP^*(t+\frac{\tilde t}{2}, u+\frac{\tilde u}{2}, t-\frac{\tilde t}{2}, u-\frac{\tilde u}{2}) \ = \ (2\pi)^{-(2n-1)} \int e^{i(\tilde t\tau + \tilde u\mu + \tilde y\eta)} a(t,u,y,\tau,\mu,\eta)\, d\tau\, d\mu d\eta\, dg_x(y)$$

and taking Fourier transform in $(\tilde t, \tilde u)$:

$$c(t,u,\tau,\mu) \ = \ \int a^\#(t,u,y,\tau,\mu - \tau y, 0)\,|dg_x(y)|.$$

Evaluating at the center of the coordinate system in the positive contact direction gives:

$$c(0,0,\tau,0) \ = \ \int a^\#(0,0,y,\tau,-\tau y,0)\,|dg_x(y)|$$

and after a change of variables $y = -\tau^{-1/2}\bar y$

$$c(0,0,\tau,0) \ = \ \tau^{-\frac{n-1}{2}} \int a^\#(0,0,-\tau^{-\frac{1}{2}}\bar y, \tau, \tau^{\frac{1}{2}}\bar y, 0)\,|dg_x(\tau^{-1/2}\bar y)|$$

Taking $\tau \to \infty$ we recover, via (3.31), the symbol in the statement. □

Proposition 4.5. *Let $A, B \in \mathcal{I}^0_+(\mathbb{S}^*X)$ be elements of the Hermite ideal. Then $AP^*PB \in \mathcal{I}^{-n+1}_+(\mathbb{S}^*X)$ and*

$$\sigma_{-n+1}(AP^*PB) = \sigma_0(A) \# Q \# \sigma_0(B) \in \Psi^{-\infty}_{\mathrm{Is}}(W, L^{\frac{n-1}{2}}) \tag{4.7}$$

where Q it the operator with Schwartz kernel identically 1 (on W), i.e. the projection on constant functions.

Proof. Since the wave front set of AP^*PB turns out to be concentrated on the contact line bundle we can localize near a point in $\mathbb{S}^*X$ and it will suffice to understand its symbol in the positive contact direction. We have modulo smoothing operators

$$(AP^*PB)(z + \frac{w}{2}, z - \frac{w}{2}) = \int A(z + \frac{w}{2}, z')\chi(y' - y'')B(z'', z - \frac{w}{2})dt'dx'dy'dy''$$

where $z' = (t', u', y')$, $z'' = (t', u', y'')$ and χ is compactly supported near 0 with $\chi(0) = 1$; also set $v = y' - y''$ and we will always identify $v = z' - z''$. Hence the symbol $p(z, \sigma)$ of AP^*PB is given via (3.32)-(3.33) by:

$$\begin{aligned} p(z,\sigma) =&(2\pi)^{-(4n-2)} \int e^{-i\phi(z+w/2,z-w/2,\sigma)+i\phi(z+w/2,z',\sigma')+i\phi(z'-v,z-w/2,\sigma'')} \\ & a(\frac{z+w/2+z'}{2},\sigma')\,\chi(v)\,b(\frac{z'-v+z-w/2}{2},\sigma'')dz'dvd\sigma'd\sigma''dw. \end{aligned} \tag{4.8}$$

Changing the variables from z', w, σ', σ'' to:

$$z_1 = \frac{-z + w/2 + z'}{2}, \quad z_2 = \frac{-z + z' - v - w/2}{2}, \quad \sigma_1 = \sigma' - \sigma, \quad \sigma_2 = \sigma'' - \sigma$$

(4.8) becomes:

$$\begin{aligned} p(z,\sigma) = \pi^{-(4n-2)} \int & e^{-iv(\eta+\eta_1+\eta_2)+2i\left(z_1\sigma_2 - z_2\sigma_1 + u_1(y+y_2)\tau_2 - u_2(y+y_1)\tau_1 + (u_1y_2 - u_2y_1)\tau\right)} \\ & a(z+z_1,\sigma+\sigma_1)\,\chi(v)\,b(z+z_2,\sigma+\sigma_2)dvdz_1dz_2d\sigma_1d\sigma_2 \end{aligned}$$

and further changing t_1, t_2 to $t_1 - u_1(y + y_2)$ and $t_2 - u_2(y + y_1)$ respectively:

$$\begin{aligned} p(z,\sigma) = \pi^{-(4n-2)} \int & e^{2i(z_1\sigma_2 - z_2\sigma_1 + (u_1y_2 - u_2y_1)\tau)}\, a(t+t_1-u_1(y+y_2), u+u_1, y+y_1, \sigma+\sigma_1) \\ & \widehat{\chi}(\eta+\eta_1+\eta_2)\, b(t+t_2-u_2(y+y_1), u+u_2, y+y_2, \sigma+\sigma_2)dz_1dz_2d\sigma_1d\sigma_2. \end{aligned}$$

We rescale σ_1 and σ_2 by a factor of τ to $\sigma_1 = \tau\sigma^1, \sigma_2 = \tau\sigma^2$ and apply the stationary phase lemma to the integral in τ^1, τ^2, z_1, z_2. The phase function

$$2(t_1\tau^2 - t_2\tau^1 + u_1 \cdot \zeta^2 - u_2 \cdot \zeta^1 + y_1 \cdot \eta^2 - y_2 \cdot \eta^1 + u_1 \cdot y_2 - u_2 \cdot y_1)\tau$$

is stationary at $t_1 = t_2 = 0$, $\tau^1 = \tau^2 = 0$, $y_1 = -\zeta^1$, $y_2 = -\zeta^2$, $u_1 = \eta^1$, $u_2 = \eta^2$. Rescaling back by a $\tau^{-1/2}$ factor:

$$\zeta^1 = \tau^{-1/2}\bar{\zeta}',\ \zeta^2 = \tau^{-1/2}\bar{\zeta}'',\ \eta^1 = \tau^{-1/2}\bar{\eta}',\ \eta^2 = \tau^{-1/2}\bar{\eta}''$$

after applying stationary phase lemma we get:

$$\begin{aligned}
&p(z,\tau,\tau^{1/2}\bar{\zeta},\tau^{1/2}\bar{\eta}) \sim 2^{2n}\pi^{-(2n-2)}\int e^{2i(\bar{\eta}'\bar{\zeta}''-\bar{\eta}''\bar{\zeta}')}\hat{\chi}(\tau^{1/2}(\bar{\eta}+\bar{\eta}'+\bar{\eta}''))\\
&a(t-\tau^{-1/2}\bar{\eta}'(y-\tau^{-1/2}\bar{\zeta}''),u+\tau^{-1/2}\bar{\eta}',y-\tau^{-1/2}\bar{\zeta}',\tau,\tau^{1/2}(\bar{\zeta}+\bar{\zeta}'),\tau^{1/2}(\bar{\eta}+\bar{\eta}'))\\
&b(t-\tau^{-1/2}\bar{\eta}''(y-\tau^{-1/2}\bar{\zeta}'),u+\tau^{-1/2}\bar{\eta}'',y-\tau^{-1/2}\bar{\zeta}'',\tau,\tau^{1/2}(\bar{\zeta}+\bar{\zeta}''),\tau^{1/2}(\bar{\eta}+\bar{\eta}''))\\
&d\bar{\zeta}'d\bar{\zeta}''d\bar{\eta}'d\bar{\eta}''
\end{aligned}$$

(and lower order terms)

Since $\hat{\chi}(\tau^{1/2}(\bar{\eta}+\bar{\eta}'+\bar{\eta}'')) \sim (2\pi)^{n-1}\tau^{-\frac{n-1}{2}}\delta(\bar{\eta}+\bar{\eta}'+\bar{\eta}'')$ as $\tau \to \infty$ we get from the above expression:

$$\begin{aligned}
&\sigma_{-n+1}(APP^*B)(z,\bar{\zeta},\bar{\eta}) = 2^{3n-1}\pi^{-n+1}\int e^{-2i\bar{\zeta}'\bar{\eta}''+2i\bar{\zeta}''\bar{\eta}'}\\
&\sigma_0(A)(z,\bar{\zeta}+\bar{\zeta}',\bar{\eta}+\bar{\eta}')\delta(\bar{\eta}+\bar{\eta}'+\bar{\eta}'')\sigma_0(B)(z,\bar{\zeta}+\bar{\zeta}'',\bar{\eta}+\bar{\eta}'')d\bar{\zeta}'d\bar{\zeta}''d\bar{\eta}'d\bar{\eta}''
\end{aligned} \tag{4.9}$$

To show (4.7) one just has to check that in the isotropic calculus the symbol of the operator $\sigma_0(A)\,Q\,\sigma_0(B)$ is given by the right-hand side of the above expression. □

Taking A and B in the previous propositions to be a generalized Szegő projection gives:

Corollary 4.6. *If S is a generalized Szegő projection, $SP^*PS \in \mathcal{I}_+^{-n+1}(\mathbb{S}^*X)$ is an elliptic Toeplitz operator on $\mathbb{S}^*X$ and $PSP^* \in \Psi^{-\frac{n-1}{2}}(X)$ is an elliptic pseudodifferential operator on X. In particular, $P : SC^\infty(\mathbb{S}^*X) \to C^\infty(X)$ is Fredholm.*

Proof. The symbol of SP^*PS, given by (4.7), is just a positive multiple of s and hence invertible (on the range of s), i.e. SP^*PS is elliptic. the principal symbol of PSP^* is, at each point in $\mathbb{S}^*X$, a positive multiple of $\int e^{-|\zeta|^2}\,d\zeta$, hence non-vanishing. □

4.4 The isomorphism

The results in the previous section, especially Corollary 4.6, extend to the adiabatic case. Consider the lift of the metric g_ϵ of §4.1 from $\mathbb{S}^*_\epsilon X$ to $\mathbb{S}^*X$ and let P_ϵ be the push-forward map of the previous section associated to the metric g_ϵ/ϵ^2 on $\mathbb{S}^*X$. We will denote by P their family and also use the same notation for the push-forward map in the translation invariant case, $\mathbb{R}^n \times \mathbb{S}^{n-1}$.

Proposition 4.7. *Let S be the Szegő projection of Theorem 4.3. Then*

$$SP^*PS \in \mathcal{I}_{a+}^{-n+1}(\mathbb{S}^*X), \quad PSP^* \in \Psi_a^{-\frac{n-1}{2}}(X)$$

and both are elliptic. Moreover, for each $x \in X$, the normal operators satisfy

$$N(SP^*PS)_x = N(S)_x P^* P N(S)_x, \quad N(PSP^*)_x = P N(S)_x P^*.$$

Proof. For $\epsilon > 0$, the above statement is contained in the previous section. The statement at and near $\epsilon = 0$ follows from the study of the Schwartz kernels near $\mathrm{ff}(M_a^2)$ and, in fact, local coordinate computations near $\mathrm{ff}(M_a^2)$ are enough. S is completely described above and P has a simple form. The same computations as in Propositions 4.4-5, but using (3.37)-(3.40) instead of (3.32)-(3.33), extend the statements from $\epsilon > 0$ to $\epsilon = 0$. □

This gives:

Theorem 4.8. ($n \geq 3$) *There exists $\epsilon_1 > 0$ such that SP^*PS and PSP^* are invertible in the adiabatic Heisenberg algebra on $\mathbb{S}^*X \times [0, \epsilon_1)$ and in the adiabatic pseudodifferential algebra on $X \times [0, \epsilon_1)$. In particular for each $0 < \epsilon < \epsilon_1$, $P_\epsilon : S_\epsilon C^\infty(\mathbb{S}^*_\epsilon X) \to C^\infty(X)$ is an isomorphism.*

Proof. Follows by Proposition 3.16 and the corresponding result for $\Psi_a^*(X)$ from the ellipticity as adiabatic operators and the invertibility of their normal operators, i.e. invertibility in the model case. □

Appendix

As an example, the adiabatic limit construction is briefly described here for the torus $X = \mathbb{R}^n/(2\pi\mathbb{Z})^n$. The complex structure on $T^*X \cong \mathbb{R}^n/(2\pi\mathbb{Z})^n \times i\mathbb{R}^n$ is the one inherited as a quotient of $\mathbb{C}^n$. We use $y_i dx_i \leftrightarrow x - iy$ to identify T^*X and $\mathbb{C}^n$, instead of the conjugate map, $y_i dx_i \leftrightarrow x + iy$, used in the previous chapters.

Since the push-forward map is an isomorphism in this case for every radius (as can be easily seen by Fourier series expansions), we will concentrate on the degeneration of the Szegő projections as the Grauert tube shrinks.

More precisely, we will recover a family of translation invariant Szegő projections on $\mathbb{R}^n \times \mathbb{S}^{n-1}$, each with Schwartz kernel on $\mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ given by (see §2.2):

$$\text{(A.1)} \qquad S(X, Y, Y') = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iX\xi} e^{-Y\xi - Y'\xi'} \frac{1}{I(2\xi)}\, d\xi, \quad |Y| = |Y'| = 1,$$
$$I(\zeta) = ||e^{-\zeta\cdot}||^2_{L^2(\mathbb{S}^{n-1})} = \int_{|\omega|=1} e^{\omega\zeta}\, d\omega$$

as the adiabatic limit at $\epsilon = 0$ of the Szegő projections S_ϵ on $\mathbb{R}^n/(2\pi\mathbb{Z})^n \times \mathbb{S}^{n-1}_\epsilon$:

$$\text{(A.2)} \qquad S_\epsilon(x, x', y, y') = \sum_{k=-\infty}^{\infty} e^{ik(x+iy)} e^{-ik(x'-iy')} \frac{1}{(2\pi)^n \epsilon^{n-1} I(2\epsilon k)}$$

with $x, x' \in \mathbb{R}^n/(2\pi\mathbb{Z})^n$, $y, y' \in \mathbb{R}^n$, $|y| = |y'| = \epsilon$.

Rescaling $y = \epsilon Y$, $y' = \epsilon Y'$, we land in the setting of the previous chapters, namely a family S_ϵ of Szegő projections on a fixed manifold, $\mathbb{R}^n/(2\pi\mathbb{Z})^n \times \mathbb{S}^{n-1}$, associated to a varying CR structure. After a second change of variables $X = (x - x')/\epsilon$, $X' = (x + x')/2$, corresponding to the blow-up in §3.2, the family S_ϵ becomes:

$$S(X, X', Y, Y', \epsilon) = \frac{\epsilon^n}{(2\pi)^n} \sum_{k=-\infty}^{\infty} \widehat{S}(\epsilon k, Y, Y')\, e^{i\epsilon k X}, \; X' \in \mathbb{R}^n/(2\pi\mathbb{Z})^n, \; Y, Y' \in \mathbb{S}^{n-1}$$

periodic in $X \in \mathbb{R}^n$ of period $2\pi/\epsilon$. Above, $\widehat{S}(\cdot, Y, Y')$ stands for the Fourier transform of (A.1) in the X-variable, i.e. the indicial family of the translation invariant Szegő projection on $\mathbb{R}^n \times \mathbb{S}^{n-1}$. Then *Poisson's summation formula* (see [10]) gives:

$$S(X, X', Y, Y', \epsilon) = \sum_{k=-\infty}^{\infty} S(X + \frac{2\pi k}{\epsilon}, Y, Y').$$

To make sense of the sum above (as a periodic tempered distribution with period $2\pi/\epsilon$ in the X variable) recall that $S(X, Y, Y')$ is smooth outside the 'diagonal' $\{X = 0, Y = Y'\}$ and rapidly decreasing as $|X| \to \infty$. In particular it can be written as a sum of a rapidly decreasing function and a distribution supported in a neighborhood of $X = 0$. This justifies the sum. Taking the limit (in $\mathcal{S}'$) as $\epsilon \to 0$, we finally get:

$$S(X, X', Y, Y', 0) = S(X, Y, Y'), \quad \forall X' \in \mathbb{R}^n/(2\pi\mathbb{Z})^n.$$

Important parts of the general construction are not apparent here, mainly because S_ϵ is translation invariant in this example even for $\epsilon > 0$.

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