

**Numerical Methods for Contingent Claims Analysis
of Investment Decisions**

by

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ABSTRACT

In this thesis I examine the numerical methods used in option valuation with analysis focusing on the more complex options associated with investment decisions. Two options implicit in many projects are identified and analyzed: i) the option to halt construction of a project, and ii) the option to shut down the production lines once the project is complete. The partial differential equations governing the values of these two options are derived, discretized, and solved using numerical techniques.

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1. Introduction

Since the publication of the seminal paper by Black and Scholes in 1972 [2], contingent claims analysis has been one of the most studied topics in finance. The purpose of my thesis is to show how contingent claims analysis can be applied to the valuation of investment decisions which often include imbedded options.

Many investment decisions have options-like characteristics which are ignored when standard discounted cash flow (DCF) methods are employed in the valuation of projects. These implicit options can often significantly impact the valuation of a project as has been shown in Majd and Pindyck [7], McDonald and Siegel [8,9], and Myers and Majd [11].

The value of this thesis lies in the extension of current models which incorporate contingent claims analysis in investment decision making. I will focus on the numerical methods used in solving the equations arising from these models. The importance of a methods paper such as this is that as models grow in complexity, analytical solutions are often unavailable, and numerical analysis must be employed to estimate solutions.

In this thesis, I will concentrate on a problem which exhibits two different types of options characteristics and explain how numerical methods are used to value the contingent claims. The problem can be broken down so that

each of the options can be analyzed with relatively straight-forward numerical techniques. The problem is also one in which the deficiencies of DCF methods are readily apparent.

In Section 2 I will describe the general nature of the problem, and I will give a specific example of a project which can be analyzed using these methods. In Sections 3 and 4 I will provide a complete analysis of each part of the problem. These two sections will be broken down into subsections each of which will focus on a specific aspect of the problem. In Subsections 3A and 4A I will derive the partial differential equations (PDEs), from which most of the analysis proceeds in the literature. Subsections 3B and 4B will focus on the transformation and discretization of the PDEs used to obtain finite difference approximations of the PDEs. In the C Subsections, I will explore the numerical methods used in solving the finite difference equations. Section 5 will conclude.

2. The Problem

Suppose a firm is deciding whether or not to invest in a project with the following characteristics:

- 1) spending decisions and associated cash outlays occur continuously over time,
- 2) there is a maximum rate at which the cash outlays can be productively spent over time,

- 3) there are no cash inflows from the project until construction is complete,
- 4) capital in place has no alternative use and thus no salvage value,
- 5) the size of the project is fixed, i.e. there exists no flexibility to build half a factory or two factories,¹
- 6) construction can be stopped and started again costlessly, and,
- 7) the capital markets are sufficiently complete so that the completed project will not affect the opportunity set available to investors.

A project which exhibits these characteristics includes an embedded option that is ignored in DCF analysis. More precisely, the decision to invest in the project at any given time is like a compound option. For example, assume that in any period, I have the option to invest one million dollars in a project that is currently ten million dollars from completion. If I invest this period, then next period I have the option to invest one million dollars in a project that is currently nine million dollars from completion. If I choose not to invest this period, then next period I would have exactly the same option as I have this period. As we

¹This is another option which can be valued using methods similar to those described below; however, we will ignore this option in our analysis.

shall see this option can be valued independently of the other underlying option that we will study.

In addition to the above assumptions, suppose that the project, once completed:

8) produces a good which has an observable price in a competitive market,

9) produces a good with a constant, known marginal cost

10) can be shut down if the price of the good falls below marginal cost, and re-started later costlessly, and

11) has a known life expectancy of T .

The economics of this option value is readily seen from assumption (10), and adds to the value of a completed factory.

The above set of assumptions may seem overly restrictive; however, some of the restrictions serve only to simplify the analysis and can be relaxed.

To frame the problem in more concrete terms, as well as to build a base case scenario with which to later test the numerical methods, consider a firm that is trying to decide whether to build a widget factory which will take a minimum of five years to complete. By a minimum completion time of five years, I mean that if construction were to proceed at the maximum possible rate and not stop, the factory would be complete in five years.

Suppose, for example, that the widget factory were only half completed and the price of widgets were to drop to one tenth of the variable cost to produce widgets. It may make sense for the firm to treat the half completed factory as a sunk cost and stop investing in the project. A value maximizing firm will value this option to halt construction in much the same fashion that financial options are valued. This implies that the firm will demand that the price of widgets be at some value above the variable cost of production before it will invest in a given period.

This option to halt construction and restart it makes the time it takes to complete a widget factory dependent on the price evolution of widgets, and thus uncertain. I will suppose, though, that a completed widget factory has a useful life of 10 years, independent of the number of years for which it is actually profitable for the factory to produce widgets. An important difference between the two parts of the problem is that the time it takes to complete the factory is uncertain due to the uncertainty of price evolution, whereas the useful life of the factory is known with certainty once construction has been completed.

Lastly, I will take the marginal cost of producing widgets to be a constant \$1, the maximum rate of investment in a widget factory to be \$1 million per year, and the maximum rate of production of widgets to be 1 million per year.

We assumed above that the price of the good, in our example the price of widgets, is exogenous. We will also assume that price evolves according to:

$$dP = (\mu - \delta)Pdt + \sigma Pdz \quad (1)$$

In other words, the price of widgets follows geometric Brownian motion, also known as the diffusion process, where dz is the increment of the Wiener process. The parameter μ is the total rate of return demanded by the market for holding widgets and includes an appropriate risk premium.

As can be seen from (1), the rate of growth of P is less than μ by an amount δ , which corresponds to the convenience yield, or rental rate, realized by holding widgets in reserve. The convenience yield on a commodity is similar to a dividend paid on common stock, as it is the return from holding the asset which when added to the capital appreciation of the asset gives the total return. The assumption that the price of widgets is exogenous, along with the need to further assume that widgets can be shorted makes the analysis especially well-suited to handle projects which produce commodities.

The option to halt construction is the same as in Majd and Pindyck's time-to-build paper [7] where the price of the underlying good has replaced the value of a completed factory as the exogenous state variable. The problem is still one of optimal control of investment decisions leading to the completion of the project. Their model estimates the

value of flexibility implicit in projects which require a period of time to be built and provides an optimal decision rule which gives the minimum value needed for investment to occur over time.

Since we assume price, not value, is exogenous, the valuation of the flexibility associated with time-to-build and the derivation of the optimal decision rule will encompass the second part of our problem. The first problem is to calculate the value of a completed factory. To do this we will draw on McDonald and Siegel's valuation model of the option to shut down [9]. They argue that since the firm can decide to shut down at any time during the life of the project, the value of a completed factory is equivalent to the value of an infinite number of options to produce. Summing the value of these options will give the value of the completed project that takes into account the option to shut down.

3. Modeling the Option to Shut Down

Given that a completed widget factory exists, how much is it worth? At every time $\tau \in [0, T]$, the firm has the option to produce widgets or to shut the factory down. I will write the value of this option for any price level and time prior to maturity as:

$$C = C(P(t), \tau - t) \quad (2)$$

On the "maturity date" τ the value of the cash flow,

$\pi(P(\tau))$, depends upon the price of widgets at time τ and the marginal cost of producing widgets MC. This future cash flow is:

$$\pi(P(\tau)) = \max[0, P(\tau) - MC] \quad (3)$$

per widget produced at time τ .

For $\delta > 0$ McDonald and Siegel [9] show that the present value of this claim on future earnings, conditional on information at time 0, corresponds to the value of a European call option on a stock paying a continuous proportional dividend of δ expiring at time τ . They also show that the value of a completed project is obtained by summing the values of the individual claims:

$$V(P(0)) = \int_0^T C(P(0), \tau) d\tau \quad (4)$$

The first subsection will go through the steps used to replicate the individual claims on the future cash flows; however, keep in mind we are really interested in the total value of the project given by (4).

3a. Replicating the Option to Shut Down

Suppose you construct a portfolio W consisting of an option to produce one widget at time τ , and short C_p widgets, where subscripts denote partial derivatives:

$$W = C - C_p P \quad (5)$$

Total differentiation of (5) gives $dW = dC - C_p dP$; however, the portfolio W includes a short position of C_p widgets. In order to short, restitution of the convenience yield must be

made to the party from which the widgets were borrowed. Thus, the amount $\delta PC_p dP$ must also appear in the equation describing the change in the value of the portfolio W:

$$dW = dC - C_p dP - \delta PC_p dt \quad (6)$$

By Ito's Lemma we know:

$$dC = -C_t dt + C_p dP + \frac{1}{2}\sigma^2 P^2 C_{p,p} dt \quad (7)$$

The first two terms in the right-hand side of (7) appear from non-stochastic differentiation of (2). The third term arises from the stochastic process assumed on P.

Substituting (7) into (6) yields:

$$dW = [\frac{1}{2}\sigma^2 P^2 C_{p,p} - \delta PC_p - C_t] dt \quad (8)$$

The important characteristic of (8) is that the terms involving dP have dropped, and with them the uncertainty involving the change in the value of the portfolio W. This implies that the portfolio should earn a riskless rate of return over any time period:

$$dW = rW dt \quad (9)$$

where r is the continuously compounded riskless rate of return.

Substituting (9) into (8) yields:

$$rW = \frac{1}{2}\sigma^2 P^2 C_{p,p} - PC_p - C_t \quad (10)$$

Substituting (5) into (10) and rearranging gives the oft seen second order PDE:

$$\frac{1}{2}\sigma^2 P^2 C_{p,p} + (r - \delta)PC_p - C_t - rC = 0 \quad (11)$$

The call price must satisfy (11) for every price and for each point in time. The equation is subject to the following boundary conditions:

$$C(P(\tau), 0) = \max[0, P(\tau) - VC] \quad (12a)$$

$$C(0, \tau - t) = 0 \quad (12b)$$

$$\lim_{P \rightarrow \infty} C_p(P(t), \tau - t) = e^{\delta(t - \tau)} \quad (12c)$$

Condition (12a) states that at expiration, the call expiring at time τ is either worthless, or worth the difference between the prevailing price $P(\tau)$ and the variable cost to produce. Condition (12b) states that if the price drops to 0 the call is worthless. Condition (12c) states that as the price of the underlying asset gets very large, the value of the option to not produce goes to zero, and the value of the option to produce the underlying asset grows at the fair market rate discounted by the convenience yield.

McDonald and Siegel [9] show that (11), subject to the boundary conditions (12a) - (12c), has a closed-form solution given by:

$$C(P(0), \tau) = P(0)e^{-\delta \tau} N(d_1) - VCe^{-r \tau} N(d_2) \quad (13)$$

$$d_1 \equiv [\ln(P(0)/VC) + (r - \delta + \frac{1}{2}\sigma^2)\tau] \sigma \sqrt{\tau} \quad (13a)$$

$$d_2 \equiv d_1 - \sigma \sqrt{\tau} \quad (13b)$$

where $N(x)$ denotes the area under the standard normal curve between $-\infty$ and x . However, no closed-form solution to (4) exists unless the project is infinitely lived. Given that T is finite, the solution to (4) would normally proceed as an

approximation of the left-hand side of (4) using the right-hand side of (13). The approximation involves numerical integration of the area under the standard normal curve.

I will present an alternative method for approximating the left-hand side of (4) using finite differencing techniques. The reason for this approach is twofold; the simplicity of (13) will provide a good example to illustrate the numerical methods before attempting more difficult models, and the approximations obtained using the numerical technique can be directly compared to the closed-form solutions to estimate how quickly the numerical method converges.

3b. Transformation and Discretization

There are two things which we hope to accomplish through a transformation of (11). The first is to write a dimensionless, or unitless, equation. By doing this we are left with a simple, but very general form devoid of unnecessary symbols and units.² To accomplish this in (11) we divide everything currently denominated in dollars, i.e. C, P, MC, by the marginal cost MC, which is analogous to the strike price of a financial option.

The second goal of the transformation is to eliminate P from the coefficients of the partial derivative terms in

² For more on dimensionless forms and a more indepth coverage of numerical methods see Ames [1].

(11). I will describe why this elimination is desirable from a numerical perspective in the next subsection. Ignoring the reason for a moment I will employ the following transformation:³

$$C(P(t), \tau - t) \equiv (MC)D(X, \tau - t) \quad (14)$$

where $X \equiv \ln(P/MC)$ (14a)

and the variable cost VC has been normalized to 1. Notice that since $C(P(t), \tau - t)$, MC, and P are denominated in dollars, $D(X, \tau - t)$ and X are dimensionless.

Differentiating (14) we find:

$$C_p = D_x MC/P \quad (15)$$

$$C_{pp} = [D_{xx} - D_x] MC/P^2 \quad (16)$$

$$C_t = D_t MC \quad (17)$$

Substitution of (15) - (17) into (11) and (12a) - (12c) gives:

$$\frac{1}{2}\sigma^2 D_{xx} + (r - \delta - \frac{1}{2}\sigma^2)D_x - D_t - rD = 0 \quad (18)$$

Subject to:

$$D(X, 0) = \max[0, e^X - 1] \quad (19a)$$

$$\lim_{X \rightarrow -\infty} D(X, \tau - t) = 0 \quad (19b)$$

$$\lim_{X \rightarrow \infty} e^{-X} D_x = e^{\delta(t - \tau)} \quad (19c)$$

Notice that all of the coefficients of the left-hand side of (18) are constant. By applying the transformation (14), we have succeeded in writing a unitless equation with

³Brennan and Schwartz use this transformation in their paper [3].

constant coefficients. An equation that is to be solved numerically should exhibit these two properties at a minimum before the methods to be described below are applied.

Recall from calculus the following Taylor expansions:

$$D(X + X) = D(X) + D_x X + \frac{1}{2}D_{xx} X^2 + (1/6)D_{xxx} X^3 + O(X^4) \quad (20)$$

$$D(X - X) = D(X) - D_x X + \frac{1}{2}D_{xx} X^2 - (1/6)D_{xxx} X^3 + O(X^4) \quad (21)$$

To obtain a finite difference approximation to (18) we will discretize the function $D(X, \tau - t)$ as follows:

$$D(X, \tau - t) = D(i\Delta X, j\Delta t) = D_{i,j} \quad (22)$$

By subtracting (21) from (20) and rearranging we obtain as an approximation to D_x :

$$D_x \approx (D_{i+1,j} - D_{i-1,j})/X \quad (23)$$

By adding (20) and (21) we can approximate D_{xx} as:

$$D_{xx} \approx (D_{i+1,j} - 2D_{i,j} + D_{i-1,j})/(\Delta X)^2 \quad (24)$$

Finally, to obtain a forward difference approximation of D_t , we can write the same Taylor expansion as in (20), but for t which gives:

$$D_t \approx (D_{i,j+1} - D_{i,j})/\Delta t \quad (25)$$

Applying these particular discretizations to (18) with some rearranging gives:

$$D_{i,j+1} = c_1 D_{i+1,j} + c_0 D_{i,j} + c_{-1} D_{i-1,j} \quad (26)$$

with $k \equiv t, \quad h \equiv X, \quad R \equiv k/h^2 \quad (26a)$

$$c_1 \equiv \frac{1}{2}R[\sigma^2 + h(r - \delta - \frac{1}{2}\sigma^2)] \quad (26b)$$

$$c_0 \equiv 1 - R\sigma^2 - kr \quad (26c)$$

$$c_{-1} \equiv \frac{1}{2}R[\sigma^2 - h(r - \delta - \sigma^2)] \quad (26d)$$

As can be seen from (26) any value $D_{i,j+1}$ can be solved in terms of $D_{i,j}$. This method is classified as an explicit method in numerical analysis since any value can be calculated explicitly from values already known. The most desirable property of the coefficients of (26) is that they do not depend on i or j . The importance of this properties will be addressed in the next section where I will focus on the numerical methods used to solve (26).

3c. Numerical Method

Perhaps the main theme of numerical analysis is convergence. Convergence requires that the solution to (26) approach the solution of (18) using boundary conditions (19a) - (19c), as both h and k go to 0. Proving the convergence of a numerical method is equivalent to showing that the method is both consistent and stable. Stability of a method is guaranteed if the error associated with the numerical solution does not grow with the number of steps required to arrive at the solution. In other words, any roundoff error introduced in computing the values of the option at the boundaries must be bounded as both k and h decrease. Numerical solutions obtained by unstable methods are of no value, since errors soon swamp the solution of the finite difference equation.

A numerical method is consistent if in the limit as h and k go to 0, the finite difference equation is no longer an approximation of the partial differential equation, but is exactly the same as the PDE. Most finite difference approximations are derived to ensure consistency; however, an excellent example of inconsistency in a reasonably derived finite difference equation is given in Ames, p. 62 [1]. In Ames' example the difference equation approximating one PDE converges to the solution of another PDE even though a stable numerical method was employed.

Our explicit method is easily shown to be consistent. If we subtract (26), which is the approximation to (18) obtained by approximating the derivative terms using (23) - (25), from the exact representation of (18) using the Taylor expansions (20) and (21), we get:

$$[-\frac{1}{2}\sigma^2/12 - (r - \delta - \frac{1}{2}\sigma^2)/6]X^2 D_{x x} + \frac{1}{2}tD_t + O(X^4 + t^2) \quad (27)$$

which approaches 0 as both X and t approach 0.

Stability is guaranteed provided that the sum of the coefficients of the right-hand side of (26) is less than or equal to one, and that the coefficients are all positive. By adding (26b), (26c), and (26d) we see that the sum of the coefficients of (26) is less than or equal to one. From (26b) - (26d) it can also be seen that two restrictions on k and h ensure that the coefficients are always positive:

$$h < \sigma^2 / |r - \delta - \frac{1}{2}\sigma^2| \quad (28)$$

$$R < (1 - r)/\sigma^2 \quad (29)$$

Notice that the stability conditions are simple functions of the parameters and not of the state variables as they would have been if the transformation had not eliminated the dependence of the coefficients on P.

A good approximation to the left-hand side of (4) can be calculated by numerically integrating (26) over the range 0 to T. The result will be used as a boundary condition for the second part of the problem to be described in the next section.

The code for implementing the algorithm given above can be found in Appendix A.⁴ Table 1 compares approximations of the value of the option to produce one widget generated by the code in Appendix A with the closed form solutions for a given set of parameter values. Table 1 includes option values for three different price levels: MC, and the nearest grid points to 10% above and below MC.

⁴All results were obtained using programs written in FORTRAN 77 for the IBM 4341. Appendix B contains an older implementation of the algorithm that I wrote in TruBasic for the IBM PC.

Table 1

$r = 0.02$ $\delta = 0.06$ $\sigma = 0.2$ $MC = 1$ $k = 0.01$

Yr	Price 0.9013		Price 1.0000		Price 1.1095	
	Actual	Estimate	Actual	Estimate	Actual	Estimate
1	0.0131	0.0250	0.0585	0.0586	0.1106	0.1168
2	0.0282	0.0381	0.0702	0.0709	0.1190	0.1212
3	0.0360	0.0451	0.0747	0.0759	0.1203	0.1209
4	0.0402	0.0488	0.0760	0.0776	0.1184	0.1183
5	0.0425	0.0506	0.0754	0.0774	0.1148	0.1146
10	0.0420	0.0471	0.0634	0.0658	0.0899	0.0903

4. Modeling the Option to Delay Investment

We have solved above for the value of a completed factory over a range of prices. These values will be used as a boundary condition in the problem of valuing the option to delay investment. First, I would like to give a general description of the second part of the problem which examines the value of the option to delay investment while in the process of investment. In the next subsection I will discuss the boundary condition details.

Firms which are considering an investment program face the following decision at any point in time: should they invest today and get one step closer to a completed factory, or should they delay investment? In our simple model this decision is based on two variables, the price of the good which the factory produces, and the amount of money per unit of good to be produced which still needs to be invested in construction, which will be designated K . It is useful to

view this decision as an option whose value I will designate as $F(P(t),K(t))$.

I will use $I(t)$ to represent the control variable which is the rate at which the firm decides to invest at any time. K is related to I in the following manner:

$$dK = -I dt \quad (30)$$

such that $K(0) = K_{max}$ (30a)

where I is bounded by 0 and some maximum rate i .

As was the case above, the economic idea motivating the application of contingent claims analysis is straightforward. In the analysis that follows, we hold that it is not one decision which is being made to invest or not invest; rather, that a stream of decisions need to be made, taking into account any changes that have occurred in the marketplace.

In short a decision to invest today does not commit one to invest in the future. What does happen; however, is that a firm which invests today relinquishes the option value which it enjoys if it does not invest today. A firm will invest in a given project if and only if the current option is worth more exercised than not exercised.

From options theory we know that a call option is always worth more unexercised than exercised unless the call is written on a dividend paying stock. As described above the parameter δ in the assumption of price evolution (2), is similar to a dividend payout. In fact, by assuming $\delta > 0$,

we have allowed for the possibility of the option to be worth more exercised than not exercised. For the case of $\delta = 0$, investment would never occur since the value of the option to invest would grow at the same risk-adjusted rate as the underlying asset.

4a. Replicating the Option to Delay Investment

Drawing to a large extent on the above analysis, consider a portfolio consisting of a long option to invest in the project and short F_p units of a completed project or a portfolio which spans the completed project:

$$W = F - F_p P \quad (31)$$

The change in the value of the portfolio W is slightly different from above. Letting I represent the control variable which is the rate of investment per unit of good to be produced we can write:

$$dW = dF - F_p dP - \delta F_p P dt - I dt \quad (32)$$

The first three terms of the right-hand side of (32) appear for the same reason as they appear in (6); the last term is the amount of investment which the firm makes during any time period.

Majd and Pindyck [7] show that a firm which can costlessly start and stop construction on a project which does not expire (i.e. in 100 years the investment opportunity will still exist) will either choose to invest at the maximum rate i or not invest at all.

Using Ito's Lemma and the above fact we see that the value of the option associated with the investment program must satisfy:

$$\frac{1}{2}\sigma^2 P^2 F_{pp} + (r - \delta)PF_p - iF_k - rF - i = 0, \quad P > P^* \quad (33a)$$

$$\frac{1}{2}\sigma^2 P^2 f_{pp} + (r - \delta)Pf_p - rf = 0, \quad P \leq P^* \quad (33b)$$

where $F(P,0) = V_0(P) \quad (34a)$

$$\lim_{P \rightarrow \infty} F_p(P,K) = e^{-\delta K}/i \quad (34b)$$

$$f(0,K) = 0 \quad (34c)$$

$$f(P^*,K) = F(P^*,K) \quad (34d)$$

$$f_p(P^*,K) = F_p(P^*,K) \quad (34e)$$

I have used $f(P,K)$ to represent the value of the investment program when P is below the critical value P^* , so that I could explicitly write the continuous condition (34d) and the differentiable condition (34e) at the free boundary P^* .

Equation (33b) has an analytic solution given by:

$$f(P) = aP^\alpha \quad (35)$$

$$\alpha \equiv [(2r\sigma^2 + (r - \delta - \frac{1}{2}\sigma^2)^2)^{\frac{1}{2}} - (r - \delta - \frac{1}{2}\sigma^2)]/\sigma^2 \quad (35a)$$

and the coefficient a must be determined along with the solution for F in the upper region using the conditions (34d) and (34e). Notice that the solution to (33b) is independent of K .

Using (34d), (34e), and (35) we can write the free boundary condition for the upper region as:

$$F(P^*,K) = P^* F_p(P^*,K)/\alpha \quad (36)$$

which will serve as the lower bound in the upper region.

4b. Transformation and Discretization

The transformation which will be employed is necessary for the same reasons as explained above; however, it is more complex and will therefore be explained in steps.

Integrating (30) subject to the initial condition (30a) gives:

$$t = (K_{max} - K)/I \quad (37)$$

Recall that (33a) describes the upper region in which investment always occurs, which implies that the control variable I is always chosen to be the maximum rate i .

Letting τ equal the minimum completion time K_{max}/i and $F(P,K) \equiv G(P,\tau - t)$ we have:

$$\frac{1}{2}\sigma^2 P^2 G_{pp} + (r - \delta)PG_p - G_t - rG - i = 0 \quad (38)$$

which is the same as (11) except for the constant term i .

The form of (38) is more often seen in the literature than (33a), and I believe it to be more natural since t has replaced K as a state variable.

I will write (38) in dimensionless form, as above, by dividing everything denominated in dollars by the marginal cost of production. Unlike the transformation used on (11), which had to be kept rather simple in order to perform the numerical integration, I further transform (34a) using:

$$G(P,\tau - t) \equiv M e^{-r\tau} H(X,\tau - t) \quad (39)$$

$$X \equiv (\sqrt{2}/\sigma)[\ln(P/MC) - (r - \delta - \frac{1}{2}\sigma^2)t] \quad (39a)$$

The same theme is apparent if you compare (39) and (39a)

with (14) and (14a), and the additional complexity will be justified below.

Differentiating (39) we obtain:

$$G_p = M Ce^{-r t} (\sqrt{2}/(\sigma P)) H_x \quad (40)$$

$$G_{p,p} = M Ce^{-r t} (\sqrt{2}/(\sigma P^2)) [(\sqrt{2}/\sigma) H_{x,x} - H_x] \quad (41)$$

$$G_t = M Ce^{-r t} [-r H + H_t - (\sqrt{2}/\sigma)(r - \delta - \frac{1}{2}\sigma^2) H_x] \quad (42)$$

Substitution of (40) - (42) into (38), (34a), (34b), (36) gives:

$$H_t = H_{x,x} - i e^{r t} / MC \quad (43)$$

$$H(X, 0) = e^{r \tau} V_0(X) / MC \quad (44a)$$

$$\lim_{X \rightarrow \infty} \exp[-rt - \sigma'^2 (X + (r - \delta - \frac{1}{2}\sigma^2)t)] H_x(X, \tau - t) = (\sigma/\sqrt{2}) \exp[\delta(t - \tau)] \quad (44b)$$

$$H(X^*, \tau - t) = \sqrt{2}/(\sigma \alpha) H_x(X^*, \tau - t) \quad (44c)$$

The most striking feature of (43) is its simplicity in relation to (33a) gained by the transformations. This simplicity allows for numerical methods which when employed lead to rapidly converging approximations, and which also have broader application.

Again, using discretizations of the form of (22), (23), (25), and the definition of the mesh ratio R in (26), yields an approximation of the form:

$$H_{i,j+1} = R H_{i+1,j} + (1 - 2R) H_{i,j} + R H_{i-1,j} - k i e^{r t} / MC \quad (45)$$

In the next subsection I will discuss the numerical method used to solve (45).

4c. Numerical Method

Equation (43) is the simple diffusion equation with the addition of the last term which is a function of time. The simple diffusion equation has been thoroughly studied in engineering disciplines, and (45) is known to be a consistent approximation to (43). To assure convergence then we only have to prove stability.

The coefficients of the H terms in the right-hand side of (45) are easily seen to sum to one. In addition, the coefficients will all be non-negative provided that the mesh ratio $R \leq \frac{1}{2}$. This simple condition assures stability, and therefore convergence. Notice that the condition on the mesh ratio R is no longer a function of the parameters of the problem. This allows for a general routine to be written which will be stable for any set of parameters. Also, Ames [1] shows that by setting $R = 1/6$, convergence is obtained at a quicker rate due to the symmetry of the numerical method.

Figure 1 shows the critical price for at every stage of the investment project. As expected the critical price is well above the marginal cost of production before the construction begins due to the lead time required for construction. As the project nears completion the critical price approaches the marginal cost of \$1.

Figure 1

$r = .02$, $\delta = .06$, $\sigma = .2$, $i = 1$

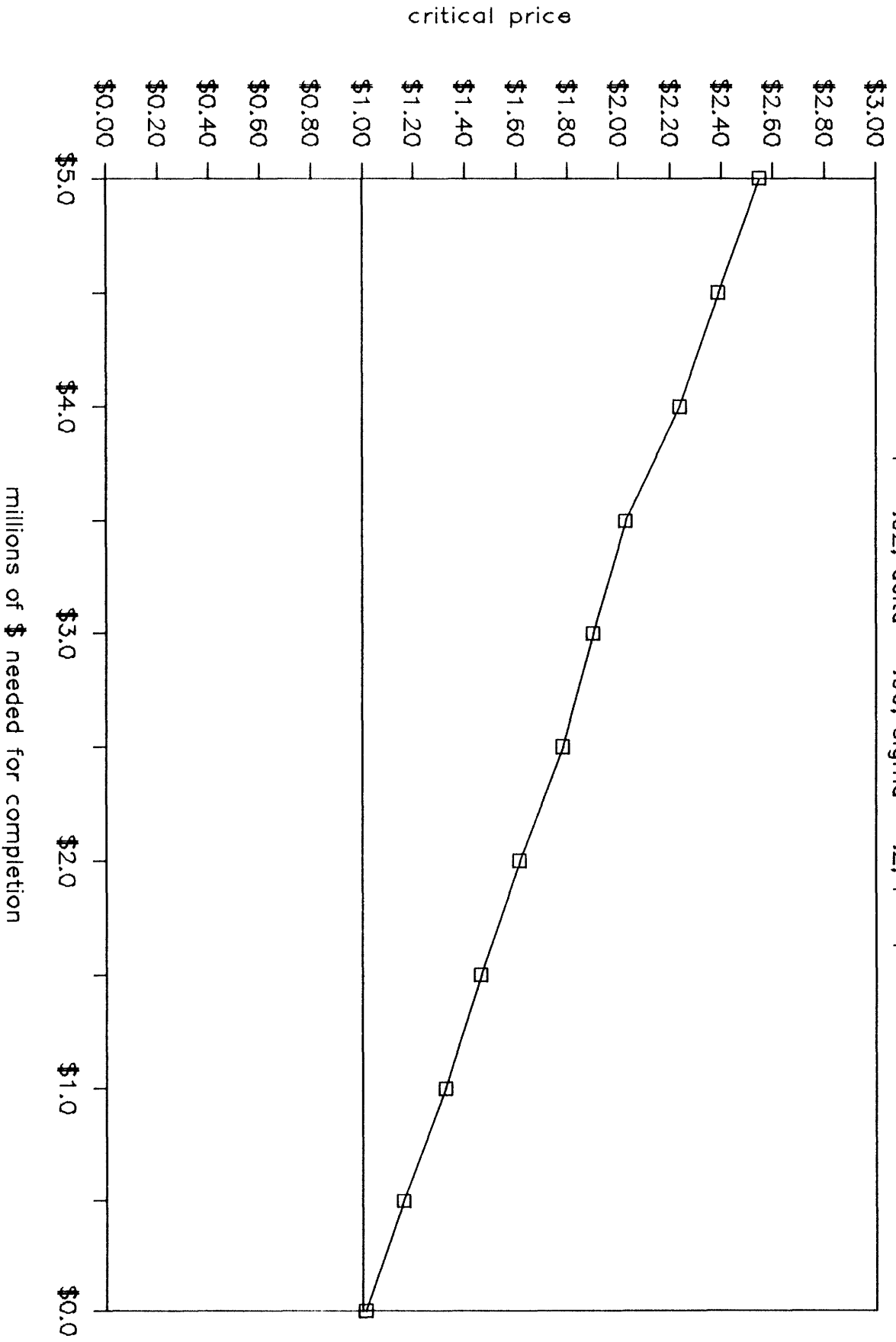


Figure 2 shows the value of a completed factory at the critical price level over the investment period. This value is always seen to be above the present value of the amount remaining to be invested. NPV analysis suggests investment if the present value of the completed project is above the present value of the costs. However, Figure 2 makes it clear that by including the value of the option to delay investment, a spread above construction costs is required for investment to be optimal.

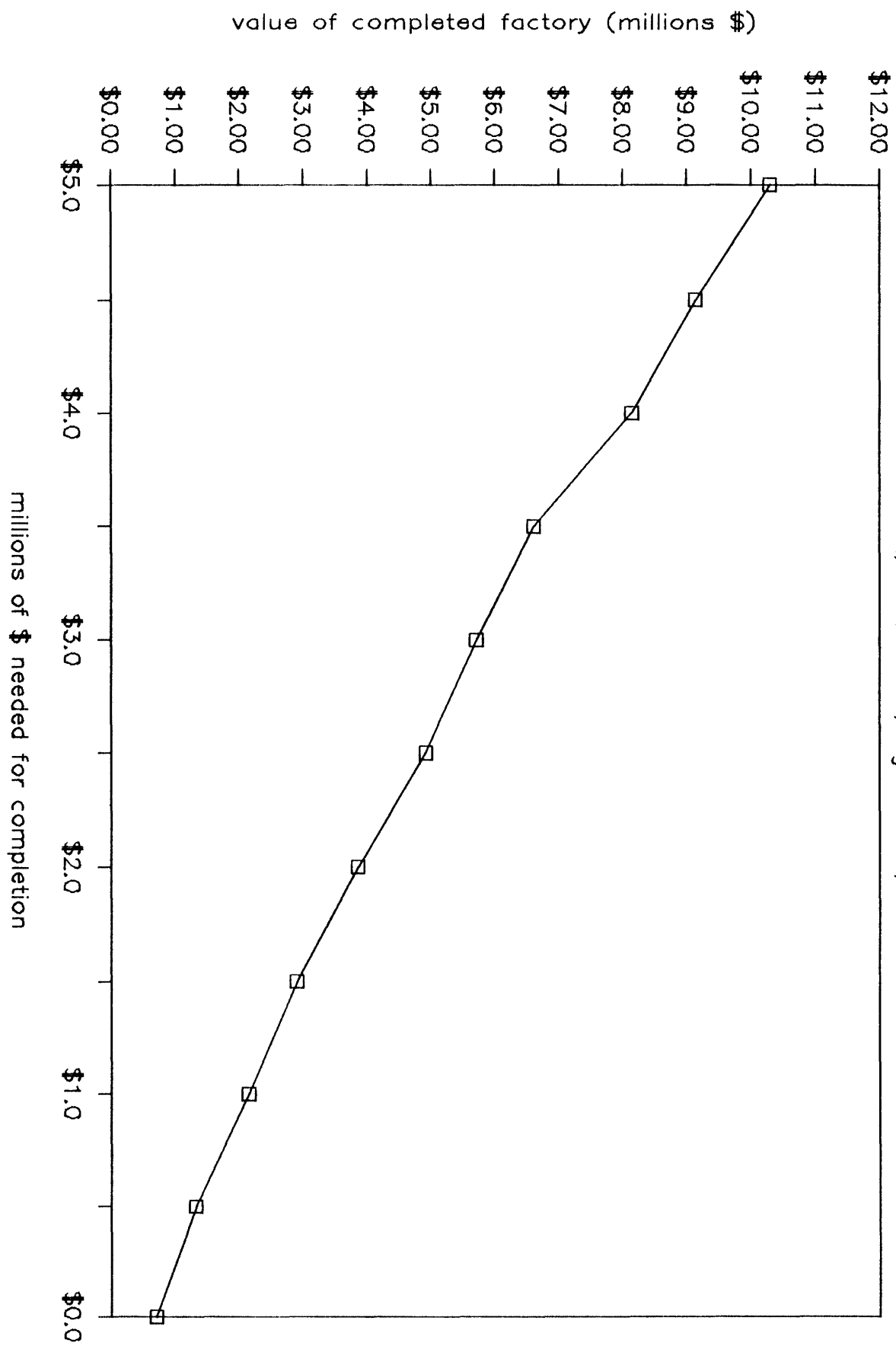
5. Conclusion

I have thus far tried to avoid discussing the practical problems encountered when applying these methods to "real life" decisions, mainly to allow the focus to be on the methods themselves. In addition, I have made some overly-restrictive assumptions which served to simplify the models so that the most important aspects of the analysis might be clarified. For example, the assumption that the maximum rate of investment for construction is constant over time can be relaxed so that different investment profiles can be accommodated.

One of the difficulties of applying the model to actual investment decisions is the estimation of the parameters. The same problem of estimating the appropriate σ in the future which exists in valuing financial options is also a major issue in applying the above models.

Figure 2

$r = .02, \text{ delta} = .06, \text{ sigma} = .2, i = 1$



Appendix A

```

C*****
C   PROGRAM TO INITIALIZE PARAMETERS AND PASS THEM TO SUBROUTINES
C   SHUTD AND BUILD FOR APPROPRIATE CRUNCHING.  ALL DESIRED OUTPUT IS
C   HANDLED BY THE SUBROUTINES.
C*****
      REAL K, MC, INV, INVR, PRICE(1000), FVAL(1000)
      DATA DEL, R, SIG, XMAX, TMAX, MC /.06, .02, .2, 5., 10., 2./
      DATA K, INV, INVR, CAPAC /.01, 500000., 100000., 100000./
      CALL SHUTD(DEL, R, SIG, XMAX, TMAX, MC, K, H, IMAX, PRICE, FVAL)
      ISHIFT = NINT(-1.*(R - DEL - .5*SIG*SIG)*INV/(INVR*H))
      H = H*SQRT(2.)/SIG
      CALL BUILD(DEL,R,SIG,INV,INVR,CAPAC,MC,ISHIFT,H,FVAL,IMAX)
      STOP
      END
C*****
C   SUBROUTINE TO CALCULATE THE VALUE OF THE OPTION TO SHUT DOWN AND
C   SUMS THEM TO OBTAIN VALUE OF A COMPLETED PLANT,
C   GIVEN A SET OF PARAMETERS
C       DEL  -- PAYOUT RATE ON AN ANNUAL BASIS
C       R    -- RISKLESS RATE OF INTEREST ON AN ANNUAL BASIS
C       SIG  -- STANDARD DEVIATION ON AN ANNUAL BASIS
C       TMAX -- USEFUL LIFE OF THE PROJECT IN YEARS
C       XMAX -- SUITABLY LARGE NUMBER USED IN PLACE OF INFINITY
C       MC   -- MARGINAL COST OF PRODUCTION
C       K    -- TIME INCREMENT
C   RETURNS
C       H    -- STEP SIZE IN X
C       PRICE -- VECTOR CONTAINING PRICES CORRESPONDING TO FVAL
C       FVAL  -- VECTOR CONTAINING VALUE OF A COMPLETED FACTORY
C       IMAX  -- MAX INDEX USED IN FVAL AND PRICE
C
C   PRINTS THE VALUE OF THE OPTION TO SHUT DOWN FOR EVERY YEAR 0
C   THRU TMAX AND FOR THREE PRICE/MC RATIOS: .9, 1.0, 1.1
C*****
      SUBROUTINE SHUTD(DEL,R,SIG,XMAX,TMAX,MC,K,H,IMAX,PRICE,FVAL)
      REAL MC,K,PRICE(1000),FVAL(1000),OLDVAL(1000),NEWVAL(1000)
      REAL MESH, TABLE(3,21)
      INTEGER II(3)
      SIGSQ = SIG*SIG
      H = SIG*SQRT(3.*K)
      MESH = 1./(3.*SIGSQ)
C*****
C   CHECK THE STABILITY CONDITIONS
C*****
      IF (H.GE.SIG/ABS(R - DEL - .5*SIGSQ).OR.MESH.GE.(1. - R)/SIGSQ)
* THEN
          IFLAG = 1
          GOTO 900
      END IF

```

```

C*****
C    CALCULATE MAX X INDEX NEEDED FOR VECTORS.  REMEMBER FORTRAN
C    DOES NOT ALLOW FOR NON-POSITIVE ARRAY SUBSCRIPTS.
C*****
      IMAX = NINT(2.*XMAX/H) + 1
      II(2) = (IMAX + 1)/2
      II(1) = NINT(LOG(.9)/H) + II(2)
      II(3) = NINT(LOG(1.1)/H) + II(2)
      DO 10 I = IMAX,1,-1
        REALI = I - (IMAX + 1)/2
        X = REALI*H
        PRICE(I) = MC*EXP(X)
        OLDVAL(I) = MAX(0.,EXP(X) - 1.)
10  CONTINUE
      DO 20 L = 1, 3
        TABLE(L,1) = OLDVAL(II(L))*MC
20  CONTINUE
C*****
C    COMPUTE COEFFICIENTS AND CALCULATE MAX T INDEX
C*****
      C1 = .5*MESHR*(SIGSQ + H*(R - DEL - .5*SIGSQ))
      C0 = 1. - MESHR*SIGSQ - K*R
      CNEG1 = .5*MESHR*(SIGSQ - H*(R - DEL - .5*SIGSQ))
      JMAX = NINT(TMAX/K)
      KINV = NINT(1./K)
      WRITE(6,401) K, H, C1, C0, CNEG1, JMAX, IMAX
401  FORMAT(5(1X,F10.6),1X,I5,1X,I5)
      DO 60 J = JMAX - 1, 0, -1
C*****
C    CALCULATE THE NEXT VALUE OF THE OPTION AT INFINITY AND ADD THE
C    TRAPEZOID TO THE ESTIMATE OF THE INTEGRAL
C*****
      REALJ = J
      NEWVAL(IMAX) = C1*2.*H*EXP(XMAX + DEL*(REALJ + 1.))*K
      * - TMAX) + C0*OLDVAL(IMAX) + (C1 + CNEG1)*OLDVAL(IMAX - 1)
      FVAL(IMAX) = FVAL(IMAX) + MC*(NEWVAL(IMAX) + OLDVAL(IMAX))*K/2.
      ICOUNT = ICOUNT + 1
      IF (ICOUNT.GE.KINV) THEN
        JFLAG = 1
        IYEAR = IYEAR + 1
      END IF
C*****
C    CALCULATE THE NEW OPTION VALUE FOR EVERY POINT DOWN THE COLUMN
C    AND ADD THE NEW TRAPEZOID TO THE ESTIMATE OF THE INTEGRAL
C*****
      DO 30 I = IMAX - 1,2,-1
        NEWVAL(I) = C1*OLDVAL(I + 1) + C0*OLDVAL(I) +
      * CNEG1*OLDVAL(I - 1)
        FVAL(I) = FVAL(I) + MC*(NEWVAL(I) + OLDVAL(I))*K/2.
30  CONTINUE

```

```

        IF (JFLAG.EQ.1) THEN
            DO 40 L = 1, 3
                TABLE(L,IYEAR + 1) = NEWVAL(II(L))*MC
40          CONTINUE
            ICOUNT = 0
            JFLAG = 0
        END IF
C*****
C          SWAP THE VALUES OF NEWVAL INTO OLDVAL
C*****
        DO 50 I = IMAX,1,-1
            OLDVAL(I) = NEWVAL(I)
50          CONTINUE
60          CONTINUE
            WRITE(6,402) (PRICE(II(I)), I = 1, 3)
            DO 70 J = 1, NINT(TMAX) + 1
                WRITE(6,402) (TABLE(I,J), I = 1, 3)
402         FORMAT(3(1X,F9.6))
70          CONTINUE
900        IF (IFLAG.EQ.1) WRITE(6,499)
499        FORMAT('AN ERROR HAS OCCURRED')
            RETURN
        END
C*****
C          SUBROUTINE TO CALCULATE THE VALUE OF THE INVESTMENT PROJECT,
C          GIVEN A SET OF PARAMETERS
C          DEL      -- PAYOUT RATE ON AN ANNUAL BASIS
C          R        -- RISKLESS RATE OF INTEREST ON AN ANNUAL BASIS
C          SIG      -- STANDARD DEVIATION ON AN ANNUAL BASIS
C          INV      -- TOTAL AMOUNT NEEDED TO BE INVESTED FOR CONSTRUCTION
C          INVR     -- MAX YEARLY RATE OF INVESTMENT
C          CAPAC    -- YEARLY CAPACITY
C          MC       -- MARGINAL COST
C          ISHIFT   -- CONSTANT TO LINE UP TWO PARTS OF PROBLEM
C          H        -- STEP SIZE IN X DIRECTION
C          FVAL     -- TERMINAL BOUNDARY CONDITION
C          IMAX     -- MAX INDEX OF FVAL VECTOR
C
C          PRINTS THE CRITICAL PRICE FOR EVERY POINT IN TIME
C*****
        SUBROUTINE BUILD(DEL,R,SIG,INV,INVR,CAPAC,MC,ISHIFT,H,FVAL,IMAX)
        REAL INV,INVR,MC,K,FVAL(1000),OLDVAL(1000),NEWVAL(1000)
        TAU = INV/INVR
        K = H*H/6.
        MESHR = 1./6.
        RIMAX = (IMAX + 1)/2
        JMAX = NINT(TAU/K)
        SIGSQ = SIG*SIG
        ALPHA = (SQRT(2.*R*SIGSQ + (R - DEL - .5*SIGSQ)**2) - (R - DEL -
*          .5*SIGSQ))/SIGSQ
        WRITE(6,401) K, H, TAU, JMAX, IMAX
401        FORMAT(3(1X,F10.6),1X,I5,1X,I5)

```



```

DO 10 I = IMAX, ISHIFT + 1, -1
  REALI = I - (IMAX + 1)/2
  PRICE = EXP(SIG*REALI*H/SQRT(2.) + (R - DEL - .5*SIGSQ)*
*      TAU)*MC
  OLDVAL(I) = FVAL(I)*EXP(R*TAU)/MC
10 CONTINUE
DO 60 J = JMAX - 1, 0, -1
  REALJ = J
  NEWVAL(IMAX) = MESHR*(SQRT(2.)*SIG*H*EXP(DEL*(K*(REALJ + 1.) -
*      TAU) + R*(REALJ + 1.)*K + SIG*(RIMAX*H - (DEL + SIG*SIG/2. -
*      R)*(REALJ + 1.)*K)/SQRT(2.))) + (1. - 2.*MESHR)*OLDVAL(IMAX)
*      + 2.*MESHR*OLDVAL(IMAX - 1)
DO 30 I = IMAX - 1, 1, -1
  REALI = I - (IMAX + 1)/2
  NEWVAL(I) = MESHR*(OLDVAL(I + 1) + OLDVAL(I - 1)) +
*      (1. - 2.*MESHR)*OLDVAL(I) - (INVR/CAPAC)*K*EXP(R*REALJ*K)/
*      MC
  IF (NEWVAL(I).GT.SQRT(2.)*NEWVAL(I + 1)/(SIG*ALPHA*H +
*      SQRT(2.))) GOTO 30
  RTIME = REALJ*K
  PRICE = EXP(SIG*REALI*H/SQRT(2.) + (R - DEL - .5*SIGSQ)*
*      RTIME)*MC
  A = EXP(-R*RTIME)*MC*NEWVAL(I)/PRICE**ALPHA
  WRITE(6,402) I,A,RTIME,PRICE,EXP(-R*RTIME)*MC*NEWVAL(I -
*      ISHIFT)
402  FORMAT(I3,4(1X,F12.9))
DO 20 L = I - 1, 1, -1
  REALL = L - (IMAX + 1)/2
  PRICE = EXP(SIG*REALL*H/SQRT(2.) + (R - DEL - .5*SIGSQ)*
*      RTIME)*MC
  NEWVAL(L) = (A*EXP(R*RTIME)*PRICE**ALPHA)/MC
20  CONTINUE
  GOTO 40
30  CONTINUE
40  DO 50 I = IMAX, 1, -1
  OLDVAL(I) = NEWVAL(I)
50  CONTINUE
60  CONTINUE
900 RETURN
END

```

Appendix B

```
SUB read_in_standard_normal
OPTION BASE 0
DIM standard_normal(30,9)
FOR i = 0 to 30
  FOR j = 0 to 9
    READ standard_normal(i,j)
  NEXT j
NEXT i
DATA .5000,.4960,.4920,.4880,.4840,.4801,.4761,.4721,.4681,.4641
DATA .4602,.4562,.4522,.4483,.4443,.4404,.4364,.4325,.4286,.4247
DATA .4207,.4168,.4129,.4090,.4052,.4013,.3974,.3936,.3897,.3859
DATA .3821,.3873,.3745,.3707,.3669,.3632,.3594,.3557,.3520,.3483
DATA .3446,.3409,.3372,.3336,.3300,.3264,.3228,.3192,.3156,.3121
DATA .3085,.3050,.3015,.2981,.2946,.2912,.2877,.2843,.2810,.2776
DATA .2743,.2709,.2676,.2643,.2611,.2578,.2546,.2514,.2483,.2451
DATA .2420,.2389,.2358,.2327,.2296,.2266,.2236,.2206,.2217,.2148
DATA .2119,.2090,.2061,.2033,.2005,.1977,.1949,.1922,.1894,.1867
DATA .1841,.1814,.1788,.1762,.1736,.1711,.1685,.1660,.1635,.1611
DATA .1587,.1562,.1539,.1515,.1492,.1469,.1446,.1423,.1401,.1379
DATA .1357,.1335,.1314,.1292,.1271,.1251,.1230,.1210,.1190,.1170
DATA .1151,.1131,.1112,.1093,.1075,.1056,.1038,.1020,.1003,.0985
DATA .0968,.0951,.0934,.0918,.0901,.0885,.0869,.0853,.0838,.0823
DATA .0808,.0793,.0778,.0764,.0749,.0735,.0721,.0708,.0694,.0681
DATA .0668,.0655,.0643,.0630,.0618,.0606,.0594,.0582,.0571,.0559
DATA .0548,.0537,.0526,.0516,.0505,.0495,.0485,.0475,.0465,.0455
DATA .0446,.0436,.0427,.0418,.0409,.0401,.0392,.0384,.0375,.0367
DATA .0359,.0351,.0344,.0366,.0329,.0322,.0314,.0307,.0301,.0294
DATA .0287,.0281,.0274,.0268,.0262,.0256,.0250,.0244,.0239,.0233
DATA .0228,.0222,.0217,.0212,.0207,.0202,.0197,.0192,.0188,.0183
DATA .0179,.0174,.0170,.0166,.0162,.0158,.0154,.0150,.0146,.0143
DATA .0139,.0136,.0132,.0129,.0125,.0122,.0119,.0116,.0113,.0110
DATA .0107,.0104,.0102,.0099,.0096,.0094,.0091,.0089,.0087,.0084
DATA .0082,.0080,.0078,.0075,.0073,.0071,.0069,.0068,.0066,.0064
DATA .0062,.0060,.0059,.0057,.0055,.0054,.0052,.0051,.0049,.0048
DATA .0047,.0045,.0044,.0043,.0041,.0040,.0039,.0038,.0037,.0036
DATA .0035,.0034,.0033,.0032,.0031,.0030,.0029,.0028,.0027,.0026
DATA .0026,.0025,.0024,.0023,.0023,.0022,.0021,.0020,.0020,.0019
DATA .0019,.0018,.0018,.0017,.0016,.0016,.0015,.0015,.0014,.0014
DATA .0013,.0013,.0013,.0012,.0012,.0011,.0011,.0010,.0011,.0010
END SUB
SUB STANDARD_NORMAL_LOOKUP(N,X(),STNTAB(,))
DIM Z(25)
FOR STN = 1 TO N
  SELECT CASE X(STN)
  CASE IS < -3.09
    LET Z(STN) = 0
  CASE IS > 3.09
    LET Z(STN) = 1
  CASE ELSE
```

```

        LET ROOT = ABS(ROUND(1000*X(STN)))
        LET I1 = INT(ROOT/100)
        LET J1 = INT((ROOT - 100*I1)/10)
        LET K1 = ROOT - 100*I1 - 10*J1
        IF J1 = 9 THEN
            LET I2 = I1 + 1
            LET J2 = 0
        ELSE
            LET I2 = I1
            LET J2 = J1 + 1
        END IF
        IF I2 = 31 THEN LET NEXT_ENTRY = 0 ELSE LET NEXT_ENTRY =
STNTAB(I2,J2)
        LET LOOK = STNTAB(I1,J1) + K1*(STNTAB(I1,J1) -
NEXT_ENTRY)/10
        IF X(STN) > 0 THEN LET Z(STN) = 1 - LOOK ELSE LET Z(STN) =
LOOK
    END SELECT
NEXT STN
END SUB
          I N V E S T   9
REM
REM This program solves both the upper and lower regions of the
REM free boundary problem similar to the one posed in the "time to
REM build" working paper of Saman Majd and Robert S. Pindyck.
REM The difference is that price (P) replaces value (V) as a state
REM variable in the formulation.
REM The code was last revised on January 21, 1987 by James C. Meehan.
REM
REM -----
OPTION BASE 0
REM
REM Specify the desired location of output file.
REM
PRINT "Where do you want the data? "
PRINT
PRINT "1. Print all output to an IBM or Epson printer"
PRINT "2. Write all output to diskette"
PRINT "(Enter 1 or 2)"
INPUT INFO
IF INFO = 1 THEN
    OPEN #1: PRINTER
    SET #1 : MARGIN 132
ELSE
    INPUT PROMPT "Enter file name for output: ": FILE$
    OPEN #1 : NAME FILE$, ACCESS OUTPUT, CREATE NEW
END IF
REM
REM          Enter the parameters of the problem.
REM R    -- annual riskless rate of interest, r
REM S    -- annual standard deviation, sigma
REM D    -- annual convenience yield, delta

```

```

REM INV  -- total investment, Ko
REM T    -- life of the project, T
REM COST -- operating cost per unit of output, C
REM Y    -- number of years investment project would take if
REM          investment proceeded at the maximum rate in each period
REM K(C) -- maximum annual rate of investment in period c, k
REM
REM Factor is used to normalize the horizontal axis.
REM
INPUT PROMPT "Enter annual riskless rate: ": R
INPUT PROMPT "Enter annual standard deviation: ": S
INPUT PROMPT "Enter annual convenience yield: ": D
INPUT PROMPT "Enter total investment, to the nearest integer: ": INV
INPUT PROMPT "Enter life of project: ": T
INPUT PROMPT "Enter operating cost: ": COST
INPUT PROMPT "Enter number of stages of investment project: ": Y
LET FACTOR = 10^INT(LOG10(INV))
DIM K(50)
DIM DD(2)
FOR C = 1 TO Y
  REM
  REM CC is used to reverse the numbering of the horizontal axis.
  REM
  LET CC = Y + 1 - C
  PRINT "Enter maximum annual rate of investment in stage"; C
  INPUT K(CC)
  LET DUM = DUM + K(CC)
  LET K(CC) = K(CC)/FACTOR
NEXT C
IF DUM >> INV THEN PRINT "WARNING: Sum of investments at each stage does
not equal total investment."
REM
REM Set dt equal to an arbitrarily small number.
REM The smaller is dt, the longer the program takes to run.
REM 1/dt must be evenly divisible by 4.
REM
LET DTINV = 12
LET DT = 1/12
REM
REM Select form of the output file.
REM
INPUT PROMPT "Would you like to see annual or quarterly output? (Enter a
or q) ": N$
SELECT CASE N$
CASE "A", "a"
  LET N = 1
CASE "Q", "q"
  LET N = 4
CASE ELSE
  PRINT "PLEASE ENTER q OR a NEXT TIME."
  PRINT "AS A PENALTY FOR NOT READING INSTRUCTIONS, YOU WILL BE
REQUIRED TO START AGAIN."

```

```

        STOP
END SELECT
PRINT
PRINT
PRINT
PRINT "RUNNING"
REM
CALL READ_IN_STANDARD_NORMAL
REM
REM
REM          Do initial computations.
REM Compute alpha from the analytical solution in the lower region.
REM           $f(P,K) = a \cdot P^{\alpha}$ 
REM Compute minimum value of dx given the values of dt and sigma.
REM Check to see if solution is stable given parameters of problem.
REM Arbitrarily set epsilon equal to half of dx.
REM Compute p+, po, and p- used in the explicit method.
REM
LET SS = S*S
LET STUFF = R - D - SS/2
LET ALPHA = (-STUFF + (STUFF*STUFF + 2*R*SS)^.5)/SS
LET DX = S*DT^.5
IF DX > SS/ABS(STUFF) THEN
    PRINT "Solution is not stable for input parameters given."
    STOP
END IF
LET EPSILON= DX/10
LET PU= DT*( SS/DX + STUFF )/ (2*DX)
LET PD= DT*( SS/DX - STUFF )/ (2*DX)
LET PF= 1-PU-PD
REM
REM Initialize arrays.
REM
DIM INDEX(200)
DIM G(200,49)
DIM DUMMY1(200)
DIM DUMMY2(200)
REM
REM Set upper and lower boundaries of vertical axis.
REM
LET B = 0
LET M = 200
REM
REM Fill in terminal boundary condition.
REM  $V(P) = \text{sum from } \tau = 1 \text{ to } T \text{ of } V_t(P)$ 
REM where  $V_t(P) = P \cdot \exp(-\delta \cdot \tau) \cdot N(d1) - C \cdot \exp(-r \cdot \tau) \cdot N(d2)$ 
REM
REM Compute upper = derivative of V w.r.t. P as P goes to infinity =
REM sum from tau = 1 to T of  $1/(1 + \delta)^{\tau}$ 
REM
FOR I=M TO B STEP -1
    LET VALU = 0

```

```

LET UPPER = 0
FOR TAU = 1 TO T
  LET UPPER = UPPER + 1/(1 + D)^TAU
  LET DD(1) = (LOG(EXP(L*DX)/COST) + STUFF*TAU)/(S*TAU^.5)
  LET DD(2) = DD(1) - S*TAU^.5
  CALL STANDARD_NORMAL_LOOKUP(2,DD,STANDARD_NORMAL)
  LET VALU = VALU + EXP(L*DX - D*TAU)*Z(1) - COST*EXP(-R*TAU)*Z(2)
NEXT TAU
LET DUMMY1(L) = VALU
LET G(L,0) = VALU
NEXT L
REM
REM Loop over entire horizontal axis, filling in column to the left
REM in dummy2(i), and saving desired columns in matrix g(i,j) which
REM corresponds to the option value expressed in the transformed
REM variable x.
REM
LET PERIODS = Y*DTINV
FOR J=1 TO PERIODS
  LET JJ = INT((J - 1)/DTINV) + 1
  LET DUMMY2(M) = PU*UPPER*2*DX* EXP( M*DX+(R - D)*(J-1)*DT) +
  PF*DUMMY1(M) + (PU+PD)*DUMMY1(M - 1) - K(JJ)*DT*EXP(R*(J - 1)*DT)
  LET L = M
  DO WHILE L > 0
    LET L = L - 1
    LET DUMMY2(L) = PU*DUMMY1(L + 1) + PF*DUMMY1(L) + PD*DUMMY1(L -
    1) - K(JJ)*DT*EXP(R*(J - 1)*DT)
    REM
    REM Check derivative condition to see if free boundary has been
    REM reached.
    REM If so, use the analytic solution on the remainder of the
  REM range.
  REM
  LOOP WHILE DUMMY2(L) - DUMMY2(L+1)/(ALPHA*DX +1) > EPSILON
  LET INDEX(J) = L
  LET A=EXP(-(R*J*DT))*DUMMY2(INDEX(J))/ EXP( INDEX(J)*DX*ALPHA )
  FOR L=INDEX(J)-1 TO 0 STEP -1
    LET DUMMY2(L)= A* EXP( L*DX*ALPHA ) * EXP(R*J*DT)
  NEXT L
  REM
  REM See if column needs to be saved in g matrix.
  REM
  IF MOD(J*N,DTINV) = 0 THEN
    LET COUNT = COUNT + 1
    FOR L = M TO B STEP -1
      LET G(L,COUNT) = DUMMY2(L)
    NEXT L
  END IF
  FOR L = M TO B STEP -1
    LET DUMMY1(L) = DUMMY2(L)
  NEXT L
NEXT J

```

```

PRINT
PRINT
PRINT
REM
REM Set number of columns to be written to 5 if writing to disk,
REM and 10 if writing in condensed mode to the printer.
REM
PRINT #1: "                SUMMARY OF INPUTS"
PRINT #1
PRINT #1
IF INFO = 1 THEN
    LET COL = 10
    PRINT #1: CHR$(27);CHR$(69);
ELSE
    LET COL = 5
END IF
PRINT #1, USING "<##### is ###.##, ": "Riskless Rate", R,
"Standard Deviation", S
PRINT #1, USING "<##### is ###.##, ": "Convenience
Yield", D, "Total Investment is", INV
PRINT #1, USING "<##### is ###.##, ": "Life of Project",
T, "Cost per Unit", COST
FOR C = 1 TO Y
    LET CC = Y + 1 - C
    PRINT #1: "In stage"; C;
    PRINT #1, USING "maximum rate of investment is ###.##,";
K(CC)*FACTOR
NEXT C
PRINT #1:
PRINT #1, USING "Calculations will use step size in x of ##.###, ": DX
PRINT #1, USING "Calculations will use step size in t of ##.###, ": DT
PRINT #1
PRINT #1
PRINT #1: "                OUTPUT"
IF INFO = 1 THEN PRINT #1: CHR$(27);CHR$(70);
PRINT #1
PRINT #1
IF INFO = 1 THEN PRINT #1: CHR$(15)
LET V = N*Y
IF V - COL + 1 > 0 THEN LET VV = V - COL + 1 ELSE LET VV = 0
FOR TABLE = 1 TO ( INT(V/COL)+1 )
    PRINT #1: "                Value of"
    PRINT #1: "                underlying    Stages remaining, and"
    PRINT #1: " Price                asset        Value of contingent claim"
    PRINT #1: "                ";
    FOR J= V TO VV STEP -1
        PRINT #1, USING "###                ": J;
    NEXT J
    PRINT #1
    PRINT #1
    FOR L = M TO B STEP -1
        IF EXP(L*DX) < 100 AND EXP(L*DX) > COST/2 THEN

```

```

        PRINT #1, USING ">####.## >#####.## ": EXP(L*DX),
G(L,0)*FACTOR;
    IF TABLE < INT(V/COL) + 1 AND INFO = 1 THEN LET CONTROL = VV
ELSE LET CONTROL = VV + 1
    FOR J=V TO CONTROL STEP -1
        IF INDEX(J/(N*DT))=L THEN LET Q$="*" ELSE LET Q$=" "
        LET DUMMY= G(L,J)* EXP(-(R*J/N))
        PRINT #1, USING ">####.## # ": DUMMY*FACTOR, Q$;
    NEXT J
    IF CONTROL >< VV THEN
        IF INDEX(VV/(N*DT))=L THEN LET Q$="*" ELSE LET Q$=" "
        LET DUMMY= G(L,0)*EXP(-(R*VV/N))
        PRINT #1, USING ">####.## # ": DUMMY*FACTOR, Q$
    END IF
END IF
NEXT L
PRINT #1
PRINT #1
PRINT #1
PRINT #1
LET V = V - COL
IF V - COL + 1 < 0 THEN LET VV = 0 ELSE LET VV = V - COL + 1
NEXT TABLE
CLOSE #1
END

```


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