

# Portfolio Choices with Taxes

by

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M.S. Applied Mathematics, Massachusetts Institute of Technology, 1996

Submitted to the Sloan School of Management  
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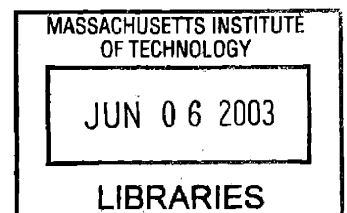
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## Abstract

I analyze the intertemporal portfolio problem of an investor who has access to both taxable and tax-deferred (retirement) accounts. In a complete-market setting, through a tax-arbitrage argument, I show that tax-deferred accounts have only a wealth effect on overall portfolio decisions through the effective tax subsidy provided, and the optimal location decision of where to place an asset is separable from the allocation decision of overall portfolio composition among different assets. Investors optimally hold *only* the asset that provides the highest effective tax subsidy in their tax-deferred accounts, and their optimal portfolio allocation is determined by reducing the two-account problem to a taxable-account-only problem with the wealth level adjusted for tax subsidies. I also provide heuristic rules to rank assets by their corresponding effective tax subsidies for application purposes.

In incomplete markets when investors face borrowing and short-selling constraints, I first solve a reduced-form version of the general model to provide conditions under which the complete-market optimal location decision of preferring the higher-taxed assets in the tax-deferred account is violated, and derive analytical solutions for the optimal portfolio allocation by transforming the two-account problem into a mixture of two single-account problems (one with only a taxable account and one with only a tax-deferred account). For financial planning purposes, I also derive convenient “rules of thumb” to approximate theoretical results. I finally solve a version of the general model numerically both to assess the performance of heuristic rules in approximating the optimal portfolio decisions, and to quantify the impact of tax-deferred investing on individual saving decisions.

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# Chapter 1

## Introduction

Tax-deferred retirement accounts (IRA, Roth-IRA, 401(k) and 403(b), etc.) have grown considerably in popularity and represent nowadays the crucial tool for retirement planning. Data from the 1998 Survey of Consumer Finances shows that 45.7% of households participate in a self-directed saving plans, up from 30.7% in 1989. As reported by Bergstrasser and Poterba (2002), “in 1998, the median household with both tax-deferred and taxable financial assets had 57.1 percent of its financial assets in a tax-deferred account. At the 25th percentile this value was 28 percent, while at the 75th percentile it was 85.6 percent.”

Given the relevance and popularity of tax-deferred accounts, the decisions of allocating wealth between taxable and tax-deferred accounts and of the portfolio composition within each are obviously very important. Following Shoven (1999), we use the terms *asset allocation* for the decision of overall portfolio composition among different assets and *asset location* for the decision of whether to hold an asset in taxable or tax-deferred accounts.

Not surprisingly, the decisions are also complex. Even with only a taxable account, there are no consensus among either the academics or the practitioners regarding the life-cycle portfolio allocation decisions. The presence of tax-deferred accounts not only doubles the size of the investment opportunity set but also introduces complicated interactions between the taxable and tax-deferred portfolio decisions, since the after-tax returns for the same asset held in taxable or tax-deferred accounts have different yet high correlated distributions, For example, if investors

have subjective views about asset returns, say they believe the stock market is going to rally relative to bonds<sup>1</sup>, they can take advantage of this view either by increasing their overall exposure to stocks (i.e., changing the allocation decision) or by placing their stock holdings in tax-deferred accounts in order to reduce their capital gains taxes (i.e., changing the location decision). Another example of interaction is when investors face the trade-off between selling their taxable assets with unrealized capital gains and withdrawing from their tax-deferred accounts to meet liquidity needs. These interactions between taxable and tax-deferred portfolio decisions make it a daunting task to isolate the impact of tax-deferred accounts on overall portfolio decisions.

While both academics and practitioners are grappling with this task, they take rather different approaches. On one hand, academics focus more on properly capturing the interaction between taxable and tax-deferred decisions through solving the two decisions simultaneously in a general framework<sup>2</sup> and so far rely heavily on numerical procedures. On the other hand, practitioners often rely on simply rules of thumb and in general ignore the interaction between these two decisions. We seek to bridge the gap between the two approaches by identifying the condition under which these two decisions are indeed separable, and deriving readily applicable approximation rules for practitioners to follow.

## 1. Main Results

### 1.1. Complete Markets

We start with the complete market case when investors are allowed to borrow and short-sell in taxable accounts. Instead of solving both the portfolio allocation and the asset location decisions simultaneously, we use a replication argument to show that the location and allocation decisions are separable. Specifically, we show that the optimal tax-deferred portfolio strategy is myopic and can be uniquely replicated by a dynamic portfolio of taxable assets. As a result, the trading strategies in tax-deferred accounts do not affect their overall risk exposure since investors can always take an offsetting position in taxable accounts using the replicating portfolio. The

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<sup>1</sup>The view could be either rational (say due to private information) or irrational (say due to some behavior biases).

<sup>2</sup>See, for example, Dammon, Spatt and Zhang (2001)

privilege to invest in tax-deferred accounts is then equivalent to receiving an ex-ante effective tax subsidy from the government while the only impact of following different tax-deferred trading strategies is the different effective tax subsidy generated. It then follows naturally that the optimal location decision is determined purely by the motive to maximize the effective tax subsidy received, while the optimization problem with both taxable and tax-deferred accounts is reduced to that with only taxable accounts by replacing the presence of tax-deferred accounts with a wealth increase equal to the corresponding effective tax-subsidy. The optimal taxable portfolio is finally determined by the taxable-account-only portfolio with the adjusted wealth level by subtracting the replicating portfolio for the optimal tax-deferred strategy.

Although the effective tax subsidy enables us to derive the simple structure of the optimal solution, it may not be a very intuitive notion. For application purposes, we define the notion of “effective tax rate” that measures the current-period effective tax subsidy that an asset generates if placed in tax-deferred accounts. The optimal location decision is simplified to a myopic decision of placing the asset with the highest effective tax rate in tax-deferred accounts each period. We further show that the effective tax rate depends mainly on asset characteristics like dividend yields and asset volatility. The only relevant individual information is their investment horizons and tax brackets. While other factors, like expected asset returns, subjective views of market conditions, individual endowments and preferences, etc., are important for overall allocation decisions, they should not influence investors’ asset location decisions.

For example, investors may differ on their views about the expected return of the stock and hence differ on their desired overall stock holding, as long as they all agree on the effective tax rates of assets, the location preference is identical across investors. For example, even if an investor prefers to hold 100% of his portfolio in stocks due to his extremely favorable view of the expected return, as long as bonds have higher effective tax rates, the investor holds only bonds in his tax-deferred account, while he borrows to lever up his stock position in taxable accounts to reflect his preference for stocks.

## A Numerical Example

To illustrate the point that the effective tax rate is the only determinant of the optimal location decision, consider the following one-period numerical example. At time 0, assume an investor in 40% tax bracket holds a portfolio of two assets in his taxable account: a risk-free bond with expected return of 5% and taxed at the 40% regular income rate; and a stock with expected return of 50%<sup>3</sup> and taxed at the 20% capital gains rate. The investor now has an additional dollar of after-tax labor income to invest and the option to open a new Roth IRA account (with no future tax liability on returns) to invest that dollar. At the end of the period, the investor withdraws money from both accounts and consumes final wealth. What is the best way to invest that dollar?

Since the expected tax payment is  $(50\%) \times (20\%) = 10\%$  for the stock and  $(5\%) \times (40\%) = 2\%$  for the bond, conventional wisdom suggests investing that dollar in stocks to minimize expected tax payments (investors may rebalance their portfolio in taxable accounts for optimal risk exposure), and the investor receives \$1.50 wealth from this additional dollar invested in his tax-deferred account at the end of the period.

Alternatively, the investor invests the additional \$1 in bonds in the Roth account while shifting \$1.25 from his bond holdings to stocks in the taxable account (he rebalances the rest of his portfolio exactly the same way as in the first strategy, if necessary). With this strategy, he receives  $\$1 \times (1+5\%) = \$1.05$  on his \$1 bond investment in his Roth account, and gains an extra after-tax return of  $(1.25) \times ((.5)(1 - .2) - (.05)(1 - .4)) = \$0.4625$  in the taxable account, for the total of \$1.5125, higher than that generated by the previous strategy. Since the after-tax risk profiles of the two strategies are the same,<sup>4</sup> the second strategy of holding bonds in the tax-deferred account clearly dominates the first strategy of holding stocks in the tax-deferred account.

So why is minimizing expected tax payment not the optimal strategy here? With an extra \$1.25 in taxable stocks relative to bonds, the second strategy actually has higher expected

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<sup>3</sup>The 50% return can either be the rational expected return on a high beta asset, or the irrational belief of an individual investor.

<sup>4</sup>The second strategy increases the after-tax stock exposure by  $(1.25)(1 - 20\%) = \$1$ , identical to that in the first strategy.

tax payments of  $1.25 \times [(.5)(.2) - (.05)(.4)] = 10\%$ . The reason is that Uncle Sam should be considered a co-investor in taxable investments. In the second strategy, where the investor holds more stocks in the taxable account and more bonds in the tax-deferred, the government has a riskier co-investment in the taxable account in the form of future tax revenue, and the 10% extra tax payment is simply the compensation for the higher risk. In the original strategy, where extra stock holding is in the tax-deferred account and does not affect tax revenue, the government has a safer co-investment and requires lower return.

## 1.2. Incomplete Markets: Heuristics

Although the complete market result provides a clean benchmark, its implementation might require large short positions in taxable accounts which are not feasible in reality. Furthermore, the predictions regarding optimal asset location are at odds with the empirical evidence according to which investors tend to hold a similar mix of stocks and bonds in both taxable and tax-deferred accounts.<sup>5</sup> To understand the impact of borrowing and short-selling constraints in taxable accounts on the optimal portfolio location and allocation decisions, and specifically whether the constraints are sufficient to account for the discrepancy between the complete market results and the empirical findings, we solve explicitly the optimization problem with the constraints.

Since the location and allocation decisions are no longer separable, the structure of the optimal solution is much more complicated. We take two complimentary approaches to deal with the complication: first we rely on intuitions generated in complete markets to solve a reduced-form version of the general problem, and derive some heuristical rules for the optimal portfolio decisions that are readily applicable; then we rely on numerical procedure to solve a specific version of the general model, both to characterize the structure of the optimal solution, and more importantly, to check the validity of the above approximation rules. The main objective of these two approaches are to gain some insights regarding how the complete market results break down under constraints and how investors should adjust their optimal strategy in practice

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<sup>5</sup>See for example, Bodie and Crane (1997), who use data from a 1996 survey of TIAA-CREF participants, Poterba and Samwick (1999), who use data from the 1995 Survey of Consumer Finances and Bergstrasser and Poterba (2002), who use data from the 1998 Survey of Consumer Finances.

to account for the constraints.

Without the ability to borrow and short-sell in the taxable account, the complete market location decision of placing *only* the highest tax rate asset in the tax-deferred account would lead to an unbalanced portfolio. An intuitive adjustment would be to rank assets according to their effective tax rates and place them in the tax-deferred account according to a *pecking order* with the highest effective tax rate asset coming first. This adjustment, however, implies interior mix of stocks and bonds in *at most* one account.<sup>6</sup> Note that this pecking order adjustment is “myopic” in the sense that it implies investors altering their unconstrained portfolio only to satisfy the binding current period constraints. We identify conditions under which, the binding future period constraints alter the optimal location decision and the pecking order is violated, implying the possibility of an interior mixture in both accounts. However, in a calibrated version of the model we find that, despite the theoretical possibility of this violation, in the absence of major liquidity shocks or jumps in the return distribution, violations of the pecking order are unlikely, and this result is robust to several parameterization of the problem. We conclude that, for practical purposes, the pecking order adjustment is a reasonable approximation for the optimal location decision, and the borrowing and short-selling constraints alone are not sufficient to explain the empirical finding of interior mixture of stocks and bonds in both accounts.

The simplicity of the pecking order adjustment does not solve, however, the problem of correctly determining the optimal portfolio allocation in the taxable and tax-deferred accounts. A natural approach would be to follow the steps of the complete market case in which the overall portfolio allocation is obtained by (i) calculating the effective tax-subsidy from holding only the highest taxed asset in the tax-deferred account; (ii) re-express the two-account problem as a taxable-account-only problem where the “effective wealth” is equal to the taxable wealth augmented by the effective tax subsidy and (iii) solve for the optimal allocation in the taxable-account-only problem. In the case of complete markets, the effective tax-subsidy in step (i) can be easily calculated as a replication cost of the highest tax rate asset. If investors follows a

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<sup>6</sup>Since bonds in general have higher tax rates, if investors hold a mixture in the tax-deferred accounts, they should hold only stocks in the taxable, and if they hold a mixture in the taxable account, they should hold only bonds in the tax-deferred.

pecking order location strategy to accommodate borrowing and short selling constraints, step (i) would imply that the effective tax subsidy is a weighted average of the replication costs of assets held in the tax-deferred account, with weights equal to the percent of asset holdings. This makes the effective tax-subsidy dependent of the overall allocation, or step (i) dependent on step (iii) above, severely complicating the adjustment needed to account for the constraints.

We break the interdependency between allocation decisions and the effective tax subsidy and derive the correct optimal allocation rule under borrowing and short selling constraints. When investors follow a pecking order rule in determining their location strategy, they hold an interior mix of stock and bonds in *at most* one account. If the interior mix is in the *taxable* account (i.e. only the highest tax rate asset is in the tax-deferred account), then the complete market results apply (this is more likely to occur when the tax-deferred balance is small). On the other hand, if the interior mix is in the *tax-deferred* account (more likely to occur for large tax-deferred balances), instead of intuitively calculating the effective tax-subsidy as a weighted average as previously discussed, investors should *change their view-point* and solve a tax-deferred-only problem by calculating the “effective tax cost” of holding assets in the taxable account relative to the tax-deferred wealth. This approach avoids the complexity of having to simultaneously determine the effective tax-subsidy and the allocation decision and, at the same time, delivers useful “rules of thumb” for optimal portfolio allocation.

### 1.3. Incomplete Markets: Numerical Results

Having derived the simple approximate portfolio rules, we solve some special versions of the general model with power utility, only one risky asset with binomial asset returns and no option to postpone capital gains to understand the structure of the optimal solution. We first measure the deviation of the constrained optimal portfolio decisions to that of the complete market results to understand when the complete markets intuition is appropriate and when some adjustments are in order, and then construct approximate portfolios rules as discussed before and assess their performance by contrasting to the constrained optimal solution. We focus our discussion both on the inputs (e.g., the size of the effective tax subsidy) and on the outputs (e.g., the optimal

portfolios) of the approximation rules.

We focus our discussion of optimal portfolio on two main results: First, although we have identified the difference in future effective tax subsidy as the driving force for the violation of the naive pecking order rule, it remains to be verified whether and when the violation occurs, and whether the ratio of future effective tax subsidies satisfies the reduced-form condition when the violation occurs. The second is whether the heuristic portfolio rules provide a reasonable approximation for the optimal portfolio, more specifically, whether applying the approximation rules using some coarse exogenous bounds on the effective tax subsidy yields reasonable bounds on the optimal portfolio.

Starting with a benchmark case with no consumption, no labor income, no liquidity needs, and no future contribution to tax-deferred accounts, we show that, first, the violation of pecking order is unlikely, consistent with the small difference in future effective tax subsidy across states; and second, although the bounds on the effective tax subsidy are very coarse, the approximation rules perform relatively well, measured both by the bounds on the optimal portfolio, and the utility loss occurred for following the approximation rules relative to following the optimal strategy. We then introduce more realistic features one at a time to understand their impacts on the optimal solution and the performance of approximation rules.

We finally derive policy implications for the provisions of tax-deferred accounts, specifically, whether and to what extent they stimulate savings. Since investors postpone paying capital gain or dividend taxes in their retirement accounts, assets have higher effective returns if placed in tax-deferred accounts. In making their savings decisions, investors trade off current consumptions with future consumptions financed by current savings. When the return on assets are higher, one dollar saved will finance more future consumption, and investors are more willing to save. We term this additional trade-off between current and future consumption induced by the higher asset returns the “return effect”. On the other hand, higher asset returns also make investors wealthier. As long as individuals consume more with higher total wealth, they increase their level of consumption after the introduction of tax-deferred accounts, or equivalently, they save less. This reduction in savings due to increased effective wealth level is termed the “wealth



effect". The optimal savings decision depends on the balance between the return and the wealth effects.

In a complete market setting, tax-deferred investment opportunities are equivalent to receiving tax subsidies from the government where the size of the subsidies are independent of investor characteristics. As a result, investors always contribute the maximum amount allowed to tax-deferred accounts in order to maximize the effective tax subsidy received, while financing their current period consumption through borrowing if necessary. Since the marginal investment decision is made in taxable accounts, only the taxable returns are relevant for the tradeoff between saving and current consumption decisions. The extra return effect induced by higher tax-deferred asset returns does not exist. Investors always reduce their savings after the introduction of tax-deferred accounts, since the only impact of tax-deferred accounts on optimal consumption decision is the wealth effect.

However, the complete-market result relies heavily on the assumption that investors are allowed to borrow in their taxable accounts. In general, without borrowing, investors may need to trade off current consumption with the decision of contributing one more dollar to tax-deferred accounts which earns the higher return. Therefore, the marginal investment decision is made in the tax-deferred accounts and the return effect induced by higher tax-deferred returns needs be considered. In general, we expect the return effect to be most pronounced for less wealthy investors who do not max out their contributions and constantly face the trade off between current consumption and an extra dollar saved in the tax-deferred accounts. For the less wealthy investors, this return effect may dominate the wealth effect and the existence of tax-deferred accounts may serve to encourage overall saving. On the other hand, wealthier investors who always contribute the maximum allowed to their tax-deferred accounts, extra saving does not generate more tax-deferred investing, the return effect is dominated by the wealth effect, and investors decrease savings in the presence of tax-deferred accounts.

The numerical results enables us to measure how much different classes of investors benefit from the opportunity to invest in tax-deferred accounts, and how much they alter their savings decision in the presence of tax-deferred accounts. For example, although less wealthy investors

increase their savings when they are young in order to increase their total contribution to tax-deferred accounts, higher net worth investors tend to reduce their saving when they have the rights to invest in tax-deferred accounts, due to the wealth effect of tax subsidy received. The overall impact of tax-deferred investing on aggregate saving is ambiguous.

## 2. Literature review

Since the pioneering work of Merton (1969, 1971) there has been an extensive body of literature addressing the problem of an investor selecting an optimal portfolio over the life-cycle. The majority of the recent work in this area focuses on a taxable-account-only problem and significant efforts has been devoted to relaxing different dimensions of the original setup, for example, considering the impact of unpredictability of asset returns (e.g., Kandel and Stambaugh (1996)); introducing labor income and liquidity needs over the life-cycle (e.g., Bodie, Merton and Samuelson (1992), Viceira (2001)); imposing borrowing, short-selling constraints and other forms of incompleteness (e.g., He and Pearson (1991)); accounting for transaction costs (e.g., Constantinides (1986), Davis and Norman (1990), Dumas and Luciano (1994)), and studying the impact of capital gains taxes (Constantinides (1983, 1984), Dammon, Spatt and Zhang (2001, 2002b)).

Tepper and Affleck (1974), Black (1980) and Tepper (1981) are the first to address the problem of wealth allocation when corporations have the choice between two accounts characterized by different tax treatments. By introducing the concept of tax-arbitrage in a static setting, it is shown that, by integrating the asset composition decision within a corporate pension fund with the portfolio decisions of shareholders, it is optimal to fully fund the pension account and hold in it only bonds which are taxed more heavily. Our approach can be viewed as a dynamic version of the tax-arbitrage argument applied to the individual investor setting. We study a general setting when investors are allowed to postpone capital gains in their taxable accounts and characterize the effective tax subsidy for investing in tax-deferred accounts which determines the optimal location decision and relate it to fundamentals of the asset properties and investor characteristics for application purposes.

The dynamic replication argument employed in the paper is also closely related to the approach initiated by Constantinides (1983, 1984) in the context of optimal capital gain realization with only a taxable account. In a complete market setting, he shows that investors always postpone capital gains and realize losses immediately whenever possible. The main contribution of this paper is to extend the replication argument to tax-deferred assets, derive the separability between location and allocation decisions, and finally map the two-account problem to a taxable-account-only problem.

Although Constantinides (1983, 1984) derived very elegant solution assuming complete markets, the setting is not very realistic. There is a growing literature that explicitly introduces frictions, like borrowing and short-selling constraints, and solve the portfolio problem with tax considerations using numerical methods. While there are some attempts to solve the taxable-account-only problem using exact tax basis (e.g., Dybvig-Koo (1996), DeMiguel-Uppal (2002)), the state space, which includes the entire trading history, grows exponentially and only very small number of trading periods can be dealt with. Adding tax-deferred accounts certainly complicates the matter further. The few papers that deal with both taxable and tax-deferred accounts rely on simplifying assumptions like using average cost basis (Dammon, Spatt and Zhang (2002a)) or assuming an exogenous portion of capital gains are realized each period (Shoven and Sialm (2002)).

In the presence of borrowing and short-selling constraints, Dammon, Spatt and Zhang (2002a) and Shoven and Sialm (2002) confirm the single-period intuition of placing the higher taxed asset first in the tax-deferred account. Dammon, Spatt and Zhang (2002a) incorporate further realistic features (labor income, treatment of capital gains tax) into a life-cycle model and derive numerically the portfolio location and allocation decisions. Shoven and Sialm (2002) focus on the impact of municipal bonds on the location decision by solving a single period model with stocks, bond and municipal bonds. They document that, when the proportion of total returns distributed by stock mutual funds as capital gains is high, investors hold mutual funds in the tax-deferred account and municipal bonds in the taxable. On the other hand, when the proportion distributed is low, investors hold bonds in the tax-deferred and the mutual fund in the

taxable. This happens because municipal bonds and taxable bonds can be viewed as substitutes for each other and the effective tax rate on bonds (used for location decisions) is the minimum between the implied tax rate on municipal bonds<sup>7</sup> and the tax rate on the bond. The location decision depends on the relationship between the effective tax rate on bonds and on the mutual fund which increases with the proportion of return distributed as capital gains.

Our approach differs substantially from these numerical approaches. We take as given the optimal taxable-account-only solution and focus our attention on characterizing the impact of tax-deferred accounts on the overall portfolio decisions in a general setting, and eventually map the two-account problem to a taxable-account-only problem.

Although our approach does not provide new insights for dealing with the high dimensionality problem induced by the trading history or the complexity induced by different factors in determining the optimal portfolio allocation with only a taxable account, it addresses a different and equally important question of how the existence of tax-deferred accounts alters our portfolio decisions. Moreover, the solution is remarkably simple: for the decision of which asset to hold in the tax-deferred account, we do not need any individual specific information except for their tax bracket, investment horizon, and the fact that they are optimizing; for the overall portfolio holding in taxable accounts, we need *only* as an input the optimal portfolio allocation when investors have only a taxable account. It is again unnecessary to learn the individual preference. This preference-independence may sound artificial, since the individual preference is presumably embedded in his single-account portfolio solution. However, in reality, it is very likely that investors decide their portfolio allocation (say, 60% stocks and 40% bonds) based on advice from their financial planners, even though it may be extremely hard to back out the assumptions and utility functions implied by such portfolio allocation. For practical purposes, our approach can be used by investors as an add-on to their existing portfolio strategy with only taxable accounts. The optimality of our two-account solution is only relative to the given single-account solution. We do not need to take a stand on whether the single-account solution is optimal.

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<sup>7</sup>The implied tax rate on the municipal bond is defined as the tax rate that makes the taxable investor indifferent between taxable and municipal bonds.

### **3. Structure of the thesis**

The thesis is structured as follows. Chapter 2 sets up a general model to be used throughout the thesis and define the specifics of the tax environment. Chapter 3 considers the complete market case when borrowing and short selling are allowed in the taxable account. Chapter 4 studies in a reduced-form setting the interaction between location and allocation decision when markets are not complete. Chapter 5 discusses the implementation of the approximate portfolio rules and assess their performance with respect to the true optimal solution. Chapter 6 concludes. Appendix A contains the proofs of all the propositions, Appendix B describes the numerical procedure, and Appendix C presents the tables and figures.



## Chapter 2

# The Model

In this chapter, we set up a discrete-time finite-horizon partial equilibrium model to determine the optimal portfolio decisions when investors hold assets in both taxable and tax-deferred accounts. Investors are risk averse, take prices as given, and solve for optimal portfolio to maximize utility from intertemporal consumptions as well as terminal bequeaths. We describe below the asset return processes, tax environments, trading constraints, and finally the individual optimization problem.

### 1. Assets and Investors

There are one risk-free bond (indexed  $i = 1$ ) and  $N-1$  risky assets (indexed  $i = 2, \dots, N$ ) traded in the economy. Let  $P_{i,t}$  and  $D_{i,t}$  be the asset price and the dividend payment for asset  $i$  at time  $t$ . Denote  $\bar{d}_{i,t} \equiv \frac{\bar{D}_{i,t}}{P_{i,t-1}}$  to be the dividend yield,  $\bar{g}_{i,t} \equiv \frac{\bar{P}_{i,t}}{P_{i,t-1}} - 1$  to be the capital gains rate, and  $\bar{r}_{i,t} = 1 + \bar{d}_{i,t} + \bar{g}_{i,t}$  to be the total return for asset  $i$ . The risk-free bond has constant interest rate over time, paid out in full each period, or,  $\bar{d}_{1,t} = r_f - 1$ , and  $\bar{g}_{1,t} = 0$  at any time  $t$ . The total return  $\bar{r}_{i,t}$  for risky asset  $i$  is binomial and i.i.d. over time. Without much loss of generality, we further assume that  $\bar{d}_i$  and  $\bar{g}_i$  evolve according to the same underlying binomial process:

$$\bar{d}_i = \begin{cases} d_i^H \\ d_i^L \end{cases}, \quad 1 + \bar{g}_i = \begin{cases} u_i & \text{with prob. } p_i \\ u_i^{-1} & \text{with prob. } 1 - p_i \end{cases} \quad (1)$$

The following vector notations are used later to set up the optimization problem:  $\mathbf{P}_t = [P_{1,t}, \dots, P_{N,t}]'$  denotes the vector of asset prices,  $\mathbf{D}_t = [D_{1,t}, \dots, D_{N,t}]'$  is the dividend payment, and  $\bar{\mathbf{r}}_t = [\bar{r}_{1,t}, \bar{r}_{2,t}, \dots, \bar{r}_{N,t}]'$  is the total return process at time  $t$ .

## 2. Tax Environments

The U.S. tax codes are very complicated and ever-changing. As much as we would like to, taking all relevant tax regulations into account is a futile exercise. We make several simplifying assumptions to capture the main features of the tax codes while keeping the model tractable. Following are descriptions of the tax environments in both taxable and tax-deferred (retirement) accounts.

### Taxable Accounts

In their taxable accounts, investors purchase financial assets using their after-tax labor income or other sources of income. The following two assumptions specify the tax treatments for investment returns over time.

**Assumption 1** *In taxable accounts, (i) dividends and interest payments are paid out and taxed annually at a rate  $\tau_d$ , and (ii) both long-term and short-term capital gains/losses are taxed upon realization at the same rate  $\tau_g \leq \tau_d$ .*

This assumption ignores some details of the tax code for tractability: for example, short term capital gains are taxed at  $\tau_d$  while long term gains at  $\tau_g$ ; investors are only allowed to realize tax credits on capital losses through offsetting their realized capital gains, dividends, or up to \$3000 of ordinary incomes, with any extra losses carried over to future years; and there is a “Wash Sale” rule specifying that investors cannot realize capital losses on an asset if they repurchase the same asset within 30 days before or after the sale (i.e., a 60-day window).<sup>1</sup>

**Assumption 2** *Investors have fixed investment horizon and on the terminal date  $T$ , they are always allowed to realize tax losses, and there are two possible tax treatments regarding unrealized*

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<sup>1</sup>Moreover, we can incorporate in the model the possible different tax treatments for dividends and interest payments as suggested in the recent Bush tax reform.



*capital gains in taxable accounts: (i) escape capital gains taxes through “step-up of tax basis”, and (ii) forced to realize all capital gains.*

The step-up of tax basis corresponds to a special provision in the current tax code that allows investors to escape capital gain tax upon their death<sup>2</sup>, which makes stocks more attractive in taxable accounts for older investors. The forced realization of capital gains approximates the tax treatment for investors who sell assets to meet their liquidity needs.

The finite horizon assumption enables us to study explicitly the horizon effect, more importantly, the solution to the optimization problem is not well defined when the investment horizon approaches infinity since tax-deferred accounts are infinitely more valuable than taxable accounts<sup>3</sup>. Although it is more realistic to consider uncertain investment horizons induced by changing life expectancy or unforeseeable liquidity events, the horizon risk presents an additional source of risk and complicates the problem significantly. For simplicity and without much loss of generality<sup>4</sup>, we assume fixed horizon.

## Tax-deferred Accounts

There are several types of tax-deferred accounts in practice. The most popular type is 401(k)/403(b) or Keogh plans, where both ordinary income taxes on initial contributions and the taxes on interests and capital gains are deferred and taxed as ordinary income only upon withdrawal. Another popular type is Roth IRA, where investors pay ordinary income taxes on initial contributions but there are no more taxes on future distributions. It turns out that as long as the future tax

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<sup>2</sup>Specifically, they can pass the taxable assets with embedded capital gains to their estates and their beneficiaries can avoid paying capital gains taxes by keeping the assets and recording the current market prices as their tax basis for the assets.

<sup>3</sup>For example, let  $R_f = r_f - \tau_d(r_f - 1)$  be the after-tax return on risk-free bonds. If investors hold only risk-free bonds in both accounts for  $T$  periods, one dollar grows to  $r_f^T$  in tax-deferred accounts and  $R_f^T$  in taxable accounts. Hence, one dollar in tax-deferred is equivalent to  $\left(\frac{r_f}{R_f}\right)^T$  dollars in taxable accounts.

<sup>4</sup>More specifically, the market with only the underlying asset and the risk-free bond is no longer complete (see, for example, Constantinides (1983).) We could explicitly assume the existence of insurance contract on the event to complete the market. But this approach complicates the solution without offering much new insights. Instead of explicitly modelling the impact of uncertain horizons, we can heuristically address this issue using fixed-horizon solutions with different terminal tax treatments. For example, if an investor has a 5% chance of encountering a large liquidity needs in 10 years upon which he is forced to liquidate all assets, and a 95% chance of living for another 50 years and bequeath his remaining assets to his beneficiaries, the optimal portfolio can be approximated by a weighted average of the solutions to the following two problems: (i) a 10-year horizon problem with forced realization of capital gains, and (ii) a 50-year horizon problem with step-up of tax basis on the terminal date.

rates are deterministic, the two types of tax-deferred accounts are equivalent within a constant factor.

The following example illustrates the point. Let  $\tau_0, \tau_T$  be the ordinary income tax rates on the initial contribution date and terminal withdrawal date, respectively, and let  $\tilde{r}_T$  be the cumulative before-tax return for any portfolio over the  $T$  periods, then the cumulative after-tax return on \$1 investment in 401(k) plans is  $\$1 \times \tilde{r}_T \times (1 - \tau_T)$ , whereas the return in Roth accounts is  $\$1 \times (1 - \tau_0) \times \tilde{r}_T$ . Clearly, each dollar invested in 401(k) plans is equivalent to  $\frac{1 - \tau_T}{1 - \tau_0}$  in Roth accounts, independent of the realized portfolio return  $\tilde{r}_T$ , as long as investors follow the same strategy in both accounts.<sup>5</sup> Given the equivalence, it suffices to consider only Roth type of tax-deferred accounts for the model.<sup>6</sup>

**Assumption 3** *In tax-deferred accounts, investors (i) never pay taxes on either dividend payments or capital gains/losses, (ii) can contribute a maximum of  $\overline{M}_t$  to their tax-deferred accounts at time  $t$ , and (iii) are allowed to withdraw from tax-deferred accounts with penalty  $\eta_t$  at time  $t$  and are forced to withdraw all tax-deferred wealth on terminal date  $T$ .*

At time  $t$ , a negative  $\overline{M}_t$  implies that investors are forced to withdraw certain amounts from their tax-deferred accounts, and investors receive  $(1 - \eta_t)$  dollars in taxable accounts for each dollar withdrawn. In practice, there are different rules about contribution limits ( $\overline{M}_t$ ) and withdrawal penalty ( $\eta_t$ ) for different types of tax-deferred accounts. For example, the annual contribution limit is \$3,000 for Roth IRA and \$11,000 for 401(k) plans. For 401(k) plans, there is a 10% penalty if investors withdraw from tax-deferred accounts before age  $59\frac{1}{2}$  and investors are required to take annual minimum distributions from tax-deferred accounts that are proportional

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<sup>5</sup>Intuitively, the income taxes are government's claim on investors' asset positions. In the case of Roth accounts, the government takes its share before an investor starts investing and decides its own portfolio, while in the case of 401(k) plans, the government leaves its share in the investor's account to follow exactly the same investment strategy as the investor and takes its share out only when the investor starts withdrawing from his accounts. Clearly, whether the government invests on its own or leaves it with the investor should have no impact on the return of the investor's portfolio or his investment decisions.

<sup>6</sup>Investors need to attach a factor of  $(1 - \tau_T)$  to their 401(k) wealth to derive the effective Roth IRA wealth before they can apply the results derived in this paper if they have in practice a 401(k) account rather than a Roth IRA. Furthermore, contributing \$1 to a Roth account is accomplished by simply moving a cash position of \$1 from the taxable to the Roth account, however, to contribute \$1 to a 401(k) account, investors increase cash by \$1 in the 401(k), but only decrease the cash position in the taxable by  $1/(1 - \tau_0)$  since they do not pay income tax on their contribution.

to their life expectancy after age 70. On the other hand, Roth IRA has no required distribution after age 70 and no penalty for some qualified early withdrawal. Assumption 3 is flexible enough to capture these features.

### 3. Trading Constraints

Investors are allowed to borrow and short-sell in their taxable accounts as described in the following assumption.

**Assumption 4** *In taxable accounts, investors are allowed (i) to borrow at risk-free rate with full tax deduction (at rate  $\tau_d$ ) for interest payments, and (ii) to short-sell risky assets by paying cash dividends when declared with full tax deduction, and are required to post 100% of the market value (“mark-to-market”) of the asset as cash collateral and earn a “rebate interest rate” ( $\widehat{r}_t - 1$ ), which is also taxed at rate  $\tau_d$ . The rebate rate  $\widehat{r}_t < r_f$  is specified in Appendix A to rule out arbitrage.*

Full tax deduction on interest payment is not too strong an assumption considering the ability for investors to take out mortgage loans. The restrictions on short-selling is rather complicated and typically institution specific. In general, investors are required to “mark-to-market” in the sense that the required amount of cash collateral moves with market prices. Institutional investors need to post 102% of the market value of the borrowed share upon transaction date as cash collateral<sup>7</sup>, and earn a “rebate interest rate” on the collateral determined by market conditions, like the volatility of the asset, the difficulty to locate the share to be shorted, and so on. On the other hand, retail investors may be required to post as much as 150% of the market value as cash collateral and they typically do not receive interest payments on their cash collateral since their shares are held in street name. In summary, the rebate interest rate is lower than the risk free rate ( $\widehat{r}_t < r_f$ ) and is *exogenously* determined based on investor characteristics and market conditions. In this paper, for simplicity, we assume  $\widehat{r}_t$  is defined *endogenously* to rule out arbitrage opportunities.<sup>8</sup>

<sup>7</sup>Duffie, Gârleanu, and Pedersen (2002) provide a nice survey of the institutional details.

<sup>8</sup>The following example illustrates the necessity to have  $\widehat{r}_t < r_f$  on the short sell proceeds in order to rule out arbitrage opportunities (See Constantinides (1983) for a detailed discussion.) If  $\widehat{r}_t = r_f$ , Investors are guaranteed

Assumption 4 is the main assumption of the paper that effectively completes the market. Although not very realistic, the assumption drastically simplifies the problem and enables us to derive analytic solutions regarding optimal portfolio decisions. Using the simple structure of the optimal portfolio as a benchmark, Garlappi and Huang(2002) derive approximate portfolio rules when investors are not allowed to borrow or short-sell.

**Assumption 5** *In tax-deferred accounts, investors are not allowed to borrow or short-sell any asset.*

Aside from being consistent with the current regulations on tax-deferred accounts, Assumption 5 is also necessary to ensure a properly functioning market, as illustrated in Chapter 3. Finally, for simplicity, we assume there are no explicit transaction costs in terms of bid-ask spreads or brokerage fees and so on.

**Assumption 6** *There is no transaction cost in either taxable or tax-deferred accounts.*

## 4. Dynamic Optimization Problem

We start describing the optimization problem by identifying the state and choice variables. Since investors are allowed to postpone realizing capital gains on all shares, the purchase price of each share in any asset  $i$  matters for the overall optimization. Let  $h_{i,s,t-1}$  be the number of shares of asset  $i$  purchased at time  $s$  that investors hold in their taxable accounts *before* they make their trading decisions at time  $t$ , and  $y_{i,s,t} \equiv P_{i,s}/P_{i,t}$  be the corresponding tax cost-basis for the share, where  $P_{i,s}$  is the price of the share at time  $s$  when it was original purchased, and  $P_{i,t}$  is the current market price.<sup>9</sup> Then the matrices  $\mathbf{H}_{t-1} = [H_{1,t-1}, \dots, H_{T,t-1}]$ , where

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positive return on zero initial investment if they form a portfolio by shorting one share of an asset and simultaneously purchasing a share of the same asset using the proceeds. If the asset price goes up, investors can close the short position to get tax credits on the losses, re-short one share at the current price (new short proceeds exactly cover the cost to close the old short position), and finally invest the amount (the tax credits on losses) at the risk-free rate. Similarly, if the price goes down, they close the long position to realize tax credits on losses, re-purchase the share (using the proceeds from closing the long position), and invest the rest (tax credit on losses) in the risk-free bond. In the end, the long and short positions offset each other and the total bond position is always higher than the capital gains tax due, because the tax credits on losses earn interest over time and the corresponding tax liability on capital gains are postponed without interest.

<sup>9</sup> $y_{i,s,t} > 1$  indicates there are capital losses on the position and  $< 1$  corresponds to capital gains.

$H_{s,t-1} = [h_{1,s,t-1}, \dots, h_{N,s,t-1}]'$ , and  $Y_t = [Y_{1,t}, \dots, Y_{T,t}]$ , where  $Y_{s,t} \equiv [y_{1,s,t}, \dots, y_{N,s,t}]'$ , denote the share holding of all assets before-trading at time  $t$  and the corresponding tax cost-basis<sup>10</sup>, and clearly are state variables. Moreover, since investors are restricted in moving money between the taxable and tax-deferred accounts, the before-trading wealth in their tax-deferred accounts (after incorporating the current period prices),  $W_t^D$ , is also a state variable. The detailed portfolio holding in tax-deferred accounts, however, is *not* a state variable, since investors can trade to any position in tax-deferred accounts given their current position without occurring any cost by Assumption 6. Finally, the current period stock price  $P_t$ , is also a state variable, since it affects the total wealth in taxable accounts. In summary, the state variables at time  $t$  are  $\{H_{t-1}, Y_t, W_t^D, P_t\}$ .

Given the current set of state variables, investors maximize overall utility by optimizing over the following choice variables at time  $s = t, \dots, T$ , (i) the dollar consumption,  $C_s$ , (ii) the dollar contribution to (or withdrawal from) tax-deferred accounts,  $M_s$ , (iii) the portfolio holding in taxable accounts after trading<sup>11</sup>,  $H_s$ , and (iv) the portfolio holding in tax-deferred accounts,  $\Theta_s = [\theta_{1,s}, \dots, \theta_{N,s}]'$ , where  $\theta_{i,s}$  is the percentage of tax-deferred wealth invested in asset  $i$  at time  $s$ .

Let  $V_t(H_{t-1}, Y_t, W_t^D, P_t)$  be the indirect utility function for investors at time  $t$  given the state variables, then investors' optimization problem can be stated as follows:

$$V_t(H_{t-1}, Y_t, W_t^D, P_t) = \max_{\{C_s, M_s, H_s, \Theta_s\}_t^T} \left\{ \mathbb{E}_t \left[ \sum_{s=t}^{T-1} [\delta^{s-t} u(C_s)] + \delta^{T-t} \Gamma(C_T) \right] \right\} \quad (2)$$

subject to the following constraints for all  $s = t, t+1, \dots, T$ ,

$$C_s = \text{diag} \left( (H_{s-1} - H_s) [(1 - \tau_g) + \tau_g Y_s]' \right) P_s + (1 - \tau_d) \mathbf{1}'_T H_{s-1} D_s - (1 - \eta_s \mathbf{1}_{\{M_s < 0\}}) M_s \quad (3)$$

$$W_{s+1}^D = (W_s^D + M_s) \Theta_s' \bar{r}_{s+1} \quad (4)$$

$$-W_s^D \leq M_s \leq \bar{M}_s, \quad \mathbf{0}_N \leq \Theta_s \leq \mathbf{1}_N \quad (5)$$

<sup>10</sup>We define  $h_{i,s,t} = y_{i,s,t} = 0$  for  $s > t$ .

<sup>11</sup>Since the *after-trading* share holding at time  $s$  is identical to the *before-trading* share holding at time  $s+1$ , to economize on the notations used, we use the vector  $H_s$  to denote both.

$$M_T = W_T^D, \quad \mathbf{H}_T = 0 \quad (6)$$

Equation (2) defines the value function, where  $\delta$  is the time discount,  $u(\cdot)$  is the utility over consumption, and  $\Gamma(\cdot)$  is the utility over terminal bequeath, which may include the terminal consumption as well. Equations (3) and (4) are the budget constraints in taxable and tax-deferred accounts, respectively, where  $\mathbf{1}_T = [1, \dots, 1]'$  is a vector of ones with  $T$  elements,  $\text{diag}(A) = [a_{11}, \dots, a_{tt}]'$  is the vector of diagonal elements for matrix  $A = [a_{ij}]$ ,  $\mathbf{1}_{\{M_s < 0\}}$  is an indicator function that equals one if and only if  $M_s < 0$ , and  $\eta_s$  is the potential withdrawal penalty for moving money from the tax-deferred to taxable accounts before the terminal date. Equation (5) specifies the limit ( $\overline{M}_t$ ) to the amount an investor can contribute to tax-deferred accounts, and that investors cannot withdraw more than the wealth in tax-deferred accounts, or there are borrowing and short-selling constraints in tax-deferred accounts, where  $\mathbf{0}_N = [0, \dots, 0]'$  is a vector of zeros with  $N$  elements. Finally, Equation (6) describes the terminal condition when investors liquidate all wealth (taxable or tax-deferred) for consumption and bequeath motive.

## 5. Solution Method and Discussion

Even with only taxable accounts, the optimal portfolio problem is very complicated since the state space ( $\mathbf{H}_{t-1}, \mathbf{Y}_t$  in particular) includes the entire trading history. Instead of solving the general two-account problem, we study the impact of tax-deferred accounts on the overall portfolio decisions by introducing the concept of *effective tax subsidy*, which is defined as the wealth transfer that would make investors, deprived of their tax-deferred accounts, indifferent to investors who have the privilege to defer taxes in retirement accounts. For simplicity, we first ignore the possibility to contribute to tax-deferred accounts in the future. Formally,

**Definition 1** *Assume investors are not allowed to contribute to tax-deferred accounts in the future, or  $\overline{M} = 0$  in Equation (5). Let  $V_t(\mathbf{H}_{t-1}, \mathbf{Y}_t, W_t^D |_{\Theta_{t:T}}, \mathbf{P}_t)$  and  $V_t(\mathbf{H}_{t-1}, \mathbf{Y}_t, 0, \mathbf{P}_t)$  be the value functions of investors who follow exogenous trading strategy  $\Theta_{t:T}$  or have no wealth in tax-*

deferred accounts, respectively, then the effective tax subsidy  $Z_t^{ce}$  solves the following equation

$$V_t(\mathbf{H}_{t-1}, \mathbf{Y}_t, \mathbf{W}_t^D | \Theta_{t,T}, P_t) = V_t(\mathbf{H}_{t-1} + Z_t^{ce} \mathbf{W}_t^D \mathbf{e}_{NT}^{1,t}, \mathbf{Y}_t, 0, P_t) \quad (7)$$

where  $\mathbf{e}_{NT}^{1,t}$  is an index matrix of size  $N \times T$  with the  $(1,t)$ -th element equals one and zero otherwise.

The quantity  $Z_t^{ce}$  in the above definition measures the *average* value of a dollar invested in tax-deferred accounts. Although in general, it incorporates information regarding future trading strategies and can only be solved dynamically, we show in the next chapter that when investors are allowed to borrow and short-sell in taxable accounts,  $Z_t^{ce}$  is independent of individual preference or future trading strategies. As a result, the taxable and tax-deferred portfolio decisions can be solved sequentially by first deriving the optimal tax-deferred portfolio ( $\Theta_t$ ) and calculating the corresponding  $Z_t^{ce}$ , and then deriving the optimal taxable portfolio ( $\mathbf{H}_t$ ) by mapping the two-account problem to a taxable-account-only problem.





## Chapter 3

# Optimal Portfolio Decisions in Complete Markets

In general, the taxable and tax-deferred portfolio decisions should be determined simultaneously and there are dynamic interactions between future- and current-period decisions. The high-dimensionality problem induced by the history dependence of taxable portfolio makes the task of solving for optimal portfolio decisions formidable. Fortunately, it is not necessary to solve the overall problem in order to characterize the impact of tax-deferred accounts. As long as investors follow the optimal trading strategy in tax-deferred accounts, the additional investment opportunity set introduced by tax-deferred asset payoffs is spanned by asset payoffs in taxable accounts, and tax-deferred assets can be dynamically replicated by a portfolio of taxable assets.

The replication argument enables us to derive the following properties of the optimal portfolio: First, the optimal tax-deferred portfolio decision is separable from the taxable portfolio decision, and the optimal location decision (of which assets to prefer in tax-deferred accounts) is fully specified by the tax-deferred portfolio while the allocation decision (of the risk exposure to each asset in the overall portfolio) is achieved through adjusting the taxable portfolio. Second, the optimal location decision can be solved myopically (without taking into account the future portfolio decision). Lastly, the optimal portfolio allocation for the two-account problem can be derived through a preference-free mapping between the two-account problem and a

taxable-account-only problem, if we take as a given the optimal taxable-account-only portfolio decision.

## 1. Replication Arguments

We start characterizing the relationship between taxable and tax-deferred portfolio decisions by showing that tax-deferred trading strategies can be dynamically replicated by portfolios of taxable assets. The cost of replication, in general, depends both on the trading strategy followed in tax-deferred accounts and on the replicating strategy in taxable accounts. We demonstrate the uniqueness of the replication strategy in three steps: First, we derive the optimal trading strategy on any asset in taxable accounts and show that the strategy is independent of investor preference or overall portfolio decision. Second, if investors follow the above trading strategy, then any taxable position with embedded capital gains can be uniquely replicated using taxable assets with no embedded capital gains. The first two steps are very similar to the results developed in Constantinides (1983). We reproduce the results here for completeness. Finally, we show that tax-deferred positions can be replicated using dynamic portfolios of taxable assets.

### 1.1. Optimal Trading Strategy in the Taxable Accounts

The following lemma shows that the optimal trading strategy on any given asset position in taxable accounts is uniquely determined by the current price level and the initial purchase price (the cost basis) of the share. Using the notation specified in the model chapter, the optimal trading strategy can be stated as follows.

**Lemma 1** *The optimal trading strategy on any existing position of asset  $i$  in taxable accounts is to realize losses immediately and to postpone gains until the terminal date. Formally, let  $h_{i,s,t-1}$  and  $y_{i,s,t}$  be the holding of asset  $i$  (purchased at time  $s$ ) and the before-trading tax cost basis at time  $t < T$ , then the optimal holding of asset  $i$  after trading is*

$$h_{i,s,t} = \begin{cases} h_{i,s,t-1}, & \text{if } \{y_{i,s,t} < 1 \text{ and } h_{i,s,t-1} > 0\} \text{ or } \{y_{i,s,t} > 1 \text{ and } h_{i,s,t-1} < 0\} \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

**Proof.** See Appendix A. ■

The first case in equation (1) states that the optimal trading strategy is to keep the current asset position if there are unrealized capital gains, that is, if the current market price is higher than the purchase prices for long positions (described by conditions in the first bracket), or if the current price is lower than the purchase prices for short positions (the second bracket). The second case states that if there are capital losses, the optimal strategy is to liquidate the entire position immediately. The optimality of postponing gains and realizing losses relies on the fact that investors can always introduce new asset holdings ( $h_{i,t,t}$ ) to achieve any desired risk profile. For example, if there are losses on a long position and the optimal portfolio requires retaining the long position, investors do strictly better by realizing the losses immediately to get tax credits and then re-purchasing the same asset, because they can earn interests on the tax credits received. Similarly, if there are capital gains on the long position and investors need to reduce total risk exposure to the asset, investors can do better by retaining the share while simultaneously shorting the same asset than by directly liquidating their long position. Again, the benefit is the interest payments saved on the capital gains.

Lemma 1 shows that the current tax basis  $y_{i,s,t}$  of any asset position is the *only* determinant for the optimal trading strategy on the position. The fact that the optimal trading strategy is independent of individual preference or their overall portfolio decisions establish the foundation for a simple replication argument, to which we now turn our attention.

## 1.2. Replicating Taxable Assets with Embedded Capital Gains

The following lemma shows that as long as investors follow the above defined optimal trading strategies in their taxable accounts, any taxable position with an embedded capital gain is equivalent to a cash position in taxable accounts. To simplify notation, we remove the subscript  $\{i, s, t\}$  for the tax basis whenever there is no risk of confusion.

**Lemma 2** *Assume investors follow the optimal trading strategy described in Lemma 1 for any asset position in their taxable accounts, let  $y$  be the tax cost basis for a long position in asset  $i$  at*

time  $t$ , then there exists a unique relative value  $z_{i,t}(y)$  such that investors are indifferent between having one dollar face value in a long position with tax basis  $y$  and having  $z_{i,t}(y)$  dollars of cash in taxable accounts. Similarly there exists a unique relative value  $s_{i,t}(y)$  for any short position in asset  $i$  with cost basis  $y$ .

**Proof.** See Appendix A. ■

The main intuition for the above lemma is that, in a binomial setting, the return space generate by any position in asset  $i$  with any cost basis is spanned by the returns on asset  $i$  with no embedded capital gains ( $y = 1$ ) and the risk-free bond. Specifically, let  $\bar{d}_i$ ,  $\bar{g}_i$  be the before-tax dividend and capital gains rates defined in equation (1), then a long position with face value \$1 and tax basis  $y$  evolves into a portfolio of  $(1 - \tau_d)\bar{d}_i$  after-tax cash dividends and a long position in asset  $i$  with  $\$(1+\bar{g}_i)$  face value and tax cost basis  $y/(1+\bar{g}_i)$ . Therefore, investors are indifferent between holding a long position in asset  $i$  from time  $t$  to  $t + 1$  and receiving the following cash payments at time  $t + 1$  if they follow optimal trading strategy from time  $t + 1$  on,

$$\bar{R}_{i,t}(y) \equiv (1 - \tau_d)\bar{d}_i + (1 + \bar{g}_i)z_{i,t+1}(y/(1 + \bar{g}_i)).$$

Let  $R_{i,t}^H(y)$  and  $R_{i,t}^L(y)$  be the realizations of the effective one-period return  $\bar{R}_{i,t}(y)$  when the return on asset  $i$  is high and low respectively, then we can define a “risk neutral” measure using a position in asset  $i$  with no embedded capital gains and the risk-free bond as the base assets,

$$q_{i,t} \equiv \frac{R_{i,t}^H(1) - R_f}{R_{i,t}^H(1) - R_{i,t}^L(1)}. \quad (2)$$

Any other position in asset  $i$  can be viewed as a derivative asset whose value can be calculated by discounting its future cash flows using the risk neutral measure:

$$z_{i,t}(y) = \frac{1}{R_f} \left[ (1 - q_{i,t})R_{i,t}^H(y) + q_{i,t}R_{i,t}^L(y) \right]. \quad (3)$$

It is important to note that, although the effective one-period return, and hence, the relative values of any taxable position are defined recursively, they are independent of investor preference

or their overall portfolio strategies in the future.

### 1.3. Replicating Tax-deferred Assets

We now extend the replication argument to assets in tax-deferred accounts following state-independent trading strategies as defined below.

**Definition 2** *Let  $\theta_{i,t}$  denote the percentage of tax-deferred wealth invested in asset  $i$  at time  $t$ ,  $\Theta_t = [\theta_{1,t}, \dots, \theta_{N,t}]'$  be the vector of portfolio holdings, then the tax-deferred trading strategy  $\Theta_{t:T} = [\Theta_t, \dots, \Theta_T]$  is defined as state-independent if  $\Theta_s$  is independent of the realizations of state variables at any time  $s = t, \dots, T$ .*

The following lemma shows that any tax-deferred position following any (not necessarily optimal) state-independent future trading strategy is equivalent to a unique cash position in taxable accounts.

**Lemma 3** *If investors follow the optimal trading strategy described in Lemma 1 for assets in their taxable accounts, there exists a relative value  $z_{t:T}^D$  such that investors are indifferent between having one dollar in their tax-deferred accounts following state-independent future trading strategy  $\Theta_{t:T}$  and having  $z_{t:T}^D$  dollars of cash in their taxable accounts.*

**Proof.** See Appendix A. ■

Lemma 3 implies that investors are indifferent between having the opportunity to invest a dollar in tax-deferred accounts or receiving a fixed lump-sum subsidy from the government, therefore, the relative value can be linked to the concept of effective tax subsidy  $Z_t^{ce}$  defined in the model chapter. Although in general, the value  $Z_t^{ce}$  incorporates information regarding overall portfolio optimization in the future and can only be solved dynamically, Lemma 3 indicates that it can be determined separately from the overall optimization if investors restrict themselves to state-independent tax-deferred strategies. Specifically,

**Corollary 1** *Let  $z_{t:T}^D$  be the relative value of a dollar in tax-deferred accounts following future trading strategy  $\Theta_{t:T}$ , then the corresponding effective tax subsidy in Definition 1 can be expressed as  $Z_t^{ce} = z_{t:T}^D - 1$ .*

Corollary 1 illustrates that when investors are allowed to borrow or short-sell in taxable accounts, the effective tax subsidy is drastically simplified and depends *only* on the future trading strategy followed in tax-deferred accounts. The independence of the subsidy on individual preference or overall portfolio decisions ensures the separability between the taxable and tax-deferred portfolio decisions in the next section.

Another implication of Lemma 3 is that the assumption of no borrowing or short-selling in tax-deferred accounts is necessary to ensure a properly functioning market. It is clearly possible to have different relative values (or replication costs) for different tax-deferred trading strategies, to fix ideas, assume the strategy  $\Theta_I$  of holding only asset  $i$  in tax-deferred accounts has a replication cost of \$3 whereas the replication cost is only \$2 for strategy  $\Theta_J$  of holding only asset  $j$ . If investors are allowed to borrow and short-sell in tax-deferred accounts, they can always short asset  $j$  to long asset  $i$  in tax-deferred accounts, while at the same time long the replicating portfolio for strategy  $\Theta_J$  and short the replicating portfolio for strategy  $\Theta_I$  in the taxable accounts. The overall risk exposure is zero by the definition of replication portfolio, and investors pocket the \$1 difference in replication costs, generating an arbitrage opportunity.

## 2. Separation between Location and Allocation Decisions

We now show that the optimal tax-deferred strategy is state-independent, hence can be dynamically replicated using a taxable portfolio. As a result, the portfolio strategies in tax-deferred accounts do not affect the overall risk exposure of investors, since they can always undo any position taken in tax-deferred accounts by including an offsetting replicating portfolios in their taxable accounts. The only impact of following different trading strategies in tax-deferred accounts is a wealth effect due to the different corresponding effective tax subsidy received ( $Z_{t:T}^D$ ). The independence of the tax subsidy on overall portfolio strategies then implies the separability between taxable and tax-deferred portfolio strategies. Formally,

**Proposition 1 (Separability)** *Let  $H_t$  and  $\Theta_{t:T}$  be the optimal portfolios in taxable and tax-deferred accounts at time  $t$ , respectively, then the optimal tax-deferred portfolio  $\Theta_{t:T}$  is independent of the taxable portfolio  $H_t$ , and is chosen only to maximize the corresponding effective tax*

subsidy  $Z_{t:T}^D$ .

**Proof.** See Appendix A. ■

The proposition states that investors can solve their optimization problem sequentially by first determining the tax-deferred portfolio ( $\Theta_{t:T}$ ) to maximize the effective tax subsidy and then choosing the taxable portfolio ( $H_t$ ) to achieve optimal risk exposure.

The significance of this separability result is illustrated in the following example. Assume two investors differ drastically in their desired exposure to certain asset  $i$  due to their different risk aversion, endowment, or different private information regarding the payoff of the asset. In particular, say investor 1 prefers to hold none of asset  $i$  and investor 2 prefers to hold 100% of his portfolio in asset  $i$ . If, however, investor 1 and 2 agree that the strategy of holding only asset  $i$  in tax-deferred accounts maximizes the effective tax subsidy received, they will hold the *same* tax-deferred portfolio, namely, 100% in asset  $i$ . Their individual preference is only reflected in their corresponding taxable portfolio, where investor 1 holds a huge short position in asset  $i$  to offset the undesired exposure to the asset in tax-deferred accounts, whereas investor 2 holds 100% in asset  $i$  to reflect his strong preference for it.

The two questions remaining from the above example is whether investors 1 and 2, given their different preferences, will agree on the ranking of assets by their corresponding effective tax subsidy, and how, in general, investors should adjust their taxable portfolios to account for their tax-deferred trading strategies. In the next section, we address these two questions by presenting the properties of the optimal portfolio decisions.

### 3. Properties of the Optimal Portfolio

The separability between taxable and tax-deferred portfolio decisions allows us to decompose the optimal portfolio decision into location and allocation decisions, where the location decision refers to the preference for holding assets in taxable vs. tax-deferred accounts, and the allocation decision describes the desired risk exposure to each assets. We show in this section that the location decision is fully specified by the tax-deferred strategy  $\Theta_{t:T}$  and the allocation decision

can be inferred from the taxable holding  $H_t$  by subtracting the effective taxable holding in the replicating portfolio for the tax-deferred strategy ( $\Delta_{t,T}^D$  in equation (7))

### 3.1. Location Decisions

Proposition 1 states that the optimal tax-deferred portfolio ( $\Theta_{t,T}$ ) is chosen to maximize the corresponding effective tax subsidy ( $Z_{t,T}^D$ ) received. By definition, investors need to optimize over tax-deferred trading strategies for all future periods to derive the asset-specific and/or individual-specific characteristics that maximize the corresponding tax subsidy. In general, there are interactions between current- and future-period decisions. However, we show that, due to dynamic replication, the optimal tax-deferred strategy is myopic. Formally, we define a quantity that depends only on the current-period asset characteristics and derive the optimal tax-deferred portfolio decision based on it.

**Definition 3** *At any time  $t$ , the current-period effective tax subsidy is defined as*

$$Z_{i,t}^D \equiv \frac{1}{R_f} [(1 - q_{i,t})r_i^H + q_{i,t}r_i^L] - 1. \quad (4)$$

*for any asset  $i$ , where  $q_{i,t}$  is the risk-neutral probability defined in equation (2).*

The following proposition shows that the current-period effective tax subsidy  $Z_{i,t}^D$  is a sufficient statistic for the optimal tax-deferred portfolio decision at time  $t$ .

**Proposition 2 (Location)** *The optimal location decision is fully specified by the optimal tax-deferred trading strategy  $\Theta_{t,T}$ , and can be myopically determined by holding only those assets that generate the highest current-period effective tax subsidies in tax-deferred accounts. Formally, at any time  $t$ , let  $Z_{i,t}^D$  and  $Z_{j,t}^D$  be the current-period tax subsidies in Definition 3 for assets  $i$  and  $j$ , respectively, then asset  $i$  is held in tax-deferred accounts if and only if  $Z_{i,t}^D \geq Z_{j,t}^D$  for any asset  $j$ .*

**Proof.** See Appendix A. ■

The current period subsidy  $Z_{i,t}^D$  is fully specified by Equations (3), (2), and (4), as long as investors always follow the optimal trading strategy to postpone gains and realize losses for



assets in their taxable accounts. The subsidy is determined mainly by the return distributions of asset  $i$  and the risk-free rate. The only characteristics of the investors that are relevant for the definition of effective tax subsidy are their investment horizon and tax status, including both the tax rates on capital gains and dividends, and the terminal tax treatments. All investors in the same tax brackets who have the same investment horizon will agree on the ranking of assets by the current-period effective tax subsidy.

Investors can determine their tax-deferred portfolio each period by first ranking all assets by their current-period effective tax subsidy, and then holding only the asset that provides the highest tax subsidy in tax-deferred accounts. To keep a balanced portfolio, they do not hold (or even need to short-sell) in taxable accounts the asset that provides the highest level of effective tax subsidy. Hence, that asset can be viewed as having a preferred location of tax-deferred accounts, while all other assets are held only in taxable accounts and hence have a preferred location of taxable accounts. In this sense, the tax-deferred portfolio decision fully specifies the asset location decision, and Proposition 1 also implies the separability between location and allocation decisions.

### 3.2. Allocation Decisions

We now calculate the optimal allocation decision by transforming the optimal portfolio problem with both taxable and tax-deferred accounts to that with only taxable accounts. Without loss of generality, we simplify the optimal portfolio problem described in Equations (2)-(6) by ignoring the consumption and contribution decisions (choice variables  $C_t$  and  $M_t$ , respectively) in this section and then discuss the two decisions specifically in Section 3.3.

Proposition 2 states that investors can uniquely determine their optimal tax-deferred portfolio  $\Theta_{t:T}$  without considering their risk preference or overall portfolio strategies. Since having a dollar in tax-deferred accounts following strategy  $\Theta_{t:T}$  is equivalent to being endowed with a taxable portfolio  $\Delta_{t:T}^D$  as defined in Equation (7), investors are indifferent between following strategy  $\Theta_{t:T}$  in tax-deferred accounts, and having no access to tax-deferred accounts while their

taxable wealth level is increased by the replication cost of the portfolio.<sup>1</sup> Moreover, to properly account for the risk exposure in tax-deferred accounts, we need to subtract the correspond replicating portfolio from the overall taxable portfolio holding. The following proposition formalizes the above intuition:

**Proposition 3 (Allocation)** *Assume  $\bar{M} = 0$  and  $u(\cdot) = 0$  in the optimization problem specified by Equations (2)-(6), let  $\Theta_{t:T}$  be the optimal tax-deferred trading strategy derived in Proposition 2,  $\Delta_{t:T}^D$  and  $Z_{t:T}^D$  be the corresponding taxable replicating portfolio and the effective tax subsidy, respectively, and  $\mathcal{H}_t(\mathbf{H}_{t-1}, \mathbf{Y}_t, \mathbf{P}_t) \equiv \mathbf{H}_t(\mathbf{H}_{t-1}, \mathbf{Y}_t, 0, \mathbf{P}_t)$  be the optimal portfolio when investors have no access to tax-deferred accounts, then the optimal taxable portfolio for the two-account problem is*

$$\mathbf{H}_t(\mathbf{H}_{t-1}, \mathbf{Y}_t, W_t^D, \mathbf{P}_t) = \mathcal{H}_t(\widehat{\mathbf{H}}_{t-1}, \mathbf{Y}_t, \mathbf{P}_t) - W_t^D \Delta_{t:T}^D e_T^t \quad (5)$$

where  $\widehat{\mathbf{H}}_{t-1} = \mathbf{H}_{t-1} + Z_{t:T}^D W_t^D e_{NT}^{1,t-1}$  is the before-trading taxable portfolio augmented by the effective tax subsidy, and  $e_T^t$  (or  $e_{NT}^{1,t-1}$ ) is an index vector (matrix) that equals to one for the  $t$ -th (or  $(1, t-1)$ -th) elements and zero otherwise.

**Proof.** See Appendix A. ■

Having applied Proposition 2 to determine the optimal tax-deferred portfolio, investors can follow Proposition 3 to derive the optimal taxable portfolio in three steps: First, calculate the corresponding effective tax subsidy received for the optimal tax-deferred portfolio (calculate  $Z_{t:T}^D$ ). Second, account for the “wealth effect” of tax-deferred investing by increasing their taxable holding of risk-free assets by the effective tax subsidy, and derive the optimal portfolio allocation ignoring the existence of tax-deferred accounts (calculate  $\mathcal{H}_t(\widehat{\mathbf{H}}_{t-1}, \mathbf{Y}_t, \mathbf{P}_t)$ , where  $\widehat{\mathbf{H}}_{t-1}$  includes the effective tax subsidy). Finally, account for the “hedging effect” of tax-deferred accounts by subtracting the taxable replicating portfolio  $\Delta_{t:T}^D$  for each dollar in tax-deferred accounts from the overall taxable portfolio derived in the previous step (subtract the term  $W_t^D \Delta_{t:T}^D e_T^t$ ). It is possible that some elements of  $\mathbf{H}_t$  is less than zero, indicating that investors borrow or short-sell

<sup>1</sup>The replication cost  $\mathbf{1}'_N \Delta_{t:T}^D$  is equal to  $1 + Z_{t:T}^D$ , where  $Z_{t:T}^D$  is the corresponding effective tax subsidy for each dollar in tax-deferred accounts.

in their taxable accounts to achieve optimal portfolio allocation.

### 3.3. Consumption and Contribution Decisions

Since the only impact of tax-deferred accounts on the overall portfolio decision is the wealth effect due to the effective tax subsidy received, the optimal contribution decision is always maximizing contribution and the optimal consumption decisions can be derived by adjusting the taxable-account-only consumption decision. Formally,

**Corollary 2** *Consider the optimization problem specified by Equations (2)-(6), let  $Z_{t:T}^D$  be the effective tax subsidy received for following optimal tax-deferred strategy in Proposition 2, and  $C_t(\mathbf{H}_{t-1}, \mathbf{Y}_t, \mathbf{P}_t) \equiv C_t(\mathbf{H}_{t-1}, \mathbf{Y}_t, 0, \mathbf{P}_t)|_{\bar{M}=0}$  be the optimal consumption decision when investors have no access to tax-deferred accounts, then (i) investors always contribute the maximum allowed to their tax-deferred accounts, or  $M_t = \bar{M}_t$  at any time  $t$ , (ii) the present value of the effective tax subsidy received for all future contributions is*

$$Z_{t:T}^M = \sum_{s=t}^T \frac{1}{R_f^{s-t}} \bar{M}_s Z_{s:T}^D, \quad (6)$$

and (iii) the optimal consumption decision for the two-account problem is

$$C_t(\mathbf{H}_{t-1}, \mathbf{Y}_t, \mathbf{W}_t^D, \mathbf{P}_t) = C_t(\widehat{\mathbf{H}}_{t-1}^M, \mathbf{Y}_t, \mathbf{P}_t) \quad (7)$$

where  $\widehat{\mathbf{H}}_{t-1}^M = \mathbf{H}_{t-1} + (Z_{t:T}^D \mathbf{W}_t^D + Z_{t:T}^M) e_{NT}^{1,t-1}$  is the before-trading taxable portfolio augmented by the effective tax subsidies received both for current tax-deferred wealth and for future contributions to tax-deferred accounts.

**Proof.** See Appendix A. ■

The optimal taxable portfolio in Proposition 3 can be adjusted to account for the future contributions by including the wealth adjustment for  $Z_{t:T}^M$ , or replacing  $\widehat{\mathbf{H}}_{t-1}$  by  $\widehat{\mathbf{H}}_{t-1}^M$  in Equation (5). Clearly, the existence of tax-deferred accounts and the opportunity to make future contributions to tax-deferred accounts are equivalent to increasing the endowment of safe assets

in taxable accounts. Assuming, with only taxable accounts, investors always increase their consumption level after receiving an additional endowment of safe assets, (that is,  $C_t$  increases with the first element of  $\widehat{H}_{t-1}^M$ ), then they will always increase the level of their consumption after the introduction of tax-deferred accounts. Therefore, the savings rate always decreases, contradicting the popular belief that tax-deferred investing opportunities encourage savings from investors.

#### 4. Implementation of the Optimal Portfolio

The optimal portfolio decisions derived so far is centered around the notion of effective tax subsidy: First, we demonstrate that granting investors the rights to invest in tax-deferred accounts is equivalent to providing them with fixed tax subsidies, independent of individual preferences or overall portfolio strategies. Then we identify in each period the assets that provide the highest current-period effective tax subsidies as defined in Equation (4) and to hold *only* those assets in tax-deferred accounts. The optimal portfolio decisions with both taxable and tax-deferred accounts is then reduced to that with only taxable accounts by replacing tax-deferred accounts with the corresponding wealth transfer equal to the effective subsidy, while the desired level of overall risk exposure is achieved through adjusting their portfolio allocation in their taxable accounts (borrow or short-sell if necessary).

From a theoretical point of view, Propositions 1-3 solves the optimal portfolio problem specified in Equations (2)-(6), with the effective tax subsidy well defined and a sufficient statistic for the information related to optimal portfolio decisions. However, for practical purposes, the effective tax subsidy is not a very intuitive notion and it is hard to generate realistic advice for investors to follow. It would be nice to identify a variable that captures similar information, but can be intuitively related to features of the underlying assets or characteristics of the individual investors. We introduce in this section one such variable, the “effective tax rate”, to better understand the properties of the optimal portfolio and to generate easy-to-follow approximate portfolio rules that accounts for the impact of tax-deferred accounts.

To construct a variable that captures similar information conveyed by the effective tax sub-

sidy, we need to first understand the properties of the subsidy and its main determinants. The following example provides some intuition: consider a risky asset  $i$  that pays out all returns as dividends<sup>2</sup> and is taxed at a rate  $\tau_i$  each period. Then the “risk neutral probability” in equation (2) reduces to  $q_{i,t} = \frac{R_S^H - R_f}{R_S^H - R_S^L}$ , where  $R_S^H = r_i^H - \tau_i(r_i^H - 1)$ ,  $R_S^L = r_i^L - \tau_i(r_i^L - 1)$ ,  $R_f = r_f - \tau_d(r_f - 1)$ , and the current period effective tax subsidy in equation (4) simplifies to  $Z_{i,t}^D = \frac{1}{R_f} \left[ 1 + \frac{R_f - 1}{1 - \tau_i} \right] - 1$ . Besides the after-tax risk-free interest rate  $R_f$ , the tax rate for the asset is the *only* determinant for effective tax subsidy, with higher tax rate assets generating higher tax subsidies if placed in tax-deferred accounts. The following definition generalizes the above intuition.

**Definition 4** Let  $Z_{i,t}^D$  be the current-period effective tax subsidy defined in equation (4), and  $\hat{\tau}_{i,t}$  solves the following equation

$$\max\{0, Z_{i,t}^D\} = \frac{1}{R_f} \left[ 1 + \frac{R_f - 1}{1 - \hat{\tau}_{i,t}} \right] - 1, \quad (8)$$

then  $\hat{\tau}_{i,t}$  is defined as the current-period effective tax rate for asset  $i$  at time  $t$ .

The variable  $\hat{\tau}_{i,t}$  is termed the “effective tax rate” since  $\hat{\tau}_{i,t} = \tau_i$  in the previous example. The max operator on the left hand side sets the effective tax rate to  $\hat{\tau}_{i,t} = 0$  when the effective tax subsidy  $Z_{i,t}^D < 0$ . This is because the negative effective tax subsidy implies that holding a dollar of asset  $i$  in the tax-deferred account is worse than holding a dollar in taxable accounts.<sup>3</sup> For the purpose of location decision, investors should never hold such asset in tax-deferred accounts, setting the effective tax rate to  $\hat{\tau}_{i,t} = 0$  is sufficient to capture this preference<sup>4</sup>.

By construction, the effective tax rate  $\hat{\tau}_{i,t}$  is a monotonic transformation of the current-period effective tax subsidy  $Z_{i,t}^D$  and hence also a sufficient statistic for optimal portfolio decisions. The notion of effective tax rates is useful only if we can describe the properties based on some

<sup>2</sup>Let  $r_i^H, r_i^L$  be the total realized return, then the capital gains component is  $g_i^H = g_i^L = 0$ , and the dividend process is  $d_i^H = r_i^H - 1$  and  $d_i^L = r_i^L - 1$ , where  $d_i^L < 0$  can be thought of as realizing losses on the asset immediately.

<sup>3</sup>As we will show later in Figure C-1 the negative subsidy is possible when investors are allowed to escape capital gains taxes through step-up of basis in taxable accounts, since investors do not pay taxes on capital gains in both accounts, while they still receive tax credits on losses in taxable accounts. Intuitively, the asset has a negative effective tax rate.

<sup>4</sup>If we do not set  $\hat{\tau}_{i,t} = 0$  through the max operator, it is possible to get positive  $\hat{\tau}_{i,t}$  for small enough  $z_{i,t}^D$  ( $< 1/R_f$ ) due to the functional form on the right hand side, violating the economic meaning of  $\hat{\tau}_{i,t}$ .

characteristics of the underlying assets or the trading strategies of the investors without explicitly solving for the effective tax rates through Equations (3), (2), (4), and (8).<sup>5</sup>

To understand the properties of effective tax rates, we first consider the case when investors are not allowed to postpone capital gains in their taxable accounts and derive the closed-form expression for the effective tax rate. Later, we rely on numerical results to identify other determinants of the effective tax rates when investors have the option to postpone capital gains.

### Without Timing Option to Postpone Capital Gains

If investors are not allowed to postpone capital gains taxes, the effective tax rate can be expressed as a weighted average of the tax rates on dividends and capital gains.

**Rules of Thumb 1 (No Postponing Capital Gains)** *If investors are not allowed to postpone capital gains taxes, let*

$$\delta_i^{RN} \equiv (1 - q_i)d_i^H + q_i d_i^L, \quad \text{where } q_i = \frac{R_S^H - R_f}{R_S^H - R_S^L} \quad (9)$$

be the “Risk-Neutral Dividend Yield”,<sup>6</sup> then the effective tax rate of asset  $i$  can be expressed as

$$\hat{\tau}_{i,t} = \omega \tau_d + (1 - \omega) \tau_g, \quad (10)$$

where

$$\omega \equiv \frac{(1 - \tau_g) \delta_i^{RN}}{(1 - \tau_d)(r_f - 1 - \delta_i^{RN}) + (1 - \tau_g) \delta_i^{RN}}. \quad (11)$$

More specifically, for an asset with linear dividend yield,  $\bar{d}_i = \beta_0(r_f - 1) + \beta_r(\bar{r}_i - 1)$ , the weight  $\omega$  reduces to

$$\omega = \frac{\beta_0 + \beta_r - \epsilon \beta_r}{1 - \epsilon(1 - \beta_0)}, \quad \text{where } \epsilon = \frac{\tau_d - \tau_g}{1 - \tau_g}. \quad (12)$$

<sup>5</sup>Otherwise, we could have solved directly for the effective tax subsidy using the first three equations and it would have been easier to apply Proposition 2 directly.

<sup>6</sup>Assets are taxed according to Assumption 1, except that capital gains/losses are taxed upon accrual at a rate  $\tau_g$ . Define  $R_f = r_f - \tau_d(r_f - 1)$ ,  $R_S^H = 1 + (1 - \tau_d)d_i^H + (1 - \tau_g)g_i^H$ , and  $R_S^L = 1 + (1 - \tau_d)d_i^L + (1 - \tau_g)g_i^L$ .

The main contribution of Rules of Thumb 1 is to identify the risk-neutral dividend yield ( $\delta_i^{RN}$ ) as the main determinants of effective tax rates. The expected return of assets, or the subjective views of investors, does not matter for the definition of effective tax rates, and hence, for the location decision of which asset to prefer in tax-deferred accounts. Most importantly, *none* of the individual characteristics, like risk preferences, investment horizons, terminal tax treatments, or overall portfolio strategies, etc., affects effective tax rates. As a result, *all* investors agree on the ranking of assets by their effective tax rates as long as they are in the same tax bracket.

The example of linear dividend yields is general enough to cover most asset classes. For example, the case  $\beta_r \rightarrow 0$  approximates dividend paying stocks, which usually keep relatively stable dividends over time (or slowly adjusting the dividends level) irrespective of their stock performance, while the case  $\beta_0 \rightarrow 0$  approximates actively managed stock mutual funds that realize most of their capital gains quickly and pass them to investors as dividends, with  $\beta_r$  representing the percentage of total returns paid out each year.<sup>7</sup>

Note that the weight  $\omega$  roughly equals to  $\beta_0 + \beta_r$ , and it is important to distinguish between the risk-neutral dividend yield and the expected dividend yield, since the expected dividend yield includes the expected return of assets which should not affect the effective tax rates. For example, assume the tax rates on dividends and capital gains are 40% and 20% respectively, then a risk-free bond with 5% interest rate that pays out all return as dividend ( $\beta_0 = 1, \beta_r = 0 \Rightarrow \omega = 1$ ) has an expected dividend yield of 5% and an effective tax rate of 40%. On the other hand, a risky asset with 20% expected return that pays out half of the net return as dividends and half as capital gains ( $\beta_0 = 0, \beta_r = .5 \Rightarrow \omega = .5$ ) has an effective tax rate of  $(.5)(40\%) + (.5)(20\%) = 30\%$  even though the expected dividend yield is as high as  $(.5)(20\%) = 10\%$ .

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<sup>7</sup>Of course, there are some details left out by this approximation, mostly noticeably the asymmetry in dividend payments for mutual funds in the high and low states of asset return, since the funds cannot pass along capital losses to investors. Moreover, the case of short-term investors who realize short-term capital gains which are taxed at the same rate as dividends falls in this category as well, even though we do not explicitly consider the difference between short- and long-term capital gains in the model.

## With Timing Option to Postpone Capital Gains

When investors are allowed to postpone capital gains, the definition of effective tax rates is more complicated since the risk neutral probability  $q_{i,t}$  in Equation (2) can only be determined recursively and it depends on individual characteristics like the remaining investment horizon and terminal tax treatments. We can no longer solve the effective tax rate in closed form, instead, we rely on numerical approach to understand its properties.<sup>8</sup>

Table C.1 summarizes the base set of parameters used in the numerical analysis. The risk-free interest rate is  $r_f = 6\%$ , and the risky asset  $i$  follows a binomial process with mean  $\mu = 10\%$  and volatility  $\sigma = 20\%$ . The dividend process on the risky asset is assumed to be proportional to the total return on the asset,  $\tilde{d} \equiv \delta \tilde{r}$ , with  $\delta = 3\%$ , implying  $\beta_0 = \delta / (r_f - 1) = .5$  and  $\beta_r = \delta = .03$  in Rules of Thumb 1. The tax rate is  $\tau_g = 20\%$  for capital gains and  $\tau_d = 40\%$  for interests and dividend payments. The maximum investment horizon is 80 years and there are two different terminal tax treatments regarding unrealized capital gains in taxable accounts: “step-up of tax basis” where investors are allowed to escape capital gains taxes, or “forced realization of capital gains” where investors are forced to pay capital gains taxes on the terminal date.

To focus on the impact of postponing capital gains taxes, we report in Figure C-1 the effective tax rate as a function of the remaining investment horizon ( $T - t$ ) and terminal tax treatments for assets that pay no dividends. Panel A and B represent the cases when the asset volatility is  $\sigma = 20\%$  and  $60\%$  respectively<sup>9</sup>. The solid lines in both panels represent the case when investors escape terminal capital gains taxes through step-up of tax basis, and indicate that the effective tax rate is always negative and increases with the remaining investment horizon. The dashed lines represent the case when investors are forced to realize capital gains on the terminal date, and are almost mirror images of the solid lines around the zero line. The dashed and solid lines converge to the same level (zero) since the terminal treatments are not important for the current-period effective tax rates when the remaining investment horizon is long enough.

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<sup>8</sup>We first solve recursively Equations (1) and (2) for the “risk neutral” probability  $q_{i,t}$  and then applying Equations (4) and (8) to derive the effective tax rates.

<sup>9</sup>For illustration purpose, we allow  $\hat{\tau}_{i,t} < 0$  by ignoring the max operation in the left-hand side of Equation (8). In all future analysis, we follow Equation (8) to trim  $\hat{\tau}_{i,t}$  at zero since negative  $\hat{\tau}_{i,t}$  does not provide additional information regarding location decisions.



The negative effective tax rate for the step-up of basis case may seem strange since intuitively, investors can at most avoid paying any taxes on capital gains, implying a zero tax rate on capital gains. However, the effective tax rate measures both the benefit (of escaping capital gains taxes) and the cost (of foregoing tax credits on capital losses) of investing in tax-deferred accounts relative to in taxable accounts. With the step-up provision, investors can escape *all* capital gains taxes in taxable accounts (by always postponing gains until the terminal date and then escape taxes through step-up of basis), while they are allowed to realize tax credits on capital losses, hence, investing in taxable accounts dominates investing in tax-deferred accounts ( $Z_{i,t}^D < 0$ ) and the corresponding effective tax rate is always negative.

The fact that the effective tax rate increases with the investment horizon and approaches zero may also be counterintuitive, since the benefit of realizing losses in taxable accounts should be higher for longer investment horizon and certainly does not approach zero. However, the *current-period* effective tax rate  $\hat{\tau}_{i,t}$  in equation (8) measures the marginal benefit of investing in tax-deferred accounts for an extra period from time  $t$  to  $t+1$ , taken as a given the benefit of investing in tax-deferred accounts from time  $t+1$  on. When the horizon is long enough, an extra investment period has little impact on the total capital losses realized since investors need to cancel out previous capital gains before they can realize a loss.

The following numerical example illustrates the point. Assume an asset has binomial return 10% or  $-10\%$  (state  $H$ ,  $L$  respectively) with equal probability. With one period to go, investors escape taxes when the return is 10% and they realize tax credits of  $.1\tau_g$  for each dollar invested if the return is  $-10\%$ . The present value of the cost of investing in tax-deferred accounts relative to investing in the taxable is  $-\frac{1}{R_f}(.1\tau_g)(1-q)$ , where  $q$  is the risk neutral probability. With two periods to go, investors can still realize  $.1\tau_g$  of tax credits in state  $L$  in the first period. In the second period, there are 4 different states,  $\{HH, HL, LH, LL\}$ , with  $HH$  indicating high returns in both periods, and similarly for the rest. Investors can realize capital losses only in state  $LL$ , since in state  $HL$ , although the current period return is negative, it is cancelled out by the gains in the previous period. Over the two periods, therefore, the total tax losses realized are  $.1\tau_g$  each at state  $L$  in period 1 and state  $LL$  in period 2, with a risk neutral present value

of  $\frac{1}{R_f}(.1\tau_g)(1-q) + \frac{1}{R_f^2}(.1\tau_g)(1-q)^2$ . Subtracting the benefit  $\frac{1}{R_f}(.1\tau_g)(1-q)$  of realizing losses in the single period case, we derive that the expected benefit of being able to realize losses on the asset for an extra period is only  $\frac{1}{R_f^2}(.1\tau_g)(1-q)^2$ , much lower than that for the single-period case, since investors need to cancel previous capital gains before they can realize losses in states like  $HL$ , and the benefit of an extra period decreases further as horizon increases.

When investors are forced to realize terminal capital gains, the results are similar to those for the step-up of basis case, except that investing in tax-deferred accounts dominates investing in taxable accounts. On the terminal date, with all capital gains/losses realized and taxed at  $\tau_g = 20\%$ , the effective tax rate is 20% by definition. When the horizon is long enough, the impact of terminal tax treatments is small and the effective tax rate also converges to zero, the same as that when investors escape terminal taxes. We summarize the properties of the effective tax rates for no-dividend-paying assets as follows:

**Rules of Thumb 2 (No Dividend)** *If investors are allowed to postpone capital gains taxes in their taxable accounts, and an asset pays no dividend ( $\tilde{d}_i = 0$ ), then the effective tax rate  $\hat{\tau}_{i,t}$  in Definition 4 can be approximated as follows:*

- *If investors are forced to realize capital gains taxes on the terminal date, the effective tax rate (i) equals  $\tau_g$  on the terminal date, (ii) decreases with the remaining investment horizon and approaches zero for long enough horizon (say for  $T - t \geq 20$  years), and (iii) the speed of convergence increases with the asset volatility  $\sigma$ .*
- *If investors are allowed to escape terminal taxes through step-up of basis, the effective tax rate roughly equals minus the effective tax rate if investors are forced to realize gains  $\hat{\tau}_{i,t}|_{step-up} \approx -\hat{\tau}_{i,t}|_{realize-gain}$ .*

When assets pay dividends, we can decompose the total return into the dividend and capital gains components where dividends are taxed each period at a rate  $\tau_d$  while capital gains can be thought of as being taxed each period at an “effective tax rate”  $\hat{\tau}_g \equiv \hat{\tau}_{i,t}|_{\tilde{d}_i=0}$  derived using Rules of Thumb 2. Therefore, the effective tax rate for a dividend-paying asset can be approximated

by a weighted average approach similar to Rules of Thumb 1 by replacing the tax rates on capital gains  $\tau_g$  with a lower “effective tax rate” on capital gains  $\hat{\tau}_g$ .

**Rules of Thumb 3 (Dividends)** *If investors are allowed to postpone capital gains taxes, let  $\hat{\tau}_g \equiv \hat{\tau}_{i,t}|_{\bar{d}_i=0}$  be the effective tax rate when assets pay no dividends, then the effective tax rate for an asset with linear dividend yield,  $\bar{d}_i = \beta_0(r_f - 1) + \beta_r(\bar{r}_i - 1)$ , can be approximated by the following weighted average of tax rates*

$$\hat{\tau}_{i,t} \approx \omega\tau_d + (1 - \omega)\hat{\tau}_g \quad (13)$$

where  $\omega$  is defined in Equation (12).

To be precise, the weight  $\omega$  should be replaced by  $\hat{\omega}$  that measures the dividend yield as a percentage of total *taxable* returns and should change over time due to the changing taxable portion of the capital gains component. We use a constant  $\omega$  for simplicity. Figure C-2 reports the effective tax rate for assets with dividend yields  $\delta = 3\%$ , where the solid and dashed lines are calculated by applying Equation (8), and the results are almost a parallel shift of those in Figure C-1. To evaluate the performance of the approximation rule, we report directly the approximate effective tax rates calculated using Rules of Thumb 3 in the dotted lines, which are quite close to the solid and dashed lines, indicating that the weighted average approach with constant  $\omega$  provides good approximation of the effective tax rates, especially when asset volatility is not too high (in Panel A). Both Figure C-1 and C-2 illustrate that the horizon effect is monotone for each given terminal tax treatment and the effective tax rates do not change much after the investment horizon reaches 30 years. Therefore, in the remainder of our analysis, we present results only for the cases of short- and long-horizon (5 and 30 years, respectively).

To further assess the performance of Rules of Thumb 3, we explore a wider range of parameter values in Figure C-3 and C-4, where Figure C-3 reports effective tax rates as functions of dividend yields when the asset volatility is set at 20%, and Figure C-4 reports effective tax rates as functions of asset volatilities when the dividend yield is set at  $\delta = 3\%$ . The approximation works the best when the dividend yield is not too high and asset volatility is not too high.

In Figure C-3, if the investment horizon is long or investors are forced to realize capital gains taxes on the terminal date, the effective tax rate is almost always positive, indicating that it is generally beneficial to be able to invest in tax-deferred accounts. If, however, investors can escape capital gains taxes in taxable accounts and the investment horizon is short, the effective tax rate is only positive for high enough dividend yield (e.g.  $\geq 1.5\%$  in the example), indicating that there are significant benefits of realizing losses in taxable accounts.

Figure C-4 indicates that the effective tax rate decreases with asset volatilities, since the option value of postponing gains and realizing losses in taxable accounts increases, reducing the benefit of investing in tax-deferred accounts. However, the impact of asset volatility is usually small except in the short horizon case when investors escape terminal capital gains taxes. For example, the effective tax rates for long-horizon investors drop from about 25% to 20% when asset volatility increases from 10% to 60%, while the corresponding drop is from 30% to 25% for short-horizon investors if they are forced to realize capital gains on terminal date. The only significant change happens for short-horizon investors who are allowed to escape terminal capital gains taxes when the effective tax rate drops from 20% to below 0% as the asset volatility increases to 50%, indicating it is no longer beneficial to place the asset in tax-deferred accounts even though the asset pays a sizeable dividend (3%). The following rule of thumb summarizes the impact of volatility on effective tax rates:

**Rules of Thumb 4 (Asset Volatility)** *If investors are allowed to postpone capital gains taxes, the effective tax rate decreases with asset volatility, but the impact is usually small enough to be safely ignored for approximation purposes, except for the case of short-horizon investors who escape capital gains taxes on the terminal date, when effective tax rates can easily drop below zero for high enough asset volatility.*

## Chapter 4

# From Complete to Incomplete markets: A Heuristic Approach

The simple structure of the optimal portfolio strategy described by Propositions 1-3 relies on the assumption that investors are allowed to borrow and short-sell any assets in taxable accounts, and as a result, the optimal tax-deferred strategy can be perfectly replicated by a taxable portfolio, equating the privilege of investing in tax-deferred accounts to receiving a government tax subsidies. In reality, however, borrowing and short selling are restricted for small investors, and the ability to deduct interest payments from taxable income is limited. These restrictions make the replication argument infeasible, and Proposition 1-3 no longer hold.

To fully address the impact of borrowing and short-selling constraints on optimal portfolio decisions, we need to solve the model specified in Equation (2)-(6) while explicitly imposing the constraint  $H_t \geq 0$  for all assets at any time  $t$ . As discussed before, this is a daunting task due to the history dependence of portfolio holdings. One common approach in the literature is to specify a version of the model and to solve it numerically.<sup>1</sup> The common problem with the numerical approach is that, without theoretical guidance, it is hard to understand the structure of the optimal solution, and even harder to generalize the results to a more realistic setting for practical purposes.

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<sup>1</sup>See for example Dammon, Spatt and Zhang (2002a,2002b), Gallmeyer, Kaniel and Stompaids (2000) among others.

In this chapter, instead of deriving the exact dynamic solution when investors are not allowed to borrow or short-sell in taxable accounts, we provide heuristic rules for investors to follow by extending the complete market results to the incomplete market case, and later use the rules to guide our numerical analysis in the next chapter. To isolate the the impact of borrowing and short-selling constraints on the optimal portfolio decisions, we first focus our attention on a simplified version of the general model where investors are not allowed to postpone capital gains on the risky asset to avoid the history dependence of the state variables. Under certain assumptions, we are able to recover the separability between tax-deferred and taxable portfolio decisions and to derive the adjustments necessary to account for the borrowing and short-selling constraints. Later on we provide further heuristic adjustments to the optimal portfolio when investors are allowed to postpone capital gains on risky asset in their taxable accounts.

## 1. Portfolio Decisions Without Postponing Capital Gains

We start by specifying the model setup for this chapter, then characterize the properties of the optimal portfolio solution under certain assumptions, and finally derive some heuristical portfolio rules for investors to follow in practice.

### 1.1. Model Setup

The model setup is similar to that specified in Chapter 2, except for the simplified return processes and tax treatments for risky assets. There are only two financial instruments available for trading at each date (or  $N = 2$ ): a risk-free bond and a risky stock that are both taxed upon accrual at different rates. Specifically, the return processes in Equation 1 and the tax treatments in Assumption 1 are replaced by the following assumption:

**Assumption 7** *There are two assets traded in both taxable and tax-deferred accounts, (i) a risky stock with i.i.d. binomial return over time, paid out in full and taxed annually at a rate  $\tau_s$  (or  $\bar{r}_s = r_S^H, r_S^L$  with equal probability, and  $\bar{d}_s = \bar{r}_s - 1$ ), and (ii) a risk-free bond with constant interest rates, also paid out in full and taxed annually at a rate  $\tau_d > \tau_s$  in taxable accounts (or  $r_B^H = r_B^L = r_f$ ).*

We denote the return on the bond as  $r_B^H, r_B^L$  since the state of the world is fully specified by the return on the stock with only two assets. The after-tax returns in taxable accounts are  $R_f = r_f - \tau_d(r_f - 1)$  and  $\tilde{R}_s = \tilde{r}_s - \tau_s(\tilde{r}_s - 1)$  for bonds and stocks respectively. This assumption abstracts away from issues of optimal capital gains realization and is introduced for tractability purposes. From the discussion of effective tax rates in Chapter 3, we can interpret  $\tau_s < \tau_d$  as reflecting the (un-modelled) optimal tax realization strategy for the risky asset under borrowing and short-selling constraints.<sup>2</sup>

Let us denote by  $W_t^T$  and  $W_t^D$  the wealth balances *after trading*<sup>3</sup> at time  $t$  in taxable and tax-deferred accounts respectively, and the total (nominal) wealth is defined as  $W_t \equiv W_t^T + W_t^D$ . It is also convenient to define the fraction  $\phi_t \equiv W_t^D/W_t$  to represent the size of tax-deferred accounts as a fraction of total wealth. Finally, let  $\theta_t, \theta_t^D$  be the percentage of total wealth  $W_t$  invested in stocks in taxable and tax-deferred accounts after trading at time  $t$ .

For simplicity we further assume that investors do not care about intertemporal consumption ( $u(C_t) = 0$ ) and cannot contribute to or withdraw from tax-deferred accounts ( $\bar{M}_t = 0$  and  $\eta_t = 1$ ). The investor's optimization problem specified in Equations (2)-(6) can be reduced to the following:

$$V_t(W_t, \phi_t) \equiv \max_{\{\theta_s, \theta_s^D\}_t^T} E_t \left[ \Gamma(\bar{W}_T, \bar{\phi}_T) \right] \quad (1)$$

where, for all  $s = t, t + 1, \dots, T$ ,

$$\bar{W}_s^T = W_{s-1}[\theta_{s-1}\tilde{R}_s + (1 - \phi_{s-1} - \theta_{s-1})R_f], \quad (2)$$

$$\bar{W}_s^D = W_{s-1}[\theta_{s-1}^D\tilde{r}_s + (\phi_{s-1} - \theta_{s-1}^D)r_f], \quad (3)$$

$$0 \leq \theta_s^D \leq \phi_s. \quad (4)$$

$$0 \leq \theta_s \leq 1 - \phi_s, \quad (5)$$

<sup>2</sup>Therefore,  $\tau_s$  should be in-between the complete-market  $\hat{\tau}_{i,t}$  from following the optimal strategy and the  $\tau_d$  from following the "worst" strategy of liquidating all capital gains immediately to be taxed at  $\tau_d$ .

<sup>3</sup>This is a slight abuse of notation since we defined wealth  $W_t^D$  as the before-trading wealth in the general model in Chapter 2 to be consistent with the notation for taxable positions. However, without considering tax cost basis, it is more intuitive to define all wealth as after-trading.

Equations (2) and (3) are the intertemporal budget constraints that describe the evolutions of the wealth in taxable and tax-deferred accounts respectively, and Equations (4) and (5) impose that investors cannot borrow or short-sell in either taxable or tax-deferred accounts. Note that Equation (5) is the only new feature introduced in this streamlined model relative to the general model.

Finally, Definition 3 for the effective tax subsidy is simplified to the following:

**Definition 5** *Let  $V_t(W_t, \phi_t)$  and  $V_t(W_t, 0)$  be the indirect utility function, at time  $t$ , of an investor who does and, respectively, does not have access to tax-deferred accounts. The effective tax subsidy per dollar in tax-deferred accounts is the value  $Z_t^{ce} = Z_t^{ce}(W_t, \phi_t)$ , such that*

$$V_t(W_t + Z_t^{ce} \phi_t W_t, 0) = V_t(W_t, \phi_t). \quad (6)$$

## 1.2. Optimal Portfolio Decisions

In general, there are interactions between the optimal location and allocation decisions under borrowing and short-selling constraints. Using intuition generated from the complete market results, we first identify conditions under which the two decisions are separable and then characterize both the location and the allocation decisions under the above conditions.

### The Location Decision

We start by applying Proposition 1-2 in complete markets to the current setting to gain some intuition about the structure of the optimal solution.

**Lemma 4** *When investors are allowed to borrow and short-sell in taxable accounts, the solution to the problem specified by Equations (1)-(4) has the following properties:*

- (i) *The optimal location decision is separable from the overall allocation decision;*
- (ii) *Investors hold only bonds in tax-deferred accounts;*



(iii) The effective tax subsidy  $Z_t^{ce}$  is state-independent and given by  $Z_t^{ce} = \left(\frac{r_f}{R_f}\right)^{T-t} - 1$ , at any time  $t = 0, 1, \dots, T$ .

This lemma can be easily proved by noting that the effective tax rates in Definition 1 for the stock and the bond are  $\tau_s$  and  $\tau_d$  respectively, and the current-period subsidy for the bond is  $r_f/R_f - 1$ , always greater than that for the stock when  $\tau_s < \tau_d$ .

When investors are not allowed to borrow or short-sell in taxable accounts, the first obvious concern is that investors may no longer be able to hold *only* bonds in tax-deferred accounts for fear of holding an extremely unbalanced portfolio, especially if the size of tax-deferred accounts is large. An intuitive remedy would be to follow a “naive pecking order” rule that places all the desired bond holding in tax-deferred accounts first, followed by stocks, if necessary, to fill up tax-deferred accounts. However, this adjustment is myopic in the sense that it takes into account only the binding current-period constraints. It is possible that the binding constraints in the future may induce investors to prefer stocks in tax-deferred accounts.

To properly address the above concern we would need to solve the whole model specified by Equations (1)-(4). Unfortunately, its complexity does not allow analytical tractability and solutions can only be obtained numerically. Instead we take a reduced-form approach to characterize the structure of the optimal solution and to understand the impact of borrowing and short-selling constraints on portfolio decisions. Using the concept of effective tax subsidy  $Z_t^{ce}$  introduced in Definition 5, we rewrite the general intertemporal problem in the following way

$$V_t(W_t, \phi_t) = \max_{(\theta_t, \theta_t^D)} E_t[V_{t+1}(\bar{W}_{t+1}, \bar{\phi}_{t+1})] \quad (7)$$

$$= \max_{(\theta_t, \theta_t^D)} E_t[V_{t+1}(\bar{W}_{t+1} + \bar{Z}_{t+1}^{ce} \bar{\phi}_{t+1} \bar{W}_{t+1}, 0)] \quad (8)$$

$$\equiv \max_{(\theta_t, \theta_t^D)} E_t[\mathcal{U}(\bar{W}_{t+1} + \bar{Z}_{t+1}^{ce} \bar{\phi}_{t+1} \bar{W}_{t+1})] \quad (9)$$

where equation (7) is the basic recursion characterizing the value function, equation (8) follows directly from Definition 5, and equation (9) follows from defining a function  $\mathcal{U}(\cdot)$  such that  $\mathcal{U}(\cdot) \equiv V_{t+1}(\cdot, 0)$ .

Lemma 4 shows that the next-period effective tax subsidy  $\tilde{Z}_{t+1}^{ce}$  is state-independent in complete markets due to replication of tax-deferred asset returns using taxable assets. With borrowing and short-selling constraints in taxable accounts, however,  $\tilde{Z}_{t+1}^{ce}$  is no longer state-independent. Specifically, it depends on individual preferences and is endogenously determined by portfolio decisions at time  $t$  as well as all future decisions embedded in future value functions. For example, consider an investor expecting a large liquidity shock in the next period. If he holds a large stock position in his taxable accounts, the investor may be forced to withdraw everything from his tax-deferred accounts to meet liquidity needs when the stock return is low, and the value of the next-period effective subsidy is zero since taxable and tax-deferred wealth become indistinguishable for the purpose of meeting liquidity needs.<sup>4</sup> On the other hand, if the stock return is high, assume his taxable wealth is enough to cover the liquidity needs and no withdrawal is necessary, then the effective tax-subsidy is clearly positive. Therefore, the next-period effective tax subsidy differs significantly across states. Contrarily, if investors hold bonds in taxable accounts for the current period, the stock return clearly has no impact on the next-period withdrawal decision to meet liquidity needs, hence, the next period effective tax subsidy is less affected by the realization of the stock returns. This example illustrates that the future effective tax subsidy is endogenously determined and clearly state-dependent under borrowing and short-selling constraints.

The endogenous nature of the future subsidy induces the interaction between location and allocation decisions for the current period. To properly account for the interaction, we again need to solve the dynamic problem exactly. Even though the current simplified return structure assumes away history dependence, the optimal solution under constraints is still very complicated. Before resorting to numerical methods for the exact solution, it is important to generate some insights regarding the impact of tax-deferred accounts on portfolio location and allocation decisions under constraints. For that purpose, we take a reduced-form approach that assumes *exogenous* but state-dependent future effective tax subsidy. Specifically, we replace the endogenous  $\tilde{Z}_{t+1}^{ce}$  with an exogenous random variable  $\bar{Z}_{t+1}$ , measurable with respect to the information

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<sup>4</sup>The subsidy may actually be less than zero if there is a withdrawal penalty.

structure of the problem. Since we assume that the return on the stock is binomial,  $\tilde{Z}_{t+1}$  will also be binomial, taking values  $Z_{t+1}^L$  and  $Z_{t+1}^H$ . The reduced form problem can hence be summarized as follows, for every  $t = 0, 1, \dots, T - 1$ :

$$v_t(W_t, \phi_t) = \max_{(\theta_t, \theta_t^D)} E_t[\mathcal{U}(\tilde{W}_{t+1} + \tilde{Z}_{t+1} \tilde{\phi}_{t+1} \tilde{W}_{t+1})] \quad (10)$$

subject to the constraints (2)-(5).

The introduction of an exogenous  $\tilde{Z}_{t+1}$  enables us to interpret the above single-period problem as a “snapshot” of our dynamic model. The following proposition shows that the separability between location and allocation decisions can be recovered under the assumption of exogenous effective tax subsidy.

**Proposition 4 (Separability)** *When investors are not allowed to borrow or short-sell in taxable accounts, if the next-period effective tax-subsidy  $\tilde{Z}_{t+1}^{ce}$  is taken as exogenous with binomial realizations  $Z_{t+1}^H$  and  $Z_{t+1}^L$  at any time  $t$ , then the optimal portfolio location decision is separable from the allocation decision.*

Recall in complete markets that the  $Z_{t+1}^H = Z_{t+1}^L = Z_{t+1:T}^*$  is state-independent and hence the optimal location decision myopically depends on the current-period effective tax subsidy only. With exogenous  $Z_{t+1}^H$  and  $Z_{t+1}^L$ , we can derive a similar sufficient statistic for the asset location decision at time  $t$  and provide conditions under which the naive pecking order rule of preferring bonds in tax-deferred accounts is violated.

**Definition 6** *At any time  $t$ , let  $z_H \equiv 1 + Z_{t+1}^H$ ,  $z_L \equiv 1 + Z_{t+1}^L$  be the relative value of a dollar in tax-deferred accounts at next period, then the adjusted effective tax subsidy is defined as*

$$\tilde{Z}_{i,t}^D \equiv \frac{1}{R_f} [(1 - q)r_i^H z_H + q r_i^L z_L] - 1. \quad (11)$$

for asset  $i = B$  or  $S$ , where  $q \equiv \frac{R_S^H - R_f}{R_S^H - R_S^L}$  is the risk-neutral probability.

**Proposition 5 (Location)** *Following conditions of Proposition 4, the optimal portfolio location is determined by a “adjusted pecking-order rule” where investors place the asset that provides*

*the highest adjusted effective tax subsidy in tax-deferred accounts first, followed by the lower adjusted tax subsidy assets, until they fill up their tax-deferred accounts.*

The proof of Proposition 5 (provided in Appendix A) is similar to that for Proposition 3 in complete markets, except that a *local* replication argument is used to highlight the borrowing and short selling constraints. Specifically, as long as investors hold an interior mix of stocks and bonds in one account, it is always possible to replicate a sufficiently small amount of wealth in the other account. For example, suppose there is an interior mix in taxable accounts, then  $\epsilon$  dollars of tax-deferred wealth invested in asset  $i$  ( $i = B$  or  $S$ ) has a payoff at time  $t + 1$  of either  $\epsilon r_i^H$  or  $\epsilon r_i^L$ . The assumption of exogenous  $\tilde{Z}_{t+1}^{ce}$  ensures that one dollar in tax-deferred accounts is always equivalent to either  $z_H$  or  $z_L$  dollars in taxable accounts. Hence  $\epsilon r_i^H$ ,  $\epsilon r_i^L$  dollars in the tax deferred account correspond to  $\epsilon r_i^H z_H$ ,  $\epsilon r_i^L z_L$  dollars in taxable accounts. If we define

$$\Delta_{i,t} \equiv \frac{r_i^H z_H - r_i^L z_L}{R_S^H - R_S^L}, \quad (12)$$

then it is easily verifiable that, for sufficiently small  $\epsilon$ , the payoff in tax-deferred accounts at time  $t + 1$  from investing  $\epsilon$  dollars in asset  $i$  can be perfectly replicated by a portfolio of  $\epsilon \Delta_{i,t}$  dollars in taxable stocks and  $\epsilon(1 + \hat{Z}_{i,t}^D - \Delta_{i,t})$  dollars in taxable bonds. To fix idea, assume the replication cost of bonds in tax-deferred accounts is higher than that of stocks ( $\hat{Z}_{B,t}^D > \hat{Z}_{S,t}^D$ ), then investors prefer bonds in tax-deferred accounts. For any positive amount of stock in tax-deferred accounts (i.e.,  $\theta_i^D \in (0, \phi_t)$ ), investors can improve their utility by decreasing  $\theta_i^D$  by a small amount  $\epsilon$  (and invest the proceeds in tax-deferred bonds) and selling bond in taxable to increase  $\theta_t$  by  $\epsilon(\Delta_{S,t} - \Delta_{B,t})$ . This adjustment increases wealth while leaving the risk exposure unaltered. In complete markets, investors can continue this adjustment as long as there are any stocks in tax-deferred. With borrow and short-selling constraints, investors may hit the borrowing constraint in taxable accounts before they remove all tax-deferred stock holdings, and as a result, they may hold an interior mixture of stocks and bonds in either taxable or tax-deferred accounts. Moreover, the local replication argument also ensures that there is interior mixture in at most one account.

Note that the adjusted pecking order suggested by Proposition 5 is different from the naive pecking order rule since the ranking of assets by their adjusted effective tax subsidy may be different from the naive ranking by their current-period tax rates. For example, if  $z_H \neq z_L$ , it is possible that the stock generates a higher adjusted effective tax subsidy than the bond, and investors prefer stocks in tax-deferred accounts, even though the stock is taxed at a lower rate than the bond. This “violation” of the naive pecking pecking rule reflects the adjustment to optimal location decisions for binding future-period constraints. Since the adjusted effective tax subsidy of the bond is higher than that of the stock ( $\widehat{Z}_{B,t}^D \geq \widehat{Z}_{S,t}^D$ ) if and only if  $\frac{z_H}{z_L} < \bar{z}$ , where

$$\bar{z} \equiv \frac{R_s^H - R_f r_f - r_L^L}{R_f - R_s^L r_s^H - r_f}, \quad (13)$$

we can restate Proposition 5 in the following corollary:

**Corollary 3** *Following conditions of Proposition 4, the optimal portfolio location can be characterized as follows:*

1. *If  $\frac{z_H}{z_L} < \bar{z}$ ,  $(\theta_t, \theta_t^D) \in ([0, 1 - \phi_t] \times \{0\}) \cup (\{1 - \phi_t\} \times [0, \phi_t])$ , i.e., investors prefer bonds in tax-deferred accounts (naive pecking order).*
2. *If  $\frac{z_H}{z_L} \geq \bar{z}$ ,  $(\theta_t, \theta_t^D) \in (\{0\} \times [0, \phi_t]) \cup ([0, 1 - \phi_t] \times \{\phi_t\})$ , i.e., investors prefer stocks in tax-deferred accounts (violation of the naive pecking order).*

The corollary states that, when the future effective tax subsidy  $\tilde{Z}_{t+1}^{ce}$  is taken as exogenous, we can limit the search of optimal portfolios from the two-dimensional space of all feasible portfolios  $(\theta_t, \theta_t^D) \in [0, 1 - \phi_t] \times [0, \phi_t]$  to the above one-dimensional subsets. In particular, if  $z_H/z_L < \bar{z}$ , the search for the optimal portfolio can be restricted either (i) to the set of portfolios investing 100% of tax-deferred accounts in bonds ( $\theta_t^D = 0$ ) and an amount  $\theta_t \in [0, \phi_t]$  of wealth in taxable stocks or (ii) to the set of portfolios investing 100% of taxable wealth in stock ( $\theta_t = 1 - \phi_t$ ) and amount  $\theta_t^D \in [0, \phi_t]$  in tax-deferred stocks. On the other hand, if  $z_H/z_L \geq \bar{z}$  the search of the optimal portfolio can be limited to the set of portfolio investing either 0% taxable wealth in stock ( $\theta_t = 0$ ) or 100% of tax-deferred wealth in stocks ( $\theta_t^D = 1 - \phi_t$ ). Contrasting to the

complete market location recipe of holding *only* bonds in tax-deferred accounts, the first case can be interpreted as a pecking order adjustment to account for borrowing and short-selling constraints in the current period and the second case indicates a *violation* of the naive pecking order due to the possibility of binding future constraints.

The simple pecking order rule by the adjusted effective tax subsidy is only a result of the *exogenous* future effective tax subsidy. Recall in the previous example with large liquidity needs, if investors hold 100% of his total portfolio in bonds (or  $\theta_t = \theta_t^P = 0$ ) then uncertainty does not affect the state in the next period and, by construction,  $z_H/z_L = 1$ . On the other hand, if investors hold large stock positions in taxable accounts, then the evolution of the uncertain stock return affects the withdrawal decision to meet liquidity needs, and the relative value of a dollar invested in tax-deferred accounts and  $z_H/z_L > 1$ . In general, asset allocation (i.e. a large or small stock holding) can affect the value of a dollar placed in tax-deferred accounts (i.e. the ratio of  $z_H/z_L$ ) and, ultimately, the asset location decision. Moreover, it is possible to have interior mixture of stocks and bonds in both accounts.<sup>5</sup>

Nonetheless, the simple location decision in the proposition provides some useful insights regarding when and why the pecking order of preferring the most taxed asset (i.e. the bond, in our model) in tax-deferred accounts might be violated. Specifically, we can think of the total effective subsidy as composed of two parts: the subsidy from investing in tax-deferred accounts for the *current* period and the subsidy from investing for all *future* periods. To isolate the current tax subsidy, let us consider  $z_H = z_L = z$ . In this case the future subsidy per tax-deferred dollar is set to  $z - 1$ , whereas the per-dollar subsidy from the current period is either  $\frac{zB_{s,t}}{z} - 1 = \frac{r_f}{R_f} - 1$  or  $\frac{zS_{s,t}}{z} - 1 = (r_f - 1)(1 - \frac{\tau_d - \tau_s}{1 - \tau_s})$  (from (11)), depending on whether bonds or stocks are invested in tax-deferred accounts. Since the future tax-subsidy is independent of the asset location ( $z_H = z_L = z$ ), only the current tax-subsidy determines the choice of which assets to put in tax-deferred accounts. The current period subsidy increases with the tax-rate of assets in tax-deferred accounts and, being  $\tau_s < \tau_d$ , investors always prefer bond in tax-deferred

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<sup>5</sup>Following the example of the previous paragraph, it is possible that for low values of  $\theta_t$ ,  $z_H/z_L < \bar{z}$  and bonds are preferred in tax-deferred accounts. However, for high values of  $\theta_t$ , the ratio  $z_H/z_L > \bar{z}$  and stocks are preferred in tax-deferred accounts. As a consequence an interior mix of stock and bonds in both accounts can emerge as an optimal portfolio.

accounts. On the other hand, if  $z_H \neq z_L$ , the future effective subsidy is no longer neutral on the current choice of which asset to prefer in the tax deferred account. For example, when  $z_H > z_L$ , a stock would return more in states with higher future subsidy and less in states with lower future subsidy. Hence, if we were to focus only on the future subsidy, stocks will be preferred in tax-deferred accounts. The overall location decision depends on the balance of current vs. future subsidy. When  $z_H/z_L < \bar{z}$ , the current subsidy is larger than the future subsidy and bonds are preferred in tax-deferred accounts; on the other hand, when  $z_H/z_L > \bar{z}$ , the future subsidy effect dominates and stocks are preferred (violating the pecking order of locating the highest tax-rate asset first in tax-deferred accounts).<sup>6</sup>

The above propositions show that, when investors are not allowed to borrow or short-sell in taxable accounts, instead of holding only the highest tax rate assets in tax-deferred accounts, investors need both to adjust their complete-market definition of effective tax rates and to follow an adjusted pecking order rule in that holds the highest adjusted effective tax rate asset in tax-deferred accounts first, followed by the lower adjusted tax rate asset, until they fill up their tax-deferred accounts. Most importantly, the adjusted effective tax rate is no longer preference independent. Moreover, the *exogenous* future effective tax subsidy  $\bar{Z}_{t+1}$  allows us to reduce the multi-period problem (2) to a sequence of single-period problems in which location and allocation decisions are separable. In the next section, we transform the general portfolio problem with taxable and tax-deferred account into a single-account-only problem and derive the solution based on any given single-account solution.

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<sup>6</sup>Alternatively, we can interpret  $z_H$  and  $z_L$  as marginal utilities in high and low state. An investor is indifferent between stocks and bonds when

$$qz_H + (1 - q)Zz_L = qz_H r_f + (1 - q)z_L r_f,$$

where  $q$  is defined in (12). Let  $\bar{z}_H$  and  $\bar{z}_L$  be the levels of  $z$  such that this equality holds. (Note that  $\bar{z}_H/\bar{z}_L = \bar{z}$ , by Proposition 5). Then if  $z_H > \bar{z}_H$  and/or  $z_L < \bar{z}_L$ ,

$$q(z_H - \bar{z}_H)(r_H - r_f) + (1 - q)(z_L - \bar{z}_L)(r_L - r_f) > 0,$$

implying that if  $z_H/z_L > \bar{z}$  stocks will be preferred in the tax-deferred account.

## The Allocation Decision

Having addressed the location decision in the presence of borrowing and short-selling constraints, we now turn our attention to the question of how to determine the optimal portfolio allocation. The two cases derived in Corollary 3 lead to two corresponding allocation decisions. For ease of exposition, we will focus the rest of our analysis only on the case in which  $z_H/z_L$  is less than  $\bar{z}$ .<sup>7</sup>

Instead of solving for the taxable and tax-deferred holding simultaneously, we address the optimal allocation problem by transforming a two-account problem into a mixture of two single-account problems for which we assume to know the desired allocation. To clarify our approach, suppose, for instance, that an investor hold  $\$X$  in his taxable account and  $\$Y$  in his tax-deferred account. As an input, we need the investor's preferred allocation between stock and bonds when only one account (taxable or tax-deferred) is available. This allows us to determine the two-account solution consistent with the preferences that justify the original single-account allocation. The advantage of this approach is that it does not require any information about investors' preference structure except for the desired single account portfolio allocation.<sup>8</sup> Furthermore, compared to the alternative of relying on numerical solutions of specific versions of the general problem in (2)-(4), our approach allows us to characterize the general impact of tax-deferred accounts on overall portfolio allocation.

We start by defining the optimal portfolios  $\bar{\theta}_t^T(W_t)$  and  $\bar{\theta}_t^D(W_t)$  when investors have only a taxable and a tax-deferred account respectively. Formally,

**Definition 7**  $\bar{\theta}_t^T(W_t)$  is the solution of the “taxable-account-only” problem (i.e., problem (2) subject to the constraints (2) to (4) where  $\phi_t = 0$ ). Similarly,  $\bar{\theta}_t^D(W_t)$  is the solution of the “tax-deferred-account-only” problem (i.e., the same problem except that  $\phi_t = 1$ ).

Before presenting our main result, let us first consider the case in which investors are free to borrow and short-sell and the evolution of the effective tax subsidy  $Z_t^{ce}$  is taken as exogenous. For analogy with the complete market environment, we will refer to this case as the “ $Z$ -complete market”. The following lemma characterizes the optimal allocation decision in a two-account

<sup>7</sup>The solution for the case  $z_H/z_L > \bar{z}$  is fully derived in Appendix A.

<sup>8</sup>In practice investors receive advice from financial planners regarding optimal portfolio mix even though it is very hard to recover the preference structure justifying such allocations.



problem given any optimal taxable-account-only portfolio  $\bar{\theta}_t^T$  and will be useful in the derivation of the optimal allocation with borrowing and short-selling constraints.

**Lemma 5 (“Z-Complete Markets”)** *Let  $\bar{\theta}_t^T(W_t)$  be any taxable-account-only portfolio strategy as in Definition 7 and assume investors are allowed to borrow and short-sell. If, at any time  $t$ , the realizations  $Z_{t+1}^H$  and  $Z_{t+1}^L$  of the effective tax-subsidy  $\tilde{Z}_{t+1}^{ce}$  are taken exogenously to satisfy  $(1 + Z_{t+1}^H)/(1 + Z_{t+1}^L) \equiv z_H/z_L < \bar{z}$ , then the optimal portfolio strategy is*

$$\hat{\theta}_t = \bar{\theta}_t^T(\widehat{W}_t) [1 + \phi_t(z_{B,t} - 1)] - \phi_t \Delta_{B,t}, \quad \hat{\theta}_t^D(W_t, \phi_t) = 0. \quad (14)$$

where  $\widehat{W}_t = W_t(1 + \phi_t(z_{B,t} - 1))$  and  $z_{B,t}$  and  $\Delta_{B,t}$  are given in equation (11).

The Lemma shows that the optimal portfolio  $\hat{\theta}_t$  in equation (14) can be decomposed into two terms. The first term,  $(1 + \phi_t(z_{B,t} - 1))$ , adjusts for the “wealth effect” due to the effective tax subsidy generated by holding bonds in tax-deferred accounts. An investor with  $(1 - \phi_t)W_t$  dollars in taxable accounts and  $\phi_t W_t$  in the tax-deferred is as well off as an investor having  $W_t[1 + \phi_t(z_{B,t} - 1)]$  and no access to a tax-deferred account. Given the optimal taxable-account-only portfolio  $\bar{\theta}_t^T(\widehat{W}_t)$ , the dollar stock holding for this investor is  $\bar{\theta}_t^T(\widehat{W}_t) \times \widehat{W}_t$  and hence the first term follows from the definition of  $\widehat{W}_t$  and from the fact that  $\hat{\theta}_t$  is expressed as a fraction of nominal wealth  $W_t$ . The second term is the “hedging demand” that accounts for the fact that the effective return from tax-deferred accounts, even if invested only in bonds, is inherently correlated with stock returns due to the correlation between the process describing the evolution of the effective tax subsidy and the asset returns. The term  $\Delta_{B,t}$  captures the stock holding already implicit in a bond position in the tax-deferred and hence needs to be netted out from the total stock holding.

In our analysis of the optimal location decision, we pointed out how, by assuming an exogenous tax subsidy  $\tilde{Z}_{t+1}$ , we can recover the separability of location and allocation decision and, ultimately rely on a pecking order adjustment for the optimal portfolio location. In light of this and of Lemma 5, it is natural to think that a natural approach to solve for the optimal allocation with borrowing and short-selling constraints would be to mimic the complete market steps of

(i) calculating the effective tax-subsidy from implementing the pecking order location strategy, (ii) re-expressing the two-account problem as a taxable-account-only problem with “effective wealth” equal to the taxable wealth augmented by the effective tax subsidy and (iii) solving for the optimal allocation in the taxable-account-only problem.

It is immediate to see that, if we try to implement this procedure when investors follow a pecking order location strategy with a mixture of stock and bonds ( $\theta_t^D$  and  $\phi_t - \theta_t^D$ ,  $\theta_t^D \in (0, \phi_t)$ ) in tax-deferred accounts (i.e. the borrowing constraint in taxable accounts is binding), the above procedure becomes considerably complex since the determination of the effective tax-subsidy in step (i) will be a weighted average of the replication cost of the stock and the bond position held in tax-deferred accounts (i.e.,  $z_{B,t}$  and  $z_{S,t}$ ), with weights equal to  $\theta_t^D$  and  $\phi_t - \theta_t^D$  respectively. But this will make the effective tax-subsidy, used to determine the optimal allocation, dependent to the allocation itself. The optimal allocation problem will therefore become a complex fixed-point problem. Moreover, and most importantly, this heuristic procedure does not lead to the correct solution.<sup>9</sup>

The complexity of the above approach arises only when there is an interior mix in tax-deferred accounts. However, from Proposition 5 we know that there can be *at most* one account with an interior mix of stock and bonds at any point in time. Specifically, when there is a mix in tax-deferred accounts, investors hold only a single asset in taxable accounts. If investors change their view-point and think of investing in a tax-deferred world, a taxable account then imposes a “tax cost”. Using the intuition of the  $Z$ -complete market, investors can calculate this effective tax cost without knowing the overall portfolio allocation.

Proposition 6 formalizes this intuition and derive the optimal portfolio allocation with borrowing and short-selling constraints by transforming the original two-account problem into two *constrained* single-account problems.

**Proposition 6 (Portfolio Allocation)** *Let  $\bar{\theta}_t^T(W_t)$  and  $\bar{\theta}_t^D(W_t)$  be any single-account portfolio*

<sup>9</sup>To note this, contrast the solution from the fixed-point when  $\phi_t = 1$  (tax-deferred only problem) and the solution in Proposition 6 below. It is easy to show in fact that the fixed point in the complete market argument when  $\phi_t = 1$  is  $\hat{\theta}_t^D = \frac{\bar{\theta}_t^D(1+z_{B,t})}{1+\bar{\theta}_t^T(z_{B,t}-z_{S,t})+(\Delta_{B,t}-\Delta_{S,t})} - \Delta_{B,t}$  which differs from the true optimum  $\bar{\theta}_t^D$  unless for a knife-edge case.

lio strategy as in Definition 7. When investors are not allowed to borrow or short-sell and, at any time  $t$ , the realizations  $Z_{t+1}^H$  and  $Z_{t+1}^L$  of the effective tax-subsidy  $\bar{Z}_{t+1}^{ce}$  are taken exogenously to satisfy  $(1 + Z_{t+1}^H)/(1 + Z_{t+1}^L) \equiv z_H/z_L < \bar{z}$ , the optimal portfolio strategy is the one between the following two that provides the highest expected utility level.

$$\begin{cases} \theta_t(W_t, \phi_t) = \min\{1 - \phi_t, \max\{0, \hat{\theta}_t\}\}, & \theta_t^D(W_t, \phi_t) = 0. \\ \theta_t(W_t, \phi_t) = 1 - \phi_t, & \theta_t^D(W_t, \phi_t) = \max\{0, \min\{\phi_t, \hat{\theta}_t^D\}\}, \end{cases} \quad (15)$$

where  $\hat{\theta}_t$  is given in equation (14),  $\hat{\theta}_t^D$  is defined as

$$\hat{\theta}_t^D = \bar{\theta}_t^D(\widehat{W}_t^D) \left[ \phi_t + (1 - \phi_t)z_{S,t}^D \right] - (1 - \phi_t)\Delta_{S,t}^D, \quad \widehat{W}_t^D = W_t(\phi_t + (1 - \phi_t)z_{S,t}^D), \quad (16)$$

with  $z_{S,t}^D$  and  $\Delta_{S,t}^D$  defined in Appendix A.

The proposition highlights the fact that the optimal portfolio problem in the presence of borrowing and short selling constraints is a mixture of two “polar cases”: the constrained taxable-account only problem (first part of equation (15) which is a constrained version of the  $Z$ -complete market solution in Lemma 5) and the constrained tax-deferred-only problem (second part of equation (15)). Intuitively, the upper case in equation (15) is more likely to occur when the balance in tax-deferred accounts is small and the lower case is more likely for larger tax-deferred account.

The two polar cases in the above proposition emerge from the fact that Proposition 5 reduces the space of optimal portfolios  $(\theta_t, \theta_t^D)$  to the two lines  $([0, 1 - \phi_t] \times \{0\}) \cup (\{1 - \phi_t\} \times [0, \phi_t])$  with each line corresponding to each polar case. For example, along the first line, investors solving problem (10) are free to choose a portfolio  $\theta_t$  in taxable accounts as long as it satisfies  $\theta_t \in [0, 1 - \phi_t]$  and are “forced” to hold only bonds in tax-deferred accounts ( $\theta_t^D = 0$ ). In Appendix A we show that this problem is equivalent to a constrained “taxable-account-only” problem with effective taxable wealth  $\widehat{W}_t = W_t^T + z_{B,t}W_t^D$  and where the optimal portfolio  $\bar{\theta}_t$

satisfies the following constraints:

$$\frac{\phi_t \Delta_{B,t}}{1 - \phi_t + \phi_t z_{B,t}} \leq \bar{\theta}_t \leq \frac{1 - \phi_t + \phi_t \Delta_{B,t}}{1 - \phi_t + \phi_t z_{B,t}}. \quad (17)$$

The main intuition is that, being forced to hold only bonds in tax-deferred accounts is equivalent to be endowed with a payoff  $r_f W_t^D z_L$  and  $r_f W_t^D z_H$  in taxable accounts which, in turn, is equivalent to holding a *non-tradeable* portfolio of  $\Delta_{B,t} W_t^D$  stocks and  $(z_{B,t} - \Delta_{B,t}) W_t^D$  bonds in taxable accounts. The constraint (17) is derived by imposing that (i) the dollar amount invested in stock  $\bar{\theta}_t \widehat{W}_t$  is at least as much as the dollar amount of stocks  $\Delta_{B,t} W_t^D$  in the equivalent non-tradeable portfolio and (ii) the dollar amount invested in bonds  $(1 - \bar{\theta}_t) \widehat{W}_t$  is at least  $(z_{B,t} - \Delta_{B,t}) W_t^D$ . The optimal portfolio in (15) follows from imposing the constraints on the solution (14) to the  $Z$ -complete market solution.

Similar argument applies for the case in which investors hold only stocks in taxable accounts ( $\theta_t = 1 - \phi_t$ ) and select a portfolio  $\theta_t^D \in [0, \phi_t]$  (i.e. the second line derived in Proposition 5). The problem investors solve in this scenario is equivalent to a constrained “tax-deferred-account-only” problem in which they are endowed with a non-tradeable portfolio of  $\Delta_{S,t}^D W_t^T$  stocks and  $(z_{S,t}^D - \Delta_{S,t}^D) W_t^T$  bonds in tax-deferred accounts. The optimal tax-deferred stock holding  $\theta_t^D$  in equation (16) can be similarly decomposed into two terms. The first term,  $1 + (1 - \phi_t) z_{S,t}^D$ , adjust for the “wealth effect” induced by the (negative) effective tax-subsidy<sup>10</sup> from holding stocks in taxable accounts. The second term is the “hedging demand” accounting from the fact that holding only stocks in taxable accounts is equivalent of holding a non-tradeable portfolio of  $\Delta_{S,t}^D W_t^T$  stocks and  $(z_{S,t}^D - \Delta_{S,t}^D) W_t^T$  bonds in tax-deferred accounts. The term  $(1 - \phi_t) \Delta_{S,t}^D$  precisely adjusts for this.

The simple structure of the solution in Proposition 5 and 6 is surprising given the complexity of the original problem. The results largely rely on the assumption of an exogenous process for the effective tax subsidy which removes the interaction both between different time periods and between location and allocation decisions. Although in general these interactions exists, the

<sup>10</sup>The effective subsidy  $(z_{S,t}^D - 1)$  is negative but the “replication cost”  $z_{S,t}^D$  is positive, and less than one. See Appendix A for details.

reduced-form approach allows us to identify the evolution of the effective tax-subsidy as the key determinant of the optimal portfolio decisions when there are borrowing and short-selling constraints. More importantly, by approximating the effective tax-subsidy, we derive in the next section rules of thumb regarding portfolio choices that can be easily implemented in practice.

### 1.3. Implementation of Optimal Portfolio Rules

To implement the results derived in Propositions 5 and 6 we need as an input the future-period effective tax subsidy  $\tilde{Z}_{t+1}^{ce}$ , which measures the wealth transfer that would make an investor deprived of his tax deferred account indifferent to an investor having the opportunity to defer taxes in a retirement account. The only way to derive the exact effective tax subsidy  $\tilde{Z}_{t+1}^{ce}$  is by solving fully the optimization problem in (1), which defeats the purpose of an approximation approach. Instead, we take an “educated guess” on the structure of the effective tax subsidy and use it to construct approximate portfolio rules.

Proposition 5 provides conditions for the validity of the pecking order location strategy of preferring bonds in tax-deferred accounts. The conditions rely on the magnitude of the ratio of the relative values of a dollar in a tax-deferred account in the two possible future states of the world ( $z_H/z_L$ ). A large (small) ratio implies that stocks (bonds) are preferred in tax-deferred accounts. Although the process for the effective tax subsidy is assumed to be exogenous and both conditions are possible, in general we expect the ratio  $z_H/z_L$  to be less than  $\bar{z}$ .<sup>11</sup> For implementation purposes we therefore take the naive pecking order rule of preferring the bond in tax-deferred accounts as an approximation of the optimal location strategy with borrowing and short selling constraints, as summarized below.

**Rules of Thumb 5 (Location)** *If borrowing and short-selling are not allowed in taxable accounts, the optimal location decision can be approximated by a naive pecking order rule where investors rank assets by their current-period effective tax rates defined in complete markets in placing assets to their tax-deferred accounts.*

Given the detailed discussion in the previous section regarding the conditions for the violation

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<sup>11</sup>Note that  $\bar{z} > 1$ .

of the naive pecking order rules, it may seem strange to rely on the above rules of thumb for general location decisions. However, in absence of an event that generates considerable asymmetry between the high and the low states of the world (like a large liquidity needs discussed previously), it is unlikely to have a large  $z_H/z_L$  ratio. For example, consider a special case in which the process driving the effective tax subsidy is described by a recombining binomial tree. Since  $z_H$  and  $z_L$  are the average expected value of future tax subsidies, in a recombining tree  $z_H$  and  $z_L$  share most of the future paths (roughly, only the extreme “ups” are unreachable from  $z_L$  and only the extreme “downs” are unreachable from  $z_H$ ). This means that in general, for recombining trees the difference between  $z_H$  and  $z_L$  is unlikely to be high.<sup>12</sup> Therefore, although it is possible for investors to violate the naive pecking order in anticipation of binding future-period constraints, the violation is rare.

To construct approximations to the optimal portfolio allocation derived in Proposition 6, we take as a given the single-account portfolio choices  $\bar{\theta}_t^T$  and  $\bar{\theta}_t^D$ , and further bound the next-period effective tax subsidy  $\bar{Z}_{t+1}^{ce}$  using the subsidy implied by the “best” and “worst” portfolio strategy, where the best strategy correspond to the complete market rule of investing the entire tax-deferred wealth in bonds while the worst portfolio rule calls for investing all tax-deferred wealth in stocks for all the remaining periods. Since the portfolio holdings are linear in the effective tax subsidy, bounding the subsidy is equivalent to bounding the portfolio holdings.

**Rules of Thumb 6 (Allocation)** *If borrowing and short-selling are not allowed in taxable accounts, the optimal allocation decision can be approximately bounded by substituting the following two sets of effective subsidies into Equations (15) and (16) in Proposition 6: (i)  $z_H = z_L = 1 + Z_t^B$ , and (ii)  $z_H = z_L = 1 + Z_t^S$ , where*

$$Z_t^B = \left( \frac{r_f}{R_f} \right)^{T-t} - 1, \text{ and } Z_t^S = \left( \frac{r_{fS}}{R_f} \right)^{T-t} - 1 \quad (18)$$

where  $r_{fS} = r_f - \frac{\tau_d - \tau_s}{1 - \tau_s} (r_f - 1)$ .

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<sup>12</sup>We do not have a characterization for non-recombining trees but the intuition is that, unless large liquidity shock are introduced, it is very difficult to generate a large wedge between  $z_H$  and  $z_L$  in a model with binomial uncertainty.

Note that if investors are risk averse and never need to withdraw from tax-deferred accounts before time  $T$ , the effective tax subsidy  $Z_t^{ce}$  can be bounded by  $Z_t^S \leq Z_t^{ce} \leq Z_t^B$  for all  $t = 0, \dots, T$ . Consistent with Rules of Thumb 5, we ignore the possible difference between  $z_H$  and  $z_L$  for simplicity. The above approximate approach does not rely on any specific characterization of preferences, as long as investors can specify their desired single-account allocation rules.

To illustrate our approach we provide an example of how to determine the optimal allocation strategy assuming that investors follow a specific allocation rule for a single account problem. For ease of exposition let us consider the popular “100–Age” rule in which a household adjust his portfolio through time by allocating  $(100 - \text{Age})\%$  of his wealth to stocks.<sup>13</sup>

Figure C-5 and C-6 report the optimal portfolio in the taxable (Panel A) and tax-deferred (Panel B) account for different size  $\phi$  of the tax deferred account along with the approximate portfolio obtained by using the mapping derived in Proposition 6. The dotted lines describe the portfolio allocation of an investor following the “100–Age” rule in both accounts. The solid lines are obtained by assuming that  $z_H = z_L = 1 + Z_t^B$  while the dash-dotted line corresponds to the case of  $z_H = z_L = 1 + Z_t^S$ . We notice that the upper and lower bounds of the portfolios obtained by following the approximate rules are relatively close to each other while differing dramatically from the ones resulting from the single account solution specified by the “100–Age” rule. This suggests that even with a coarse estimate of the future effective tax subsidy we can improve upon a naive application of single-account rules for the two-account solution. In the next section we further assess the performance of these approximate rules by comparing them against the true optimal solution for a particular version of the general model.

## 2. Portfolio Decisions With the Option to Postpone Gains

Relying on the simple structure of heuristic portfolio rules under borrowing and short-selling constraints when investors are not allowed to postpone capital gains, we now derive further adjustments to the optimal portfolio rules in order to account for the option to postpone capital gains in taxable accounts. The main objective of this analysis is to provide useful rules of

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<sup>13</sup>See Burton Malkiel (1990).

thumb for practical purposes. Although the option to postpone capital gains can be easily accounted for by a lower effective tax rate on capital gains in complete markets, the impact on the optimal portfolio decisions is more complicated under borrowing and short-selling constraints. Specifically, investors may be forced to liquidate positions with embedded capital gains whenever they need to reduce their risk exposure, and as a result, the size of effective tax subsidy and the corresponding effective tax rate depend on individual characteristics like existing taxable positions, risk aversion, and future trading strategies. For example, if risk averse investors have large existing taxable positions in asset  $i$ , they are more likely to reduce these taxable positions to unload risk if they choose to also hold asset  $i$  in tax-deferred accounts, therefore, the effective tax subsidy for holding asset  $i$  should be lower for these investors when there are embedded capital gains on the taxable positions. To be able to use the simple approximation rules derive previously, we follow the complete market logic of adjusting the definition of effective tax rates both for the option to postpone gains and for the existing embedded capital gains and future trading strategies.

**Rules of Thumb 7 (Embedded Capital Gains)** *If borrowing and short-selling are not allowed in taxable accounts, and an investor needs to liquidate an extra  $\Delta$  dollars of asset  $j$  with cost basis  $y$  for every dollar of asset  $i$  introduced in tax-deferred accounts for risk re-balancing purposes, then the adjusted current-period effective tax rate,  $\hat{\tau}_{i,t}$ , for asset  $i$  solves the following equation:*

$$\max \left\{ 0, Z_{i,t}^D - \Delta [z_{j,t}(y) - (1 - \tau_g(1 - y))] \right\} = \frac{1}{R_f} \left[ 1 + \frac{R - 1}{1 - \hat{\tau}_{i,t}} \right] - 1 \quad (19)$$

where  $z_{j,t}(y)$  and  $Z_{i,t}^D$  are defined in Equation (2) and (4) respectively.

In theory, the above  $\Delta$  measures the *extra* re-balancing needs induced by adding tax-deferred holding of asset  $i$  and should be endogenously determined by trading off the utility costs of an unbalanced portfolio and the tax cost of realizing the embedded capital gains prematurely ( $\Delta [z_{j,t}(y) - (1 - \tau_g(1 - y))]$ ). Deriving the exact  $\Delta$  would require solving the overall portfolio problem, contradicting the purpose of the approximation approach. Moreover, the complete



market  $z_{j,t}(y)$  is an overstatement of the value of the current taxable position since investors may optimally choose to liquidate the position in the future even if they do not liquidate it immediately. The rule of thumb can be viewed as a shortcut to approximate the impact of existing taxable positions on the optimal location decision and there are several intuitive guidelines for the implementation, for example, asset  $j$  is usually the same as asset  $i$ , or at least they are highly correlated, such that investors need to reduce their taxable exposure to asset  $j$  in order to maintain a balanced portfolio if they increase their holding of asset  $i$  in tax-deferred accounts. Moreover,  $\Delta$  is usually higher for more risk-averse investors since the cost of an unbalanced portfolio is higher, and lower for larger embedded capital gains on the existing positions, since it is more costly to reduce the taxable position.

Figure C-7 presents the adjusted effective tax rates defined in the above rule of thumb as functions of the embedded capital gains on existing assets when  $i = j$  and  $\Delta$  is exogenously given. Panel A and B reports the case when  $\Delta = 1$  and  $.5$  respectively. The adjusted effective tax rates reduces significantly with the embedded capital gain, especially for short term investors who can escape terminal capital gains taxes, for example, the dotted line in Panel B indicates that even if investors only need to liquidate half a share of taxable position for every share of asset  $i$  held in tax-deferred accounts ( $\Delta = .5$ ), the effective tax rate drops from 15% to below zero when the embedded gain is only about 20% ( $y = .8$ ). Investors prefer not to hold asset  $i$  in tax-deferred accounts since the perspective of escaping the embedded taxes is attractive enough to outweigh the benefit of saving the taxes on future dividend payments. When investors are forced to realize capital gains on the terminal dates, liquidating the asset in order to buy in the tax-deferred is less costly, since investor only lose the time value of the tax payment, for example, the solid line in Panel B indicates that investors only decide against liquidating existing taxable position to invest in tax-deferred accounts when the embedded capital gains is close to 100% ( $y = 0$ ). Given the significant impact of existing taxable positions on the effective tax rate, it is safe to conclude that when investors have a large position in an asset with sizeable embedded capital gains, they are less likely to hold that asset in tax-deferred accounts. Moreover, it is more likely to have  $\hat{\tau}_{i,t} < 0$  for a given asset, implying that it is no longer beneficial to place the

asset in tax-deferred accounts if investors need to sell the same asset in taxable accounts with embedded capital gains to balance the risk exposure.

The future trading strategy for an investor also matters for the optimal location decision when borrow and short-sell are not allowed. For example, if an investor plans to follow a buy-and-hold strategy for asset A to eventually bequeath it to their beneficiaries, and to buy asset B for speculative purpose to be liquidated within a year. Intuitively, the appropriate investment horizon for asset A is the expected life expectancy and the terminal tax treatment is step-up of tax basis, on the other hand, the appropriate investment horizon for asset B is less than one year and the terminal tax treatment is forced realization of capital gains taxes. It is easy to see that the effective tax rate for asset A is in general lower than that for asset B and investors should prefer asset B in tax-deferred accounts. However, if the investor also expects a large liquidity need in 5 years which may require him to liquidate all taxable wealth, the asset A held in taxable accounts will then be liquidated in 5 years and the appropriate investment horizon for asset A is actually five years and tax treatment is forced realization of capital gains. Investors again need to compare the effective tax rates on A and B to derive the optimal location decision. In summary,

**Rules of Thumb 8 (Exogenous Trading Strategy)** *If borrowing and short-selling are not allowed in taxable accounts, and investors follow different trading strategies for different assets, then the adjusted current-period effective tax rate of each asset  $i$  should be calculated using the corresponding investment horizon and terminal tax treatment for that specific asset.*

As indicated by the previous example, the rule of thumb is not perfectly specified and investors may need to go through several iterations to derive the appropriate effective tax rates and the optimal location decision. Nonetheless, the above approximation provides a good starting point to incorporate the individual trading strategy into their optimal location decisions.

The ranking of the assets by their effective tax rates differs among investors due to individual characteristics like investment horizon, terminal tax treatments, existing taxable positions and future trading strategies. Moreover, the ranking is defined period by period and the same investor may have different ranking over time due to changes in the above characteristics and

the optimal location and allocation decisions should be adjusted over time correspondingly.



## Chapter 5

# Optimal Portfolio in Incomplete Markets: A Numerical Approach

We have derived closed-form solutions in Chapter 3 for the impact of tax-deferred accounts on optimal portfolio decisions when investors are allowed to borrow and short-sell in their taxable accounts, and have provided heuristic adjustments in Chapter 4 to the optimal portfolio when borrowing and short-selling are not allowed in taxable accounts. The implementation of the heuristic rules, however, relies on the knowledge of the value of tax-deferred accounts under constraints, which can only be obtained after solving explicitly the optimization problem under constraints.

In this chapter we solve numerically a version of the general model to understand the impact of borrowing and short-selling constraints on individual decisions in a dynamic setting. We first back out from the indirect utility function the implied effective tax-subsidy that investors receive for being able to invest in tax-deferred accounts, relating the results to the closed-form solutions whenever possible. By understanding the impact of constraints on the effective subsidy for this specific version of the general model, we are able to generate insights regarding the implementation of heuristic portfolio rules in more realistic situations. We then present the structure of the optimal portfolio and evaluate the performance of the heuristical rules by comparing the range of approximate portfolios to the optimal solution and by comparing

the utility loss of following these approximate rules relative to following some intuitive sub-optimal rules. Finally, we simulate time-series of stock returns to illustrate the dynamics of the consumption pattern and portfolio choices over the life-cycle and to address some public finance issues like who benefits most from the existence of tax-deferred accounts, and whether and to what extent tax-deferred investing encourages saving.

## 1. Methodology

We first specify the model as a simplified version of the general model in Chapter 2, then choose the set of parameters values used in the calculation, and finally describe the numerical procedure.

### 1.1. Model Setup

To solve the model numerically, we need to make simplifying assumptions regarding asset return processes and investor preferences in the general model. For realism, we also introduce new features that are omitted in the general model, like inflation rate, labor income, liquidity needs, and random investment horizon due to mortality rates.

We follow Assumption 7 in Chapter 2 to assume that stocks are taxed upon accrual at a fixed rate. As illustrated in the literature<sup>1</sup>, solving the general model with exact tax basis is formidable, if not impossible. One common approach<sup>2</sup> is to use the average cost basis of risky assets to avoid the path-dependency issue. However, using average cost under-estimates the value of tax timing option and it is hard to assess the performance of this approximation. Moreover, Chapter 3 has provided some heuristic rules to account for the value of postponing capital gains in complete markets and Chapter 4 focuses on the impact of borrowing and short-selling constraints on the optimal portfolio taking as a given the adjustment for the option to postpone gains. Although, when investors are allowed to postpone gains, they would prefer to short new asset instead of liquidating assets with embedded capital gains, making the borrowing and short-selling constraint more binding, and the complete market definition of effective tax

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<sup>1</sup>For example, Dybvig-Koo (1996), DeMiguel-Uppal (2002).

<sup>2</sup>For example, Dammon-Spatt-Zhang(2002)

rate only provides a lower bound on the realized effective tax rate, the computational complexity introduced in order to properly account for the interactions between these factors does not justify the explicit modelling of the tax basis.

## Preferences and Value Function

Although theoretically, it is hard to deal with mortality risk since we need to explicitly trade insurance contracts on the risk in order to complete the market and to derive analytical solutions, it is rather easy to include the mortality risk in a numerical setting. Let  $\lambda_t$  denote the probability that an investor survives at time  $t$ . Incorporating the simplified stock return process and the mortality rates, we can modify the basic recursion for value functions in Equations (2) to derive the following:

$$V_t(W_t, \phi_t) \equiv \max_{\{C_t, \theta_s, \theta_s^D, M_s\}_t^T} E_t \left[ \lambda_t \left( u \left( \frac{C_t}{(1+i)^t} \right) + \delta E_t[V_{t+1}(W_{t+1}, \phi_{t+1})] \right) \right] + (1 - \lambda_t) \Gamma \left( \frac{W_t}{(1+i)^t} \right) \quad (1)$$

where  $i$  is the inflation rate. We further assume that investors have power utility over both intertemporal consumptions and terminal bequests in real terms,

$$u(C_t) \equiv \begin{cases} \rho \frac{C_t^{1-\gamma}}{1-\gamma}, & \text{if } \gamma > 1 \\ \rho \ln C_t, & \text{if } \gamma = 1 \end{cases}, \quad \Gamma(W_t) \equiv \begin{cases} (1-\rho) \frac{W_t^{1-\gamma}}{1-\gamma}, & \text{if } \gamma > 1 \\ (1-\rho) \ln W_t, & \text{if } \gamma = 1. \end{cases}$$

where  $\gamma$  is risk aversion coefficient and  $\rho$  characterizes the relative importance of consumption over bequest motive. If  $\rho = 0$ , investors care only about bequest (equivalent to only terminal consumption), and if  $\rho = 1$ , the investors care only about intertemporal consumption.

## Budget Constraints

We denote labor income by  $L_t$  and liquidity needs by  $Q_t$ , which measures large cash outflow that exceeds labor income, for example, housing down-payments, college and medical expenses. For simplicity, both  $L_t$  and  $Q_t$  are assumed to be exogenous deterministic functions of time, and

$\widehat{L} \equiv (1 - \tau_d)L_t - Q_t$  is the total non-financial income/expenditure. Then the budget constraints in Equations (3)-(6) reduce to the following:

$$W_s^T = W_{s-1}[\theta_{s-1}R_s + (1 - \phi_{s-1} - \theta_{s-1})R_f] - C_s + \widehat{L}_s - (1 - \eta_s)M_s, \quad (2)$$

$$W_s^D = W_{s-1}[\theta_{s-1}^D r_s + (\phi_{s-1} - \theta_{s-1}^D)r_f] + M_s, \quad (3)$$

$$-W_s^D \leq M_s \leq \overline{M}_s, \quad M_T = W_T^D, \quad C_T = W_T \quad (4)$$

Equation (2) describes the evolution of the wealth in taxable accounts, where  $\eta_s = \eta \mathbf{1}_{\{M_s < 0, s \leq J\}}$  is the withdrawal penalty if investors withdraw ( $M_t < 0$ ) from tax-deferred accounts before the retirement year ( $t \leq J$ ). Equation (3) describes the evolution of the wealth in taxable accounts. Equation (3) describes the contribution limit and terminal conditions. Finally, we impose the borrowing and short-selling constraint in taxable and tax-deferred accounts as described in Equation (4).

## 1.2. Parameter Values

We consider two sets of parameters. The base set describes the simple scenario where investors are not allowed to contribute to tax-deferred accounts ( $\overline{M} = 0$ ), do not care about intertemporal consumption ( $\rho = 0$ ), have no labor income or liquidity needs ( $\widehat{L} = L = Q = 0$ ), and have no mortality risk ( $\lambda_t = 1$  for all  $t < T$ ). This case is the same as the simple version solved heuristically in Section 4 and can be used to evaluate the performance of the approximation rules derived. Moreover, it serves as a benchmark case for further numerical analysis.

The second set of parameters describes a more realistic case. The results can serve as a robustness check for the heuristic rules derived in previous chapter. Moreover, the fixed dollar contribution to tax-deferred accounts generates a wealth effect and allows us to address some public finance issues like whether the poor or the rich benefits more from the existence of tax-deferred accounts and whether the tax-deferred investing encourages saving.

Top part of Table C.2 summarizes the parameters used in the benchmark case. The (nominal) return on the stock is modelled as a binomial process with an expected return of 11% and a



standard deviation of 20% (all percentages have to be intended as p.a.). The bond yields a safe (nominal) return of 6% per year. These parameters imply a risk premium of 5% for the stock. We assume a tax rate of  $\tau_d = 50\%$  on the income from bonds and of  $\tau_s = 20\%$  on capital gains on stocks. As we mentioned above, the lower tax rate on stock is supposed to capture optimal capital gain tax realizations.<sup>3</sup> The annualized inflation rate is set to 3.5%. The initial age of the investor is 20 and his terminal age is 100. Retirement occurs at age  $K = 65$ . The investor has a risk aversion coefficient  $\gamma = 3$ , a time discount rate of  $\delta = 0.96$ .

Bottom part of Table C.2 reports the additional parameters used for the more realistic case. The investor experiences a liquidity shock of size  $Q = \$100,000$  at age 35. The maximum annual allowed contribution to a tax deferred account is  $M = \$10,000$  and the penalty for earlier withdrawal is set to  $\eta = 10\%$ . The labor income is taken from the deterministic component of the labor income process in Cocco, Gomes and Maenhout (1999). The annual mortality rate is calibrated to match the mortality table from the Society of Actuaries for the U.S. population. The investor has a bequest motive  $\rho = 0.044$ . The bequest motive is obtained by matching the parameters used by Dammon, Spatt and Zhang (2002).<sup>4</sup>

### 1.3. Numerical Procedure

We solve the optimization problem using dynamic programming. Details of the numerical implementation are discussed in Appendix B. The main procedure is to discretize the state space  $(W_t, \phi_t)$  into a grid and recursively solve for the optimal decisions on each grid point starting from the terminal date  $T$ .

There are two state variables  $[W_t, \phi_t]$  and four choice variables  $[M_t, C_t, \theta_t, \theta_t^P]$  at any time  $t$ . Instead of solving all the optimal choices simultaneously, we develop a two-step “projection” method to separate the contribution decision  $M_t$  and the consumption decisions  $C_t$  from the

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<sup>3</sup>The value of the tax rate on the stock is consistent with the effective tax rate calculated in Figure C-3 of Chapter 3 for a similar risky asset with  $\delta = 3\%$  dividend yield when the remaining investment horizon is long enough.

<sup>4</sup>If we choose  $\rho \equiv \frac{1-\delta}{A_H^{1-\gamma} + 1 - \delta}$ , where  $\delta$  is the time preference and  $A_H \equiv \frac{r^*(1+r^*)^{H-1}}{(1+r^*)^{H-1}}$  is the  $H$ -year annuity factor with  $r^* \equiv \frac{(1-\tau_d)(r-1)-i}{1+i}$  adjusted for inflation rate  $i$ , then  $\Gamma(W_t) = \sum_{s=t}^{t+H-1} \delta^{s-t} u(A_H W_t)$ , or the above bequest function is equivalent to the expected utility for a beneficiary with the same utility function as the decedent and receives an  $H$ -year annuity with annual real payment  $A_H W_t$  starting year  $t$ .

portfolio decision  $\theta_t, \theta_t^P$ , reducing the four-dimensional choice set to that with only two dimensions. The details of the two-step method is also explained in Appendix B.

## 2. Performance of the Approximation Rules

We now characterize the structure of the optimal solution, starting with the base set of parameters, and gradually introducing realistic features described before. For each parameter set, after solving for the optimal portfolio using dynamic programming we first evaluate the assumptions used in our implementation by inferring the true effective tax-subsidy  $Z_t^{ce}$  from the indirect utility function. The size of effective tax subsidies is an important quantity in public finance that measures how much each type of investors benefit from the existence of tax-deferred accounts. Moreover, the subsidy can be used to generate approximate portfolio rules. We assess the performance of the approximate portfolios by comparing them with the true optimal solution and compare the welfare losses (relative to the optimum) incurred by implementing these approximations relative to a “naive” two-account rule.

We start with the benchmark case of no labor income ( $L = 0$ ), no liquidity shock ( $Q = 0$ ), no future contribution to tax-deferred accounts ( $\bar{M} = 0$ , even though the state space allows for the existence of a tax-deferred account of any size), no intermediate consumption ( $\rho = 0$ ), and fixed investment horizon ( $\lambda_t = 1$  for all  $t < T$ ). This case is the same as the simple version solved heuristically in Chapter 4 and is used to evaluate the performance of the approximation rules derived. Moreover, it serves as a benchmark case for our further numerical analysis.

### 2.1. Effective Tax Subsidy

Using Definition 1, we can back out the effective tax subsidy for investing in tax-deferred accounts using the indirect utility function calculated at each grid point. We start our investigation of the effective tax subsidy by looking at the effect of time in Figure C-9. Since the implementation of Rules of Thumb 6 relies on the bounds of the effective tax subsidy, to visually confirm the validity of the bounds, we also compare the true effective subsidy from the numerical solution with the theoretical bounds. The solid lines (labelled  $Z_t^B$  and  $Z_t^S$ ) represent the upper and

lower bound described in Rules of Thumb 6. The two other lines report the implied effective tax subsidy  $Z_t^{ce}$  obtained from the optimal solution of the portfolio problem. The dash-dotted line refers to the case of  $\phi = 0.2$  and the dashed refers to the case of  $\phi = 0.8$ . As expected, the true tax subsidy falls in-between the theoretical bounds. From the figure we can infer that time and the effective tax rate have crucial effect on the magnitude of the effective tax subsidy. Moreover, the difference between upper and lower bound can be quite dramatic. For example, for an individual who is 20 years old with a fixed investment horizon of 80 years, investing one dollar in tax deferred bonds is equivalent to receiving a subsidy of almost 9 dollars. If he invests in tax-deferred stocks the subsidy decreases to less than 1 dollar.

## 2.2. Optimal Portfolio Strategy

We now report the optimal portfolio strategy and use it as benchmark to assess the performance of the heuristic rules in the previous chapter. Figure C-10 reports the percentage of total wealth invested in stock in taxable (Panel A) and tax-deferred (Panel B) accounts for different size  $\phi$  of the tax deferred account along with the approximate portfolio obtained by using the mapping derived in Proposition 6. The dashed lines describe the optimal portfolio allocation in both accounts. The solid lines are obtained by assuming that  $z_H = z_L = 1 + Z_t^B$  while the dash-dotted line corresponds to the case of  $z_H = z_L = 1 + Z_t^S$ . It is important to point out that the figure does not represent a time series of portfolio holdings. At every point in time  $t$  the figure represent the portfolio holding of an investor who at time  $t$  has a tax deferred account of size  $\phi = .2$  or  $\phi = 0.8$  (Figure C-10). In other words, due to the stochastic evolution through time of the state variable  $\phi$  the portfolio at time  $t + 1$  most likely refers to a different individual from to whom the portfolio time  $t$  refers. The optimal portfolio holding decrease over time, due to the effect of the tax subsidy which is itself decreasing through time. Moreover, the optimal portfolio always lies between the two approximate portfolios, indicating that our myopic implementation of the portfolio allocation rules can be a good approximation for the intertemporal solution. Finally, although  $Z_t^B$  and  $Z_t^S$  differ drastically (as we saw in Figure C-9) the upper and lower bounds on the portfolios are relatively close to each other.

One of the properties we can notice from the analysis of the portfolio strategies is that investors never hold a interior mixture of stocks and bonds in both taxable and retirement accounts, consistent with the prediction from Rules of Thumb5. To confirm the condition for naive pecking order rule provided in Proposition 5, we calculate the value of the next-period effective tax subsidy in high and low states of the world ( $z_H, z_L$ ) and report in Figure C-11 the maximum value of the ratio  $z_H/z_L$  across all possible states at any time. Also reported in the figure is the threshold value  $\bar{z}$  that determines when a violation of the naive pecking order occurs. We note that the ratio  $z_H/z_L$  is always considerably lower than the threshold  $\bar{z}$ . Consistent with the prediction of Proposition 5, the investor in this example always prefer bonds in the tax-deferred account and no violation of the pecking order is observed.

We finally use the optimal solution as a benchmark to compute the welfare loss incurred by applying the approximate portfolio rules as well as other sub-optimal strategies, often used by investor in practice.<sup>5</sup> Figure C-12 plots the welfare loss of following the approximate portfolio rule over time. The solid line is obtained by assuming that  $z_H = z_L = 1 + Z_t^B$  while for the dash-dotted line  $z_H = z_L = 1 + Z_t^S$ . For comparison purposes, we also plot the welfare loss incurred by following a (naive) strategy that forces investors to hold the same fraction of stocks in both accounts, where this fraction is chosen optimally each period.<sup>6</sup> Panel A refers to the case of small tax-deferred account ( $\phi = 0.2$ ) while Panel B consider the case of a large tax-deferred account ( $\phi = 0.8$ ). For the case of a small tax deferred account the wealth loss incurred by the approximate rules is between 1 and 3% of wealth at age 20 and drops dramatically as time elapses. In comparison, a strategy that hold the same portfolio in both accounts generates a welfare loss of approximately 35% at age 20 and decreases gradually over time. Similar conclusions can be drawn from the case of  $\phi = 0.8$  (Panel B). The naive rule has an 8% loss at age 20 while the optimal approximate rule generates a loss much less than 1%. We conclude that the proposed approximate rules perform quite well both relative to the naive rule and to the optimal solution.

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<sup>5</sup>Welfare losses are computed as the certainty equivalent wealth cost of a suboptimal strategy.

<sup>6</sup>This strategy seems to be consistent with empirical evidence. For example Bodie and Crane (1997), using data from a survey of TIAA-CREF participants and Bergstrasser and Poterba (2002), using data from the Survey of Consumer Finances, find that investor choose similar asset allocation in their taxable and tax-deferred accounts.

### 2.3. Extensions to Realistic Cases

Having shown that the heuristic portfolio rules in the previous chapter provides reasonable approximation to the optimal portfolio in the benchmark case, we now extend the analysis to more realistic situation with the following additional features: uncertain investment horizon, labor income, intermediate consumption, future contribution to tax-deferred accounts, and liquidity shocks.

#### Mortality Risk

As discussed in Chapter 2, even if borrowing and short-selling are allowed in taxable accounts, it is rather complicated to properly account for the mortality risk, since an insurance contract needs to be traded in order to fully lay off the mortality risk. Recall that  $\lambda_t$  is the probability that an investor survives at time  $t$ , then  $F(s) \equiv \prod_{i=1}^s \lambda_i$  is the probability that an investor survives up to time  $s$ , and the life expectancy at time  $t$  can be calculated as  $\sum_{s=t}^T [(s-t)(F(s) - F(s-1))]$ . For approximation purpose, it is sufficient to replace the fixed investment horizon by the remaining life expectancy of the investor in the current model.

#### Labor Income

We consider a simple example when investors differ in their current wealth level, while they share the same exogenous labor income process over their lifetime. From previous literature (e.g., Bodie, Merton and Samuelson (1992)), since labor income is assumed to be deterministic, and investors cannot borrow against future labor income, they are effectively endowed with a non-tradable bond position. Hence, they should optimally hold more stocks in their financial portfolio, and the effect is usually stronger for younger investors.<sup>7</sup> Since future labor income represents a much larger portion of their total wealth for poor investors, other things equal, their borrowing and short-selling constraints are more likely to bind and the size of effective tax subsidy is smaller from previous discussion.

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<sup>7</sup>But the impact of labor income on optimal portfolio may not be monotone over time due to the hump-shaped labor income process over time.

## Contributions to Tax-deferred Accounts

So far we have analyzed the effective subsidy that investors receive for their existing tax-deferred assets assuming that they are not allowed to contribute in the future ( $\bar{M} = 0$ ). We now consider the impact of allowing for future annual contribution  $\bar{M} > 0$  on the total subsidy that investors receive. Following the same logic in Definition 5 for  $Z_t^{ce}$ , we define a  $Z_{total}$  that solves the following condition:<sup>8</sup>

$$V(W + Z_{total}, \phi = 0)|_{M=0} = V(W, \phi)|_M. \quad (5)$$

$Z_{total}$  is the certainty equivalent wealth that makes an investor with neither an existing tax-deferred account nor the privilege to contribute to the tax-deferred in the future indifferent to someone with both. We can further decompose  $Z_{total} = Z_\phi \phi W + Z_M$ , where  $Z_M$  is the benefit from future contribution and  $Z_\phi$  the effective subsidy for each dollar in current tax-deferred accounts.

Figure C-13 reports the total effective subsidy  $Z_{total}$  when investors can contribution an upper limit of  $\bar{M} = \$10,000$  each year to the tax-deferred accounts. Panel A presents the results when investors have no current tax-deferred accounts ( $\phi = 0$ ) and Panel B is for a large current tax-deferred accounts ( $\phi = .9$ ). The solid line in each panel is for the wealthy investors who has current wealth of \$1m and the dashed line is for the less wealthy with \$100,000 of total wealth. The dotted line represent the subsidy received in the complete market case.<sup>9</sup> For example, an individual who starts at age 20 with \$1m in the taxable account and the privilege

<sup>8</sup>In order to assure a fair comparison between the  $\bar{M} = 0$  and the  $\bar{M} > 0$  case, we impose that after retirement, the investor in the  $\bar{M} = 0$  case is forced to withdraw from his tax-deferred account. This is done for two reasons. First, the investor in the  $\bar{M} > 0$  case is forced to withdraw after retirement. Second, even if contribution is not allowed in the  $\bar{M} = 0$  case, the state space still allows tax-deferred account of size  $\phi$  at any time.

<sup>9</sup>This is computed as follows. Let  $Z(t+1)$  be the subsidy at time  $t+1$  for each dollar in the tax-deferred and let  $\lambda(t+1)$  the probability that investor will survive from period  $t$  to  $t+1$ . The value of the subsidy at time  $t$  solves the following iteration

$$Z(t) = [(1 - \lambda(t+1)) \times 1 + \lambda(t+1)Z(t+1)] \times \frac{r_f}{R_f}$$

If the investor contribute  $\bar{M}(t)$  dollars per year, the total subsidy from future contributions in year  $t$  is

$$Z_M(t) = \sum_{k=t}^T \frac{M(k)Z(k)}{R_f^{k-t}}$$

to contribute up to \$10,000 per year to a tax-deferred account is effectively getting a subsidy of \$150,000 over his life time and the subsidy reduces to only \$10,000 for someone who starts with 100K of total wealth. Similarly, if an investor has total 1M of wealth at age 50, of which 900K is already in the tax-deferred account, he is effectively receiving a subsidy of \$250,000 from the government for the privilege of keeping his tax-deferred account and be able to further contribute for another 15 years. In summary, the wealthier investors are receiving more benefit in general, since they are less constrained and more likely to contribute up to the maximum allowed and to keep the higher tax-rate assets in the tax-deferred account to take advantage of the full benefit of tax-deferring. We can conclude a tax system that grants the privilege to invest in a tax-deferred account is effectively providing a tax subsidy whose magnitude increases with the current wealth level of the contributor and whose dynamics through time.

Since the contribution limit is an important instrument in the hands of the legislator, assessing the impact of a change in its level on the subsidy received by the investors can be useful from the public policy point of view. We will return to this point later when we analyze whether and when the introduction of a tax-deferred account stimulate savings.

### 3. Saving Decisions

So far we have been looking at our numerical solution from a static point of view. Both the value investing in a tax-deferred account and the properties of the optimal portfolio solution have been analyzed as a function of the values of the state variables  $(W, \phi)$  at every point in time. Although this is useful to study the properties of the general solution, it does not allow us to analyze the evolution of portfolio, consumption and contribution decisions of an investor over the life-cycle. By simulating the model, we can both address these issues and further touch on public finance implications like whether the existence of tax-deferred accounts stimulates individual saving, or equilibrium pricing issues like whether allowing tax-deferred investing increases or decreases equity premium. For all these purpose, realism is very important and we focus all our simulation on the most realistic set of parameters possible (with all the additional features).

Uncertainty in the model derives from the evolution of the risky asset. We simulate 100,000

time series of the state variables  $(W, \phi)$  by assuming that the realized return on the stock are obtained from a binomial distribution with parameters as described in Table C.2. For each time series, by using the law of motions described in equations (4) and (5) and the optimal solution computed in the previous chapter we can obtain the evolution of the state variables as well as the dynamics of the portfolio, consumption and contribution decision of a particular investor.

In most of the following analysis we will assume that the investor start at age 20 with an initial wealth of \$10,000. At times, (See Figures C-16 and C-17) we will also consider the case of an investor starting with a wealth of \$1m. Figures C-14 and C-15 represent, for each date, the average across 100,000 path of the realized value of wealth ( $W$ ) and size of the tax-deferred account ( $\phi$ ) respectively. Panels A and B refers to an investor who starts with an initial wealth of \$1m, while Panels C and D refers to the case of an investor starting with an initial wealth of \$10,000. Moreover, Panels A and C, refers to the case of no liquidity shock ( $Q = 0$ ) while Panels B and D include the possibility of a liquidity shock at when the investor is 35 years old.

We will now rely on our simulation results to analyze the characteristic of the portfolio location over time, with particular emphasis on the effect of liquidity needs on the hedging demand of investors. We conclude by addressing the question of whether and under what conditions the existence of a tax-deferred account might stimulate savings.

An important component of portfolio decision is the how much to contribute to tax-deferred account over the life cycle. Corollary 2 shows that in complete markets, investors always max out their contribution to maximize the benefit extracted from tax-sheltering in the retirement account. This strategy is possible in the absence of frictions such as borrowing and short-selling constraint. When these frictions are considered, the contribution decision is no longer trivial since investors optimally trade off the benefit of the effective tax subsidy from investing in tax-deferred accounts to the cost of reduced current-period consumption and potential future withdrawal due to liquidity needs. We now use our simulation results to study the evolution of the contribution decision over the life-time of an investor and the effect that the presence of a retirement account has on the trade off between consumption and savings.

Figure C-16 compares the contribution decisions for individuals across wealth (top panels



versus bottom panels) and for different values of the liquidity shock (left panel refers to the no-liquidity shock case and the right panels refer to the case of a liquidity shock of fixed size). In each graph we report the average path of contribution (dotted line) and an example of an individual path (solid line). Following a single path is more informative than taking averages since it allows us to see directly the evolution of the optimal contribution decision over time. This is information that is somehow lost during the process of taking average across paths. Looking at the path for the wealthy investor (top panels), we note that early in his age he is maxing out his contribution to the tax-deferred account and that he withdraw to finance consumption around age 45. It is interesting to point out that the switch between maximum contribution and withdrawal is not gradual but rather dramatic as the path shows. This is due to the fact that in general it pays to max out the contribution, even if there will be chances of future withdrawal at a penalty as shown by the back-of-the-envelope calculation in Chapter 4.

For the low-wealth investor (bottom panels), we note first that there is no instances in which he is maxing out on his tax-deferred account contribution. Moreover, in the case of liquidity shock (south-east plot) we observe withdrawal from the tax deferred account in correspondence of the liquidity shock years. More interestingly, this withdrawal is preceded by years in which the investor is contributing more than \$5,000 to the tax-deferred account. Again, we observe the non-smoothness of the contribution decision which is consistent with the fact that investors try to get the maximum subsidy from investing in a tax-deferred account.

One of the most interesting public policy issues concerning retirement accounts is whether and to what extent they stimulate savings. Recall that in complete markets, Corollary 2 states that, the only impact of tax-deferred accounts on optimal consumption decision is the wealth effect due to effective subsidy received, and as long as their optimal consumptions increase with overall wealth level, investors always increase their consumption as a fraction of their nominal wealth after the introduction of tax-deferred accounts, or equivalently, they always *reduce* their savings, contradicting the popular belief that tax-deferred investing opportunities encourage savings from investors. When investors are not allowed to borrow in taxable accounts, they may need to trade off current consumption with the ability to contribute one more dollar to the

tax-deferred accounts and hence earn one more effective tax subsidy. This substitution effect is most pronounced for less wealthy investors who in general do not max out their contributions and constantly face the trade off between current consumption and an extra dollar saved in the tax-deferred accounts which finances future consumption. It is for this class of investors that the marginal saving increases the effective tax subsidy received and the existence of tax-deferred accounts may serve to encourage overall saving. The optimal saving decision can only be solved by balancing the wealth and substitution effects. In general, for wealthier investors who always contribute up to the maximum allowed to tax-deferred accounts, extra saving does not generate more subsidy, and the substitution effect is dominated by the wealth effect and investors *always decrease* savings in the presence of tax-deferred accounts. On the other hand, the substitution effect could dominate for poorer investors and they *may increase* their total saving after the introduction of tax-deferred accounts.

Our model represents a natural environment to address this question. We simulate the model and report the typical consumption pattern for investors starting with different initial wealths. Figure C-17 (top panels) reports the average path of consumption in dollar terms when the investor is not allowed to contribute ( $\bar{M} = 0$ ) and when he is allowed to contribute up to \$10,000 each year. We also report (lower panels) the difference between the two consumption choices. The figure shows an interesting pattern. For the low wealth individual (right panels), we note that the consumption level early on is lower when the contribution limit is \$10,000 and higher in later years. This is due to the interaction of two forces. The first force is a “wealth effect”. The tax-deferred account is providing a subsidy. The individuals, feeling wealthier will consume more (this is exactly what happens in the complete market case). The second force is a “substitution effect”, i.e., the trade-off between the marginal benefit of consumption and the marginal benefit of saving. In complete markets, since borrowing is allowed, the opportunity cost of investment is independent on the consumption decision. When investors are not allowed to borrow, the opportunity cost of investing is higher (due to the lost subsidy) if investors need to cut back on savings in the tax-deferred account to finance consumption. The fact that low wealth individual consume less early on in their life reflect the fact that the substitution

effect is dominating in early years. The presence of a retirement account induces savings (low consumption) early on. For the high wealth individual, on the other side, we note that the path of consumption when  $\bar{M} = \$10,000$  is uniformly higher than when  $\bar{M} = \$0$ . This is consistent with the fact that the wealth-effect dominates for an investor who is less constrained, and this translates in a higher consumption uniformly over his lifetime, consistent with the complete market result. Combined with the demographic distribution of US population, we can further estimate the impact of tax-deferred accounts on aggregate savings.

#### 4. Equilibrium Asset Price

The current model also sheds light on the impact of tax-deferred accounts on equilibrium asset prices. Consider an economy with only a stock and a bond where the bond is taxed more heavily than the stock and there are no tax-deferred investment opportunities. How would the equilibrium prices change if the government introduces tax-deferred accounts where neither the stock nor the bond is taxed? Investors are assumed to fund tax-deferred accounts using their existing taxable wealth and there is a preset contribution limit on the amount they are allowed to invest in tax-deferred accounts.

If investors are allowed to borrow and short-sell in taxable accounts, Proposition 2 and 3 suggest that investors hold only bonds in tax-deferred accounts and the only change in their taxable stock holding is driven by the increased effective wealth level due to effective tax subsidy received. As long as the optimal *dollar* amount of stock holding increases with individual wealth level, the total dollar demand for the stock increases, and the dollar demand for the bond has to decrease, since the *nominal* wealth in the economy is not changed after the introduction of tax-deferred accounts.<sup>10</sup> As a result, the bond price decreases relative to the stock price, or the equity premium decreases after the introduction of tax-deferred accounts.

The result seems counter-intuitive, since the introduction of tax-deferred accounts allows investors to hold the heavily-taxed bonds in tax-deferred accounts, making the bond more attractive, and the bond price should increase relative to the stock price. However, this argument

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<sup>10</sup>We ignore the potential increase in personal savings after introducing tax-deferred accounts.

fails to recognize that holding one dollar of bond in tax-deferred accounts is equivalent to holding  $Z_{t:T}^D > 1$  dollars of bonds in taxable accounts. Although the effective holding (accounting for the Z-effect) of bonds is higher, the nominal dollar demand for bonds turns out to always decrease after the introduction of tax-deferred accounts, since investors hold *only* bonds in tax-deferred accounts.

On the other hand, when investors are not allowed to borrow or short-sell in taxable accounts, the above intuition that bonds become more attractive *does* enter into equilibrium asset prices. For example, in an extreme case when there are no upper limits on the size of tax-deferred accounts, investors would simply move all their assets to tax-deferred accounts.<sup>11</sup> Or equivalently, we shift into a new economy with no taxes on either the stock or the bond. The overall demand for the bond has to be higher in this new economy relative to the old, and the bond price has to increase relative to the stock price, generating a higher equity premium. We term this change in opportunity set the substitution effect.

In general, the impact of introducing tax-deferred accounts on equity premium depends on the maximum allowed contribution to tax-deferred accounts. For small tax-deferred accounts, the wealth effect in the complete market case dominates, and the equity premium decreases, whereas for large enough tax-deferred accounts, the substitution effect dominates and the equity premium increases.

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<sup>11</sup> Assume capital gains are taxed upon accrual for the stock, for example.

## Chapter 6

# Conclusion

Saving for retirement is one of the most important financial decisions in an individual's life. The opportunity to invest in tax-sheltered retirement accounts (e.g. IRA, 401(k)) complicates the traditional life-cycle portfolio decisions by introducing the interaction between the taxable and tax-deferred portfolio holdings.

In a complete market setting, we show that the optimal taxable and tax-deferred portfolio decisions are separable in that the tax-deferred portfolio can be determined myopically without considering overall portfolio allocation. The rights to invest in tax-deferred accounts is equivalent to receiving a fixed effective tax subsidy from the government, which depends only on the characteristics of assets placed in tax-deferred accounts. The optimal location decision is determined by placing only the assets that yields the highest effective tax subsidy in tax-deferred accounts, while the overall allocation can be determined by mapping the two-account problem to a taxable-account-only problem with overall wealth level augmented by the tax subsidy.

The existence of constraints on borrowing and short selling in taxable accounts further increases the complexity of the task of selecting the optimal mix of security to hold in both taxable and tax-deferred accounts. Building on the concept of effective tax-subsidy generated by the possibility of investing in a tax-deferred account, we analyze the effect of borrowing and short-selling constraints on the optimal location and allocation strategies. We show that, the location strategy of preferring the higher-taxed asset (pecking order) can be violated and provide

conditions under which this might occur. Our numerical analysis, however, seems to indicate that this violation is unlikely, hence suggesting that the intuitive pecking order adjustment to account for borrowing and short-selling constraints in the location decision is in general valid. The optimal allocation is then derived by mapping the two-account problem into a mixture of two-single account problems each of which delivers useful portfolio rules that can be easily implemented. Compared to popular naive allocation rules, the proposed approximate rules provide welfare gains in the order of 10 to 30%, depending on the size of the tax-deferred account.

We finally study the impact of tax-deferred accounts on individual savings decisions. Although less wealthy investors increase their savings when they are young in order to increase their total contribution to tax-deferred accounts, higher net worth investors reduce their saving when they are presented the opportunity to invest in tax-deferred accounts.

# Appendix A

## Proofs

### Proofs of Lemmas 1 and 2.

We prove simultaneously the following three induction assumptions using backward induction on the remaining investment horizon  $T - t$ .

1. the optimal trading strategy is to postpone gains and realize losses immediately (Lemma 1),
2. a long position with basis  $y$  is equivalent to  $z_{i,t}(y)$  of cash, and a short position is equivalent to  $s_{i,t}(y)$  of cash (Lemma 2),
3. it is possible to define a short rebate rate  $\hat{r}_t$  such that  $z_{i,t}(1) = -s_{i,t}(1) = 1$ .

Note that part three of the induction assumption is necessary to rule out arbitrage. We need to define  $\hat{r}_t$  recursively at each period.

On the terminal date  $T$ , Lemma 1 holds trivially since investors always realize losses and they either postpone or realize gains depending on the terminal tax treatment. For Lemma 2, depending on the terminal tax treatment, any long taxable position in asset  $i$  is equivalent to the following cash positions

$$z_{i,T}(y) = \begin{cases} 1 - \tau_g(1 - y), & \text{if forced to realize capital gains} \\ 1 - 1_{\{y > 1\}}\tau_g(1 - y), & \text{if escape tax through step-up of basis} \end{cases} \quad (1)$$

and any short position is equivalent to

$$s_{i,T}(y) = \begin{cases} -1 + \tau_g(1 - y), & \text{if forced to realize capital gains} \\ -1 + 1_{\{y < 1\}}\tau_g(1 - y), & \text{if escape tax through step-up of basis} \end{cases} \quad (2)$$

where  $1_{\{y > 1\}}$  is an indicator function that equals to 1 when  $y > 1$ , and 0 otherwise, and similarly for  $1_{\{y < 1\}}$ . Part three of the induction assumption is clearly true for any short rebate rate on the terminal date.

We now assume all three parts hold at time  $t + 1$  and prove the results for time  $t$ .

First we define the effective one-period return from time  $t$  to  $t + 1$  for any long or short taxable positions assuming investors retain the position until time  $t + 1$  and follow optimal trading strategy (defined in Lemma 1) from time  $t + 1$  on. Let  $\tilde{d}_i$ ,  $\tilde{g}_i$  be the before-tax dividend and capital gains rates defined in equation (1), then a long position with face value \$1 and tax

basis  $y$  evolves into a portfolio of  $(1 - \tau_d)\bar{d}_i$  after-tax cash dividends and a long position in asset  $i$  with  $\$(1 + \bar{g}_i)$  face value and tax cost basis  $y/(1 + \bar{g}_i)$ . By induction assumption, the new long position is equivalent to a cash holding at time  $t + 1$ , therefore, holding the long position in asset  $i$  at time  $t$  is equivalent to receiving the following cash payments at time  $t + 1$

$$\bar{R}_{i,t}(y) \equiv (1 - \tau_d)\bar{d}_i + (1 + \bar{g}_i)z_{i,t+1}(y/(1 + \bar{g}_i)),$$

We define the above  $\bar{R}_{i,t}(y)$  as the effective one-period return for the long position from time  $t$  to  $t + 1$ , which follows a binomial distribution with realizations

$$R_{i,t}^H(y) \equiv (1 - \tau_d)d_i^H + u_i z_{i,t+1}(u_i^{-1}y), \quad R_{i,t}^L(y) \equiv (1 - \tau_d)d_i^L + u_i^{-1}z_{i,t+1}(u_i y).$$

The effective one-period return on short positions can be derived similarly, except that we need to account for the difference between the short rebate rate on the cash collateral and the risk-free rate. According to Assumption 4, investors are required to post \$1 as cash collateral<sup>1</sup> on short-sell proceeds and they earn a rebate interest rate  $\hat{r}_{f,t} - 1$  at time  $t$ . Let  $\hat{R}_{f,t} - 1 = (1 - \tau_d)(\hat{r}_{f,t} - 1)$  be the after-tax rebate interest rate, then  $(R_f - \hat{R}_{f,t})$  measures the cost of short selling and should be included in the effective one-period return, specifically,

$$\begin{aligned} R_{-i,t}^H(y) &\equiv -(1 - \tau_d)d_i^H + u_i s_{i,t+1}(u_i^{-1}y) - (R_f - \hat{R}_{f,t}) \\ R_{-i,t}^L(y) &\equiv -(1 - \tau_d)d_i^L + u_i^{-1} s_{i,t+1}(u_i y) - (R_f - \hat{R}_{f,t}). \end{aligned}$$

The short rebate rate  $\hat{r}_{f,t}$  is defined to rule out tax arbitrage by setting the return on the following zero-cost portfolio to zero: short \$1 of asset  $i$  at market price and earn  $\hat{r}_{f,t} - 1$  on the short proceeds, borrow \$1 to buy asset  $i$  at market price, and follow optimal trading strategy on both the long and short positions in all future periods. From induction assumption, the long position in asset  $i$  is equivalent to  $R_{i,t}^H(1)$  (or  $R_{i,t}^L(1)$ ) of cash next period, whereas the short position, along with the difference on rebate interest rate and the risk free rate on the borrowed amount, is equivalent to a cash position of  $R_{-i,t}^H(1)$  (or  $R_{-i,t}^L(1)$ ). If  $R_{i,t}^H(1) + R_{-i,t}^H(1) \geq R_{i,t}^L(1) + R_{-i,t}^L(1)$ , then the return on the above portfolio can be replicated by a portfolio of long position in asset  $i$  and bonds, or effectively, we can define a “risk-neutral” probability measure  $q_{i,t}$

$$q_{i,t} \equiv \frac{R_{i,t}^H(1) - R_f}{R_{i,t}^H(1) - R_{i,t}^L(1)}$$

and the present value of the portfolio returns can be expressed as

$$(1 - q_{i,t})(R_{i,t}^H(1) + R_{-i,t}^H(1)) + q_{i,t}(R_{i,t}^L(1) + R_{-i,t}^L(1)).$$

If on the other hand,  $R_{i,t}^H(1) + R_{-i,t}^H(1) \leq R_{i,t}^L(1) + R_{-i,t}^L(1)$ , then the return on the above portfolio can be replicated by a portfolio of short positions in asset  $i$  and bonds, and we can define a “risk-neutral” probability measure

$$q_{-i,t} \equiv \frac{R_{-i,t}^H(1) + R_f}{R_{-i,t}^H(1) - R_{-i,t}^L(1)},$$

<sup>1</sup>Even though the relative value of a short position with face value \$1 is in general  $\neq -1$  due to unrealized capital gains/losses, only \$1 is required as cash collateral.



The present value of the portfolio returns can be expressed as

$$(1 - q_{-i,t})(R_{i,t}^H(1) + R_{-i,t}^H(1)) + q_{-i,t}(R_{i,t}^L(1) + R_{-i,t}^L(1)).$$

It turns out  $q_{i,t} = q_{-i,t}$  under the rebated rate  $\widehat{r}_t$  defined below. Therefore, it does not matter which replication we use.

Since the net cost of the above portfolio is zero, to rule out arbitrage, the above present value has to be zero as well. Effectively, under the “risk-neutral measure”, the present value of shorting one share (including the cost from lower rebate rate) is the same as minus the present value of buying one share at current market price, or  $s_{i,t}(1) = -z_{i,t}(1) = -1$ . We can solve for the rebate rate from the above equation,

$$\widehat{R}_t \equiv R_f - [(1 - q_{i,t})u_i(z_{i,t+1}(u_i^{-1}) + s_{i,t+1}(u_i^{-1})) + q_{i,t}u_i^{-1}(z_{i,t+1}(u_i) + s_{i,t+1}(u_i))].$$

since  $\widehat{R}_t$  depends only on  $z_{i,t+1}$  and  $s_{i,t+1}$ , it is well defined using induction assumptions. Note that the after-tax rebate rate  $\widehat{R}_t \leq R_f$  since  $z_{i,t+1}(u_i^{-1}) + s_{i,t+1}(u_i^{-1}) \geq 0$  and  $z_{i,t+1}(u_i) + s_{i,t+1}(u_i) \geq 0$ . Intuitively, being able to postpone gains and realize losses immediately on both the long and short positions gives investors an arbitrage opportunity if they can make full use of the short sell proceeds. Equivalently, the before-tax short rebate rate  $\widehat{r}_t < r_f$  is necessary to rule out that arbitrage. This proves the third part of the induction assumptions.

To prove the first part, we need to consider four possible cases: long or short positions with gains or losses, respectively. We prove only the two cases for long positions. The short positions can be proved similarly. First, if there are losses on the long position, or  $y > 1$ , we need to show that it is optimal to realize losses immediately. Assume the result is not true and investors retain the position and eventually sell it at time  $s > t$  ( $s$  could be the terminal date). Without loss of generality, we ignore dividend payments. Let  $P_t, P_s$  be the current and future stock prices, then for each share of long position with current tax basis  $y$ , investors would obtain an after-tax proceeds of  $(1 - \tau_g)P_s + \tau_g P_t y$  if they are required to pay capital gains taxes upon liquidation. On the other hand, if the investor liquidate the position today, he obtains an after-tax proceeds of  $P_t + \tau_g P_t (y - 1)$  on the share. He can purchase a new share using  $P_t$  of the proceeds and invest the remaining proceeds  $\tau_g P_t (y - 1)$  in risk-free bonds. The value of this portfolio at time  $s$  will be

$$(1 - \tau_g)P_s + \tau_g P_t + \tau_g P_t (y - 1)R_f^{s-t} = (1 - \tau_g)P_s + \tau_g P_t y + \tau_g P_t (y - 1)(R_f^{s-t} - 1),$$

higher than the after-tax proceeds if the investor retains the long position, with the difference  $\tau_g P_t (y - 1)(R_f^{s-t} - 1)$  equals to the interest earned on the tax credits over the period. Similarly, if the investor escapes terminal taxes on date  $s$ , then the total proceeds is  $P_s$  if he retains the share, and  $P_s + \tau_g P_t (y - 1)R_f^{s-t}$  if he liquidates today, purchases a new share and invests the rest in bonds. The different is the total future value of tax credits. No matter what happens to the share in the future, investors do better if they realize the losses immediately.

To show that it is optimal to postpone gains on long positions, consider an investor who liquidates a long position of an asset with cost basis  $y$  to follow certain strategy on the proceeds that yields  $\bar{r}$  over the next period. Then the total proceeds of the share at time  $t + 1$  is  $[P_t - \tau_g P_t (1 - y)]\bar{r}$ . We show that the investor can do strictly better if he retains the long position while short-sells one share of the same asset at market price. The proof relies on the definition of the short rebate rate where the rebate interest rate plus the option value to postpone gains in the

future generates an effective return of the risk-free rate on the short proceeds. Equivalently, we can simplify the calculation by setting the short rebate rates to the risk-free rate while assuming investors liquidate both the long and the short positions on the asset next period to forego the potential option value. Specifically, the investor retains the original long position, short a new share with proceeds  $P_t$  and places  $[P_t - \tau_g P_t(1 - y)]$  of the proceeds in the same strategy that yields  $\tilde{r}$  while investing the rest  $\tau_g P_t(1 - y)$  in risk-free bonds. Then the total proceeds at time  $t + 1$  on the portfolio is

$$\begin{aligned} & [(1 - \tau_g)P_{t+1} + \tau_g P_t y] - [(1 - \tau_g)P_{t+1} + \tau_g P_t] + [P_t - \tau_g P_t(1 - y)]\tilde{r} + \tau_g P_t(1 - y)R_f \\ = & [P_t - \tau_g P_t(1 - y)]\tilde{r} + \tau_g P_t(1 - y)(R_f - 1) \end{aligned}$$

higher than that on the original strategy of liquidating immediately, with the difference  $\tau_g P_t(1 - y)(R_f - 1)$  equals to the interest saved on the capital gains taxes. This concludes the proof for the second part of the induction assumption.

Finally, we show that the relative values of the long position at time  $t$  is uniquely defined. The result holds trivially for the case of  $y > 1$ , since we have just shown that investors optimally liquidate the asset position to realize losses immediately, and the long position is equivalent to  $z_{i,t}(y) = 1 - \tau_g(1 - y)$  of cash. If  $y \leq 1$ , the optimal trading strategy is to retain the long position, and we have just derived the corresponding effective one-period return  $R_{i,t}^H(y)$  and  $R_{i,t}^L(y)$ . The present value under risk neutral measure  $q_{i,t}$  provides the replication cost for the above return using taxable asset  $i$  and bonds purchased in the spot market.

$$z_{i,t}(y) = \frac{1}{R_f} \left[ (1 - q_{i,t})R_{i,t}^H(y) + q_{i,t}R_{i,t}^L(y) \right]. \quad (3)$$

The proof for short positions is identical since the definition of short rebate rates ensures that, at current market price, a new short position is equivalent to minus a new long position. If  $y \leq 1$ , investors realize losses on the short positions immediately, and the relative value is  $s_{i,t}(y) = -1 + \tau_g(1 - y)$ . If  $y > 1$ , investors retain the short position and the relative value  $s_{i,t}(y)$  can also be defined as the present value of future cash flows under the ‘‘risk neutral’’ measure  $q_{-i,t}$ ,

$$s_{i,t}(y) = \frac{1}{R_f} \left[ (1 - q_{-i,t})R_{-i,t}^H(y) + q_{-i,t}R_{-i,t}^L(y) \right].$$

where short positions in asset  $i$  with basis  $y = 1$  and the risk free bonds are used as the base assets. ■

### Proof of Lemma 3.

The proof also uses backward induction on the remaining investment horizon  $T - t$ . The induction assumptions are (i) the relative value  $z_{t:T}^D$  is uniquely defined for any *state-independent* tax-deferred strategy  $\Theta_{t:T}$  from time  $t$  on, such that having a dollar in tax-deferred accounts following a state-independent trading strategy  $\Theta_{t:T}$  is equivalent to having  $z_{t:T}^D$  dollars of cash in taxable accounts, and (ii)  $z_{t+1:T}^D$  is also state-independent.

On the terminal date  $T$ , by Assumption 3, investors are forced to liquidate all tax-deferred assets and move the proceeds to taxable accounts for terminal consumption. As a result, one dollar in tax-deferred accounts is equivalent to  $z_{T:T}^D = 1 - \eta_T$  dollars in taxable accounts, where

$\eta_T \geq 0$  is the penalty for withdrawal on the terminal date. Clearly, the relative value  $z_{T:T}^D$  is identical across states. We now assume both results hold at time  $t+1$  and prove the same for time  $t$ .

Consider first a special example of state-independent tax-deferred trading strategies where investors hold only asset  $i$  in tax-deferred accounts at time  $t$ . We denote such strategy as  $\Theta_{t:T} = [e_N^i, \Theta_{t+1:T}]$ , where  $e_N^i = [0, \dots, 1, \dots, 0]$  is an index vector of size  $N$  with the  $i$ -th element equals one and zero otherwise, and  $\Theta_{t+1:T}$  is a state-independent strategy from time  $t+1$  on. By induction assumption, at time  $t+1$ , having \$1 in tax-deferred accounts that follows trading strategies  $\Theta_{t+1:T}$  is equivalent to having  $z_{t+1:T}^D$  in taxable accounts and  $z_{t+1:T}^D$  is identical across states. Hence, one dollar in tax-deferred accounts following strategy  $\Theta_{t:T}$  evolves into  $r_i^H$  or  $r_i^L$  dollars in tax-deferred accounts at time  $t+1$ , when each dollar in tax-deferred accounts is equivalent to  $z_{t+1:T}^D$  in taxable accounts. Therefore, having one dollar in tax-deferred accounts following strategy  $\Theta_{t:T}$  is equivalent to receiving the following return at time  $t+1$  in taxable accounts:

$$R_{i,t}^{DH} \equiv r_i^H z_{t+1:T}^D, \quad R_{i,t}^{DL} \equiv r_i^L z_{t+1:T}^D. \quad (4)$$

The above return can be replicated by a portfolio of  $\$ \Delta_{i,t}^D$  asset  $i$  (with no embedded capital gains, or  $y = 1$ ) and  $\$ B_{i,t}^D$  risk-free bonds, where

$$\Delta_{i,t}^D \equiv \frac{R_{i,t}^{DH} - R_{i,t}^{DL}}{R_{i,t}^H(1) - R_{i,t}^L(1)}, \quad B_{i,t}^D = \frac{1}{R_f} \frac{R_{i,t}^H(1)R_{i,t}^{DL} - R_{i,t}^{DH}R_{i,t}^L(1)}{R_{i,t}^H(1) - R_{i,t}^L(1)}. \quad (5)$$

Holding \$1 in tax-deferred accounts following strategy  $\Theta_{t:T}$  is equivalent to having  $z_{t:T}^D = \Delta_{i,t}^D + B_{i,t}^D$  dollars in taxable accounts following the replicating strategy. Another way to calculate the replication cost  $z_{t:T}^D$  is to discount the equivalent one-period cash flows  $R_{i,t}^{DH}$  and  $R_{i,t}^{DL}$  under the "risk-neutral" measure  $q_{i,t}$  defined in equation (2):

$$z_{t:T}^D = \frac{1}{R_f} \left[ (1 - q_{i,t})R_{i,t}^{DH} + q_{i,t}R_{i,t}^{DL} \right]. \quad (6)$$

The replicating portfolio for the above tax-deferred strategy can be used as a base to construct the replicating portfolios for any general tax-deferred strategies. Specifically, for a tax-deferred dollar following any general strategy  $\Theta_{t:T} = [\Theta_t, \Theta_{t+1:T}]$ , where  $\Theta_t = [\theta_{1,t}, \dots, \theta_{N,t}]'$ , the taxable replicating portfolio (in dollars) can be written in the following vector form:

$$\Delta_{t:T}^D = \left[ \sum_{i=1}^N \theta_{it} B_{i,t}^D + \theta_{1t} \Delta_{1t}^D, \theta_{2t} \Delta_{2t}^D, \dots, \theta_{Nt} \Delta_{Nt}^D \right]', \quad (7)$$

and the replication cost ( $z_{t:T}^D = \mathbf{1}'_N \Delta_{t:T}^D$ ) can be expressed as

$$z_{t:T}^D = \Theta_t' [z_{t_0:T}^D, \dots, z_{t_N:T}^D]'. \quad (8)$$

Clearly, as long as  $\Theta_t$  is state-independent, the replicating cost  $z_{t:T}^D$  is identical across states. ■

## Proof of Proposition 1.

Let  $X_t \equiv [H_{t-1}, Y_t, W_t^D, P_t]$  be the realization of state variables at time  $t$ ,  $V_t(X_t)$  be the corresponding value function, and  $H_t^*(X_t)$ ,  $\Theta_t^*(X_t)$  be the optimal taxable and tax-deferred portfolio strategies at time  $t$ , respectively. The proof again relies on backward induction on remaining investment horizon  $T - t$ .

There are three induction assumptions at time  $t$ ,

1.  $\Theta_s^*$  is state-independent for any time  $s \geq t$ , Hence,  $\Theta_{t:T}^* \equiv [\Theta_t^*, \dots, \Theta_T^*]$  is uniquely defined over all realizations of states at time  $t$ .
2.  $\Theta_{t:T}^*$  maximizes the corresponding replication cost  $z_{t:T}^*$  defined in Lemma 3.
3.  $\Theta_t^*$  is independent of  $H_t^*$ ,

On the terminal date  $T$ , by Assumption 3, investors are forced to liquidate and withdraw all tax-deferred assets for consumption. The tax-deferred trading strategy is clearly state-independent, maximizes replication cost (among the only feasible solution), and is independent of the taxable strategy.

Assume all three induction assumptions are true from time  $t + 1$  on. We need to show the same for time  $t$ . Let  $\Theta_{t+1:T}^*$  be the optimal tax-deferred strategy from time  $t + 1$  on and  $z_{t+1:T}^*$  be the corresponding replication cost. By induction assumption,  $\Theta_{t+1:T}^*$  is state-independent. Hence, any optimal tax-deferred strategy  $\Theta_{t:T}^*(X_t)$  can always be written as  $[\tilde{\Theta}_t(X_t), \Theta_{t+1:T}^*]$ , since the optimality of  $\Theta_{t+1:T}^*$  implies that investors can improve their utility by switching to  $\Theta_{t+1:T}^*$  from time  $t + 1$  on had they not done so already. The optimal strategy  $\tilde{\Theta}_t(X_t)$  at time  $t$ , however, may depend on realizations of state variables  $X_t$  at time  $t$ .

Given that  $\Theta_{t+1:T}^*$  and its corresponding replication costs  $z_{t+1:T}^*$  are state-independent, we can follow the same steps as in the proof of Lemma 3 while replacing  $z_{t+1:T}^D$  with  $z_{t+1:T}^*$  in Equation (4) and the following derivations, and define  $z_{t:T}^*$  as the replication cost for strategy  $[e_N^i, \Theta_{t+1:T}^*]$  where investors hold only asset  $i$  in tax-deferred for current period and following the optimal strategy  $\Theta_{t+1:T}^*$  for future periods. The replication cost  $z_{t:T}^*$  is well-defined and also state-independent at time  $t$ .

Let  $\Theta_t$  be any feasible current-period tax-deferred strategy, then a tax-deferred dollar following any pre-specified strategy  $[\Theta_t, \Theta_{t+1:T}^*]$  can be replicated by a taxable portfolio with costs  $\Theta_t'[z_{t_0:T}^*, \dots, z_{t_n:T}^*]'$  (following the derivation of Equation (8)). Define  $\Theta_t^*$  to be the strategy that maximizes this replication cost<sup>2</sup>, or

$$\Theta_t^* = \operatorname{argmax}_{\Theta_t} \Theta_t'[z_{t_0:T}^*, \dots, z_{t_n:T}^*]'$$

We now prove, by contradiction, that the optimal tax-deferred trading strategy at time  $t$  is  $\Theta_t^*$  across all states. Assume not, then there exists at least one state  $X_t$  such that the optimal tax-deferred strategy is different from  $\Theta_t^*$ , or  $\Theta_{X_t} \equiv \tilde{\Theta}_t(X_t) \neq \Theta_t^*$ . Assume  $H_t(X_t)$  is the corresponding optimal taxable portfolio in that state. Let  $\Delta_{t:T}^*$ ,  $\Delta_{t:T}^D(X_t)$  be the replicating portfolios and  $z_{t:T}^*$ ,  $z_{t:T}^D(X_t)$  be the replication costs (similar to those defined in Equations (7)

<sup>2</sup> $\Theta_t^*$  may not be unique, in which case the optimal strategy is any linear combination of these  $\Theta_t^*$ , and investors are indifferent between these strategies. Any one of these strategy can be defined as the optimal and it will still be state-independent. Without loss of generality, we assume  $\Theta_t^*$  is unique.

and( 8)) for tax-deferred strategies  $[\Theta_t^*, \Theta_{t+1:T}^*]$  and  $[\Theta_{X_t}, \Theta_{t+1:T}^*]$  respectively. We show that investors are better off if they switch to strategy  $\Theta_t^*$  in tax-deferred accounts, while at the same time shorting the replicating portfolio for strategy  $[\Theta_t^*, \Theta_{t+1:T}^*]$ , longing the replicating portfolio for strategy  $[\Theta_{X_t}, \Theta_{t+1:T}^*]$  and finally holding the difference between the replication costs in bonds. Formally, instead of holding  $H_t(X_t)$  in taxable and  $\Theta_{X_t}$  in tax-deferred accounts, investors hold  $\Theta_t^*$  in tax-deferred accounts while change their taxable holding to

$$\widehat{H}_t(X_t) = H_t(X_t) - W_t^D [\Delta_{t:T}^* - \Delta_{t:T}^D(X_t) - (z_{t:T}^* - z_{t:T}^D(X_t))e_N^1] e_T',$$

where  $e_T^t = [0, \dots, 1, \dots, 0]'$  is an index vector of size  $T$  that equals to one for the  $t$ -th element and zero otherwise. By the definition of replicating portfolios, the total return from following  $\Theta_t^*$  in tax-deferred accounts and  $H_t(X_t) - W_t^D [\Delta_{t:T}^* - \Delta_{t:T}^D(X_t)] e_T'$  in taxable accounts is equivalent to that of following the original strategy of  $\Theta_{X_t}$  and  $H_t(X_t)$ . The definition of  $\Theta_t^*$  ensures that  $z_{t:T}^* \geq z_{t:T}^D(X_t)$ , and the new strategy of following  $\Theta_t^*$  in tax-deferred accounts dominates the original strategy by recovering the original risk profile, while increasing the taxable wealth by the difference in the replication costs. Hence, the optimal tax-deferred strategy is state-independent and maximizes the replication costs, or  $\Theta_{t:T}^* = [\Theta_t^*, \Theta_{t+1:T}^*]$ . It is also clear that  $\Theta_{t:T}^*$  is independent of  $H_t$ .

## Proof of Proposition 2.

The proof follows directly from those of Lemma 3 and Proposition 1. After replacing  $z_{t+1:T}^D$  in Equation (4) by  $z_{t+1:T}^*$ , it is easy to see that the optimal tax-deferred strategy can be determined myopically:

$$z_{i:T}^* = \frac{1}{R_f} [(1 - q_{i,t})r_i^H + q_{i,t}r_i^L] \times z_{t+1:T}^* = z_{i,t}^D z_{t+1:T}^*. \quad (9)$$

and

$$z_{t:T}^* = \Theta_i^* [z_{t_0:T}^*, \dots, z_{t_n:T}^*] = \Theta_i^* [z_{0,t}^D, \dots, z_{N,t}^D] z_{t+1:T}^* \quad (10)$$

Given the linear structure,  $z_{t:T}^*$  is maximized by holding only assets that provide the highest current-period effective tax subsidy  $z_{i,t}^D$ . ■

## Proofs of Proposition 3 and Corollaries 1, 2.

First, we show that investors always contribute the maximum allowed to their tax-deferred accounts, or  $M_t = \overline{M}_t$  at any time  $t$ . If not, they can increase their contribution by  $\epsilon$ , following the exogenously specified optimal tax-deferred strategy, while shorting the replicating portfolio in taxable accounts. Clearly, they match the risk of the original profile while increase their taxable wealth by  $\epsilon(z_{t:T}^* - 1) \geq 0$  as long as  $z_{t:T}^* > 1$  (or as long as there exists an asset with positive tax rate).

Second, We map the two-account problem to that with only a taxable account. Let  $V_t(H_{t-1}, Y_t, W_t^D, P_t)$  be the indirect utility function for investors at time  $t$ , defined in Equation (2), we show that there exists an  $\widehat{H}_{t-1}^M$  such that

$$V_t(H_{t-1}, Y_t, W_t^D, P_t) = V_t(\widehat{H}_{t-1}^M, Y_t, 0, P_t).$$

Since the optimal tax-deferred strategy is well-defined in Proposition 2, the replication argument in Lemma 3 indicates that investors are indifferent between having  $W_t^D$  in tax-deferred accounts or having  $W_t^D z_{t:T}^*$  dollars of cash in taxable accounts, while each dollar holding in tax-deferred accounts is equivalent to a holding of taxable replicating portfolio  $\Delta_{t:T}^D$ . Therefore, the total current  $W_t^D$  and future contribution are equivalent to  $Z_{t:T}^D W_t^D + Z_{t:T}^M$  in taxable accounts, where  $Z_{t:T}^M = \sum_{s=t}^T \frac{1}{R_f^{s-t}} \bar{M}_s Z_{s:T}^D$  is the present value of all future contributions.

The optimal consumption can simply be determined using the adjusted wealth level, since investors can always borrow to finance their consumption.

The overall risk exposure can be determined using the taxable-account-only solution with the adjusted wealth level. Since the tax-deferred holding is equivalent to a replicating portfolio in taxable, the optimal taxable portfolio is then equal to the total risk exposure minus the replicating portfolio for tax-deferred holdings. ■

## Proof of Lemma 4.

The proof is by replication and induction. Let us consider a single-period problem first. Since markets are complete, any asset held in the tax-deferred account can be replicated by stocks and bonds in the taxable account. Let us consider an investor who holds 1 dollar worth of bonds in the tax-deferred account for a period. At the end of the period, the value of this position will be  $r_f$ . To achieve the same return, the investor can hold  $\frac{r_f}{R_f} > 1$  bonds in the taxable account. A bond in the tax-deferred account can hence be replicated by  $\frac{r_f}{R_f}$  bonds in the taxable account. The cost of this replication strategy is  $z_B = \frac{r_f}{R_f}$ . An investor who only has access to a taxable account and is endowed of 1 dollar of wealth to invest, will need a *subsidy* of  $z_B - 1$  to be as well off as an investor who can invest his 1-dollar endowment in tax-deferred bonds. Similarly, let us now consider the case of an investor who holds 1 dollar worth of stocks in the tax-deferred account for a period. At the end of the period the value of this position will be  $r_H$  or  $r_L$  depending on the realization of the prevailing state of the world. This return can be replicated by holding  $\frac{R_H r_L - R_L r_H}{R_f (R_H - R_L)}$  stocks and  $\frac{r_H - r_L}{R_H - R_L}$  bonds in the taxable account. The cost of this replication strategy is  $z_S = \frac{1}{R_f} [(1 - q)r_H + qr_L]$ , where  $q = \frac{R_H - R_f}{R_H - R_L}$ . An investor who only has access to a taxable account and is endowed of 1 dollar of wealth to invest, will need a *subsidy* of  $z_S - 1$  to be as well off as an investor who can invest his 1-dollar endowment in bond held in a tax-deferred account. Notice that the replication cost  $z_S$  is the discounted expected return of a tax-exempt stock under the martingale measure  $q$  implied by the distribution of asset returns in the taxable account.

The location decision therefore reduces to the decision to choose the asset to hold in the tax-deferred account that maximizes the effective subsidy (or, similarly, that has the highest replication cost). Using the fact that  $\tau_d > \tau_s$  and the definitions of  $R_f = r_f - \tau_d(r_f - 1)$  and  $R_i = r_i - \tau_s(r_i - 1)$ ,  $i = H, L$ , it is immediate to show that,  $z_B > z_S$ . This implies that (i) the one period effective tax subsidy  $Z^{ce} = z_b - 1 = \frac{r_f}{R_f}$ , (ii) investor hold only bonds in the tax deferred account, i.e.  $\theta^D(W, \phi) = 0$ . To determine  $\theta(\phi)$  note that an investor with  $(1 - \phi)W$  in the taxable accounts and  $\phi W$  in the tax-deferred should be as well off as having a taxable wealth of  $\bar{W} \equiv W [1 + \phi(z_B - 1)]$  and no access to tax-deferred accounts. This reduces the problem to a traditional portfolio problem with only one account. Let  $\bar{\theta}(\bar{W})$  be the optimal allocation for an investor holding one account only. Since the portfolio  $\theta(\phi)$  is expressed as a fraction of nominal

wealth  $W$  we obtain  $\theta(W, \phi) \equiv \frac{\bar{\theta}(\widehat{W})\widehat{W}}{W} = \bar{\theta}(\widehat{W}) [1 + \phi(z_B - 1)]$ , proving (iii) for the single period case.

To show the result for any multiperiod problem we proceed by induction. Let us assume that (i), (ii) and (iii) hold at time  $t + 1$  in a  $T$  period problem, i.e., (i)  $Z_{t+1}^{ce} = z_{B,t+1} - 1$ ,  $z_{B,t+1} = \left(\frac{r_f}{R_f}\right)^{T-t-1}$ ; (ii) investors hold only bonds in the tax-deferred account from time  $t + 1$  to  $T$  ( $\theta_k^D(W_k, \phi_k) = 0$ ,  $k = t + 1, \dots, T$ ) and (iii)  $\theta_{t+1}(\phi) = \bar{\theta}_{t+1}(\widehat{W}_{t+1}) [1 + \phi(z_{B,t+1} - 1)]$ , where  $\widehat{W}_{t+1} = W_{t+1}(1 + \phi_{t+1}Z_{t+1}^{ce})$ . Consider an investor selecting his portfolio at time  $t$ . Following the above steps it is possible to show that the replication cost of placing 1 dollar of tax deferred wealth in bonds is  $z_B \times z_{b,t+1} = \left(\frac{r_f}{R_f}\right)^{T-t} \equiv z_{B,t}$  and the replication cost of placing 1 dollar of tax-deferred wealth in stock is  $z_S \times z_{B,t+1}$ . Since  $z_S < z_B$  it follows that  $z_{S,t} < z_{B,t}$  and therefore it is optimal to hold bonds in the tax deferred account at time  $t$  and that the tax-subsidy received from implementing this strategy for the remaining  $T - t$  periods is  $Z_t^{ce} = z_{B,t} - 1$ . We conclude that, for all  $t = 0, 1, \dots, T - 1$  (i) the effective tax subsidy at time  $t$  is  $Z_t^{ce} = z_{B,t} - 1 = \left(\frac{r_f}{R_f}\right)^{T-t}$ ; (ii) investors only hold bonds in the tax-deferred account, i.e.  $\theta_t^D(W_t, \phi_t) = 0$  and (iii) given a single account solution  $\bar{\theta}_t(\widehat{W}_t)$ ,  $\theta_t(W_t, \phi_t) = \bar{\theta}_t^T(\widehat{W}_t) [1 + \phi_t Z_t^{ce}]$ , where  $\widehat{W}_t = W_t(1 + \phi_t Z_t^{ce})$ . ■

## Proof of Proposition 5.

We prove only the case in which  $\frac{z_H}{z_L} < \frac{R_S^H - R_f}{R_f - R_S^L} \frac{r_f - r_L^L}{r_S^H - r_f} \equiv \bar{z}$ . The case of  $\frac{z_H}{z_L} \geq \bar{z}$  follows similar steps.

Let  $(\theta, \theta^D) \in (0, 1) \times (0, 1)$  be a generic interior portfolio and consider a perturbation  $(\theta + \xi, \theta^D - \zeta)$  of it with  $\xi, \zeta > 0$ . Starting with an initial wealth  $W_t$  and given an exogenous  $\bar{z} = \{z_H, z_L\}$ , the expected utility under the perturbed portfolio is

$$E[\mathcal{U}(\bar{W}_{t+1}(\theta, \theta^D) + W_0(\xi(\bar{R}_s - R_f) - \zeta\bar{z}(\bar{r}_s - r_f)))] \quad (11)$$

where  $\bar{W}_{t+1}(\theta, \theta^D) = W_0[\theta\bar{R}_s + (1 - \phi - \theta)R_f + \bar{Z}_1(\theta^D\bar{r}_s + (\phi - \theta^D)r_f)]$  is the next period wealth from the original portfolio  $(\theta, \theta^D)$ .<sup>3</sup> Notice that

$$\bar{z}\bar{r}_s = \Delta_S\bar{R}_s + (z_S - \Delta_S)R_f \quad (12)$$

$$\bar{z}\bar{r}_f = \Delta_B\bar{R}_s + (z_B - \Delta_B)R_f \quad (13)$$

where

$$\Delta_S \equiv \frac{z_H r_S^H - z_L r_L^L}{R_S^H - R_S^L} \quad z_S \equiv \frac{1}{R_f} [(1 - q)z_H r_S^H + qz_L r_L^L], \quad (14)$$

$$\Delta_B \equiv \frac{r_f(z_H - z_L)}{R_S^H - R_S^L} \quad z_B \equiv \frac{1}{R_f} [(1 - q)r_f z_H + q r_f z_L] \quad (15)$$

and  $q \equiv \frac{R_S^H - R_f}{R_S^H - R_S^L}$ . Let us substituting (12) and (13) in (11) and choose  $\xi$  such that  $\xi = \zeta(\Delta_S - \Delta_B)$ . Since  $(\Delta_S - \Delta_B) > 0$  and  $\zeta > 0$ ,  $\xi > 0$ . The expected utility under the perturbed strategy is

$$E[\mathcal{U}(\bar{W}_{t+1}(\theta, \theta^D) + W_t(R_f(z_B - z_S)))]$$

<sup>3</sup>Recall that we are ignoring contributions  $M$  to the tax-deferred account.

It is easy to verify that if  $z_H/z_L < \frac{R_S^H - R_f}{R_f - R_S^L} \frac{r_f - r_L^L}{r_S^H - r_f} \equiv \bar{z}$ ,  $z_B > z_S$  and therefore the perturbation  $(\theta + \xi, \theta^D - \zeta)$  with  $\zeta > 0$  and  $\xi = \zeta(\Delta_S - \Delta_B) > 0$  improve the expected utility without adding any risk. We can continue to perturb as far as the borrowing and short selling constraint do not bind. This means that one of the following three cases will prevail when  $z_H/z_L < (\bar{z})$ : (i)  $\theta_t \in (0, 1 - \phi_t)$ ,  $\theta_t^D = 0$  (Only bonds in the tax-deferred account); (ii)  $\theta_t = 1 - \phi_t$ ,  $\theta_t^D \in (0, 1)$  (Stocks in tax-deferred only) when 100% of the taxable account is invested in stocks; (iii)  $\theta_t = 1 - \phi_t$ ,  $\theta_t^D = 0$ . In summary,  $(\theta_t, \theta_t^D) \in ([0, 1 - \phi_t] \times \{0\}) \cup (\{1 - \phi_t\} \times [0, \phi_t])$ . The case of  $z_H/z_L \geq \bar{z}$  is dealt with similarly by choosing a perturbed portfolio  $(\theta + \xi, \theta^D - \zeta)$  with  $\zeta < 0$  and  $\xi = \zeta(\Delta_S - \Delta_B) < 0$ . This yields an optimal portfolio  $(\theta_t, \theta_t^D) \in (\{0\} \times [0, \phi_t]) \cup ([0, 1 - \phi_t] \times \{\phi_t\})$ . ■

## Proof of Rules of Thumb 6.

Consider an investor with  $(1 - \phi_t)W_t$  dollars in the taxable account and  $\phi_t W_t$  in the tax-deferred. The second inequality follows by noticing that, from Lemma 4, when markets are complete and  $\tau_{s,s} < \tau_{s,b}$ , it is optimal to place the whole taxable wealth in bonds. This is equivalent of receiving a non tradeable riskless portfolio of  $\Delta_{B,t}$  bonds in the taxable account. The effective tax subsidy in this case is  $Z_t^{ce} = z_{B,t} - 1$ . If markets are incomplete, the inequality  $1 + Z_t^{ce} \leq z_{B,t}$  follows trivially. To show the first inequality,  $z_{s,t} \leq 1 + Z_t^{ce}$ , we need to show that investing all the tax-deferred wealth in stocks is the worst possible strategy. To see this, notice that, by investing all his taxable wealth in stocks, it is as if the investor is adding a non-tradeable risky portfolio of  $\Delta_{S,t}$  dollars in stocks and  $z_{s,t} - \Delta_{S,t}$  dollars in bonds in the taxable account. Since  $\Delta_{S,t} > \Delta_{B,t}$ ,  $z_{s,t} \leq z_{B,t}$  and investors are risk averse it must be that the value of this strategy is less than the optimal complete market strategy. Moreover, any other strategy in the taxable account would be equivalent to a larger portfolio in the taxable account containing less risky assets and therefore will be preferred to a strategy of investing the whole tax-deferred wealth in stocks. ■

## Proof of Lemma 5 and Proposition 6.

From Proposition 5, if  $z_H/z_L < \bar{z}$  the optimal portfolio  $(\theta_t, \theta_t^D)$  can lie on either the set  $[0, 1 - \phi_t] \times \{0\}$  or on the set  $\{1 - \phi_t\} \times [0, \phi_t]$ .

Let us consider the case  $(\theta_t, \theta_t^D) \in [0, 1 - \phi_t] \times \{0\}$  first. Assume an initial wealth  $W_t$  and a tax-deferred account  $\phi_t$ . Given that  $\theta_t^D = 0$ , the investor face the following two account problem

$$V_t(W_t, \phi_t) = \max_{\theta_t} E_t[\mathcal{U}(W_t(\theta_t(\bar{R}_S - R_f) + (1 - \phi_t)R_f) + \bar{z}\phi_t r_f)] \quad (16)$$

s.t.  $0 \leq \theta_t \leq 1 - \phi_t$ . By Proposition 5, each dollar in tax-deferred bonds is equivalent to a portfolio of  $\$ \Delta_B$  in taxable stocks and  $\$(z_B - \Delta_B)$  in taxable bonds, where  $\Delta_B$  and  $z_B$  are defined in (15). Since all of the tax deferred wealth is invested in bonds, the problem of selecting a portfolio  $\theta_t$  with  $(1 - \phi_t)W_t$  wealth in the taxable account and  $\phi_t W_t$  wealth in the tax-deferred account is equivalent to the problem of selecting a portfolio  $\bar{\theta}_t$  with only a taxable account of size  $\widehat{W}_t = W_t(1 - \phi + z_B \phi_t)$  provided that

$$\bar{\theta}_t \widehat{W}_t \geq \phi_t W_t \Delta_B \quad (17)$$



$$\widehat{W}_t(1 - \bar{\theta}) \geq \phi_t W_t(z_B - \Delta_B). \quad (18)$$

Constraints (17) and (18) follow from the fact that a portfolio of only bonds in the tax-deferred account is equivalent to a portfolio of  $\$ \Delta_B$  in taxable stocks and  $\$(z_B - \Delta_B)$  in taxable bonds. Therefore, in order to “accommodate” this tax-deferred portfolio in the taxable account, any taxable-account only portfolio  $\bar{\theta}$  should allow for at least  $\Delta_B \phi W_t$  stocks (constraint (17)) and at least  $\phi_t(z_B - \Delta_B)$  bonds (constraint (18)). Substituting the expression for  $\bar{z}r_f$  from (13) into (16) and using  $\widehat{W}_t = W_t(1 - \phi + z_B \phi_t)$  we obtain the following taxable account-only formulation

$$\begin{aligned} V_t(W_t, \phi_t) &= \max_{\bar{\theta}_t} E_t[\mathcal{U}(W_t((\bar{\theta}_t + \phi_t \Delta_B)(\bar{R}_S - R_f) + (1 - \phi_t + \phi_t z_{B,t})R_f))] \\ &= \max_{\bar{\theta}} E_t[\mathcal{U}(\widehat{W}_t(\bar{\theta}(\bar{R}_S - R_f) + R_f))] \\ &= V(\widehat{W}_t) \end{aligned} \quad (19)$$

where we defined

$$\theta_t + \phi_t \Delta_B \equiv \bar{\theta}(1 - \phi_t + \phi_t z_{B,t}), \quad \widehat{W}_t = W_t(1 + \phi_t(z_{B,t} - 1)) \quad (20)$$

and, by (17) and (18),

$$\frac{\phi_t \Delta_{B,t}}{1 - \phi_t + \phi_t z_{B,t}} \leq \bar{\theta}_t \leq \frac{1 - \phi_t + \phi_t \Delta_{B,t}}{1 - \phi_t + \phi_t z_{B,t}} \quad (21)$$

By assumption the solution to the taxable-account-only problem with initial wealth  $\widehat{W}_t$  is  $\bar{\theta}_t^T(\widehat{W}_t)$ . Therefore, substituting (20) in the constraint (21), we obtain  $\theta_t(W_t, \phi_t) = \min\{1 - \phi_t, \max\{0, \bar{\theta}_t\}\}$  where the unconstrained solution  $\bar{\theta}_t$  satisfies (20), i.e.,  $\bar{\theta}_t = \bar{\theta}_t^T(\widehat{W}_t)[1 - \phi_t + \phi_t z_{B,t}] - \phi_t \Delta_{B,t}$ . This completes the proof of the first candidate solution in (15).

Consider now the case  $(\theta_t, \theta_t^D) \in 1 - \phi_t \times [0, 1 - \phi_t]$ . Following similar steps, it is possible to show that the two-account problem with  $(1 - \phi_t)W_t$  wealth in the taxable account and  $\phi_t W_t$  wealth in the tax-deferred account is equivalent to a tax-deferred-account-only problem with initial wealth equal to  $\widehat{W}_t^D = W_t(1 + (1 - \phi_t)z_{S,t}^D)$ . This follows since each dollar in the taxable stocks is equivalent to  $\Delta_{S,t}^D$  dollars in tax-deferred stocks and  $z_{S,t}^D - \Delta_{S,t}^D$  dollars in tax-deferred bonds, where  $\Delta_{S,t}^D$  and  $z_{S,t}^D$  are defined as

$$\Delta_{S,t}^D \equiv \frac{\frac{R_S^H}{z_H} - \frac{R_S^L}{z_L}}{r_S^H - r_L^L}, \quad z_{S,t}^D \equiv \frac{1}{r_f} \left[ (1 - q^D) \frac{R_S^H}{z_H} + q^D \frac{R_S^L}{z_L} \right]$$

and  $q^D \equiv \frac{r_S^H - r_f}{r_S^H - r_L^L}$ . Following similar steps as before, we can rewrite the two-account problem into a constrained tax-deferred-account-only problem as follows

$$V_t(W_t, \phi_t) = \max_{\bar{\theta}_t^D} E_t[\mathcal{U}(W_t(\frac{1}{z}(1 - \phi_t)\bar{R}_S + \bar{\theta}_t^D(\bar{r}_S - r_f) + \phi_t r_f))] \quad (22)$$

$$= \max_{\bar{\theta}_t^D} E_t[\mathcal{U}(W_t((\bar{\theta}_t^D + (1 - \phi_t)\Delta_{S,t}^D)(\bar{r}_S - r_f) + (\phi_t + (1 - \phi_t)z_{S,t}^D)r_f))] \quad (23)$$

$$= \max_{\bar{\theta}^D} E_t[\mathcal{U}(\widehat{W}_t^D(\bar{\theta}^D(\bar{r}_S - r_f) + r_f))] \quad (23)$$

$$= V(\widehat{W}_t^D)$$

where we defined

$$\theta_t^D + (1 - \phi_t)\Delta_{S,t}^D \equiv \bar{\theta}^D(\phi_t + (1 - \phi_t)z_{S,t}^D), \quad \widehat{W}_t^D = W_t(\phi_t + (1 - \phi_t)z_{S,t}^D) \quad (24)$$

and, by the equivalent of (17) and (18),

$$\frac{1 - \phi_t\Delta_{S,t}^D}{\phi_t + (1 - \phi_t)z_{S,t}^D} \leq \bar{\theta}_t^D \leq \frac{\phi_t + (1 - \phi_t)\Delta_{S,t}^D}{\phi_t + (1 - \phi_t)z_{S,t}^D} \quad (25)$$

Since the optimal solution to the tax-deferred account only problem with initial wealth  $\bar{\theta}_t^D(\widehat{W}_t^D)$  is known, the solution for the two account is hence given by  $\theta_t^D(W_t, \phi_t) = \max\{0, \min\{\phi_t, \widehat{\theta}_t^D\}\}$  where the unconstrained solution  $\widehat{\theta}_t^D$  is obtained from (24), i.e.,  $\widehat{\theta}_t^D = \bar{\theta}_t^D(\widehat{W}_t^D) [\phi_t + (1 - \phi_t)z_{S,t}^D] - (1 - \phi_t)\Delta_{S,t}^D$ .

If  $z_H/z_L > \bar{z}$ , following similar steps as above, it is easy to obtain the following characterization of the optimal portfolio:

$$\begin{cases} \theta_t(W_t, \phi_t) = \min\{1 - \phi_t, \max\{0, \widehat{\theta}_t^D\}\}, & \theta_t^D(W_t, \phi_t) = \phi_t. \\ \theta_t(W_t, \phi_t) = 0, & \theta_t^D(W_t, \phi_t) = \max\{0, \min\{\phi_t, \widehat{\theta}_t^D\}\}, \end{cases} \quad (26)$$

where

$$\widehat{\theta}_t = \bar{\theta}_t^T(\widehat{W}_t) [1 - \phi_t + \phi_t z_{S,t}] - \phi_t \Delta_{S,t}, \quad \widehat{W}_t = W_t(1 + \phi_t(z_{S,t} - 1)) \quad (27)$$

$$\widehat{\theta}_t^D = \bar{\theta}_t^D(\widehat{W}_t^D) [\phi_t + (1 - \phi_t)z_{B,t}^D] - (1 - \phi_t)\Delta_{B,t}^D, \quad \widehat{W}_t^D = W_t(\phi_t + (1 - \phi_t)z_{B,t}^D), \quad (28)$$

where  $z_{S,t}$  and  $\Delta_{S,t}$  are given in equation (12), and  $\Delta_{B,t}^D$  and  $z_{B,t}^D$  are defined as

$$\Delta_{B,t}^D \equiv \frac{R_f - R_f}{z_H - z_L}, \quad z_{B,t}^D \equiv \frac{R_f}{r_f} \left[ (1 - q^D) \frac{1}{z_H} + q^D \frac{1}{z_L} \right]$$

with  $q^D \equiv \frac{r_S^H - r_f}{r_S^H - r_L}$ . This completes the proof of the proposition. ■

## Appendix B

# Numerical Implementations

### Implementation of asset returns

Gross stock returns are assumed to follow an exogenously specified binomial process. Let  $\bar{r}_i = \bar{d}_i + (1 + \bar{g}_i)$  and  $1 + \bar{g}_i = u_i$  and  $u_i^{-1}$  with probability  $p_i$  and  $1 - p_i$  respectively, then the conditional mean and variance of the logarithmic asset return are

$$\begin{aligned}\mu_i &\equiv E[\ln \bar{r}_i] = p_i \ln r_i^H + (1 - p_i) \ln r_i^L \\ \sigma_i^2 &\equiv \text{Var}[\ln \bar{r}_i] = p_i(1 - p_i)[\ln r_i^H - \ln r_i^L]^2\end{aligned}$$

If we further assume  $\bar{d}_i = \delta_i \bar{r}_i$ , or that the dividend yield is a constant proportion of the gross return, then we can eliminate  $p_i$  and get  $u_i = e^{\sqrt{(\mu_i + \ln(1 - \delta_i))^2 + \sigma_i^2}}$ . The binomial total return (including dividends) process has a limiting log-normal distribution  $\ln \bar{r}_i \sim N(\mu_i, \sigma_i)$ .

### Calculation of relative values of taxable assets

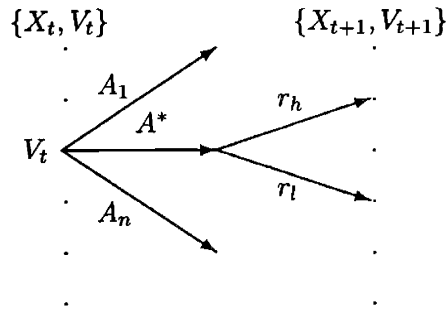
Instead of solving  $z_{i,t}(y)$  for all  $y$ , we focus on a special example for  $y = u^{-k}$ ,  $k = -1, 0, 1, \dots, n$  and  $d_h = \frac{\delta}{1-\delta}u$ ,  $d_l = \frac{\delta}{1-\delta}u^{-1}$  to get the main intuition. The function of  $z_{i,t}(y)$  can be reduced to a recursive equation, when we define  $f(n, k) \equiv z_{i,t}(y)$  for  $y = u^{-k}$

$$f(n, k) = \delta(1 - \tau_d) + \frac{1}{R_f} [(1 - q_n)u f(n - 1, k + 1) + q_n u^{-1} f(n - 1, k - 1)]$$

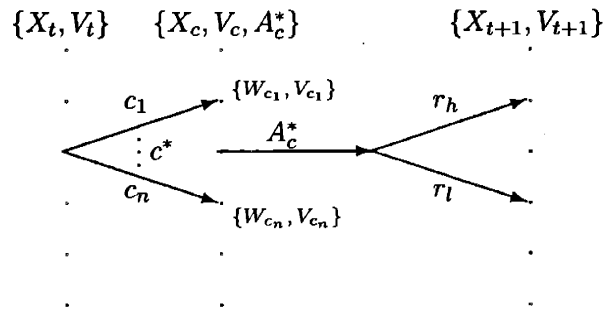
the boundary conditions are  $f(n, -1) = [1 - \tau_g(1 - u)]f(n, 0)$ ,  $f(n, 0) = 1$  and initial conditions are  $f(0, -1) = 1 - \tau_g(1 - u)$  and  $f(0, k) = 1, k \geq 0$  when investors are allowed to escape capital gains tax at death, or  $f(0, k) = 1 - \tau_g(1 - u^{-k}), \forall k$  when investors are forced to realize all capital gains on terminal date. We can then interpolate the result to find  $V$  for any  $y \in [u^{-k}, u^{-k+1}]$ .

### Two-Step Projection Method

Let  $X_t, V_t$  be the state vector and value function, then the following figure illustrates the typical grid methods,



where  $V_T$  is calculated on terminal date  $T$  and then we solve backwards for  $V_t$  at time  $t$ , taking  $X_{t+1}, V_{t+1}$  as given, investors choose action  $A_i$  from the choice set  $A \equiv \{m_t, c_t, \theta_t, \theta_t^D\}$ , the stock returns  $\tilde{r}_{t+1} = r_h$  or  $r_l$ , directing the transition to a state at time  $t + 1$  and the corresponding value function. Average over possible stock returns yields the value function for a specific action  $A_i$ , and  $A^*$  is the optimal action that maximizes the expected value function for a given state  $X_t$ . The two-step method is illustrated as following:



Instead of choosing a four-dimensional  $A_i$ , a hypothetical intermediate step  $X_c$  is introduced, where consumption  $c_t$  is assumed to be chosen already. At state  $X_c$ , following the same method as above, the investor solves the optimal  $A_c^*$  for each  $X_c$ . Then, go back to state  $X_t$  and solve for optimal  $c^*$ . Each  $X_t$  is mapped uniquely to a hypothetical state  $X_c$  for any given  $c_i$ , and there is also a change in the wealth level, for example,  $W_c = 1 - c$  if the investor simply liquidates some bonds to meet consumption. In general, the  $X_c$  space is very dispersed and introducing the extra step does not help much, in this case, however, since  $V_c$  is homogeneous in  $W_c$  for power utility function, we can take the  $W_c$  factor out and restrict the  $X_c$  to a similar bounded space as the original  $X_t$ , successfully reducing the dimension of the choice set. We effectively project the  $\{W_c, X_c\}$  to a smaller set  $X_c$  under the assumed utility. It is necessary that the choice variables are relatively separable for this method to work properly, as in the case of  $c_t$ , or  $m_t$ , however, we should not try to solve  $\theta_t$  and  $\theta_t^D$  separately.

## Appendix C

# Tables and Figures

Table C.1: Parameter Values: Complete Markets.

<i>Parameter</i>	<i>Value</i>
Riskless rate ( $r_f$ )	6%
Stock risk premium ( $\mu - r_f$ )	4%
Dividend yield ( $\delta \equiv \bar{d}/\bar{r}$ )	3%
Stock volatility ( $\sigma$ )	20%
Tax rate on capital gains ( $\tau_g$ )	20%
Tax rate on dividends and interests ( $\tau_d$ )	40%

Table C.2: Parameter Values: Numerical Solutions.

<i>Parameter</i>	<i>Value</i>
Riskless rate ( $R_f$ )	6%
Stock Risk premium ( $\mu - R_f$ )	5%
Stock Volatility ( $\sigma$ )	20%
Tax rate on Bond income ( $\tau_d$ )	50%
Tax rate on Stock income ( $\tau_s$ )	20%
Inflation	3.5%
Retirement Age ( $K$ )	65
Risk Aversion ( $\gamma$ )	5.00
Time Discount ( $\delta$ )	0.96
Maximum allowed annual contribution ( $M$ )	\$10,000
Withdrawal Penalty( $\eta$ )	10%
Liquidity Year	35
Liquidity shock ( $Q$ )	\$100,000
Bequest Motive( $\rho$ )	0.044

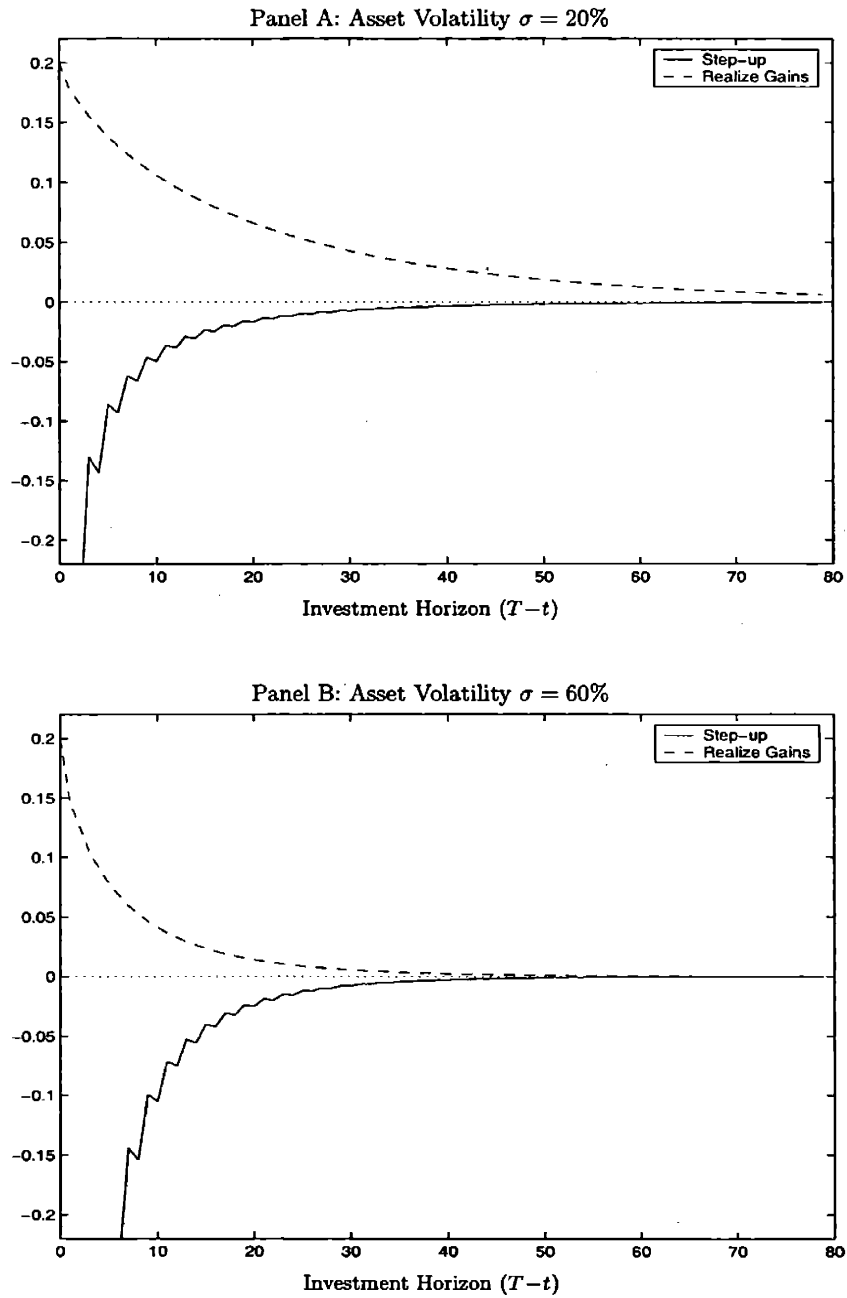


Figure C-1: Effective tax rates  $\hat{\tau}$  as functions of investment horizon for assets with no dividends ( $\bar{d}_i = 0$ ). Panel A and B represents the cases when asset volatility  $\sigma = 20\%$  and  $60\%$ , respectively. The solid lines are for the case when investors are allowed to escape capital gains tax through step-up of tax basis in their taxable accounts on the terminal date, and the dashed lines are for the case when investors are forced to realize all capital gains taxes. The other parameters are given in Table C.1.

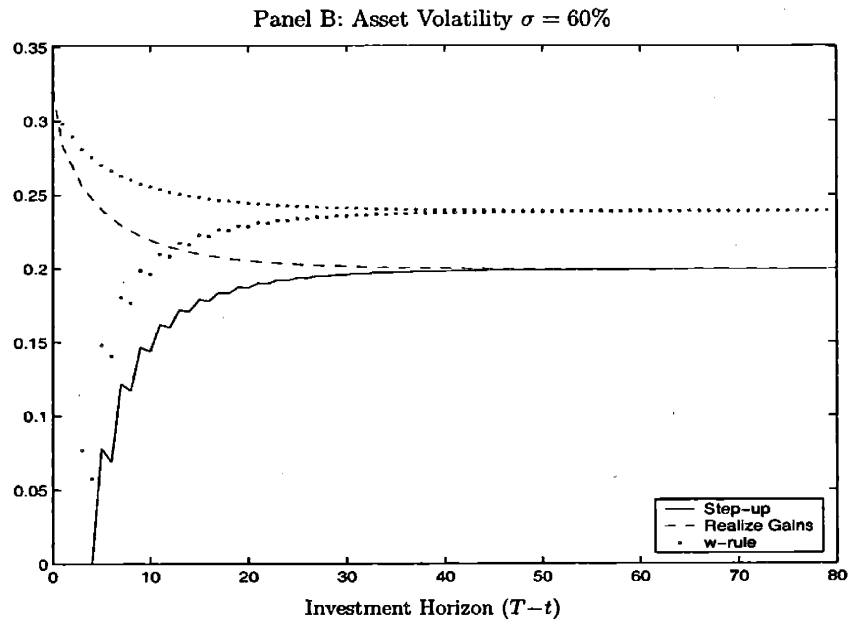
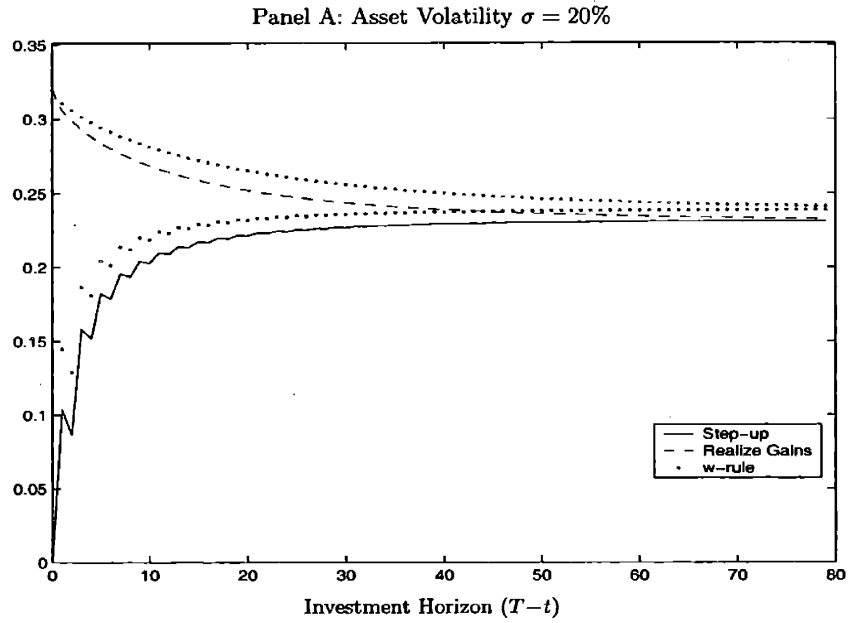


Figure C-2: Effective tax rate  $\hat{\tau}$  as functions of investment horizon when dividend yield is  $\delta = 3\%$ . Panel A and B represents the cases when asset volatility  $\sigma = 20\%$  and  $60\%$ , respectively. The solid lines are for the case when investors are allowed to escape capital gains tax through step-up of tax basis in their taxable accounts on the terminal date, the dashed lines are for the case when investors are forced to realize all capital gains taxes, and the dotted lines are the approximate effective tax rates calculated using Rules of Thumb 3. The other parameters are given in Table C.1.



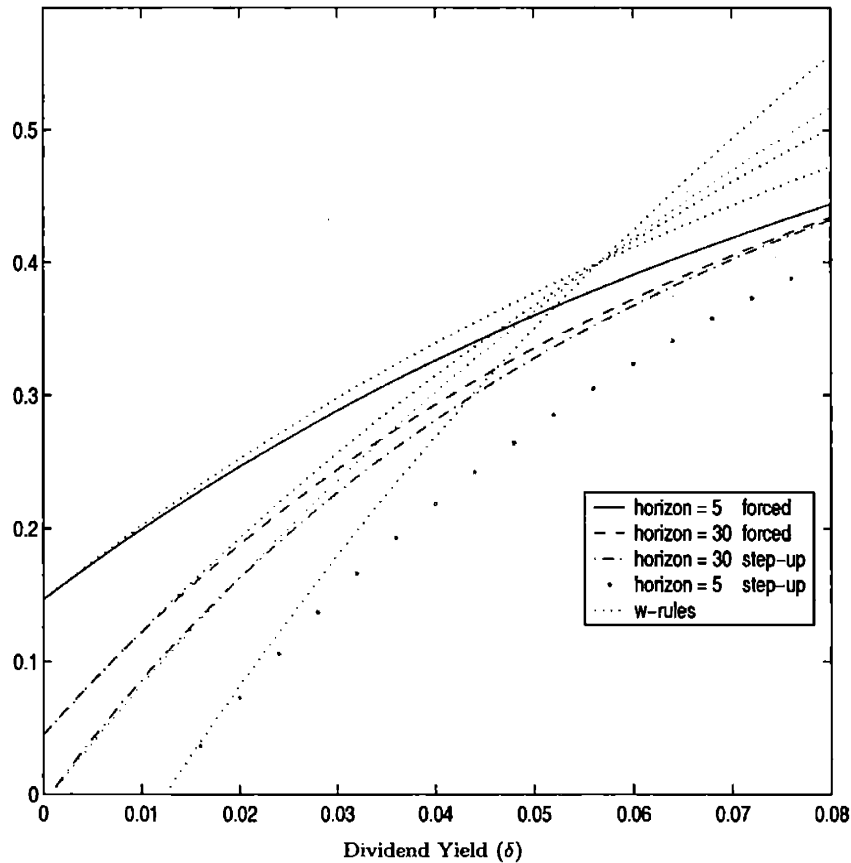


Figure C-3: Effective tax rates  $\hat{\tau}$  as function of dividend yields. The solid and the dashed line report the cases when investors are forced to realize capital gains taxes in their taxable accounts on the terminal date and the remaining investment horizons are  $T-t=5$  and 30 years respectively. The large dotted and dash-dotted lines are for the cases when investors are allowed to escape capital gains tax through step-up of tax basis. The four small dotted lines are calculated by applying the approximate “ $\omega$ ”-rules in Rules of Thumb 3 for each of the above four cases. The four small dotted lines are quite close to the other lines, indicating that the approximation rule works well especially for lower asset volatility and dividend yields. The other parameters are given in Table C.1.

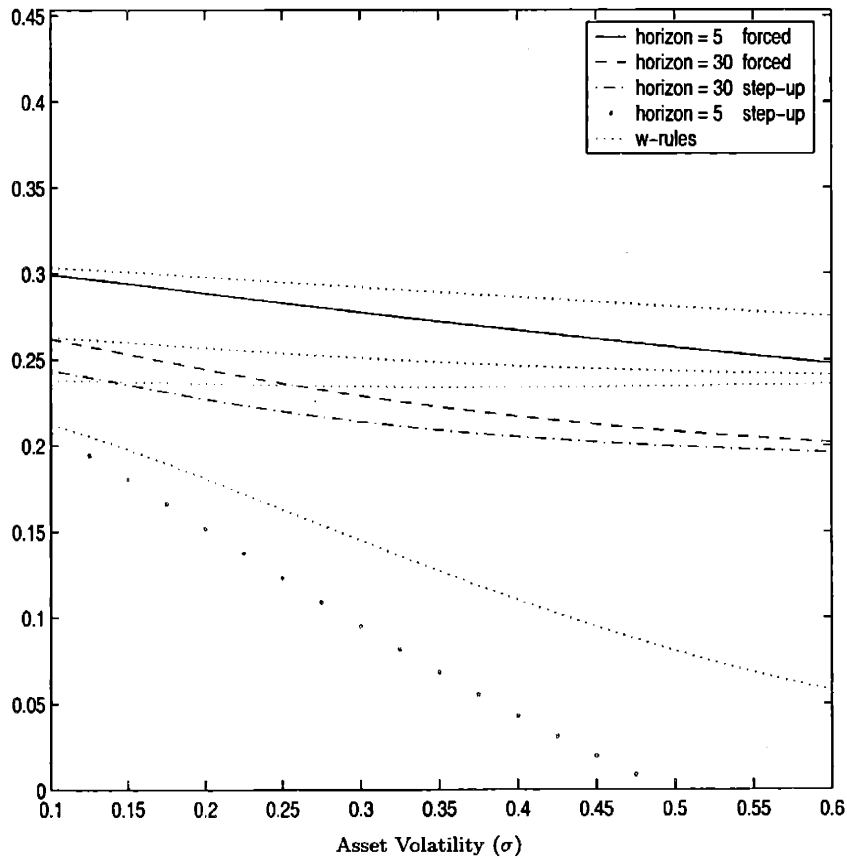


Figure C-4: Effective tax rates  $\hat{\tau}$  as function of asset volatility when  $\delta = 3\%$ . The two solid and dashed lines represent the two different terminal tax treatments when the remaining investment horizon is  $T-t=5$  and 30 years respectively, where the top solid and dashed lines are for the cases when investors are forced to realize all capital gains taxes in their taxable accounts on the terminal date, and the bottom solid and dashed lines are for the case when investors are allowed to escape capital gains tax through step-up of tax basis. The other parameters are given in Table C.1.

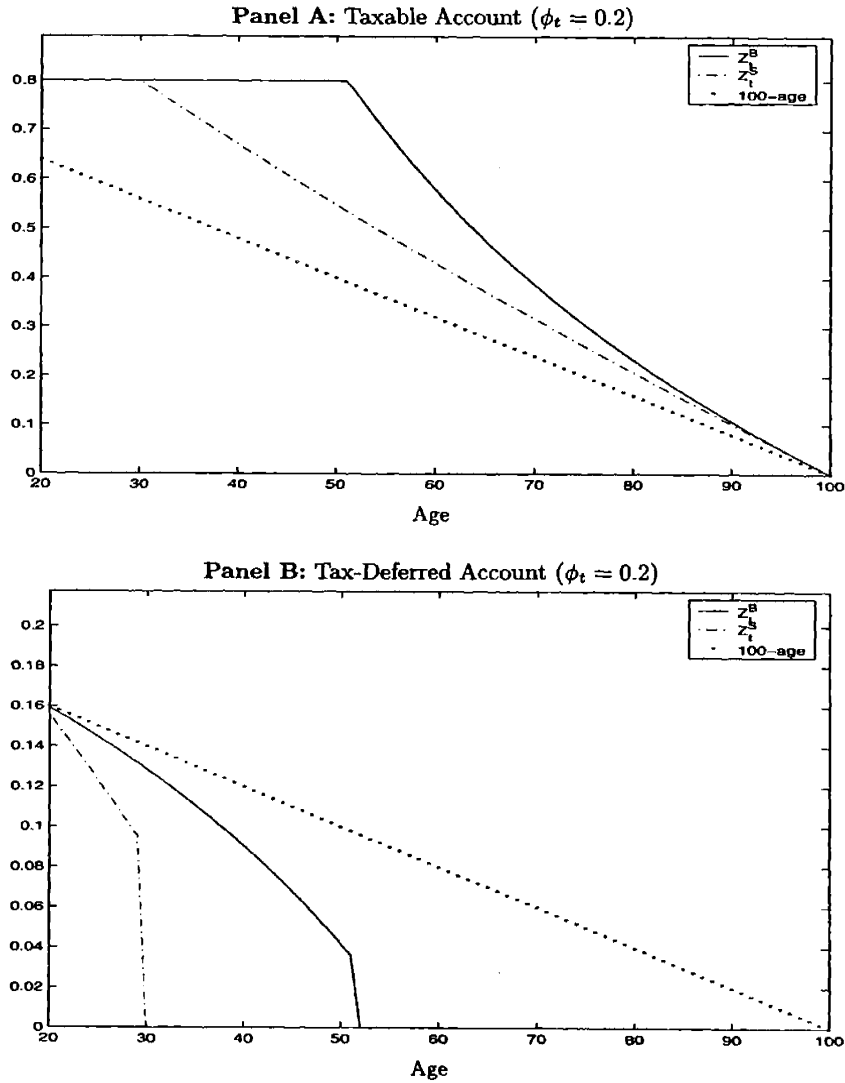


Figure C-5: "100-Age" Rule. Small Tax-Deferred Account. The figure reports the fraction of total wealth invested in taxable stocks (Panel A) and tax-deferred stocks (Panel B) when the size of the tax-deferred account is  $\phi = 0.2$ . The solid line is the upper bound for the optimal portfolio and is obtained by imposing  $z_{B,t} = z_{S,t} = Z_t^B$  in Proposition 6 (see equation (15)). The dashed-dotted line is the lower bound for the optimal portfolio and is obtained by imposing  $z_{B,t} = z_{S,t} = Z_t^S$  in Proposition 6, equation (15). The dotted line, in both panels represents the portfolio chosen by an investor implementing the 100-Age rule in both accounts.

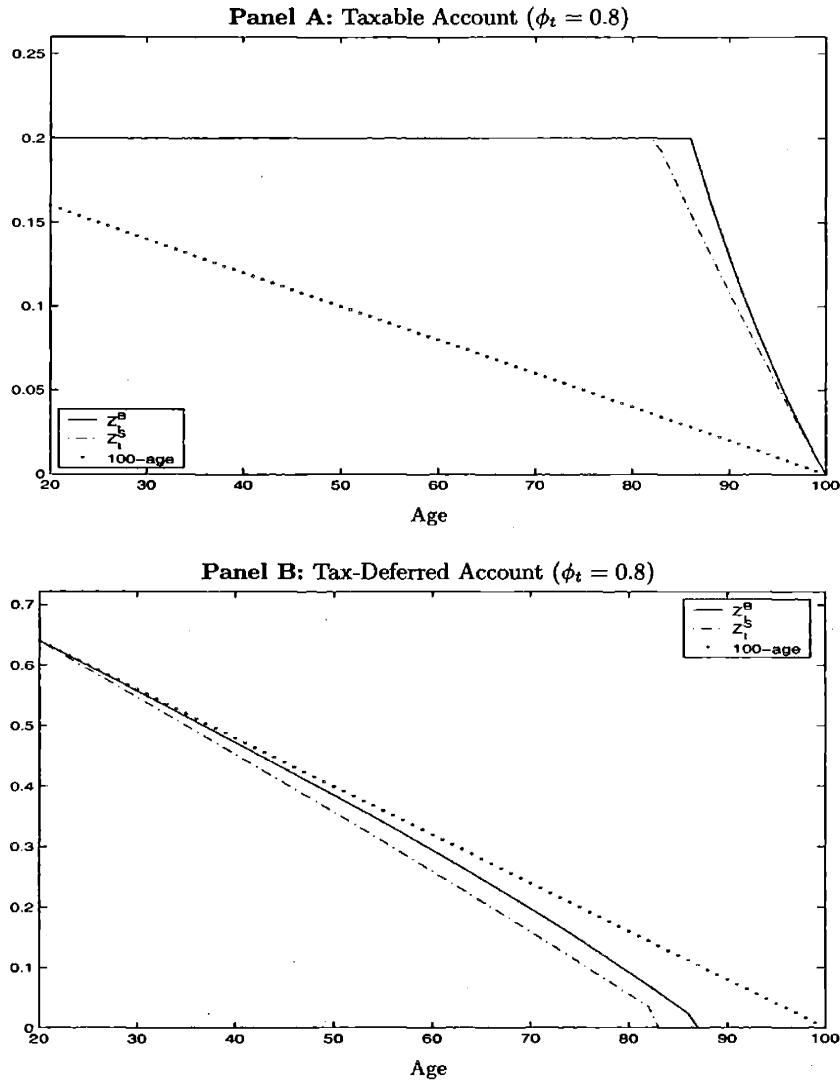


Figure C-6: “100–Age” Rule. **Large Tax-Deferred Account.** The figure reports the fraction of total wealth invested in taxable stocks (Panel A) and tax-deferred stocks (Panel B) when the size of the tax-deferred account is  $\phi = 0.8$ . The solid line is the upper bound for the optimal portfolio and is obtained by imposing  $z_{B,t} = z_{S,t} = Z_t^B$  in Proposition 6 (see equation (15)). The dashed-dotted line is the lower bound for the optimal portfolio and is obtained by imposing  $z_{B,t} = z_{S,t} = Z_t^S$  in Proposition 6, equation (15). The dotted line, in both panels represents the portfolio chosen by an investor implementing the 100–Age rule in both accounts.

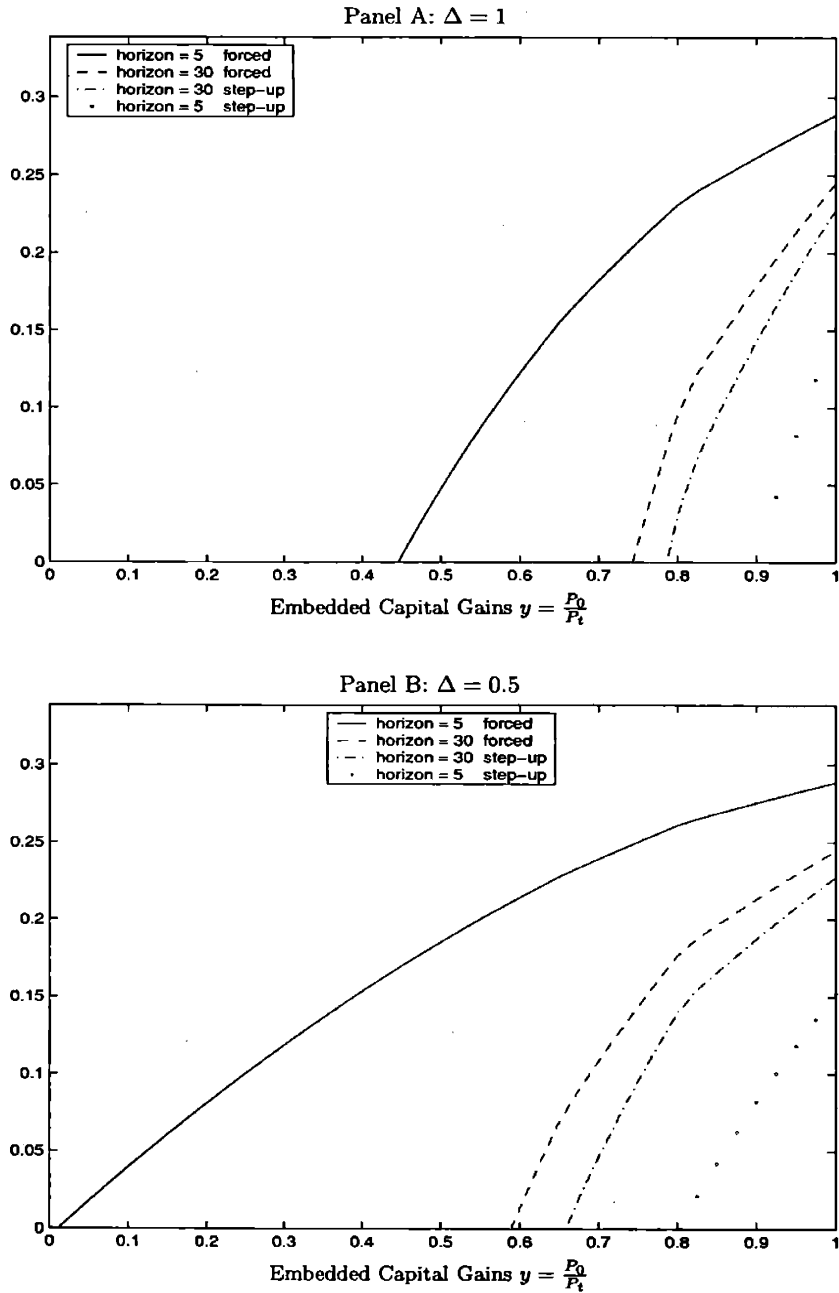


Figure C-7: Effective tax rates  $\hat{\tau}$  as function of embedded capital gains. The assumption is that investors need to sell a share of taxable asset with embedded capital gains ( $y < 1$ ) in order to purchase that asset in the tax-deferred account (to maintain a balanced portfolio). Panel A and B represent two different tax treatments in taxable accounts regarding unrealized capital gains on the terminal date. In each Panel, there are three lines reporting the results for different investment horizons: the solid lines are for the cases with  $T-t=5$  periods before the terminal date, the dashed lines are for 20 periods before, and the dotted lines are for 80 periods before. The other parameters are given in Table C.1.

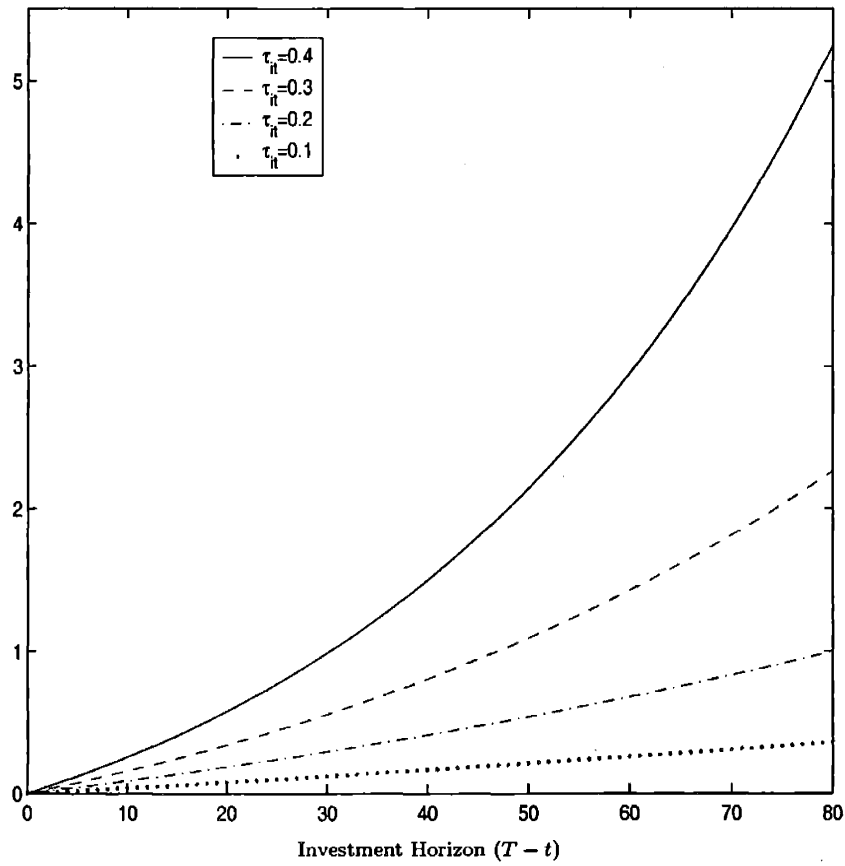


Figure C-8: Effective tax Subsidy  $Z_{t,T}^D$  as functions of investment horizon ( $T-t$ ) when investors hold assets with the following constant effective tax rates  $\hat{\tau}_{i,t}$  in tax-deferred accounts for the remaining investment horizon:  $\hat{\tau}_{i,t} = 40\%$  for the solid line,  $30\%$  for the dashed line,  $20\%$  for the dash-dotted line, and  $10\%$  for the dotted lines, respectively. The risk-free interest rate and tax rates are given in Table C.1.

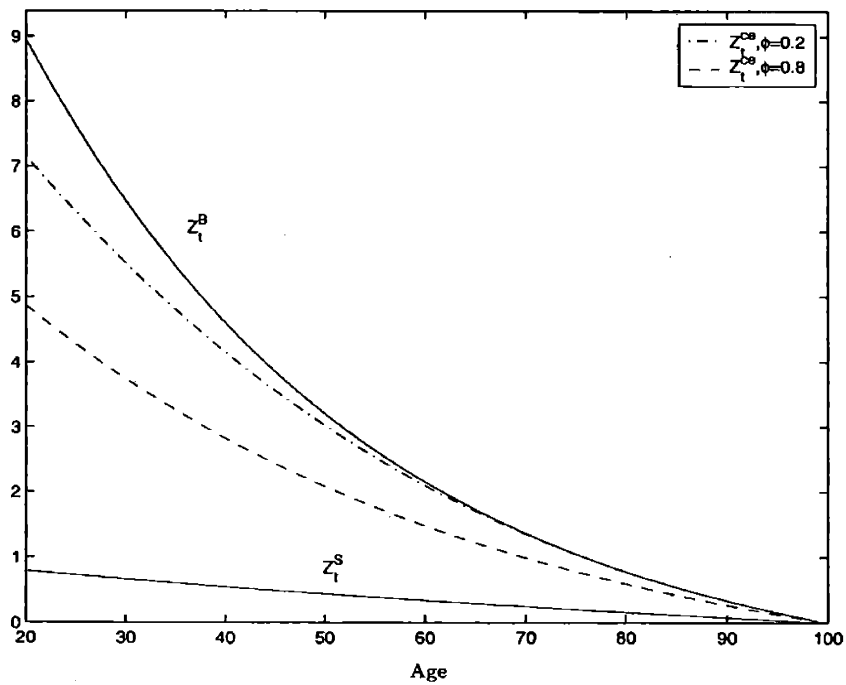


Figure C-9: **Bounds on the Effective Tax-Subsidy  $Z_t^{ce}$ .** The solid line represent the upper bound  $Z_t^B$  and the lower bound  $Z_t^S$  on the effective tax subsidy  $Z_t^{ce}$  as described in equation (18). The inner lines plot the true effective tax subsidy  $Z_t^{ce}$  for two different values of the tax-deferred account ( $\phi_t = 0.2, 0.8$ ) obtained by solving the problem numerically.

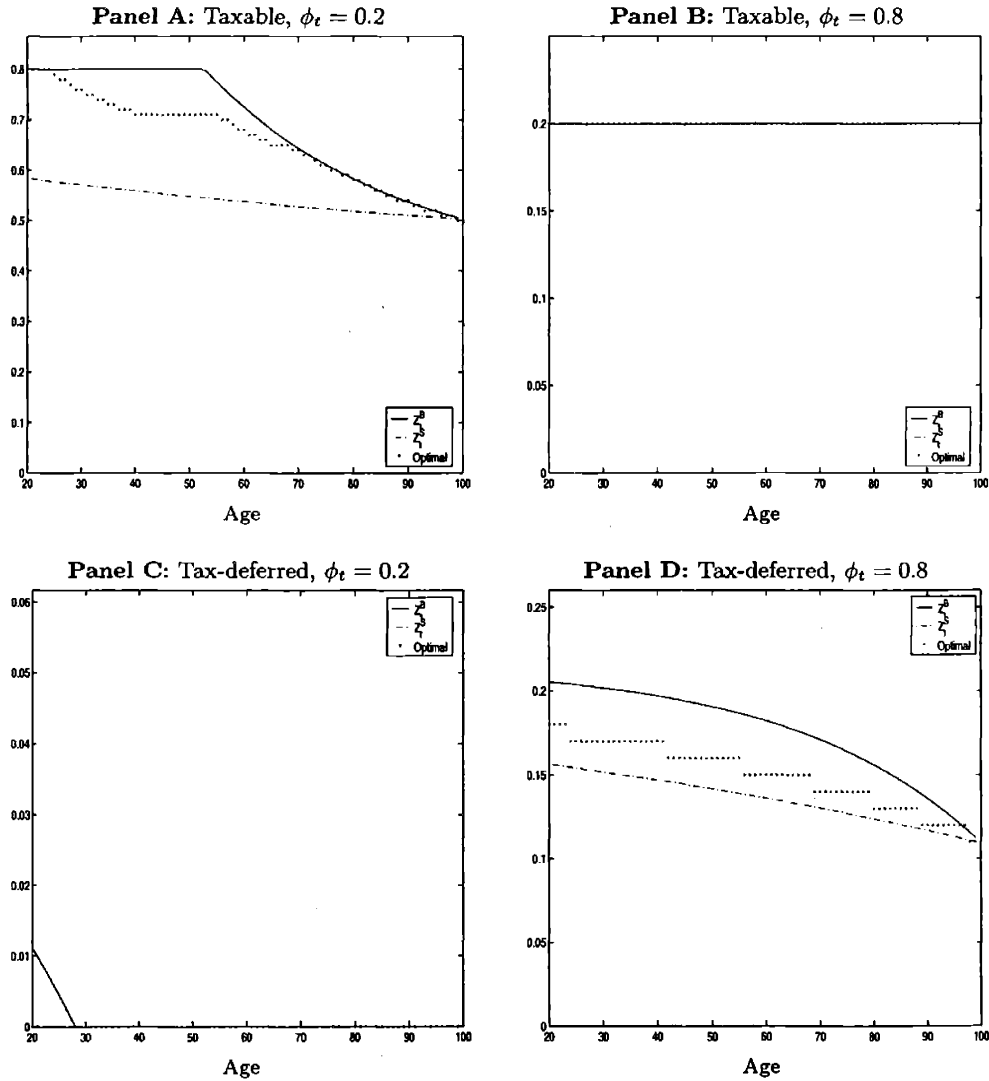


Figure C-10: **Optimal Portfolio vs. Approximate Rules.** The figure reports the fraction of total wealth invested in taxable stocks (Panels A and B) and tax-deferred stocks (Panels C and D) when the size of the tax-deferred account is small ( $\phi = 0.2$ , Panels A and C) and large ( $\phi = 0.8$ , Panels B and D). The solid lines are the upper bounds for the optimal portfolio and are obtained by imposing  $z_{B,t} = z_{S,t} = Z_t^B$  in Proposition 6, equation (15). The dashed-dotted lines are the lower bounds for the optimal portfolio and are obtained by imposing  $z_{B,t} = z_{S,t} = Z_t^S$  in Proposition 6, equation (15). The dotted lines represent the optimal portfolio obtained by solving problem (2) numerically for the parameters described in Table C.2.



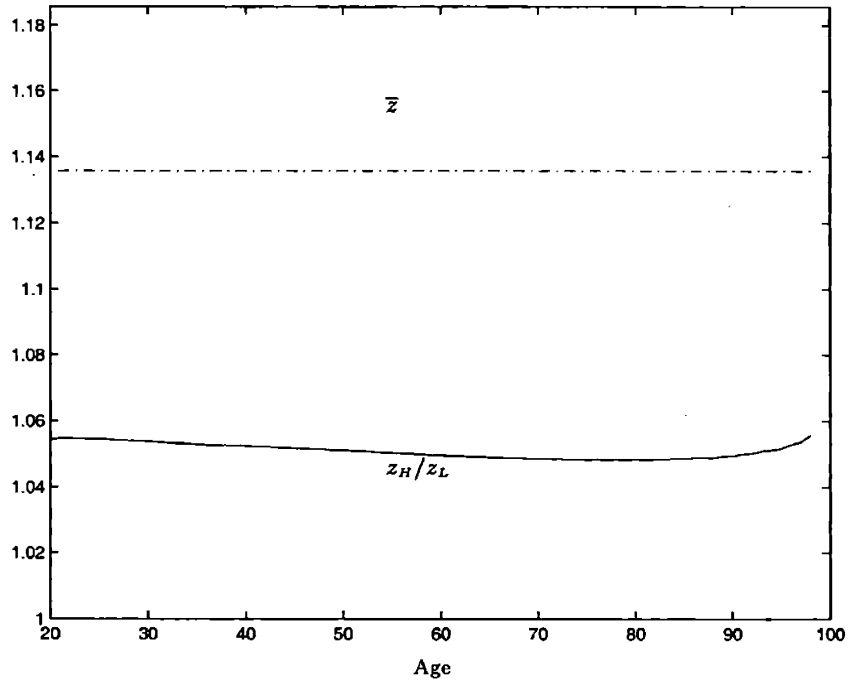


Figure C-11: **Conditions for Pecking Order.** The figure reports, for each date, the highest value achieved by the implied ratio  $z_H/z_L$  over the entire state space  $\phi_t$ . The implied ratio is obtained by solving problem (2) for a CRRA utility function with parameters described in Table C.2. The horizontal line represent the cutoff value  $\bar{z}$ . According to Proposition 5, if  $z_H/z_L < \bar{z}$  investors prefer bonds in the tax-deferred account and prefer stocks otherwise.

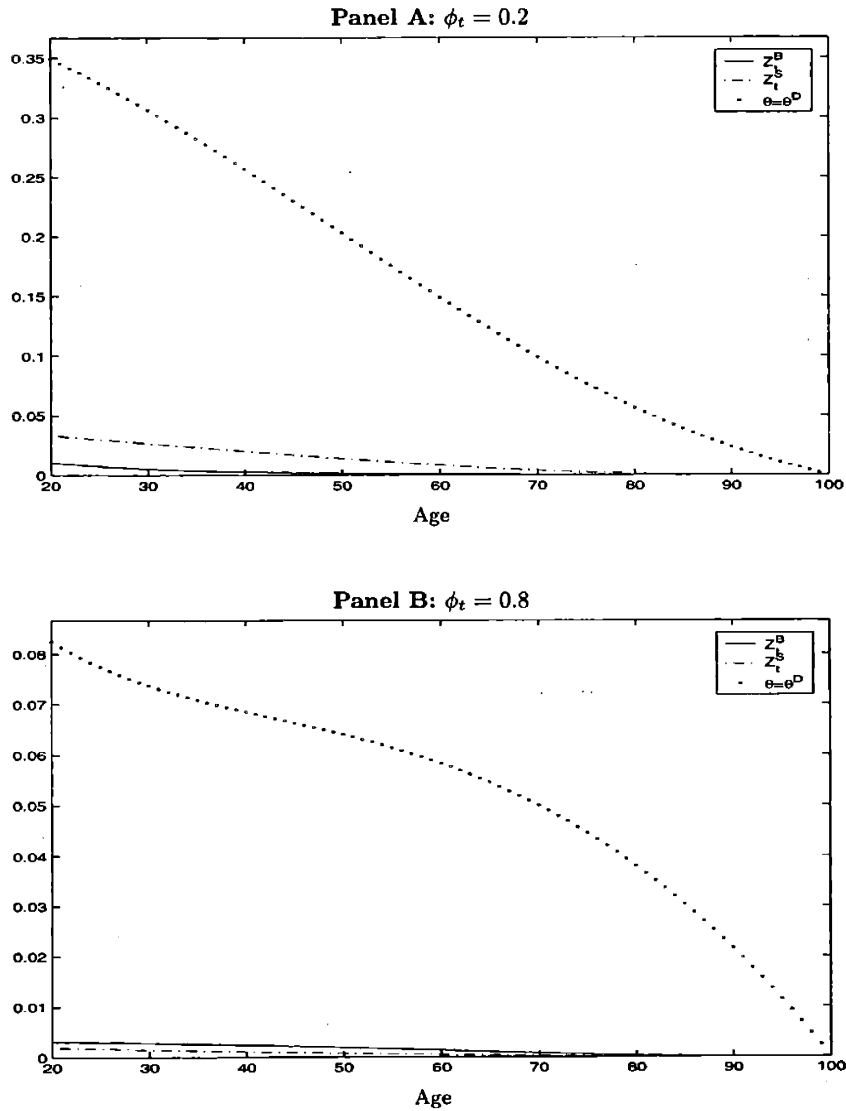


Figure C-12: **Welfare Analysis.** The solid and dash-dotted lines report the welfare loss incurred by implementing the approximate rules instead of the optimal portfolio solution. The solid line refers to the case in which the approximate are implemented by imposing  $z_{B,t} = z_{S,t} = Z_t^B$  in Proposition 6 (see equation (15)) while the dash dotted line refers to the case in which the approximate rules are implemented by imposing  $z_{B,t} = z_{S,t} = Z_t^S$  in Proposition 6 (see equation (15)). Also reported is the welfare loss incurred from holding the same portfolio in both accounts (dotted line).

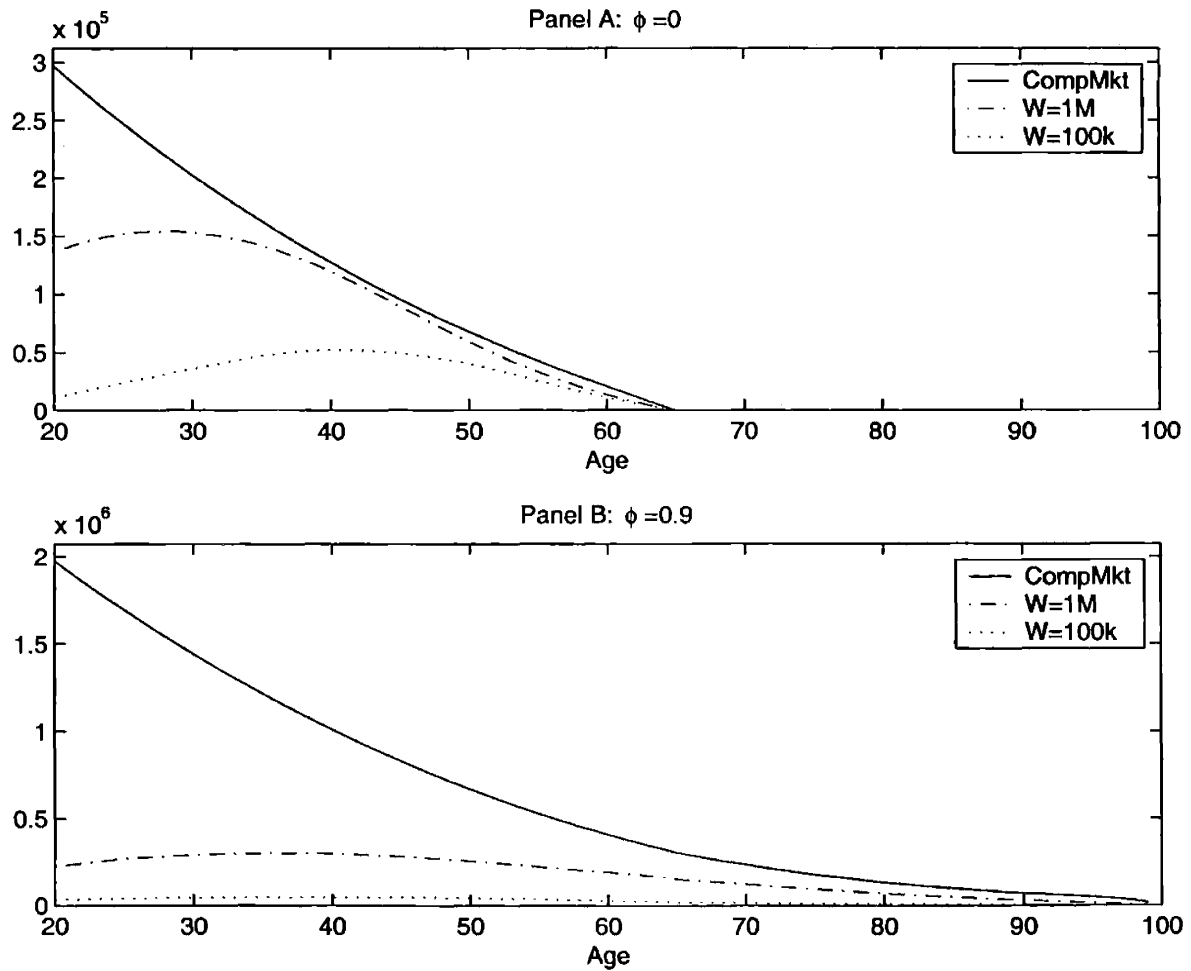


Figure C-13: **Value of Future Contributions.** This Figure shows the value of the tax subsidy as a function of the age of the investor. Two wealth level are represented, \$1m and \$100,000. The solid line refers to the case of no liquidity shocks ( $Q = 0$ ) while the dotted line refers to the case of a liquidity shock of fixed size equal to \$100,000. Note that the lines do not represent time series of the value of the tax subsidy.

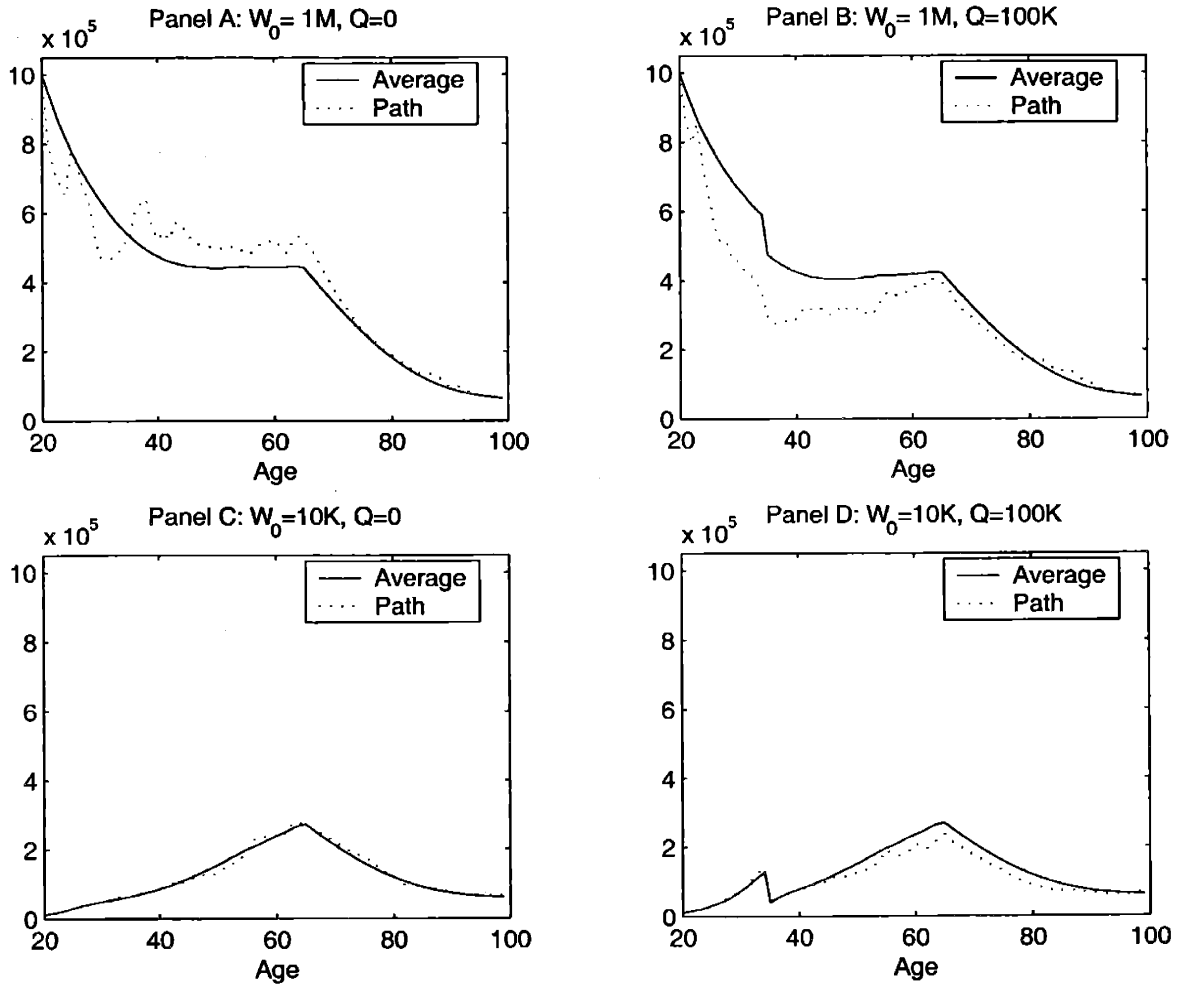


Figure C-14: **Wealth.** This Figure shows a simulated path (solid line) indicating the optima contribution decision of an investor starting with \$10,000 at age 20. The dotted line represent the average over 100,000 paths.

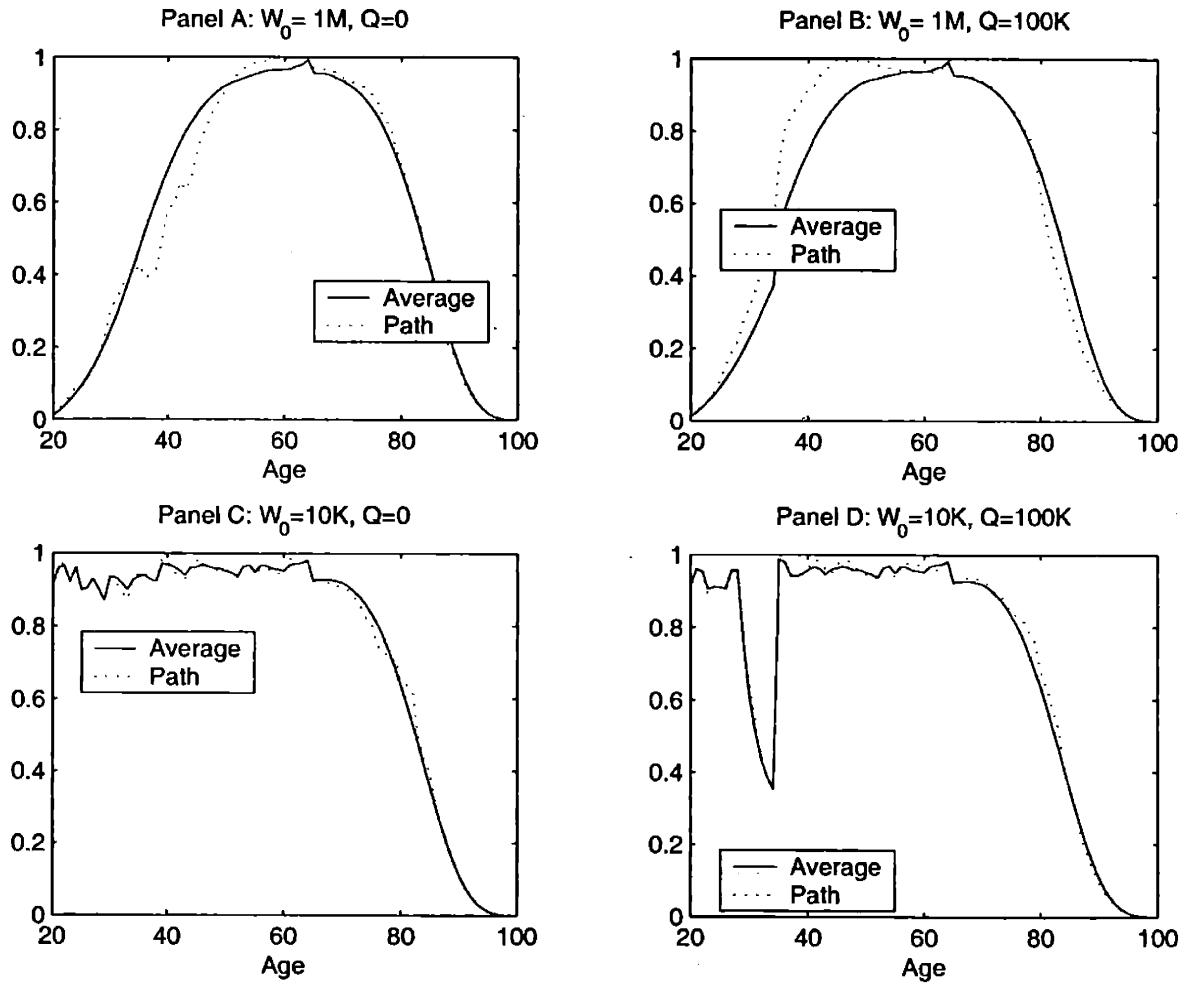


Figure C-15:  $\phi_c$ . This Figure shows a simulated path (solid line) indicating the optima contribution decision of an investor starting with \$10,000 at age 20. The dotted line represent the average over 100,000 paths.

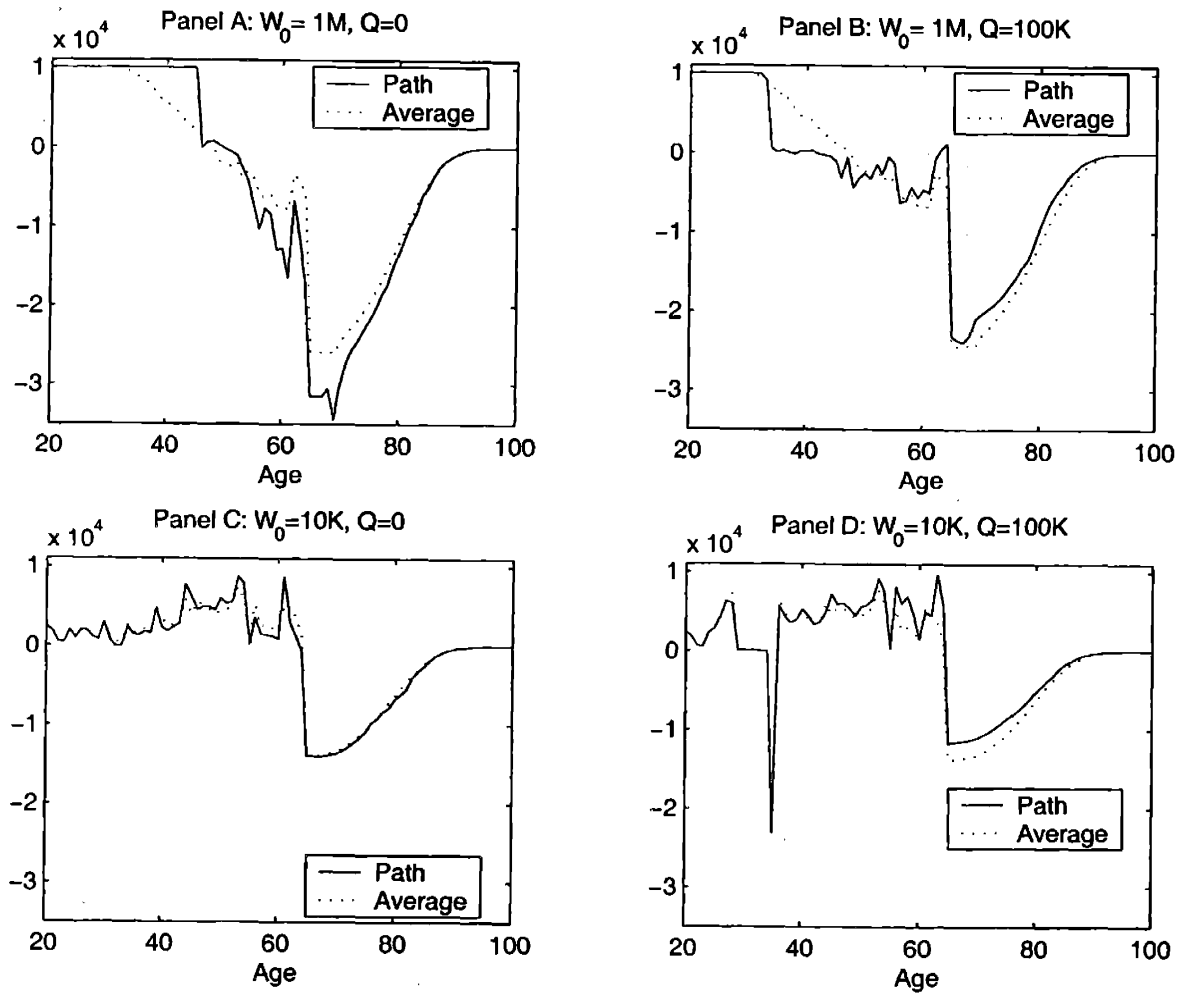


Figure C-16: Monte Carlo Simulation of Optimal Contribution Decisions. This Figure shows a simulated path (solid line) indicating the optima contribution decision of an investor starting with \$10,000 at age 20. The dotted line represent the average over 100,000 paths.

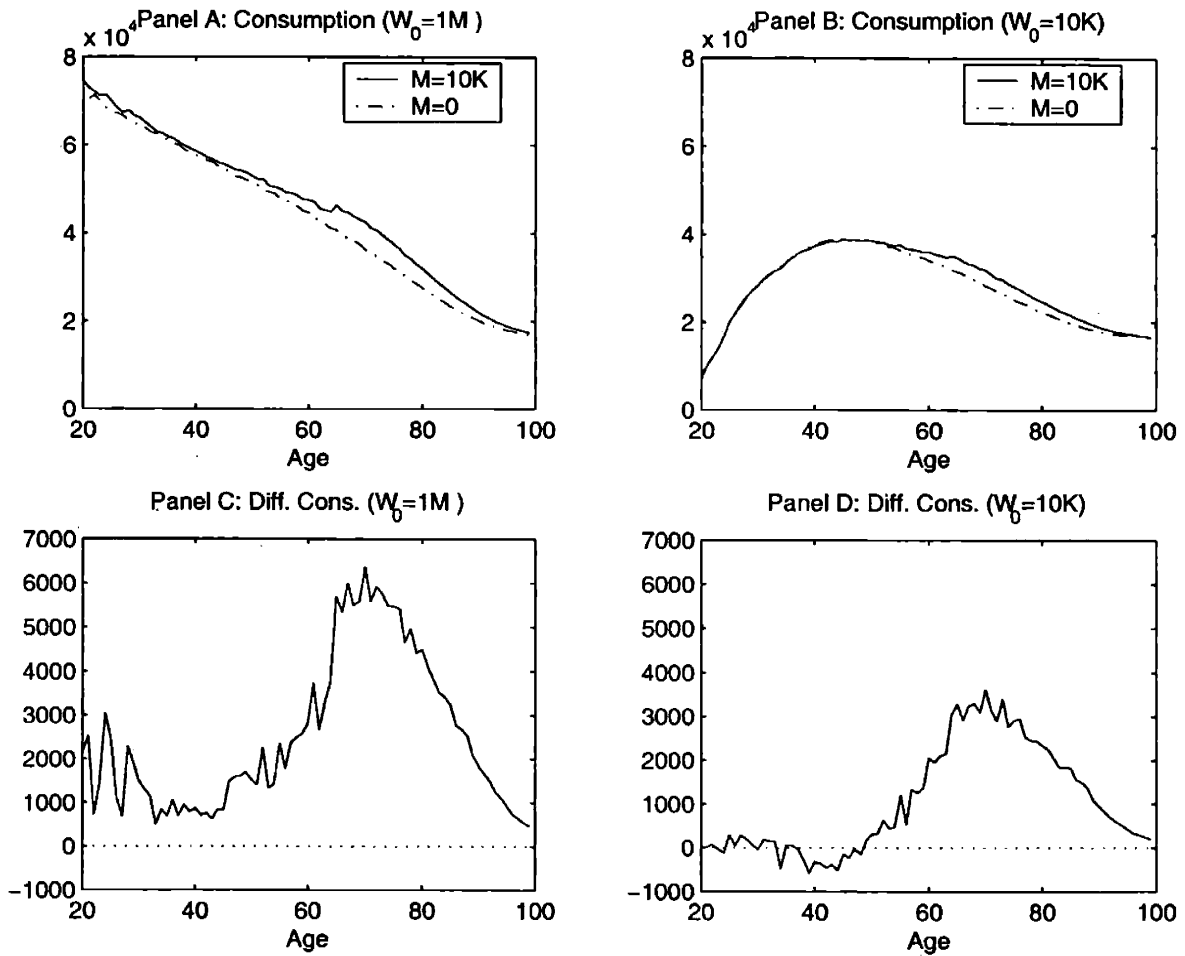


Figure C-17: **MonteCarlo Simulation of Optimal Consumption.** This Figure represent the average optimal consumption over 100,000 paths for an investor starting with initial wealth of \$1m (Panel A) and \$10,000 (Panel B). The dashed lines represent the case in which no contribution is allowed while the solid lines represent the case in which investor can contribute up to \$10,000 in their tax-deferred account. Panel C and D represent the difference between the average consumption when contribution is allowed and when contribution is not.





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