A Profit Maximization Model in a Two-Echelon Supply Chain Management: Distribution And Pricing Strategies

by

YE MAO
B.S. in Civil Engineering (2000)
Tsinghua University, Beijing, P.R. China

Submitted to the Department of Civil and Environmental Engineering and the Sloan School of Management in Partial Fulfillment of the Requirements for the Degrees of

Master of Science in Transportation
and
Master of Science in Operations Research

at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY
June 2003

C⃝ 2003 Massachusetts Institute of Technology. All rights reserved.
A Profit Maximization Model for a Two-Echelon Supply Chain Management: Distribution and Pricing Strategies

by

YE MAO

Submitted to the Department of Civil and Environmental Engineering and the Sloan School of Management on May 6, 2003 in Partial Fulfillment of the Requirements for the Degrees of

Master of Science in Transportation

and

Master of Science in Operations Research

Abstract

Distribution and pricing strategies play a central role in the field of supply chain management. Heuristic approaches to the vehicle routing problem (VRP) are usually used to design optimal delivery routes to serve geographically dispersed customers, who are price elastic. There is a rich literature discussing either the manufacturer’s distribution strategy or its pricing initiatives. The purpose of this thesis is to develop a profit maximization model that presents an integrated distribution and pricing strategy for any company facing such issues. We first examine a simplified scenario when all customers are located in the same delivery region and their demand is deterministic. Both truckload (TL) and less-than-truckload (LTL) shipment strategies are analyzed and compared. We later extend our findings to multiple delivery regions and discuss the impact of the manufacturer’s pricing flexibility on its profit. Then we relax the assumption of deterministic customer demand and introduce the safety stock cost. Finally the application on across delivery region situations is shown. Although some of our assumptions simplify our model, we believe that it provides insight into more complex supply chain management problems.

Thesis Supervisor: Donald B. Rosenfield
Title: Senior Lecture, Sloan School of Management
Director, Leaders for Manufacturing Fellows Program

Thesis Reader: Joseph M. Sussman
Title: JR East Professor
Professor of Civil and Environmental Engineering
Engineering Systems Division
Acknowledgements

I would like to thank my advisor, Don Rosenfield, for his continuous guidance, advice and encouragement throughout my three years at MIT. His valuable suggestions and comments are essential to the completion of this thesis. From our interaction, I learned way beyond Transportation and Operations Research. It is my tremendous fortune and honor to have had the opportunity of working with him.

Many thanks to my thesis reader, professor Joe Sussman, for his advice and time in reviewing this thesis. He lectured the very first class I took at MIT and that was a superb experience.

My gratitude to all the faculties, students and administrative staffs at the CTL for making the center an inviting, collaborative and interesting place. Especially professor Nigel Wilson, who has been my academic advisor, and professor Cynthia Barnhart, who supports me to pursue the Master of Science in Operations Research. My sincere thanks to Yasmine El Alj and Alessandra Cappiello, whose friendship I shall treasure.

I would also like to thank the Leaders for Manufacturing Program at MIT for funding my research and study.

Finally, as always, I am grateful to my family – my parents, sister and finance – for their never-ending patience, love and support.
Contents

1 Introduction .................................................................................. 13
  1.1 Problem Specifications.......................................................... 16
    1.1.1 Problem Definitions.................................................... 16
    1.1.2 Model Assumptions....................................................... 17
  1.2 Literature Reviews.................................................................. 17
    1.2.1 Literature Reviews for Basic Initiatives............................ 17
    1.2.2 Literature Reviews for Pricing in Supply Chain Management... 18
    1.2.3 Literature Reviews for Combined Pricing and Inventory Management in Supply Chain Management........................................... 19
    1.2.4 Literature Reviews for the Vehicle Routing Problem (VRP) and Production-Distribution Integration........................................... 20

2 Model with Deterministic Demand and a Single Delivery Region ........ 23
  2.1 Model Assumptions................................................................... 25
  2.2 Customer Characteristics....................................................... 26
  2.3 Model for Truckload (TL) Shipment......................................... 27
    2.3.1 Model Objective.............................................................. 27
    2.3.2 Variable Definitions........................................................ 27
    2.3.3 Decision Variables......................................................... 28
    2.3.4 Model Formulation.......................................................... 28
    2.3.5 Solutions........................................................................... 29
    2.3.6 Computing Procedures.................................................... 31
    2.3.7 Sensitivity Analysis.......................................................... 31
2.3.7.1 What If the Demand Increases?.................................31
2.3.7.2 What If the Demand Becomes More Elastic?...............33
2.3.7.3 What If the Demand Becomes Less Elastic (More Inelastic)?...35
2.3.7.4 What If the Customer Density Increases?......................36
2.3.7.5 What If the Distance Increases?.................................37
2.3.8 Special Case: Direct Shipping........................................39
2.3.9 What If $r_u < 1$?.......................................................39
2.4 Model for Less-Than-Truckload (LTL) Shipment......................40
2.4.1 Model Objective.......................................................40
2.4.2 Variable Definitions...................................................40
2.4.3 Decision Variables....................................................40
2.4.4 Model Formulation....................................................40
2.4.5 Solutions.....................................................................41
2.5 An Integrated TL/LTL Shipment Model.................................42
2.5.1 Objective Function.....................................................42
2.5.1.1 Profit Functions....................................................42
2.5.1.2 Transportation Cost Component..............................43
2.5.1.3 Inventory Cost Component....................................43
2.5.1.4 Local Peddling Distance.......................................44
2.5.1.5 The Integrated Profit Functions for TL and LTL Shipments.....45
2.5.2 Solutions.....................................................................45

3 A Convenient Delivery Strategy: Dispatch One Truckload at a Time 47
3.1 A Criteria for Determining the Optimality..........................47
3.2 Derive a Lower Bound....................................................49
3.3 Numerical Examples.......................................................54
3.3.1 Example 1...............................................................54
3.3.2 Example 2...............................................................55
## 4 A Discussion on Deterministic Demand-Price Functions

4.1 Linear Demand Price Function .............................................................. 57
4.2 Constant Price Elasticity Demand Function ........................................ 59
4.3 Demand Function of New Durable Products ...................................... 60
   4.3.1 Bass Diffusion Model ............................................................... 60
   4.3.2 Combining Bass Diffusion Model with the Linear Demand Function .... 62
   4.3.3 Combining Bass Diffusion Model with the Constant Price Elasticity Demand Function .............................................................. 63

## 5 Application on across Delivery Region Situations

5.1 Model Assumptions .............................................................................. 65
5.2 Scenario One: Prices are Allowed to Be Variable for Different Delivery Regions .............................................................. 66
5.3 Scenario Two: Prices Need to Be Kept Constant across Delivery Regions .... 67
   5.3.1 Model Objective ........................................................................ 67
   5.3.2 Variable Definitions ................................................................... 67
   5.3.3 Decision Variables ...................................................................... 69
   5.3.4 Model Formulation ..................................................................... 69
   5.3.5 Solutions .................................................................................... 71
      5.3.5.1 Linear Demand-Price Function Scenario .............................. 72
      5.3.5.2 Constant Price Elasticity Demand Function Scenario .......... 73
   5.3.6 Algorithm for Optimal Price ....................................................... 75

## 6 Model with Stochastic Demand and a Single Delivery Region

6.1 Model Assumptions .............................................................................. 77
6.2 Demand and Safety Stock Statistics .................................................. 78
   6.2.1 Variable Definitions ................................................................... 78
   6.2.2 Demand Statistics ...................................................................... 79
   6.2.3 Inventory Management Policies and Safety Stock ........................ 81
6.3 Model with Stochastic Demand and One Delivery Region ................. 81
   6.3.1 Model Objective ........................................................................ 82
6.3.2 Variable Definitions .............................................................82
6.3.3 Decision Variables .............................................................82
6.3.4 Model Formulation .............................................................82
6.3.5 Solutions ........................................................................83
   6.3.5.1 Linear Demand-Price Function Scenario .......................83
   6.3.5.2 Constant Price Elasticity Demand Function Scenario ........84
6.3.6 Computing Procedures ........................................................85
6.4 Model Results Interpretation .....................................................86

7 Model with Stochastic Demand and Multiple Delivery Regions 89
7.1 Model Assumptions .................................................................89
7.2 Scenario One: Prices are Allowed to Be Variable for Different Delivery
              Regions .........................................................................90
7.3 Scenario Two: Prices Need to Be Kept Constant across Delivery Regions ....91
   7.3.1 Demand and Safety Stock Statistics ......................................91
      7.3.1.1 Variable Definitions .................................................91
      7.3.1.2 Demand Statistics .....................................................93
      7.3.1.3 Safety Stock Statistics ..............................................95
   7.3.2 Model Objective ...............................................................97
   7.3.3 Variable Definitions ........................................................97
   7.3.4 Decision Variables ..........................................................97
   7.3.5 Model Formulation ..........................................................97
   7.3.6 Solutions .......................................................................99
      7.3.6.1 Linear Demand-Price Function Scenario .....................100
      7.3.6.2 Constant Price Elasticity Demand Function Scenario ......101
   7.3.7 Algorithm for Optimal Price ..............................................103

8 Conclusions and Recommendations 105
8.1 Summary of Results ...............................................................105
8.2 Recommendations for Future Research Areas .............................107
## List of Figures

1-1  Distribution System.................................................................16

2-1  Customer Geographical Distribution & Delivery Region Representation........24

2-2  Linear Demand-Price Function..................................................26

2-3  A Typical Peddling Situation...................................................29

2-4  Parallel Shift in Linear Demand-Price Function..............................32

2-5  Price Elasticity Change in Linear Demand-Price Function....................33

2-6  Special Case: Direct Shipping....................................................39

4-1  Constant Price Elasticity Demand Function.....................................59
Chapter 1

Introduction

The topic of increasing a manufacturing company’s profit has been in discussion since 1950s. Different elements that directly contribute to the manufacturer’s profit include its distribution channel structure, its revenue and its cost.

A firm’s choice of distribution network is more of a strategic issue than tactical operations. Companies may have various types of distribution channels. A traditional channel setup would be the following. The producer ships finished goods to wholesalers, who purchase with large volume and often with discounts. Retailers buy from wholesalers and sell to end users at a premium. However, the structure of distribution channels has been changed continuously as technology greatly facilitates the information dispersion among interested parties. It has been common for us to see customers bypassing immediate upstream echelons in the distribution network. For example, in the airline industry, customers used to book their tickets from those agencies which then booked tickets with airlines. This is still a popular mode in today’s airline seats selling (distribution) network and represents a significant portion of the total revenue. Nevertheless, there is a growing trend for end customers (individual fliers) to book tickets directly from airlines as their fare promotions become more and more easily accessible.
On the other hand, powered by the internet search engines for low fares, a group of companies such as Expedia.com represents a new echelon in the distribution channel and is gaining an increasing portion of the total revenue. We are expecting new and exciting changes in the distribution network.

It is a common pitfall for manufacturing companies to view the revenue generation as a marketing problem. Consequently it would not be surprising for us to find that this problem is managed separately from the firm’s distribution channel selection and its operations strategies. Selling prices and quantities however determine the total revenue. Any change in the selling price would incur a consequent change in the product quantity sold. The end users’ demand shift will be ultimately captured by the upstream echelons. The manufacturer, near the upstream end of the supply chain, is certain to encounter a much larger demand change as explained by the famous bullwhip effect. Lee HL, Padmanabhan V, Whang S qualitatively addressed the bullwhip effect in the paper “The bullwhip effect in supply chains” in 1997[21]. Therefore, it tailors its production and distribution operations to satisfy the exaggerated demand change, which affects the operations cost. Thus we argue that pricing decisions should not be treated as a separate marketing issue. A good pricing strategy such as temporary rebate policy could stimulate enough excess demand, which generates a higher revenue for a period of time. More often, a carefully reasoned price discriminate policy can benefit the company with a stable revenue streams and steady daily operations. Thus pricing should be incorporated within a total cost and revenue equation.

Total cost is the other factor that is part of the company’s profit equation. Manufacturing cost, transportation cost and inventory holding cost are the three most frequently used components when deriving the total operations cost. When there does not exist an explicit evidence of economies of scale or diseconomies of scale in the current production volume, the effect of manufacturing cost is often negligible. There is a rich literature discussing either the transportation cost or the inventory holding cost alone or the trade-off between the two. On the transportation cost side, the vehicle routing
problem (VRP) and the traveling salesman problem (TSP) focus on finding an optimal delivery strategy at minimum cost while satisfying various constraints. On the inventory cost side, we have seen different policies such as periodic and continuous review or more complicated models. The economic order quantity (EOQ) approach is first introduced in 1913 by Ford W. Harris to solve the trade-off between the inventory cost and the setup cost and set the manufacturing lot sizes [16].

The objective of this thesis is to develop a model that maximizes a manufacturer’s overall profit given its customer distribution. When designing the model, we try to consider all three elements, distribution network, pricing and distribution strategies. As distribution network design is a high-level strategic issue and usually already pre-chosen by the firm, our focus is on a two echelon distribution channel setup. We believe that a simple base network would well illustrate our key findings on simultaneously deploying price discriminating and distribution strategies.

The thesis is organized as follows. In chapter 1, we introduce the problem and present an overview, which includes the problem specifications and related literature review. In chapter 2, we discuss the model with assumptions that the demand is deterministic and our customers are located in a single delivery region. Both truckload (TL) and less-than-truckload (LTL) delivery strategies are compared with sensitivity analysis on parameter changes. A combined TL/LTL formulation is presented, which is followed by numerical examples. In chapter 3, we introduce a delivery strategy that is convenient for companies to employ – serve customers as frequently as possible with TL shipments. Examples are given to illustrate its optimality. In chapter 4, we investigate three types of deterministic demand-price functions to simulate customers’ price sensitivity, and their applications to our model are studied. In chapter 5, we relax one of our assumptions in chapter 2 and allow customers to occupy different delivery regions. We discuss the assumption’s impact on the manufacturer’s pricing flexibility and present a modified formulation. In chapter 6, we further relax the deterministic demand assumption presented in chapter 2 and extend our model to incorporate stochastic demand. In chapter 7, we apply the
results of chapter 6 to across delivery region settings. We shall summarize our results and suggest future research directions in chapter 7.

1.1 Problem Specifications

1.1.1 Problem Definitions

We consider a manufacturing company whose distribution network consists of two echelons: its central manufacturing facility and a channel of wholesalers/retailers/end customers. Without loss of generality, the manufacturing facility can be replaced by a warehouse, and the second echelon may be occupied by retailers or end customers instead of wholesalers. Figure 1-1 illustrates our focus on a typical supply chain. The company owns a fleet of trucks that delivers finished products to customers. Customers are located at different delivery regions with unique distribution density. Furthermore, they are price sensitive and respond to pricing initiatives. Our objective is to use price discrimination and design a distribution strategy to maximize the company’s profit. Besides the revenue side, we assume that the total cost consists of production cost, transportation cost and inventory holding cost.

![Distribution System](image)

Figure 1-1: Distribution System
1.1.2 Model Assumptions

For simplicity, we have made the following assumptions.

- In one typical delivery region, we have \( n \) customers who are of the same demand type. In this case, they share exactly the same demand-price curve.
- Every customer in the delivery regions has to be visited per shipment. In other words, trucks make stops at all customers for each shipment.
- The company owns the fleet of trucks, and it pays the transportation cost.
- The company owns its finished products until they reach the customers.
- Prices are constant within the delivery region.
- Customers are subject to a minimum replenishment frequency.
- The manufacturer uses algorithms based on the Vehicle Routing Problem to schedule its truck fleet.

1.2 Literature Reviews

1.2.1 Literature Reviews for Basic Initiatives

Our initiative for this thesis stems from the paper of Burns L., Hall R., Blumenfeld D. and Daganzo C. (1985). This paper develops and evaluates an analytic method that aims in minimizing the sum of transportation and inventory costs for a supplier who distributes items to many customers. The analytic approach focuses on the spatial density of customers and on the distribution of customer demand, rather than on the demand of specific customers in precise locations. Their results indicate that, for direct shipping, the optimal shipment size is given by the economic order quantity (EOQ) model, while for peddling, the optimal shipment size is a full truck. Direct shipping and peddling are two specific distribution strategies. Direct shipping involves shipping separate loads from the supplier directly to each customer. Peddling involves dispatching trucks that deliver items to more than one customer per load [5].

Blumenfeld D., Burns L., Diltz J., and Daganzo C. (1985) discuss optimal shipping strategies (i.e. routes and shipment sizes) on freight networks by analyzing trade-offs
between transportation, inventory, and production setup costs. A simple optimization method is developed that simultaneously determines optimal routes and shipment sizes [3].


1.2.2 Literature Reviews for Pricing in Supply Chain Management

Most of the existing literatures on pricing issues in supply chain management focus on quantity discount, manufacturer's return policy, the importance of coordination within the supply chain, and transfer pricing.

Burwell T., Dave D., Fitzpatrick K., and Roy M. (1997) incorporate quantity and freight discounts in inventory decision making and develop an algorithm to determine the optimal lot size and selling price for a class of demand functions, including constant price-elasticity and linear demand [7]. Corbett C. and Groote X. (2000) drop the assumption that the supplier has full information about the buyer's cost structure and derive the supplier’s optimal quantity discount policy for the joint economic lot-sizing problem under asymmetric information and compare it to the situation where the supplier has full information [12].

Marvel H. and Peck J. (1995) model the manufacturer's decision to accept returns, showing that this decision depends crucially on the nature of the demand uncertainty. Uncertainty over customer arrivals favors returns, while uncertainty over consumers' valuation of the manufacturer's product leads distributors to set retail prices too high (from the manufacturer's standpoint) when returns are allowed. It is shown that returns can be expected to raise retail prices, while maintaining or shrinking distributor margins [22]. Emmons H. and Gilbert S. (1998) identify the role of returns policy in pricing and inventory decisions for catalogue goods [13]. Lau H. and Lau A. (1999) discuss a manufacturer’s pricing strategy and return policy for a single-period commodity [20].
Thomas D. and Griffin P. (1996) address coordinated planning between two or more stages of the supply chain, and suggest coordination of procurement, production and distribution within the supply chain [26]. Gilbert S. and Ballou R. (1999) develop a model that quantifies the benefits to the supplier from obtaining advanced commitments from downstream customers. This model can be used to suggest the maximum price discount that can be offered to customers to encourage them to commit to their orders in advance. Careful balancing of the advanced ordering time with the price discount can lead to cost reductions for both members of the supply channel [15].

Vidal C. and Goetschalckx M. (2001) present a global supply chain model with transfer pricing and transportation cost allocation [27].

1.2.3 Literature Reviews for Combined Pricing and Inventory Management in Supply Chain Management

Kunreuth H. and Schrage L. (1973) discuss joint pricing and inventory decisions for constant priced items [19]. Gallego G. and Vanryzin G. (1994) formulate the dynamic pricing problem of selling a given stock of items by a deadline. Demand is price sensitive and stochastic and the firm’s objective is to maximize expected revenues. For a particular exponential family of demand functions, they find the optimal pricing policy in closed form. For general demand functions, they find an upper bound on the expected revenue based on analyzing the deterministic version of the problem [14].

Weng Z. (1999) derives closed-form multi-attribute measures of performance for the supply chain consisting one manufacturer and one distributor in the presence of and in the absence of coordination of pricing and production/ordering decisions [29].

Bhattacharjee S. and Ramesh R. (2000) present a heuristic multi-period inventory and pricing model for a single product, where the product has a fixed life perishability for a certain number of periods. The profit maximization problem is modeled as a dynamic
program, and the Wagner-Whitin dynamic programming recursions are developed for both perishable and non-perishable products [1].

Ingene C. and Parry M. (1995) discuss wholesale pricing behavior within a two-level vertical channel consisting of a manufacturer selling through multiple independent retailers [18].

Chen F., Federgruen A., and Zheng Y. (2001) address a two-echelon distribution system in which the sales volumes of the retailers are endogenously determined on the basis of known demand functions. They derive an optimal strategy, maximizing total systemwide profits in a centralized system, but only if coordination is achieved via periodically charged, fixed fees, and a nontraditional discount pricing scheme under which the discount given to a retailer is the sum of three discount components based on the retailer’s (i) annual-sales volume, (ii) order quantity, and (iii) order frequency, respectively. The authors also show that no (traditional) discount scheme, based on order quantity only, suffices to optimize channelwide profits when there are multiple nonidentical retailers [9].

1.2.4 Literature Review for the Vehicle Routing Problem (VRP) and Production-Distribution Integration

Clarke G. and Wright J. proposed a most widely used VRP heuristic algorithm in 1964 for vehicle scheduling from a central depot to a number of delivery points [11].

Later treatments usually apply the vehicle routing problem to the bigger picture of integrating production and distribution systems. Blumenfeld D., Burns L., and Daganzo C. (1991), for example, examine the problem of whether it is cost-effective to synchronize production and transportation schedules on a product network, which consists of one origin and many destinations. Tradeoffs between production set-up, freight transportation, and inventory costs on the network are analyzed, and total costs are
compared for synchronized and independent schedules. This paper focuses on a simple production system and the cost savings from synchronization is very large [2].

Chien T. (1993) discusses direct shipping policies to maximize profit where the demands that a company are facing are stochastic but follow a known probability distribution [10].

Chandra P. and Fisher M. (1994) compare two approaches, one in which the production scheduling and vehicle routing problems are solved separately, and another in which they are coordinated within a single model. The two approaches are applied to 132 distinct test cases with different values of the basic model parameters, which include the length of the planning horizon, the number of products and retail outlets, and the cost of setups, inventory holding and vehicle travel. The reduction in total operating cost from coordination ranges from 3% to 20%, which indicates the conditions under which companies should consider the organizational changes necessary to support coordination of production and distribution [8].

Pyke D. and Cohen M (1994) develop a model of an integrated production-distribution system comprised of a single station model of a factory, a stockpile of finished goods, and a single retailer. The distributions of key random variables are approximated to compute costs and service levels for all products across the supply chain [24].

Viswanathan S. and Mathur K. (1997) consider distribution systems with a central warehouse and many retailers that stock a number of different products. Demands are deterministic at retailers for each product. The warehouse acts as a break-bulk center and does not keep any inventory. The products are delivered via peddling. The objective is to determine replenishment policies that specify the delivery quantities and the vehicle routes used for the delivery, so as to minimize the long-run average inventory and transportation costs. A heuristic approach is developed as a stationary nested joint replenishment policy for the problem. The proposed heuristic is capable of solving
problems involving distribution systems with multiple products, which differs from many existing methods [28].

In “A review of integrated analysis of production-distribution systems” by Sarmiento A. and Nagi R. (1999), recent work on integrated analysis of production-distribution systems is reviewed [25].
Chapter 2

Model with Deterministic Demand and a Single Delivery Region

As discussed in chapter one, our objective is to derive certain pricing strategy in coordination with a company’s distribution strategy. By the appropriate price discrimination integrated in the efficient distribution strategy, we show that we can optimize an overall profit function. We first discuss the ownership of the transportation cost, the definition of delivery regions, and the Vehicle Routing Problem (VRP) application to our problem.

Either a company or its customers assume the cost of transportation for finished goods shipped from the manufacturer’s central facility or warehouse to the customer’s door. If in the contract, the customers actually pay for this transportation cost, the manufacturer can simply use some third party logistics company to delivery the goods and does not need to worry about this cost component directly. However, if the manufacturer is instead responsible for the transportation cost, the company must also address the issue of the distribution strategy. Another possible practice would be for the manufacturing company to charge a calculated amount of shipping and handling fee on its customers,
and on the other hand, manage its own trucking fleet or contract out a third party logistics company. This type of cost structure could in fact earn a premium on shipping cost. From the supply chain’s perspective, it is also the manufacturer’s concern to keep the transportation cost low even if it is not directly associated with this cost component. Therefore without loss of generality, we assume in our model that the company owns its fleet of trucks and pays for the transportation cost.

We further assume that customers are grouped into regions by their geographical location and their demand level. Each region is treated independently, and we refer it as a delivery region throughout this thesis. Figure 2-1 gives a graphic view. The company is allowed to set only one specific price within each delivery region for various practical reasons. However, the selling prices across different delivery regions can vary.

![Figure 2-1: Customer Geographical Distribution & Delivery Region Representation](image)
Since each delivery region is treated independently, the truck dispatch for every delivery region is also scheduled independently. First let us define direct shipping and peddling again. Direct shipping and peddling are two specific distribution strategies. Direct shipping involves shipping separate loads from the supplier directly to each customer. Peddling involves dispatching trucks that deliver items to more than one customer per load. Since a truck has to visit all the customers assigned to it exactly once, in the local peddling scenario, it is the well-known Vehicle Routing Problem (VRP). In this case, a company would apply the VRP results to schedule its trucks.

In this chapter, we focus on developing a profit maximization model given that the customer demand is deterministic and the company groups all of its customers into one delivery region. We will relax the deterministic demand assumption in chapter 6, and the single delivery region assumption will be relaxed in chapter 5.

An outline of this chapter is as follows. After listing the assumptions in section 2.1, we discuss the customer characteristics in section 2.2. Then a model with truckload (TL) shipments is presented in details in section 2.3, followed by a model with less-than-truckload (LTL) shipments in section 2.4. Within the section of TL shipments, various sensitivity analyses are discussed and compared. Section 2.5 seeks to generalize the model with a combined TL/LTL shipping method.

2.1 Model Assumptions

Based on the assumptions we have discussed in section 1.1, we assume the following.

- Customer demand is deterministic.
- All customers are located in one delivery region.
- In one typical delivery region, we have \( n \) customers who are of the same demand type. In this case, they share exactly the same demand-price curve. We use the linear demand-price function as our base discussion throughout this chapter.
- Every customer in the delivery regions has to be visited per shipment. In other words, trucks make stops at all customers for each shipment.
- The company owns the fleet of trucks, and it pays the transportation cost.
- The company owns its finished products until they reach the customers.
- Prices are constant within the delivery region.
- Customers are subject to a minimum replenishment frequency.
- The manufacturer uses algorithms based on the Vehicle Routing Problem to schedule its truck fleet.

### 2.2 Customer Characteristics

Customer demand pattern and their distribution density are the two aspects that characterize the company's customers in this model. We define density as the number of customers per square unit of area, and denote it by \( p \). As discussed before, we further assume that all customers in this delivery region share the same demand-price curve and therefore incur only one demand level \( q \) for each given price. Figure 2-2 below shows one commonly used demand-price function: linear demand-price function.

\[ q = f(p) = -a \cdot p + b \]

where \( a > 0 \), \( b > 0 \).

![Figure 2-2: Linear Demand-Price Function](image-url)
2.3 Model for Truckload (TL) Shipment

Generally, because of the scale economies, truckload (TL) shipments are more attractive than less-than-truckload (LTL) shipments. When the optimal solution dictates shipping one TL to each customer, we refer to this as the Direct Shipping Method. And when the aggregate customer demand is so low that we could not afford to serve them by peddling one TL per shipment, we will be forced to use LTL Shipments. The LTL shipment method is presented in section 2.4, and the direct shipping method will be explained in chapter 3. Special cases are discussed later in more details in this chapter.

2.3.1 Model Objective

The objective of this model is to maximize the total profit for the manufacturer.

2.3.2 Variable Definitions

\( p \) Offering price per unit in this region ($/unit)

\( q \) Demand per customer (units/(week*customer))

\( h \) Manufacturing cost per unit ($/unit)

\( R \) Capital of interest ($/week)

\( D \) Average round-trip distance from the warehouse to the delivery region (miles)

\( d \) Average local peddling delivery distance per truck (miles)

\( T \) Average transit time from the warehouse to the delivery region per truck (weeks)

\( t \) Average local peddling transit time per truck (weeks)

\( V \) Shipment Size (units/shipment)

\( W \) Full truckload (units/truck)

\( \gamma \) Fixed cost of initiating one truck dispatch ($/dispatch)

\( \alpha \) Transportation cost per unit direct shipping distance ($/mile)

\( \beta \alpha \) Transportation cost per unit peddling distance ($/mile)

where \( \beta \) is a scale factor.

\( \sigma \) Fixed cost of a customer stop ($/stop)

\( n \) Number of customers per delivery region (i.e. delivery region size)
\[ m \quad \text{Number of stops per truck} \]
\[ \rho \quad \text{Customer density (customer per square mile)} \]
\[ \rho = n / A, \text{ where } A \text{ is the area of the region.} \]
\[ f \quad \text{Shipment frequency (number of shipments/week)} \]
\[ K \quad \text{Approximation constant for calculating } d \]

### 2.3.3 Decision Variables

\[ p \quad \text{Selling price per unit in this region ($/unit)} \]
\[ r \quad \text{Number of TLs per shipment} \]

where \( r \) takes only positive integer values.

### 2.3.4 Model Formulation

The company’s profit is its revenue net its costs. We consider the total cost consisting of the manufacturing cost, the transportation cost and the inventory cost. Each component is addressed as follows.

\[ \text{Revenue (per unit)} = p \quad \text{(2.1)} \]
\[ \text{Manufacturing Cost (per unit)} = h \quad \text{(2.2)} \]
\[ \text{Transportation Cost (per unit)} = \frac{\gamma + \alpha d + \beta d + \sigma m}{W} \quad \text{(2.3)} \]

(\text{Note that } D, d, m, T \text{ and } t \text{ are all based on each truck.})

\[ \text{Inventory Cost (per unit)} = hR^* \left( \frac{V}{2nq} + T + t \right) = hR^* \left( \frac{rW}{2nq} + T + t \right) \quad \text{(2.4)} \]

In the inventory cost component, the term \( \frac{V}{2nq} \) represents the assumption that the manufacturer owns the finished goods for approximately half of their lifetime, from manufacturing to delivery. The question is how much peddling distance there is, based on typical heuristic approaches to the Vehicle Routing Problem. One can estimate the local peddling distance from the number of stops per truck, the customer distribution
density, etc. See, for example the thesis of Brenda P. Choy, Analyzing Total Distance in Vehicle Routing Problems, 1996 [4].

A typical peddling situation is shown in Figure 2-3 below.

![Figure 2-3: A Typical Peddling Situation](image)

Therefore, the number of stops per truck is \( m = \left\lfloor \frac{n}{r} \right\rfloor \) (2.5), and the peddling distance per truck is

\[
d = K \sqrt{\frac{mn}{r \rho}} = K \sqrt{\frac{n}{r}} \sqrt{\frac{n}{r \rho}} \quad (K=0.6)
\]

Profit (per unit) = \( M_U \)

\[
M_U = p - hR \left( \frac{rW}{2nq} + T + t \right) - \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) - h
\]

Profit = \( M_U * nq = M = f(p,r) \)

\[
M = npq - hR \left( \frac{rW}{2} + (T + t) * nq \right) - \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) * nq - hnq
\]

### 2.3.5 Solutions

As defined in section 2.3.2, the shipment size is \( V = rW \), and the shipment frequency is \( f \). In order to meet all the demand, we make shipment size times shipment frequency
equals demand. That is, \( rW \cdot f = nq \) with both the RHS (right hand side) and the LHS (left hand side) conditioned on a fixed period of time. And we get \( r = \frac{nq}{fW} \) (2.9).

Using equation (2.9) above, we can find the upper bound of \( r \) by imposing a minimum shipment frequency, denoted by \( f_{\text{min}} \). Therefore, the upper bound of \( r \), the maximum number of TLs per shipment is, \( r_U = \left\lfloor \frac{nq}{f_{\text{min}} \cdot W} \right\rfloor \) (2.10).

Thus a vector of \( r \) such that \( \bar{r} = \{1, 2, 3, ..., r_U\} \) could be set up. And for any given \( r \), we can find the optimal offering price of \( p^* \) according to equation (2.11) below. Plugging the linear demand-price function into the equation of the total profit, we get the total profit as a function of the selling price and \( r \).

\[
M = npq - hR \left( \frac{rW}{2} + (T + t) \right) nq - \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) nq - hnq
\]

\[
= npq - \left[ hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] nq - hR \cdot \frac{rW}{2}
\]

\[
= np(-ap + b) - \left[ hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] n(-ap + b) - hR \cdot \frac{rW}{2}
\]

Take the partial derivative of the total profit with respect to the selling price \( p \).

\[
\frac{\partial}{\partial p} M(p, r) = 0
\]

\[
-2anp + bn + an \left[ hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] = 0
\]

\[
p^* = \frac{1}{2} \left[ \frac{b}{a} + hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right]
\]

Therefore, the optimal selling price based on a given \( r \) is,

\[
p^* = \frac{1}{2} \left[ \frac{b}{a} + hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] ^{(2.11)}
\]
A notable result in the equation (2.11) is that \( r \) directly affects \( d \) and \( m \), and indirectly affects \( t \) in deriving the optimal offering price \( p^* \). In fact, if \( r \) increases, the inventory cost increases as well but the peddling transportation cost decreases. In chapter 3, we will look at the scenario where the customers are served by one TL shipment at a time.

### 2.3.6 Computing Procedures

The procedures for computing an optimal profit and its associated optimal number of TLs per shipment are given as follows.

1. For any given \( r \), the optimal offering price \( p^*_i \) is given above in equation (2.11).
2. By substituting \( p^*_j \) into equation (2.8), we can get the maximum profit \( M^*_j \).
3. Repeat 1,2 for the valid vector of \( r \), i.e., \( r = \{r_1, r_2, ..., r_U\} \), and get a vector of \( M^* \), i.e., \( M^* = \{M^*_1, M^*_2, ..., M^*_U\} \).
4. The optimized (maximized) profit is \( M_{\text{max}} = \text{Max}(M^*) \). Accordingly, the optimal offering price is \( p^*_j \); the optimal shipment size is \( V_j = r_j * W \); and the optimal shipment frequency is \( f_j = \frac{nq_j}{r_j * W} = \frac{(-ap^*_j + b) * n}{r_j * W} \).

### 2.3.7 Sensitivity Analyses

#### 2.3.7.1 What If the Demand Increases?

Suppose that customer demand pattern can be captured by parallel linear demand-price curves. Thus, when customers’ demand becomes larger, it implies a larger \( b \) value, say \( b_1 \), where \( b_1 > b \), \( b_1 = b + \Delta b \). Figure 2-4 illustrates such a demand change.
For the scenario where the customer demand increases, i.e., $\Delta b > 0$, let subscript 0 denotes state 0 (initial state/base case), and subscript 1 denotes state 1 (the new scenario with increased demand). In this example, customer demand is larger in state 1 than in state 0.

\[
q_0 = -a^* p_0 + b_0 \quad (2.12)
\]

\[
q_1 = -a^* p_0 + b_1 = -a^* p_0 + (b_0 + \Delta b) \quad (2.13)
\]

\[
\Delta b > 0 \quad (2.14)
\]

According to equation (2.10),

\[
r_{U,1} = \left[ n^* \left( \frac{-ap_0 + b_0 + \Delta b}{f_{\min}^* W} \right) \right] \geq r_{U,0} = \left[ n^* \left( \frac{-ap_0 + b_0}{f_{\min}^* W} \right) \right]. \quad (2.15)
\]

This implies that as the demand increases, the upper bound of the number of TL per shipment also increases or at least keeps the same. Therefore, it is possible for us to get a larger optimal shipment size in state 1, but it is not guaranteed.

According to equation (2.11),

\[
p_{0}^{*} = \frac{1}{2} \left[ \frac{b_0}{a} + hR^*(T + t) + \left( \frac{\gamma + \alpha D + \beta \sigma d + \sigma m}{W} \right) + h \right] \quad (2.16)
\]

\[
p_{1}^{*} = \frac{1}{2} \left[ \frac{b_0 + \Delta b}{a} + hR^*(T + t) + \left( \frac{\gamma + \alpha D + \beta \sigma d + \sigma m}{W} \right) + h \right] \quad (2.17)
\]
It is obvious that $p^*_1 > p^*_0$, which suggests that when demand increases, if we stick to the same shipment size ($rW$), we should offer a higher price to maximize the total profit. However, if we change the shipment size to make it larger, for instance, from three TLs per shipment to four TLs per shipment, we should re-visit the optimal offering price by doing the computing procedures illustrated in section 2.3.6. In a special case, if the iteration results in a same $r$ in both state 0 and state 1, the optimal offering price will increase by exactly $\frac{\Delta b}{2a}$ from equations (2.16) and (2.17).

### 2.3.7.2 What If the Demand Becomes More Elastic?

In this section and the following section, we discuss the effect of customer price elasticity change on the total profit. When demand becomes more price elastic, a larger $a$ value in the demand-price function should be used to estimate the demand, as $b$ value keeps the same. Let’s analyze the scenario when the customer demand becomes more elastic. Similar to before, subscript 0 denotes state 0 (initial state/base case), and subscript 1 denotes state 1 (the new scenario with increased demand elasticity). In this example, customer demand elasticity is larger in state 1 than in state 0. Figure 2-5 shows a graphic view of the demand curve change.

![Figure 2-5: Price Elasticity Change in Linear Demand-Price Function](image-url)
By definition of the linear demand-price function, we have
\[ q_0 = -a_0 \cdot p_0 + b \]  
(2.18)
\[ q_1 = -a_1 \cdot p_0 + b = (a_0 + \Delta a) \cdot p_0 + b \]  
(2.19)
\[ \Delta a > 0 \]  
(2.20)

According to equation (2.10),
\[ r_{U,1} = \left| n^* \left( \frac{-(a_0 + \Delta a) p_0 + b}{f_{\min} * W} \right) \right| \leq r_{U,0} = \left| n^* \left( \frac{-a_0 p_0 + b}{f_{\min} * W} \right) \right|. \]  
(2.21)

This implies that as the demand becomes more elastic, the upper bound of the number of TLs per shipment decreases or at most keeps the same. Therefore, it is possible for us to get a smaller optimal shipment size in state 1, but again it is not guaranteed.

According to equation (2.11),
\[ p_0^* = \frac{1}{2} \left[ \frac{b}{a_0} + hR^* (T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha D + \sigma m}{W} \right) + h \right] \]  
(2.22)
\[ p_1^* = \frac{1}{2} \left[ \frac{b}{a_0 + \Delta a} + hR^* (T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha D + \sigma m}{W} \right) + h \right] \]  
(2.23)

For any \( \Delta a \) that is strictly positive, we have \( p_1^* < p_0^* \). This suggests that when demand becomes more elastic, if we stick to the same shipment size \( rW \), we should offer a lower price to maximize the total profit. However, if we change the shipment size to make it smaller, for instance, from four TLs per shipment to three TLs per shipment, we should re-visit the optimal offering price by doing the computing procedures illustrated in section 2.3.6. In a special case, if the iteration results in a same \( r \) in both state 0 and state 1, the optimal offering price should decrease by \[ \frac{1}{2} \left( \frac{b}{a_0} - \frac{b}{a_0 + \Delta a} \right) = \frac{\Delta a * b}{2a_0(a_0 + \Delta a)} \]

from equations (2.22) and (2.23).
2.3.7.3 What If the Demand Becomes Less Elastic (More Inelastic)?

In the opposite case of the previous section, when demand becomes less elastic, a smaller \( a \) value in the demand-price function should be used to estimate the demand, as \( b \) value keeps the same. We now analyze the scenario when the customer demand becomes more inelastic. Subscript 0 denotes state 0 (initial state/base case), and subscript 2 denotes state 2 (the new scenario with decreased demand elasticity). In this example, customer demand elasticity is larger in state 0 than in state 2. Again Figure 2-4 provides a graphic view.

Similarly, we get

\[
q_0 = -a_0 \cdot p_0 + b \quad (2.24)
\]

\[
q_2 = -a_2 \cdot p_0 + b = -(a_0 - \Delta a) \cdot p_0 + b \quad (2.25)
\]

\(
\Delta a > 0 \quad (2.26)
\)

According to equation (2.10),

\[
r_{U,2} = \left\lfloor n \cdot \left( \frac{-(a_0 - \Delta a)p_0 + b}{f_{\text{min}} \cdot W} \right) \right\rfloor \geq r_{U,0} = \left\lfloor n \cdot \left( \frac{-a_0 p_0 + b}{f_{\text{min}} \cdot W} \right) \right\rfloor \quad (2.27)
\]

This implies that as the demand becomes more inelastic, the upper bound of the number of TLs per shipment increases or at least keeps the same. Therefore, it is possible for us to get a larger optimal shipment size in state 2, but it is not guaranteed.

According to equation (2.11),

\[
p_0^* = \frac{1}{2} \left[ \frac{b}{a_0} + hR \cdot (T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] \quad (2.28)
\]

\[
p_2^* = \frac{1}{2} \left[ \frac{b}{a_0 - \Delta a} + hR \cdot (T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] \quad (2.29)
\]

For any \( \Delta a \) that is strictly positive, we have \( p_2^* > p_0^* \). This suggests that when demand becomes more inelastic, if we stick to the same shipment size (\( rW \)), we should offer a
higher price to maximize the total profit. However, if we change the shipment size to make it larger, for instance, from three TLs per shipment to four TLs per shipment, we should re-visit the optimal offering price by doing the computing procedures illustrated in section 2.3.6. In a special case, if the iteration results in a same \( r \) in both state 0 and state 2, the optimal offering price should increase by \( \frac{1}{2} \left( \frac{b}{a_0 - \Delta a} - \frac{b}{a_0} \right) = \frac{\Delta a \cdot b}{2a_0(a_0 - \Delta a)} \) from equations (2.28) and (2.29).

### 2.3.7.4 What If the Customer Density Increases?

If the density \( \rho \) increases, equation (2.6) shows that \( d \) (peddling distance per truck) decreases and therefore \( t \) (average peddling traveling time per truck) also decreases as long as \( r \) (number of TLs per shipment) keeps the same. Let's analyze the scenario when the customer density increases. Subscript 0 denotes state 0 (initial state/base case), and subscript 1 denotes state 1 (the new scenario with increased customer density). Therefore, customers' density is larger in state 1 than in state 0.

\[
d_0 = K \sqrt{mn/r \rho_0} = K \sqrt{\frac{n}{r}} \frac{n}{r \rho_0} \tag{2.30}
\]

\[
t_0 = \frac{d_0}{v} = \frac{K}{v} \sqrt{\frac{n}{r}} \frac{n}{r \rho_0} \tag{2.31}
\]

\[
d_1 = K \sqrt{mn/r \rho_1} = K \sqrt{\frac{n}{r}} \frac{n}{r(\rho_0 + \Delta \rho)} \tag{2.32}
\]

\[
t_1 = \frac{d_1}{v} = \frac{K}{v} \sqrt{\frac{n}{r}} \frac{n}{r(\rho_0 + \Delta \rho)} \tag{2.33}
\]

\( \Delta \rho > 0 \) \tag{2.34}

where \( v \) is the truck's average peddling speed while in the delivery region.

For any \( \Delta \rho \) that is strictly positive, we have \( d_0 > d_1 \) and \( t_0 > t_1 \).

According to equation (2.11), the optimal offering prices for both states are
A smaller $d$ will give us a smaller transportation cost, and a smaller $t$ will give us a smaller inventory cost. The combined cost reduction in state 1 will allow us to choose a lower offering price in state 1 compared to state 0 as long as we stick to the same shipment size ($rW$). The optimal shipment size and the optimal offering price can be found by doing the computing procedures in section 2.3.6.

In a special case, if the iteration results in a same $r$ in both state 0 and state 1, the optimal offering price should decrease by

\[
\frac{1}{2} \left[ \left( \frac{hR}{v} + \frac{\beta \alpha}{W} \right) \star K \sqrt{\frac{n}{r}} \star \sqrt{\frac{1}{\rho_0} - \sqrt{\frac{1}{\rho_0 + \Delta \rho}}} \right],
\]

as equation (2.37) shows.

\[
p_1^* - p_0^* = \frac{1}{2} \left( \frac{hR(t_1 - t_0)}{v} + \frac{\beta \alpha}{W} (d_1 - d_0) \right)
\]

\[
= \frac{1}{2} \left[ \left( \frac{hR}{v} + \frac{\beta \alpha}{W} \right) \star K \left( \frac{n}{r} \right) \star \left( \frac{1}{\rho_0 + \Delta \rho} - \frac{1}{\rho_0} \right) \right] < 0
\]  

\[\text{(2.37)}\]

### 2.3.7.5 What If the Distance Increases?

An increase of the distance from the warehouse to the delivery region would result in an increased $D$ and $T$, the per truck average round-trip distance and transit time from the warehouse to the delivery region respectively. Note that $d$ and $t$ are only associated with the density, instead of the distance we are discussing here. Thus the distance change does not have any impact on $d$ and $t$. Let’s analyze the scenario when the distance from the delivery region to the warehouse increases. Subscript 0 denotes state 0 (initial state/base case), and subscript 1 denotes state 1 (the new scenario with increased distance). In this example, the distance is larger in state 1 than in state 0.
\[ D_1 = D_0 + \Delta D \]  \hspace{1cm} (2.38)

\[ T_0 = \frac{D_0}{v'} \]  \hspace{1cm} (2.39)

\[ T_1 = \frac{D_1}{v'} \]  \hspace{1cm} (2.40)

\[ \Delta D > 0 \]  \hspace{1cm} (2.41)

where \( v' \) is the average speed of the truck from the warehouse to the delivery region (not the peddling speed).

According to equation (2.11) above, the optimal offering prices for both states are

\[ p_0^* = \frac{1}{2} \left[ \frac{b}{a} + hR(T_0 + t) + \left( \frac{\gamma + \alpha D_0 + \beta \alpha d + \sigma m}{W} \right) + h \right] \]  \hspace{1cm} (2.42)

\[ p_1^* = \frac{1}{2} \left[ \frac{b}{a} + hR(T_1 + t) + \left( \frac{\gamma + \alpha D_1 + \beta \alpha d + \sigma m}{W} \right) + h \right] \]  \hspace{1cm} (2.43)

According to equation (2.3), the transportation cost goes up with the increased \( D \). And the inventory cost also goes up with the increased \( T \), by equation (2.4). Therefore, if we stick to the same shipment size \( (rW) \), the combined cost increasing in state 1 will enforce us to choose a higher offering price. Again, the optimal shipment size and the optimal offering price can be found by following the computing procedures in section 2.3.6.

In a special case, if the iteration results in a same \( r \) in both state 0 and state 1, the optimal offering price should increase by \( \frac{1}{2} \left( \frac{hR}{v'} + \frac{\alpha}{W} \right) * \Delta D \), as in equation (2.44).

\[ p_1^* - p_0^* = \frac{1}{2} \left( \frac{hR}{v'} + \frac{\alpha}{W} \right) * (D_1 - D_0) \]  \hspace{1cm} (2.44)

\[ = \frac{1}{2} \left[ \left( \frac{hR}{v'} + \frac{\alpha}{W} \right) * (D_1 - D_0) \right] = \frac{1}{2} \left( \frac{hR}{v'} + \frac{\alpha}{W} \right) * \Delta D > 0 \]
2.3.8 Special Case: Direct Shipping

If the result shows that \( r = n \), we have the number of truckloads per shipment to be exactly the same as the number of customers in this region. The optimal solution suggests us direct ship every customer in this region. In other words, each customer is served by exactly one truckload per shipment, as shown in Figure 2-6 above. We will resort to the module of direct shipping to find the optimal price.

2.3.9 What If \( r_U < 1 \)?

If the result shows that the upper bound of \( r \) is less than one, \( r_U < 1 \), we are in a less than truckload situation. In this case, a low demand density and minimum shipment requirements limit deliveries in size.

To deal with this situation, on one hand, we can group more customers into this delivery region if the physical situation allows us to do so. That is we increase \( n \), the number of customers in a delivery region. But if we cannot group more customers due to certain constraints, we are forced to the option of serving the customers in this delivery region with LTL shipments. Thus the customer shipment frequency is maintained at the manufacturer’s cost of dispatching less-than-truckload shipment. The shipment size is decided by \( f_{\text{min}} \), such that \( V_{\text{max}} = \frac{nq}{f_{\text{min}}} \) (2.45). This leads us to the LTL shipment model below in section 2.4.
2.4 Model for Less-Than-Truckload (LTL) Shipment

2.4.1 Model Objective
The objective of this model is to maximize the total profit for the manufacturer.

2.4.2 Variable Definitions
All variables used in the formulation below have the same definition as in the section 2.3.2.

2.4.3 Decision Variables
\( p \)       Selling price per unit in this region ($/unit)
\( V \)       Shipment Size (units/shipment)
Since only one truck is dispatched at a time, if we set the optimal shipment size to be
\[ V^* = V_{\text{max}} = \frac{nq}{f_{\text{min}}} \] (2.45), we will therefore have only one decision variable \( p \) left and the problem becomes easy to solve.

2.4.4 Model Formulation
The company’s profit is its revenue net its costs. As before, we consider the total cost consisting of the manufacturing cost, the transportation cost and the inventory cost. Each component is addressed as follows.

Revenue (per unit) = \( p \) \hspace{1cm} (2.1)
Inventory Cost (per unit) = \( hR^* \left( \frac{V}{2nq} + T + t \right) \) \hspace{1cm} (2.46)
Transportation Cost (per unit) = \( \frac{\gamma + \alpha D + \beta \rho d + \sigma m}{V} \) \hspace{1cm} (2.47)
Manufacturing Cost (per unit) = \( h \) \hspace{1cm} (2.2)
Therefore, we can write the profit as

\[
\text{Profit} = M = f(p, V)
\]

\[
M = \left( p - hR \left( \frac{V}{2} + T + t \right) \frac{\gamma + \alpha D + \beta \alpha d + \sigma n}{V} - \frac{h}{n} \right) nq
\]

(2.48)

where \( d = K \sqrt{mn/\rho} = 0.6 \sqrt{mn/\rho} = 0.6 \sqrt{n^2/\rho} = \frac{0.6n}{\sqrt{\rho}} \)

(2.49)

2.4.5 Solutions

Set \( V^* = V_{\text{max}} = \frac{nq}{f_{\text{min}}} \) (2.45) and plug it into equation (2.48) to get \( M = f(p) \).

\[
M = npq - hR \left( \frac{V}{2} + (T + t) nq \right) - \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma n}{nq} \right) f_{\text{min}} nq - hnq
\]

\[
= npq - hR \left( \frac{1}{2f_{\text{min}}} + (T + t) nq \right) - \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma n}{f_{\text{min}}} \right) nq - hnq
\]

\[
= npq - hR \left( \left[ hR \left( \frac{1}{2f_{\text{min}}} + (T + t) nq \right) + h \right] f_{\text{min}} \right)
\]

\[
= np(-ap + b) - h \left[ hR \left( \frac{1}{2f_{\text{min}}} + (T + t) + h \right) f_{\text{min}} \right)
\]

By taking the derivative of \( M \) with respect to \( p \), we can get the optimal selling price

\[
p^* = \frac{\partial}{\partial p} M(p) = 0
\]

\[
-2anp + bn + an \left[ hR \left( \frac{1}{2f_{\text{min}}} + (T + t) + h \right) f_{\text{min}} \right) = 0
\]

\[
p^* = \frac{1}{2} \left[ \frac{b}{a} + hR \left( \frac{1}{2f_{\text{min}}} + (T + t) + h \right) \right]
\]
Therefore, the optimal offering price in the LTL shipment scenario is

\[ p^* = \frac{1}{2} a \left[ b + hR \left( \frac{1}{2f_{\min}} + (T + t) \right) + h \right] \]  

(2.50)

2.5 An Integrated TL/LTL Shipment Model

In this section, we try to derive an integrated model that considers both the truckload and the less-than-truckload shipment scenarios, discussed in the section 2.3 and 2.4.

2.5.1 Objective Function

After comparing the cost components for each scenario, an integrated objective function is derived at the end of this section.

2.5.1.1 Profit Functions

When \( 1 \leq r < n \), we refer to the objective function as \( M_{TL} \): Truckload peddling.

\[ M_{TL} = f(p, r) = npq - hR \left( \frac{rW}{2} + (T + t)nq \right) - \left( \frac{\gamma + D + \sigma}{W} \right) * nq - nhq \]  

(2.51)

When \( r = n \), we direct ship every customer. Now \( d = 0 \), \( t = 0 \) and \( m = 1 \). We refer to the objective function as \( M_{TLd} \): Truckload direct shipping.

\[ M_{TLd} = npq - hR \left( \frac{nW}{2} + T * nq \right) - \left( \frac{\gamma + D + \sigma * 1}{W} \right) * nq - nhq \]  

(2.52)

When \( r < 1 \), we refer to the objective function as \( M_{LTL} \): Less-than-truckload shipping.

\[ M_{LTL} = f(p, r) = \left( p - hR \left( \frac{V}{2nq} + T + t \right) - \frac{\gamma + D + \sigma}{rW} \right) * nq \]  

(2.48)
2.5.1.2 Transportation Cost Component

We first compare the formulation of the transportation cost component, and then derive an integrated formulation for both scenarios.

For the TL peddling shipment case \((1 \leq r < n)\), the transportation cost per unit is

\[
\text{Transpn Cost} = \frac{\gamma + \alpha D + \beta \alpha \sigma}{W} = \frac{\gamma + \alpha D + \beta \alpha K \sqrt{\frac{n}{r}} \cdot \frac{n}{r p} + \sigma \sqrt{\frac{n}{r}}}{W}
\]

For the TL direct shipping case \((r = n)\), the transportation cost per unit is

\[
\text{Transpn Cost} = \frac{\gamma + \alpha D + \sigma \cdot 1}{W}
\]

For the LTL shipment case \((r < 1)\), the transportation cost per unit is

\[
\text{Transpn Cost} = \frac{\gamma + \alpha D + \beta \alpha \sigma}{r W} = \frac{\gamma + \alpha D + \beta \alpha K n / \sqrt{\rho} + \sigma n}{r W}
\]

Thus, a general formula for the transportation cost per unit is

\[
\text{Transpn Cost} = \frac{\gamma + \alpha D + \beta \alpha K \cdot \sqrt{\frac{n}{\max(r,1)}} \cdot \sqrt{\frac{n}{\max(r,1) \cdot \rho}} \cdot \frac{n - r}{n} + \sigma \cdot \frac{n}{\max(r,1)}}{\min(r,1) \cdot W}
\]

\[
\gamma + \alpha D + \beta \alpha K \cdot \sqrt{\frac{n}{\min(r,1)}} \cdot \sqrt{\frac{n}{\min(r,1) \cdot \rho}} \cdot \frac{1}{r} \cdot \frac{n - r}{n} + \sigma \cdot \min\left(\frac{n}{r}, n\right)
\]

\[
= \frac{\gamma + \alpha D + \beta \alpha K \cdot \sqrt{\frac{n}{\min(r,1)}} \cdot \sqrt{\frac{n}{\min(r,1) \cdot \rho}} \cdot \frac{1}{r} \cdot \frac{n - r}{n} + \sigma \cdot \min\left(\frac{n}{r}, n\right)}{\min(r,1) \cdot W}
\]

(2.55)

2.5.1.3 Inventory Cost Component

As in the previous section, we first compare the formulation of the inventory cost component, and then derive an integrated formulation for both scenarios.

For the TL peddling shipment case \((1 \leq r < n)\), the inventory cost per unit is

\[
\text{Inventory Cost} = h R \left(\frac{r W}{2} + (T + t) q\right)
\]

(2.56)
For the TL direct shipping case \((r = n)\), the inventory cost per unit is

\[
\text{Inventory Cost} = hR * \left( \frac{nW}{2} + Tnq \right)
\]

(2.57)

For the LTL shipment case \((r < 1)\), the inventory cost per unit is

\[
\text{Inventory Cost} = hR * \left( \frac{V}{2} + (T + t)nq \right) = hR * \left( \frac{rW}{2} + (T + t)nq \right)
\]

(2.58)

Thus, a general formula of the Inventory Cost is

\[
\text{Inventory Cost} = hR * \left( \frac{rW}{2} + (T + t)\left[ \frac{n-r}{n} \right] * nq \right)
\]

(2.59)

### 2.5.1.4 Local Peddling Distance

For the TL peddling shipment case \((1 \leq r < n)\), the average local peddling distance per truck predicted by the VRP is

\[
d = K \sqrt{mn/r\rho} = K * \sqrt{\frac{m}{r}} \frac{n}{r\rho}
\]

(K=0.6).

(2.6)

For the LTL shipment case \((r < 1)\), the average local peddling distance per truck is zero. The formulations of the integrated cost components already capture this property.

For LTL shipment case \((r < 1)\), since the number of stops per truck is the same as the number of customers in this delivery region, we have \(m = n\), and we can write the average local peddling distance per truck as

\[
d = K \sqrt{mn/\rho} = K * \sqrt{n} \frac{n}{\rho} = K * \sqrt{\frac{n^2}{\rho}}
\]

(K=0.6)

(2.60)
2.5.1.5 The Integrated Profit Functions for TL and LTL Shipments

Therefore, a general formula of the manufacturer’s overall profit is,

\[
M = npq - hR * \left( \frac{rW}{2} + (T + t * \left\lceil \frac{n-r}{n} \right\rceil) * nq \right) \\
\quad - \left( \gamma + \alpha D + \beta \alpha K * \sqrt{\min \left( \left\lceil \frac{n}{r} \right\rceil, n \right)} * \sqrt{\frac{n}{\rho} * \min \left( \frac{1}{r}, 1 \right) * \left\lfloor \frac{n-r}{n} \right\rfloor} + \sigma * \min \left( \left\lceil \frac{n}{r} \right\rceil, n \right) \right) * nq - nhq
\]

(2.61)

2.5.2 Solutions

Since the upper bound of \( r \), the number of TLs per shipment is,

\[
r_u = \left\lfloor \frac{nq}{f_{\min} * W} \right\rfloor
\]

(2.10)

- If \( r_u \geq 1 \), follow the computing procedures for computing an optimal profit and its associated optimal number of TLs per shipment.
  1. For any given \( r_i \), the optimal offering price \( p_i^* \) is given above in equation (2.11).
  2. By substituting \( p_i^* \) into equation (2.8), we can get the maximum profit \( M_i^* \).
  3. Repeat 1,2 for the valid vector of \( r \), i.e., \( r = \{r_1, r_2, ..., r_u\} \), and get a vector of \( M^* \), i.e., \( M^* = \{M_1^*, M_2^*, ..., M_u^*\} \)
  4. The optimized (maximized) profit is \( M_{\max} = \max \left( M^* \right) = M_j^* \). Accordingly, the optimal offering price is \( p_j^* \); the optimal shipment size is \( V_j = r_j * W \); and the optimal shipment frequency is \( f_j = \frac{nq_j}{r_j * W} = \frac{(-a)p_j + b)*n}{r_j * W} \).

- If \( r_u < 1 \), set \( r = \frac{nq}{f_{\min} * W} \) and \( V = V_{\max} = \frac{nq}{f_{\min}} \), and we get

45
\[ p^* = \frac{1}{2} \left[ \frac{b}{a} + h R^* \left( \frac{1}{2 f_{\min}} + (T + t) \right) + h \right] \quad (2.50) \]

\( p^* \) is the optimal selling price and the optimal shipment size is \( V = \frac{nq}{f_{\min}} \) (2.45). By plugging \( p^* \) into the equation (2.48), we get the maximum profit \( M^* \).
Chapter 3

A Convenient Delivery Strategy: Dispatch One Truckload at a Time

We introduce an alternative delivery strategy that may be easier to deploy. Instead of trying to find an optimal number of truckload shipments for each fixed time horizon, the company may simply choose to dispatch exactly one truckload at a time whenever the demand over a delivery region reaches one truckload. This is equivalent to setting $r = 1$ in our model derived in chapter two. Besides the convenience of this strategy, it may increase customer satisfaction due to the company’s frequent service/shipments. This chapter discusses whether this delivery strategy is optimal from the company’s perspective, and gives a lower bound to determine its optimality.

3.1 A Criteria for Determining the Optimality

We first discuss the impact that the variable $r$ has on the total profit. Recall that the equation (2.8) shows that the total profit is

$$\text{Profit} = M_U * nq = M = f(p,r)$$

$$M = npq - hR * \left( \frac{rW}{2} + (T + t) * nq \right) - \left( \frac{\gamma + \alpha D + \beta \alpha D + \sigma m}{W} \right) * nq - hnq$$

(2.8)
Note that the total profit is a function of $p$ and $r$. Also recall equation (2.11), which gives the optimal selling price based on a given $r$.

$$p^* = \frac{1}{2} \left[ \frac{b}{a} + hR(T + t) + \left( \frac{y + \alpha D + \beta a d + \sigma m}{W} \right) + h \right]$$

(2.11)

As mentioned in the section 2.3.5, $r$ directly affects $d$ and $m$, and indirectly affects $t$ in deriving the optimal offering price $p^*$. Specifically, as $r$ increases, $m$ decreases since each truck needs to visit less number of customers. And $d$ and $t$ also decrease as a consequence that each truck has less stops to make. In short,

$$r \uparrow \implies d \downarrow, \ t \downarrow, \ m \downarrow$$

Equations (2.3) and (2.4) illustrate the expression of the transportation cost and the inventory cost. We find that part of the transportation cost are associated with $d$ and $m$, and part of the inventory cost are associated with $r$ and $t$. We subtract the respective cost components and add them up. Furthermore, we define RRC (r Related Cost) as the summation times $nq$, the total demand over one delivery region.

Transportation Cost (per unit) = $\frac{y + \alpha D + \beta a d + \sigma m}{W}$

(2.3)

Inventory Cost (per unit) = $hR^* \left( \frac{V}{2nq} + T + t \right) = hR^* \left( \frac{rW}{2nq} + T + t \right)$

(2.4)

Summation of associated cost components = $hR^* \left( \frac{rW}{2nq} + t \right) + \frac{\beta a d + \sigma m}{W}$

$$RRC = hR^* \left( \frac{rW}{2} + tnq \right) + \left( \frac{\beta a d + \sigma m}{W} \right) * nq$$

(3.1)

We want to find out the change of RRC in terms of the change of $r$. Since we have already known that $r$ can only take integer values (except for the LTL shipments scenario), we explore $\frac{\Delta RRC}{\Delta r}$ instead of $\frac{\partial RRC}{\partial r}$.
\[
\frac{\Delta RRC}{\Delta r} = \frac{RRC(r+1) - RRC(r)}{(r+1) - r} = RRC(r+1) - RRC(r)
\]  
(3.2)

If \( \frac{\Delta RRC}{\Delta r} < 0 \), as \( r \) increases incrementally, \( \Delta RRC \) decreases. Hence the transportation cost and the inventory cost decrease if we choose a larger \( r \). On the contrary, if \( \frac{\Delta RRC}{\Delta r} \geq 0 \), as \( r \) increases incrementally, \( \Delta RRC \) also increases. Thus the transportation cost and the inventory cost increase if we choose a larger \( r \). This implies that we can use \( \frac{\Delta RRC}{\Delta r} \) as a criteria to determine whether the delivery strategy of serving the customers as frequently as possible with one full truckload, with the assumption that \( r = 1 \) in the base case.

### 3.2 Derive a Lower Bound

In this section, we show that we can derive a lower bound for \( \frac{\Delta RRC}{\Delta r} \).

\[
\frac{\Delta RRC}{\Delta r} = RRC(r+1) - RRC(r)
\]

\[
= hR*\left(\frac{(r+1)W}{2} + tr_{r+1}nq\right) + \left(\frac{\beta\alpha d_{r+1} + \sigma m_{r+1}}{W}\right)*nq - hR*\left(\frac{rW}{2} + tr,nq\right) - \left(\frac{\beta\alpha d_{r} + \sigma m_{r}}{W}\right)*nq
\]

\[
= hR*\frac{W}{2} + hRnq*(t_{r+1} - t_{r}) + \frac{nq}{W}*[\beta\alpha*(d_{r+1} - d_{r}) + \sigma*(m_{r+1} - m_{r})]
\]

According to equation (2.6),

\[
d = K\sqrt{mn/\rho} = K*\sqrt{\frac{n}{r}}*\frac{n}{\rho r} \quad (K=0.6)
\]  
(2.6)

We write the expression for \( d_{r+1} \) and \( d_{r} \) as

\[
d_{r+1} = K*\sqrt{\frac{n}{r + 1}}*\frac{n}{(r+1)\rho}
\]

(3.4)
\( d_r = K^* \sqrt{\frac{n}{r}} \cdot \frac{n}{r \rho} \)  \hspace{1cm} (3.5)

Therefore, we write \( d_{r+1} - d_r \) as

\[
d_{r+1} - d_r = K^* \sqrt{\frac{n}{r+1}} \cdot \frac{n}{(r+1) \rho} - K^* \sqrt{\frac{n}{r}} \cdot \frac{n}{r \rho}
\]

\[
= K^* \sqrt{\frac{n}{\rho}} \left( \sqrt{\frac{n}{r+1}} \cdot \frac{1}{r+1} - \sqrt{\frac{n}{r}} \cdot \frac{1}{r} \right) - K^* \sqrt{\frac{n}{\rho}} \left( \sqrt{\frac{n}{r+1}} \cdot \frac{1}{r+1} - \sqrt{\frac{n}{r}} \cdot \frac{1}{r} \right)
\]

\[
= -K^* \sqrt{\frac{n}{\rho}} \left( \sqrt{\frac{n}{r+1}} \cdot \frac{1}{r+1} - \sqrt{\frac{n}{r}} \cdot \frac{1}{r} \right)
\]

In order to derive the lower bound for \( \frac{\Delta R R C}{\Delta r} \), we first present and then prove three lemmas.

**Lemma 1**

\( d_{r+1} - d_r \geq -\frac{Kn}{2\sqrt{\rho}} \)  \hspace{1cm} (3.7)

**Proof.** According to the equation (2.32),

\[
d_{r+1} - d_r = -K^* \sqrt{\frac{n}{\rho}} \left( \sqrt{\frac{n}{r+1}} \cdot \frac{1}{r+1} - \sqrt{\frac{n}{r}} \cdot \frac{1}{r} \right).
\]

We write the part in parenthesis explicitly and get the following result.

\[
\sqrt{\frac{n}{r}} \cdot \frac{1}{r} - \sqrt{\frac{n}{r+1}} \cdot \frac{1}{r+1} \leq \sqrt{\frac{n}{r}} \cdot \frac{1}{r} - \sqrt{\frac{n}{r+1}} \cdot \frac{1}{r+1} = \sqrt{n}^* \left( \frac{1}{r} - \frac{1}{r+1} \right) = \sqrt{n}^* \left( 1 - \frac{r}{r+1} \right)
\]

This is a monotonic decreasing function with respect to \( r \). Since that in the base case, \( r = 1 \), we show that

\[
\sqrt{\frac{n}{r}} \cdot \frac{1}{r} - \sqrt{\frac{n}{r+1}} \cdot \frac{1}{r+1} \leq \sqrt{n}^* \left( 1 - \frac{r}{r+1} \right) \leq \sqrt{n}^* \left( 1 - \frac{1}{1+1} \right) = \frac{\sqrt{n}}{2}
\]
\[
\sqrt{\frac{n}{r}} \frac{1}{r} - \sqrt{\frac{n}{r+1}} \frac{1}{(r+1)} \leq \frac{\sqrt{n}}{2}.
\]

Therefore
\[
d_{r+1} - d_r = -K \sqrt{\frac{n}{\rho}} \left( \sqrt{\frac{n}{r}} \frac{1}{r} - \sqrt{\frac{n}{r+1}} \frac{1}{(r+1)} \right) \geq -\frac{K_n}{2\sqrt{\rho}}.
\]

\[\textbf{Lemma 2} \quad t_{r+1} - t_r \geq -\frac{Kn}{2v\sqrt{\rho}} \quad (3.8)\]

\textbf{Proof.} Define } v \text{ to be the average local peddling speed (miles/hr), therefore
\[
t_{r+1} - t_r = \frac{d_{r+1}}{v} - \frac{d_r}{v} = \frac{1}{v} (d_{r+1} - d_r).
\]

According to Lemma 1, \(d_{r+1} - d_r \geq -\frac{Kn}{2\sqrt{\rho}}\), it is a consequence that
\[
t_{r+1} - t_r = \frac{1}{v} (d_{r+1} - d_r) \geq -\frac{Kn}{2v\sqrt{\rho}} \quad \text{since } v \text{ should be strictly positive.}
\]

\[\textbf{Lemma 3} \quad m_{r+1} - m_r \geq -\frac{n}{2} \quad (3.10)\]

\textbf{Proof.} According to equation (2.5), \(m = \left[ \frac{n}{r} \right] \), we can write \(m_{r+1}\) and \(m_r\) explicitly as
\[
m_{r+1} = \left[ \frac{n}{r+1} \right] \quad (3.11)
\]
\[
m_r = \left[ \frac{n}{r} \right] \quad (3.12)
\]

Thus we have
\[
-(m_{r+1} - m_r) = m_r - m_{r+1} = \left[ \frac{n}{r} \right] - \left[ \frac{n}{r+1} \right] \leq \frac{n}{r} - \frac{n}{r+1} = n \left( 1 - \frac{r}{r+1} \right).
\]

This is a monotonic decreasing function with respect to \(r\). Since in the base case \(r = 1\), we show that
\[-(m_{r+1} - m_r) = \frac{n}{r} \left(1 - \frac{r}{r+1}\right) \leq n - \left\lfloor \frac{n}{2} \right\rfloor \leq n - \frac{n}{2} = \frac{n}{2} \] (3.14)

Therefore, \( m_{r+1} - m_r \geq -\frac{n}{2} \)

Now let’s try to analyze the equation (3.3).

\[
\frac{\Delta RRC}{\Delta r} = RRC(r+1) - RRC(r) = hR \frac{W}{2} + hRnq*(t_{r+1} - t_r) + \frac{nq}{W} [\beta \alpha * (d_{r+1} - d_r) + \sigma * (m_{r+1} - m_r)]
\]

\[
\frac{\Delta RRC}{\Delta r} = hR \frac{W}{2} + hRnq*(t_{r+1} - t_r) + \frac{nq}{W} [\beta \alpha * (d_{r+1} - d_r) + \sigma * (m_{r+1} - m_r)]
\]

\[
\frac{\Delta RRC}{\Delta r} \geq hR \frac{W}{2} - hRnq * \frac{Kn}{2\sqrt{\rho}} - \frac{nq}{W} [\beta \alpha * \frac{Kn}{2\sqrt{\rho}} + \sigma * \frac{n}{2}]
\]

(3.15)

Thus we have shown that \( hR \frac{W}{2} - hRnq * \frac{Kn}{2\sqrt{\rho}} - \frac{nq}{W} \left[\beta \alpha * \frac{Kn}{2\sqrt{\rho}} + \sigma * \frac{n}{2}\right] \) is a lower bound for \( \frac{\Delta RRC}{\Delta r} \). If we can prove that the lower bound is nonnegative, we can argue that \( \frac{\Delta RRC}{\Delta r} \) is also nonnegative, which suggests that the transportation cost and the inventory cost increase as \( r \) increases incrementally. Therefore, if we can prove that the following inequation (3.16) holds, we can safely argue that it is always our best interest to serve the delivery region with one full truckload shipment at a time.

\[
hR \frac{W}{2} - hRnq * \frac{Kn}{2\sqrt{\rho}} - \frac{nq}{W} \left[\beta \alpha * \frac{Kn}{2\sqrt{\rho}} + \sigma * \frac{n}{2}\right] \geq 0
\] (3.16)
There are two issues that we want to mention there. The first one is that

\[ \frac{hR* W}{2} - hRnq* \frac{Kn}{2\sqrt{\rho}} - \frac{nq*}{W} \left[ \beta \alpha* \frac{Kn}{2\sqrt{\rho}} + \sigma* \frac{n}{2} \right] \]

is a lower bound, but may not be a tight lower bound.

If \( hR* W - hRnq* \frac{Kn}{2\sqrt{\rho}} - \frac{nq*}{W} \left[ \beta \alpha* \frac{Kn}{2\sqrt{\rho}} + \sigma* \frac{n}{2} \right] \geq 0, \frac{\Delta RRC}{\Delta r} \geq 0 \), the delivery strategy discussed in this chapter is optimal.

If \( hR* W - hRnq* \frac{Kn}{2\sqrt{\rho}} - \frac{nq*}{W} \left[ \beta \alpha* \frac{Kn}{2\sqrt{\rho}} + \sigma* \frac{n}{2} \right] < 0, \frac{\Delta RRC}{\Delta r} \) is free (it can be positive, negative or zero), we are uncertain about the optimality of the delivery strategy discussed in this chapter.

Hence, if in some case, \( hR* W - hRnq* \frac{Kn}{2\sqrt{\rho}} - \frac{nq*}{W} \left[ \beta \alpha* \frac{Kn}{2\sqrt{\rho}} + \sigma* \frac{n}{2} \right] \) turns out to be strictly negative, it may still be possible that \( r = 1 \) is the best solution as \( \frac{\Delta RRC}{\Delta r} \) could be positive.

The second issue is that the inventory cost while the truck is peddling in the delivery region \( hRnq* \frac{Kn}{2\sqrt{\rho}} \) is always very small compared to the half-truckload inventory holding cost per year \( hR* \frac{W}{2} \) because the average local peddling time is usually in the magnitude of hours.
3.3 Numerical Examples

In this section, we give two examples to show numerically that the lower bound we have derived could be strictly negative, which implies that we are uncertain about the optimality of the delivery strategy.

3.3.1 Example 1

<table>
<thead>
<tr>
<th>[Example 1]</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(h = $100) per item</td>
<td>(\alpha = $1.4) per mile</td>
</tr>
<tr>
<td>(R = 25%) per year</td>
<td>(\beta = 2)</td>
</tr>
<tr>
<td>(W = 200) items per load</td>
<td>(\sigma = $40) per stop</td>
</tr>
<tr>
<td>(n = 30) customers</td>
<td>(\nu = 20) miles per hr</td>
</tr>
<tr>
<td>(q = 25) item per week</td>
<td>(K = 0.6)</td>
</tr>
<tr>
<td>(\rho = 0.01) per square mile</td>
<td></td>
</tr>
</tbody>
</table>

Assumptions:
1) The time horizon is half a year.
2) The offering price is fixed here, therefore the demand is constant.

\[
\begin{align*}
\frac{hR}{2} \cdot \frac{W}{2} &= 100 \cdot 0.25 \cdot \frac{200}{2} = 2500 \\
\frac{hRnq}{2\sqrt{\rho}} \cdot \frac{Kn}{2\nu} &= 100 \cdot 0.25 \cdot 30 \cdot 25 \cdot 26 \cdot \frac{0.6 \cdot 30}{2 \cdot 20 \cdot \sqrt{0.01}} \cdot \frac{1}{365 \cdot 24} = 250.43 \\
\frac{nq}{W} \left[ \frac{\beta \alpha \cdot Kn}{2\sqrt{\rho}} - \sigma \cdot \frac{n}{2} \right] &= \frac{30 \cdot 25 \cdot 26}{200} \cdot \left[ 2 \cdot 1.4 \cdot \frac{0.6 \cdot 30}{2 \cdot \sqrt{0.01}} + 40 \cdot \frac{30}{2} \right] = 83070 \\
\frac{hR}{2} - \frac{hRnq}{2\sqrt{\rho}} \cdot \frac{Kn}{2\nu} - \frac{nq}{W} \left[ \frac{\beta \alpha \cdot Kn}{2\sqrt{\rho}} + \sigma \cdot \frac{n}{2} \right] &= 2500 - 250.43 - 83070 = -80820.43 < 0
\end{align*}
\]
### 3.3.2 Example 2

**[Example 2]**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$ = $30$ per item</td>
<td>$\alpha = $1.4 per mile</td>
</tr>
<tr>
<td>$R$ = 25% per year</td>
<td>$\beta = 2$</td>
</tr>
<tr>
<td>$W$ = 2000 items per load</td>
<td>$\sigma = $40 per stop</td>
</tr>
<tr>
<td>$n$ = 30 customers</td>
<td>$\nu = 20$ miles per hr</td>
</tr>
<tr>
<td>$q$ = 50 item per week</td>
<td>$K = 0.6$</td>
</tr>
<tr>
<td>$\rho = 0.01$ per square mile</td>
<td></td>
</tr>
</tbody>
</table>

Assumptions are the same as in example 1.

$$hR \frac{W}{2} = 30 \times 0.25 \times \frac{2000}{2} = 7500$$

$$hRnq \frac{Kn}{2\nu\sqrt{\rho}} = 30 \times 0.25 \times 30 \times 50 \times 26 \times \frac{0.6 \times 30}{2 \times 20 \times \sqrt{0.01}} \times \frac{1}{365 \times 24} = 150.26$$

$$\frac{nq}{W} \left[ \frac{\beta \alpha \frac{Kn}{2\sqrt{\rho}} + \sigma \frac{n}{2}}{2} \right] = \frac{30 \times 50 \times 26}{2000} \times \left[ \frac{2 \times 1.4 \times \frac{0.6 \times 30}{2 \times \sqrt{0.01}} + 40 \times \frac{30}{2}}{2} \right] = 16614$$

$$hR \frac{W}{2} - hRnq \frac{Kn}{2\nu\sqrt{\rho}} - \frac{nq}{W} \left[ \frac{\beta \alpha \frac{Kn}{2\sqrt{\rho}} + \sigma \frac{n}{2}}{2} \right] = 7500 - 150.26 - 16614 = -9264.26 < 0$$

Since the lower bound of $\frac{\Delta RRC}{\Delta r}$ is strictly negative based on the analyses above, we are uncertain about the sign of $\frac{\Delta RRC}{\Delta r}$, and thus cannot argue that $r = 1$ will return the maximum profit. The two examples above show that the increase of transportation cost is too high that it is not necessarily optimal for companies to choose $r = 1$ and serve customers as frequently as possible.
Chapter 4

A Discussion on Deterministic
Demand-Price Functions

As setting up optimal prices is a key component of this thesis, we need to develop assumptions for demand functions to reflect the demand change as the price fluctuates. In this chapter, we present three types of deterministic demand-price functions. We first discuss two commonly used functions that describe customers’ demand-price relationship: linear demand-price function and constant price elasticity. Then we introduce the Bass Diffusion model, which is the standard form of modeling the sales of new durable products. The Bass Diffusion model is further integrated with the assumption of linear demand-price function and the demand function with constant price elasticity respectively.

4.1 Linear Demand-Price Function

Linear demand-price function is the most frequently made assumption in describing how the customers respond to price changes. The demand can be expressed as an affine function of the price, as shown in the equation (4.1) below, where $a$ and $b$ are positive factors. Any change in the price $\Delta p$ results in a change in the demand $\Delta q$, which is
linear to the price change. The equation (4.1) suggests that \( \Delta q = -a \Delta p \). Refer to Figure 2-2 in chapter two for a graphic view: the demand decreases linearly as the price increases; there is a saturated demand \( q = b \) when the price equals to zero, and when the price gets high enough, the demand drops to zero.

\[
q = -a \cdot p + b \tag{4.1}
\]

We derive the elasticity of demand with respect to price as the resulted fractional incremental change of the demand divided by the fractional incremental change of the price.

\[
E_{q,p} = \frac{\partial q}{\partial p} \cdot \frac{p}{Q} = -a \cdot \frac{p}{-ap + b} = \frac{ap}{ap - b} \tag{4.2}
\]

It is notable that \( E_{q,p} \) varies in terms of the \( p \) value. In other words, at various price levels, the corresponding elasticity is different. Recall that the equation (2.11) gives the optimal selling price based on a given \( r \).

\[
p^* = \frac{1}{2} \left[ \frac{b}{a} + hR^* (T + t) + \left( \frac{\gamma + \alpha d + \beta \alpha d_1 + \sigma n}{W} \right) + h \right] \tag{2.11}
\]

If the company adopts the delivery strategy discussed in chapter three, which dispatches one truckload at a time, we plug in \( r = 1 \) and get

\[
p^* = \frac{1}{2} \left[ \frac{b}{a} + hR^* (T + t_1) + \left( \frac{\gamma + \alpha d + \beta \alpha d_1 + \sigma n}{W} \right) + h \right] \tag{4.3}
\]

where \( d_1 = K \cdot \sqrt{\frac{n \cdot n}{\rho}} = Kn / \sqrt{\rho} \)

\[
t_1 = \frac{K}{\nu} \cdot \sqrt{\frac{n \cdot n}{\rho}} = \frac{Kn}{\nu \sqrt{\rho}} \tag{4.4}
\]
4.2 Constant Price Elasticity Demand Function

Another common assumption about the demand is to assume that elasticity of the demand to the price is constant at any price level. Consistent with what discussed so far in section 4.1, the elasticity of demand with respect to price is defined as the resulted incremental change of the demand divided by the incremental change of the price. The demand function is often expressed as in equation (4.6) or (4.7), where $e$ and $f$ are positive factors.

\[ q = e^* p^f \]  
\[ \ln q = \ln e + f^* \ln p \]

Thus the demand price elasticity is constant at $f$. Figure 4-1 shows the demand price function.

\[ E_{q,p} = \frac{\partial Q}{\partial p} \frac{p}{Q} = \frac{ef^* p^{f-1} * p}{e^* p^f} = f \]  

*Figure 4-1: Constant Price Elasticity Demand Function*

Recall the profit function in equation (2.8). In chapter two, we used the linear demand-price function as described in section 4.1. At this time, we plug in the demand function that has constant price elasticity as in equation (4.6). The optimal offering price changes to the form in equation (4.9).

\[ M = npq - hR * \left( \frac{rW}{2} + (T + t) * nq \right) - \left( \frac{v + aD + \beta cD + \sigma m}{W} \right) * nq - hnq \]  

(2.8)
\[ p^* = \frac{f}{f+1} \left[ hR(T+t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] \]  

Compare equation (4.9) to (2.11), it is interesting to see that the expressions are quite similar except that the optimal offering price derived using the linear demand-price function has a multiplier of one half, while the optimal price derived using the constant demand price elasticity has a multiplier of \( \frac{f}{f+1} \). And the latter does not have the constant term of \( \frac{b}{a} \).

\[ p^* = \frac{1}{2} \left[ \frac{b}{a} + hR(T+t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] \]  

If we adopt the delivery strategy of dispatching one truckload at a time, we set \( r = 1 \) and get the optimal offering price as

\[ p^* = \frac{f}{f+1} \left[ hR(T+t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] \]  

where \( d_i = K \sqrt{\frac{n}{\rho}} = Kn / \sqrt{\rho} \)  

\[ t_i = \frac{K}{\nu} \sqrt{\frac{n}{\rho}} = \frac{Kn}{\nu \sqrt{\rho}} \]  

### 4.3 Demand Function of New Durable Products

#### 4.3.1 Bass Diffusion Model

First we introduce the Bass Diffusion model, which is a well-known parametric approach to estimating new product demand trajectory over time. It is the standard form of modeling the sales of new durable products. Bass Diffusion model has been discussed in
a rich literature. For example, Ho TH, Savin S., and Terwiesch C. (2002) study how a firm should manage its supply processes in a new product diffusion environment with backorders and lost sales [17]. Meixell MJ and Wu SD (2001) propose an approach to analyzing demand scenarios in technology-driven markets where product demands are volatile, but follow a few identifiable lifecycle patterns [23].

The demand function of the Bass Diffusion model is determined by the product’s innovation and imitation factors, the total number of potential sales over the product’s life cycle, and the current cumulative sales. The demand is dynamically reevaluated by the cumulative sales already took place. Equation (4.11) presents the mathematical terms.

\[
q = \frac{dx(t)}{dt} = g(S - x) + h(S - x)x
\]  

(4.11)

where

- \( q \) is the sales rate (or demand rate)
- \( S \) is the total number of potential sales over the life cycle of the product
- \( g \) is the coefficient of innovation
- \( h \) is the coefficient of imitation
- \( x \) is the cumulative sales

The model is designed to answer the question: when will customers adopt a new product or technology? The term innovation is used to refer to external influences, and the term imitation is used to refer to internal influences. The assumptions made for the Bass Diffusion model are as follows.

- Diffusion process is binary (consumer either adopts, or waits to adopt).
- Constant maximum potential number of buyers (N).
- Eventually, all \( N \) will buy the product.
- No repeat purchase, or replacement purchase.
- The impact of the word-of-mouth is independent of adoption time.
- Innovation is considered independent of substitutes.
- The marketing strategies supporting the innovation are not explicitly included.
Equation (4.11) can be written conceptually as:

Sales at time \( t \) = Innovation Effect + Imitation Effect

\[ = g \times \text{Remaining Potential} + h \times \text{Adopters} \times \text{Remaining Potential}. \]

where Remaining Potential = Total Potential - Number of Adopters (up to time \( t \)).

### 4.3.2 Combining Bass Diffusion Model with the Linear Demand Function

In this section, we integrate the linear demand price function to the Bass Diffusion model such that the demand becomes the consequence of the price. Suppose that both the innovation coefficient and the imitation factor can be expressed as a linear pattern to the price.

Let

\[ g = -a_1 \times p + b_1 \]  \hspace{1cm} (4.12)

and

\[ h = -a_2 \times p + b_2 \]  \hspace{1cm} (4.13)

where \( a_1, b_1, a_2, b_2 \) are carefully chosen coefficients that well reflect the linear relationship between the innovation factor \( g \) and the price \( p \), and between the imitation factor \( h \) and the price \( p \).

Plugging equations (4.12) and (4.13) to equation (4.11), we get

\[ q = g(S - x) + h(S - x)x = (-a_1 \times p + b_1)(S - x) + (-a_2 \times p + b_2)(S - x)x \]  \hspace{1cm} (4.14)

\[ q = -[a_1(S - x) + a_2(S - x)x] \times p + [b_1(S - x) + b_2(S - x)x] \]  \hspace{1cm} (4.15)

Now the demand becomes a function of the price \( p \), the total number of potential sales over the life cycle of the product \( S \), and the cumulative sales \( x \).

Comparing (4.15) with the linear demand function as in (4.1),

\[ q = -a \times p + b \]  \hspace{1cm} (4.1)

\[ q = -[a_1(S - x) + a_2(S - x)x] \times p + [b_1(S - x) + b_2(S - x)x] \]  \hspace{1cm} (4.15)

we substitute

\[ a = a_1(S - x) + a_2(S - x)x \]  \hspace{1cm} (4.16)

\[ b = b_1(S - x) + b_2(S - x)x \]  \hspace{1cm} (4.17)
According to equation (4.2), equation (4.18) below shows the elasticity of the demand to the price in the Bass Diffusion/Linear Demand model.

\[ E_{q,p} = \frac{\partial Q}{\partial p} \cdot \frac{p}{Q} = -a^* \frac{p}{-ap + b} = \frac{ap}{ap - b} \] (4.2)

\[ E_{q,p} = \frac{\partial Q}{\partial p} \cdot \frac{p}{Q} = \frac{ap}{ap - b} = \frac{[a_1(S - x) + a_2(S - x)x]*p}{[a_1(S - x) + a_2(S - x)x]*p - [b_1(S - x) + b_2(S - x)x]} \] (4.18)

Consequently, we have the updated optimal offering price as

\[ p^* = \frac{1}{2} \left[ \frac{b_1(S - x) + b_2(S - x)x}{a_1(S - x) + a_2(S - x)x} + hR*(T + t) + \left( \frac{\gamma + \alpha d + \beta d + \sigma m}{W} \right) + h \right] \] (4.19)

### 4.3.3 Combining Bass Diffusion Model with the Constant Price Elasticity Demand Function

**Elasticity Demand Function**

In this section, we integrate the demand function with constant price elasticity to the Bass Diffusion model. Suppose that both the innovation coefficient and the imitation factor can be expressed as a constant elasticity function to the price.

Let \( g = e_1 * p^{f_1} \) \hspace{1cm} (4.20)

and \( h = e_2 * p^{f_2} \) \hspace{1cm} (4.21)

where \( e_1, f_1, e_2, f_2 \) are carefully chosen coefficients that well reflect the relationship between the innovation factor \( g \) and the price \( p \), and between the imitation factor \( h \) and the price \( p \).

Plugging equations (4.20) and (4.21) to equation (4.11), we get

\[ q = g(S - x) + h(S - x)x = (e_1 * p^{f_1})*(S - x) + (e_2 * p^{f_2})*(S - x)x \] \hspace{1cm} (4.22)

\[ q = e_1(S - x)*p^{f_1} + e_2(S - x)x* p^{f_2} \] \hspace{1cm} (4.23)

Now the demand becomes a function of the price \( p \), the total number of potential sales over the life cycle of the product \( S \), and the cumulative sales \( x \).

Compare (4.23) with the demand function with constant price elasticity as in (4.6).
\[ q = e^* p' \]  
\[ q = e_1(S-x)^* p^h + e_2(S-x)x^* p^f \]  

(4.6)  
(4.23)

We derive the elasticity of the demand to the price in the Bass Diffusion/Constant Price Elasticity Demand model as shown in equation (4.25).

\[ E_{Q,p} = \frac{\partial Q}{\partial p} \frac{p}{Q} = \left[ e_1f_1(S-x)^* p^{h-1} + e_2f_2(S-x)x^* p^{f-1} \right] \frac{p}{e_1(S-x)^* p^h + e_2(S-x)x^* p^f} \]  

(4.24)

\[ E_{Q,p} = \frac{e_1f_1(S-x)^* p^{h} + e_2f_2(S-x)x^* p^{f}}{e_1(S-x)^* p^{h} + e_2(S-x)x^* p^{f}} \]  

(4.25)

Recall the profit function in equation (2.8).

\[ M = npq - hR \left( \frac{rW}{2} + (T + t) * nq \right) - \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) * nq - hnq \]  

(2.8)

By setting \( \frac{\partial M}{\partial p} = 0 \), we find that the optimal offering price has to satisfy the following

\[ \frac{e_1(f_1 + 1)(S-x)^* p^{h} + e_2(f_2 + 1)(S-x)x^* p^{f}}{e_1f_1(S-x)^* p^{h} + e_2f_2(S-x)x^* p^{f}} * p = hR * (T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \]  

(4.26)

or equivalently

\[ \left( 1 + \frac{1}{E_{Q,p}} \right) * p = hR * (T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \]  

(4.27)

And thus the result for \( p \) is

\[ p = \frac{E_{Q,p}}{1 + E_{Q,p}} * hR * (T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \]  

(4.28)
Chapter 5

Application on across Delivery Region Situations

In chapter two, we discussed a model for determining the integrated pricing and delivery strategy in a two-echelon supply chain. One of the assumptions that we have made there is that the manufacturer serves only one delivery region. In other words, all customers are grouped into one delivery region. In this chapter, we will try to relax this assumption to accommodate the situation where customers are distributed in different delivery regions. The model developed later in this chapter presents an integrated pricing and delivery strategy based on this new setting. The results arrived in chapter two nevertheless built a foundation for the discussion in this chapter. Here we distinguish between two scenarios. While in the first scenario, prices are allowed to be variable for different delivery regions, only one price should be offered in the second scenario. Their implications are also discussed.

5.1 Model Assumptions

Considering the relaxation of the number of delivery regions and the assumptions we have made in section 2.1 of chapter two, we assume the following.

- Customer demand is deterministic.
- Customers are located in more than one delivery region.
- In one typical delivery region, we have \( n \) customers who are of the same demand type. In this case, they share exactly the same demand-price curve. But demand curves can vary between different delivery regions.
- We use either the linear demand-price function or the constant price elasticity demand function to represent the demand types of all customers in our discussion.
- Each customer in every delivery region has to be visited each shipment. In other words, trucks make stops at all customers for each shipment.
- Assignments of trucks to each delivery region are independent. That is, trucks assigned to delivery region \( A \) cannot help ship some products to delivery region \( B \) even if they have extra capacity.
- The company owns the fleet of trucks, and it pays the transportation cost.
- The company owns its finished products until they reach the customers.
- Prices are constant within the same delivery region.
- Customers are subject to a minimum replenishment frequency.
- The manufacturer uses algorithms based on the Vehicle Routing Problem to schedule its truck fleet.

5.2 **Scenario One: Prices are Allowed to Be Variable for Different Delivery Regions**

If we can vary the offering prices across delivery regions, we can simply treat each delivery region as independent entities, and apply our previous results of pricing and delivery strategies. This flexibility of pricing allows the manufacturer to first group its customers to distinct delivery regions and then derive the locally optimal pricing and distribution strategies for each delivery region. Fortunately, it happens that the package of the locally optimal strategies turns out to be globally optimal, which maximizes the manufacturer's overall profit for all of its customer base. Therefore, the optimal price and the shipment frequency for each delivery region hold true for the across delivery region scenario, given that the prices are allowed to be variable.
However, for various practical reasons, a company may be restricted to offer only one constant price for all of its customers. As certain parameters change, such as the customer demand or the number of customers in a delivery region (growing end user population), the company can roll out a new set of pricing and distribution strategies from time to time. However, we here assume that at any given time, it chooses to stick to a single price for all of its delivery regions. The following section presents a systematic discussion of the changes according to this assumption.

5.3 Scenario Two: Prices Need to Be Kept Constant across Delivery Regions

For the discussion below, we assume that the manufacturer does not have pricing flexibility and can only offer one single price to all delivery regions. This yields differences from the models of chapter two. Without loss of generality, we further assume that there are only two delivery regions. The insights presented at the end also apply to situations with multiple delivery regions.

5.3.1 Model Objective

The objective of this model is to maximize the total profit for the manufacturer.

5.3.2 Variable Definitions

In the purpose of clarity, we here define all variables that will be used in the model. Note that some variables are already defined in section 2.3.2 of chapter two.

\[
\begin{align*}
 p & \quad \text{Offering price per unit in this region ($/unit)} \\
 q_A & \quad \text{Demand per customer in delivery region A (units/(week*customer))} \\
 q_B & \quad \text{Demand per customer in delivery region B (units/(week*customer))} \\
 h & \quad \text{Manufacturing cost per unit ($/unit)} \\
 R & \quad \text{Capital of interest ($/week)}
\end{align*}
\]
$D_A$ Average round-trip distance from the warehouse to delivery region A (miles)

$D_B$ Average round-trip distance from the warehouse to delivery region B (miles)

$d_A$ Average local peddling delivery distance per truck in delivery region A (miles)

$d_B$ Average local peddling delivery distance per truck in delivery region B (miles)

$T_A$ Average transit time per truck from the warehouse to delivery region A (weeks)

$T_B$ Average transit time per truck from the warehouse to delivery region B (weeks)

$t_A$ Average local peddling transit time per truck in delivery region A (weeks)

$t_B$ Average local peddling transit time per truck in delivery region B (weeks)

$V$ Shipment Size (units/shipment)

$W$ Full truckload (units/truck)

$\gamma$ Fixed cost of initiating one truck dispatch ($/dispatch$)

$\alpha$ Transportation cost per unit direct shipping distance ($/mile$)

$\beta\alpha$ Transportation cost per unit peddling distance ($/mile$) where $\beta$ is a scale factor.

$\sigma$ Fixed cost of a customer stop ($/stop$)

$n_A$ Number of customers in delivery region A (i.e. the size of delivery region A)

$n_B$ Number of customers in delivery region B (i.e. the size of delivery region B)

$m_A$ Number of stops per truck in delivery region A

$m_B$ Number of stops per truck in delivery region B

$\rho_A$ Customer density in delivery region A (customer per square mile)

$\rho_A = \frac{n_A}{A_A}$, where $A_A$ is the area of delivery region A.

$\rho_B$ Customer density in delivery region B (customer per square mile)

$\rho_B = \frac{n_B}{A_B}$, where $A_B$ is the area of delivery region B.

$f_A$ Shipment frequency for delivery region A (number of shipments/week)

$f_B$ Shipment frequency for delivery region B (number of shipments/week)

$K$ Approximation constant for calculating $d_A$ and $d_B$
5.3.3 Decision Variables

\( p \)  Selling price per unit in delivery region A and B ($/unit)

\( r_A \)  Number of TLs per shipment for delivery region A

\( r_B \)  Number of TLs per shipment for delivery region B

where \( r_A \) and \( r_B \) take only positive integer values.

5.3.4 Model Formulation

The company’s profit is its revenue net its costs. As before, we consider the total cost consisting of the manufacturing cost, the transportation cost and the inventory cost. Each component is addressed as follows.

Revenue (per unit) = \( p \)  \hspace{1cm} (2.1)

Manufacturing Cost (per unit) = \( h \)  \hspace{1cm} (2.2)

Transportation Cost (per unit) for delivery region A = \[
\frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma n_A}{W}
\]  \hspace{1cm} (5.1)

Transportation Cost (per unit) for delivery region B = \[
\frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma n_B}{W}
\]  \hspace{1cm} (5.2)

(Note that \( D_A, D_B, d_A, d_B, m_A, m_B, T_A, T_B \) and \( t_A, t_B \) are all based on each truck.)

Inventory Cost (per unit) for delivery region A = \[
hR^*\left(\frac{V_A}{2n_Aq_A} + T_A + t_A\right) = hR^*\left(\frac{r_AW}{2n_Aq_A} + T_A + t_A\right)
\]  \hspace{1cm} (5.3)

Inventory Cost (per unit) for delivery region B = \[
hR^*\left(\frac{V_B}{2n_Bq_B} + T_B + t_B\right) = hR^*\left(\frac{r_BW}{2n_Bq_B} + T_B + t_B\right)
\]  \hspace{1cm} (5.4)

We calculate the peddling distance based on typical heuristic approaches to the Vehicle Routing Problem. Recall equation (2.6) in chapter two.
\[ d = K \sqrt{\frac{mn}{r \rho}} = K \sqrt{\frac{n}{r} \frac{n}{r \rho}} \quad (K=0.6) \] 

Therefore, the average local peddling distance per truck for delivery region A is

\[ d_A = K \sqrt{\frac{m_A n_A}{r_A \rho_A}} = K \sqrt{\frac{n_A}{r_A} \frac{n_A}{r_A \rho_A}} \]  

(5.5)

and that for delivery region B is

\[ d_B = K \sqrt{\frac{m_B n_B}{r_B \rho_B}} = K \sqrt{\frac{n_B}{r_B} \frac{n_B}{r_B \rho_B}} \]  

(5.6)

We derive the profit term by taking into account all of its components.

Profit (per unit) in delivery region A = \( M_{UA} \)

\[ M_{UA} = p - hR \left( \frac{r_A W}{2n_A q_A} + T_A + t_A \right) - \left( \frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma n_A}{W} \right) - h \]  

(5.7)

Profit (per unit) in delivery region B = \( M_{UB} \)

\[ M_{UB} = p - hR \left( \frac{r_B W}{2n_B q_B} + T_B + t_B \right) - \left( \frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma n_B}{W} \right) - h \]  

(5.8)

Profit in delivery region A = \( M_{UA} * n_A q_A = M_A = f(p, r_A) \)

\[ M_A = n_A p q_A - hR \left( \frac{r_A W}{2} + (T_A + t_A) * n_A q_A \right) - \left( \frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma n_A}{W} \right) * n_A q_A - h n_A q_A \]  

(5.9)

Profit in delivery region B = \( M_{UB} * n_B q_B = M_B = f(p, r_B) \)

\[ M_B = n_B p q_B - hR \left( \frac{r_B W}{2} + (T_B + t_B) * n_B q_B \right) - \left( \frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma n_B}{W} \right) * n_B q_B - h n_B q_B \]  

(5.10)

Thus \( M \), the total profit over two delivery regions A and B is the summation of \( M_A \) and \( M_B \).
\[ M = M_A + M_B = n_A p_A - hR \left( \frac{r_A W}{2} + (T_A + t_A) n_A q_A \right) - \left( \frac{\gamma + \alpha d_A + \beta c d_A + \sigma m_A}{W} \right) n_A q_A - h n_A q_A \]
\[ + n_B p_B - hR \left( \frac{r_B W}{2} + (T_B + t_B) n_B q_B \right) - \left( \frac{\gamma + \alpha d_B + \beta c d_B + \sigma m_B}{W} \right) n_B q_B - h n_B q_B \]

(5.11)

### 5.3.5 Solutions

By definition, the shipment size is \( V = rW \), and the shipment frequency is \( f \). We now drop the subscripts on \( n, q, r \) and \( f \) to denote the number of customers, customer demand, the number of truckloads per shipment and shipment frequency respectively for each delivery region. In order to meet all the demand, we make shipment size times shipment frequency equals demand. That is, \( rW * f = nq \) with both the RHS and the LHS conditioned on a fixed period of time. And we get \( r = \frac{nq}{fW} \) (2.9).

Since the customers are subject to a minimum shipment frequency, we find the upper bound of \( r \), the maximum number of TLs per shipment as, \( r_u = \left\lfloor \frac{nq}{f_{\min} * W} \right\rfloor \) (2.10).

Therefore, we have for delivery region A

\[ r_{UA} = \left\lfloor \frac{n_A q_A}{f_{A_{\min}} * W} \right\rfloor \]  

(5.12)

and for delivery region B

\[ r_{UB} = \left\lfloor \frac{n_B q_B}{f_{B_{\min}} * W} \right\rfloor \]  

(5.13)

Thus a vector of \( r_A \) and a vector of \( r_B \) such that \( r_A = \{1, 2, 3, ..., r_{UA}\} \), \( r_B = \{1, 2, 3, ..., r_{UB}\} \) could be set up. And for any given \( r_A \) and \( r_B \), we can find the optimal offering price of \( p \) according to equation (5.18) below for the linear demand-price function case. For the case of constant price elasticity demand function, a more complex procedure is required.
5.3.5.1 Linear Demand-Price Function Scenario

If the customer demand can be captured by the linear demand-price function, the individual demand of the customers in delivery region A can be expressed as

\[ q_A = -a_A \cdot p + b_A \]  
(5.14)

and that in delivery region B as

\[ q_B = -a_B \cdot p + b_B \]  
(5.15)

where \( a_A = a_B \) is assumed in most cases so that the demand-price curves of different demand types are parallel.

Plugging the linear demand-price functions into the equation of the total profit (5.11), we get the total profit as a function of the selling price \( p \) and the number of truckloads per shipment \( r_A, r_B \).

\[
M = M_A + M_B = M(p, r_A, r_B) \\
= n_A p q_A - h R \left( \frac{r_A W}{2} + (T_A + t_A) * n_A q_A \right) - \left( \frac{\gamma + \alpha D_A + \beta \delta A + \sigma n_A}{W} \right) * n_A q_A - h n_A q_A \\
+ n_B p q_B - h R \left( \frac{r_B W}{2} + (T_B + t_B) * n_B q_B \right) - \left( \frac{\gamma + \alpha D_B + \beta \delta B + \sigma n_B}{W} \right) * n_B q_B - h n_B q_B \\
= n_A p (-a_A * p + b_A) - h R \left( \frac{r_A W}{2} + (T_A + t_A) * n_A (-a_A * p + b_A) \right) \\
- \left( \frac{\gamma + \alpha D_A + \beta \delta A + \sigma n_A}{W} \right) * n_A (-a_A * p + b_A) - h n_A (-a_A * p + b_A) \\
+ n_B p (-a_B * p + b_B) - h R \left( \frac{r_B W}{2} + (T_B + t_B) * n_B (-a_B * p + b_B) \right) \\
- \left( \frac{\gamma + \alpha D_B + \beta \delta B + \sigma n_B}{W} \right) * n_B (-a_B * p + b_B) - h n_B (-a_B * p + b_B) \\
\]  
(5.16)

Take the partial derivative of the total profit with respect to the selling price \( p \).

\[
\frac{\partial}{\partial p} M(p, r_A, r_B) = 0 , \text{ or} 
\]
\[(n_Ab_A + n_bb_B) + a_A n_A \left[ hR*(T_A + t_A) + \left( \frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma m_A}{W} \right) + h \right] \]
\[+ a_B n_B \left[ hR*(T_B + t_B) + \left( \frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma m_B}{W} \right) + h \right] = 2*(n_A a_A + n_B a_B)* p \]

(5.17)

Therefore, the optimal selling price based on any given \( r_A \) and \( r_B \) is,

\[ p^* = \frac{1}{2*(n_Aa_A + n_Ba_B)} \left\{ \left( n_Ab_A + n_bb_B \right) \right. \]
\[ + a_A n_A \left[ hR*(T_A + t_A) + \left( \frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma m_A}{W} \right) + h \right] \]
\[ + a_B n_B \left[ hR*(T_B + t_B) + \left( \frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma m_B}{W} \right) + h \right] \]

(5.18)

### 5.3.5.2 Constant Price Elasticity Demand Function Scenario

On the other hand, if the customer demand can be approximated by the constant price elasticity demand function, the individual demand of the customers in delivery region A is

\[ q_A = e_A * p^\gamma \]

(5.19)

and that in delivery region B is

\[ q_B = e_B * p^\gamma \]

(5.20)

Thus the total profit can be written as a function of the selling price \( p \) and the number of truckloads per shipment \( r_A, r_B \) by plugging in the demand functions.
Taking the partial derivative of the total profit with respect to the selling price $p$, we get

$$\frac{\partial}{\partial p} M(p, r_A, r_B) = 0$$

$$n_A e_A (f_A + 1)p^{f_A} + n_B e_B (f_B + 1)p^{f_B}$$

$$- n_A e_A f_A p^{f_A - 1} \left[ hR * (T_A + t_A) + \left( \frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma m_A}{W} \right) + h \right]$$

$$- n_B e_B f_B p^{f_B - 1} \left[ hR * (T_B + t_B) + \left( \frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma m_B}{W} \right) + h \right] = 0$$

(5.22)

In this case, the optimal selling price $p$ cannot be expressed in a closed form formula. However, it can be found by an iterative procedure.
5.3.6 Algorithm for Optimal Price

The procedures for computing an optimal profit and its associated optimal number of truckloads per shipment are given as follows.

1. For any given $r_{Aj}$ and $r_{Bi}$, the optimal offering price $p^*_i$ is given above in equation (5.18), or by computation using (5.22).

2. By substituting $p^*_i$ into equation (5.11), we can get the maximum profit $M^*_i$.

3. Repeat 1,2 for the combination of the valid values of $r_A$ and $r_B$, where

\[ r_A = \{1,2,3,...,r_{UA}\}, \quad r_B = \{1,2,3,...,r_{UB}\}, \]

and get a vector of $M^*$, i.e.,

\[ M^* = \{M^*_1, M^*_2,...,M^*_U\} \]

4. The optimized (maximized) profit is \( M_{\text{max}} = \max(M^*) = M^*_j \). Accordingly, the optimal offering price is $p^*_j$. The optimal shipment size for delivery region A is

\[ V_{Aj} = r_{Aj}^* \cdot W. \]

And the optimal shipment frequency for delivery region A is

\[ f_{Aj} = \frac{n_A q_A}{r_{Aj}^* \cdot W}. \]

The optimal shipment size for delivery region B is $V_{Bj} = r_{Bj}^* \cdot W$.

And the optimal shipment frequency for delivery region B is

\[ f_{Bj} = \frac{n_B q_B}{r_{Bj}^* \cdot W}. \]

Note that in step 3 above, most often we don’t need to try all the combinations of $r_A$ and $r_B$. Typically we can establish a subset of $r_A$ from $\{1,2,3,...,r_{UA}\}$ and a subset of $r_B$ from $\{1,2,3,...,r_{UB}\}$ that are appropriate. The combinations of the elements in the subsets can then be used to derive the vector of $M^*$. As an example, if the company is only interested in shipping its customers as often as possible, as discussed in chapter three, we only need to set $r_A = 1$ and $r_B = 1$, plug them into equation (5.18) in the linear demand-price function case and get the optimal price directly. This may largely reduce the number of calculations.
Chapter 6

Model with Stochastic Demand and a Single Delivery Region

We have already discussed in chapter two the distribution and pricing strategies in one delivery region given deterministic customer demand. The results in chapter two set the foundation for further analysis and provide valuable insights into our new problem settings. However, in reality, the customer demand is not deterministic. It usually follows a pattern with stochastic nature. In this chapter, we relax the assumption of deterministic demand. By stipulating stochastic demand, we develop a modified model in section 6.3 and discuss its implications. We follow by comparing the optimal offering price in this new demand setting to that found in chapter two. Throughout this chapter, we restrict our analysis to one delivery region.

6.1 Model Assumptions

Considering the relaxation of customer demand and the assumptions we have made in section 2.1 of chapter two, we further assume the following.

- Customer demand is stochastic.
- In one typical delivery region, we have $n$ customers.
In one delivery region, the individual customer demand is independently, normally distributed with mean \( \mu_i \) and variance \( \sigma_i^2 \) \( (i = 1...n) \).

We use either the linear demand-price function or the constant price elasticity demand function to represent the demand types of all customers in our discussion.

### 6.2 Demand and Safety Stock Statistics

We first introduce a series of notation and statistics to get a better understanding of the individual customer demand, the average demand, and the aggregate demand. Based on that, two inventory management policies are presented and their corresponding safety stock levels are derived.

#### 6.2.1 Variable Definitions

- \( X_i \): A random variable representing the individual customer \( i \)'s demand (units/week) \( (i = 1...n) \)
- \( \mu_i \): The mean of the individual customer \( i \)'s demand (units/week) \( (i = 1...n) \)
- \( \sigma_i \): The standard deviation of the individual customer \( i \)'s demand (units/week) \( (i = 1...n) \)
- \( Y \): A random variable representing the average customer demand (units/week)
- \( \mu_Y \): The mean of the average customer demand (units/week)
- \( \sigma_Y \): The standard deviation of the average customer demand (units/week)
- \( Z \): A random variable representing the aggregate (total) customer demand (units/week)
- \( \mu_Z \): The mean of the aggregate customer demand (units/week)
- \( \sigma_Z \): The standard deviation of the aggregate customer demand (units/week)
- \( \phi \): The standard deviation to mean ratio of the random variable \( Y \)
- \( \bar{\mu} \): Same as \( \mu_Y \), for notation convenience
- \( \bar{\sigma} \): Same as \( \sigma_Y \), for notation convenience
\( n \)  Number of customers in the delivery region (i.e. delivery region size)
\( z \)  Safety stock factor
\( L \)  The lead time from a product being ordered to it being ready for shipping (weeks)
\( l \)  The review period for periodic review inventory management policy (weeks)

### 6.2.2 Demand Statistics

We define two random variables \( Y \) and \( Z \) to represent the average and aggregate demand in the delivery region, respectively. We set

\[
Y = \frac{1}{n} (X_1 + X_2 + \ldots + X_n) = \frac{1}{n} \sum_{i=1}^{n} X_i, \tag{6.1}
\]

and \( Y \) is a random variable representing the average demand. Since \( X_1, X_2, \ldots, X_n \) are independent, normally distributed with mean \( \mu_i \) and variance \( \sigma_i^2 \) (\( i = 1 \ldots n \)), the expected value (mean) and the variance of \( Y \) are as follows.

\[
E(Y) = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{1}{n} \sum_{i=1}^{n} \mu_i, \tag{6.2}
\]

\[
Var(Y) = Var\left[ \frac{1}{n} (X_1 + X_2 + \ldots + X_n) \right] = \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2, \tag{6.3}
\]

We also set \( Z = X_1 + X_2 + \ldots + X_n = \sum_{i=1}^{n} X_i \), \( \sum_{i=1}^{n} X_i \), \( (6.4) \)
and $Z$ is a random variable representing the aggregate demand. Since $X_1, X_2, ..., X_n$ are independent, normally distributed with mean $\mu_i$ and variance $\sigma_i^2$ ($i = 1..n$), the expected value (mean) and the variance of $Z$ are as follows.

$$E(Z) = E(X_1 + X_2 + ... + X_n)$$
$$= E(X_1) + E(X_2) + ... + E(X_n)$$

$$E(Z) = \sum_{i=1}^{n} \mu_i$$

(6.5)

$$Var(Z) = Var(X_1 + X_2 + ... + X_n)$$
$$= Var(X_1) + Var(X_2) + ... + Var(X_n)$$

$$Var(Z) = \sum_{i=1}^{n} \sigma_i^2$$

(6.6)

For the convenience of notation, we introduce the terms $\bar{\mu}$, $\bar{\sigma}$ and $\phi$, and set

$$\bar{\mu} = \mu_Y$$
$$\bar{\sigma} = \sigma_Y$$

Therefore, we have

$$\mu_Y = \frac{1}{n} \sum_{i=1}^{n} \mu_i = \bar{\mu}$$

(6.7)

$$\sigma_Y^2 = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 = \bar{\sigma}^2$$

(6.8)

$$\sigma_Y = \bar{\sigma}$$

(6.9)

$$\mu_Z = \sum_{i=1}^{n} \mu_i = n \bar{\mu}$$

(6.10)

$$\sigma_Z^2 = \sum_{i=1}^{n} \sigma_i^2 = n^2 \cdot \bar{\sigma}^2 = (n \bar{\sigma})^2$$

(6.11)

$$\sigma_Z = n \bar{\sigma}$$

(6.12)

Since $\phi$ is defined as the standard deviation to mean ratio of the random variable $Y$, we derive $\phi$ as $\phi = \frac{\sigma_Y}{\mu_Y} = \frac{\bar{\sigma}}{\bar{\mu}}$ (6.13). Consequently we get $\bar{\sigma} = \phi \bar{\mu} = \phi \mu$ (6.14). Recall that $q$ is defined in chapter two as the average demand per customer per week, so we have

$q = \bar{\mu}$. An important assumption is that for a specific delivery region, $\phi$ is constant. It is
useful to measure the delivery region’s $\bar{\sigma}$ when its average demand $\bar{\mu}(q)$ changes as price varies.

6.2.3 Inventory Management Policies and Safety Stock

The two commonly used inventory management policies are the Continuous Review policy (also called the QR model) and the Periodic Review policy (also called the Base Stock model). The continuous review policy requires order fixed quantity (Q) when the total inventory at-hand drops below a reorder point (R). And the periodic review policy requires order at fixed time intervals to raise the total inventory at-hand to a certain preset order-up-to-level. The continuous review method is usually used to monitor A-items because it is more expensive compared to the periodic review alternative. Because of the different nature of how these two policies manage inventory, the safety stocks associated with them are therefore different. The safety stock we keep in our central production facility to hedge the customer demand uncertainty is as follows.

Continuous Review Policy (QR Model)
Safety Stock = $z \cdot \sigma \cdot \sqrt{L} = zn\bar{\sigma}\sqrt{L} = zn\phi q\sqrt{L} \quad (6.15)$

Periodic Review Policy (Base Stock Model)
Safety Stock = $z \cdot \sigma \cdot \sqrt{L + l} = zn\bar{\sigma}\sqrt{L + l} = zn\phi q\sqrt{L + l} \quad (6.16)$

To summarize, the safety stock level is

$SS = zn\phi q\sqrt{L + l} \quad (6.17)$, where $l = 0$ if the QR Model is used in managing the inventory. In the model development below, we will include the safety stock consideration into the inventory cost component to reflect the added cost due to the demand uncertainty.

6.3 Model with Stochastic Demand and One Delivery Region

In this section, we develop a model that handles the stochastic demand. We assume that all customers reside in one delivery region.
6.3.1 Model Objective
The objective of this model is to maximize the total profit for the manufacturer.

6.3.2 Variable Definitions
All variables have been defined in either section 2.3.2 of chapter two or section 6.2.1 above. But note that \( q \) is now the average demand per customer in the delivery region (units/(week*customer)).

6.3.3 Decision Variables
\( p \)  
Selling price per unit in this region ($/unit)

\( r \)  
Number of TLs per shipment

where \( r \) takes only positive integer values.

6.3.4 Model Formulation
The company’s profit is its revenue net its costs. As before, we consider the total cost consisting of the manufacturing cost, the transportation cost and the inventory cost. In the discussion below, the safety stock factor is included in the total inventory cost. Each component is addressed as follows.

Revenue (per unit) = \( p \) \hspace{2cm} (2.1)

Manufacturing Cost (per unit) = \( h \) \hspace{2cm} (2.2)

Transportation Cost (per unit) = \( \frac{\gamma + \alpha D + \beta c d + \sigma m}{W} \) \hspace{2cm} (2.3)

(Note that \( D, d, m, T \) and \( t \) are all based on each truck.)

Inventory Cost (per unit)

\[
= hR \left( \frac{\gamma}{2nq} + T + t \right) + \frac{hR \cdot z n \phi q \sqrt{L + l}}{2nq}
\]

\[
= hR \left( \frac{rW}{2nq} + T + t \right) + hR \cdot \phi \sqrt{L + l}
\]  \hspace{2cm} (6.18)
The new inventory cost component shown in equation (6.18) above is the old inventory cost component (equation (2.4): \( hR \left( \frac{rW}{2nq} + T + t \right) \)) plus the safety stock.

As discussed in chapter two, the number of stops per truck is \( m = \left\lfloor \frac{n}{r} \right\rfloor \) (2.5), and the peddling distance per truck is

\[
d = K \sqrt{mn/r\rho} = K \sqrt{\left( \frac{n}{r} \right) \cdot \frac{n}{r\rho}} \quad (K=0.6)
\]

Therefore, Profit (per unit) = \( M_U \)

\[
M_U = p - hR \left( \frac{rW}{2nq} + T + t \right) - hR \left( z\phi \sqrt{L + l} \right) - \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) - h
\]

(6.19)

Profit = \( M_U \cdot nq = M = f(p,r) \)

\[
M = npq - hR \left( \frac{rW}{2} + (T + t) \cdot nq \right) - hR \left( z\phi nq \sqrt{L + l} \right) - \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) \cdot nq - hnq
\]

(6.20)

### 6.3.5 Solutions

As discussed before, we have \( r = \frac{nq}{fW} \) (2.9) and \( r_U = \left\lfloor \frac{nq}{f_{\min} W} \right\rfloor \) (2.10). Thus a vector of \( r \) such that \( \tilde{r} = \{1,2,3,\ldots,r_U\} \) could be set up. We assumed that the customer demand is stochastic. But at the same time, it is also a function of the offering price. We keep safety stock at the central facility to cope with the stochastic nature of the demand, and still express the average demand per customer \( q \) as a function to the offering price of \( p \).

Based on what kind of demand function we assume, for any given \( r \), we can find the optimal offering price of \( p^\star \) according to equation (6.22) or (6.24) below.

#### 6.3.5.1 Linear Demand-Price Function Scenario

Suppose that the average customer demand is a linear function to the price, as stated in equation (4.1) below.
\[ q = -a^* p + b \]  

(4.1)

Plugging the linear demand-price function into the equation of the total profit, we get the total profit as a function of the selling price \( p \) and the number of TLs per shipment \( r \).

\[
M = npq - hR^* \left( \frac{rW}{2} + (T + t) * nq \right) - hR^* z\phi nq \sqrt{L + l} - \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right)^* nq - hR^* \frac{rW}{2}
\]

\[
= npq - \left[ hR(T + t) + hR^* z\phi \sqrt{L + l} + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] * nq - hR^* \frac{rW}{2}
\]

\[
= np(- ap + b) - \left[ hR(T + t) + hR^* z\phi \sqrt{L + l} + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] * n(- ap + b)
\]

\[
- hR^* \frac{rW}{2}
\]

(6.21)

Take the partial derivative of the total profit with respect to the selling price \( p \).

\[
\frac{\partial}{\partial p} M(p, r) = 0
\]

\[
- 2anp + bn + an^* \left[ hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] + anhRz\phi \sqrt{L + l} = 0
\]

\[
P_{ss} = \frac{1}{2} \left[ \left( \frac{b}{a} \right) + hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h + hRz\phi \sqrt{L + l} \right]
\]

Therefore, the optimal selling price based on a given \( r \) is,

\[
P_{ss} = \frac{1}{2} \left[ \left( \frac{b}{a} \right) + hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h + hRz\phi \sqrt{L + l} \right]
\]

(6.22)

6.3.5.2 Constant Price Elasticity Demand Function Scenario

If the average customer demand price elasticity is constant, as stated in equation (4.6) below,

\[ q = e^* p' \]  

(4.6)

we plug the constant price elasticity demand function into the equation of the total profit, and get the total profit as a function \( p \) and \( r \).
\[ M = npq - hR \left( \frac{rW}{2} + (T + t) * nq \right) - hR * z \phi nq \sqrt{L + l} - \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) * nq - hnq \]

\[ = npq - \left[ hR(T + t) + hR * z \phi \sqrt{L + l} + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] * nq - hR * \frac{rW}{2} \]

\[ = np(e * p') - \left[ hR(T + t) + hR * z \phi \sqrt{L + l} + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] * n(e * p') - hR * \frac{rW}{2} \]

\[ = ne * p'^{f-1} - \left[ hR(T + t) + hR * z \phi \sqrt{L + l} + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] * ne * p' - hR * \frac{rW}{2} \]

(6.23)

Take the partial derivative of the total profit with respect to the selling price \( p \).

\[ \frac{\partial}{\partial p} M(p, r) = 0 \]

\[ (f + 1) * ne * p' - \left[ hR(T + t) + hR * z \phi \sqrt{L + l} + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] * ne * f * p'^{f-1} = 0 \]

\[ p_{ss}^* = \frac{f}{f + 1} \left[ hR(T + t) + hR * z \phi \sqrt{L + l} + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h \right] \]

Therefore, the optimal selling price based on a given \( r \) is,

\[ p_{ss}^* = \frac{f}{f + 1} \left[ hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \alpha d + \sigma m}{W} \right) + h + hRz \phi \sqrt{L + l} \right] \]

(6.24)

### 6.3.6 Computing Procedures

The procedures for computing an optimal profit and its associated optimal number of TLs per shipment are given as follows.

1. For any given \( r_i \), the optimal offering price \( p_i^* \) is given above in equation (6.22) or (6.24).

2. By substituting \( p_i^* \) into equation (6.21) or (6.23), we can get the maximum profit \( M_i^* \).
3. Repeat 1,2 for the valid vector of $r$, i.e., $\tilde{r} = \{r_1, r_2, \ldots, r_u\}$, and get a vector of $M^*$, i.e., $\tilde{M}^* = \{M_1^*, M_2^*, \ldots, M_u^*\}$

4. The optimized (maximized) profit is $M_{\text{max}} = \text{Max}(\tilde{M}^*) = M_j^*$. Accordingly, the optimal offering price is $p_j^*$; the optimal shipment size is $V_j = r_j^* W$; and the optimal shipment frequency is $f_j = \frac{nq_j}{r_j^* W} = \frac{(-ap_j^* + b)^* n}{r_j^* W}$ in the linear demand-price function scenario, and $f_j = \frac{nq_j}{r_j^* W} = \frac{(ep_j^*)^* n}{r_j^* W}$ for the constant price elasticity demand function case.

### 6.4 Model Results Interpretation

For linear demand-price function, compare equation (6.22) above with our previous result derived in chapter two without the safety stock consideration (equation (2.11)).

\[
p^* = \frac{1}{2} \left[ \frac{b}{a} + hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \sigma D + \sigma m}{W} \right) + h \right]
\]

\[
p_{ss}^* = \frac{1}{2} \left[ \frac{b}{a} + hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \sigma D + \sigma m}{W} \right) + h + hRz\phi \sqrt{L + 1} \right]
\] (6.22)

For constant price elasticity demand function, compare the equation (6.24) with our previous result derived in chapter four without the safety stock consideration (equation (4.9)).

\[
p^* = \frac{f}{f + 1} \left[ hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \sigma D + \sigma m}{W} \right) + h \right]
\]

\[
p_{ss}^* = \frac{f}{f + 1} \left[ hR(T + t) + \left( \frac{\gamma + \alpha D + \beta \sigma D + \sigma m}{W} \right) + h + hRz\phi \sqrt{L + 1} \right]
\] (6.24)
The only difference is that with safety stock consideration, the optimal price should be increased by \( \Delta_1 = \frac{1}{2} h R z \phi \sqrt{L + l} \) (6.25) in the linear demand-price function scenario, and
\[
\Delta_2 = \frac{f}{f + 1} h R z \phi \sqrt{L + l} \quad (6.26)
\]
in the constant price elasticity demand function scenario, with all other parameters held the same. This result also makes intuitive sense. Because our total variable cost increases if we add the safety stock, it is then reasonable to raise our prices given a higher cost.

Also note that the increase in optimal price is a linearly increasing function of \( \phi \).

\[
\Delta_1 = \frac{1}{2} h R z \phi \sqrt{L + l} = \left( \frac{1}{2} h R z \sqrt{L + l} \right) \ast \frac{\sigma}{\mu} \\
\Delta_2 = \frac{f}{f + 1} h R z \phi \sqrt{L + l} = \left( \frac{f}{f + 1} h R z \sqrt{L + l} \right) \ast \frac{\sigma}{\mu}
\]

As we mentioned before, \( \phi \) is a constant for a specific delivery region. It measures the delivery region’s average demand standard deviation per one unit of its mean value. If \( \phi \) is large, we anticipate more fluctuation in the demand change (larger standard deviation per one unit of mean demand). Hence we need to keep more safety stock to hedge the greater demand uncertainty. Consequently, we are forced to offer a higher price to offset our cost increase due to the safety stock increase.

A closer look at the safety stock also gives us more insights into this problem. Recall equation (6.17)
\[
SS = z n \phi q \sqrt{L + l} = z n \sqrt{L + l} \quad (6.17)
\]
The safety stock is a linearly increasing function of \( n \) and \( \sqrt{L + l} \). This implies that as the number of customers \( n \) gets larger, or as the average customer demand gets more volatile, we should keep a higher level of safety stock.
Chapter 7

Model with Stochastic Demand and Multiple Delivery Regions

The previous chapter outlines a model for determining the integrated pricing and delivery strategy in a two-echelon supply chain under stochastic customer demand. We assumed that the manufacturer serves only one delivery region. In other words, all customers are grouped into one delivery region. This assumption is relaxed in this chapter to accommodate multiple delivery regions. The model developed later in this chapter presents an integrated pricing and delivery strategy based on this new setting. As we have done in chapter five when deriving the across delivery region applications for deterministic demand, two important scenarios are distinguished. In the first scenario, prices are allowed to be variable for different delivery regions, and only one price should be offered in the second scenario. A discussion of their implications follows at the end of this chapter.

7.1 Model Assumptions

Considering stochastic customer demand setting, the relaxation of the number of delivery regions and the assumptions of section 2.1 of chapter two, we assume the following.
Customer demand is stochastic.

Customers are located in more than one delivery region.

In one typical delivery region, we have \( n \) customers.

In one delivery region, the individual customer demand is independently, normally distributed with mean \( \mu_i \) and variance \( \sigma_i^2 (i = 1...n) \).

The average demand can vary between different delivery regions.

We use either the linear demand-price function or the constant price elasticity demand function to represent the demand price relationship of the average demand in each delivery region.

Each customer in a delivery region has to be visited each shipment. In other words, trucks make stops at all customers in the delivery region for every shipment.

Assignments of trucks to each delivery region are independent. That is, trucks assigned to delivery region \( A \) cannot help ship some products to delivery region \( B \) even if they have extra capacity.

The company owns the fleet of trucks, and it pays the transportation cost.

The company owns its finished products until they reach the customers.

Prices are constant within the same delivery region.

Customers are subject to a minimum replenishment frequency.

The manufacturer uses algorithms based on the Vehicle Routing Problem to schedule its truck fleet.

### 7.2 Scenario One: Prices are Allowed to Be Variable for Different Delivery Regions

If we can vary the offering prices across delivery regions, we can simply treat each delivery region as independent entities, and apply our previous results of pricing and delivery strategies derived in chapter six. This flexibility of pricing allows the manufacturer to first group its customers to distinct delivery regions and then derive the locally optimal pricing and distribution strategies for each delivery region. And the
package of the locally optimal strategies turns out to be globally optimal, which maximizes the manufacturer's overall profit for all of its customer base. Therefore, the optimal price and the shipment frequency for each delivery region hold true for the across delivery region scenario, given that the prices are allowed to be variable.

However when a company is restricted to offer only one constant price for all of its customers, a more complicated model should be followed. The following section develops a model based on this constraint of unique pricing.

7.3 Scenario Two: Prices Need to Be Kept Constant across Delivery Regions

For the discussion below, we assume that the manufacturer does not have pricing flexibility and can only offer one single price to all delivery regions. This yields differences from the models of chapter six. Without loss of generality, we further assume that there are only two delivery regions. The insights presented at the end also apply to situations with multiple delivery regions.

7.3.1 Demand and Safety Stock Statistics

Before we begin to build the model, first identify some demand and safety stock statistics that will be helpful in the model development.

7.3.1.1 Variable Definitions

\( q_A \) Average demand per customer in delivery region A (units/(week*customer))

\( q_B \) Average demand per customer in delivery region B (units/(week*customer))

\( X_{iA} \) A random variable representing the individual customer \( i \)'s demand in delivery region A (units/week) \( (i = 1...n_A) \)

\( \mu_{iA} \) The mean of the individual customer \( i \)'s demand in delivery region A (units/week) \( (i = 1...n_A) \)
\( \sigma_{iA} \)  The standard deviation of the individual customer \( i \)'s demand in delivery region A (units/week) \((i = 1..n_A)\)

\( Y_A \)  A random variable representing the average customer demand in delivery region A (units/week)

\( \mu_{YA} \)  The mean of the average customer demand in delivery region A (units/week)

\( \sigma_{YA} \)  The standard deviation of the average customer demand in delivery region A (units/week)

\( Z_A \)  A random variable representing the aggregate customer demand in delivery region A (units/week)

\( \mu_{ZA} \)  The mean of the aggregate customer demand in delivery region A (units/week)

\( \sigma_{ZA} \)  The standard deviation of the aggregate customer demand in delivery region A (units/week)

\( \phi_A \)  The standard deviation to mean ratio of the random variable \( Y_A \)

\( \bar{\mu}_A \)  Same as \( \mu_{YA} \), for notation convenience

\( \bar{\sigma}_A \)  Same as \( \sigma_{YA} \), for notation convenience

\( z \)  Combined safety stock factor for both delivery region A and B

\( L_A \)  The lead time from a product being ordered to it being ready for shipping for delivery region A (weeks)

\( l_A \)  The review period for periodic review inventory management policy for delivery region A (weeks)

\( X_{jB} \)  A random variable representing the individual customer \( j \)'s demand in delivery region B (units/week) \((j = 1..n_B)\)

\( \mu_{jB} \)  The mean of the individual customer \( j \)'s demand in delivery region B (units/week) \((j = 1..n_B)\)

\( \sigma_{jB} \)  The standard deviation of the individual customer \( j \)'s demand in delivery region B (units/week) \((j = 1..n_B)\)
**7.3.1.2 Demand Statistics**

Since $X_{i1}, X_{i2}, \ldots, X_{in_A}$ are independent, normally distributed with mean $\mu_{iA}$ and variance $\sigma_{iA}^2$ ($i = 1 \ldots n_A$), the expected value (mean) and the variance of $Y_A$ and $Z_A$ are as follows.

$$Y_A = \frac{1}{n_A} \left( X_{i1} + X_{i2} + \ldots + X_{in_A} \right) = \frac{1}{n_A} \sum_{i=1}^{n_A} X_{iA}$$  \hspace{1cm} (7.1)

$$E(Y_A) = \frac{1}{n_A} \sum_{i=1}^{n_A} \mu_{iA}$$  \hspace{1cm} (7.2)

$$Var(Y_A) = \frac{1}{n_A^2} \sum_{i=1}^{n_A} \sigma_{iA}^2$$  \hspace{1cm} (7.3)
\[
Z_A = X_{i1} + X_{i2} + \ldots + X_{in_A} = \sum_{i=1}^{n_A} X_{iA}
\]  \hspace{1cm} (7.4)

\[
E(Z_A) = \sum_{i=1}^{n_A} \mu_{iA}
\]  \hspace{1cm} (7.5)

\[
\text{Var}(Z_A) = \sum_{i=1}^{n_A} \sigma_{iA}^2
\]  \hspace{1cm} (7.6)

Also because \(X_{j1}, X_{j2}, \ldots, X_{jn_B}\) are independent, normally distributed with mean \(\mu_{jB}\) and variance \(\sigma_{jB}^2\) (\(j = 1 \ldots n_B\)), the expected value (mean) and the variance of \(Y_B\) and \(Z_B\) are as follows.

\[
Y_B = \frac{1}{n_B} (X_{j1} + X_{j2} + \ldots + X_{jn_B}) = \frac{1}{n_B} \sum_{i=1}^{n_B} X_{jB}
\]  \hspace{1cm} (7.7)

\[
E(Y_B) = \frac{1}{n_B} \sum_{j=1}^{n_B} \mu_{jB}
\]  \hspace{1cm} (7.8)

\[
\text{Var}(Y_B) = \frac{1}{n_B^2} \sum_{j=1}^{n_B} \sigma_{jB}^2
\]  \hspace{1cm} (7.9)

\[
Z_B = X_{j1} + X_{j2} + \ldots + X_{jn_B} = \sum_{j=1}^{n_B} X_{jB}
\]  \hspace{1cm} (7.10)

\[
E(Z_B) = \sum_{j=1}^{n_B} \mu_{jB}
\]  \hspace{1cm} (7.11)

\[
\text{Var}(Z_B) = \sum_{j=1}^{n_B} \sigma_{jB}^2
\]  \hspace{1cm} (7.12)

For convenience, we set \(\mu_A = \mu_{YA}\), \(\sigma_A = \sigma_{YA}\) and \(\mu_B = \mu_{YB}\), \(\sigma_B = \sigma_{YB}\). Therefore, we have

\[
\mu_{YA} = \frac{1}{n_A} \sum_{i=1}^{n_A} \mu_{iA} = \mu_A
\]  \hspace{1cm} (7.13)

\[
\sigma_{YA}^2 = \frac{1}{n_A^2} \sum_{i=1}^{n_A} \sigma_{iA}^2 = \sigma_A^2
\]  \hspace{1cm} (7.14)

\[
\sigma_{YA} = \sigma_A
\]  \hspace{1cm} (7.15)
\[
\mu_{Z_A} = \sum_{i=1}^{n_A} \mu_{iA} = n_A \overline{\mu_A} \tag{7.16}
\]

\[
\sigma_{Z_A}^2 = \sum_{i=1}^{n_A} \sigma_{iA}^2 = n_A \overline{\sigma_A}^2 = (n_A \overline{\sigma_A})^2 \tag{7.17}
\]

\[
\sigma_{Z_A} = n_A \overline{\sigma_A} \tag{7.18}
\]

\[
\mu_{YB} = \frac{1}{n_B} \sum_{j=1}^{n_B} \mu_{jB} = \overline{\mu_B} \tag{7.19}
\]

\[
\sigma_{YB}^2 = \frac{1}{n_B} \sum_{j=1}^{n_B} \sigma_{jB}^2 = \overline{\sigma_B}^2 \tag{7.20}
\]

\[
\sigma_{YB} = \overline{\sigma_B} \tag{7.21}
\]

\[
\mu_{ZB} = \sum_{j=1}^{n_B} \mu_{jB} = n_B \overline{\mu_B} \tag{7.22}
\]

\[
\sigma_{ZB}^2 = \sum_{j=1}^{n_B} \sigma_{jB}^2 = n_B \overline{\sigma_B}^2 = (n_B \overline{\sigma_B})^2 \tag{7.23}
\]

\[
\sigma_{ZB} = n_B \overline{\sigma_B} \tag{7.24}
\]

For a specific delivery region, \( \phi \) is constant. \( \phi \) measures the delivery region’s average demand’s standard deviation when its mean changes as the price changes. Therefore, we have

\[
\phi_A = \frac{\sigma_{Y_A}}{\mu_{Y_A}} = \frac{\overline{\sigma_A}}{\mu_A} \tag{7.25}
\]

\[
\overline{\sigma_A} = \phi_A \overline{\mu_A} = \phi_A q_A \tag{7.26}
\]

\[
\phi_B = \frac{\sigma_{Y_B}}{\mu_{Y_B}} = \frac{\overline{\sigma_B}}{\mu_B} \tag{7.27}
\]

\[
\overline{\sigma_B} = \phi_B \overline{\mu_B} = \phi_B q_B \tag{7.28}
\]

### 7.3.1.3 Safety Stock Statistics

In this example, we use only one central facility to serve both delivery region A and B.

To hedge the lead-time demand uncertainty of those delivery regions, some safety stock
needs to be kept in the central facility. Even though prices are uniform, the delivery regions are independent since we have different number of deliveries in the regions. We assume that the customer demand in delivery region A is independent of that in delivery region B. As mentioned before, the individual customer demand in each delivery region is also independent from each other. Thus based on the two different inventory management policies presented in section 6.2.3, the safety stock level is as follows.

Continuous Review Policy (QR Model)

$$SS_{Combined} = z \sqrt{L_A \cdot \sigma_{ZA}^2 + L_B \cdot \sigma_{ZB}^2}$$

$$= z \sqrt{L_A \cdot n_A \sigma_A^2 + L_B \cdot n_B \sigma_B^2}$$

$$= z \sqrt{L_A n_A \sigma_A^2 n_A + L_B n_B \sigma_B^2 n_B}$$

$$= z \sqrt{L_A \left(\sigma_A n_A\right)^2 + L_B \left(\sigma_B n_B\right)^2}$$

$$= z \sqrt{L_A (\phi_A q_A n_A)^2 + L_B (\phi_B q_B n_B)^2}$$

Periodic Review Policy (Base Stock Model)

$$SS_{Combined} = z \sqrt{(L_A + l_A) \cdot \sigma_{ZA}^2 + (L_B + l_B) \cdot \sigma_{ZB}^2}$$

$$= z \sqrt{(L_A + l_A) \cdot n_A \sigma_A^2 + (L_B + l_B) \cdot n_B \sigma_B^2}$$

$$= z \sqrt{(L_A + l_A) \cdot n_A \sigma_A^2 n_A + (L_B + l_B) \cdot n_B \sigma_B^2 n_B}$$

$$= z \sqrt{(L_A + l_A) \left(\sigma_A n_A\right)^2 + (L_B + l_B) \left(\sigma_B n_B\right)^2}$$

$$= z \sqrt{(L_A + l_A) (\phi_A q_A n_A)^2 + (L_B + l_B) (\phi_B q_B n_B)^2}$$

To summarize, the safety stock level is

$$SS_{Combined} = z \sqrt{(L_A + l_A) (\phi_A q_A n_A)^2 + (L_B + l_B) (\phi_B q_B n_B)^2}$$  (7.31), where \(l_A = l_B = 0\) if the QR Model is used in managing the inventory.

Based on the discussion above, we develop a model that determines the integrated pricing and delivery strategy in a two-echelon supply chain given that the customer demand is stochastic, and the manufacturer serves two delivery regions.
7.3.2 Model Objective

The objective of this model is to maximize the total profit for the manufacturer.

7.3.3 Variable Definitions

All variables have been defined in section 5.3.2 of chapter five and in section 7.3.1.1 above.

7.3.4 Decision Variables

\( p \)  
Selling price per unit in delivery region A and B ($/unit)

\( r_A \)  
Number of TLs per shipment for delivery region A

\( r_B \)  
Number of TLs per shipment for delivery region B

where \( r_A \) and \( r_B \) take only positive integer values.

7.3.5 Model Formulation

The company’s profit is its revenue net its costs. As before, we consider the total cost consisting of the manufacturing cost, the transportation cost and the inventory cost, which includes the safety stock cost. Each component is addressed as follows.

Revenue (per unit) = \( p \)  \hspace{1cm} (2.1)

Manufacturing Cost (per unit) = \( h \)  \hspace{1cm} (2.2)

Transportation Cost (per unit) for delivery region A = \[
\frac{\gamma + \alpha D_A + \beta \omega d_A + \sigma m_A}{W}
\] \hspace{1cm} (5.1)

Transportation Cost (per unit) for delivery region B = \[
\frac{\gamma + \alpha D_B + \beta \omega d_B + \sigma m_B}{W}
\] \hspace{1cm} (5.2)

(Note that \( D_A, D_B, d_A, d_B, m_A, m_B, T_A, T_B \) and \( t_A, t_B \) are all based on each truck.)

Inventory Cost (per unit) for delivery region A excluding safety stock
Inventory Cost (per unit) for delivery region B excluding safety stock

\[ = hR^* \left( \frac{V_B}{2n_Bq_B} + T_B + t_B \right) = hR^* \left( \frac{r_BW}{2n_Bq_B} + T_B + t_B \right) \] (7.33)

Safety Stock Cost at the central facility

\[ SS_{\text{Combined}} = hR^* z \sqrt{(L_A + l_A)^* (\phi_A q_A n_A)^2 + (L_B + l_B)^* (\phi_B q_B n_B)^2} \] (7.34)

We have the number of stops per truck as \( m = \left\lceil \frac{n}{r} \right\rceil \) (2.5), and the peddling distance per truck as

\[ d = K \sqrt{mn/r} = K \sqrt{\left\lceil \frac{n}{r} \right\rceil \frac{n}{r} \rho} \quad (K=0.6) \] (2.6)

Therefore, the average local peddling distance per truck for delivery region A is

\[ d_A = K \sqrt{m_A n_A/r_A} = K \sqrt{\left\lceil \frac{n_A}{r_A} \right\rceil \frac{n_A}{r_A} \rho_A} \] (5.5)

and that for delivery region B is

\[ d_B = K \sqrt{m_B n_B/r_B} = K \sqrt{\left\lceil \frac{n_B}{r_B} \right\rceil \frac{n_B}{r_B} \rho_B} \] (5.6)

We derive the profit term by taking into account all its components except the safety stock cost.

Profit (per unit) in delivery region A = \( M_{UA} \)

\[ M_{UA} = p - hR^* \left( \frac{r_A W}{2n_A q_A} + T_A + t_A \right) \quad \left( \frac{\gamma + \alpha A + \beta \alpha d_A + \sigma m_A}{W} \right) - h \] (7.35)

Profit (per unit) in delivery region B = \( M_{UB} \)

\[ M_{UB} = p - hR^* \left( \frac{r_B W}{2n_B q_B} + T_B + t_B \right) \quad \left( \frac{\gamma + \alpha A + \beta \alpha d_B + \sigma m_B}{W} \right) - h \] (7.36)
Profit in delivery region A = \( M_{UA} * n_A q_A = M_A = f(p, r_A) \)

\[
M_A = n_A p q_A - hR \left( \frac{r_A W}{2} + (T_A + t_A) * n_A q_A \right) - \left( \frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma m_A}{W} \right) * n_A q_A - h n_A q_A
\]

(7.37)

Profit in delivery region B = \( M_{UB} * n_B q_B = M_B = f(p, r_B) \)

\[
M_B = n_B p q_B - hR \left( \frac{r_B W}{2} + (T_B + t_B) * n_B q_B \right) - \left( \frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma m_B}{W} \right) * n_B q_B - h n_B q_B
\]

(7.38)

By considering the safety stock kept in the central facility, we derive the total profit over two delivery regions A and B as \( M \). \( M \) is the summation of \( M_A \) and \( M_B \), minus the safety stock cost.

\[
M = M_A + M_B - \text{SafetyStockCost} =
\]

\[
n_A p q_A - hR \left( \frac{r_A W}{2} + (T_A + t_A) * n_A q_A \right) - \left( \frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma m_A}{W} \right) * n_A q_A - h n_A q_A
\]

\[
+ n_B p q_B - hR \left( \frac{r_B W}{2} + (T_B + t_B) * n_B q_B \right) - \left( \frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma m_B}{W} \right) * n_B q_B - h n_B q_B
\]

\[
- hR * z * \sqrt{(L_A + l_A) * (\phi_A q_A n_A)^2 + (L_B + l_B) * (\phi_B q_B n_B)^2}
\]

(7.39)

7.3.6 Solutions

As before, we have \( r = \frac{nq}{fW} \) (2.9), and \( r_{U} = \left[ \frac{nq}{f_{min} * W} \right] \) (2.10). Therefore, for delivery region A

\[
r_{UA} = \left[ \frac{n_A q_A}{f_{A_{min}} * W} \right]
\]

(5.12)

and for delivery region B

\[
r_{UB} = \left[ \frac{n_B q_B}{f_{B_{min}} * W} \right]
\]

(5.13)
Thus a vector of $r_A$ and a vector of $r_B$ such that $\overrightarrow{r_A} = \{1, 2, 3, \ldots, r_{UA}\}$, $\overrightarrow{r_B} = \{1, 2, 3, \ldots, r_{UB}\}$ could be set up. And for any given $r_A$ and $r_B$, we can find the optimal offering price of $p^*$ using an iterative procedure. Equation (7.41) shows the linear demand-price function case, and equation (7.43) illustrates the constant price elasticity demand function case.

7.3.6.1 Linear Demand-Price Function Scenario

If the customer demand can be captured by the linear demand-price function, the individual demand of the customers in delivery region A can be expressed as

$$q_A = -a_A \cdot p + b_A \quad (5.14)$$

and that in delivery region B as

$$q_B = -a_B \cdot p + b_B \quad (5.15)$$

where $a_A = a_B$ is assumed in most cases so that the demand-price curves of different demand types are parallel.

Plugging the linear demand-price functions into the equation of the total profit (7.39), we get the total profit as a function of the selling price $p$ and the number of truckloads per shipment $r_A$, $r_B$, for delivery region A and B, respectively.

$$M = M(p, r_A, r_B)$$

$$= n_A pq_A - hR \left( \frac{r_A W}{2} + (T_A + t_A) \cdot n_A q_A \right) - \left( \frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma n_A}{W} \right) \cdot n_A q_A - h n_A q_A$$

$$+ n_B pq_B - hR \left( \frac{r_B W}{2} + (T_B + t_B) \cdot n_B q_B \right) - \left( \frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma n_B}{W} \right) \cdot n_B q_B - h n_B q_B$$

$$- hR \cdot z \cdot \sqrt{(L_A + l_A) \cdot (\phi_A q_A n_A)^2 + (L_B + l_B) \cdot (\phi_B q_B n_B)^2}$$

$$= n_A p \cdot (-a_A \cdot p + b_A) + n_B p \cdot (-a_B \cdot p + b_B)$$

$$- \left[ hR \left( \frac{r_A W}{2} + (T_A + t_A)n_A \right) + \left( \frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma n_A}{W} \right)n_A + h n_A \right] \cdot (-a_A \cdot p + b_A)$$

$$- \left[ hR \left( \frac{r_B W}{2} + (T_B + t_B)n_B \right) + \left( \frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma n_B}{W} \right)n_B + h n_B \right] \cdot (-a_B \cdot p + b_B)$$

$$- hR \cdot z \cdot \sqrt{(L_A + l_A) \cdot (\phi_A n_A)^2 \cdot (-a_A \cdot p + b_A)^2 + (L_B + l_B) \cdot (\phi_B n_B)^2 \cdot (-a_B \cdot p + b_B)^2}$$

(7.40)
Take the partial derivative of the total profit with respect to the selling price \( p \), we get

\[
\frac{\partial}{\partial p} M(p, r_A, r_B) = 0
\]

\[-2(a_A n_A + a_B n_B) * p
\]

\[+
 hRz \left[ \left( L_A + l_A \right) * (\phi_A n_A)^2 * (-a_A * p + b_A)^2 + \left( L_B + l_B \right) * (\phi_B n_B)^2 * (-a_B * p + b_B)^2 \right]^{1/2}
\]

\[
\left[ a_A \left( L_A + l_A \right) * (\phi_A n_A)^2 * (-a_A * p + b_A) + a_B \left( L_B + l_B \right) * (\phi_B n_B)^2 * (-a_B * p + b_B) \right]
\]

\[+
 n_A b_A + n_B b_B
\]

\[+
 \left[ hR * \left( \frac{r_A W}{2} + (T_A + t_A)n_A \right) + \left( \frac{\gamma D_A + \beta c d_A + \sigma n_A}{W} \right) n_A + hn_A \right] * a_A
\]

\[+
 \left[ hR * \left( \frac{r_B W}{2} + (T_B + t_B)n_B \right) + \left( \frac{\gamma D_B + \beta c d_B + \sigma n_B}{W} \right) n_B + hn_B \right] * a_B
\]

\[= 0
\]

(7.41)

The optimal selling price \( p \) cannot be expressed in a closed form formula. However, it can be found by an iterative procedure.

### 7.3.6.2 Constant Price Elasticity Demand Function Scenario

On the other hand, if the customer demand can be approximated by the constant price elasticity demand function, the individual demand of the customers in delivery region A can be expressed as

\[ q_A = e_A * p^{f_A} \]  

(5.19)

and that in delivery region B as

\[ q_B = e_B * p^{f_B} \]  

(5.20)

Thus the total profit can be written as a function of the selling price \( p \) and the number of truckloads per shipment \( r_A, r_B \) by plugging in the demand functions.
\[ M = M(p,r_A,r_B) \]
\[ = n_A p q_A - hR \left( \frac{r_A W}{2} + (T_A + t_A) * n_A q_A \right) - \left( \frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma m_A}{W} \right) * n_A q_A - h n_A q_A \]
\[ + n_B p q_B - hR \left( \frac{r_B W}{2} + (T_B + t_B) * n_B q_B \right) - \left( \frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma m_B}{W} \right) * n_B q_B - h n_B q_B \]
\[ - hR \left( \phi_A q_A n_A \right)^2 + \left( L_B + l_B \right) * (\phi_B q_B n_B)^2 \]
\[ = n_A p \left( e_A * (f_A + 1) \right) + n_B p \left( e_B * (f_B + 1) \right) \]
\[ - \left[ hR \left( \frac{r_A W}{2} + (T_A + t_A) n_A \right) + \left( \frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma m_A}{W} \right) n_A + h n_A \right] * e_A f_A * p^{f_A-1} \]
\[ - \left[ hR \left( \frac{r_B W}{2} + (T_B + t_B) n_B \right) + \left( \frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma m_B}{W} \right) n_B + h n_B \right] * e_B f_B * p^{f_B-1} \]
\[ - hR \left[ (L_A + l_A) * (\phi_A n_A)^2 * e_A * (p^{f_A})^2 + (L_B + l_B) * (\phi_B n_B)^2 * e_B * (p^{f_B})^2 \right]^{\frac{1}{2}} \]
\[ = 0 \]

(7.43)

Taking the partial derivative of the total profit with respect to the selling price \( p \), we get

\[ \frac{\partial}{\partial p} M(p,r_A,r_B) = 0 \]

\[ n_A e_A (f_A + 1) * p^{f_A} + n_B e_B (f_B + 1) * p^{f_B} \]
\[ - \left[ hR \left( \frac{r_A W}{2} + (T_A + t_A) n_A \right) + \left( \frac{\gamma + \alpha D_A + \beta \alpha d_A + \sigma m_A}{W} \right) n_A + h n_A \right] * e_A f_A * p^{f_A-1} \]
\[ - \left[ hR \left( \frac{r_B W}{2} + (T_B + t_B) n_B \right) + \left( \frac{\gamma + \alpha D_B + \beta \alpha d_B + \sigma m_B}{W} \right) n_B + h n_B \right] * e_B f_B * p^{f_B-1} \]
\[ - hR \left[ (L_A + l_A) * (\phi_A n_A)^2 * e_A * (p^{f_A})^2 + (L_B + l_B) * (\phi_B n_B)^2 * e_B * (p^{f_B})^2 \right]^{1/2} \]
\[ * \left[ f_A (L_A + l_A) * (\phi_A n_A e_A)^2 * p^{2 f_A-1} + f_B (L_B + l_B) * (\phi_B n_B e_B)^2 * p^{2 f_B-1} \right] \]

(7.43)

In this case, the optimal selling price \( p \) cannot be expressed in a closed form formula either. But still it can be found by an iterative procedure.
7.3.7 Algorithm for Optimal Price

The procedures for computing an optimal profit and its associated optimal number of truckloads per shipment are given as follows.

1. For any given \( r_{Ai} \) and \( r_{Bi} \), the optimal offering price \( p^* \) can be found by computation using equation (7.41) or (7.43).
2. By substituting \( p^* \) into equation (7.39), we can get the maximum profit \( M^*_j \).
3. Repeat 1,2 for the combination of the valid values of \( r_A \) and \( r_B \), where \( r_A = \{1,2,3,...,r_{UA}\} \), \( r_B = \{1,2,3,...,r_{UB}\} \), and get a vector of \( M^* \), i.e., \( \tilde{M}^* = \{M^*_1,M^*_2,...,M^*_u\} \).
4. The optimized (maximized) profit is \( M_{\text{max}} = \text{Max}(\tilde{M}^*) = M^*_i \). Accordingly, the optimal offering price is \( p^*_j \). The optimal shipment size for delivery region A is \( V_{Ai} = r_{Ai} \cdot W \). And the optimal shipment frequency for delivery region A is \( f_{Ai} = \frac{n_{Ai}q_{Ai}}{r_{Ai} \cdot W} \). The optimal shipment size for delivery region B is \( V_{Bi} = r_{Bi} \cdot W \). And the optimal shipment frequency for delivery region B is \( f_{Bi} = \frac{n_{Bi}q_{Bi}}{r_{Bi} \cdot W} \).

Similar as before, most often we don’t need to try all the combinations of \( r_A \) and \( r_B \) in step 3 above. We can establish a subset of \( r_A \) from \( \{1,2,3,...,r_{UA}\} \) and a subset of \( r_B \) from \( \{1,2,3,...,r_{UB}\} \) that are appropriate, and then use the combinations of the elements in the subsets to derive the vector of \( M^* \). As an example, if the company is only interested in shipping its customers as often as possible, we only need to set \( r_A = 1 \) and \( r_B = 1 \), use equation (7.41) or (7.43) and get the optimal price. This may largely reduce the number of calculations.
Chapter 8

Conclusions and Recommendations

8.1 Summary of Results

Cost reduction plays a central role in the field of supply chain management. Logistics costs typically include inventory costs and distribution costs. In this thesis, we focus on establishing a model that maximizes a manufacturer’s overall profit across his delivery regions. Further we define the profit as the revenue net the total cost, which consists of the logistics cost and the manufacturing cost. As a widely observed phenomena, customer demand fluctuates in respond to the product price changes. Several demand price relationships have been discussed and results have been derived for scenarios with the linear demand-price function and the constant price elasticity demand function. By looking at the profit maximization problem and linking the price and demand together, we develop an integrated distribution and pricing strategy in a two-echelon supply chain.

In the first half of the thesis, we studied the more restrictive setting where the customer demand is deterministic. We have shown that for a single delivery region and with linear demand-price relationship, the optimal offering price for truckload (TL) shipment is
And the optimal offering price for the less-than-truckload (LTL) shipment is
\[
p^* = \frac{1}{2} \left[ \frac{b}{a} + hR(T + t) + \left( \frac{\gamma + \alpha d + \beta ad + \sigma m}{W} \right) + h \right]
\]  
(2.11)

The results are followed by sensitivity analyses. An integrated TL/LTL shipment model is presented at the section 2.5 in chapter two. We also discussed a convenient delivery strategy of dispatching one truckload at a time in chapter three. A lower bound is derived to determine the optimality of this strategy.

\[
\frac{\Delta RRC}{\Delta r} = hR^* \frac{W}{2} + hRnq^* (t_{r+1} - t_r) + \frac{nq^*}{W} \left[ \beta \alpha^* (d_{r+1} - d_r) + \sigma^* (m_{r+1} - m_r) \right]
\]
(3.3)

The optimal offering price given constant price elasticity demand function is
\[
p^* = \frac{f}{f + 1} \left[ hR(T + t) + \left( \frac{\gamma + \alpha d + \beta ad + \sigma m}{W} \right) + h \right]
\]
(4.9)

To apply the findings to across delivery regions, we first investigate the manufacturer’s pricing flexibility. If the company can set different prices for different delivery regions, we find each delivery region to be independent units and apply the results for single delivery region. If the company cannot vary the price and is obligated to offer the same price to all of its delivery regions, an iterative procedure needs to be followed to derive the optimal offering price for either demand function setting.

In the second half of the thesis, we relax the assumption of the deterministic customer demand and developed a model for stochastic demand. In order to hedge the demand uncertainty, we need to keep a certain amount of safety stock in the central facility. The level of the safety stock is determined by the inventory management policy that the company adopts. We investigate the continuous review policy (QR model) and the periodic review policy (base stock model). For a single delivery region and with linear demand-price relationship, the optimal offering price is
\[ p_{SS}^* = \frac{1}{2} \left[ \frac{b}{a} + hR(T + t) + \left( \frac{\gamma + \alpha D + \beta c d + \sigma m}{W} \right) + h + hRz\phi \sqrt{L + l} \right] \]  

(6.22)

And with constant price elasticity demand function, the optimal offering price is

\[ p_{SS}^* = \frac{f}{f + 1} \left[ hR(T + t) + \left( \frac{\gamma + \alpha D + \beta c d + \sigma m}{W} \right) + h + hRz\phi \sqrt{L + l} \right] \]  

(6.24)

In the application to the across delivery region scenarios, we again distinguish between the company’s pricing flexibility. If the company can set different prices for different delivery regions, we treat each delivery region to be independent units and apply the results for single delivery region. If the company is obligated to offer the same price to all of its delivery regions, we derive the optimal offering price as follows based on two delivery regions and with linear demand price relationship.

\[ p^* = \frac{1}{2(n_A a_A + n_B b_B)} \left[ hR \left( T_A + t_A + z_A\phi A \sqrt{L_A + l_A} \right) + \left( \frac{\gamma + \alpha D_A + \beta c d_A + \sigma m_A}{W} \right) + h \right] 
+ a_A n_A \left[ hR \left( T_A + t_A + z_A\phi A \sqrt{L_A + l_A} \right) + \left( \frac{\gamma + \alpha D_A + \beta c d_A + \sigma m_A}{W} \right) + h \right] 
+ a_B n_B \left[ hR \left( T_B + t_B + z_B\phi B \sqrt{L_B + l_B} \right) + \left( \frac{\gamma + \alpha D_B + \beta c d_B + \sigma m_B}{W} \right) + h \right] \]  

(7.41)

An iterative procedure needs to be followed to derive the optimal offering price if the constant price elasticity demand function is used instead.

Our results demonstrate how a manufacturer can integrate its distribution and pricing strategies to maximize his overall profit. We believe the results are useful in the study of more complex supply chain problems.

### 8.2 Recommendations for Future Research Areas

We suggest that the following three areas will be good topics for future research.

First, we assume that the company has only one central facility to serve all of its delivery regions. A relaxation of this assumption is worthwhile to show the interaction between
different central facilities/warehouses and the assignment of delivery regions. This is a very interesting topic but also complicates the situation considerably.

Second, our focus has been on a two-echelon supply chain. One can certainly integrate additional echelons into the picture to extend either the distribution problem or the pricing problem. We further did not investigate the bullwhip effect given our supply chain setting. But this becomes an issue with a larger supply chain and more manufacturing channels.

Third, since the customer demand is constantly changing and actually it can evolve into a different pattern as time passes. A dynamic pricing strategy will be necessary to handle this situation and compliments the results derived in this thesis.

As the supply chain grows and more elements are added, it becomes a fairly complex system and requires the good interaction and coordination between its components. Many of the problems we have discussed are heuristically analyzed. In this thesis, we assume that the manufacturer uses heuristic algorithms based on the Vehicle Routing Problem to schedule its truck fleet. Also empirical studies will provide insights into this problem and evaluate our results with data. Although an optimal strategy cannot be easily identified, each piece of new findings will certainly unveil part of the whole picture and contributes to a system-wide optimization.
Bibliography


