# Regularity of Neumann Solutions to an Elliptic Free Boundary Problem

by

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## Abstract

We examine the regularity properties of solutions to an elliptic free boundary problem, near a Neumann fixed boundary. Consider a nonnegative function $u$ which minimizes the functional

$$J[v] = \int_{\Omega} \left(|\nabla u|^2 + Q^2(x)\chi_{\{u>0\}}\right)$$

on a bounded, convex domain $\Omega \subset \mathbb{R}^n$. This function $u$ is harmonic in its positive phase and satisfies $|\nabla u(x)| = Q(x)$ along the free boundary $\partial\{u > 0\}$, in a weak sense. We prove various basic properties of such a minimizer near the portion of the boundary $\Gamma \subset \partial\Omega$ on which $\frac{\partial u}{\partial \nu} = 0$ weakly. These results include up-to-the boundary gradient estimates on harmonic functions with Neumann boundary conditions on convex domains. The main result is that the minimizer $u$ is Lipschitz continuous. The proof in dimension 2 is by means of conformal mapping as well as a simplified monotonicity formula. In higher dimensions, the proof is via a maximum principle estimate for $|\nabla u|$.


Thesis Supervisor: David S. Jerison
Title: Professor of Mathematics

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# Chapter 1

# Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain and let $Q$ be a smooth, bounded, nonnegative function on $\Omega$. The subject of this thesis will be a function $u$ with the following properties:

$$\begin{aligned} u(x) &\geq 0 && \forall\, x \in \Omega \\ \Delta u(x) &= 0 && \forall\, x \in \{u > 0\} \cap \Omega \\ |\nabla u(x)| &= Q(x) && \forall\, x \in \partial\{u > 0\} \cap \Omega \end{aligned} \tag{1.1}$$

The free boundary is the set $\Lambda = \partial\{u > 0\} \cap \Omega$. This free boundary problem has several applications, particularly to the study of jets and cavities. See, for example, [3] and the other papers on the subject by Alt, Caffarelli, and Friedman referred to in that paper, as well as the book on free boundary problems by Friedman[10].

Let $\Omega$ be a convex domain, such that $\partial\Omega$ has Lipschitz constant $L$. Let $S$ be a closed subset of $\partial\Omega$ and let $u_0$ be a smooth nonnegative function on $\mathbb{R}^n$. Let the functional $J$ be defined by

$$J[v] = \int_\Omega (|\nabla v|^2 + Q^2(x)\chi_{\{v>0\}})\mathrm{d}x \tag{1.2}$$

and let $u$ be the minimizer of $J$ in $K = \{v \in H^1 : v = u_0 \text{ on } S\}$. The function $u$ is a solution of the free boundary problem described above, in a weak sense. Let $\Gamma = \partial\Omega \setminus S$. Note that $u$ satisfies Neumann boundary conditions along $\Gamma$ in a suitable sense, as is standard fo variational problems in which the boundary condition is not prescribed. The behavior of $u$

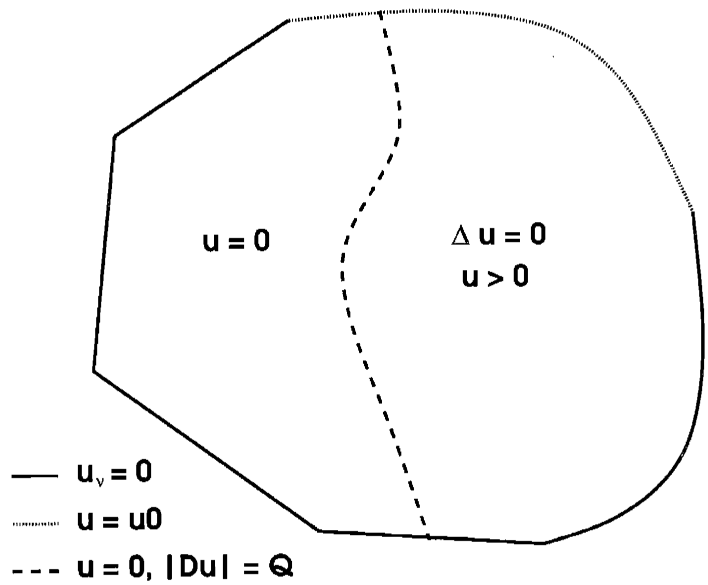


Figure 1-1: The Domain of the Free Boundary Problem

near this Neumann boundary $\Gamma$ will be the focus of this work. The main result is:

**Theorem 1** *Let $r_0 > 0$ and define $\Omega_{r_0} = \{x \in \Omega : \text{dist}(x, S) > r_0\}$. Let $A = \sup_S u_0$. Suppose that $\partial\Omega$ is convex in a neighborhood of $\overline{\Gamma}$, and let $L$ be the Lipschitz constant of the boundary $\partial\Omega$. Let $Q$ be a measurable function with $0 \leq Q \leq M$. Then there is a $C$ depending only on $n, L, M, A$ and $r_0$ such that if $u$ is the minimizer of $J$ in $K$ then for almost every $x \in \Omega_{r_0}$,*

$$|\nabla u(x)| \leq C.$$

We make several comments about this theorem:

1. We are interested only in the behavior of $u$ near a Neumann boundary. Nontrivial Dirichlet conditions must be assigned somewhere in order to prevent $u$ from being trivial. We choose to exclude a neighborhood of the places where Dirichlet conditions hold in order to prevent confusion. This does not mean Lipschitz continuity does not hold there, merely that it has not been studied.

2. $\Omega$ must be convex. Without this extra condition the theorem fails. Indeed, even harmonic functions in non-convex Lipschitz domains fail to be Lipschitz continuous. However, if $\Omega$ is non-convex one can still find a minimizer $u$, and that minimizer

is Hölder continuous for some $0 < \alpha < 1$ determined solely by the fixed boundary regularity.

3. The variational approach taken here allows one to consider a $u$ which is a priori not smooth enough for the free boundary condition in (1.1) to make sense pointwise. The free boundary condition can be recovered almost everywhere after some regularity results are obtained, if $Q$ is strictly positive and smooth [2]. This approach also enables one to obtain existence and basic properties of $u$ fairly easily. On the other hand, energy minimization is a stronger property than simply solving the PDE; the results of this approach do not apply to solutions which are not at least local minimizers of $J$. Examples of global saddle point solutions do exist [2], and they can have lower free boundary regularity than local minimizers.

4. The proof of this theorem is different in two dimensions than in higher dimensions. Both cases begin by extending the average control lemma of [2] to the Neumann boundary case. In two dimensions, the proof proceeds by methods of conformal mapping and the use of a monotonicity formula as in [4]. In higher dimensions, we use an interior gradient estimate of [9], modified in the convex Neumann case as in [14]. It is likely that this proof applies also to two dimensions, but the separate two-dimensional proof is included because we find it to be instructive. The methods available in two dimensions provide more precise information about $u$, and, we expect, will make a study of the shape of the free boundary much easier in that case.

The one-phase free boundary problem treated here was first considered by Alt and Caffarelli in [2]. They first concluded that a minimizer exists for the functional $J$ with nonnegative boundary conditions, then computed the basic properties of local minima, including that they are globally subharmonic and harmonic in their positive phase. They also proved that the free boundary condition holds in a weak sense. Alt and Caffarelli proceeded to examine the question of interior regularity, and concluded that a local minimum $u$ is locally Lipschitz continuous. They found that, with $0 < m \leq Q$ and $Q$ smooth, the free boundary is a $C^{1,\alpha}$ surface except at a set of 0 surface measure. Moreover, for $n = 2$ the free boundary is analytic if $Q$ is. They finally produced a non-minimizing global solution in dimension 3 with a point singularity in the free boundary at the origin, and several other examples.

In 1984, Alt, Caffarelli and Friedman continued the study of the problem by examining the two-phase problem [4]. They studied a minimizer $u$ of

$$I[v] = \int_{\Omega} (|\nabla v|^2 + q^2(x)\lambda^2(v))\mathrm{d}x, \tag{1.3}$$

where $q$ is a smooth positive function and

$$\lambda(v) = \begin{cases} \lambda_1 & v > 0 \\ \lambda_2 & v < 0 \cdot \\ \min(\lambda_1, \lambda_2) & v = 0 \end{cases}$$

They minimized $I$ in $\tilde{K} = \{v \in H^1 : v = w_0 \text{ on } S \subset \partial\Omega\}$, where $w_0$ is not required to be nonnegative. In that paper, they introduced their well-known *monotonicity lemma*. They proved that, for a minimizer $u$, if $u(x_0) = 0$ then the function

$$\phi(r) = \left(\frac{1}{r^2}\int_{B_r(x_0)} |x - x_0|^{2-n}|\nabla u_+|^2 dx\right) \cdot \left(\frac{1}{r^2}\int_{B_r(x_0)} |x - x_0|^{2-n}|\nabla u_-|^2 dx\right)$$

is increasing in $r$. This allowed them to concluded that $u$ is locally Lipschitz continuous in $\Omega$. A similar technique will be used in Chapter 4. They also concluded that, for $n = 2$, the free boundary is continuously differentiable.

Further work on the regularity of the free boundary was done by Weiss, who concluded that singularities in the free boundary have codimension 3 [16]. The study of the regularity of the free boundary continues; in a recent preprint, Caffarelli, Jerison, and Kenig prove that the free boundary of a minimizer is smooth in dimension 3. Since a non-smooth critical point is known to exist in three dimensions, this shows that minimizers have higher regularity than general solutions. They conjecture that the free boundaries of minimizers are in fact smooth up to dimension 7 [7].

Fixed boundary regularity of $u$ has not been approached using variational techniques.

However, uniform regularization methods have been used to study the regularity of solutions near a smooth boundary. In this technique, the free boundary problem is modeled by a smooth semilinear PDE:

$$\Delta u_\epsilon = Q^2(x)\beta_\epsilon(u_\epsilon)$$

where $\beta(z) \geq 0$, $\beta(z) = 0$ for $z \leq 0$ and for $z \geq 1$, and $\int_0^1 \beta(z)dz = 1$. $\beta_\epsilon(z) = \frac{1}{\epsilon}\beta(\frac{z}{\epsilon})$. Uniform estimates on $u_\epsilon$ are obtained as $\epsilon \to 0$, and the $u_\epsilon$ are shown to converge uniformly to a solution to the free boundary problem in an appropriate sense. (See, for example, [6].) This method has the advantage of applying to a larger class of solutions, but cannot be used if the regularity of $\partial\Omega$ is too low, as is the case in the current work. Uniform regularization methods have been used to study the regularity of solutions near a smooth Neumann boundary. In [6], Berestycki, Caffarelli, and Nirenberg used these methods to establish uniform Lipschitz continuity up to a smooth Neumann boundary. They establish that these solutions do indeed converge to a Lipschitz continuous weak solution to the free boundary problem, and obtain some control over the shape of the free boundary as well. They also apply their results to a related free boundary problem of combustion theory.

In [11], Gurevich examined the uniform regularity of this singular perturbation of the problem, near a boundary with smooth non-trivial dirichlet data $u_0$. He concluded that an extra condition ($|\nabla u_0| = 0$ when $u_0 = 0$) is necessary and sufficient to obtain uniform Lipschitz continuity of the $u_\epsilon$, which implies Lipschitz continuity of the uniform limit $u$. This condition is automatically satisfied in the one-phase problem if $u_0$ is smooth, so Lipschitz continuity does hold in that case; if $u_0$ is not assumed to be smooth, the extra condition may not hold. Without the extra condition, the best regularity that can be expected is of the form

$$|u_\epsilon(x) - u_\epsilon(y)| \leq C|x-y|(1 + |\log|x-y||).$$

This regularity is still enough to guarantee that a weak solution $u$ to the free boundary problem exists as a uniform limit of solutions to the singularly perturbed equation, but Lipschitz regularity of $u$ does not hold. The question of the shape of the free boundary itself remains open near a Dirichlet boundary.

The structure of this thesis is as follows. Chapter 2 provides detailed definitions and basic properties of the minimizer $u$. Chapter 3 gives some necessary properties of positive harmonic functions with Neumann boundary conditions. It includes up-to-the-boundary gradient control lemmas in both two and higher dimensions, which will be used significantly in the proof of the main theorem. Chapter 4 contains the proof of the main theorem, including the average control lemma which is its main step. Chapter 5 concludes with comments on directions for future work.

# Chapter 2

# Preliminaries

Consider a domain $\Omega \subset \mathbb{R}^n$. Let $\Omega$ be bounded and connected, with $\partial\Omega$ locally a Lipschitz graph with Lipschitz constant $L$. In general, $\Omega$ will be convex. However, for some of the basic properties this condition will not be necessary. Let $\nu$ be the outer unit normal to $\partial\Omega$, where defined.

Let

$$J[v] = \int_\Omega \left( |\nabla v|^2 + Q^2(x)\chi_{\{v>0\}} \right) dx,$$

where $Q(x)$ is a smooth function with $0 \leq m \leq Q(x) \leq M$. Let $S$ be a closed, proper, nonempty subset of $\partial\Omega$. Let $u_0$ be a smooth, nonnegative function on $\mathbb{R}^n$. Let $A = \sup_S u_0$. We minimize $J$ over the set

$$K = \{v \in H^1 : v = u_0 \text{ on } S\}.$$

We now list some notations that we will use throughout this thesis. Let $a > 0$. Then $\Omega_a$ will be the open set $\{x \in \Omega : d(x,S) > a\}$. Let $D \subset \mathbb{R}^n$ be a domain. For a function $f \in H^1(D)$, and a measurable set $T \subset \partial D$ of positive Hausdorff measure, $f|_T$ will denote the trace of the function $f$ along $T$, which is in $L^2(T)$. $\operatorname{supp}(f)$ will denote the support of $f$. Finally, $|D|$ denotes the Lebesgue measure of $D$ and

$$\fint_D f \mathrm{d}x = \frac{1}{|D|} \int_D f \mathrm{d}x.$$

Let $u$ be a minimizer of $J$ in $K$. We begin by listing some basic properties of $u$, as presented in [2].

**Lemma 1** *A minimizer $u$ exists, and for any such minimizer (or local minimizer):*

1. $\Delta u \geq 0$, *in a distributional sense.*

2. $0 \leq u \leq A$.

3. $\forall\, \Omega' \Subset \Omega$, $\forall\, 0 < \alpha < 1$, $u \in C^\alpha(\Omega')$.

4. $\{u > 0\}$ *is open.*

5. $\Delta u = 0$ *in* $\{u > 0\}$.

The minimizer $u$ is also Hölder continuous up to the boundary, although the exponent $\alpha$ is now controlled by the Lipschitz constant of $\partial\Omega$, and may not be close to 1. For this lemma, as with the main theorem, only points of $\Omega$ which are far from $S$ are considered. Once again, this is not because we do not expect $u$ to be Hölder up to $S$, provided $u_0$ is Hölder, but rather because we are not interested in the behavior up to the Dirichlet boundary.

**Lemma 2** *Let $r_0 > 0$. Then $\exists\, \alpha > 0$ such that $u \in C^\alpha(\Omega_{r_0})$, with $\alpha$ depending on $n$, and $L$, and $\|u\|_{C^\alpha}$ depending on $n, L, M,\ A,$ and $r_0$.*

Proof Let $x_0 \in \Omega$. $\exists\, s > 0$ depending only on the Lipschitz character of $\Omega$ such that one can cover $\partial\Omega$ with balls of radius $s$ such that for each such ball $B$, $B \cap \partial\Omega$ is a Lipschitz graph. Let $r = \frac{1}{2}\min(1, r_0, s)$. Cover $\Omega$ with balls of radius $r$ such that for each ball $B_r(x)$, $B_{2r}(x)$ is either an interior ball in $\Omega$ or $\partial\Omega \cap B_{2r}(x)$ is a Lipschitz graph as above. Let $x_0$ be the center of one of these balls. We will show that $u \in C^\alpha(B_r(x_0) \cap \Omega)$ for some $\alpha$ independent of $x_0$. This will suffice, as $\Omega$ is bounded.

First, suppose $B_{2r}(x_0) \subset \Omega$. Then $u \in C^\alpha(B_r)$ for any $\alpha < 1$, as in ([4], Theorem 2.1). We may therefore suppose that $B_r(x_0) \not\subset \Omega$. Let $x \in B_r(x_0)$ and let $r_x = \mathrm{dist}(x, \partial B_{2r}(x_0))$. Let $t < r_x$. Let $D_t = B_t(x) \cap \Omega$. Let $\Gamma_{D,t} = (\partial B_t(x)) \cap \Omega$, $\Gamma_{N,t} = B_t(x) \cap \partial\Omega$. If $\Gamma_{N,t} = \emptyset$, let

$v_t$ be the harmonic function on $D_t$ such that $v_t = u$ on $\partial D_t$. In this case, as in [4] Theorem 2.1, we may conclude from the minimality of $u$ that, for each $t < r$,

$$\int_{D_t} |\nabla(u - v_t)|^2 \leq C(r_x)t^n.$$

Moreover, we may conclude that

$$\int_{D_t} |\nabla(v_{2t} - v_t)|^2 \leq C(r_x)t^n. \tag{2.1}$$

as well.

If $\Gamma_{N,t} \neq \emptyset$, define $v_t$ to be the harmonic function on $D$ with $v_t = u$ on $\Gamma_{D,t}$ and $\frac{\partial v}{\partial \nu} = 0$ on $\Gamma_{N,t}$. As above, for each $t < r_x$,

$$\int_{D_t} |\nabla(u - v_t)|^2 \leq C(r_x)t^n.$$

Now, because $\Omega$ is Lipschitz, there is a bilipschitz map

$$F : D_t \longrightarrow B_t^+(0)$$

with $\{x : x_n = 0\} = F(\Gamma_{N,t})$ and $(\partial B_{N,t}(0))^+ = F(\Gamma_{D,t})$. The Lipschitz constants of $F$ and $F^{-1}$ are controlled solely by $L$. Let

$$\left(a^{ij}(y)\right) = |\det(\nabla F^{-1}(y))| \left((\nabla F)^{\mathrm{T}}\nabla F\right)(F^{-1}(y)),$$

whenever $y_n \geq 0$. When $y_n < 0$, let

$$\left(a^{ij}(y_1, \ldots, y_n)\right) = \begin{cases} a^{ij}(y_1, \ldots, -y_n) & i \neq n \text{ and } j \neq n \\ -a^{ij}(y_1, \ldots, -y_n) & i = n \text{ or } j = n \text{ but } i \neq j \\ a^{ij}(y_1, \ldots, -y_n) & i = j = n \end{cases}.$$

Note that $a^{ij}$ is uniformly elliptic with bounded, measurable coefficients on all of $B_t(0)$, and that the bounds on $a^{ij}$ depend only on $L$.

Define $\tilde{v}_t(y) = v_t(F^{-1}y)$ on $B_t$ when $y_n \geq 0$. For $y \in B_t$ with $y_n < 0$, let $\tilde{v}\ (y_1, \ \ldots, \ y_n) = \tilde{v}\ (y_1, \ \ldots, \ -y_n)$.

Then, on $B_t(0)$, I will show that $\tilde{v}_t$ satisfies the equation

$$\sum_{i,j} \frac{\partial}{\partial x_j}(a^{ij}\frac{\partial \tilde{v}_t}{\partial x_i})\mathrm{d}x = 0$$

in the weak sense, i.e. for every $\phi \in C_c^\infty(B_t(0))$,

$$\int_{B_t(0)} \langle a^{ij}\nabla\tilde{v}_t, \nabla\phi\rangle = 0.$$

Compute:

$$\begin{aligned}\int_{B_t(0)} \langle a^{ij}\nabla\tilde{v}_t, \nabla\phi\rangle \mathrm{d}y &= \int_{B_t^+(0)} \langle a^{ij}\nabla\tilde{v}_t, \nabla\phi\rangle \mathrm{d}y + \int_{B_t^-(0)} \langle a^{ij}\nabla\tilde{v}_t, \nabla\phi\rangle \mathrm{d}y \\ &= \int_{B_t^+(0)} \langle |\det(\nabla F^{-1}(x))| \left((\nabla F)^{\mathrm{T}}\nabla F\right)(F^{-1}(x))\nabla\tilde{v}_t, \nabla\phi\rangle \mathrm{d}y \\ &\quad + \int_{B_t^+(0)} \langle |\det(\nabla F^{-1}(x))| \left((\nabla F)^{\mathrm{T}}\nabla F\right)(F^{-1}(x))\nabla\tilde{v}_t, \nabla\psi\rangle \mathrm{d}y.\end{aligned}$$

Here, $\psi$ is defined on $B_t(0)$ by $\psi(x_1, \ldots, x_n) = \phi(x_1, \ldots, -x_n)$, and the last line is true by the definition of the $a^{ij}$, using the change of variables formula on the map $(x_1, \ldots, x_n) \to (x_1, \ldots, -x_n)$. Note that both $\phi,\ \psi \in C^\infty(B_t \cap \{x_n \geq 0\})$ and $\phi,\ \psi = 0$ on $(\partial B_t) \cap \{x \geq 0\}$. Hence, if we define $\tilde{\phi}(y) = \phi(F(y))$ and analogously for $\tilde{\psi}$, then both $\tilde{\phi}$ and $\tilde{\psi}$ are valid test functions in $\{f \in H^1(B_t \cap \Omega) : f = 0 \text{ on } (\partial B_t) \cap \Omega\}$.

Hence, we conclude, after another change of variables, that

$$\int_{B_t(0)} \langle a^{ij}\nabla\tilde{v}_t, \nabla\phi\rangle \mathrm{d}y = \int_{B_t^+(0)} \langle \nabla F(F^{-1}(y))\nabla\tilde{v}_t, \nabla F(F^{-1}(y))\nabla\phi\rangle \,|\det(\nabla F^{-1}(y))|\,\mathrm{d}y$$

$$+ \int_{B_t^+(0)} \langle \nabla F(F^{-1}(y))\nabla\tilde{v}_t, \nabla F(F^{-1}(y))\nabla\psi\rangle \,|\det(\nabla F^{-1}(y))|\,\mathrm{d}y$$

$$= \int_{B_t(x)\cap\overline{\Omega}} \langle \nabla v_t, \nabla\tilde{\phi}\rangle \,\mathrm{d}x + \int_{B_t(x)\cap\overline{\Omega}} \langle \nabla v_t, \nabla\tilde{\psi}\rangle \,\mathrm{d}x$$

$$= 0,$$

using the energy definition of $v_t$.

Because $v_t$ satisfies an elliptic equation of the appropriate form, by Theorem 5.3.6 of [13], there exists $C, \mu_0$, with $0 < \mu_0 < 1$, depending only on $n$ and the $a^{ij}$ (which in turn depend only on the Lipschitz constant of $\partial\Omega$) such that, for any $l < t$,

$$||\nabla v_t||_{L^2(B_l\cap\Omega)} \leq C||\nabla v_t||_{L^2(B_t\cap\Omega)} \left(\frac{l}{t}\right)^{(\frac{n}{2}-1+\mu_0)} \tag{2.2}$$

This regularity holds for any $l < t < r$.

Now, we return to the consideration of $u$. Choose some $t < r_x$. Recall that we have

$$\int_{B_t\cap\Omega} |\nabla(u - v_t)|^2 \mathrm{d}x \leq Ct^n,$$

and note that this implies that

$$\int_{B_{2^{(i-1)}t}\cap\Omega} |\nabla(v_{2^{i-1}t} - v_{2^it})|^2 \mathrm{d}x \leq C(2^{i-1}t)^n.$$

Applying the result of our previous calculation to the function $v_{2^{i-1}t} - v_{2^it}$ on $B_{2^{(i-1)}t} \cap \Omega$, we find that, if $\Gamma_{N,2^{i-1}t} \neq \emptyset$,

$$\int_{B_t\cap\Omega} |\nabla(v_{2^{i-1}t} - v_{2^it})|^2 \mathrm{d}x \leq Ct^n(2^{(2-2\mu_0)i}).$$

Recall that if $\Gamma_{N,2^{i-1}t} = \emptyset$, then the bound we get in (2.1) is even better.

We have, by the triangle inequality,

$$\begin{aligned} \|\nabla u\|_{L^2(B_t\cap\Omega)} &\leq \|\nabla(u - v_t)\|_{L^2(B_t\cap\Omega)} + \sum_{i=1}^{-\log(\frac{t}{r})} \|\nabla(v_{2^{i-1}t} - v_{2^i t})\|_{L^2(B_t\cap\Omega)} + \|\nabla v_r\|_{L^2(B_t\cap\Omega)} \\ &\leq Ct^{\frac{n}{2}}(1 + \sum_{i=1}^{-\log(\frac{t}{r})} (2^{(1-\mu_0)i})) + t^{\frac{n}{2}-1+\mu_0} \end{aligned}$$

After summation, we conclude that, for every $x \in B_r(x_0)$, for every $t < r_x$

$$\int_{B_t(x)\cap\Omega} |\nabla u|^2 \, dx \leq C(r_x)t^{n-2+2\mu_0}.$$ [1]

Therefore, by Theorem 3.5.2 of [13], $u \in C^{\mu_0}(B_r(x_0) \cap \Omega)$, with $\mu_0$ and $\|u\|_{C^{\mu_0}}$ depending only on the given constants. □

**Corollary 1** *$\{u > 0\} \cap \Gamma$ is open in $\partial\Omega$.*

Finally, we consider the sense in which Neumann boundary conditions hold for $u$. Note that $\frac{\partial u}{\partial \nu}$ may not be defined pointwise along $\partial\Omega$, and in fact $\nu$ is not defined pointwise. Throughout this thesis, when we say that Neumann boundary conditions hold "in a weak sense" for a harmonic function on a set $U$, we mean that, $\forall \phi \in C^\infty(U)$,

$$0 = \int_U \nabla u \cdot \nabla \phi.$$

Comparison with the smooth case, where $\frac{\partial u}{\partial \nu}$ exists pointwise and we can integrate by parts, indicates that this condition is equivalent to $u$ being a harmonic function with Neumann boundary conditions. To indicate that $\frac{\partial u}{\partial \nu} = 0$ only on a closed subset of $\partial\Omega$, we will require $\phi = 0$ on $\partial\Omega \setminus \Gamma$ but otherwise can be any smooth function. Once again, integration by parts indicates that this is the appropriate condition.

**Lemma 3** $\frac{\partial u}{\partial \nu} = 0$ *weakly along* $\Gamma$.

[1] Note that, if $\mu_0 = 1$ then when we sum we will get $C(r_x)t^n|log(\frac{t}{r})|$, instead of the quantity computed above. This argument cannot be used to obtain the estimate with $\mu_0 = 1$, i.e. Lipschitz continuity.

PROOF Let $x_0 \in \Gamma$ with $u(x_0) > 0$. Choose $s > 0$ such that $u > 0$ on $B_s(x_0) \cap \Omega$, and $B_s(x_0) \cap S = \emptyset$. Let $\phi \in C_0^\infty(B_s(x_0))$. On $B_s(x_0) \cap \Omega$, we have

$$J[u + \epsilon\phi] - J[u] = \int_{B_s(x_0)\cap\Omega} |\nabla(u + \epsilon\phi)|^2 - |\nabla u|^2,$$

and

$$\frac{d}{d\epsilon} J[u + \epsilon\phi]\Big|_{\epsilon=0} = 0.$$

Since $u > 0$ on $B_s(x_0)$, $J$ is smooth in $\epsilon$, for $\epsilon$ sufficiently small,

$$0 = \int_{B_s(x_0)\cap\Omega} \nabla u \cdot \nabla\phi.$$

Since this holds for all such $x_0$ and $s$, we conclude that $\frac{\partial u}{\partial \nu} = 0$ weakly along $\Gamma$ in the sense defined above. □

# Chapter 3

# Properties of Harmonic Functions

The proof of the main theorem will use several properties of harmonic functions on convex domains with Neumann boundary conditions, which are proven in this chapter. In Lemma 4, we prove a weak maximum principle for harmonic functions with mixed boundary conditions on a Lipschitz domain. We also present a Harnack inequality up to the Neumann boundary in Lemma 5. The chapter concludes with up-to-the-Neumann-boundary gradient bounds for harmonic functions in convex domains; the cases of two dimensions and higher dimensions are considered separately.

**Lemma 4** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain, and let $\Gamma_D \subset \partial\Omega$ be a measurable set of positive Hausdorff measure in $\partial\Omega$. Let $u \in H^1(\Omega)$ satisfy:*

1. $\int_\Omega \nabla u \cdot \nabla \phi \geq 0 \qquad \forall \phi \in P = \{f \in H^1 | f \geq 0 \text{ and } f|_{\Gamma_D} = 0\}$
2. $\exists\ u_0 \in L^2(\partial\Omega)$ *such that* $u|_{\Gamma_D} = u_0 \geq 0$ *on* $\Gamma_D$.

*Then $u \geq 0$ in $\Omega$.*

Proof Consider the function $u^-(x) = \max(0, -u(x))$. $u^-$ is in $H^1(\Omega)$ and $u^-|_{\Gamma_D} = 0$ because $-u|_{\Gamma_D} \leq 0$ and $u^- \geq 0$. Therefore, $u^- \in P$. Hence,

$$\int_\Omega \nabla u \cdot \nabla u^- \geq 0.$$

Because $u \in H^1$, $\nabla u = \nabla u^+ \chi_{\{u>0\}} - \nabla u^- \chi_{\{u<0\}}$. Since $\text{supp}(u^+) \cap \text{supp}(u^-) = \emptyset$, $\nabla u \cdot \nabla u^- = \nabla u^- \cdot \nabla u^-$. Therefore, we may conclude that

$$-\int_\Omega |\nabla u^-|^2 \geq 0$$

which implies that $\nabla u^- \equiv 0$, so, since $\Omega$ is connected, $u^- \equiv C \geq 0$. But $u^-|_{\Gamma_D} = 0$ on $\Gamma_D$, so $C = 0$. Therefore, $u^- \equiv 0$, so $u = u^+ \geq 0$. □

**Lemma 5** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, with Lipschitz constant $L$. Suppose $\Gamma_N$ is a closed subset of $\partial\Omega$. Let $u$ be a positive harmonic function on $\Omega$ such that $\frac{\partial u}{\partial \nu} = 0$ along $\Gamma_N$, in a weak sense. Then, for any $x_0 \in \Omega$, for any $r > 0$ such that $B_r(x_0) \cap \partial\Omega \subset \Gamma_N$,*

$$\sup_{B_{\frac{r}{2}}(x_0)} u(x) \leq C \inf_{B_{\frac{r}{2}}(x_0)} u(x)$$

*where the constant $C$ depends only on $n$ and the Lipschitz character of $\partial\Omega$.*

Proof

It suffices to proof this inequality locally near $\Gamma_N$. Hence, we may choose $x_0 \in \Gamma_N$ and $s > 0$ such that:

1. $B_{2s}(x) \cap (\partial\Omega \setminus \Gamma_N) = \emptyset$
2. $\Gamma_N$ is simply connected in $B_{2s}$ and
3. $\Gamma_N$ is a Lipschitz graph in the $x_n$-direction in $B_{2s}$, possibly after a rotation of coordinates.

Then there exists a bilipschitz map $F$ from $B_{2s}(x_0) \cap \Omega$ to $B^+_{2s}(0)$ such that $F$ extends continuously to the boundary and $F(\Gamma_N) = \{y | |y| \leq 2s \text{ and } y_n = 0\}$. The Lipschitz constants of $F$ and $F^{-1}$ depend only on the Lipschitz constant $L$ of $\Gamma_N$. Let

$$\left(a^{ij}(y)\right) = |\det(\nabla F^{-1}(y))| \left((\nabla F)^{\mathrm{T}} \nabla F\right) (F^{-1}(y)),$$

whenever $y_n \geq 0$. When $y_n < 0$, let

$$(a^{ij}(y_1, \dots, y_n)) = \begin{cases} a^{ij}(y_1, \dots, -y_n) & i \neq n \text{ and } j \neq n \\ -a^{ij}(y_1, \dots, -y_n) & i = n \text{ or } j = n \text{ but } i \neq j \\ a^{ij}(y_1, \dots, -y_n) & i = j = n \end{cases}.$$

Note that $a^{ij}$ is uniformly elliptic with bounded, measurable coefficients on all of $B_{2s}(0)$, and that the bounds on $a^{ij}$ depend only on $L$.

Define $\tilde{u}(y) = u(F^{-1}y)$ on $B_{2s}$ when $y_n \geq 0$. For $y \in B_{2s}$ with $y_n < 0$, let $\tilde{u}\ (y_1, \ \dots, \ y_n) = \tilde{u}\ (y_1, \ \dots, \ -y_n)$. By the same calculation as in Lemma 2, $\tilde{u}$ satisfies the equation

$$\sum_{i,j} \frac{\partial}{\partial x_j}(a^{ij}\frac{\partial \tilde{u}}{\partial x_i}) = 0$$

in the weak sense, i.e. for every $\phi \in C_c^\infty(B_{2s}(0))$,

$$\int_{B_{2s}(0)} \langle a^{ij}\nabla\tilde{u}, \nabla\phi\rangle dx = 0.$$

Then, by ([12], Theorem 8.20), $\tilde{u}$ satisfies

$$\sup_U \tilde{u} \leq C \inf_U \tilde{u},$$

for any $U \Subset B_{2s}(0)$, where $C$ depends only on the eigenvalues of $a^{ij}$ and the distance from $\partial U$ to $\partial B_{2s}$. Clearly, the same inequality holds when we restrict to the upper half ball because $\tilde{u}$ was created by an even reflection. Choose $U = F(B_s(x))$. Then, by returning to $B_{2s}(x) \cap \Omega$ along $F^{-1}$, we conclude that $\sup_{B_s(x)\cap\Omega} u \leq C \inf_{B_s(x)\cap\Omega} u$, where $C$ depends only on $n$ and the Lipschitz constant $L$ of $\partial\Omega$. □

We next present a gradient bound up to the Neumann boundary on a convex domain in $\mathbb{R}^2$. This lemma uses conformal mapping to compare $u$ to a harmonic function on the upper half plane. The availability of this tool makes the 2-dimensional case much easier to handle than higher dimensions. This lemma will be applied to bounded domains $\Omega$. However, for simplicity, in this lemma $\Omega$ is unbounded. Since this change does not affect the part of the domain on which the function is defined, it will not affect its applicability.

**Lemma 6** *Let $\Omega \subset \mathbb{R}^2$ be a convex domain, such that $\partial\Omega = \{y_2 > f(y_1)\}$, $f$ a Lipschitz function with Lipschitz constant $L$, $f(0) = 0$, and $\min_{y\in\mathbb{R}} f(y) = 0$. Let $u$ be a harmonic function on $B_4(0) \cap \Omega$, with $0 \le u \le A$, such that $\frac{\partial u}{\partial \nu} = 0$ along $\partial\Omega \cap B_4(0)$. Then*

$$|\nabla u| \le C(L)A$$

*on $B_2(0) \cap \Omega$.*

PROOF Let $\Phi : H = \{(x_1, x_2) : x_2 > 0\} \longrightarrow \Omega$ be the conformal map such that $\Phi(\infty) = \infty$ and $\Phi(i) = i$. Let $\Psi = \Phi^{-1}$. Let $D_2 = \Psi(B_2 \cap \Omega)$ and $D_4 = \Psi(B_4 \cap \Omega)$. On $D_4$, let $v = u \circ \Phi$, so that $u = v \circ \Psi$. Then

$$|\nabla u(x)| = |\nabla \Psi(x)||\nabla v(\Psi(x))|.$$

$v$ is a harmonic function on $D_4$ with $0 \le v \le A$ and $\frac{\partial v}{\partial \nu} = 0$ along $\{(x_1, x_2) \in \overline{D_4} : x_2 = 0\}$, because conformal mapping preserves harmonicity and angles. Therefore, by even extension $v$ can be considered as a harmonic function on $U_4 = \{ z : z \text{ or } \overline{z} \in D_4\}$.

Because $\partial\Omega$ is Lipschitz, it satisfies Ahlfors' three-point condition with a constant depending only on the Lipschitz constant $L$ of $\partial\Omega$. Therefore, the mapping $\Psi$ can be extended to a quasiconformal mapping of $\mathbb{C}$ onto itself, with $\Psi(\partial\Omega) = \mathbb{R}$, and the quasiconformal constant of the map once again depends only on $L$ ([1], Chapter 4, Section D and Section E Theorem 5). Recall that $\Psi$ has the normalization $\Psi(i) = i$. By applying Corollary 5.4 of [5] several times, we may conclude that there exists $c$ depending only on $L$ such that, if $|w| = 2$ and $|z| = 4$, then $|\Psi(w) - \Psi(z)| \ge c$. Therefore, $v$ can be considered as a harmonic function on $B_c(a) \subset U_4$ for each $a \in D_2$. We may conclude that, for each such $a$,

$$|\nabla v(a)| \le \frac{A}{c}$$

by the interior regularity of harmonic functions.

We will therefore be able to conclude that $|nabu(x)| = |\nabla\Psi(x)||\nabla v(\Psi(x))| \leq \tilde{C}$ for every $x \in B_2(0) \cap \Omega$ if we can prove that $|\nabla(\Psi)| \leq C(L)$ on $B_2$. First note that, again by Corollary 5.4 of [5], there exists a $C > 0$ such that, for every $z \in B_4(0)$, $|\Psi(z)| \leq C$. Define $G(w)$ to be the positive Green's function on $\Omega$ with pole at $\infty$, i.e.

$$\begin{aligned} \Delta G(x) &= 0 && \forall x \in \Omega \\ G &\geq 0 && \forall x \in \Omega \\ G &= 0 && \forall x \in \partial\Omega \end{aligned}$$

Normalize $G$ by $G(i) = 1$.

Because $\Phi$ and $\Psi$ are conformal, they preserve Green's function. so $\tilde{G} = G \circ \Phi$ is the Green's function on the upper half plane with pole at $\Psi(\infty) = \infty$, and with normalization $\tilde{G}(\Psi(i)) = G(i) = 1$. This function is just $\text{Im}(z)$. So $G(w) = \text{Im}(\Psi(w))$. By the Cauchy-Riemann equations, $|\Psi'(w)| = |\nabla G(w)|$, so to control $|\Psi'|$ we only need to control $|\nabla G|$.

Recall from above that there is a constant $C$ such that, $\forall z \in B_4(0)$, $|\Psi(z)| \in B_C(0)$. This means that $G(z) = \text{Im}(\Psi(z)) \leq C$. Let $w \in B_2(0)$. If $\text{dist}(w, \partial\Omega) \geq 1$, then by the interior regularity of $G$ on $B_1(w)$, $|\nabla G| \leq C$. On the other hand, if $\text{dist}(w, \partial\Omega) \leq 1$, let $w_0 \in \partial\Omega$ be a point such that $\text{dist}(w, \partial\Omega) = \text{dist}(w, w_0) = r$. On $B_{2r}(w_0) \cap \Omega$, $G \leq C$. Moreover, $G = 0$ on $\partial\Omega \cap B_{2r}(w_0)$. Let $L = \{ w : w_2 = l(w_1) \}$ be a support plane for $\partial\Omega$ at $w_0$. Let $D = B_{2r}(w_0) \cap \{ w : w_2 > l(w_1) \} \supset B_{2r}(w_0) \cap \Omega$. Let $h$ be the function on $D$ satisfying:

$$\begin{aligned} \Delta h(z) &= 0 && \forall z \in D \\ h(z) &= 0 && \forall z \in L \cap B_{2r}(w_0) \\ h(z) &= C && \forall z \in \partial(B_{2r}(w_0)) \cap \{w : w_2 > l(w_1)\} \end{aligned}$$

Then, since $G \leq h$ on $\partial(B_{2r}(w_0) \cap \Omega)$, by the maximum principle $G \leq h$ on $B_{2r}(w_0) \cap \Omega$. But $h \leq C\text{dist}(\cdot, L)$ on $B_r(w_0)$. Therefore, $G(w) \leq h(w) \leq Cr$ since $r = \text{dist}(w, w_0) = \text{dist}(w, L)$. But then, by the interior regularity of $G$ on $B_r(w)$, $|\nabla G(w)| \leq C$. We may conclude that $|\nabla G| \leq C$ on all of $B_2(0) \cap \Omega$, so $|\nabla\Psi| \leq C$ on $B_2(0) \cap \Omega$ as well. □

Note that, by scaling, this lemma implies that, for $\Omega$ as in the statement of the lemma, for every $u$ which is harmonic on $B_{2r} \cap \Omega$, with Neumann boundary conditions along $\partial\Omega$, we have on $B_r \cap \Omega$ that

$$|\nabla u| \leq C(L)\frac{A}{r}.$$

We now provide a similar result for higher dimensions. The function $\Phi$ was introduced in [9], and the idea for using convexity to control the Neumann boundary is similar to that used in [14]. The result is proven first on smooth domains, then a limiting procedure generalizes it to all convex domains.

**Lemma 7** *Let $\Omega \subset \mathbb{R}^n$ be a domain such that $\partial\Omega$ is the graph of a smooth, convex function $f$. Suppose $0 \in \Omega$ and let $r = \operatorname{dist}(0, \partial\Omega)$. Let $R > 2r$ and let $D = B_R(0) \cap \Omega$. Let $\Gamma = B_R \cap \partial\Omega$ and let $S = \partial B_R \cap \Omega$. Let $u$ be a harmonic function on $D$ with $0 \leq u \leq A$ and $\frac{\partial u}{\partial \nu} = 0$ along $\Gamma$. Then $\exists C > 0$ depending only on $n$ such that $|\nabla u| \leq C\frac{A}{R}$ on $B_{\frac{R}{2}}$.*

Proof Let

$$\Phi(x) = \frac{(R^2 - |x|^2)^2\, |\nabla u|^2}{(9A^2 - (u - 2A)^2)^2}. \tag{3.1}$$

Then:

1. $\Phi = 0$ on $S$.

2. $\Phi > 0$ inside $D$.

3. $\Phi$ is smooth in $D \cup \Gamma$ because $u$ and $\nabla u$ are smooth and the denominator of $\Phi$ cannot approach 0.

4. $\max_{x \in \Gamma} \Phi(x) < \max_{x \in \overline{D}} \Phi(x)$.

PROOF

$$\begin{aligned}\frac{\partial\Phi}{\partial\nu} &= 2\left(\frac{(R^2-|x|^2)^2}{(9A^2-(u-2A)^2)^2}\right)(\nabla u\cdot\frac{\partial}{\partial\nu}\nabla u)\\ &\quad -2\left(\frac{|\nabla u|^2(R^2-|x|^2)}{(9A^2-(u-2A)^2)^2}\right)(\vec{x}\cdot\frac{\partial\vec{x}}{\partial\nu})\\ &\quad -2\left(\frac{|\nabla u|^2(R^2-|x|^2)^2}{(9A^2-(u-2A)^2)^3}\right)(u-2A)\frac{\partial u}{\partial\nu}\\ &= (a)+(b)+(c)\end{aligned}$$

Note that $(c)=0$ because $\frac{\partial u}{\partial\nu}=0$. For $(b)$, note that by the convexity of $\Omega$, $\vec{x}\cdot\frac{\partial\vec{x}}{\partial\nu}>0$, so $(2)<0$. Finally, consider (a). Since $2\frac{(R^2-|x|^2)^2}{(9A^2-(u-2A)^2)^2}>0$, we consider only $\nabla u\cdot\frac{\partial}{\partial\nu}\nabla u$, at a point $x$. After rotation, suppose that $\nu(x)=e_n$, so we can use $e_1,\ldots,e_{n-1}$ as local coordinates for $\Gamma$. Then

$$\nabla u\cdot\frac{\partial}{\partial\nu}\nabla u=\nabla u\cdot\nabla(\frac{\partial u}{\partial\nu})-\sum_{i,j=1}^{n-1}\frac{\partial\nu_i}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}=-\sum_{i,j=1}^{n-1}\frac{\partial\nu_i}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$$

because $\frac{\partial u}{\partial\nu}=0$. But the matrix $\frac{\partial\nu_i}{\partial x_j}$ is just the second fundamental form of $\Gamma$ in local coordinates, and therefore by convexity it is positive definite. So $\nabla u\cdot\frac{\partial}{\partial\nu}\nabla u<0$ along $\Gamma$. We conclude that $\frac{\partial\Phi}{\partial\nu}<0$ along $\Gamma$, so the maximum of $\Phi$ cannot occur there. □

So the maximum of $\Phi$ occurs at a point $x_0\in D$. At $x_0$, we have:

$$0=\nabla\Phi=\left(\frac{-2\nabla(|x|^2)}{(R^2-|x|^2)}+\frac{\nabla(|\nabla u|^2)}{|\nabla u|^2}+\frac{2\nabla((u-2A)^2)}{(9A^2-(u-2A)^2)}\right)\Phi \tag{3.2}$$

and

$$\begin{aligned}0\geq\Delta\Phi=\Big(&\frac{-2\Delta(|x|^2)}{(R^2-|x|^2)}+\frac{-2|\nabla(|x|^2)|^2}{(R^2-|x|^2)^2}+\frac{\Delta(|\nabla u|^2)}{|\nabla u|^2}-\frac{|\nabla(|\nabla u|^2)|^2}{|\nabla u|^4}\\ &+\frac{2\Delta((u-2A)^2)}{(9A^2-(u-2A)^2)}+\frac{2|\nabla((u-2A)^2)|^2}{(9A^2-(u-2A)^2)^2}\Big)\Phi\end{aligned} \tag{3.3}$$

Note that $\nabla(|x|^2) = 2\vec{x}$ and $\nabla((u-2A)^2) = 2(u-2A)\nabla u$. Plugging into (3.2), we have

$$0 = \frac{-4\vec{x}}{R^2 - |x|^2} + \frac{\nabla(|\nabla u|^2)}{|\nabla u|^2} + \frac{4(u-2A)\nabla u}{(9A^2 - (u-2A)^2)}$$

which implies that

$$\frac{|\nabla(|\nabla u|^2)|^2}{|\nabla u|^4} \le \frac{16|x|^2}{(R^2 - |x|^2)^2} + \frac{16(u-2A)^2|\nabla u|^2}{(9A^2 - (u-2A)^2)^2} + \frac{32|x||u-2A||\nabla u|}{(R^2 - |x|^2)(9A^2 - (u-2A)^2)} \tag{3.4}$$

Because $\Delta u = 0$, we have $\Delta(|\nabla u|^2) = 2\sum_{i,j}(\frac{\partial^2 u}{\partial x_i \partial x_j})^2$, and

$$|\nabla(|\nabla u|^2)|^2 = 4\sum_{i,j,k} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_k}.$$

Therefore, comparing $2|\nabla u|^2 \Delta(|\nabla u|^2)$ with $|\nabla(|\nabla u|^2)|^2$, we have:

$$\frac{\Delta(|\nabla u|^2)}{|\nabla u|^2} \ge \frac{1}{2}\frac{|\nabla(|\nabla u|^2)|^2}{|\nabla u|^4}. \tag{3.5}$$

We also have $\Delta(|x|^2) = 2n$ and $\Delta((u-2A)^2) = 2|\nabla u|^2$. Plugging these and (3.5) into 3.3, we get:

$$\begin{aligned} 0 &\ge \frac{-4n}{R^2 - |x|^2} - \frac{8|x|^2}{(R^2 - |x|^2)^2} - \frac{1}{2}\frac{|\nabla(|\nabla u|^2)|^2}{|\nabla u|^4} + \frac{4|\nabla u|^2}{9A^2 - (u-2A)^2} + \frac{8(u-2A)^2|\nabla u|^2}{(9A^2 - (u-2A)^2)^2} \\ &\ge \frac{-4n}{R^2 - |x|^2} - \frac{8|x|^2}{(R^2 - |x|^2)^2} - \frac{8|x|^2}{(R^2 - |x|^2)^2} - \frac{8(u-2A)^2|\nabla u|^2}{(9A^2 - (u-2A)^2)^2} + \\ &\quad - \frac{16|x||u-2A||\nabla u|}{(R^2 - |x|^2)(9A^2 - (u-2A)^2)} + \frac{4|\nabla u|^2}{9A^2 - (u-2A)^2} + \frac{8(u-2A)^2|\nabla u|^2}{(9A^2 - (u-2A)^2)^2} \end{aligned}$$

Therefore,

$$\frac{4|\nabla u|^2}{9A^2 - (u-2A)^2} \le \frac{16|x||u-2A||\nabla u|}{(R^2 - |x|^2)(9A^2 - (u-2A)^2)} + \frac{4n(R^2 - |x|^2) + 16|x|^2}{(R^2 - |x|^2)^2}.$$

Multiplying by $(R^2 - |x|^2)^2$, dividing by $9A^2 - (u-2A)^2$, and recalling the definition of $\Phi$ given in (3.1), we have:

$$4\Phi(x_0) \leq \frac{16RB\sqrt{(\Phi(x_0))}}{9A^2 - (u-2A)^2} + \frac{(4n+16)R^2}{9A^2 - (u-2A)^2}$$

Recall that $5A^2 \leq 9A^2 - (u-2A)^2 \leq 8A^2$. Let $z = \sqrt{\Phi(x_0)}$. Then the quadratic formula applied to $z$ implies that

$$\Phi(x_0) \leq C_n \frac{R^2}{A^2}.$$

Since $x_0$ is the maximum of $\Phi$, we infer that

$$\Phi(x) \leq C_n \frac{R^2}{A^2}$$

on all of $B_R(0) \cap \Omega$. Hence, on $B_{\frac{R}{2}}(0) \cap \Omega$, where $R^2 - |x|^2 \sim R^2$ and $(u - 2A) \sim A$, we find

$$|\nabla u|^2 \leq c_n \frac{A^2}{R^2}.$$

□

**Lemma 8** *Let $\Omega \subset \mathbb{R}^n$ be a convex domain, with $0 \in \Omega$. Let $r = \text{dist}(0, \partial\Omega)$ and let $R > 2r$. Let $D = B_R(0) \cap \Omega$, and let $u$ be a harmonic function on $B_{2R}(0) \cap \Omega$ with $0 \leq u \leq A$, and $\frac{\partial u}{\partial \nu} = 0$ weakly along $\partial\Omega \cap B_R(0)$. Then $\exists C > 0$ such that, on $B_{\frac{R}{2}}(0) \cap \Omega$,*

$$|\nabla u| \leq C \frac{A}{R}.$$

Proof Let $\Omega_i$ be a collection of smooth, convex domains increasing to $\Omega$, with $0 \in \Omega_i$. Let $D_i = B_R(0) \cap \Omega_i$, and let $u_i$ be the function on $D_i$ satisfying

$$\begin{aligned} \Delta u_i(x) &= 0 && \forall x \in \Omega_i \\ u_i(x) &= u(x) && \forall x \in \partial B_R(0) \cap \Omega_i \\ \frac{\partial u_i}{\partial \nu}(x) &= 0 && \forall x \in B_R(0) \cap \partial\Omega_i \end{aligned}$$

Note that $0 \le \inf_{D_i} u_i$ and $\sup_{D_i} u_i \le \sup_D u = A$.

Then, there is a subsequence $u_{i_j}$ and a $u_0$ such that:

1. $u_{i_j} \to u_0$ uniformly on $\overline{D \cap B_{\frac{3R}{4}}(0)}$.

   By Theorem 5.3.7 of [13], $u_i \in C^\beta(D_i \cap B_{\frac{3R}{4}})$ for some $\beta > 0$ depending only on $L$ and moreover $\|u_i\|_{C^\beta} \le C(n, L, B)$. Since the $u_i$ are uniformly bounded in $C^\beta$, a subsequence $u_{i_j}$ converges uniformly to a function $u_0$ on $D \cap B_{\frac{3R}{4}}(0)$.

2. We may assume that the $u_{i_j}$ converge to $u_0$ in $C^\infty$ on compact subsets of $D \cap B_{\frac{3R}{4}}(0)$.

3. Recall that there exists a bounded extension operator from $H^1(D_i)$ to $H^1(D)$. Consider the function $\tilde{u}_i$ given by this extension of $u_i$ to $D$; $\|\tilde{u}_i\|_{H^1(D)} \le C\|u_i\|_{H^1(D_i)}$.

   Consider $U_i = \Omega_i \cap B_{2R}$. Let $w_i$ be the minimizer of

$$\int_{U_i} |\nabla v|^2 dx$$

   in $K_i = \{v \in H^1 : v = u \text{ on } \overline{U_i \setminus D_i}\}$. Then $w_i = u_i$ on $D_i$ because, $\forall \phi \in C_0^\infty(B_R)$,

$$\int_{D_i} (\nabla w_i \cdot \nabla \phi) dx,$$

   so $w_i$ is harmonic and $\frac{\partial w_i}{\partial \nu} = 0$ on $\partial D_i \setminus \partial B_R$. In addition, $w_i = u$ on $\partial B_R \cap \Omega$, so $w_i = u_i$ on $D_i$. We conclude that $u_i$ satisfies

$$\int_{U_i} |\nabla u_i|^2 dx \le \int_{U_i} |\nabla u|^2 dx \le \|u\|_{H^1(B_{2R} \cap \Omega)}.$$

   Since, in addition,

$$\int_{U_i} u_i^2 dx \le (2R)^n B^2,$$

   we conclude that $\|u_i\|^2_{H^1(D_i)} \le C(n, B, R)$.

   Therefore, $\|\tilde{u}_i\|_{H^1(D)} \le C(n, B, R)$, so we may assume that the sequence $\tilde{u}_{i_j}$ also converges weakly in $H^1(D)$. Moreover, this weak limit function must also be the weak limit of the $\tilde{u}_{i_j}$ on any fixed $D_{i_0}$, but for $i_j > i_0$, $\tilde{u}_{i_j} = u_{i_j}$ on $D_{i_0}$. Therefore, since

$u_{i_j} \to u_0$ uniformly on $D_{i_0}$, the weak limit of $\tilde{u}_{i_j}$ must also be $u_0$.

By interior $C^\infty$ convergence, we know $u_0$ is harmonic in $D$, and by construction $u_0 = u$ on $\partial B_R(0) \cap \Omega$. Moreover, for any $\phi \in C_0^\infty(B_R)$, we have:

$$\int_D \nabla u_0 \cdot \nabla\phi = \int_{D_{i_j}} (\nabla u_0 - \nabla u_{i_j}) \cdot \nabla\phi \;+\; \int_{D_{i_j}} \nabla u_{i_j} \cdot \nabla\phi \;+\; \int_{D\setminus D_{i_j}} \nabla u_0 \cdot \nabla\phi$$
$$= (1) + (2) + (3)$$

As $i_j \to \infty$, (1) $\to 0$ by weak-$H^1$ convergence of the $u_{i_j}$ to $u_0$. By construction of the $u_i$, (2) $= 0$. Finally, (3) $\leq \|u_0\|_{H^1}\|\phi\|_{H^1}|D \setminus D_{i_j}| \to 0$ by construction of the $D_i$, since $\|u_0\|_{H^1} \leq \|u\|_{H^1}$.

We conclude that $\frac{\partial u_0}{\partial\nu} = 0$ weakly along $B_R \cap \partial\Omega$. Hence, by the uniqueness of harmonic functions on these domains (Lemma 4), $u_0 = u$ on $D$. So, $u$ is the uniform limit of the $u_{i_j}$ on $D \cap B_{\frac{3R}{4}}(0)$. Note that, by Lemma 7, the $u_{i_j}$ satisfy the gradient bound

$$|\nabla u_{i_j}|^2 \leq c_n \frac{A^2}{R^2}$$

for each $i_j$ on $D \cap B_{\frac{R}{2}}(0)$. Therefore, by uniform convergence, $u$ also satisfies the bound

$$|\nabla u|^2 \leq c_n \frac{A^2}{R^2}.$$

on $D \cap B_{\frac{R}{2}}(0)$. □

Note that this lemma has some interest in its own right, as a boundary regularity result for harmonic functions. One example of an application of this is to the size of the first nontrivial Neumann eigenvalue of the spherical laplacian on (geodesically) convex subsets of the sphere. Let $f(\theta)$ be such an eigenfunction on $V \subset S^{n-1}$, with eigenvalue $\lambda$. Let $\alpha$ be given by $\alpha(\alpha + n - 2) = \lambda$. Then the function $\tilde{f} = |x|^\alpha f(\frac{x}{|x|})$ is harmonic in the set $\tilde{V} = \{x \in \mathbb{R}^n : |x| \leq 1, \frac{x}{|x|} \in V\}$. The geodesic convexity of $V$ implies that $\tilde{V}$ is convex in $\mathbb{R}^n$, and the Neumann boundary conditions on $V$ in $S^{n-1}$ correspond to Neumann boundary conditions for $\tilde{f}$ along $\partial\tilde{V} \cap \{x : |x| < 1\}$. Therefore, the above lemma applies to $\tilde{f}$ on

$\tilde{V}$, so $\tilde{f}$ must be Lipschitz up to the boundary on $\frac{1}{2}\tilde{V}$, including the origin. But Lipschitz continuity of $\tilde{f}$ at 0 is equivalent to $\alpha \geq 1$, which is equivalent to $\lambda \geq n-1$.

This lower bound for eigenvalues is known on manifolds of positive Ricci curvature with convex boundary ([15], [8]). However, as the above lemma is proved using a maximum principle argument, it represents a new, entirely non-variational proof of this result for the special case of the sphere.

# Chapter 4

# Lipschitz Regularity

This chapter contains the proof of the main theorem of this thesis, namely that solutions of the free boundary problem are Lipschitz continuous up to convex Neumann boundaries. The problem is as defined in Chapter 2. The main tool is a lemma that gives an average growth rate of $u$ away from the free boundary which is compatible with Lipschitz regularity. This result follows from a generalization of the techniques used in [2], so that they apply close to a convex boundary with Neumann boundary conditions. The next step is to prove Lipschitz regularity on the Neumann boundary itself. In two dimensions, this is done via a monotonicity formula argument; in higher dimensions we apply the gradient control of Lemma 8. After these lemmas, we are able to give a complete proof of Lipschitz continuity in all dimensions via the maximum principle.

**Lemma 9** *There is a $C$ depending only on $n, L$, and $M$ such that $\forall x \in \partial\Omega$, $\forall B_r(x) \subset \mathbb{R}^n$ such that $B_{2r}(x) \cap S = \emptyset$and $B_r(x) \cap \partial\Omega$ is a Lipschitz graph,*

$$\frac{1}{r}(u(x)) > C \Rightarrow u > 0 \text{ in } B_r(x) \cap \Omega.$$

PROOF

Define $D = B_r(x) \cap \Omega$. Let $\Gamma_D = \partial B_r(x) \cap \Omega$ and $\Gamma_N = B_r(x) \cap \partial\Omega$. Then, let $v \in H^1(B_r \cap \Omega)$ be the minimizer of the functional $\int_{B_r \cap \Omega} |\nabla f|^2$ on the set

$$K = \{ f \in H^1(D) | f = u \text{ on } \Gamma_D \}.$$

Then $v$ is harmonic, $v = u \geq 0$ along $\Gamma_D$ and $v$ satisfies:

$$\int_W \nabla v \cdot \nabla \phi = 0$$

for every $\phi \in H^1(B_r \cap \Omega)$ such that $\phi \geq 0$ and $\phi = 0$ on $\Gamma_D$. In this weak sense, $v$ satisfies the Neumann boundary condition $\frac{\partial v}{\partial \nu} = 0$ along $\Gamma_N$.

We can conclude by Lemma 4 that $v \geq 0$ on $B_r \cap \Omega$, and therefore, by the usual strong maximum principle, $v > 0$ on the interior.

Moreover, $v$ is a valid competitor for $u$ as minimizer of $J$, so:

$$\int_{B_r \cap \Omega} (|\nabla v|^2 + Q^2) \geq \int_{B_r \cap \Omega} (|\nabla u|^2 + Q^2 \chi_{\{u>0\}}),$$

which implies that

$$\int_{B_r \cap \Omega} |\nabla (v-u)|^2 \leq \int_{B_r \cap \Omega} Q^2 \chi_{\{u=0\}}. \tag{4.1}$$

Now, we need to obtain an estimate in the opposite direction, namely we want to prove that

$$(\frac{1}{r} u(x))^2 \int_{B_r \cap \Omega} \chi_{\{u=0\}} \leq C \int_{B_r \cap \Omega} |\nabla (v-u)|^2.$$

Comparing this estimate with (4.1) will imply the claim of the lemma.

To prove this, first let $x = 0$. Note also that if we dilate by the formula $u_r(y) = \frac{1}{r} u(ry)$ then everything scales the same way, so we may assume that $r = 1$. So we may assume that we are in $D = B_1(0) \cap \Omega$. In addition, we assume (possibly after a rotation) that $\partial\Omega \cap B_1(0)$ is a Lipschitz graph in the $x_n$-direction, with Lipschitz constant $L$. Then, there exists an $\epsilon(L)$ such that $B_{2\epsilon}(0, 0, \ldots, 0, \frac{1}{2}) \subset D$. Note that $\epsilon \leq \frac{1}{4}$. For each $z \in B_\epsilon((0, 0, \ldots, 0, \frac{1}{2}))$, $D$

is star-shaped with respect to $z$.

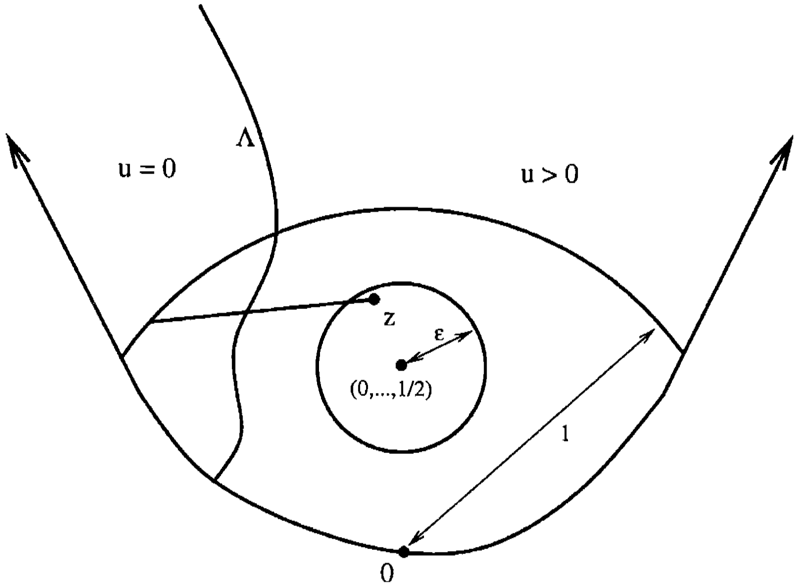


Figure 4-1: The shape of the domain near the intersection of free and fixed boundary

Note that since $0 \in \partial\Omega$ and $\Omega$ is convex, $\partial\Omega \cap B_1(0)$ is simply connected, and $\Omega \cap B_1(0)$ is contained in $\{x \in B_1(0) : x_n > 0\}$. Let $F$ be a bilipschitz map from $D = \Omega \cap B_1(0)$ to $D' = B_1(0) \setminus \overline{D}$, such that $F$ extends continuously to a map from $\overline{D}$ to $\overline{D'}$ with $F\Big|_{\partial\Omega} = \mathrm{Id}$, and $F(\Omega \cap \partial B_1(0)) = (\partial B_1(0)) \setminus \Omega$ . The Lipschitz constants of $F$ and $F^{-1}$ depend only on $L$. Define the function $\tilde{u}$ on $B_1(0)$ by

$$\tilde{u}(x) = \begin{cases} u(x) & \forall x \in \overline{\Omega} \cap B_1(0) \\ u(F^{-1}x) & \forall x \in B_1(0) \setminus \overline{\Omega} \end{cases}$$

and define $\tilde{v}$ similarly.

For every $\xi \in S^{n-1}$, we define

$$R_\xi = \sup\{r \mid r\xi + z \in B_1(0)\}$$
$$r_\xi = \inf\{r \mid \frac{\epsilon}{2} \leq r \leq R_\xi \text{ and } u(r\xi + z) = 0\}$$

If $\{r \mid \frac{\epsilon}{2} \leq r \leq R_\xi \text{ and } u(r\xi+z) = 0\} = \emptyset$, let $r_\xi = R_\xi$. Define $\tau_\xi(t) = z+t\xi$ for $r_\xi \leq t \leq R_\xi$. Note that $\tilde{v}(\tau(R_\xi)) = \tilde{u}(\tau(R_\xi))$ because we are on the boundary of $B_1(0)$: if $\tau(R_\xi) \in$

$\Omega\cap B_1(0)$, then $\tilde{v}(\tau(R_\xi)) = v(\tau(R_\xi)) = u(\tau(R_\xi)) = \tilde{u}(\tau(R_\xi))$ by construction of $v$. Otherwise, $v(\tau(R_\xi)) = v(x)$ for $x = F^{-1}(\tau(R_\xi)) \in \Omega \cap \partial B_1(0)$. Since $v(x) = u(x) = \tilde{u}(F(x))$, we have $\tilde{v}(\tau(R_\xi)) = \tilde{u}(\tau(R_\xi))$ as before. Also note that the path $\tau$ has unit speed at all times, and recall that $\tilde{u}(\tau(r_\xi)) = 0$ . Then

$$
\begin{aligned}
\tilde{v}(r_\xi\xi + z) &= \tilde{v}(r_\xi\xi + z) - \tilde{u}(r_\xi\xi + z) \\
&= \tilde{v}(\tau(r_\xi)) - \tilde{u}(\tau(r_\xi)) \\
&= \tilde{v}(\tau(R_\xi)) - \tilde{u}(\tau(R_\xi)) - \int_{\tau_\xi} \frac{\partial}{\partial t}\left((\tilde{v} - \tilde{u})(\tau_\xi(t))\right) \mathrm{d}t \\
&= 0 + \int_{\tau_\xi} \frac{\partial}{\partial t}\left((\tilde{u} - \tilde{v})(\tau_\xi(t))\right) \mathrm{d}t \\
&\leq \int_{\tau_\xi} |\nabla(\tilde{v} - \tilde{u})| \mathrm{d}t \\
\tilde{v}(r_\xi\xi + z) &\leq \sqrt{|\tau_\xi|}\left(\int_{\tau_\xi} |\nabla(\tilde{v} - \tilde{u})|^2 \mathrm{d}t\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.2}
$$

Now, $|\tau_\xi| = R_\xi - r_\xi$. Define $s_\xi$ to be the unique $s < R_\xi$ such that $\tau(s) \in \partial\Omega$ if such an $s$ exists. Otherwise, let $s_\xi = R_\xi$. Note that, for all $\xi$, $r_\xi \leq s_\xi$. Then

$$
\int_{\tau_\xi} |\nabla(\tilde{v} - \tilde{u})|^2 \mathrm{d}t = \int_{r_\xi}^{s_\xi} |\nabla(\tilde{v} - \tilde{u})|^2 \mathrm{d}t + \int_{s_\xi}^{R_\xi} |\nabla(\tilde{v} - \tilde{u})|^2 \mathrm{d}t.
$$

Now we will estimate $v(r_\xi\xi + z) = \tilde{v}(r_\xi\xi + z)$ from below. We know that $v$ is harmonic on $B_{\frac{1}{4}}(0) \cap \Omega$ and moreover that this domain is far from any Dirichlet boundary pieces of $D = B_1(0) \cap \Omega$ which is the domain of definition of $v$. Hence, by the modified Harnack inequality (Lemma 5), $v(x) \geq cv(0)$ for every $x$ in $B_{\frac{1}{8}}(0)$.[1] Now, let $V = D \setminus B_{\frac{1}{8}}(0)$, and on $V$ define

$$
H(x) = \frac{-cv(0)}{\log 8} \log|x|.
$$

Then, on $V$, the function $v - H$ has the following properties:

[1]In the lemma, the center point of the ball is required to be inside the domain rather than on the boundary, but this is irrelevant. For example, one can consider instead the result of the lemma on $B_{\frac{1}{8}+\epsilon}(0, \ldots, 0, \epsilon)$

$$\int_V \nabla(v-H)\cdot\nabla\phi \geq 0 \qquad \forall \phi \in \{f \in H^1(V) \,|\, f \geq 0 \text{ and } f = 0 \text{ on } (\partial B_{\frac{1}{8}}(0) \cup \partial B_1(0)) \cap \Omega)\}$$

$$v - H \geq 0 \qquad \text{on } \partial B_{\frac{1}{8}}(0) \cap \Omega \text{ and } \partial B_1(0) \cap \Omega$$

The first property holds because $v-H$ is weakly harmonic, and in addition $\frac{\partial H}{\partial \nu} < 0$ by the convexity of $D$, so $\frac{\partial(v-H)}{\partial \nu} \geq 0$ weakly. Hence, by Lemma 4, $v - H \geq 0$ on $V$, so

$$v(x) \geq cv(0)(1-|x|) \qquad \forall x \in B_1(0)^2$$

So,

$$v(r_\xi \xi + z) \geq cv(0)(1 - |r_\xi \xi + z|).$$

But $1 - |r_\xi \xi + z| \geq c(R_\xi - r_\xi)$. To check this we may suppose first that $(1 - |r_\xi \xi + z|) < \frac{1}{4}$, because otherwise, since $R_\xi - r_\xi \leq 2$ we are trivially done.

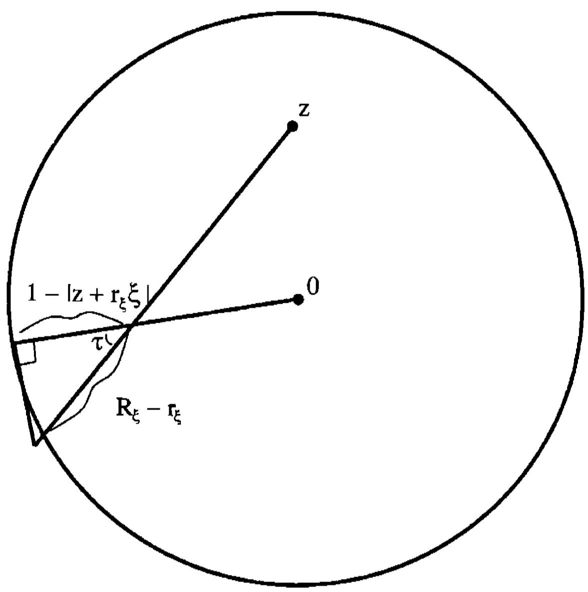


Figure 4-2: Comparision of $R_\xi - r_\xi$ and $1 - |r_\xi \xi + z|$.

Because $|z| \leq \frac{3}{4}$, if $|r_\xi \xi + z| > \frac{3}{4}$, the ray from 0 to the point $r_\xi \xi + z$ and the ray from $z$ to that point must meet at an angle less than $\tau_0 < 90°$. But then, by definition of cosine

[2]Note that, if $|x| \leq \frac{1}{8}$, we already know that $v(x) \geq cv(0) \geq cv(0)(1 - |x|)$.

on the triangle shown in the figure, $R_\xi - r_\xi \leq \frac{1}{\cos(\tau)}(1-|r_\xi\xi+z|) \leq \frac{1}{\cos(\tau_0)}(1-|r_\xi\xi+z|) \leq C(1-|r_\xi\xi+z|)$. Hence,

$$cv(0)(R_\xi - r_\xi) \leq v(r_\xi\xi+z).$$

We conclude, by (4.2), that:

$$cv(0)(R_\xi - r_\xi) \leq C'(R_\xi - r_\xi)^{\frac{1}{2}}\Big(\int_{r_\xi} |\nabla(v-u)|^2 \mathrm{d}t\Big)^{\frac{1}{2}}$$

which implies that

$$C\,v(0)^2(R_\xi - r_\xi) \leq \int_{r_\xi} |\nabla(\tilde{v}-\tilde{u})|^2 \mathrm{d}t \leq \int_{r_\xi}^{s_\xi} |\nabla(v-u)|^2 \mathrm{d}t + \int_{s_\xi}^{R_\xi} |\nabla(\tilde{v}-\tilde{u})|^2 \mathrm{d}t$$

Integrating in $\xi$, we obtain, for the left-hand side:

$$\int_{S^{n-1}} (R_\xi - r_\xi)\mathrm{d}\xi = \int_{S^{n-1}} \int_{r_\xi}^{R_\xi} \mathrm{d}r\mathrm{d}\xi \geq \frac{1}{2^{n-1}} \int_{S^{n-1}} \int_{r_\xi}^{R_\xi} r^{n-1}\mathrm{d}r\mathrm{d}\xi$$
$$\geq \frac{1}{2^{n-1}} \int_{B_1(0)\setminus B_{\frac{\xi}{2}}(z)} \chi_{\{\tilde{u}=0\}}\mathrm{d}x \geq \frac{1}{2^{n-1}} \int_{D\setminus B_{\frac{\xi}{2}}(z)} \chi_{\{u=0\}}\mathrm{d}x.$$

and, for the right-hand side:

$$\int_{S^{n-1}} \int_{r_\xi}^{s_\xi} |\nabla(\tilde{v} - \tilde{u})|^2 \mathrm{d}r\mathrm{d}\xi \leq \frac{2^{n-1}}{\epsilon(L)^{n-1}} \int_{S^1} \int_{r_\xi}^{s_\xi} |\nabla(v-u)(r\xi + z)|^2 r^{n-1} \mathrm{d}r\mathrm{d}\xi$$
$$\leq \frac{2^{n-1}}{\epsilon(L)^{n-1}} \int_D |\nabla(v-u)|^2 \mathrm{d}x$$

$$\int_{S^{n-1}} \int_{s_\xi}^{R_\xi} |\nabla(\tilde{v} - \tilde{u})|^2 \mathrm{d}y\mathrm{d}\xi \leq \frac{2^{n-1}}{\epsilon(L)^{n-1}} \int_{S^1} \int_{s_\xi}^{R_\xi} |\nabla(\tilde{v} - \tilde{u})|^2 r \mathrm{d}r\mathrm{d}\xi$$
$$\leq \frac{2^{n-1}}{\epsilon(L)^{n-1}} \int_{D'} |\nabla(\tilde{v} - \tilde{u})|^2 \mathrm{d}x$$
$$\leq \frac{2^{n-1}}{\epsilon(L)^{n-1}} \int_D |\nabla F|^2 |\nabla(v-u)|^2 |\mathrm{Det} F^{-1}| \mathrm{d}x$$
$$\leq C(L) \int_D |\nabla(v-u)|^2 \mathrm{d}x$$

We combine to find that:

$$v(0)^2 \int_{D \setminus B_{\frac{\xi}{2}}(z)} \chi_{\{u=0\}} \leq C \int_D |\nabla(v-u)|^2.$$

Finally. we integrate over $z \in B_{\epsilon(L)}((0, \ldots, 0, \frac{1}{2}))$ to conclude that

$$v(0)^2 \int_D \chi_{\{u=0\}} \leq C(L) \int_D |\nabla(v-u)|^2$$

which, when we combine with equation (4.1), yields:

$$v(0)^2 \int_D \chi_{\{u=0\}} \leq C(n, L, M) \int_D \chi_{\{u=0\}}.$$

and we conclude that if $v(0) > \sqrt{C(n, L, M)}$ then $\{u = 0\}$ has measure zero in $D$. But then Equation (4.1) implies that $u$ is identical with $v$, i.e. $u$ is a positive harmonic function in $D$, so $u$ is strictly positive in $D$. Recall that $u(0) \leq v(0)$, so we may conclude that, if $\{u = 0\} \cap B_r \neq \emptyset$, then $u(0) \leq C$. □

The next step is to check the Lipschitz gradient bound on the fixed boundary, near the free boundary. The two dimensional case will again use different methods than the case of general dimension. The two dimensional lemma employs a monotonicity formula to obtain the gradient control, whereas the higher dimensional lemma depends upon the gradient estimate in Lemma 8.

**Lemma 10** *Let $\Omega \subset R^2$ be a bounded, convex domain. Let $r_0 > 0$. Let $x \in \partial\Omega$ with* $\text{dist}(x, \Lambda) < r_0$ *where $\Lambda$ is the free boundary, and $d(x, S) \geq r_0$. Then*

$$|\nabla u(x)| \leq C.$$

PROOF

The proof will require several steps:

1. For each $x$ in $\partial\Omega$, for any $r > 0$,

$$|\nabla u(x)|^2 \leq \frac{C}{r^2} \int_{B_r(x)\cap\Omega} |\nabla u|^2. \tag{4.3}$$

PROOF

Define

$$\phi(r) = \frac{1}{r^2} \int_{B_r(x)\cap\Omega} |\nabla u|^2.$$

and let

$$c(r) = \fint_{S_r(x)\cap\Omega} u.$$

Then,

$$\begin{aligned}
\phi'(r) &= \frac{1}{r^2}\int\limits_{S_r(x)\cap\Omega}|\nabla u|^2 - \frac{2}{r^3}\int\limits_{B_r(x)\cap\Omega}|\nabla u|^2 \\
&= \frac{1}{r^2}\int\limits_{S_r(x)\cap\Omega}|\nabla(u-c(r))|^2 - \frac{2}{r^3}\int\limits_{B_r(x)\cap\Omega}|\nabla(u-c(r))|^2 \\
&= \frac{1}{r^2}\int\limits_{S_r(x)\cap\Omega}|\nabla(u-c(r))|^2 - \frac{2}{r^3}\Bigg(-\int\limits_{B_r(x)\cap\Omega}(u-c(r))\Delta(u-c(r)) \quad + \\
&\quad + \int\limits_{S_r(x)\cap\Omega}(u-c(r))\frac{\partial}{\partial\nu}(u-c(r)) \quad + \int\limits_{\partial\Omega\cap B_r(x))}(u-c(r))\frac{\partial}{\partial\nu}(u-c(r))\Bigg) \\
&= \int\limits_{S_r(x)\cap\Omega}\left(\frac{1}{r^2}|\nabla(u-c(r)|^2 - \frac{2}{r^3}(u-c(r))\frac{\partial}{\partial\nu}(u-c(r))\right) \\
&= \frac{1}{r^2}\int\limits_{S_r(x)\cap\Omega)}\left(\left(\frac{\partial}{\partial\theta}(u-c(r))\right)^2 + \left(\frac{\partial}{\partial r}(u-c(r))\right)^2 - 2\frac{u-c(r)}{r}\frac{\partial}{\partial r}(u-c(r))\right) \\
&= \frac{1}{r^2}\int\limits_{S_r(x)\cap\Omega)}\left(\left(\frac{\partial}{\partial\theta}(u-c(r))\right)^2 - \left(\frac{u-c(r)}{r}\right)^2 + \left(\frac{\partial}{\partial r}(u-c(r)) - \frac{u-c(r)}{r}\right)^2\right) \\
&\geq \frac{1}{r^2}\int\limits_{S_r(x)\cap\Omega)}\left(\frac{\partial}{\partial\theta}(u-c(r))\right)^2 - \left(\frac{u-c(r)}{r}\right)^2.
\end{aligned}$$

But, since $\Omega$ is convex, for $x \in \partial\Omega$ the arc $S_r \cap \Omega$ has length at most $\pi r$, which implies that the first Neumann eigenvalue of the laplacian on this arc is at least $\frac{1}{r}$. In addition, the function $(u-c(r))|_{S_r\cap\Omega}$ is a valid competitor for the Rayleigh quotient corresponding to this eigenvalue because its average value is zero on the arc, and for almost every $r$,$(u-c(r))|_{S_r\cap\Omega}$ is in $H^1$ of the arc by Fubini's theorem. So we may conclude that

$$\int\limits_{S_r\cap\Omega}\left(\frac{\partial}{\partial\theta}(u-c(r))\right)^2 \geq \frac{1}{r^2}\int\limits_{S_r\cap\Omega}(u-c(r))^2$$

and hence $\phi'(r) \geq 0$. Moreover, for almost every choice of center point $x$,

$$\lim_{r \to 0} \phi(r) = \lim_{r \to 0} \frac{1}{r^2} \int_{B_r(x) \cap \Omega} |\nabla u|^2 = C\,|\nabla u(x)|^2$$

where here $C$ is the inverse of the limiting size of $\Omega$ in $B_r$ as $r$ goes to zero, which is bounded above because $\Omega$ is a Lipschitz domain. Hence, for almost every $x$ in $\partial\Omega$, for positive $r$, we conclude that

$$|\nabla u(x)|^2 \leq \frac{C}{r^2} \int_{B_r(x) \cap \Omega} |\nabla u|^2$$

as desired. □

2. Suppose $u$ is a positive harmonic function in $B_r(x) \cap \Omega$. Then

$$\int_{B_{\frac{r}{4}}(x) \cap \Omega} |\nabla u|^2 \leq \frac{C}{r^2} \int_{B_{\frac{r}{2}}(x) \cap \Omega} u^2. \tag{4.4}$$

Proof Let $\phi$ be a smooth cutoff function on the unit ball, i.e. $\phi \equiv 1$ on $B_{\frac{1}{2}}$ and $\phi \in C_c^\infty(B_1)$, and let $\phi_r = \phi(\frac{x}{r})$. Then,

$$\begin{aligned} \int_{B_{\frac{r}{4}}(x) \cap \Omega} |\nabla u|^2 \mathrm{d}y &\leq \int_{B_{\frac{r}{2}}(x) \cap \Omega} |\nabla u|^2 \; \phi_{\frac{r}{2}}^2(y - x) \mathrm{d}y \\ &= \int_{B_{\frac{r}{2}}(x) \cap \Omega} (\nabla \cdot (\phi_{\frac{r}{2}}^2 \, \nabla u)) \; u \mathrm{d}y \end{aligned}$$

by integration by parts. Note that all the boundary terms are zero, because on each boundary either $\phi = 0$ or $\frac{\partial u}{\partial \nu} = 0$. Moreover, since u is harmonic,

$$\int_{B_{\frac{r}{2}}(x)\cap\Omega} |\nabla u|^2 \, \phi^2_{\frac{r}{2}}(y-x)\mathrm{d}y = -2\int_{B_{\frac{r}{2}}(x)\cap\Omega} \phi \, u \, (\nabla\phi_{\frac{r}{2}} \cdot \nabla u)\mathrm{d}y$$

$$\leq 2\left(\int_{B_{\frac{r}{2}}(x)\cap\Omega} \phi^2 \, |\nabla u|^2 \mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{B_{\frac{r}{2}}(x)\cap\Omega} u^2 \, |\nabla\phi_{\frac{r}{2}}(y-x)|^2 \mathrm{d}y\right)^{\frac{1}{2}}.$$

Dividing through by the first term, and recalling that $\nabla\phi_{\frac{r}{2}}(y-x) = \nabla\phi(\frac{2(y-x)}{r}) = \frac{2}{r}\nabla\phi \leq \frac{C}{r}$, we obtain

$$\int_{B_{\frac{r}{4}}(x)\cap\Omega} |\nabla u|^2 \leq \int_{B_{\frac{r}{2}}(x)\cap\Omega} |\nabla u|^2 \phi^2 \leq \frac{C}{r^2} \int_{B_{\frac{r}{2}}(x)\cap\Omega} u^2$$

which concludes the proof. □

We will now complete the proof of the lemma. Let $x \in \Gamma$, the Neumann part of the boundary of $\Omega$. Let

$$r_x = \inf\{r | B_r(x) \cap \{u = 0\} \neq \emptyset\} < r_0.$$

By (4.3), we know that, for almost every such $x$,

$$|\nabla u(x)| \leq C \left( \fint_{B_{\frac{r}{4}}(x)\cap\Omega} |\nabla u|^2 \right)^{\frac{1}{2}}$$

and, by (4.4) we know that

$$C\left( \fint_{B_{\frac{r}{4}}(x)\cap\Omega} |\nabla u|^2 \right)^{\frac{1}{2}} \leq \frac{C}{r_x} \left( \fint_{B_{\frac{r}{2}}(x)\cap\Omega} u^2 \right)^{\frac{1}{2}}$$

$$\leq \frac{C}{r_x} \sup_{B_{\frac{r}{2}}(x)\cap\Omega} u.$$

Now, $u$ is positive in $B_{r_x}(x) \cap \Omega$, and is hence also harmonic there. We can therefore apply our modified Harnack principle (Lemma 5) to conclude that

$$\sup_{B_{\frac{r}{2}}(x)\cap\Omega} u \leq C \inf_{B_{\frac{r}{2}}(x)\cap\Omega} u \leq C\, u(x).$$

Putting everything together, we find that

$$|\nabla u(x)| \leq \frac{C}{r_x} u(x).$$

Finally, by Lemma 9, for any $\delta > 0$,

$$\frac{1}{r_x + \delta} u(x) \leq \tilde{C}$$

because $\{u = 0\} \cap B_{r_x+\delta}(x)$ has positive measure. Allowing $\delta$ to approach zero, we conclude that

$$\frac{1}{r_x} u(x) \leq \tilde{C}$$

and we can finally conclude that, for almost every $x$ in $\Gamma$,

$$|\nabla u| \leq \frac{C}{r_x} u(x) \leq C.$$

□

We now provide the corresponding result for higher dimensions:

**Lemma 11** *Let $\Omega \subset R^n$ be a bounded, convex domain. Let $r_0 > 0$. Let $x \in \partial\Omega$ with $d(x, \Lambda) < r_0$ where $\Lambda$ is the free boundary, and $d(x, S) \geq r_0$. Then*

$$|\nabla u(x)| \leq C.$$

Proof Define $r_x = \inf\{r > 0 : B_r(x) \cap \{u = 0\} \neq \emptyset\} < r_0$. Note that $u$ is a positive harmonic function on $B_{r_x}(x) \cap \Omega$. Therefore, by Lemma 5, there is a $C > 0$ such that $\sup_{B_{\frac{r_x}{2}}}(x) \leq Cu(x)$.

Now, for any $\delta > 0$, $B_{r_x+\delta}(x) \cap \{u = 0\}$ has positive measure. So by Lemma 9

$$u(x) \leq C(n, L, M)\frac{1}{r_x + \delta}.$$

Therefore,

$$u(x) \leq \frac{C(n, L, M)}{r_x}.$$

So, $\forall y \in B_{\frac{r_x}{2}}$,

$$u(y) \leq C(n, L, M) r_x.$$

Now, $u$ is again a positive harmonic function on $B_{\frac{r_x}{2}} \cap \Omega$. Moreover, $0 \leq u \leq C(n, L, M)\, r_x$ on $B_{\frac{r_x}{2}} \cap \Omega$. We may therefore apply Lemma 8, with $A = C(n, L, M) r_x$ to conclude that

$$|\nabla u(x)| \leq C\frac{C(n, L, M) r_x}{r_x} \leq C(n, L, M).$$

□

Finally, we come to the main result. I will here combine the two separate cases, $n = 2$ and $n > 2$, as the method is the same and the differences occur only in which lemmas are referred to.

**Theorem 2** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $S$ be a closed subset of $\partial\Omega$ and let $\Gamma = \partial\Omega \setminus S$. Suppose $\partial\Omega$ is convex in a neighborhood of $\overline{\Gamma}$. Let $r_0 > 0$. Then there is a constant $C(n, L, r_0, M, A)$ such that for almost every $x \in \Omega_{r_0}$, $|\nabla u(x)| \leq C$.*

PROOF

Let $U = \{\, x \mid u(x) > 0 \,\}$ and let $\Lambda = \Omega \cap \partial U$ be the free boundary. Let $x \in \Omega_{r_0}$. There are five cases (These cases may partially overlap.):

1. $x \in (\tilde{\Omega} \setminus U)$. $|\nabla u| = 0$ for almost every such $x$.[3]

2. $x \in U$ and $d(x, \partial U) \geq 1$. Then, by the interior regularity of harmonic functions on $B_1(x)$,

$$|\nabla u(x)| \leq \sup_{\Omega} u \leq \sup_{S} u = A.$$

[3] $\Lambda$ itself has measure zero, so we do not need to consider $|\nabla u|$ there.

3. $x \in U$ and $d(x, \partial\Omega) > d(x, \Lambda)$. Then, let $r = d(x, \Lambda)$. As in ([2], Corollary 3.3), we can conclude that

$$|\nabla u(x)| \leq \frac{C}{r} \fint_{\partial B_r(x))} u \leq C(M).$$

4. $x \in U$ and $d(x, \Lambda) \geq 1$. Let $r = \min(r_0, 1)$. Then $u$ is a positive, harmonic function on $B_r(x) \cap \Omega$, bounded by $A$. Therefore, by Lemma 6 or Lemma 8 (depending on dimension), $|\nabla u(x)| \leq C(n, L)\frac{A}{r_0}$.

5. Finally, consider $x \in U \cap \Omega_{r_0}$ such that $d(x, \Gamma) \leq d(x, \Lambda) \leq 1$.

   Recall from Case 3, that, for every $x \in U$ such that $\operatorname{dist}(x, \Lambda) < \operatorname{dist}(x, \partial\Omega)$, $|\nabla u| \leq C(n, M)$. Therefore, we can create a set $U' \subset U$ such that $|\nabla u| \leq C(n, M)$ on $(\partial U') \cap \Omega$ and $\partial U'$ is Lipschitz.

   Since $u$ is in $H^1$, $\nabla u$ is in $L^2$. Therefore, by Fubini's Theorem, there exists a radius $r$ such that $\frac{3}{4}r_0 \leq r \leq r_0$ and $\nabla u \in L^2(\partial B_r(x) \cap U)$ with $\|\nabla u\|_{L^2(\partial B_r(x))} \leq \|u\|_{H^1}$.

   Consider, $D = B_r(x) \cap U'$. Then $u$ is a positive harmonic function on $D$. Moreover, $\partial D$ has three parts: $\Gamma_1 = B_r \cap \Gamma$, $\Gamma_2 = B_r \cap \Omega \cap \partial U'$, and $\Gamma_3 = U' \cap \partial B_r$. These parts may not each be connected, but they are disjoint and the union of their closures is the entire boundary of $\partial D$.

   Note that $\Gamma_1$ is a convex Lipschitz hypersurface and, by Lemma 10 or Lemma 11 depending on dimension, $|\nabla u| \leq C$ on $\Gamma_1$. Note also that $\Gamma_2$ is a Lipschitz curve on which $|\nabla u| \leq C$ by construction. Finally, $\Gamma_3$ is a smooth curve.

   We define the function $v$ on $B_r(x)$ by:

$$\begin{aligned} \Delta v &= 0 && \text{on } B_r(x) \\ v &= C^2 + (|\nabla u|)^2 \chi_{\Gamma_3} && \text{on } \partial B_r(x). \end{aligned}$$

   Then, on $\Gamma_1$ and $\Gamma_2$, $v \geq C^2 \geq |\nabla u|^2$, and on $\Gamma_3$, $v = C^2 + |\nabla u|^2 > |\nabla u|^2$. Hence, since $|\nabla u|^2$ is subharmonic, by the maximum principle $v \geq |\nabla u|^2$ on $D$. So $v(x) > |\nabla u(x)|^2$.

   But, using the Poisson kernel, we find that

$$
\begin{aligned}
v(x) &= \int\limits_{\partial B_r(x)} P * v(y)\, \mathrm{d}\sigma(y) \\
&= \fint\limits_{\partial B_r(x)} v(y) \mathrm{d}\sigma(y) \\
&= C^2 + \frac{1}{\omega_n r^{n-1}} \left( \int\limits_{\partial B_r(x)} v(y)\ \chi_{\Gamma_3}(y) \mathrm{d}\sigma(y) \right) \\
&\leq C^2 + \frac{1}{\omega_n r^{n-1}}\ \|v\|_{L^2} |\Gamma_3|^{\frac{1}{2}} \\
&\leq C^2 + \frac{1}{\omega_n r^{n-1}} \|\nabla u\|_{L^2(\Omega)} |\Gamma_3|^{\frac{1}{2}} \\
&\leq C^2(n, L, M, r_0, A)
\end{aligned}
$$

And we conclude that $|\nabla u(x)| \leq C(n, L, M, r_0, A)$.

So, we can finally conclude that for almost every $x \in \Omega_{r_0}$, $|\nabla u(x)| \leq C(r_0, A, M, L)$, and, hence $u \in C^{0,1}(\Omega_{r_0})$. □

# Chapter 5

# Conclusion

There is a variety of further work which remains to be done on this problem, now that the Lipschitz bound is known to hold up to the boundary. We briefly outline a list of future questions which we plan to address soon:

1. The Two-Phase Problem:

   One can also ask for a Lipschitz bound in the two-phase case, the problem described by (1.3) and addressed in [4]. That is, suppose that $u_0$ is not required to be nonnegative. Then $u$ will satisfy $\Delta u = 0$ in both $\{u > 0\}$ and $\{u < 0\}$. Along the free boundary $\partial\{u > 0\}$, the condition will be $|\nabla u_+(x)|^2 - |\nabla u_-(x)|^2 = Q^2(x)$. In this case, one wants to prove the same result as in Chapter 4, i.e.

   $$|\nabla u| \leq C$$

   for some $C$ depending only on the given constants.[1]

   We believe this bound to be correct. At the intersection of free boundary and fixed boundary in the two-phase case, one can repeat the monotonicity formula calculation of Lemma 10 and it works, without regard to dimension, up to the point where one compares the functions to eigenfunctions of the spherical laplacian. We have two subdomains of the sphere whose union is geodesically convex. Let $\lambda_+$ be the first

[1] In this context, note that since $u \in H^1$, $\nabla u = \nabla u_+ \chi_{\{u>0\}} + \nabla u_- \chi_{\{u<0\}}$, so it suffices to check the Lipschitz bound for each of $u_+$ and $u_-$ separately.

eigenvalue of the laplacian on one of the subdomains, with appropriate mixed boundary conditions, and $\lambda_-$ is the corresponding eigenvalue on the other subdomain. The boundary conditions required are Neumann on the restriction of the fixed boundary to the sphere, and Dirichlet on the restriction of the free boundary, which cuts the domain into the two subdomains. Then, the proof of the monotonicity formula case reduces to proving the Friedland-Hayman inequality

$$\alpha_+ + \alpha_- \geq 2$$

for this case, where $\alpha_+(\alpha_+ + n - 2) = \lambda_+$ and $\alpha_-(\alpha_- + n - 2) = \lambda_-$. We conjecture the desired bound to be correct, however it is unproven at this time. Once this monotonicity formula is proven, it should be straighforward to complete the proof of Lipschitz continuity for the two-phase case. In particular, since the Friedland-Hayman inequality clearly holds for this case in dimension 2, we should be able to conclude Lipschitz continuity there.

2. The Shape of the Free Boundary:

   Once we have the Lipschitz bound, the natural next question (in both the one- and two-phase cases) is what the shape of the free boundary is as it comes into the fixed boundary. In both cases, the free boundary is known to be smooth up to a set of codimension 3 on the interior of the domain. Does it hit the fixed boundary smoothly? Does it have to come in at a particular angle? Can it hit non-smooth boundary points, and if so is there any restriction on which such points it can hit?

   Some preliminary calculations suggest that in two dimensions the free boundary cannot hit the fixed boundary at a corner of less than $90°$. We do not know whether it can hit an obtuse angle, or what the picture might be like in higher dimensions.

3. Stability with Respect to Boundary Data:

   The question of how the free boundary behaves near corners in the fixed boundary leads naturally to the question of stability with respect to the boundary data. That is, suppose that the boundary data is such that the free boundary is forced to land near but not at such a corner. Suppose it is then perturbed to move the free boundary towards the corner. Will the free boundary move continuously? Will it jump across the corner? If we conclude that the free boundary cannot hit narrow angles, then it follows that there must be a jump in this flow. How close to the corner can it get?

4. Non-Minimizing Solutions:

   A related question is whether, if energy minimizers cannot hit the corner, does that mean there are no solutions that hit the corner, or only that such solutions have a larger, or even infinite, energy. In general, we would like to know more about what non-minimizing solutions look like near a rough boundary, although we do not know what methods we might use to study such a question.

5. Other Elliptic Operators:

   We would finally like to generalize our result to other elliptic operators. We do not know what is the most general type of operator to which it should apply, but it should at least apply to other smooth elliptic operators besides the laplacian. If indeed this result also holds for other smooth elliptic operators, then it would most likely follow that it also holds on domains with an exterior ball condition rather than strict convexity. We are also interested in other boundary conditions, and what type of regularity might hold in more complex boundary situations.

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