

**TOWARDS CHARACTERIZING MORPHISMS
BETWEEN HIGH DIMENSIONAL HYPERSURFACES**

by

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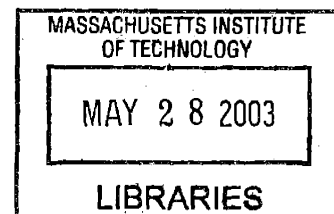
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Abstract

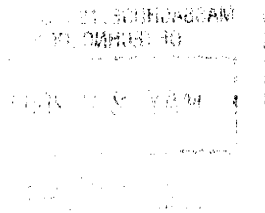
This thesis is organized into two papers. All results are proven over an algebraically closed field of characteristic zero.

Paper 1 concerns morphisms between hypersurfaces in \mathbb{P}^n , $n \geq 4$. We show that if the two hypersurfaces involved in the morphism are of general type, then the morphism of hypersurfaces extends to an everywhere-defined endomorphism of \mathbb{P}^n . A corollary is that if $X \rightarrow Y$ is a nonconstant morphism of hypersurfaces of large dimension and large degree, then $\deg Y$ divides $\deg X$. The main tool used to analyze morphism between hypersurfaces is an inequality of Chern classes analogous to the Hurwitz-inequality.

Paper 2 is a long example. We check that every morphism from a quintic hypersurface in \mathbb{P}^4 to a nonsingular cubic hypersurface in \mathbb{P}^4 is constant. In the process, we classify morphisms from the projective plane to nonsingular cubic threefolds.

Thesis Supervisor: Aise Johan de Jong

Title: Professor of Mathematics



SERVING

**PAPER 1:
TOWARDS CHARACTERIZING MORPHISMS
BETWEEN HIGH DIMENSIONAL HYPERSURFACES**

DAVID C. SHEPPARD

ABSTRACT

We prove the following theorem over an algebraically closed field of characteristic zero. Let $f : X \rightarrow Y$ be a nonconstant morphism of hypersurfaces in \mathbb{P}^n , $n \geq 4$. If Y is nonsingular and of general type, then there is a morphism $F : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $F|_X = f$ and $F^{-1}(Y) = X$. As a corollary, we see that $\deg Y$ divides $\deg X$ with quotient m , and f is given by polynomials of degree m .

SURVEY OF THE LITERATURE

Classically, algebraic geometry has sought to classify varieties. Recently focus has expanded to include the classification of morphisms between algebraic varieties. Our goal in this paper is to shed some light on what type of morphisms can occur between hypersurfaces. Let us give some results from the literature.

In [11], Paranjape and Srinivas show that every nonconstant morphism between smooth quadric hypersurfaces in \mathbb{P}^n is an isomorphism for $n \geq 4$.

Schuhmann shows in [12] that the degree of a morphism from a smooth hypersurface of degree d in \mathbb{P}^4 to a smooth quadric threefold is bounded from above in terms of d . For $d = 3$ she obtains a very good bound on the degree of possible morphisms and proves that every morphism from a smooth cubic threefold to a smooth quadric threefold is constant.

Amerik proves in [1] that the degree of a morphism from a smooth hypersurface in \mathbb{P}^n to a smooth quadric hypersurface in \mathbb{P}^n is bounded from above for $n \geq 4$ using a different argument than [12].

Beauville shows in [3] that every endomorphism of a smooth hypersurface in \mathbb{P}^n of dimension at least 2 and degree at least 3 is an automorphism. For this he uses a Hurwitz-type inequality from [2]. We will generalize this inequality in Section 1.

MAIN RESULTS AND POINT OF VIEW

The main result of this paper is the following Theorem, which we prove in Section 2.

Theorem 2. *Assume the base field is algebraically closed of characteristic zero. Let $f : X \rightarrow Y$ be a morphism of hypersurfaces in \mathbb{P}^n , $n \geq 4$. If Y is nonsingular and of general type, then there is a morphism $F : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $f = F|_X$ and $F^{-1}(Y) = X$.*

The proof of Theorem 2, and our study of morphisms between hypersurfaces in general, relies on the following definition.

Definition. Let $f : X \rightarrow Y$ be a morphism of projective k -varieties with specified very ample invertible sheaves $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ such that $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(m)$. Assume X is positive dimensional so that there is a unique such m . We will refer to m as the *polynomial degree* of f , because f is given by polynomials of degree m .

As a consequence of the Grothendieck-Lefschetz Theorem on the Picard group, every morphism $f : X \rightarrow Y$ between hypersurfaces of dimension at least 3 has a polynomial degree, cf. Lemma 2.1. We will see that the conclusion of Theorem 2 is equivalent to the following statement: $\deg Y$ divides $\deg X$ with quotient equal to the polynomial degree of f .

In light of this restatement, our first goal will be to bound the polynomial degree of possible morphisms between two given hypersurfaces. To do this, we generalize the Hurwitz-Type Inequality of [2] in the case of morphisms between complete intersections. This inequality gives good bounds on the polynomial degree of possible morphisms between hypersurfaces of high degree. In particular, if $f : X \rightarrow Y$ is a morphism between hypersurfaces in \mathbb{P}^n and Y is nonsingular of degree at least $n + 2$, i.e. Y is of general type, then the bound on the polynomial degree of f is good enough to prove Theorem 2.

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1. A HURWITZ-TYPE INEQUALITY

In this section the ground field is algebraically closed of arbitrary characteristic. The main result of this section is the following Theorem.

Theorem 1. *Let X be a complete intersection variety in \mathbb{P}^n and Y a nonsingular projective variety of the same dimension as X . Fix a very ample invertible sheaf $\mathcal{O}_Y(1)$ on Y . If $f : X \rightarrow Y$ is a morphism such that $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(m)$ for some positive integer m and the extension of function fields $k(Y) \rightarrow k(X)$ is separable, then*

$$f^*c_{top}(\Omega_Y^1(2)) \leq c_{top}(\Omega_X^1(2m)).$$

Theorem 1 is more general than the Hurwitz-Type Inequality of [2] in the sense that we do not assume the ground field has characteristic zero or that X is smooth. However, it is less general in the sense that we assume X is a complete intersection.

It is worth noting that Theorem 1 can fail if Y is singular. For example, Y could be the image of a hyperplane in \mathbb{P}^n .

Also note that if X is singular, then $\Omega_X^1(2m)$ is not locally free. However, the top Chern class $c_{top}(\Omega_X^1(2m))$ is defined via a finite locally free resolution of $\Omega_X^1(2m)$, such as the conormal sequence for $X \subset \mathbb{P}^n$.

We give some preliminary lemmas before proving Theorem 1.

Lemma 1.1. *If $X \subset \mathbb{P}^n$ is a positive dimensional complete intersection and $f : X \rightarrow \mathbb{P}^N$ is a morphism such that $f^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_X(m)$ for some positive integer m , then f extends to a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ defined on a Zariski open set containing X .*

Proof. Let ξ_1, \dots, ξ_c be the homogeneous polynomials that generate the homogeneous ideal of X , where c is the codimension of X in \mathbb{P}^n . Let ξ_i have degree d_i , and let $X_i := V(\xi_1, \dots, \xi_i)$ so that $X = X_c \subset \dots \subset X_0 = \mathbb{P}^n$. The morphism f is given by sections $f_0, \dots, f_N \in H^0(X_c, \mathcal{O}_{X_c}(m))$. To lift f from X_c to a rational map on X_{c-1} we need to see that the restriction map

$$H^0(X_{c-1}, \mathcal{O}_{X_{c-1}}(m)) \longrightarrow H^0(X_c, \mathcal{O}_{X_c}(m))$$

is surjective. So it suffices to check that $H^1(X_{c-1}, I(m)) = 0$, where $I \subset \mathcal{O}_{X_{c-1}}$ is the ideal sheaf of X_c in X_{c-1} . Since $I = \mathcal{O}_{X_{c-1}}(-d_c)$ is a twisted structure sheaf and X_{c-1} is a complete intersection in \mathbb{P}^n of dimension at least 2,

$$H^1(X_{c-1}, I(m)) = 0.$$

So the global sections f_i lift from $\mathcal{O}_{X_c}(m)$ to $\mathcal{O}_{X_{c-1}}(m)$. Continuing, we lift the f_i to global sections F_i of $\mathcal{O}_{\mathbb{P}^n}(m)$. Set $F = (F_0, \dots, F_N) : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$, and note that F is undefined on $V(F_1, \dots, F_N)$, which is disjoint from X because $F|_X = f$ is a morphism. \square

The following positivity result essentially appears in [6].

Lemma 1.2. *Consider the following fiber square*

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ S & \xrightarrow{\rho} & T \end{array}$$

where $\rho : S \rightarrow T$ is a regular imbedding of varieties of codimension i and V is a k -dimensional variety mapping to T . If $N_S T$ is globally generated, then

$$S \cdot V = \sum_j \mu_j[R_j] + P$$

where the R_j are the reduced structures on the $(k-i)$ -dimensional irreducible components of W , the μ_j are positive integers, and P is an effective $(k-i)$ -cycle on W .

Proof. We apply a positivity result to the Basic Construction in [6, Chapter 6].

Let N be the pullback of $N_S T$ to W . Then N is a globally generated vector bundle of rank i on W , and the normal cone $N_W V \hookrightarrow N$ is a purely k -dimensional closed subscheme of N . If σ is the zero section of N , then $\rho^*[V]$ is defined to be $\sigma^*[N_W V]$. Recall that $[N_W V]$ is the sum of the k -cycles associated to the irreducible components of $N_W V$ taken with appropriate multiplicities.

If Z_j is a $(k-i)$ -dimensional irreducible component of W , then $N_W V$ has an irreducible component C_j that dominates Z_j . If R_j is the reduced structure on Z_j , then $N|_{R_j}$ is the reduced structure on C_j because $N|_{R_j}$ is reduced and irreducible, $\dim C_j = k = \dim N|_{R_j}$, and $N|_{R_j}$ contains the reduced structure $(C_j)_{\text{red}}$. So by definition, $[C_j] = \mu_j[N|_{R_j}]$, where μ_j is the length of the stalk of \mathcal{O}_{C_j} at the generic point of C_j . Therefore $\sigma^*[C_j] = \mu_j[R_j]$ because $\sigma^*[N|_{R_j}] = [R_j]$. This accounts for the term $\sum \mu_j[R_j]$ in the formula for $\rho^*[V]$.

Moreover, σ^* takes effective cycles to effective cycles because N is globally generated, cf. [6, Theorem 12.1(a)]. So if C is an irreducible component of $N_W V$ other than one of the C_j described above, then $\sigma^*[C]$ is effective. These other components C account for P . \square

Lemma 1.3. *Let E be a globally generated vector bundle over a variety X . Let $K \subseteq E$ be a closed subscheme with $\dim K < \text{rank } E$. Then there is a section σ of E such that $\sigma(X) \cap K$ is empty.*

Proof. Since E is globally generated, there is a surjective morphism of vector bundles $\pi : X \times \mathbb{A}^{h^0} \rightarrow E$, where $h^0 := h^0(X, E)$. All the fibers of π are affine spaces of the same dimension. So $\dim \pi^{-1}(K) = \dim K + (h^0 - \text{rank } E)$. In other words, $h^0 - \dim \pi^{-1}(K) = \text{rank}(E) - \dim K > 0$, whence there is a constant global section τ of $X \times \mathbb{A}^{h^0}$ over X that does not intersect $\pi^{-1}(K)$. Take $\sigma = \pi \circ \tau$. \square

Lemma 1.4. *Let $0 \rightarrow L \rightarrow E \rightarrow F \rightarrow 0$ be a short exact sequence of vector bundles on a complete variety X such that $\text{rank } F = \dim X$. If E is globally generated, then for any morphism of vector bundles $i : L \rightarrow E$ we have*

$$\sigma^*[i(L)] \leq c_{\text{top}}(F)$$

where σ is the zero section of E and $i(L)$ is the scheme-theoretic image of $i : L \rightarrow E$. Equality holds if i is a closed immersion.

Proof. First assume that i is the given closed immersion $L \rightarrow E$. Consider the fiber diagram

$$\begin{array}{ccc} L & \longrightarrow & X \\ \downarrow i & & \downarrow \tau \\ X & \xrightarrow{\sigma} & E \xrightarrow{\phi} F \end{array}$$

where τ is the zero section and ϕ is the quotient map. Calculate

$$\begin{aligned} \sigma^*[i(L)] &= \sigma^*\phi^*[X] \\ &= (\phi \circ \sigma)^*[X] \\ &= c_{top}(F). \end{aligned}$$

If $i : L \rightarrow E$ is any closed immersion of vector bundles on X with quotient bundle F_i , then the last statement of the lemma follows from

$$c_{top}(F) = \left\{ \frac{c(E)}{c(L)} \right\}_0 = c_{top}(F_i).$$

Now assume that $i : L \rightarrow E$ is any morphism of vector bundles on X . If $\dim i(L) < \dim L$, then $\sigma^*[i(L)] = 0$. So $\sigma^*[i(L)] \leq c_{top}(F)$ because F is globally generated, cf. [6, Theorem 12.1(a)]. Therefore, we may assume $\dim i(L) = \dim L$.

Let Σ be the zero section of $E \times \mathbb{A}^1$ over $X \times \mathbb{A}^1$, and let $\sigma_t := \Sigma|_{X \times t}$ be the zero section of $E \times t$ over $X \times t$. Let $x_t : X \times t \rightarrow X \times \mathbb{A}^1$ and $e_t : E \times t \rightarrow E \times \mathbb{A}^1$ be the inclusion maps. These maps fit together in the following fiber square.

$$\begin{array}{ccc} X \times t & \xrightarrow{\sigma_t} & E \times t \\ x_t \downarrow & & \downarrow e_t \\ X \times \mathbb{A}^1 & \xrightarrow{\Sigma} & E \times \mathbb{A}^1 \end{array}$$

Therefore, $x_t^*\Sigma^*\alpha = \sigma_t^*e_t^*\alpha$ for any cycle α on $E \times \mathbb{A}^1$.

Let $i_0 = i$, and let i_1 be the given closed immersion $L \rightarrow E$ with quotient F . Consider the morphism

$$\begin{aligned} I : L \times \mathbb{A}^1 &\longrightarrow E \times \mathbb{A}^1 \\ (v, t) &\longrightarrow t \cdot i_1(v) + (1-t) \cdot i_0(v) \end{aligned}$$

of vector bundles on $X \times \mathbb{A}^1$, and let $i_t = I|_{L \times t} : L \times t \rightarrow E \times t$. Let Z denote the scheme-theoretic image of I , and define

$$\lambda_t := x_t^*\Sigma^*[Z] = \sigma_t^*e_t^*[Z].$$

The remainder of the proof will rest on the fact that $e_1^*[Z] = [i_1(L)]$. To see this, note that I is a closed immersion away from its degeneracy locus, which is a Zariski closed subset of $X \times \mathbb{A}^1$ disjoint from $X \times 1$. So I is a closed immersion above some Zariski open neighborhood U of $X \times 1$ in

$X \times \mathbb{A}^1$. Consider the following fiber diagram

$$\begin{array}{ccc} e_1^{-1}(Z) & \longrightarrow & L \times \mathbb{A}^1 \\ \downarrow & & \downarrow I \\ E \times 1 & \xrightarrow{e_1} & E \times \mathbb{A}^1 \\ \downarrow & & \downarrow \\ X \times 1 & \longrightarrow & X \times \mathbb{A}^1 \end{array}$$

Base change the diagram from $X \times \mathbb{A}^1$ to $U \subset X \times \mathbb{A}^1$. Now all the arrows in the diagram are closed immersions. Since the outer square is a fiber square, we obtain

$$e_1^{-1}(Z) \cong L \times 1$$

In other words,

$$\begin{aligned} e_1^{-1}(Z) &= I(L \times 1) \\ &= i_1(L) \end{aligned}$$

Because $e_1^{-1}(Z)$ is reduced at the generic point, we have $e^*[Z] = [i_1(L)]$, which implies that

$$\lambda_1 = \sigma_1^*[i_1(L)]$$

Now that we have analyzed λ_1 , we will analyze λ_0 .

Note that $i_0(L)$ is an irreducible component of $e_0^{-1}(Z)$ because $i_0(L) \subset e_0^{-1}(Z)$ and

$$\dim e_0^{-1}(Z) + 1 = \dim Z = \dim i_0(L) + 1.$$

So by Lemma 1.2, $e_0^*[Z] = [i_0(L)] + P$ for some effective cycle P . Therefore,

$$\lambda_0 = \sigma_0^*[i_0(L)] + \sigma_0^*P$$

Moreover, σ_0^*P is an effective 0-cycle because E is globally generated.

If α is any cycle on $X \times \mathbb{A}^1$, then since $X \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is proper, the degree of the restriction $x_t^*\alpha$ of α to the fiber $X \times t$ does not depend on t , cf. [6, Proposition 10.2]. Take $\alpha = \Sigma^*[Z]$ to see that the 0-cycles λ_t on X all have the same degree. Now we can calculate

$$\begin{aligned} \deg \sigma_0^*[i_0(L)] &\leq \deg \lambda_0 \\ &= \deg \lambda_1 \\ &= \deg \sigma_1^*[i_1(L)] \\ &= c_{top}(F). \end{aligned}$$

□

Proof of Theorem 1. Let $f : X \rightarrow Y$ be a morphism of projective varieties. Assume X is a complete intersection in \mathbb{P}^n and Y is nonsingular of the same dimension as X . Fix a very ample invertible sheaf $\mathcal{O}_Y(1)$ on Y with corresponding projective embedding $Y \hookrightarrow \mathbb{P}^N$. Assume $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(m)$ for some positive integer m , which implies that f is finite and surjective. Assume also that f is a separable morphism.

By Lemma 1.1, there is a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ defined on a Zariski open subset of \mathbb{P}^n containing X such that $f = F|_X$. This extended

map F induces a morphism $f^*(\Omega_{\mathbb{P}^n}^1|_Y) \rightarrow \Omega_{\mathbb{P}^n}^1|_X$, which gives the following commutative diagram of sheaves on X :

$$(1.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & f^*(I_Y/I_Y^2(2)) & \longrightarrow & f^*(\Omega_{\mathbb{P}^n}^1|_Y(2)) & \longrightarrow & f^*(\Omega_Y^1(2)) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I_X/I_X^2(2m) & \longrightarrow & \Omega_{\mathbb{P}^n}^1|_X(2m) & \longrightarrow & \Omega_X^1(2m) \longrightarrow 0 \end{array}$$

where $I_X \subset \mathcal{O}_{\mathbb{P}^n}$ and $I_Y \subset \mathcal{O}_{\mathbb{P}^n}$ are the ideal sheaves of X and Y . The bottom row is exact because I_X is the ideal sheaf of a reduced complete intersection.

To apply our intersection-theoretic lemmas, we transform diagram (1.1) of sheaves on X into a diagram of schemes over X by applying the covariant functor

$$\Phi : \{\text{coherent sheaves on } X\} \longrightarrow \{\text{schemes of finite type over } X\} \\ \mathcal{F} \longrightarrow \text{Spec}(\text{Sym}_{\mathcal{O}_X}[\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)])$$

where $\text{Sym}_{\mathcal{O}_X}(-)$ denotes the symmetric algebra of an \mathcal{O}_X -module. If \mathcal{F} is a locally free sheaf, then $\Phi(\mathcal{F})$ is the vector bundle whose sheaf of sections is \mathcal{F} . Apply Φ to diagram (1.1), and denote the resulting diagram of X -schemes by

$$(1.2) \quad \begin{array}{ccccccc} L_Y & \xrightarrow{i_Y} & E_Y & \rightarrow & F_Y & & \\ \downarrow & & \downarrow \psi & & \downarrow & & \\ L_X & \xrightarrow{i_X} & E_X & \rightarrow & F_X & & \end{array}$$

Note that every scheme in (1.2) is a vector bundle on X , except for F_X if X is singular. Also note that i_Y is a closed immersion because Y is nonsingular, and that E_X, E_Y are generated by global sections because $\Omega_{\mathbb{P}^n}^1(a)$ is globally generated for $a \geq 2$.

Let σ be the zero section of E_Y so that $\psi \circ \sigma$ is the zero section of E_X . By Lemma 1.4,

$$(1.3) \quad f^*c_{top}(\Omega_Y^1(2)) = \sigma^*[i_Y(L_Y)].$$

Let $i_X(L_X)$ be the scheme-theoretic image of $i_X : L_X \rightarrow E_X$. By equation (1.3), it suffices to show

$$(1.4) \quad \sigma^*[i_Y(L_Y)] \leq (\psi \circ \sigma)^*[i_X(L_X)]$$

$$(1.5) \quad \leq c_{top}(\Omega_X^1(2m)).$$

To prove (1.4) it is enough to show $\psi^*[i_X(L_X)] = [i_Y(L_Y)] + P_Y$ for some effective cycle P_Y on E_Y . Indeed, σ^*P_Y is effective because E_Y is globally generated, whence

$$\begin{aligned} \sigma^*[i_Y(L_Y)] &\leq \sigma^*[i_Y(L_Y)] + \sigma^*P_Y \\ &= \sigma^*\psi^*[i_X(L_X)] \\ &= (\psi \circ \sigma)^*[i_X(L_X)]. \end{aligned}$$

Consider the fiber diagram

$$\begin{array}{ccccc} \psi^{-1}(i_X(L_X)) & \longrightarrow & E_Y \times_X i_X(L_X) & \longrightarrow & i_X(L_X) \\ \downarrow & & \downarrow & & \downarrow \\ E_Y & \xrightarrow{\Gamma_\psi} & E_Y \times_X E_X & \xrightarrow{\pi_2} & E_X \end{array}$$

where Γ_ψ is the graph of ψ . Then $\psi = \pi_2 \circ \Gamma_\psi$, whence

$$\begin{aligned} \psi^*[i_X(L_X)] &= \Gamma_\psi^* \pi_2^*[i_X(L_X)] \\ &= \Gamma_\psi^*[E_Y \times_X i_X(L_X)]. \end{aligned}$$

Since Γ_ψ is a section of the vector bundle $E_Y \times_X E_X$ over E_Y

$$\Gamma_\psi^*(N) \cong E_Y \times_X E_X$$

where N is the normal bundle of $\Gamma_\psi(E_Y)$ in $E_Y \times_X E_X$. Therefore, N is globally generated because E_X is globally generated over X . So it suffices by Lemma 1.2 to show that $i_Y(L_Y)$ is an irreducible component of $\psi^{-1}(i_X(L_X))$.

By the assumption that $k(Y) \hookrightarrow k(X)$ is a separable field extension, the stalk of $\Omega_{X/Y}^1$ at the generic point of X is $\Omega_{k(X)/k(Y)}^1 = 0$. Hence there is some nonempty open U in X such that the restriction of $f^*\Omega_Y^1 \rightarrow \Omega_X^1$ to U is an isomorphism of locally free sheaves. So when diagram (1.1) is restricted to U , the morphism $f^*(\Omega_Y^1(2)) \rightarrow \Omega_X^1(2m)$ becomes an isomorphism. Hence $F_Y \rightarrow F_X$ is an isomorphism when restricted to U . It follows that $\psi^{-1}(i_X(L_X))$ and $i_Y(L_Y)$ coincide over U . Therefore, $i_Y(L_Y)$ is an irreducible component of $\psi^{-1}(i_X(L_X))$. This establishes equation (1.4).

To prove equation (1.5), it suffices by Lemma 1.4 to show that there is a closed immersion $L_X \rightarrow E_X$ of vector bundles, i.e. that there is a morphism of locally free sheaves $I_X/I_X^2 \rightarrow \Omega_{\mathbb{P}^n}^1|_X$ with empty degeneracy locus.

Let X be cut out by homogeneous polynomials ξ_1, \dots, ξ_c where c is the codimension of X in \mathbb{P}^n . Let $a_i = \deg \xi_i$ so that

$$I_X/I_X^2 \cong \bigoplus_{i=1}^c \mathcal{O}_X(-a_i).$$

By decreasing n if necessary, we assume $a_i \geq 2$ for each a_i . If $c = 0$, then there is nothing to prove, so assume $c > 0$. We will construct a morphism $\bigoplus \mathcal{O}_X(-a_i) \rightarrow \Omega_{\mathbb{P}^n}^1|_X$ with empty degeneracy locus one summand at a time.

Since $a_1 \geq 2$, the locally free sheaf $\Omega_{\mathbb{P}^n}^1|_X(a_1)$ is globally generated. By Lemma 1.3, the rank n vector bundle $\Phi(\Omega_{\mathbb{P}^n}^1|_X(a_1))$ has a section that avoids the zero section. Hence there is a morphism $\sigma_1 : \mathcal{O}_X \rightarrow \Omega_{\mathbb{P}^n}^1|_X(a_1)$ with empty degeneracy locus. Tensoring σ_1 with $\mathcal{O}_X(-a_1)$ gives a morphism $\phi_1 : \mathcal{O}_X(-a_1) \rightarrow \Omega_{\mathbb{P}^n}^1|_X$ with empty degeneracy locus. If $c = 1$, we are done.

If $c \geq 2$, then let $\phi'_1 : \mathcal{O}_X(a_2 - a_1) \rightarrow \Omega_{\mathbb{P}^n}^1|_X(a_2)$ denote the morphism obtained from ϕ_1 by tensoring with $\mathcal{O}_X(a_2)$. Since $n > n - c + 1$ and the image of $\Phi(\phi'_1)$ has dimension $n - c + 1$, Lemma 1.3 implies that there is

a section of $\Phi(\Omega_{\mathbb{P}^n|X}^1(a_2))$ that avoids the image of $\Phi(\phi'_1)$. In other words, there is a morphism $\sigma_2 : \mathcal{O}_X \rightarrow \Omega_{\mathbb{P}^n|X}^1(a_2)$ such that

$$\phi'_1 \oplus \sigma_2 : \mathcal{O}_X(a_2 - a_1) \oplus \mathcal{O}_X \rightarrow \Omega_{\mathbb{P}^n|X}^1(a_2)$$

has empty degeneracy locus. If $\phi_2 : \mathcal{O}_X(-a_2) \rightarrow \Omega_{\mathbb{P}^n|X}^1$ is obtained from σ_2 by tensoring with $\mathcal{O}_X(-a_2)$, then tensoring the above morphism with $\mathcal{O}_X(-a_2)$ yields a morphism

$$\phi_1 \oplus \phi_2 : \mathcal{O}_X(-a_1) \oplus \mathcal{O}_X(-a_2) \rightarrow \Omega_{\mathbb{P}^n|X}^1$$

with empty degeneracy locus.

Continuing like this, we obtain a morphism $\bigoplus \phi_i : \bigoplus \mathcal{O}_X(-a_i) \rightarrow \Omega_{\mathbb{P}^n|X}^1$ with empty degeneracy locus. This completes the proof of Theorem 1.

2. MORPHISMS BETWEEN HYPERSURFACES

We will apply Theorem 1 to the case of hypersurfaces in \mathbb{P}^n . We fix the notation and assumptions of the following discussion for the rest of the paper.

Let $f : X_d \rightarrow Y_e$ be a nonconstant morphism of hypersurfaces of the indicated degrees in \mathbb{P}^n , $n \geq 4$. Assume X is integral and Y is nonsingular. We also assume $e \geq 3$ because the inequality of Chern classes in Theorem 1 only gives good information in this range.

The Grothendieck-Lefschetz Theorem, [8, Theorem 4.3.2], states that $\text{Pic } X$ is generated by $\mathcal{O}_X(1)$. Therefore, $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(m)$ for some non-negative integer m . Since f is not constant, the polynomial degree m of f is positive.

As $f^*\mathcal{O}(1)$ is ample, f is finite. Therefore, f induces a finite extension of function fields. We assume that the extension $k(Y) \rightarrow k(X)$ is separable.

Now we introduce a hypersurface H that will be central to our study of $f : X \rightarrow Y$. By Lemma 1.1, the morphism $f : X \rightarrow Y$ of Theorem 2 extends to a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined at all but finitely many points away from X . Since $e \geq 3$, Y is not the image of a hyperplane in \mathbb{P}^n , because the only smooth variety that is the image of a morphism from a projective space is projective space itself, cf. [10]. Therefore, Y is not the image of $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Hence, F is dominant because its image is irreducible and contains Y . It follows that $F^{-1}(Y)$ is a hypersurface in \mathbb{P}^n . Since $X \subset F^{-1}(Y)$, we may define the hypersurface H in \mathbb{P}^n as the difference of divisors

$$H := F^{-1}(Y) - X.$$

We will study H because $F^{-1}(Y) = X$ if and only if $H = 0$ as a divisor on \mathbb{P}^n , i.e. H is empty.

2.1. First Calculations. The ground field will be algebraically closed of arbitrary characteristic in this subsection. We will derive closed formulas for $c_{n-1}(\Omega_X^1(2m))$ and $f^*c_{n-1}(\Omega_Y^1(2))$. So consider the short exact sequences

$$0 \rightarrow \mathcal{O}_X(-d) \rightarrow \Omega_{\mathbb{P}^n}^1 \rightarrow \Omega_X^1 \rightarrow 0$$

$$0 \longrightarrow \Omega_{\mathbb{P}^n}^1 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

Let $h := c_1(\mathcal{O}_X(1))$, and calculate the total Chern class of Ω_X^1 to be

$$\begin{aligned} c(\Omega_X^1) &= \frac{(1-h)^{n+1}}{1-dh} \\ &= \left(\sum_{i=0}^{n-1} \binom{n+1}{i} (-h)^i \right) \cdot \left(\sum_{j=0}^{n-1} (dh)^j \right) \end{aligned}$$

The i^{th} Chern class of Ω_X^1 is therefore given by

$$c_i(\Omega_X^1) = h^i \cdot \sum_{j=0}^i (-1)^j \binom{n+1}{j} d^{i-j}$$

The usual calculation with Chern roots shows

$$(2.1) \quad c_{n-1}(\Omega_X^1(2m)) = \sum_{i=0}^{n-1} c_i(\Omega_X^1)(2mh)^{n-1-i}$$

$$(2.2) \quad = h^{n-1} \sum_{i=0}^{n-1} \sum_{j=0}^i (-1)^j \binom{n+1}{j} d^{i-j} (2m)^{n-1-i}$$

Notice that for each pair of integers a, b such that $a \geq 0$, $b \geq 0$, and $a + b \leq n - 1$, the monomial $d^{a+1}(2m)^b$ has coefficient $(-1)^N \binom{n+1}{N}$ in (2.2), where $N = n - 1 - a - b$. Therefore, we introduce the notation

$$\Phi_N(x, y) := x^N + x^{N-1}y + \cdots + xy^{N-1} + y^N$$

and use the observation $h^{n-1} = d$ to obtain

$$c_{n-1}(\Omega_X^1(2m)) = d \sum_{k=0}^{n-1} (-1)^k \binom{n+1}{k} \Phi_{n-1-k}(d, 2m).$$

We continue the calculation of $c_{n-1}(\Omega_X^1(2m))$ as follows.

$$\begin{aligned}
c_{n-1}(\Omega_X^1(2m)) &= \sum_{k=0}^n (-1)^k \binom{n+1}{k} d \left(\frac{d^{n-k} - (2m)^{n-k}}{d-2m} \right) \\
&= \frac{1}{d-2m} \left\{ \sum_{i=0}^n (-1)^i \binom{n+1}{i} d^{n+1-i} \right. \\
&\quad \left. - d \sum_{j=0}^n (-1)^j \binom{n+1}{j} (2m)^{n-j} \right\} \\
&= \frac{1}{(2m)(d-2m)} \left\{ 2m \sum_{i=0}^n (-1)^i \binom{n+1}{i} d^{n+1-i} \right. \\
&\quad \left. - d \sum_{j=0}^n (-1)^j \binom{n+1}{j} (2m)^{n+1-j} \right\} \\
&= \frac{1}{(2m)(d-2m)} \left\{ 2m \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} d^{n+1-i} \right. \\
&\quad \left. - d \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} (2m)^{n+1-j} \right. \\
&\quad \left. + (-1)^{n+1} (d-2m) \right\} \\
&= \frac{2m(d-1)^{n+1} - d(2m-1)^{n+1} + (-1)^{n+1}(d-2m)}{2m(d-2m)}
\end{aligned}$$

Introducing $x = 2m - 1$ and $y = d - 1$, we calculate $c_{n-1}(\Omega_X^1(2m))$ to be

$$\begin{aligned}
c_{n-1}(\Omega_X^1(2m)) &= \frac{(x+1)y^{n+1} - (y+1)x^{n+1} + (-1)^{n+1}(y-x)}{(x+1)(y-x)} \\
&= \frac{xy(y^n - x^n) + (y^{n+1} - x^{n+1}) + (-1)^{n+1}(y-x)}{(x+1)(y-x)} \\
&= \frac{xy\Phi_{n-1}(x,y) + \Phi_n(x,y) + (-1)^{n+1}}{x+1} \\
&= \frac{xy\Phi_{n-1}(x,y) + x(\Phi_{n-1}(x,y) + \frac{y^n}{x}) + (-1)^{n+1}}{x+1} \\
&= \frac{x(y+1)\Phi_{n-1}(x,y) + y^n + (-1)^{n+1}}{x+1}.
\end{aligned}$$

Therefore we obtain the formula

(2.3)

$$\boxed{c_{n-1}(\Omega_X^1(2m)) = \frac{d(2m-1)\Phi_{n-1}(2m-1, d-1) + (d-1)^n + (-1)^{n+1}}{2m}}$$

By taking $m = 1$ and substituting e for d in formula (2.3), we have a formula for $c_{n-1}(\Omega_Y^1(2))$. Therefore, we can use the equations

$$f^*c_{n-1}(\Omega_Y^1(2)) = \deg f \cdot c_{n-1}(\Omega_Y^1(2)) \quad \text{and} \quad \deg f = \frac{dm^{n-1}}{e}$$

to derive the following formula for $f^*c_{n-1}(\Omega_Y^1(2))$

$$(2.4) \quad \boxed{f^*c_{n-1}(\Omega_Y^1(2)) = \frac{dm^{n-1}}{e} \left(\frac{e \Phi_{n-1}(1, e-1) + (e-1)^n + (-1)^{n+1}}{2} \right)}$$

We will need the following polynomial fact in the proof of Proposition 2.2.

Lemma 2.1. *If x, y are positive real numbers with $x \geq 3$, $N \geq 3$ is an integer, and $\Phi_N(y, 2) > (x+1)^N + 1$, then $y > x$.*

Proof. Since $\Phi_N(y, 2)$ increases with respect to y it suffices to show that if $x \geq 3$, then $\Phi_N(x, 2) \leq (x+1)^N + 1$. Notice that the coefficients of the polynomial $P(x) = (x+1)^N + 1 - \Phi_N(x, 2)$ have only one sign change. So by Descartes's rule of signs, $P(x)$ has only one positive real root. Therefore, since $P(0) < 0$, it suffices to check that $P(3) \geq 0$. One easily checks this for $N \geq 3$. \square

Proposition 2.2. *In the notation established at the beginning of this Section:*

- (1) *For each triple (d, e, n) there is an integer $M = M(d, e, n)$ such that $m \leq M$.*
- (2) *$d \geq e$.*
- (3) *If $d = e$, then $m = 1$.*
- (4) *If $e \geq 5$, then $d - 1 > m(e - 2)$.*

Remarks. If the base field is \mathbb{C} and X is nonsingular, then (2) has the following proof, which is independent of Theorem 1. There is an injection of singular cohomology rings $H^*(Y, \mathbb{C}) \rightarrow H^*(X, \mathbb{C})$. So in this case, (2) can be proved by computing the dimension of the middle cohomology groups of X and Y .

Part (3) is a generalization of the result in [3] that in characteristic zero every endomorphism of a smooth hypersurface of degree at least 3 and dimension at least 2 is an automorphism. We only assume X and Y have the same degree, not that $X = Y$, and we do not assume characteristic zero, only that the morphism is separable. The case $n = 3$ can be checked without much work using Theorem 1.

Part (4) will be needed for the proof of Theorem 2.

Proof. Theorem 1 states (2.3) \geq (2.4). Dividing both sides of this inequality by dm^{n-1} results in

$$\frac{2m-1}{2m} \Phi_{n-1} \left(\frac{d-1}{m}, \frac{2m-1}{m} \right) + \frac{d-1}{d} \frac{1}{2m} \left(\frac{d-1}{m} \right)^{n-1} + \frac{(-1)^{n+1}}{2m^n d}$$

$$\geq \frac{1}{2}\Phi_{n-1}(e-1, 1) + \frac{(e-1)^n + (-1)^{n+1}}{2e}$$

Using $\frac{d-1}{d} < 1$, combine the first two terms in the above inequality to see

$$\begin{aligned} \Phi_{n-1}\left(\frac{d-1}{m}, 2\right) &> \frac{1}{2}\Phi_{n-1}(e-1, 1) + \frac{(e-1)^n + (-1)^{n+1}}{2e} \\ &= \frac{1}{2}\left(\frac{(e-1)^n - 1}{(e-1) - 1}\right) + \frac{(e-1)^n + (-1)^{n+1}}{2e} \\ &= \frac{2(e-1)^{n+1} - e + (-1)^{n+1}(e-2)}{2(e-2)} \\ &\geq \frac{(e-1)^{n+1} - (e-1)}{e(e-2)}. \end{aligned}$$

Since we assume $e \geq 3$, this implies

$$(2.5) \quad \Phi_{n-1}\left(\frac{d-1}{m}, 2\right) > (e-1)^{n-1} + 1.$$

Suppose m were not bounded from above. Taking the limit of (2.5) as $m \rightarrow \infty$ shows $2^{n-1} \geq (e-1)^{n-1} + 1$. This contradiction proves (1).

To prove (4), notice that if $e \geq 5$, then Lemma 2.2 and inequality (2.5) imply that $\frac{d-1}{m} > e-2$.

If $m = 1$, then $d = e$, as follows. Use Lemma 1.1 to extend f to a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ with $F^*\mathcal{O}(1) = \mathcal{O}(1)$. The image of F is a linear subspace of \mathbb{P}^n that contains Y , namely \mathbb{P}^n itself. So F is an automorphism of \mathbb{P}^n , and $d = e$.

To prove (2) and (3) we assume $m \geq 2$ and prove $d > e$. If $e \geq 5$, then $d > e$ by (4). The cases $e = 3$ and $e = 4$ can be checked by hand in case $n = 4$, and it suffices to check (2) and (3) for the case $n = 4$ because the upper bounds on m given by the inequality of Theorem 1 improve as n increases. \square

Corollary 2.3. *Let $f : X \rightarrow Y$ be a nonconstant separable morphism of hypersurfaces in \mathbb{P}^n , $n \geq 4$, such that Y is nonsingular and $\deg X = \deg Y \geq 3$. There is an automorphism $F : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $f = F|_X$.*

Proof. By Proposition 2.2(3), $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(1)$. By Lemma 1.1, there is a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $f = F|_X$. Since $F^*\mathcal{O}(1) = \mathcal{O}(1)$, the image of F is a linear subspace of \mathbb{P}^n containing Y . So F is in fact an automorphism. \square

2.2. Hypersurfaces of General Type. We now assume the ground field is algebraically closed of characteristic zero. The purpose of this subsection is to prove the following Theorem.

Theorem 2. *Assume the base field is algebraically closed of characteristic zero. Let $f : X \rightarrow Y$ be a morphism of hypersurfaces in \mathbb{P}^n , $n \geq 4$. If Y*

is nonsingular and of general type, then there is a morphism $F : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $f = F|_X$ and $F^{-1}(Y) = X$.

The proof will rely on Proposition 2.4 below, which is an inequality that will bound the polynomial degree m of f from below.

To prove Theorem 2, we focus our attention on the hypersurface $H := F^{-1}(Y) - X$ in \mathbb{P}^n defined at the beginning of Section 2. In particular, we wish to show that H is the 0 divisor, i.e. that $F^{-1}(Y) = X$.

Define Σ to be an irreducible component of a general hyperplane section of H , taken with its reduced structure. Then F is defined at every point of Σ because F is undefined at only finitely many points in \mathbb{P}^n . We will analyze the morphism $F|_\Sigma : \Sigma \rightarrow Y$ using the following Proposition.

Proposition 2.4. *Let Σ be an integral hypersurface in \mathbb{P}^{n-1} and Y be a smooth hypersurface in \mathbb{P}^n , $n \geq 4$. Let $\delta = \deg \Sigma$, and $e = \deg Y$. If $g : \Sigma \rightarrow Y$ is a morphism with $g^* \mathcal{O}_Y(1) = \mathcal{O}_\Sigma(m)$ for some positive integer m , then*

$$n - \delta + m(e - n) \leq 0.$$

Proof. First we claim that there is a canonical morphism

$$(2.6) \quad \bigwedge^{n-2} \Omega_\Sigma^1 \rightarrow \omega_\Sigma^o$$

that is an isomorphism on the nonsingular locus of Σ , where ω_Σ^o is the dualizing sheaf of Σ . Let I denote the ideal sheaf of Σ in \mathbb{P}^n . Since Σ is a reduced local complete intersection in \mathbb{P}^n , there is a short exact sequence

$$(2.7) \quad 0 \rightarrow I/I^2 \rightarrow \Omega_{\mathbb{P}^n}^1|_\Sigma \rightarrow \Omega_\Sigma^1 \rightarrow 0.$$

Therefore the morphism

$$\Phi : \left(\bigwedge^2 I/I^2 \right) \otimes \bigwedge^{n-2} \Omega_\Sigma^1 \rightarrow \bigwedge^n \Omega_{\mathbb{P}^n}^1|_\Sigma$$

$$\xi_1 \wedge \xi_2 \otimes d\bar{\phi}_1 \wedge \cdots \wedge d\bar{\phi}_{n-2} \mapsto d\xi_1 \wedge d\xi_2 \wedge d\phi_1 \wedge \cdots \wedge d\phi_{n-2}$$

is well-defined. Using the formula

$$\omega_\Sigma^o \cong \left(\bigwedge^2 I/I^2 \right)^{-1} \otimes \bigwedge^n \Omega_{\mathbb{P}^n}^1|_\Sigma$$

tensor Φ with the dual of the invertible sheaf $\bigwedge^2 I/I^2$ to obtain the morphism (2.6). This is an isomorphism when restricted to Σ_{reg} , because all the sheaves in (2.7) are locally free on Σ_{reg} .

Since $g^* \mathcal{O}_Y(1) = \mathcal{O}_\Sigma(m)$ is ample, g has finite fibers. So the canonical morphism $g^* \Omega_Y^1 \rightarrow \Omega_\Sigma^1$ is a surjection at the generic point of Σ by the characteristic zero assumption of this subsection. By taking exterior powers and composing with (2.6), we obtain a composite morphism

$$\bigwedge^{n-2} g^* \Omega_Y^1 \rightarrow \bigwedge^{n-2} \Omega_\Sigma^1 \rightarrow \omega_\Sigma^o$$

that is a surjection at the generic point of Σ . Since Σ is a hypersurface in \mathbb{P}^{n-1} of degree δ , $\omega_\Sigma^2 = \mathcal{O}_\Sigma(\delta - n)$. So dualizing the above morphism gives the exact sequence

$$(2.8) \quad 0 \longrightarrow \mathcal{O}_\Sigma(n - \delta) \longrightarrow \bigwedge^{n-2} g^* T_Y.$$

This is an injection because it is an injection at the generic point of Σ and $\mathcal{O}_\Sigma(n - \delta)$ is torsion-free. Tensoring (2.8) with $\mathcal{O}_\Sigma(m(e - n) - 1)$ and applying the formula $\bigwedge^{n-2} T_Y = \Omega_Y^1(-K_Y)$ yields the exact sequence

$$(2.9) \quad 0 \longrightarrow \mathcal{O}_\Sigma(n - \delta + m(e - n) - 1) \longrightarrow (g^* \Omega_Y^1)(m - 1).$$

Tensoring with $\mathcal{O}_Y(-K_Y) = \mathcal{O}_Y(n + 1 - e)$, the conormal sequence for Y in \mathbb{P}^n and the Euler sequence for \mathbb{P}^n give the following short exact sequences, respectively:

$$(2.10) \quad 0 \longrightarrow \mathcal{O}_Y(n + 1 - 2e) \longrightarrow \Omega_{\mathbb{P}^n}^1 \otimes \mathcal{O}_Y(-K_Y) \longrightarrow \Omega_Y^1(-K_Y) \longrightarrow 0$$

$$(2.11) \quad 0 \longrightarrow \Omega_{\mathbb{P}^n}^1 \otimes \mathcal{O}_Y(-K_Y) \longrightarrow \mathcal{O}_Y(n - e)^{\oplus n+1} \longrightarrow \mathcal{O}_Y(n + 1 - e) \longrightarrow 0$$

Tensor (2.10) and (2.11) with $\mathcal{O}_Y(e - n)$, apply g^* , then tensor with $\mathcal{O}_\Sigma(-1)$ to obtain the following short exact sequences:

$$(2.12) \quad 0 \longrightarrow \mathcal{O}_\Sigma(m(1 - e) - 1) \longrightarrow (g^* \Omega_{\mathbb{P}^n}^1)(m - 1) \longrightarrow (g^* \Omega_Y^1)(m - 1) \longrightarrow 0$$

$$(2.13) \quad 0 \longrightarrow (g^* \Omega_{\mathbb{P}^n}^1)(m - 1) \longrightarrow \mathcal{O}_\Sigma(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_\Sigma(m - 1) \longrightarrow 0$$

Since $H^0(\Sigma, \mathcal{O}_\Sigma(-1)) = 0$, (2.13) yields $H^0(\Sigma, (g^* \Omega_{\mathbb{P}^n}^1)(m - 1)) = 0$. Therefore (2.12) implies $H^0(\Sigma, (g^* \Omega_Y^1)(m - 1)) = 0$ because $H^1(\Sigma, \mathcal{O}_\Sigma(m(1 - e) - 1)) = 0$. Hence $n - \delta + m(e - n) - 1 < 0$ by (2.9), as desired. \square

Proof of Theorem 2. Suppose that $F^{-1}(Y) \neq X$. Then H is not empty, and Proposition 2.4 implies

$$n + m(e - n) \leq \deg \Sigma \leq \deg H = em - d.$$

Therefore $d \leq n(m - 1)$. If Y is of general type, i.e. $e \geq n + 2$, then Proposition 2.2(4) implies $d > mn$. This contradiction finishes the proof.

Remark. Suppose the ground field k has positive characteristic. If the characteristic is large, say $\text{char } k > \alpha$, where $\alpha := \frac{em-d}{e}m^{n-2}$, then the morphism $F|_\Sigma : \Sigma \rightarrow F(\Sigma)$ is separable. Indeed, the Grothendieck-Lefschetz Theorem, [8, Theorem 4.3.2], implies that the divisor $F(\Sigma) \subset Y$ is the intersection of Y with another hypersurface. So one can check that $\deg F|_\Sigma \leq \alpha$ by applying the projection formula to $F|_\Sigma \rightarrow F(\Sigma)$.

It follows that if $\text{char } k > \alpha$, then the proof of Proposition 2.4 is still valid. Hence, Theorem 2 will also hold in positive characteristic if $\text{char } k > \alpha$.

Corollary 2.5. *If $f : X \rightarrow Y$ is a nonconstant morphism between hypersurfaces in \mathbb{P}^n , $n \geq 4$, such that Y is nonsingular and of general type, then $\deg Y$ divides $\deg X$ with quotient m such that $f^* \mathcal{O}_Y(1) = \mathcal{O}_X(m)$.*

Proof. By Theorem 2, there is a morphism $F : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $X = F^{-1}(Y)$ and $F^*\mathcal{O}(1) = \mathcal{O}(m)$. It follows that X is a hypersurface of degree $m \cdot \deg Y$. \square

2.3. Hypersurfaces Not of General Type. The ground field will have characteristic zero unless indicated otherwise. We will show that if $3 \leq e \leq n + 1$ and d is not too much larger than e , then the conclusion of Theorem 2 still holds. The following definition will be central to our point of view.

Definition. If Z is any scheme and $F : Z \dashrightarrow \mathbb{P}^n$ is a rational map given by sections F_0, \dots, F_n of some line bundle on Z , then let $\text{indet}(F)$ denote the scheme of common vanishing of the F_i in Z :

$$\text{indet}(F) := V(F_0, \dots, F_n) \subset Z.$$

Lemma 2.6. *Using the notation at the beginning of Section 2:*

- (1) *If $H \neq 0$, then $\deg H = em - d \geq e$. This holds for $e \geq 2$.*
- (2) *If $p \in \text{indet}(F)$, then H has order at least e at p , regardless of the characteristic of the ground field.*

Proof. Suppose $p \in \text{indet}(F)$ is a reduced closed point. If $Y = V(G)$ for a homogeneous polynomial $G = G(y_0, \dots, y_n)$ of degree e , then $F^{-1}(Y) := V(G(F_0, \dots, F_n))$ has order at least e at p because the F_i are all zero at p . But $F^{-1}(Y) = X + H$, and p is not contained in X . So H has order at least e at p . This proves (2), and it proves (1) in case F is not defined at some point of H .

If $F|_H$ is a morphism, then (1) follows from Proposition 2.2(2) in case $e \geq 3$. And in the case $e = 2$, we need only see that $d \neq 1$. However, Lazarsfeld shows in [10] that if a smooth variety Y is the image of a morphism from a projective space, then Y is itself a projective space. \square

Proposition 2.7. *If $m = 1, 2$, then the conclusion of Theorem 2 holds, i.e. there is a morphism $F : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $f = F|_X$ and $X = F^{-1}(Y)$.*

Proof. If $m = 1$, then the image of $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a linear subspace that contains Y . So F is an automorphism of \mathbb{P}^n , and $d = e$.

Suppose $m = 2$ and $d \neq 2e$. Then X and H both have degree e by Lemma 2.6 and Proposition 2.2(2). If $e \geq 3$, then $m = 1$ by Proposition 2.2(3), which is a contradiction. If $e = 2$, then $m = 1$ because every nonconstant morphism of smooth quadrics in \mathbb{P}^n is an isomorphism for $n \geq 4$, cf. [11]. This contradiction shows $d = 2e$ after all. \square

Proposition 2.8. *Fix d, e, m with $e \geq 3$, and assume one of the following three conditions holds:*

- (i) $d < e^2$
- (ii) $d > (m - 1)^2$
- (iii) $m \leq e$

Then the conclusion of Theorem 2 holds for n sufficiently large.

Proof. Using Theorem 1 and formulas (2.3) and (2.4), let n tend to infinity and get $d - 1 \geq m(e - 1)$. If $H \neq 0$, then $em - d \geq e$ by Lemma 2.6(1). Together, these two inequalities contradict each of the three conditions above. \square

Examples in Characteristic Zero. Theorem 1 gives an upper bound on the polynomial degree m of f whenever $e \geq 3$. Using these explicit upper bounds, along with Proposition 2.4 and Lemma 2.6(1), one can check that the conclusion of Theorem 2 holds for the following cases in \mathbb{P}^4 :

$$\begin{aligned} e = 3 & \quad d \leq 4 \\ e = 4 & \quad d \leq 10 \\ e = 5 & \quad d = 1, \dots, 23, 25, 26, 29 \end{aligned}$$

Examples in Positive Characteristic. Theorem 1 and formulas (2.3), (2.4) hold in arbitrary characteristic. So we may compute upper bounds on the polynomial degree m of f in positive characteristic as well.

If $em \neq d$ and $\deg H = 1$, then $F|_H$ is a morphism by Lemma 2.6(2). This is impossible because Y is not the image of a morphism from \mathbb{P}^{n-1} , as shown in [10]. So if $H \neq 0$, then $\deg H > 1$.

Using the fact $em - d > 1$ and the explicit upper bounds on m that we obtain from Theorem 1, we see that the conclusion of Theorem 2 holds for the following cases in \mathbb{P}^4 :

$$\begin{aligned} e = 3 & \quad d \leq 3 \\ e = 4 & \quad d \leq 8 \\ e = 5 & \quad d = 1, \dots, 11, 14 \\ e = 6 & \quad d = 1, \dots, 14, 17, 18 \\ e = 7 & \quad d = 1, \dots, 17, 20, 21, 22, 27 \end{aligned}$$

Question. One can ask if the general type hypothesis of Theorem 2 is too strong. The results of Section 2.3 seem to indicate that this is indeed the case. To be precise, if $f : X \rightarrow Y$ is a nonconstant separable morphism of hypersurfaces in \mathbb{P}^n , $n \geq 4$, such that Y is nonsingular of degree at least 2, is it true that there is an endomorphism $F : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $f = F|_X$ and $X = F^{-1}(Y)$?

More generally, suppose Y is a nonsingular complete intersection in \mathbb{P}^n of hypersurfaces of degrees $e_1, \dots, e_k \geq 2$ where k is the codimension of Y in \mathbb{P}^n , and $f : X \rightarrow Y$ is a nonconstant morphism from a complete intersection $X \subset \mathbb{P}^n$ to Y with $\dim X = \dim Y \geq 3$. Is it necessarily true that X is a complete intersection in \mathbb{P}^n of hypersurfaces of degrees $e_1 m, \dots, e_k m$ and that f extends to a morphism $F : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $F^{-1}(Y) = X$?

PAPER 2:
MORPHISMS FROM QUINTIC THREEFOLDS
TO CUBIC THREEFOLDS ARE CONSTANT

ABSTRACT

We show that every morphism from a quintic threefold in \mathbb{P}^4 to a nonsingular cubic threefold in \mathbb{P}^4 is constant in characteristic zero. In the process, we classify morphisms from \mathbb{P}^2 to nonsingular cubic hypersurfaces in \mathbb{P}^4 given by degree 3 polynomials.

INTRODUCTION

The author shows in [14] that if $f : X \rightarrow Y$ is a morphism of hypersurfaces in \mathbb{P}^4 such that $\deg Y = 3$ and $\deg X \leq 4$, then f is either constant or $\deg X = 3$ and f is an isomorphism. The purpose of this paper is to extend this result by proving the following theorem.

Theorem 1. *If $f : X \rightarrow Y$ is a morphism of hypersurfaces in \mathbb{P}^4 over an algebraically closed field of characteristic zero such that $\deg X = 5$, $\deg Y = 3$, and Y is nonsingular, then f is constant.*

The motivation for investigating such morphisms to cubic hypersurfaces is the expectation that if $f : X \rightarrow Y$ is a morphism between hypersurfaces in \mathbb{P}^n , $n \geq 4$, such that Y is nonsingular of degree at least 2, then $\deg Y$ divides $\deg X$ with quotient q , and f is given by polynomials of degree q , i.e. $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(q)$. This result is proven in [14] when $\deg Y \geq n + 2$. It is also proven in some cases where $3 \leq \deg Y \leq n + 1$. The fact that morphisms from quintic threefolds to cubic threefolds are necessarily constant is the first nontrivial case of morphisms to cubics.

1. OUTLINE OF PROOF

Throughout the paper, the base field will be algebraically closed of characteristic zero, and $f : X \rightarrow Y$ will denote a nonconstant morphism of hypersurfaces in \mathbb{P}^4 , such that $\deg X = 5$, $\deg Y = 3$, and Y is nonsingular.

The Grothendieck-Lefschetz Theorem, [8, Theorem 4.3.2], states that $\text{Pic } X$ is generated by $\mathcal{O}_X(1)$. So $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(m)$ for some positive integer m . In terms of m , [14, Theorem 1] states that $\deg f^*c_3(\Omega_Y^1(2)) \leq \deg c_3(\Omega_X^1(2m))$. By computing the Chern classes in this inequality, one checks that $m \leq 3$.

We claim there is a rational map $F : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ undefined at only finitely many points disjoint from X such that $f = F|_X$. To see this, consider the short exact sequence

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m)) \longrightarrow H^0(X, \mathcal{O}_X(m)) \longrightarrow H^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m-5))$$

Since $H^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m-5)) = 0$, the sections of $\mathcal{O}_X(m)$ that define $f : X \rightarrow \mathbb{P}^4$ lift to \mathbb{P}^4 and define a rational map $F : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$, as claimed.

We introduce a hypersurface H in \mathbb{P}^4 that will play a key role. Lazarsfeld proves in [10] that \mathbb{P}^k is the only smooth k -dimensional variety that is the image of a morphism from \mathbb{P}^k . In particular, Y is not the image of a morphism from \mathbb{P}^3 . It follows by considering a general hyperplane in \mathbb{P}^4 that the image of $F : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ is not Y . So F is dominant because the image of F is irreducible and contains Y . Hence $F^{-1}(Y) = X + H$ for some hypersurface H of degree $3m - 5$.

This shows $m \neq 1$. We see that $m \neq 2$ as follows. Suppose $m = 2$. Then H is a hyperplane in \mathbb{P}^4 . Therefore $F|_H : H \dashrightarrow Y$ is not a morphism. Let $p \in H$ be a point of indeterminacy of F . This means that $F = (F_0, \dots, F_4)$ for homogeneous polynomials F_i of degree 2 that all vanish at p . If Y is defined by the homogeneous cubic polynomial $G = G(y_0, \dots, y_4)$, then $F^{-1}(Y) = X \cup H$ is defined by the homogeneous sextic polynomial $G(F_0, \dots, F_4)$. Since the F_i all vanish at p and G is homogeneous of degree 3, $G(F_0, \dots, F_4)$ vanishes to order at least 3 at p . In other words, p is a triple point of $F^{-1}(Y) = X \cup H$. This is impossible because p is contained in the hyperplane H but not contained in X . Therefore, $m \neq 2$.

It is considerably more difficult to check that $m = 3$ is also not possible. That task will occupy us for the remainder of the paper. We focus our attention on the degree 4 hypersurface H and the rational map $F|_H : H \dashrightarrow Y$.

We prove Theorem 1 by considering the various possibilities for H . If H contains many copies of \mathbb{P}^2 , then we will use the resulting maps $\mathbb{P}^2 \rightarrow Y$, which would be given by degree 3 polynomials. So our first task is to classify such morphisms from \mathbb{P}^2 . This is done in Section 2. Then we return to the map $H \dashrightarrow Y$ in Section 3 and work through several cases using the geometry of rational maps from threefolds to three dimensional cubics.

2. MORPHISMS FROM \mathbb{P}^2 TO CUBIC THREEFOLDS

Let $g : \mathbb{P}^2 \rightarrow Y$ be a morphism given by degree 3 polynomials, i.e. $g^*\mathcal{O}_Y(1) = \mathcal{O}_{\mathbb{P}^2}(3)$. The purpose of this section is to prove the following result about g .

Theorem 2. *We can choose coordinates x_0, x_1, x_2 on \mathbb{P}^2 and coordinates on \mathbb{P}^4 such that*

$$g = (x_0^3, x_1^3, x_2^3, x_0x_1x_2, 0) : \mathbb{P}^2 \longrightarrow Y \subset \mathbb{P}^4.$$

Let S be the image of g . Since Y is nonsingular, $S \subset Y$ is a Cartier divisor. So by the Grothendieck-Lefschetz Theorem, [8, Theorem 4.3.2], S is the zero locus $V(s)$ of a section $s \in H^0(Y, \mathcal{O}_Y(a))$ for some $a > 0$. To see that S is the scheme-theoretic intersection of Y with another hypersurface Y' in \mathbb{P}^4 , consider the following piece of a long exact sequence of cohomology groups.

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(a)) \longrightarrow H^0(Y, \mathcal{O}_Y(a)) \longrightarrow H^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(a-3))$$

Since $H^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(a-3)) = 0$, s is the image of some $s' \in H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(a))$. So $S = V(s') \cap Y$. Take $Y' := V(s')$.

Calculate $g^*c_1(\mathcal{O}_Y(1))^2$ to be $9 = \deg g \cdot \deg S$ to see that $3 = \deg g \cdot \deg Y'$. Hence we will break the proof of Theorem 2 into two cases depending on whether $\deg Y' = 3$ or $\deg Y' = 1$. First, we recall the following result from [4], which we will use repeatedly.

Proposition 2.1. *Let Δ be the space of lines on the smooth cubic threefold Y . Then Δ is a complete nonsingular surface that does not contain a rational curve.*

Case 1. Assume Y' has degree 3. We will derive a contradiction.

By assumption, $g : \mathbb{P}^2 \rightarrow S$ has degree 1, so it is a finite birational morphism. Our strategy is to analyze the double point class $\mathbb{D}(g)$ of $g : \mathbb{P}^2 \rightarrow Y$. See [6, Section 9.3] for the construction and computation of $\mathbb{D}(g)$. The construction will be used implicitly in the proof of Proposition 2.6 and Corollary 2.7. Following [6], we calculate the double point class of g :

$$\begin{aligned} \mathbb{D}(g) &= g^*g_*[\mathbb{P}^2] - c_1(g^*T_Y) + c_1(T_{\mathbb{P}^2}) \\ &= g^*c_1(\mathcal{O}_Y(3)) - g^*c_1(\mathcal{O}_Y(2)) + c_1(\mathcal{O}_{\mathbb{P}^2}(3)) \\ &= c_1(\mathcal{O}_{\mathbb{P}^2}(6)). \end{aligned}$$

An important fact for our purposes is that the construction of the cycle $\mathbb{D}(g)$ not only gives a 1-cycle modulo rational equivalence, it actually constructs a Weil divisor in \mathbb{P}^2 . We will denote this Weil divisor by $D(g)$ and consider it as a closed subscheme of \mathbb{P}^2 . This notation differs slightly from [6] in that we use $D(g)$ to denote a divisor, not just a set. Roughly speaking, $D(g)$ is the curve in \mathbb{P}^2 consisting of the closed points x such that either $g(x) = g(y)$ for some $y \neq x$ or such that $T_x\mathbb{P}^2 \rightarrow T_{g(x)}Y$ is not injective. The scheme structure of $D(g)$ comes from the fact that an integral curve D in $D(g)$ will appear with multiplicity if for a general point $x \in D$ there is more than one other point $y \in \mathbb{P}^2$ with $g(x) = g(y)$ or if g ramifies to high order along D .

The following result tells us that the image of $D(g)$ under g is equal to the singular locus of S as a set.

Lemma 2.2. *Let $g : V \rightarrow W$ be a finite surjective birational morphism of varieties with V regular. Then for $w \in W$ a closed point, w is a nonsingular point of W if and only if the scheme-theoretic preimage $g^{-1}(w)$ is a single reduced point.*

Proof. Suppose $g^{-1}(w)$ is a single reduced point v . Let $D \subset W$ be a general curve in W through w such that D is a local complete intersection in W at w and

$$\dim T_w W - \dim T_w D = \dim W - 1.$$

By assumption, the pullback of the maximal ideal of w generates the maximal ideal of v . So the curve $C := g^{-1}(D)$ is nonsingular at v because C is cut out near v by the $\dim W - 1$ equations that define D near w . The morphism $g|_C$ is an isomorphism in a neighborhood of v because $g^{-1}(w) = v$ and $g|_C$ is birational in a neighborhood of v . So D is regular at w , whence W is too by the above equation.

Conversely, suppose w is a regular closed point of W . Let A be the ring of regular functions on an open affine U of W containing w , and let B be the ring of regular functions on the open affine $g^{-1}(U)$. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to the point w , and let M denote the multiplicative set $A \setminus \mathfrak{m}$. Then $M^{-1}B$ is the integral closure of $A_{\mathfrak{m}} = M^{-1}A$ in its field of fractions. But $A_{\mathfrak{m}}$ is integrally closed because it is regular. Therefore $M^{-1}B = A_{\mathfrak{m}}$, whence $M^{-1}B$ is a local ring whose maximal ideal is generated by \mathfrak{m} . In other words, $g^{-1}(w)$ is a single reduced point. \square

Now let $C \subset S$ denote the image of $D(g)$ with its reduced structure:

$$C := g(D(g))_{\text{red}}$$

By Lemma 2.2, C is the maximal reduced curve contained in the singular locus of S . We will rule out the case $\deg Y' = 3$ by comparing $D(g)$ to $g^{-1}(C)$, where $g^{-1}(C)$ denotes the scheme-theoretic preimage. So our strategy is to compare the multiplicity of integral curves in $D(g)$ and $g^{-1}(C)$.

Lemma 2.3. *There is a cubic hypersurface Y'' in \mathbb{P}^4 such that $S = Y \cap Y''$, and Y'' is singular along C .*

Proof. Let $\{Y_t\}$ be the pencil of cubic hypersurfaces spanned by Y and Y' . Since $Y_t \cap Y = S$ so long as $Y_t \neq Y$, it suffices to show that one of the Y_t is singular along C .

Let C_0 be an irreducible component of C . For every point $p \in C_0$ we have $T_p S = T_p Y$ because S is singular at p . Therefore $T_p S = T_p Y_t$ for t general, and there is a unique t , say t_p , such that Y_{t_p} is singular at p . Only finitely many of the Y_t are singular, so one of the Y_t , say Y_{t_0} , is singular at infinitely many points of C_0 . Hence Y_{t_0} is singular along C_0 .

If C_1 is another irreducible component of C , then C_0 and C_1 meet at some point p because C_0 and C_1 are both images of curves in \mathbb{P}^2 . If Y_{t_1} is singular along C_1 , then Y_{t_0} and Y_{t_1} are both singular at p , so $Y_{t_0} = Y_{t_1}$. The Lemma follows. \square

We now assume Y' to be singular along C .

Lemma 2.4. *The scheme-theoretic preimage $g^{-1}(C)$ does not contain a curve of degree 6 or more.*

Proof. Let $G' = G'(y_0, \dots, y_4)$ be the homogeneous equation for Y' in \mathbb{P}^4 . Set $C_i := S \cap V(\frac{\partial G'}{\partial y_i})$ and $D_i := g^{-1}(C_i)$ for $i = 0, \dots, 4$. The D_i are all degree 6 plane curves containing $g^{-1}(C)$. So it suffices to show that the D_i are not all equal.

Suppose they were. Then the equations for the D_i in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$ are all scalar multiples of each other. These equations are the images of the partials $\frac{\partial G'}{\partial y_i}$ under the composition

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \xrightarrow{\rho} H^0(S, \mathcal{O}_S(2)) \xrightarrow{g^*} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$$

where ρ is restriction. This composition is an injection because S is not contained in a quadric. So the $\frac{\partial G'}{\partial y_i}$ are also scalar multiples of each other, say $\frac{\partial G'}{\partial y_i} = \alpha_i \frac{\partial G'}{\partial y_0}$ for scalars α_i . By Euler's formula,

$$3G' = \sum_{i=0}^4 y_i \frac{\partial G'}{\partial y_i} = \frac{\partial G'}{\partial y_0} \sum_{i=0}^4 \alpha_i y_i$$

which implies Y' is not integral. So $S = Y \cap Y'$ is not integral. This contradiction finishes the proof. \square

Now we make several observations, which we list in the following Lemma.

Lemma 2.5.

- (1) If $\Lambda \subset \mathbb{P}^4$ is any 2-plane, then $g^{-1}(\Lambda \cap S)$ does not contain a curve of degree 3 or more.
- (2) For $C' \subset C$ any curve, the secant variety of C' is contained in Y' .
- (3) Every curve in C that lies in a hyperplane is a plane curve.
- (4) C does not contain a conic curve.
- (5) For every $x \in \mathbb{P}^2$, g induces a nonzero map $T_x \mathbb{P}^2 \rightarrow T_{g(x)} Y$.

Proof. (1) follows from the fact that the preimage of every hyperplane containing Λ is a distinct degree 3 curve in \mathbb{P}^2 . The preimages are distinct because S is not contained in a hyperplane, which is our assumption for Case 1.

(2) holds because every line meeting Y' at two singular points is contained in Y' , and Y' is singular along C' .

(3) follows from (2) because Y' does not contain a hyperplane. This is because $\deg Y' = 3$ and $Y \cap Y' = S$ is integral.

(4) Suppose $C' \subset C$ is a conic curve. Then $g^{-1}(C')$ contains a curve of degree at least 2 because lines in \mathbb{P}^2 map to either lines or cubics. If Λ is the 2-plane containing C' , then $\Lambda \subset Y'$ by (2). So $\Lambda \cap S = \Lambda \cap Y$ is a degree 3 plane curve consisting of C' and a line. This line is the image of a curve in \mathbb{P}^2 . Therefore, $g^{-1}(\Lambda)$ contains a curve of degree at least 3, contradicting (1).

(5) If $T_x \mathbb{P}^2 \rightarrow T_{g(x)} Y$ is the zero map, then every line in \mathbb{P}^2 through x maps to either a line or a cuspidal plane cubic with cusp at $g(x)$. In particular, the image of every line in \mathbb{P}^2 through x is contained in $T_{g(x)} Y$, which is

impossible by the assumption for this subsection that S is not contained in a hyperplane. \square

We will describe the integral curves D that might occur in $D(g)$ in terms of their multiplicity in $D(g)$ and in the scheme-theoretic preimage $g^{-1}g(D)$. For this purpose, we recall the following definition.

Definition. If D is an integral curve contained in a one dimensional scheme Z , then $\text{mult}_D Z$ will denote the length of the Artin ring obtained by localizing O_Z at the generic point of D .

Proposition 2.6. *If $D \subset D(g)$ is an integral curve with $n := \deg g|_D$, then one of the following four cases holds:*

- (1) $n = 1$, $\text{mult}_D g^{-1}g(D) = 2$, $\text{mult}_D D(g) = 1$, D is a nonsingular conic, and D is the only integral curve in $g^{-1}g(D)$.
- (2) $n = 1$, $\text{mult}_D g^{-1}g(D) = 1$, $\text{mult}_D D(g) = 1$, and $g^{-1}g(D)$ contains two distinct nonsingular conics D, D' .
- (3) $n = 2$, $\text{mult}_D g^{-1}g(D) = 1$, $\text{mult}_D D(g) = 1$, D is a nonsingular conic, and $g(D)$ is a singular plane cubic.
- (4) $n = 3$, $\text{mult}_D g^{-1}g(D) = 1$, $\text{mult}_D D(g) = 2$, and $D, g(D)$ are both lines.

Proof. We calculate $\text{mult}_D D(g)$ as follows. First suppose that g is unramified along every curve in $g^{-1}g(D)$ in the sense that $T_x \mathbb{P}^2 \rightarrow T_{g(x)} Y$ is injective for x a general point on any irreducible curve in $g^{-1}g(D)$. In this case, if D_1, \dots, D_r are the distinct integral curves contained in $g^{-1}g(D)$, then

$$(2.1) \quad \text{mult}_D D(g) = -1 + \sum_{i=1}^r \deg g|_{D_i}$$

because the number of preimage points of a general point in $g(D)$ is the sum of the $\deg g|_{D_i}$. On the other hand, if D is the only integral curve contained in $g^{-1}g(D)$ and g is simply ramified along D in the sense that $2D$ is contained in $g^{-1}g(D)$ but $3D$ is not, then

$$(2.2) \quad \text{mult}_D D(g) = 1.$$

These are the only cases that will arise in our discussion here. See [6, Section 9.3] for the construction of the Weil divisor $D(g)$ representing $\mathbb{D}(g)$.

We claim that if $D \subset D(g)$ is a line, then $g(D)$ is a line too. So suppose $D \subset D(g)$ is a line. By Lemma 2.5(3), $g(D)$ is not a twisted cubic. So it suffices to show that $g(D)$ is not a plane cubic. If it were, then by Lemma 2.5(2), Y' contains the 2-plane Λ spanned by $g(D)$. Choose any hyperplane Γ containing $g(D)$, and let $\Sigma := \Gamma \cap Y'$. If $p \in \Gamma \cap Y'$ is a singular point of Y' away from Λ , then $\Gamma \cap Y'$ contains the cone Σ_p over $g(D)$ with vertex p . Since $\Lambda \cup \Sigma_p \subset \Sigma$ and Σ and Σ_p both have degree 3, no such p can exist. In particular, $C \subset \Lambda$. By Lemma 2.5(1), $g^{-1}(C)$ is either D, D with multiplicity 2, or D and another line. So $D(g)$ will have respectively

degree 0, 1, or 2, according to the formulas for $\text{mult}_D D(g)$ above. However, $\deg D(g) = 6$. So $g(D)$ is not a plane cubic. Therefore, if $D \subset D(g)$ is a line, then $g(D)$ is too.

Now let $D \subset D(g)$ be any integral curve. Using $(n-1)D \subset D(g)$ and taking degrees, we obtain the inequality $(n-1)\deg D \leq 6$. The projection formula yields $3\deg D = n\deg g(D)$. Therefore, if $n \geq 2$, then

$$(2.3) \quad \frac{n}{3}\deg g(D) = \deg D \leq \frac{6}{n-1}.$$

This shows $n \leq 3$ because the left hand side is an integer. Now we analyze what happens for each value $n = 1, 2, 3$.

Suppose $n = 3$, so that $\deg D = \deg g(D) \leq 3$. If $\deg D = 3$, then Lemma 2.5(1) and (3) imply that $g(D)$ is neither a plane cubic nor a twisted cubic. So $\deg g(D) \leq 2$. By Lemma 2.5(4), $\deg D \neq 2$. So $D, g(D)$ are lines, as claimed in Case (4).

To see $\text{mult}_D g^{-1}g(D) = 1$, suppose $2D \subset g^{-1}g(D)$. By Lemma 2.5(5); for $x \in D$ a general point, the map $T_x\mathbb{P}^2 \rightarrow T_{g(x)}Y$ has rank 1. Choose $p \in g(D)$ general so that $(g|_D)^{-1}(p)$ consists of three distinct points $x_1, x_2, x_3 \in D$. Through each x_i there is a unique line l_i in \mathbb{P}^2 such that $T_{x_i}l_i \rightarrow T_pY$ is the zero map. Each $g(l_i)$ is either a line through p or a cuspidal plane cubic with cusp at p . In both cases, $g(l_i)$ is contained in T_pY . Now the degree 3 curve $g^{-1}(T_pY)$ contains $2D + l_1 + l_2 + l_3$. This contradiction proves $\text{mult}_D g^{-1}g(D) = 1$.

To show $\text{mult}_D D(g) = 2$, it suffices by formula (2.1) to show there is no integral curve $D' \subset g^{-1}g(D)$ other than D . If D' were such a curve, then $\deg g|_{D'} = 3\deg D'$ because $g(D)$ is a line. So for $p \in g(D)$ a general point, $g^{-1}(p)$ would contain 3 points on D and $3\deg D'$ points on D' . There would be $9\deg D'$ lines joining one of the points on D to one of the points on D' , and these lines would map to either lines through p or nodal plane cubics with node at p . These image curves would be contained in T_pY , so we would have the impossible situation that these $9\deg D'$ lines would all be contained in the degree 3 plane curve $g^{-1}(T_pY)$. So there is no D' , and $\text{mult}_D D(g) = 2$. This gives Case (4).

Suppose $n = 2$. If $2D \subset g^{-1}g(D)$, then $2D \subset g^{-1}C$, which implies $2\deg D < 6$ by Lemma 2.4. By formula (2.3), $g(D)$ can not be a line. So D is not a line, from above. Hence $\deg D = 2$ and $\deg g(D) = 3$ by (2.3) and the fact $\deg D < 3$. Since $g(D)$ has degree 3 and can not be a twisted cubic, $g(D)$ is a singular plane cubic. By Lemma 2.5(1), $2D$ can not be contained in $g^{-1}g(D)$, so $\text{mult}_D g^{-1}g(D) = 1$. Likewise, D is the only integral curve contained in $g^{-1}g(D)$, so $\text{mult}_D D(g) = 1$ by formula (2.1). This gives Case (3).

Suppose $n = 1$. By (2.3), $\deg g(D) = 3\deg D$. Since $g(D)$ is not a line, D is not a line. So $\deg g(D) \geq 6$. If D' is another integral curve contained in $g^{-1}g(D)$, then $D' \subset D(g)$. By Cases (3) and (4), $\deg g|_{D'} = 1$ because $\deg g(D') \geq 6$. Therefore, $\deg D' = 3\deg g(D) = \deg D$.

If D is the only integral curve in $g^{-1}g(D)$, then $2D \subset g^{-1}g(D)$ because $D \subset D(g)$ and $n = 1$. By Lemma 2.4, $g^{-1}g(D)$ does not contain a curve of degree 6. So the only possibility is $\text{mult}_D g^{-1}g(D) = 2$ and $\text{deg } D = 2$ because $\text{deg } D > 1$. Since D is integral, D is a nonsingular conic. This gives Case (1) by formula (2.2).

The same reasoning shows that if D' is another integral curve in $g^{-1}g(D)$, then the maximal curve contained in $g^{-1}g(D)$ is $D + D'$, and D, D' are both nonsingular conics. This gives Case (2) by formula (2.1). \square

Corollary 2.7. *There are distinct lines l_1, l_2, l_3 in \mathbb{P}^2 such that*

$$D(g) = 2l_1 + 2l_2 + 2l_3$$

and the $g(l_i)$ are distinct lines in S .

Proof. In cases (1), (2), (3) of Proposition 2.6, D has multiplicity 1 in $D(g)$. So if $D(g)$ does not contain a line, then $D(g)$ is reduced and $D(g) \subset g^{-1}(C)$. This contradicts Lemma 2.4. So D contains a line D . If $L = g(D)$ is the image line of D , then D is the only curve contained in $g^{-1}L$ by Case (4) of Proposition 2.6.

Now suppose $D' \neq D$ is another integral curve in $D(g)$. We will show that D' is also a line. So suppose $\text{deg } D' > 1$. Then $g(D')$ is not a line by Proposition 2.6. Since D and D' meet at a point, so do $L = g(D)$ and $g(D')$. If $g(D')$ is contained in a hyperplane, then by Lemma 2.5(3), $g(D')$ is contained in a 2-plane. So either L and $g(D')$ are contained in the same 2-plane, which is impossible by Lemma 2.5(1), or the curve $L \cup g(D')$ spans a hyperplane, which is impossible by Lemma 2.5(3). So $g(D')$ is not contained in a hyperplane.

Let $p \in g(D')$ be a general point, and let Λ_p be the 2-plane spanned by p and L . Since the cubic Y' is singular along L and at p , Λ_p is contained in Y' . Therefore, $\Lambda_p \cap S = \Lambda_p \cap Y$ is equal to $L + Q_p$ for some conic curve Q_p contained in Λ_p .

If L is contained in Q_p , then $2L$ is contained in $\Lambda_p \cap Y$. So for every point $q \in L$, $T_q \Lambda_p = T_q 2L \subset T_q Y$. Since $g(D')$ is not contained in any hyperplane, there is no hyperplane that contains Λ_p for every $p \in g(D')$. Therefore, $\dim T_q Y = 4$ for every $q \in L$ because $T_q \Lambda_p$ is contained in $T_q Y$ for $p \in g(D')$ a general point. Since Y is nonsingular, one concludes that L is not contained in Q .

If Q_p is not a double line, then $g^{-1}Q_p$ contains a curve of degree at least 2. This is impossible by Lemma 2.5(1), because Λ_p contains L and Q_p . So for every $p \in g(D')$, $Q_p = 2L_p$ for some line $L_p \neq L$.

Therefore, $g(D')$ parametrizes a one dimensional family of lines on Y . By Proposition 2.6, every curve in $D(g)$ is rational, so $g(D')$ is rational. However, the space of lines on Y does not contain a rational curve by Proposition 2.1.

This proves that every integral curve in $D(g)$ is a line, and every line in $D(g)$ occurs with multiplicity 2 by Proposition 2.6. \square

Proposition 2.8. *The case $\deg Y' = 3$ does not occur.*

Proof. We will use the l_i from Corollary 2.7 to give an explicit formula for g and derive a contradiction using this formula.

Let $L_i := g(l_i)$. Each of the L_i intersect the other two, but they are not all contained in a plane by Lemma 2.5(1). So they all meet at some point p in S , and $T_p Y$ is the unique hyperplane containing all the L_i .

I claim the l_i do not all meet at a point x . So suppose there were such a point x , and note $g(x) = p$. By Lemma 2.5(5), at most one of the $g|_{l_i}$, say $g|_{l_1}$, ramifies at x . Therefore there are points $x_2 \in l_2$ and $x_3 \in l_3$ different from x such that $g(x_2) = g(x_3) = p$, and g maps the line l_{23} containing x_2 and x_3 to either a line or a nodal plane cubic with node at p . In particular, $g(l_{23})$ is contained in $T_p Y$. Now the degree 3 curve $g^{-1}(T_p Y)$ contains $l_1 + l_2 + l_3 + l_{23}$, which is impossible. So the l_i do not all meet at x .

Let $x_{ij} := l_i \cap l_j$, and note $g(x_{ij}) = p$. If $g|_{l_1}$ is not ramified at either x_{12} or x_{13} , then there is a point x' on l_1 different from x_{12}, x_{13} such that $g(x') = p$, and the line from x' to x_{23} gives a contradiction just as the line l_{23} did above. Hence each $g|_{l_i}$ ramifies at exactly one of the x_{ij} . So the scheme-theoretic preimage $g^{-1}(p)$ consists of three copies of $\text{Spec } k[\epsilon]/(\epsilon^2)$, one supported at each of the x_{ij} . The scheme $g^{-1}(p)$ is contained in the scheme $l_1 + l_2 + l_3$, but with no two copies of $\text{Spec } k[\epsilon]/(\epsilon^2)$ contained in the same l_i because $(g|_{l_i})^{-1}(p)$ has length 3, as $g|_{l_i}$ is a morphism of degree 3 of nonsingular curves, and is therefore flat. Hence, we can choose homogeneous coordinates x_0, x_1, x_2 on \mathbb{P}^2 such that $l_i = V(x_{i-1})$ and

$$g^{-1}(p) = V(x_0^2 x_1, x_1^2 x_2, x_2^2 x_0, x_0 x_1 x_2).$$

There is a 3 dimensional space of hyperplanes in \mathbb{P}^4 containing p , and these hyperplanes pull back to the linear system of cubics on \mathbb{P}^2 spanned by

$$x_0^2 x_1, x_1^2 x_2, x_2^2 x_0, x_0 x_1 x_2.$$

Set $p = (0, 0, 0, 0, 1)$ so that for a suitable choice of coordinates y_0, \dots, y_4 on \mathbb{P}^4 we have

$$(2.4) \quad g = (x_0^2 x_1, x_1^2 x_2, x_2^2 x_0, x_0 x_1 x_2, g_4) : \mathbb{P}^2 \rightarrow \mathbb{P}^4$$

for some homogeneous polynomial g_4 of degree 3.

Since $g^{-1}V(y_3) = l_1 + l_2 + l_3$, one sees that $T_p Y = V(y_3)$. So if G is the homogeneous equation for Y , then

$$(2.5) \quad G = y_4^2 y_3 + y_4 G_2 + G_3$$

where $G_i = G_i(y_0, \dots, y_3)$ is homogeneous of degree i .

By equation (2.4), the fact that g is defined at $(1, 0, 0) \in \mathbb{P}^2$ tells us that x_0^3 has nonzero coefficient in g_4 . Considering equation (2.4), we see that $x_0^7 x_1 x_2$ has nonzero coefficient in $g^*(y_4^2 y_3)$. Since $g(\mathbb{P}^2) \subset Y$, g^*G is the zero polynomial, so $x_0^7 x_1 x_2$ has nonzero coefficient in $g^*(y_4 G_2 + G_3)$ by equation (2.5). The highest power of x_0 that can appear in g^*G_3 is x_0^6 by (2.4). Therefore, $x_0^7 x_1 x_2$ has nonzero coefficient in $g^*(y_4 G_2)$.

The highest power of x_0 that can appear in a monomial of g^*G_2 is x_0^4 , and this necessarily occurs in the monomial $g_0^2 = x_0^4x_1^2$. So if x_0^7 appears in a monomial of $g^*(y_4G_2) = g_4g^*G_2$, then that monomial is $x_0^7x_1^2$, not $x_0^7x_1x_2$. This contradiction shows that $\deg Y' \neq 3$. \square

Case 2. By Proposition 2.8, the image of $g : \mathbb{P}^2 \rightarrow Y$ is the surface $S := Y \cap Y'$, where Y' is a hyperplane in \mathbb{P}^4 . We will derive a contradiction, thus proving Theorem 2.

The morphism $\delta : Y \rightarrow Y^*$ sending a point p to the hyperplane T_pY is given by an ample invertible sheaf on Y , since $\mathcal{O}_Y(1)$ generates $\text{Pic } Y$ by the Grothendieck-Lefschetz Theorem, [8, Theorem 4.3.2]. So δ is a finite morphism. In other words, no hyperplane in \mathbb{P}^4 can be tangent to Y at infinitely many points. Therefore S is a cubic surface with isolated singularities.

For $s \in S$ a general closed point, let $\{E_t\}$ be a general pencil of hyperplane sections of S containing s , and let $E'_t := g^{-1}(E_t)$. Then E_t and E'_t are both smooth plane cubics for $t \in \mathbb{P}^1$ general.

Since $g : \mathbb{P}^2 \rightarrow S$ has degree 3, we can set $g^{-1}(s) = \{a, b, c\}$. Then $(E'_t, a) \rightarrow (E_t, s)$ is an isogeny of elliptic curves of degree 3 for t general. Note that b is a 3-torsion point of E'_t .

Let $E' \subset \mathbb{P}^1 \times \mathbb{P}^2$ and $E \subset \mathbb{P}^1 \times S$ be the total spaces of the families $\{E'_t\}$ and $\{E_t\}$ over the base \mathbb{P}^1 , and consider the diagram

$$\begin{array}{ccc} E' & \longrightarrow & \mathbb{P}^2 \\ \downarrow g' & & \downarrow g \\ E & \longrightarrow & S \end{array}$$

where g' is the restriction of $\text{id}_{\mathbb{P}^1} \times g$ to $E' \subset \mathbb{P}^1 \times \mathbb{P}^2$.

Let ζ be the generic point of \mathbb{P}^1 , and let E'_ζ, E_ζ be the fibers of E', E over $\zeta \in \mathbb{P}^1$. Then $A := \zeta \times a$ and $B := \zeta \times b$ are ζ -valued points of E'_ζ , and B is a 3-torsion point of the elliptic curve (E'_ζ, A) . Hence $P \mapsto P + B$ gives an automorphism of E'_ζ over E_ζ of order 3 that extends to some birational self-map $\psi : E' \dashrightarrow E'$ of E' over E . In other words, $g' = g' \circ \psi$ as rational maps. The horizontal arrows in the diagram above are birational morphisms. So ψ induces an order 3 birational self-map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of \mathbb{P}^2 such that $g = g \circ \phi$ as rational maps. Therefore ϕ is an automorphism of \mathbb{P}^2 by the following lemma.

Lemma 2.9. *If $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a birational map and $g : \mathbb{P}^2 \rightarrow S$ is a dominant morphism to a surface S such that $g \circ \phi = g$ as rational maps, then ϕ extends to an automorphism of \mathbb{P}^2 .*

Proof. Let $U \subset \mathbb{P}^2$ be the domain of definition of ϕ . Then U is the complement of finitely many points in \mathbb{P}^2 , and $\text{Pic } U$ is generated by $\mathcal{O}_U(1)$, the restriction of $\mathcal{O}_{\mathbb{P}^2}(1)$ to U . From $g \circ \phi = g$ it follows that $\phi^*g^*\mathcal{O}_S(1) = g^*\mathcal{O}_S(1)|_U$. In other words, $\phi^*\mathcal{O}_{\mathbb{P}^2}(3) = \mathcal{O}_U(3)$. Therefore $\phi^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_U(1)$.

Since ϕ is dominant, $\phi : U \rightarrow \mathbb{P}^2$ is given by three linearly independent sections $\sigma_0, \sigma_1, \sigma_2$ of $\mathcal{O}_U(1)$, and these sections are the restriction to U of sections τ_0, τ_1, τ_2 of $\mathcal{O}_{\mathbb{P}^2}(1)$. The τ_i are linearly independent, and hence there is no point where they all vanish. So ϕ extends to the automorphism (τ_0, τ_1, τ_2) of \mathbb{P}^2 . \square

Since ϕ has order 3, its matrix is one of the following after being placed in Jordan form and scaling:

$$\phi = \begin{bmatrix} 1 & & \\ & \rho & \\ & & \rho \end{bmatrix} \quad \text{or} \quad \phi = \begin{bmatrix} 1 & & \\ & \rho & \\ & & \rho^2 \end{bmatrix}$$

where $\rho^3 = 1, \rho \neq 1$. Fix an identification of Y' with \mathbb{P}^3 so that we can write $g : \mathbb{P}^2 \rightarrow S \subset Y'$ as (g_0, \dots, g_3) for some degree 3 polynomials g_i . Then $(g_0, \dots, g_3) = (\phi^* g_0, \dots, \phi^* g_3)$. So each of the g_i are eigenvectors of $\phi^* : H^0(\mathbb{P}^2, \mathcal{O}(3)) \rightarrow H^0(\mathbb{P}^2, \mathcal{O}(3))$ with the same eigenvalue.

Suppose

$$\phi = \begin{bmatrix} 1 & & \\ & \rho & \\ & & \rho \end{bmatrix}$$

Then the following are a basis for each of the eigenspaces of ϕ^* :

$$\begin{aligned} 1 &: x_0^3, x_1^3, x_2^3, x_1^2 x_2, x_1 x_2^2 \\ \rho &: x_0^2 x_1, x_0^2 x_2 \\ \rho^2 &: x_0 x_1^2, x_0 x_2^2, x_0 x_1 x_2 \end{aligned}$$

Since S is not a 2-plane in \mathbb{P}^3 , the eigenspace containing the g_i has dimension at least 4. So the g_i all have eigenvalue 1.

Now consider the morphism

$$h = (x_0^3, x_1^3, x_2^3, x_1^2 x_2, x_1 x_2^2) : \mathbb{P}^2 \rightarrow \mathbb{P}^4.$$

The image surface S' of h is a cone over a twisted cubic curve T' with vertex $(1, 0, 0, 0, 0)$. The morphism $g : \mathbb{P}^2 \rightarrow Y' \cong \mathbb{P}^3$ is the morphism h followed by projection π_p from some point p . Projection maps lines to lines, so the image S of g is a cone over some cubic plane curve T . The curve T is singular because it is the image of a rational twisted cubic curve. Therefore, S is singular along a line. We already saw that S has only finitely many singularities. This contradiction rules out the first possibility for ϕ .

Therefore

$$\phi = \begin{bmatrix} 1 & & \\ & \rho & \\ & & \rho^2 \end{bmatrix}$$

So the following are bases for the eigenspaces of ϕ^* :

$$\begin{aligned} 1 &: x_0^3, x_1^3, x_2^3, x_0x_1x_2 \\ \rho &: x_0^2x_1, x_0x_2^2, x_1^2x_2 \\ \rho^2 &: x_0^2x_2, x_0x_1^2, x_1x_2^2 \end{aligned}$$

The eigenspace containing the g_i has dimension at least 4. So $g : \mathbb{P}^2 \rightarrow Y' = \mathbb{P}^3$ is the morphism $(x_0^3, x_1^3, x_2^3, x_0x_1x_2)$. Embed Y' in \mathbb{P}^4 as $V(y_4)$ so that $g : \mathbb{P}^2 \rightarrow \mathbb{P}^4$ has the form claimed in Theorem 2.

3. MORPHISMS FROM QUINTIC TO CUBIC THREEFOLDS

In Section 3.1 we discuss preimages of lines on Y under the rational map $H \dashrightarrow Y$. Section 3.2 gives some information about H in case it has a component that does not map dominantly onto Y . In Section 3.3, we consider the various possibilities for H and rule them out case by case. When H is integral, the results of Section 3.1 will be the main tool. When H is more degenerate, Theorem 2 will play a central role.

3.1. Preimages of Lines on Y . First we will prove a basic fact about how the dualizing sheaf of a curve behaves under normalization. Then we will discuss the family of lines on Y and what can be said about the preimage in H of a general line in Y . The main result is Corollary 3.6.

Lemma 3.1. *Let C be a projective Gorenstein scheme of pure dimension 1 that is reduced at the generic point of one of its irreducible components C' . If $\nu : \tilde{C} \rightarrow C'$ is the normalization map of the reduced structure on C' and N is the number of points $p \in \tilde{C}$ such that the map of local rings $\mathcal{O}_{C, \nu(p)} \rightarrow \mathcal{O}_{\tilde{C}, p}$ fails to be an isomorphism, then*

$$\deg \nu^* \omega_C^o \geq \deg \omega_{\tilde{C}} + N.$$

If $N = 0$, then equality holds.

Proof. Following [9, Ex. 3.7.2], we compute

$$\begin{aligned} \omega_{\tilde{C}} &\cong \mathrm{Hom}_C(\nu_* \mathcal{O}_{\tilde{C}}, \omega_C^o) \\ &\cong \mathrm{Hom}_C(\nu_* \mathcal{O}_{\tilde{C}}, \mathcal{O}_C) \otimes_{\mathcal{O}_C} \omega_C^o \end{aligned}$$

as $\mathcal{O}_{\tilde{C}}$ -modules. Since $\omega_{\tilde{C}}$ and ω_C^o are invertible, $\mathrm{Hom}_C(\nu_* \mathcal{O}_{\tilde{C}}, \mathcal{O}_C)$ is an invertible sheaf of $\mathcal{O}_{\tilde{C}}$ -modules. So it is enough to show that $\mathrm{Hom}_C(\nu_* \mathcal{O}_{\tilde{C}}, \mathcal{O}_C)$ is isomorphic to an ideal sheaf in $\mathcal{O}_{\tilde{C}}$ corresponding to a closed subscheme of \tilde{C} supported at the points where $\nu : \tilde{C} \rightarrow C$ fails to be an isomorphism.

Consider the map of local rings

$$A := \mathcal{O}_{C, \nu(p)} \longrightarrow B := \mathcal{O}_{\tilde{C}, p}$$

for some closed point $p \in \tilde{C}$. We will show that $\mathrm{Hom}_A(B, A)$ is an ideal in B and is the unit ideal if and only if $A \rightarrow B$ is an isomorphism.

Note that B is the normalization of A/P in its field of fractions, where $P \subset A$ is the prime ideal of $C' \subset C$. Consider the map of B -modules

$$\Phi : \mathrm{Hom}_A(B, A) \longrightarrow B$$

that sends $\psi \in \text{Hom}_A(B, A)$ to the equivalence class $\overline{\psi(1)}$ of $\psi(1)$ in A/P , which injects into B .

If $\Phi(\psi) = 0$, then $\psi(1) \in P$, whence $\psi(B)$ is contained in the ideal $P \subset A$. In other words, ψ is a local section of the \mathcal{O}_C -module $\text{Hom}_C(\nu^*\mathcal{O}_{\tilde{C}}, \mathcal{O}_C)$ defined in a neighborhood of $\nu(p)$ in C such that ψ vanishes on the reduced structure of C' . If we consider $\text{Hom}_C(\nu^*\mathcal{O}_{\tilde{C}}, \mathcal{O}_C)$ as an $\mathcal{O}_{\tilde{C}}$ -module, then ψ is a local section defined on a neighborhood of p in \tilde{C} , and ψ vanishes at all but finitely many points in the neighborhood. Since $\text{Hom}_C(\nu^*\mathcal{O}_{\tilde{C}}, \mathcal{O}_C)$ is an invertible $\mathcal{O}_{\tilde{C}}$ -module, $\psi = 0$. Therefore Φ is an injection of B -modules, whence $\text{Hom}_A(B, A)$ is realized as an ideal in B . It remains to check that $A \rightarrow B$ is an isomorphism if and only if Φ is.

If $A \rightarrow B$ is an isomorphism, then Φ is clearly an isomorphism. Conversely, suppose Φ is an isomorphism. Then there is some $\psi \in \text{Hom}_A(B, A)$ such that $\Phi(\psi) = 1$, i.e. $\psi(1) - 1 \in P$. Therefore $\psi(1)$ is a unit in A because A is local. Hence $\Psi : \text{Hom}_A(B, A) \rightarrow A$ given by $\psi \mapsto \psi(1)$ is surjective because it is a morphism of A -modules. Note that Φ factors as Ψ followed by $A \rightarrow B$. Since Ψ is surjective and Φ is injective, we conclude that $A \rightarrow B$ is injective. And since Φ is surjective, $A \rightarrow B$ is surjective. \square

Lemma 3.2. *There are only finitely many closed points $p \in Y$ such that there are infinitely many lines on Y through p .*

Proof. Suppose not. Then there is a curve $C \subset Y$ such that for every point $p \in C$ there are infinitely many lines on Y through p . So for every $p \in C$ there is an irreducible component Σ_p of $Y \cap T_p Y$ such that Σ_p is a cone over a plane curve with vertex p .

Recall from [7] or [13] that the family Δ_1 of lines on Y is an irreducible surface, so a dimension count shows that every line on Y lies on one of the surfaces Σ_p for some $p \in C$. Also recall from [13] that a general line L in Y has normal bundle $N_{L/Y} = \mathcal{O}_L \oplus \mathcal{O}_L$.

Fix a general line L on Y , lying on Σ_p . There is some 2-plane Λ that is tangent to Σ_p at every point of L on account of Σ_p being a cone. Therefore $T_q \Lambda \subset T_q Y$ for every $q \in L$. It follows from Nakayama's Lemma that these pointwise inclusions give an injection $T_\Lambda|_L \rightarrow T_Y|_L$ of \mathcal{O}_L -modules. Note that $T_\Lambda|_L = \mathcal{O}_L(1) \oplus \mathcal{O}_L(2)$, and consider the normal bundle sequence

$$0 \longrightarrow T_L \longrightarrow T_Y|_L \xrightarrow{\phi} \mathcal{O}_L \oplus \mathcal{O}_L \longrightarrow 0.$$

By the above description of $T_\Lambda|_L$, the composition

$$T_\Lambda|_L \longrightarrow T_Y|_L \xrightarrow{\phi} \mathcal{O}_L \oplus \mathcal{O}_L$$

is the zero morphism. So $\ker \phi$ has rank at least 2, contradicting $\ker \phi = T_L$. \square

We will need the following modification of [13, Lemma 2.1].

Lemma 3.3. *Let $g : Z \rightarrow Y$ be a morphism from a purely 3 dimensional separated scheme of finite type over the ground field. If L is a general line on Y , then $g^{-1}(L)$ has pure dimension 1.*

Moreover, if Z is integral and g is dominant, then $g^{-1}(L)$ is singular at only finitely many points. In other words, $g^{-1}(L)$ is reduced at the generic point of each irreducible component.

Proof. Let \mathbb{F} be the total space of the family of lines on Y with base Δ . From [13], Δ is a smooth surface, and there is some open subscheme Δ_0 in Δ with preimage \mathbb{F}_0 in \mathbb{F} such that \mathbb{F}_0 is a locally trivial fiber bundle over Δ_0 , whose fibers are lines in Y .

Let Z_0 be the union of the surfaces in Z that are mapped to points in Y , let Z_1 be the union of the curves on Z that are mapped to points in Y , and let Z_2 be the set of points in Z at which g fails to induce an injection on tangent spaces. Note that Z_2 contains the singular locus of Z .

Note that $g(Z_0)$ has dimension at most 1, and $g(Z_1)$ has dimension at most 2. Since the canonical morphism $\mathbb{F}_0 \rightarrow Y$ is dominant, a general line will meet $g(Z_1)$ in only finitely many points. Also, if a general line meets $g(Z_0)$, then $g(Z_0)$ is a curve such that for every point $p \in g(Z_0)$ there are infinitely many lines on Y through p . This contradicts Lemma 3.2. So a general line does not meet $g(Z_0)$. Therefore, $g^{-1}(L)$ has pure dimension 1 for L a general line on Y .

Now suppose that Z is integral and g is dominant. Then $\dim g(Z_i) \leq i$ for $i = 0, 1, 2$. We use the characteristic zero assumption of this section to get $\dim g(Z_2) \leq 2$. So a general line will meet $g(Z_2)$ in only finitely many points, and will not meet $g(Z_0)$ and $g(Z_1)$ at all. Hence $g^{-1}(L)$ is nonsingular away from the preimage of $L \cap g(Z_2)$, which consists of only finitely many points. \square

Definition. If Z is any scheme and $F : Z \dashrightarrow \mathbb{P}^n$ is a rational map given by sections F_0, \dots, F_n of some line bundle on Z , then let $\text{indet}(F)$ denote the scheme of common vanishing of the F_i in Z :

$$\text{indet}(F) := V(F_0, \dots, F_n) \subset Z.$$

Lemma 3.4. *Take F is as in the definition, and let $\pi : \tilde{Z} \rightarrow Z$ be the blowup of Z in $\text{indet}(F)$. Then there is a canonical morphism $\tilde{F} : \tilde{Z} \rightarrow \mathbb{P}^n$ such that $\tilde{F} = F \circ \pi$ as rational maps. Moreover, if $p \in \text{indet}(F)$ is a closed point, then \tilde{F} induces a closed immersion of $\pi^{-1}(p)$ into \mathbb{P}^n .*

Proof. Recall that \tilde{Z} is isomorphic to the closure of the graph of F , $\Gamma_F \subset Z \times \mathbb{P}^n$. Thus projection onto \mathbb{P}^n induces the desired morphism $\tilde{F} : \tilde{Z} \rightarrow \mathbb{P}^n$. If $p \in \text{indet}(F)$ is a closed point, then $\pi^{-1}(p)$ is a closed subscheme of $p \times \mathbb{P}^n$, which maps isomorphically onto \mathbb{P}^n by projection. \square

Lemma 3.5. *If $p \in \text{indet}(F)$ is a closed point, then $p \in H$ is a point of order at least 3.*

Proof. Let $F = (F_0, \dots, F_4)$, and let Y have homogeneous equation G . Since the F_i vanish at p and G has degree 3, $G(F_0, \dots, F_4)$ vanishes to order at least 3 at p . In other words, $p \in F^{-1}(Y)$ is a point of order at least 3. But p is contained in H and not X . The lemma follows. \square

If L is a general line in Y with $L := V(\xi_1, \xi_2, \xi_3)$ for some linear forms ξ on \mathbb{P}^4 , then define

$$\begin{aligned} F^{-1}(L) &:= V(F^*\xi_1, F^*\xi_2, F^*\xi_3) \\ C &:= F^{-1}(L) \cap H \\ D &:= F^{-1}(L) \cap X \end{aligned}$$

If $\tilde{H} \rightarrow H$ is the blowup of H at the indeterminacy scheme $\text{indet}(F|_H)$ and $h: \tilde{H} \rightarrow Y$ is the resulting morphism, then Lemma 3.3 says that $h^{-1}(L)$ is purely one dimensional, so the same holds for C . Lemma 3.3 implies D is also purely one dimensional. So $F^{-1}(L)$ is a complete intersection in \mathbb{P}^4 .

Lemma 3.6. *With C as above, $\omega_C^o = \mathcal{O}_C(-1)$.*

Proof. Since $F^{-1}(L)$ is the complete intersection of three cubic hypersurfaces in \mathbb{P}^4 ,

$$\begin{aligned} \omega_{F^{-1}(L)}^o &= \mathcal{O}_{F^{-1}(L)}(-5 + 3 + 3 + 3) \\ &= \mathcal{O}_{F^{-1}(L)}(4). \end{aligned}$$

Following [9, Ex. 3.7.2], compute

$$\begin{aligned} \omega_C^o &= \text{Hom}_{F^{-1}(L)}(\mathcal{O}_C, \omega_{F^{-1}(L)}^o) \\ &= \text{Hom}_{F^{-1}(L)}(\mathcal{O}_C, \mathcal{O}_{F^{-1}(L)}) \otimes \mathcal{O}_{F^{-1}(L)}(4). \end{aligned}$$

Therefore, if $I_C, I_D \subset \mathcal{O}_{F^{-1}(L)}$ denote the ideals of $C, D \subset F^{-1}(L)$, then it suffices to carry out the following computation

$$(3.1) \quad \text{Hom}_{F^{-1}(L)}(\mathcal{O}_C, \mathcal{O}_{F^{-1}(L)}) \cong (0 : I_C)$$

$$(3.2) \quad \cong I_D$$

$$(3.3) \quad \cong \mathcal{O}_C(-C \cap D)$$

$$(3.4) \quad \cong \mathcal{O}_C(-5).$$

The isomorphism (3.1) is given by $\psi \mapsto \psi(1)$.

To see (3.2), note that D is Cohen-Macaulay since it is a local complete intersection in X and X is C.M., as L is a l.c.i. in Y . It follows that C and D are linked because $F^{-1}(L)$ is Gorenstein, cf. [5, Theorem 21.23].

To see (3.3), we will show that

$$I_D \longrightarrow \frac{I_D + I_C}{I_C}$$

is an isomorphism of \mathcal{O}_C -modules, i.e. $I_D \cap I_C = 0$. So let a be a local section of $I_D \cap I_C$, which is necessarily supported on $C \cap D$. We will show $a = 0$.

Let $A = \mathcal{O}_{F^{-1}(L),p}$ for some closed point $p \in C \cap D$ with maximal ideal m . Let b denote the image of a in A . Some power of m annihilates b because b is supported at p . Therefore m is the annihilator of some nonzero multiple of b , provided $b \neq 0$. So m is an associated prime ideal of A . But A is C.M. and one dimensional, so every associated prime is minimal. Hence $b = 0$ after all. Therefore $a = 0$ because its localization is zero at every point.

The isomorphism (3.4) follows from $C \cap D = C \cap X$. \square

Corollary 3.7. *If $C := F^{-1}(L) \cap H$ is reduced at the generic point of an irreducible component C' , then C' is either a smooth plane conic disjoint from the rest of C , or C' is a line meeting the rest of C at only one point.*

Proof. This is immediate from Lemma 3.1 and Lemma 3.5 because the degree of the dualizing sheaf of a smooth curve is at least -2 . \square

3.2. A Multiplicity Result. The following result holds in arbitrary characteristic and will be used in the next subsection.

Lemma 3.8. *Let Z be the reduced structure on an irreducible component of H . Assume $m \leq d$. If $F|_Z : Z \dashrightarrow Y$ is not dominant, then $2Z \subset H$ as divisors in \mathbb{P}^n .*

Proof. Suppose F does not map Z dominantly onto Y . Then Z is covered by curves that are mapped to points under F . So suppose that $C \subset Z$ is an integral curve with $F(C) = (1, 0, \dots, 0)$ for simplicity, and let $I \subset \mathcal{O}_{\mathbb{P}^n}$ be the ideal sheaf of C . If $F = (F_0, \dots, F_n)$, then $F_i \in H^0(\mathbb{P}^n, I(m))$ for $i > 0$. We show that H is singular along C .

Let $K \in H^0(\mathbb{P}^n, I(em - d))$ be the homogeneous equation for H . Let $I^{(2)} \subset \mathcal{O}_{\mathbb{P}^n}$ be the ideal such that $I/I^{(2)} = (I/I^2)/\text{torsion}$. Then K induces a section $\bar{K} \in H^0(\mathbb{P}^n, I/I^{(2)}(em - d))$. We will show $\bar{K} = 0$.

If $G = G(y_0, \dots, y_n)$ is the homogeneous equation of Y , then we can write $G = y_0^{e-1}G_1 + \dots + y_0G_{e-1} + G_e$ where the $G_i = G_i(y_1, \dots, y_n)$ are homogeneous of degree i . So the homogeneous equation of $F^{-1}(Y) = X + H$ is

$$(3.5) \quad F^*G = F_0^{e-1}F^*G_1 + F_0^{e-2}F^*G_2 + \dots$$

where $F^*G = G(F_0, \dots, F_n)$.

Let D be the Cartier divisor $V(F_0^{e-1}) \cap C$ on C . Note that D is disjoint from X because D is supported on $C \cap V(F_0, \dots, F_n)$ and $V(F_0, \dots, F_n)$ is disjoint from X . So from (3.5) it follows that \bar{K} restricts to the zero section on D because $F_0^{e-i}F^*G_i$ restricts to zero on C for $i \geq 2$, and F_0^{e-1} restricts to zero on D .

Tensor the exact sequence

$$0 \longrightarrow \mathcal{O}_C(-D) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_D \longrightarrow 0$$

with $I/I^{(2)}(em - d)$ and use $\mathcal{O}_C(-D) = \mathcal{O}_C(-m(e-1))$ to obtain

$$I/I^{(2)}(m-d) \xrightarrow{\tau} I/I^{(2)}(em-d) \xrightarrow{\rho} I/I^{(2)} \otimes \mathcal{O}_D(em-d) \longrightarrow 0$$

To see that τ is an injection, note that τ is multiplication by F_0^{e-1} , which is a unit in the local ring of almost all the points of C . Therefore τ could only have torsion elements in its kernel. But $I/I^{(2)}(em-d)$ is a torsion-free sheaf. So τ is injective.

Since $\rho(\overline{K}) = 0$, \overline{K} is the image of a section

$$\tilde{K} \in H^0(C, I/I^{(2)}(m-d)).$$

From the conormal sequence of C in \mathbb{P}^n , we get a morphism $I/I^{(2)}(m-d) \rightarrow \Omega_{\mathbb{P}^n}^1|_C(m-d)$ that is an injection on the regular locus of C , and is therefore an injection because $I/I^{(2)}(m-d)$ is torsion-free.

The sheaf $\Omega_{\mathbb{P}^n}^1|_C(m-d)$ has no nonzero global sections because of the injection $\Omega_{\mathbb{P}^n}^1|_C(m-d) \rightarrow \mathcal{O}_{\mathbb{P}^n}(m-d-1)^{\oplus n+1}$ from the Euler sequence. Therefore, $I/I^{(2)}(m-d)$ has no nonzero global sections. So $\tilde{K} = 0$, whence $\overline{K} = 0$. Therefore $K \in H^0(\mathbb{P}^n, I^{(2)})$, which implies H is singular along C . Since Z is covered by such curves, H is singular at every point of Z . This is only possible if $2Z \subset H$ as divisors. \square

3.3. Morphisms from Quintic to Cubic Threefolds. Recall the decomposition $F^{-1}(Y) = X + H$ from Section 1. We will consider the various possibilities for H and rule them out one at a time, thus proving Theorem 1.

Proposition 3.9. *H does not contain a hyperplane in \mathbb{P}^4 that maps dominantly onto Y .*

Proof. Suppose $Z \subset H$ is a hyperplane that maps dominantly onto Y . If we restrict F to any \mathbb{P}^2 contained in Z , then $F|_{\mathbb{P}^2}$ is described by Theorem 2. In particular, $F(\mathbb{P}^2)$ is the intersection of Y with a hyperplane tangent to Y at 3 points. The family of 2-planes contained in Z and the family of tangent planes to Y both have dimension 3, so a general tangent plane to Y is tangent to Y at 3 points. However, since Y is a nonsingular hypersurface, a general tangent plane to Y is tangent at only one point, cf. [4, Lemma 5.15]. \square

Corollary 3.10. *If K is a hyperplane contained in H , then $2K \subset H$, and we can choose coordinates x_0, \dots, x_4 on the \mathbb{P}^4 containing K such that $K = V(x_4)$ and $F|_K$ is given by the formula*

$$F|_K = (x_0^3, x_1^3, x_2^3, x_0x_1x_2, 0)$$

Proof. By Lemma 3.7 and Proposition 3.8, $2K \subset H$.

Since $F|_K$ is not dominant and $K \cong \mathbb{P}^3$, $F|_K$ can not be a morphism. Choose a point p in the indeterminacy locus of $F|_K$. Let $\pi: \tilde{K} \rightarrow K$ be the blowup of K at the indeterminacy scheme $\text{indet}(F|_K)$, so that the rational map $F|_K$ extends to a morphism $\Phi: \tilde{K} \rightarrow S$, where S is the image surface of K under F . By Lemma 3.4, $\pi^{-1}(p)$ maps isomorphically onto S under the morphism Φ . So for $s \in S$ a general point, $\Phi^{-1}(s)$ is a curve in \tilde{K} that meets $\pi^{-1}(p)$. Therefore the preimage of s under $F|_K$ is a curve in K through p .

If Λ is any 2-plane in K disjoint from $\text{indet}(F|_K)$, then $F|_\Lambda$ is described by Theorem 2. In particular, for $s \in S$ any nonsingular point, the preimage of s in Λ consists of three reduced points. Therefore the preimage of s in K consists of three distinct lines that are reduced away from $\text{indet}(F|_K)$. Indeed, if the preimage of s in K contained a curve other than a line, we could choose Λ such that $F|_\Lambda^{-1}(s)$ was not 3 reduced points by choosing Λ to be tangent to $F|_\Lambda^{-1}(s)$ at a point but not contain any component of $F|_\Lambda^{-1}(s)$.

So there is a two parameter family of lines in K that are mapped by F to points in S . Every such line meets $\text{indet}(F|_K)$, which consists of finitely many points. So there is some point $p \in \text{indet}(F|_K)$ such that there is a two parameter family of lines in K through p that are each mapped to a point in S under F .

A line L is mapped to a point by F exactly when the scheme $L \cap \text{indet}(F)$ has length 3 because $F^*\mathcal{O}(1) = \mathcal{O}(3)$. Since a general line in \mathbb{P}^4 through p meets $\text{indet}(F)$ in a scheme of length 3, the same holds for every line through p . So every line through p is mapped to a point. Therefore, $F|_K$ is determined by $F|_\Lambda$ for any 2-plane Λ in K not containing p . Theorem 2 determines $F|_\Lambda$. If one takes $p = V(x_0, x_1, x_2)$ and $\Lambda = V(x_3)$, where x_0, \dots, x_3 are homogeneous coordinates on K , then $F|_K$ has the desired form. \square

Proposition 3.11. *H is not equal to $Q + 2K$ for an integral quadric Q and a hyperplane K .*

Proof. Suppose $H = Q + 2K$. By Corollary 3.9, $F|_K$ factors through projection from some point $p \in K$. By Lemma 3.5, H has order at least 3 at every point in $\text{indet}(F)$. So every point in $\text{indet}(F)$ is in $K \cap Q$. Hence, p is the only point in $\text{indet}(F)$ because it is the only point on K where F is undefined.

The map $F|_Q$ is dominant by Lemma 3.7. Let $\pi : \tilde{Q} \rightarrow Q$ be the blowup of Q at the indeterminacy scheme $\text{indet}(F|_Q)$, and let $q : \tilde{Q} \rightarrow Y$ be the resulting morphism extending $F|_Q$. Apply Lemma 3.3 to q to see that for L a general line, $q^{-1}(L)$ is reduced at the generic point of each of its irreducible components, whence the same is true for $C_Q := F^{-1}(L) \cap Q$.

By Lemma 3.4, q is a closed immersion when restricted to $\pi^{-1}(p)$. Since $q(\pi^{-1}(p))$ is an effective divisor on Y , it is ample by the Grothendieck-Lefschetz Theorem, [8, Theorem 4.3.2]. So every line in Y meets $q(\pi^{-1}(p))$. Therefore, C_Q has an irreducible component C_1 containing the point p . By the same argument, $F^{-1}(L) \cap K$ has an irreducible component C_2 that contains p . Since L is a general line on Y , which is covered by lines, L is not contained in the image of $K \cap Q$. So $C_1 \neq C_2$. By Lemma 3.6, $\omega_C^2 = \mathcal{O}_C(-1)$. Since C has more than one irreducible component than contains p , Lemma 3.1 shows that every irreducible component of C_Q that contains p is a line because the dualizing sheaf of every smooth curve is at least -2 . So Q is covered by lines through p , and these lines are parametrized by a general hyperplane section $Q \cap \mathbb{P}^3$ of Q . Because the line $L \subset Y$ is

general, $\dim Q \cap \mathbb{P}^3 = 2$, and $\dim \Delta = 2$, where Δ is the space of lines on Y , we conclude that every general line on Q through p maps to a line on Y . Hence the rational map $F|_Q : Q \dashrightarrow Y$ induces a rational map $Q \cap \mathbb{P}^3 \dashrightarrow \Delta$. However, Δ does not contain a rational curve by Proposition 2.1. This contradiction finishes the proof. \square

Lemma 3.12. *If L is a line in \mathbb{P}^4 such that $L \cap \text{indet}(F)$ is nonempty and is not a single reduced point, then L is contained in H .*

Proof. Suppose L intersects $\text{indet}(F)$. Then $L \cap \text{indet}(F)$ is zero dimensional, so its structure sheaf has finite dimension λ over the ground field k . So the rational map $F|_L$ is given by $\mathcal{O}_{\mathbb{P}^1}(3-\lambda)$ after $F|_L$ is extended over the points of indeterminacy. The intersection $L \cap \text{indet}(F)$ is a single reduced point exactly when $\lambda = 1$.

If $\lambda = 2$, then F maps L isomorphically onto a line. This is impossible if L is not contained in H because L would meet X in a scheme of length 5 while $F(L)$ would meet Y in a scheme of length 3. If $\lambda = 3$, then $F(L)$ is a point in Y , so $L \subset H$. This completes the proof because $\lambda \leq 3$. \square

Proposition 3.13. *H is not $2K_1 + 2K_2$ for distinct hyperplanes K_1 and K_2 .*

Proof. Suppose $H = 2K_1 + 2K_2$. According to the formula of Corollary 3.10, there are points $p_i \in K_i$ such that $F|_{K_i}$ factors through projection from p_i , and the tangent space to the indeterminacy scheme $\text{indet}(F|_{K_i})$ at p_i is equal to the tangent space of K_i at p_i .

We claim $p_1 \neq p_2$. Indeed, if $p = p_1 = p_2$, then $T_p K_1 \neq T_p K_2$ because the K_i are distinct hyperplanes. So $\text{indet}(F)$ would have a four dimensional tangent space at p . But now every line in \mathbb{P}^4 through p meets $\text{indet}(F)$ in a scheme that is nonreduced at p . So Lemma 3.12 implies that every line in \mathbb{P}^4 through p is contained in H . This is impossible because H is not all of \mathbb{P}^4 , so $p_1 \neq p_2$.

Let L be the line containing p_1, p_2 . By Lemma 3.5, H has order at least 3 at both p_i , so $p_1, p_2 \in K_1 \cap K_2$. Therefore, $L \subset K_1 \cap K_2$, so that L meets $\text{indet}(F)$ in a scheme of length 6, which is impossible because $L \cap \text{indet}(F)$ can have length at most 3. \square

Proposition 3.14. *H does not contain a hyperplane.*

Proof. The only case left to rule out is $H = 4K$. So suppose $H = 4K$. We will lift the polynomials that give $F|_K$ from K to its second infinitesimal neighborhood $2K$ and then derive a contradiction.

Using Corollary 3.10, we choose homogeneous coordinates x_0, \dots, x_4 on \mathbb{P}^4 such that $K = V(x_4)$ and

$$(3.6) \quad F = (x_0^3 + x_4 q_0, x_1^3 + x_4 q_1, x_2^3 + x_4 q_2, x_0 x_1 x_2 + x_4 q_3, x_4 q_4)$$

for some homogeneous polynomials q_i of degree 2 in the x_i . Let y_0, \dots, y_4 be homogeneous coordinates on the target \mathbb{P}^4 . Since $Y \cap Y'$ is the image of

$K = V(x_4)$ we see $Y' = V(y_4)$, and $Y \cap Y' = V(y_4, y_0y_1y_2 - y_3^3)$. So Y is given by the equation

$$G := y_0y_1y_2 - y_3^3 + y_4G_2$$

for some homogeneous G_2 of degree 2 in the y_i .

Since $2K \subset H$, x_4^2 divides the pullback

$$\begin{aligned} F^*G &= (x_0^3 + x_4q_0)(x_1^3 + x_4q_1)(x_2^3 + x_4q_2) \\ &\quad - (x_0x_1x_2 + x_4q_3)^3 + x_4q_4F^*G_2 \\ &= x_4(q_0x_1^3x_2^3 + q_1x_0^3x_2^3 + q_2x_0^3x_1^3 - 3q_3x_0^2x_1^2x_2^2 + q_4F^*G_2) \\ &\quad + x_4^2(\text{other terms}) \end{aligned}$$

Therefore x_4 divides

$$(3.7) \quad q_0x_1^3x_2^3 + q_1x_0^3x_2^3 + q_2x_0^3x_1^3 - 3q_3x_0^2x_1^2x_2^2 + q_4F^*G_2$$

Using the notation $\partial_i := \frac{\partial}{\partial y_i}$, the partial derivatives of G are

$$\begin{aligned} \partial_0G &= y_1y_2 + y_4\partial_0G_2 \\ \partial_1G &= y_0y_2 + y_4\partial_1G_2 \\ \partial_2G &= y_0y_1 + y_4\partial_2G_2 \\ \partial_3G &= -3y_3^2 + y_4\partial_3G_2 \\ \partial_4G &= G_2 + y_4\partial_4G_2 \end{aligned}$$

It follows that $V(y_0y_1, y_0y_2, y_1y_2, y_3, y_4, G_2)$ is empty because it is contained in the singular locus of Y . Hence $V(G_2)$ does not contain the point $(1 : 0 : 0 : 0 : 0)$. In other words, y_0^2 appears with nonzero coefficient in G_2 . Therefore by equation (3.6), x_0^6 has nonzero coefficient in F^*G_2 .

Moreover, $q_4F^*G_2$ is the only term in formula (3.7) in which x_0^6 can occur because the q_i have degree 2. Since (3.7) is zero (mod x_4), the x_0^6 term in (3.7) disappears when considered (mod x_4). This can only happen if x_4 divides q_4 . Further consideration of (3.7) shows that there are scalars a_0, a_1, a_2 such that the following equations hold (mod x_4):

$$\begin{aligned} q_0 &\equiv a_0x_0^2 \\ q_1 &\equiv a_1x_1^2 \\ q_2 &\equiv a_2x_2^2 \\ q_3 &\equiv \frac{1}{3}(a_0x_1x_2 + a_1x_0x_2 + a_2x_0x_1) \\ q_4 &\equiv 0 \end{aligned}$$

Therefore, equation (3.6) yields

$$F_0 = x_0^3 + a_0 x_4 x_0^2 + x_4^2 h_0$$

$$F_1 = x_1^3 + a_1 x_4 x_1^2 + x_4^2 h_1$$

$$F_2 = x_2^3 + a_2 x_4 x_2^2 + x_4^2 h_2$$

$$F_3 = x_0 x_1 x_2 + \frac{1}{3} x_4 (a_0 x_1 x_2 + a_1 x_0 x_2 + a_2 x_0 x_1) + x_4^2 \tilde{h}_3$$

$$F_4 = x_4^2 h_4$$

for some homogeneous linear polynomials h_i in x_0, \dots, x_4 .

As $H = 4K$ is contained in $F^{-1}(Y)$, x_4^4 divides

$$F^*G = F_0 F_1 F_2 - F_3^3 + F_4 F^* G_2.$$

Since F^*G has degree 9, the monomial $x_0^6 x_4^4$ can not appear with nonzero coefficient in F^*G . On the other hand, the monomial x_0^6 does have nonzero coefficient in F^*G_2 . So x_0^6 must appear in a monomial with nonzero coefficient in

$$F^*G - F_4 F^* G_2 = F_0 F_1 F_2 - F_3^3.$$

However, by the formulas for F_0, \dots, F_3 given above, the highest power of x_0 that can appear is x_0^5 . This contradiction finishes the proof. \square

Lemma 3.15. *Every point $p \in \text{indet}(F)$ is a point of order 3 on H .*

Proof. Let T_p denote the tangent space of the indeterminacy scheme $\text{indet}(F)$ at some closed point $p \in \text{indet}(F)$. If L is a line in \mathbb{P}^4 tangent to $\text{indet}(F)$ at p , then L is contained in H by Lemma 3.12. Proposition 3.14 asserts that H does not contain a hyperplane, so T_p can have dimension at most 2.

By Lemma 3.5, H has order at least 3 at p . So we only have to rule out the case $\text{ord}_p H = 4$.

Suppose $p \in \text{indet}(F)$ is a point of order 4 in H , so that H is a cone over p . Let $\pi : \tilde{\mathbb{P}}^4 \rightarrow \mathbb{P}^4$ be the blowup of \mathbb{P}^4 at $\text{indet}(F)$, and let $\tilde{F} : \tilde{\mathbb{P}}^4 \rightarrow Y$ be the resulting morphism that extends F . Let $E_p = \pi^{-1}(p)$ be the preimage of the reduced point p . Since the dimension of T_p is at most 2, if l_1 and l_2 are two general lines in \mathbb{P}^4 through p , then the scheme $l_1 \cup l_2$, which is contained in a 2-plane, meets $\text{indet}(F)$ in the reduced point p . So the strict transform of $l_1 \cup l_2$ is the blowup of $l_1 \cup l_2$ at p , which is the scheme-theoretic intersection $l_1 \cap l_2$. Therefore the strict transform of $l_1 \cup l_2$ is the disjoint union of the strict transforms \tilde{l}_1 and \tilde{l}_2 of l_1 and l_2 . In particular, a general line l in \mathbb{P}^4 through p determines a point on E_p , and distinct general lines yield distinct points in E_p . Hence E_p has an irreducible component E'_p that is the birational image of the space \mathbb{P}^3 of lines in \mathbb{P}^4 through p .

If \tilde{l} is the strict transform of a general line l in \mathbb{P}^4 through p , then \tilde{l} meets $\tilde{F}^{-1}(Y)$ in a scheme of length 6. Indeed, the morphism $\tilde{F}|_{\tilde{l}}$ is given by sections of the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$ because the rational map $F|_l$ is given by $\mathcal{O}_{\mathbb{P}^1}(3)$ and has indeterminacy scheme equal to a single reduced point. When the indeterminacy is resolved the resulting line bundle is $\mathcal{O}_{\mathbb{P}^1}(2)$. The

Cartier divisor $\tilde{F}^{-1}(Y)$ is the pullback of the degree 3 divisor Y . So $\tilde{F}^{-1}(Y)$ has intersection product 6 with the curve \tilde{l} .

Moreover, E'_p maps onto Y , as follows. If \tilde{H} is the strict transform of H in $\tilde{\mathbb{P}}^4$, then \tilde{H} intersects E_p in a scheme of dimension 2. In particular, E'_p is not contained in $E_p \cap \tilde{H}$. For l a general line in \mathbb{P}^4 through p , l only meets H at p because H is a cone over p . So the strict transform \tilde{l} of l does not meet \tilde{H} .

Since \tilde{l} meets the strict transform \tilde{X} of X in a scheme of length 5 and \tilde{l} does not meet \tilde{H} , the other point of $\tilde{l} \cap \tilde{F}^{-1}(Y)$ lies in E_p and therefore in E'_p . So \tilde{F} maps E'_p isomorphically onto Y because \tilde{F} gives a closed embedding of E_p into \mathbb{P}^4 by Lemma 3.4. This is a contradiction because E'_p is rational and Y is not. Therefore every point $p \in \text{indet}(F)$ is a point of order 3. \square

Proposition 3.16. *H is not $2Q$ for an integral quadric Q .*

Proof. By [14, Proposition 2.2(2)], Y is not the image of a morphism from a hypersurface in \mathbb{P}^4 of degree 2, so $F|_Q$ is not a morphism. Let $p \in \text{indet } F$ be a closed point. If $H = 2Q$, then every closed point in Q has order 2 or 4 in H . By Lemma 3.5, p is a point on H of order at least 3. So p has order 4, which is impossible by Lemma 3.15. We conclude $H \neq 2Q$. \square

Proposition 3.17. *H is not $Q_1 + Q_2$ for distinct integral quadrics Q_1 and Q_2 .*

Proof. Suppose $H = Q_1 + Q_2$. By Lemma 3.8, the Q_i map dominantly onto Y . Therefore, by Lemma 3.3 applied to the blowups of Q_1 and Q_2 at the indeterminacy schemes of $F|_{Q_1}$ and $F|_{Q_2}$, for L a general line in Y , $C := F^{-1}(L) \cap H$ is reduced at the generic point of each of its irreducible components.

By Lemma [14, Proposition 2.2(2)], $F|_{Q_1}$ can not be a morphism because $\deg Y > 2$. Pick $p \in \text{indet}(F|_{Q_1})$. By Lemma 3.15, p is a point of order 3 in H , so Q_2 contains p . I claim that C has an irreducible component contained in each of the Q_i through p . Indeed, if $\pi_i : \tilde{Q}_i \rightarrow Q_i$ is the blowup at $\text{indet}(F|_{Q_i})$, and $q_i : \tilde{Q}_i \rightarrow Y$ is the resulting morphism, then a general line L in Y will meet $q_i(\pi_i^{-1}(p))$. So the preimage of L in Q_i passes through p .

Therefore, by Corollary 3.7, the connected component of C containing p is the union of two lines through p , one line contained in Q_1 and the other in Q_2 . But now each of the Q_i contain a two parameter family of lines through p because Y is covered by a two parameter family of lines. So both the Q_i are cones over p , so that p is a point of order 4 in H . This contradicts Lemma 3.15. \square

The following Lemma will be needed to analyze the preimage in H of a general line in Y in the case where H is integral.

Lemma 3.18. *Let $\pi : \mathbb{F} \rightarrow B$ be a projective morphism with B integral. Suppose $\sigma : B \rightarrow \mathbb{F}$ is a section of π , and for each $b \in B$ the connected component \mathbb{F}_b of $\pi^{-1}(b)$ that contains $\sigma(b)$ is irreducible. Then $\bigcup_b \mathbb{F}_b$ is an irreducible component of \mathbb{F} .*

Proof. Let $\mathbb{F} \xrightarrow{\pi'} B' \xrightarrow{g} B$ be the Stein factorization of π . So π' has connected fibers, and g is finite. Then $\pi' \circ \sigma$ is a section of g , and $(\pi')^{-1}(\pi' \circ \sigma)(b) = \mathbb{F}_b$. Notice that $(\pi' \circ \sigma)(B)$ is an irreducible component of B' because they have the same dimension. Since $(\pi')^{-1}(\pi' \circ \sigma(B))$ has irreducible fibers \mathbb{F}_b and B is irreducible, $(\pi')^{-1}(\pi' \circ \sigma(B)) = \bigcup_b \mathbb{F}_b$ is irreducible. \square

The only case left to rule out is when H is integral.

Theorem 3.19. *H is not integral.*

Proof. Suppose H is integral. By Lemma 3.8, $F|_H$ is dominant because H has multiplicity 1 in $F^{-1}(Y)$. However, $F|_H$ can not be a morphism, because if it were then Table 1 in the Appendix would imply that the polynomial degree m of F would be at most 2, not 3. Choose $p \in \text{indet}(F)$ a reduced point.

Let L be a general line in Y , cut out by the linear forms ξ_1, ξ_2, ξ_3 in \mathbb{P}^4 . Define

$$\begin{aligned} F^{-1}(L) &:= V(F^*\xi_1, F^*\xi_2, F^*\xi_3) \\ C(L) &:= F^{-1}(L) \cap H \\ D(L) &:= F^{-1}(L) \cap X \end{aligned}$$

By Lemma 3.3, $C = C(L)$ and $D = D(L)$ are reduced at the generic point of each of their irreducible components, and

Since the linear forms $F^*\xi_i$ vanish on the indeterminacy scheme $\text{indet}(F) := V(F_0, \dots, F_4)$, $\text{indet}(F)$ is contained in $F^{-1}(L) = C \cup D$. Therefore, $\text{indet}(F)$ is contained in C because $\text{indet}(F)$ does not intersect D , as D is contained in X . By Corollary 3.7, the connected component of C that contains p is either a smooth quadric curve or the union of two lines meeting at some point. According to Lemma 3.15, p is triple point on H , so that H is not a cone over p . By Corollary 3.7, the connected component of C that contains p is a plane conic. Since H is not a cone over p , Y has a two dimensional family of lines, and L is a general line on Y , the connected component of C that contains p is a smooth plane conic.

Let B be an open subscheme of the space of lines in Y such that for every $L \in B$, every component of $C(L)$ that meets $\text{indet}(F)$ is a smooth plane conic. For $L \in B$, $F^{-1}(L)$ has degree 27, and $D(L)$ has degree 15. So $C(L)$ has degree 12 and is the disjoint union of 6 reduced plane conics.

Choose a general $L_0 \in B$, and let $C_1(L_0), \dots, C_6(L_0)$ be the connected components of $C(L_0)$. Let $\zeta_i := C_i(L_0) \cap \text{indet}(F)$, and let $\lambda(\zeta_i)$ denote the length of the zero dimensional scheme ζ_i . The polynomial degree of F is $m = 3$, $C_i(L_0)$ has degree 2, and L_0 has degree 1. So the restriction

$C_i(L_0) \dashrightarrow L_0$ of F to $C_i(L_0)$ has degree $6 - \lambda(\zeta_i)$. Therefore $\deg F|_H$ is the sum of the $\deg F|_{C_i(L_0)}$:

$$(3.8) \quad \deg F|_H = \sum_{i=1}^6 6 - \lambda(\zeta_i).$$

Now number the $C_i(L_0)$ so that $\lambda(\zeta_1) \geq \dots \geq \lambda(\zeta_6)$, and let p be a closed point of ζ_1 . Let $\mathbb{F} \subset B \times H$ be the total space of the family $\pi : \mathbb{F} \rightarrow B$ whose fiber over $L \in B$ is $C(L)$. The closed subscheme $B \times p \subset \mathbb{F}$ is a section of π . So

$$\mathbb{F}_1 := \bigcup_{L \in B} C_1(L)$$

is an irreducible component of \mathbb{F} by Lemma 3.18.

Notice that there is a morphism $\mathbb{F} \rightarrow B_H$, where B_H is the space of quadric plane curves in H that contain p , given by sending a point $(L, x) \in \mathbb{F} \subset B \times H$ to the irreducible component of $C(L)$ that contains x . The fibers of $\mathbb{F} \rightarrow B_H$ are one dimensional and $\dim \mathbb{F} = 3$. So the image in B_H of the intersections of the various irreducible components of \mathbb{F} has dimension at most 1. Hence there is at most a one dimensional space of lines in Y such that $C(L)$ has a component corresponding to a point in B_H whose fiber in \mathbb{F} lies in more than one irreducible component. So by generality of $L_0 \in B$, \mathbb{F}_1 is the only irreducible component of \mathbb{F} that has nonempty fiber over the point $C_1(L_0) \in B_H$. Therefore, the irreducible component \mathbb{F}_1 of \mathbb{F} did not depend on the choice of $p \in \zeta_1$ because for any $p \in \zeta_1$ the connected component of $C(L_0)$ that contains p is $C_1(L_0)$.

Since \mathbb{F}_1 did not depend on the choice of $p \in \zeta_1$, for every line $L \in B$, the component $C_1(L) := \pi^{-1}([L]) \cap \mathbb{F}_1$ of $C(L)$ that is contained in \mathbb{F}_1 has the property that $C_1(L) \cap \text{indet}(F) = \zeta_1$. This is because for every $q \in \zeta_1$, $B \times q$ is contained in $\mathbb{F}_1 \subset B \times H$.

Consider the composite morphism $\phi : \mathbb{F}_1 \rightarrow H \xrightarrow{F|_H} Y$. Let $y \in Y$ be a general point. There are six lines L_1, \dots, L_6 on Y through y , and $\phi^{-1}(L_i) = C_1(L_i)$. By counting preimage points of y we see that $\deg \phi$ is the sum of the degrees of the $C_1(L_i) \dashrightarrow L_i$:

$$(3.9) \quad \deg \phi = \sum_{i=1}^6 6 - \lambda(\zeta_1).$$

Since $\deg \phi \geq \deg F|_H$, equations (3.8) and (3.9) show that all the $\lambda(\zeta_i)$ are equal by maximality of $\lambda(\zeta_1)$. Therefore $\mathbb{F}_1 \rightarrow H$ is a birational morphism.

Let $p \in \text{indet}(F)$. By Lemma 3.15, p is a triple point on H , and so H is rational. Therefore \mathbb{F}_1 is rational and dominates the surface B . However, B does not contain a rational curve by Proposition 2.1. This contradiction shows that H can not be integral. \square

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