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SAMPLING THEOREMS FOR 2-D ISOTROPIC RANDOM FIELDS

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Abstract

In this note we develop new sampling theorems for isotropic random fields and their associated Fourier coefficient processes. We consider a wave-number-limited isotropic random field $z(\vec{r})$ whose spectral density function is zero outside a disk of radius B centered at the origin of the wave-number plane. It is shown that $z(\vec{r})$ can be reconstructed in the mean-square sense from its observation on the countable number of circles of radii $r_i = i\pi/B$, $i \in \mathbf{N}$, or of radii $r_i = a_{i,n}/B$, $i \in \mathbf{N}$, where $a_{i,n}$ denotes the ith zero of the nth order Bessel function $J_n(x)$, and where n is arbitrary.

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1 INTRODUCTION

Spatially-distributed random processes arise in various fields including image processing, meteorology, geophysical signal processing, oceanography and optical processing. Since it is not possible to obtain observations at every point in space, one is led in practice to deal with sampled versions of these processes. In [1], Petersen and Middleton extended the 1-D Shannon sampling theorem to m-dimensional Euclidean spaces. In particular, they developed efficient *point* sampling and reconstruction techniques for wave-number limited homogeneous random fields which minimize the number of sample points required per unit area to reconstruct a given field in the sense of a vanishing mean-square error.

Here, by contrast, we present a new reconstruction procedure for 2-D wave-number-limited *isotropic* random fields sampled along *circles* in the Euclidean plane, where the reconstruction is to be understood in a mean-square sense. Isotropic fields are characterized by the fact that their mean value is a constant independent of position and their autocovariance function is invariant under all rigid body motions, i.e. under translations and rotations. As will be shown in Section 2, the invariance of isotropic covariance functions under all rigid body motions implies that the power spectra of such fields are circularly symmetric in the wave number plane and some authors, have used this latter fact to define the notion of isotropy (e.g. [1].) In some sense, isotropy is the natural extension of the notion of stationarity in one dimension. Furthermore, isotropic random fields arise in a number of practical problems such as the black body radiation problem [2], the study of underwater ambient noise in horizontal planes parallel to the surface of the ocean [3], and the investigation of temperature and pressure distributions at constant altitude in the atmosphere [4]. The importance of the sampling techniques that we develop stems from the fact that in some applications such as the mapping of the gravitational fields of planets using orbiting satellites [5], sampling along circles is more natural than sampling at discrete points on a fixed lattice. Furthermore, these new sampling schemes have been found to be useful in developing high resolution spectral estimation techniques for isotropic random fields [6]. Finally in practice, as will be explained later, our sampling techniques can be implemented with a small mean-square error by sampling the isotropic fields at a discrete set of points along circles rather than continuously along each circle.

An important property of isotropic fields is that when they are expanded in a Fourier series in terms of the polar coordinate angle θ, the Fourier coefficient processes of different orders are uncorrelated [7]. We consider here a wave-number limited isotropic random field $z(\vec{r})$ [1], whose spectral density function is zero outside a disk of radius B centered at the origin of the wave-number plane. By developing sampling and reconstruction techniques for the Fourier coefficient processes $z_k(r)$ associated to $z(\vec{r})$, we prove that the process $z(\vec{r})$ can be reconstructed in a mean-square sense from its observation on the countable set of circles of radii $r_i = i\pi/B$, $i \in \mathbf{N}$, or of radii $r_i = a_{i,n}/B$, $i \in \mathbf{N}$, where $a_{i,n}$ is the ith zero of the nth order Bessel function $J_n(x)$, and where the index n of $J_n(x)$ can be selected arbitrarily.

Section 2 presents some standard properties of the Fourier expansions of isotropic random fields. The main results of this note are described in Section 3 where we develop two procedures for sampling wave-number limited isotropic random fields. Finally, Section 4 contains some conclusions.

[1]Throughout this paper we use $\vec{r}$ to denote a point in 2-D Cartesian space. The polar coordinates of this point are denoted by r and θ.

2 FOURIER SERIES FOR ISOTROPIC FIELDS

The covariance function

$$K(\vec{r}) = E[z(\vec{v})z(\vec{v}+\vec{r})] \tag{2.1}$$

of any zero-mean isotropic random field $z(\vec{r})$ is a function of r only, so that, by abuse of notation we can write

$$K(\vec{r}) = K(r). \tag{2.2}$$

Such a field can be expanded into a Fourier series of the form [7],[8]

$$z(\vec{r}) = \sum_{n=-\infty}^{\infty} z_n(r)\, e^{jn\theta}, \tag{2.3}$$

$$z_n(r) = \frac{1}{2\pi}\int_0^{2\pi} z(\vec{r})e^{-jn\theta}\, d\theta, \tag{2.4}$$

where the Fourier coefficient processes of different orders are uncorrelated, i.e.

$$E[z_n(r)z_m(s)] = 0, \tag{2.5}$$

for $n \neq m$. If we assume that $K(r)$ has a Hankel transform [9], i.e. if $K(r) \in L_1(r\,dr)$, then it can be shown that the covariance function $k_n(r,s)$ of the nth order Fourier coefficient $z_n(r)$ is given by [7], [8]

$$\begin{aligned} k_n(r,s) &= E[z_n(r)z_n(s)] \\ &= \int_0^{\infty} J_n(\lambda r)J_n(\lambda s)S(\lambda)\lambda\, d\lambda. \end{aligned} \tag{2.6}$$

In (2.6) $J_n(\cdot)$ is the Bessel function of order n and $S(\lambda)$ is the power spectrum associated with $z(\vec{r})$, i.e.

$$\begin{aligned} S(\vec{\lambda}) &= \int_{\mathbf{R}^2} K(\vec{r})e^{-j\vec{\lambda}\cdot\vec{r}}\, d\vec{r} \\ &= 2\pi\int_0^{\infty} K(r)J_0(\lambda r)r\, dr \\ &= S(\lambda), \end{aligned} \tag{2.7}$$

where $\lambda = |\vec{\lambda}|$ is the magnitude of the wave vector $\vec{\lambda}$, and where we have taken advantage of the circular symmetry of $K(\vec{r})$. Note that (2.6) implies that $S(\lambda)$ can be recovered from $k_n(r,s)$ for an arbitrary value of n by taking the nth order Hankel transform [9] of $k_n(r,s)$ with respect to the variable r and dividing by $J_n(\lambda s)/2\pi$. This fact will be useful in explaining the results presented in the next section. We now turn our attention to the issue of sampling the Fourier coefficient processes of different orders.

3 SAMPLING THEOREMS

In this section, we shall develop two different procedures for sampling and reconstructing the Fourier coefficient processes associated with a given isotropic random field. Using the sampling theorems for the Fourier processes, we show that a wave-number limited isotropic random field can be reconstructed from its observations on a countably infinite number of concentric circles with a vanishing mean-square error.

A Sampling the covariance functions of the Fourier processes

Let us begin by presenting two different sampling procedures for the covariance function of the nth order Fourier coefficient process.

Theorem 1 *The* nth *order Fourier coefficient process covariance function $k_n(r,s)$ of an isotropic random field $z(\vec{r})$ whose spectral density function $S(\lambda)$ is wave-number limited to the region $\lambda < B$, can be reconstructed exactly from the sample values of the* mth *order Fourier coefficient process covariance function $k_m(r,s)$ taken over a lattice of points $\{(a_{i,m}/B, a_{j,m}/B);\ i,j \in \mathbf{N}\}$, where $a_{i,m}$ is the ith zero of the mth order Bessel function $J_m(x)$.*

Proof

Over the interval $0 < \lambda < B$, $J_n(\lambda r)$ can be expanded into a Fourier-Bessel series of the form [10]

$$J_n(\lambda r) = \sum_{i=1}^{\infty} b_{i,m}^n(r) J_m(\frac{a_{i,m}\lambda}{B}),\ \ 0 < \lambda < B \tag{3.1}$$

where

$$b_{i,m}^n(r) = \frac{2\int_0^B J_n(\lambda r) J_m(\frac{a_{i,m}\lambda}{B})\lambda\, d\lambda}{B^2 J_{m+1}(a_{i,m}) J_{m-1}(a_{i,m})}. \tag{3.2}$$

Substituting (3.1) for $J_n(\lambda r)$ and $J_n(\lambda s)$ into (2.6) yields the desired result

$$k_n(r,s) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b_{i,m}^n(r) b_{j,m}^n(s) k_m(\frac{a_{i,m}}{B}, \frac{a_{j,m}}{B}). \tag{3.3}$$

Note that, according to Theorem 1, it is possible to reconstruct the covariance functions of the Fourier coefficient processes of *all* orders given sampled values of the covariance function of a *single* Fourier coefficient process of *any* order. This should not come as a surprise: if one can reconstruct $k_m(r,s)$ exactly from its sample values on the grid $\{(\frac{a_{i,m}}{B}, \frac{a_{j,m}}{B});\ i,j \in \mathrm{N}\}$ then, as mentioned earlier, one can easily compute $S(\lambda)$. Given $S(\lambda)$, one can then evaluate $k_n(r,s)$ for all n through (2.6). We now state and prove a second sampling theorem for the covariance function $k_n(r,s)$ of the nth order Fourier coefficient process. In this case we use samples of $k_n(r,s)$, rather than of $k_m(r,s)$.

Theorem 2 *The nth order Fourier coefficient process covariance function $k_n(r,s)$ of an isotropic random field $z(\vec{r})$ whose spectral density function $S(\lambda)$ is wave-number limited to the region $\lambda < B$, can be reconstructed exactly from its* own *sample values taken over a lattice of points $\{i\pi/B, j\pi/B);\ i,j \in \mathrm{N}\}$, or over a lattice of points $\{(a_{i,m}/B, a_{j,m}/B);\ i,j \in \mathrm{N}\}$, where $a_{i,m}$ is the ith zero of the mth order Bessel function $J_m(x)$.*

Proof

Consider the identity (see the Appendix)

$$J_n(\lambda r) = \sum_{i=0}^{\infty} d_i^n(r) J_n(\frac{i\pi}{B}\lambda), \ 0 < \lambda < B \tag{3.4}$$

where

$$d_i^n(r) = \frac{1}{(1+\delta_{0,n})}((-1)^n \frac{sin(B(r+\frac{i\pi}{B}))}{B(r+\frac{i\pi}{B})} + \frac{sin(B(r-\frac{i\pi}{B}))}{B(r-\frac{i\pi}{B})}), \tag{3.5}$$

and where $\delta_{0,n}$ denotes the Kronecker delta function, i.e. $\delta_{0,n} = 1$ if $n = 0$ and $\delta_{0,n} = 0$ otherwise. Substituting (3.4) into (2.6) yields

$$k_n(r,s) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d_i^n(r) d_j^n(s) k_n(\frac{i\pi}{B}, \frac{j\pi}{B}). \tag{3.6}$$

Similarly, by substituting the identity (see the Appendix)

$$J_n(\lambda r) = \sum_{i=1}^{\infty} c_{i,m}^n(r) J_n(\frac{a_{i,m}}{B}\lambda), \ 0 < \lambda < B \tag{3.7}$$

where

$$c_{i,m}^{n}(r) = \frac{2a_{i,m}J_m(Br)}{(a_{i,m}^2 - B^2r^2)J_{m+1}(a_{i,m})}(\frac{a_{i,m}}{r})^{|m-n|}, \tag{3.8}$$

into (2.6) we obtain

$$k_n(r,s) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,m}^{n}(r)c_{j,m}^{n}(s)k_n(\frac{a_{i,m}}{B}, \frac{a_{j,m}}{B}). \tag{3.9}$$

□ □

Observe that Theorem 2 asserts that the same sampling grid can be used for *all* of the Fourier coefficient process covariance functions. The sampling grid can be selected to be $\{(i\pi/B, j\pi/B)\}$ or $\{(a_{i,m}/B, a_{j,m}/B)\}$ where m is fixed but arbitrary. This fact will prove useful in deriving sampling theorems for isotropic random fields.

B Sampling Isotropic Fields

Theorem 2 can now be used to prove the following important result.

Theorem 3 *The nth order Fourier coefficient process $z_n(r)$ corresponding to an isotropic random field $z(\vec{r})$ with a wave-number limited spectrum $S(\lambda)$, where $S(\lambda) = 0$ for $\lambda > B$, can be reconstructed with zero mean-square error from its samples $\{z_n(\frac{a_{i,m}}{B});\ \ i \in \mathbf{N}\}$, where $a_{i,m}$ is the ith zero of the mth order Bessel function $J_m(x)$ as,*

$$z_n(r) = \sum_{i=1}^{\infty} c_{i,m}^{n}(r)z_n(\frac{a_{i,m}}{B}), \tag{3.10}$$

and from its samples $\{z_n(\frac{i\pi}{B});\ \ i \in \mathbf{N}\}$ as,

$$z_n(r) = \sum_{i=0}^{\infty} d_i^n(r)z_n(\frac{i\pi}{B}), \tag{3.11}$$

where (3.10) and (3.11) hold in a mean-square sense, and where $d_i^n(r)$ and $c_{i,m}^{n}(r)$ are defined respectively by (3.5) and (3.8).

Proof

To show that the series on the right hand side of (3.10) converges in the mean to $z_n(r)$ let us denote by $\hat{z}^B_{n,N}(r)$ the series

$$\hat{z}^B_{n,N}(r) = \sum_{i=1}^{N} c^n_{i,m}(r) z_n(\frac{a_{i,m}}{B}). \tag{3.12}$$

From (3.1) and (2.6), it can be shown that

$$k_n(r,s) = \sum_{i=1}^{\infty} c^n_{i,m}(s) k_n(r, \frac{a_{i,m}}{B}), \tag{3.13}$$

and that

$$k_n(\frac{a_{i,m}}{B}, r) = \sum_{j=1}^{\infty} c^n_{j,m}(r) k_n(\frac{a_{i,m}}{B}, \frac{a_{j,m}}{B}). \tag{3.14}$$

Using the above two equations we obtain

$$\begin{aligned} \lim_{N\to\infty} E[z_n(r)(z_n(r) - \hat{z}^B_{n,N}(r))] &= \lim_{N\to\infty} (k_n(r,r) - \sum_{i=1}^{N} c^n_{i,m}(r) k_n(r, \frac{a_{i,m}}{B})) \\ &= 0, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \lim_{N\to\infty} E[\hat{z}^B_{n,N}(r)(z_n(r) - \hat{z}^B_{n,N}(r))] &= \lim_{N\to\infty} (\sum_{i=1}^{N} \sum_{j=N+1}^{\infty} c^n_{i,m}(r) c^n_{j,m}(r) k_n(\frac{a_{i,m}}{B}, \frac{a_{j,m}}{B})) \\ &= 0. \end{aligned} \tag{3.16}$$

Combining (3.15) with (3.16), it follows that

$$\lim_{N\to\infty} E[|z_n(r) - \hat{z}^B_{n,N}(r)|^2] = 0. \tag{3.17}$$

By using a similar approach it can be shown that

$$\lim_{N\to\infty} E[|z_n(r) - \hat{z}^F_{n,N}(r)|^2] = 0, \tag{3.18}$$

where

$$\hat{z}^F_{n,N}(r) = \sum_{i=0}^{N} d^n_i(r) z_n(\frac{i\pi}{B}). \tag{3.19}$$

□ □

Note that Theorem 3 shows that the *non-stationary* 1-D process $z_n(r)$ can be reconstructed from its sample values at the points $\{a_{i,m}/B : \ i \in \mathbf{N}\}$ or $\{i\pi/B : \ i \in \mathbf{N}\}$. Observe also that the weighting functions $c^n_{i,m}(r)$ and $d^n_i(r)$ used in the reconstruction of $z_n(r)$ from its sample values (see (3.10)-(3.11)) are orthogonal in the sense of [1], i.e.

$$c^n_{i,m}(\frac{a_{l,m}}{B}) = 0 \qquad \text{for } l \neq i$$
$$d^n_i(\frac{l\pi}{B}) = 0 \qquad \text{for } l \neq i$$

which guarantees the linear independence of the sample values of $z_n(r)$. As pointed out in [1], the weighting functions need not be orthogonal to achieve zero mean square error. However, it seems that the convergence of the series is more rapid near the sampling points when orthogonal weights are used [1].

Now recall that the knowledge of $z(\vec{r})$ on a circle of radius r' is sufficient to compute *all* of the Fourier coefficient processes $z_n(r)$ at the location $r = r'$. Hence, we have the following important result.

Theorem 4 *Any isotropic random field $z(\vec{r})$ with a wave-number limited spectrum $S(\lambda)$, where $S(\lambda) = 0$ for $\lambda > B$, can be reconstructed with zero mean-square error from its samples on the countable set of circles of radii $r_i = \frac{a_{i,m}}{B}$, $i \in \mathbf{N}$, where $a_{i,m}$ is the ith zero of the mth order Bessel function $J_m(x)$, as*

$$z(\vec{r}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{i=1}^{\infty} c^n_{i,m}(r) \int_0^{2\pi} z(\frac{a_{i,m}}{B}, \phi) e^{-jn\phi} \, d\phi \; e^{jn\theta}, \tag{3.20}$$

and from its samples on the countable number of circles of radii $r_i = \frac{i\pi}{B}$, $i \in \mathbf{N}$, as

$$z(\vec{r}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{i=0}^{\infty} d^n_i(r) \int_0^{2\pi} z(\frac{i\pi}{B}, \phi) e^{-jn\phi} \, d\phi \; e^{jn\theta}, \tag{3.21}$$

where equations (3.21) and (3.22) are to be understood in a mean-square sense.

Theorem 4 follows directly from Theorem 3 and equations (2.1)-(2.2), and is a generalization of a result in [6]. A natural question to be asked here, is which of the above two sampling schemes –i.e. the *Bessel sampling scheme* involving sampling on circles of radii $r_i = a_{i,m}/B$, or the *uniform sampling scheme* using circles of radii $r_i = i\pi/B$– is more efficient in terms of minimizing the number of sampling circles per unit radial length? This leads us to examine the distribution of the zeros $a_{i,m}$ of the mth order Bessel function $J_m(x)$, along the positive real axis. For large i, and a fixed value m, the zeros of the mth order Bessel function are approximately given by the McMahon expansion [11]

$$a_{i,m} \approx (i + \frac{1}{2}m - \frac{1}{4})\pi - \frac{(4m^2 - 1)}{8\pi(i + \frac{1}{2}m - \frac{1}{4})} \cdots, \tag{3.22}$$

which shows that the separation $\Delta_{i,m}$ between two successive large zeros $a_{i+1,m}$ and $a_{i,m}$ of $J_m(x)$, with $i >> m$, is approximately equal to

$$\Delta_{i,m} = a_{i+1,m} - a_{i,m} \approx \pi + \frac{(4m^2 - 1)}{8i^2} + O(i^{-3}). \tag{3.23}$$

In particular, two successive large zeros of $J_0(x)$ are separated by a distance slightly less than π, whereas two successive large zeros of $J_m(x)$ for $m \neq 0$, are separated by a distance slightly larger than π. As the order i of the zeros $a_{i,m}$ of $J_m(x)$ tends to infinity the separation $\Delta_{i,m}$ between successive zeros is asymptotically equal to π, for all m. Furthermore, examination of the small zeros of the mth order Bessel function reveals that even for $i = 2$, $\Delta_{i,m}$ is approximately equal to π. Hence, the Bessel sampling scheme is slightly more efficient than the uniform sampling scheme if the zeros of a large order Bessel function are used to generate the nonuniform circular sampling grid. However, the Bessel sampling scheme is primarily of theoretical interest, while the uniform sampling scheme is of more practical value since it does not require the knowledge of a large number of zeros of one of the Bessel functions.

Finally, observe that in practice one does not need to sample the field $z(\vec{r})$ continuously as a function of θ along any of the circles r_i. Note that along any of these circles $z(r_i, \theta)$ is a stationary process with the covariance function

$$\begin{aligned} K(r_i; \theta, \phi) &= E[z(r_i, \theta)(z(r_i, \phi)] \\ &= \sum_{n=-\infty}^{\infty} k_n(r_i, r_i)\, e^{jn(\theta-\phi)}. \end{aligned} \tag{3.24}$$

Examination of a plot of $J_n(x)$ [11] reveals that

$$J_n(x) \approx 0 \quad \text{for} \quad x \gg 1 \quad \text{and} \quad n > x. \tag{3.25}$$

Hence, by using (2.6) and the Lebesgue dominated convergence theorem to interchange the operations of limit and integration, we obtain

$$k_n(r_i, r_i) \approx 0 \quad \text{for} \quad Br_i \gg 1 \quad \text{and} \quad n > Br_i. \tag{3.26}$$

Equation (3.26) implies that along any circle of radius r_i, $z(r_i, \theta)$ can be approximated with a small mean square error by the finite sum

$$z(r_i, \theta) = \sum_{n=-N}^{N} z_n(r_i)\, e^{jn\theta} \tag{3.27}$$

where $N \geq Br_i$. In particular, (3.20)and (3.21) can be approximated in the mean square sense as

$$z(\vec{r}) = \sum_{i=1}^{\infty} \sum_{n=-N_i}^{N_i} f_{i,n}(r) z_n(r_i)\, e^{jn\theta} \tag{3.28}$$

where

$$r_i = \frac{i\pi}{B}, \qquad f_{i,n}(r) = d_i^n(r) \qquad \text{and} \qquad N_i \gg i\pi \tag{3.29}$$

or

$$r_i = \frac{a_{i,m}}{B}, \qquad f_{i,n}(r) = c_{i,m}^n(r) \qquad \text{and} \qquad N_i \gg a_{i,m}. \tag{3.30}$$

The coefficients $z_n(r_i)$, $-N_i \leq n \leq N_i$, can be determined by sampling $z(r_i, \theta)$ at $2N_i + 1$ points.

4 CONCLUSION

In this note we have shown that a wave-number limited isotropic random field, with a power spectrum that is zero outside of a disk of radius B centered at the origin of the wave-number plane, can be reconstructed in a mean-square sense from its observation on the countable set of circles of radii $r_i = a_{i,m}/B$, $i \in \mathbf{N}$, where $a_{i,m}$ is the ith zero of the mth order Bessel function $J_m(x)$, or of radii $r_i = i\pi/B$, $i \in N$. This result is a direct consequence of the sampling theorems that we derived for the Fourier coefficient processes associated with the given isotropic random field.

The sampling schemes developed in this note can easily be extended to isotropic random fields in higher dimensions, provided that the random fields are expanded in spherical harmonics instead of Fourier series. Finally, observe the parallel between our sampling procedures and the correponding one-dimensional results. In one dimension, a stationary process can be reconstructed in the mean-square sense from its observation on a countable number of spheres in a space of dimension one, i.e. at a countably infinite number of points. In the general m-D case, an isotropic random field can be reconstructed in the mean-square sense from its observation on a countably infinite number of spheres in the m-D space.

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APPENDIX

Proof of (3.4)

Let $p_B(\lambda)$ be the function

$$p_B(\lambda) = \begin{cases} 1 & \lambda < B \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

By using the identity [9], p. 43 and p. 99

$$\int_{-\infty}^{\infty} J_n(\lambda r) e^{-jur}\, dr = \begin{cases} 2(-j)^n \frac{T_n(\frac{u}{\lambda})}{(\lambda^2-u^2)^{1/2}} & 0 < |u| < \lambda \\ 0 & \text{otherwise,} \end{cases} \tag{A.2}$$

where $T_n(x)$ is a Chebyshev polynomial of order n, it can be shown that the Fourier transform of $J_n(\lambda r)p_B(\lambda)$ with respect to r is bandlimited to B radians per unit distance. Hence, $J_n(\lambda r)p_B(\lambda)$ can be written as [10]

$$J_n(\lambda r)p_B(\lambda) = \sum_{i=0}^{\infty} d_i^n(r) J_n(\frac{i\pi}{B}\lambda), \;\; 0 < \lambda < B \tag{A.3}$$

where

$$d_i^n(r) = \frac{1}{(1+\delta_{0,n})}((-1)^n \frac{sin(B(r+\frac{i\pi}{B}))}{B(r+\frac{i\pi}{B})} + \frac{sin(B(r-\frac{i\pi}{B}))}{B(r-\frac{i\pi}{B})}), \tag{A.4}$$

and where $\delta_{0,n}$ denotes the Kronecker delta function.

Proof of (3.7)

Consider the identity [11], p. 72

$$J_l(\lambda r) = \sum_{i=1}^{\infty} c_{i,l}(r) J_l(\frac{a_{i,l}}{B}\lambda), \;\; 0 < \lambda < B \tag{A.5}$$

where

$$c_{i,l}(r) = \frac{2a_{i,l}J_l(Br)}{(a_{i,l}^2 - B^2r^2)J_{l+1}(a_{i,l})}. \tag{A.6}$$

By repeatedly differentiating both sides of (A.5) with respect to λ, and using the identity

$$\frac{d}{d\lambda}J_m(\lambda r) = rJ_{m-1}(\lambda r) - \frac{m}{\lambda}J_m(\lambda r) \tag{A.7}$$

for the case where $n < l$, and the identity

$$\frac{d}{d\lambda}J_m(\lambda r) = -rJ_{m+1}(\lambda r) + \frac{m}{\lambda}J_m(\lambda r) \tag{A.8}$$

for the case where $n > l$, we obtain

$$J_n(\lambda r) = \sum_{i=1}^{\infty} c_{i,l}^n(r) J_n(\frac{a_{i,l}}{B}\lambda), \;\; 0 < \lambda < B \tag{A.9}$$

where

$$c_{i,l}^n(r) = c_{i,l}(r)(\frac{a_{i,l}}{r})^{|l-n|}. \tag{A.10}$$