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# Boundary-Value Descriptor Systems: Well-Posedness, Reachability, and Observability[1]

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## Abstract

In this paper we introduce the class of two-point boundary-value descriptor systems (TPBVDS), discrete-time systems described by possibly linear dynamics and a set of boundary conditions constraining the values of the system "state" at the two endpoints of the system's interval of definition. By introducing a standard form for regular pencils we obtain a new and simple generalized Cayley-Hamilton theorem that simplifies our investigation of well-posedness, Green's function solution, and reachability and observability for TPBVDS. There are two distinct notions of reachability and observability that one can define for TPBVDS, associated with processes that propagate inward from and outward toward the boundaries. We investigate each of these in detail, obtaining, among other things, far simpler forms for the reachability and observability results found previously in literature. In addition we describe several methods for the efficient solution of TPBVDS, one involving recursions from each end of the interval toward the other and two others involving recursions that proceed outward toward and inward from the boundaries.

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## I. Introduction

The class of descriptor systems has been the subject of numerous studies in recent years (see, for example [1-9,20,21,23,24,26,27]). The fundamental property with which all of these studies have had to deal, in some form or another, is the fact that the system function matrix for a descriptor system need not be proper, leading to impulsive behavior in continuous time and giving rise to noncausal responses in discrete time. The noncausality of these models makes them a natural choice for modeling spatially, (rather than temporally) varying phenomena. Indeed, if one considers generalizations of descriptor models to more than one independent variable, one finds that these models arise in many contexts such as in describing random fields, electromagnetic problems, gravitational anomalies, etc.

In the context just described it is natural to consider descriptor models together with boundary conditions. While it has been recognized in the literature that discrete-time descriptor models are often not well-posed when initial conditions are specified, the implications of using general boundary conditions have not been investigated for these systems. This paper presents the initial steps in such an investigation.

There have been two principal stimuli for our work. The first is the work of Krener [12-14] who developed a system theory for standard (i.e. not descriptor) continuous-time linear systems with boundary conditions. (See also the related work in [15.16]). Krener's results expose the richness of boundary value models and a number of important concepts such as new notions of recursion that are more natural for such systems. The development in this paper parallels Krener's, with some important differences required to deal with the possible singularity of the system matrices involved.

The second stimulus for the study presented here has come from our work on estimation for noncausal process [10,11,22]. In particular in [22] we have examined the estimation problem for boundary-value descriptor systems. In addition to producing, among other things, both algorithms and new types of generalized Riccati equations, this study also produced a number of questions. Is the optimal estimator stable and how is stability related to reachability and observability? Do reachability and observability guarantee existence and uniqueness of positive definite solutions to the generalized Riccati equations? Stepping back we see that there are more fundamental questions. What do reachability and observability mean for boundary-value descriptor systems? What does stability mean for a boundary-value process defined on a bounded interval? In this and in subsequent papers we provide answers to these questions.

In the next section we introduce the class of two-point boundary-value descriptor systems and investigate their well-posedness. This leads us to the introduction of a normalized form for these systems. This form not only normalizes the boundary conditions in a manner analogous to that of Krener but it also brings the system matices into a form that leads to statements of a generalized Cayley-Hamilton theorem and of reachability and observability conditions that are significantly simpler than ones found in the literature. In Section III we introduce the two notions of recursion, namely inward from and outward towards the boundary, that were first used by Krener, and we investigate the processes associated with each. These provide the basis for defining two concepts of reachability and of observability which are then examined in detail in the following two sections.

Finally in Section VI we discuss the efficient solution of boundary-value descriptor equations and then close with a brief discussion in Section VII.

## II. Well-Posedness and Normalized Form

The two-point boundary-value descriptor system (TPBVDS) considered in this paper satisfies the difference equation

$$Ex(k+1) = Ax(k) + Bu(k), \qquad k = 0,\ldots,N-1 \tag{2.1}$$

with the two-point boundary condition

$$V_i x(0) + V_f x(N) = v \tag{2.2}$$

and output

$$y(k) = Cx(k), \qquad k = 0,\ldots,N \tag{2.3}$$

Here x and v are n-dimensional, while u and y are m- and p-dimensional, respectively.

As in [2], we can rewrite (2.1), (2.2) as a single set of equations

$$\mathcal{S}x = \mathcal{B}u \tag{2.4a}$$

where

$$x' = (x'(0)\ldots.x'(N)) \tag{2.4b}$$

$$u' = (u'(0),\ldots,u'(N-1),\ v') \tag{2.4c}$$

$$\mathcal{S} = \begin{bmatrix} -A & E & 0 & \cdots & & 0 \\ 0 & -A & E & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & & 0 & -A & E \\ V_i & 0 & \cdots & & 0 & V_f \end{bmatrix} \qquad (2.5a)$$

$$\mathcal{B} = \text{diag}\ (B,\ldots,B,\ I) \qquad (2.5b)$$

We see from this immediately that the well-posedness of (2.1), (2.2) -- i.e. the existence of a unique solution x(k), k = 0, 1,...,N, for any choice of v and u(k), k = 0, 1,...,N-1 --is equivalent to the invertibility of $\mathcal{S}$. Note that the invertibility of $\mathcal{S}$ implies that the submatrix consisting of all but its last block of rows has full row rank. This in turn implies that a necessary condition for well-posedness is that {E,A} comprise a regular pencil [17], i.e. that $\alpha E + \beta A$ is invertible for some and therefore for "most" $\alpha$ and $\beta$. Consequently throughout this paper we assume that this is the case.

An important aspect of regular pencils is that they can be transformed into a form that greatly simplifies the answering of numbers of questions.

<u>Definition 2.1</u>: A regular pencil {E,A} is in <u>standard form</u> if for some $\alpha$ and $\beta$

$$\alpha E + \beta A = I \qquad (2.6)$$

Note that any standard linear system (with E = I) is in standard form (take $\alpha = 1$, $\beta = 0$). Furthermore any well-posed TPBVDS can be transformed to standard form. Specifically, find $\alpha$ and $\beta$ so that $|\alpha E + \beta A| \neq 0$ and premultiply (2.1) by $(\alpha E + \beta A)^{-1}$. This does not change the system or the "state" variable x, but the new E and A matrices now satisfy (2.6). It is worth noting that one can also deduce that any regular pencil can be put into standard form by examination of its Kronecker canonical form [17], although that construction involves a similarity transformation on x as well (see Section VI).

A pencil in standard form has a number of important properties a few of which are summarized in

Proposition 2.1: Suppose that $\{E,A\}$ is in standard form. Then

(1) E and A commute and thus have a common set of generalized eigenvectors (which we refer to as generalized system eigenvectors).

(2) The pencil $\{E^k,A^k\}$ is regular for all $k > 0$.

(3) For any $k,\ell > 0$, there exist coefficients $\alpha_0,\ldots,\alpha_{n-1}$ so that

$$E^kA^\ell = \sum_{i=0}^{n-1} \alpha_i A^{n-i-1} E^i \tag{2.7}$$

Proof: Suppose without loss of generality that $\alpha \neq 0$ in (2.6). Then $E = \gamma I + \delta A$ where $\gamma = 1/\alpha$ and $\delta = -\beta/\alpha$. The commutativity of E and A then follows immediately. The remainder of (1) follows from the fact that E and A can be put into Jordan form by the same similarity transformation. Indeed the Jordan blocks must be of commensurate dimensions (i.e. no block of E or A can straddle rows of several blocks of the other without extending to include all of the rows of those blocks).[2]

Assume then that E and A are in Jordan form. Since $\{E,A\}$ is regular, E and A cannot have a zero eigenvalue associated with a common eigenvector. This in turn implies statement (2). Finally to prove (3), take any $E^kA^\ell$ and replace E by $\gamma I + \delta A$. Then apply the usual Cayley-Hamilton theorem to all powers of A higher than n-1. Finally, multiply each $A^k$ in the resulting expression by $I = (\alpha E + \beta A)^{n-k-1}$. Expanding yields an expression of the form of (2.7)

Statement (3), which states that $\{A^{n-1}, EA^{n-2}, \ldots E^{n-1}\}$ span the same subspace as $\{A^kE^\ell | k,\ell \geq 0\}$, is a generalization of the Cayley-Hamilton theorem. Note that this statement is considerably simpler than those in the literature [6,8,28] for pencils not in standard form.

Standard form also provides us with a simpler well-posedness condition:

Theorem 2.1: Suppose that $\{E,A\}$ is in standard form. Then the system (2.1), (2.2) is well-posed if and only if

$$V_iE^N + V_fA^N \tag{2.8}$$

is invertible.

Proof: One method for deriving this result is to apply row elimination to solve for x(0) and x(N) from (2.4). Methods similar to this will be used in the next section in defining inward and outward processes. In this proof we use a different method that provides some computations we can use immediately.

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[2]For example, two 4 x 4 matrices in Jordan form, one with two 2 x 2 Jordan blocks and the other with one 3 x 3 and one 1 x 1 Jordan block, don't commute.

To begin, let $\omega$ be any number such that

$$\Gamma = \omega E^{N+1} - A^{N+1} \tag{2.9}$$

is invertible (this can always be done since $\{E^{n+1}, A^{n+1}\}$ is regular). Then we can express $\mathcal{S}$ as

$$\mathcal{S} = \mathcal{S}_1\mathcal{S}_2 \tag{2.10}$$

where

$$\mathcal{S}_1 = \begin{bmatrix} I & 0 & \cdots & & 0 \\ 0 & I & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & I & 0 \\ S_{N0} & S_{N1} & \cdots & S_{N,N-1} & S_{NN} \end{bmatrix} \tag{2.11}$$

with

$$S_{Nk} = (V_iA^{N-k}E^k + \omega V_fA^{N-k-1}E^{k+1})\Gamma^{-1}, \quad k=0,\ldots,N-1 \tag{2.12a}$$

$$S_{NN} = (V_iE^N + V_fA^N)\Gamma^{-1} \tag{2.12b}$$

and

$$\mathcal{S}_2 = \begin{bmatrix} -A & E & 0 & \cdots & 0 & 0 \\ 0 & -A & E & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -A & E \\ \omega E & 0 & 0 & \cdots & 0 & -A \end{bmatrix} \tag{2.13}$$

Note that $\mathcal{S}_2$ is invertible, with

$$\mathcal{S}_2^{-1} = \begin{bmatrix} A^N\Gamma^{-1} & EA^{N-1}\Gamma^{-1} & \cdots & E^{N-1}A\Gamma^{-1} & E^N\Gamma^{-1} \\ \omega E^N\Gamma^{-1} & A^N\Gamma^{-1} & \cdots & E^{N-2}A^2\Gamma^{-1} & E^{N-1}A\Gamma^{-1} \\ \omega E^{N-1}A\Gamma^{-1} & \omega E^N\Gamma^{-1} & \cdots & E^{N-3}A^3\Gamma^{-1} & E^{N-2}A\Gamma^{-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \omega EA^{N-1}\Gamma^{-1} & \omega E^2A^{N-2}\Gamma^{-1} & \cdots & \omega E^N\Gamma^{-1} & A^N\Gamma^{-1} \end{bmatrix} \tag{2.14}$$

Consequently $\mathcal{S}$ is invertible if and only if $\mathcal{S}_1$ is invertible. Examining (2.11), (2.12) we see that this is the case if and only if the matrix in (2.8) is invertible.

Definition 2.2: The system (2.1), (2.2) is in normalized form if {E,A} is in standard form and if

$$V_i E^N + V_f A^N = I \tag{2.15}$$

This form is the counterpart of Krener's standard form in [12-14] Note that any well-posed system can be put in normalized form by left multiplication of (2.1) and (2.2). Specifically we first transform {E,A} to standard form as described previously, to obtain new E and A matrices, and we then multiply (2.2) by $(V_i E^N + V_f A^N)^{-1}$ to obtain new $V_i$ and $V_f$ matrices satisfying (2.15). From this point on we assume that (2.1), (2.2) is in standard form.

Next, note that if (2.8) is invertible, the inverse of $\mathcal{S}_1$ has the same form as (2.11) except that the last block row of $\mathcal{S}_1^{-1}$ is

$$(-S_{NN}^{-1}S_{NO},\ -S_{NN}^{-1}S_{N1},\ldots,\ -S_{NN}^{-1}S_{N,N-1},\ S_{NN}^{-1})$$

Using the expressions for $\mathcal{S}_1^{-1}$ and $\mathcal{S}_2^{-1}$ we can then write down the Green's function solution of (2.1), (2.2):

$$x(k) = A^k E^{N-k} v + \sum_{\ell=0}^{N-1} G(k,\ell) B u(\ell) \tag{2.16}$$

where

$$G(k,\ell) = \begin{cases} A^k \, [A - E^{N-k}(V_i A + \omega V_f E) E^k] E^{\ell-k} A^{N-\ell-1} \Gamma^{-1} , & \ell \geq k \\ E^{N-k} [\omega E - A^k (V_i A + \omega V_f E) A^{N-k}] E^{\ell} A^{k-\ell-1} \Gamma^{-1} , & \ell < k \end{cases} \tag{2.17}$$

Here $G(k,\ell)$ is called the Green's function of the TPBVDS. When E and A are both invertible, (2.17) can be simplified.

$$G(k,\ell) = \begin{cases} -A^k E^{N-k} V_f E^{\ell-N} A^{N-\ell-1} , & \ell \geq k \\ A^k E^{N-k} V_i E^{\ell} A^{-\ell-1} , & \ell < k \end{cases} \tag{2.18}$$

For simplicity, in the rest of the paper we assume that $\Gamma$ is invertible for $\omega = 1$ and use the expression (2.17) for G with $\omega$ set equal to 1. This assumption is equivalent to assuming that no $(N+1)^{st}$ root of unity is an eigenmode of the system (where $\sigma$ is an eigenmode if $|\sigma E - A| = 0$). All of the results in the paper have obvious extensions to the case of an arbitrary value of $\omega$, as we simply must carry $\omega$ along in the various expressions.

## III. Inward and Outward Processes

One of Krener's most important observations in his work was that boundary-value systems admit two notions of recursion, namely expanding inward from or outward toward the bounderies. In this section we introduce the counterparts to these notions for TPBVDS. As we will see, the possible singularity of both E and A leads to several differences in our context.

Each of the processes associated with these recursions have interpretations as state processes: the outward process summarized all that one needs to know about the input inside any interval in order to determine x outside the interval, while the inward process simply uses input values near the boundary to propagate the boundary condition inward. In Krener's context the outward process represented a "jump", i.e. the difference between x at one end of any interval and the value predicted for x at that point given x at the other end interval and assuming zero input inside the interval. In our context we cannot necessarily predict in either direction (because of the possible singularity of E and A) and therefore must use a slightly modified definition of the outward process:

$$z_o(k,\ell) = E^{\ell-k}x(\ell) - A^{\ell-k}x(k), \quad k<\ell \tag{3.1}$$

Note that this definition agrees with Krener's if $E = I$. However, in general $z_o(k,\ell)$ can only be propagated outward whereas in Krener's case the outward process could be propagated inward as well. An explicit expression for $z_o(k,\ell)$ in terms of the inputs between k and $\ell$ can be obtained by premultiplying (2.4) by

$$[0,\dots,0,\ A^{k-\ell-1}, EA^{k-\ell-2},\dots,E^{k-\ell-1},0,\dots,0]$$

This yields

$$z_o(k,\ell) = \sum_{j=k}^{\ell-1} E^{j-k}A^{\ell-j-1}Bu(j) \tag{3.2}$$

Also, we have the recursive relations

$$z_o(k-1,\ell) = Ez_o(k,\ell) + A^{\ell-k}Bu(k-1) \tag{3.3}$$

$$z_o(k,\ell+1) = Az_o(k,\ell) + E^{\ell-k}Bu(\ell) \tag{3.4}$$

Furthermore, as in [14] it is straightforward to show that the four-point boundary-value system

$$Ex(k+1) = Ax(k) + Bu(k) \tag{3.5}$$

$$V_i x(0) + V_f x(N) = v \tag{3.6}$$

$$E^{L-K}x(L) - A^{L-K}x(K) = z_o(K,L) \tag{3.7}$$

has the same solution as (2.1), (2.2) for $k \in [0,N] \setminus [K+1,L-1]$ (i.e. over $[0,K]$ and $[L,N]$), so $z_o(K, L)$ does indeed summarize all we need to know about inputs between K and L.

The inward process $z_i(k,\ell)$ can also be defined in a manner analogous to [14]. Unfortunately in the present context $z_i(k,\ell)$ is a complex function of the boundary matrices, the boundary value v, and the inputs u(j), $j \in [0,N-1] \setminus [k,\ell-1]$. Specifically, as we demonstrate below, for $k<\ell$, $z_i(k,\ell)$

has the form

$$z_i(k,\ell) = W_i(k,\ell)x(k) + W_f(k,\ell)x(\ell)$$
$$= F_{k\ell}[u(0),u(1),\ldots,u(k-1),u(\ell),u(\ell+1),\ldots,u(N-1),v] \qquad (3.8)$$

and, in addition

$$z_i(0,N) = v,\ W_i(0,N) = V_i,\ W_f(0,N) = V_f \qquad (3.9)$$

$$z_i(k,k) = x(k) = F_{kk}(u(0),\ldots,u(N-1),\ v) \qquad (3.10)$$

where the $F_{k\ell}$ are linear functions of their arguments. Furthermore the TPBVDS

$$Ex(k+1) = Ax(k) + Bu(k) \qquad (3.11)$$

$$W_i(K,L)x(K) + W_f(K,L)x(L) = z_i(K,L), \qquad (3.12)$$

has the same solution as (2.1), (2.2) for $k\epsilon[K,L]$, so $z_i(K,L)$ does indeed represent an inwardly-propagated boundary condition for the original system.

Let us first indicate how (3.8) - (3.10) can be computed in a recursive manner. The basic idea here is to eliminate values of x near the boundary from (2.4) in order to obtain a reduced set of equations. The resulting right-hand side will then involve the remaining u's and a new boundary condition (see (2.4c)). Specifically, suppose we wish to propagate one step in from the left, i.e. to compute $z_i(1,N)$. Note that for $\mathcal{S}$ in (2.5a) to be invertible it is necessary for

$$\begin{bmatrix} -A \\ V_i \end{bmatrix}$$

to have full column rank. Consequently we can find a block matrix [T P] of full row rank so that

$$[T \quad P] \begin{bmatrix} -A \\ V_i \end{bmatrix} = 0 \tag{3.13}$$

Premultiplying (2.4) by the matrix

$$\Omega = \begin{bmatrix} 0 & I \ldots\ldots 0 & 0 \\ \vdots & \vdots \qquad \vdots & \vdots \\ 0 & 0 \ldots\ldots I & 0 \\ T & 0 \ldots\ldots 0 & P \end{bmatrix} \tag{3.14}$$

then eliminates x(0) and leaves us with the following TPBVDS on [1,N]:

$$Ex(k+1) = Ax(k) + Bu(k) \tag{3.15}$$

$$TEx(1) + PV_f x(N) = Pv + TBu(0) \tag{3.16}$$

It is easy to see that this system is well-posed, since rank $(\Omega\mathcal{S})$ = rank $(\Omega)$ = rank $(\mathcal{S})$ - n and the system is defined over an interval with one less time step. The boundary matrices in (3.16) are not in necessarily in normalized form, so we then need to premultiply (3.16) by

$$\Lambda = (TE^N + PV_f A^{N-1})^{-1} \tag{3.17}$$

yielding

$$W_i(1,N) = \Lambda TE \quad , \; W_f(1,N) = \Lambda PV_f \tag{3.18}$$

$$F_{1N}[u(0),v] = \Lambda Pv + \Lambda TBu(0) \tag{3.19}$$

In a similar fashion we can move the right boundary inward, in this case premultiplying (2.4) by

$$\begin{bmatrix} I & 0 \ldots\ldots & 0 & 0 \\ 0 & I \ldots\ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \ldots\ldots & I & 0 \\ 0 & 0 \ldots\ldots & S & Q \end{bmatrix} \tag{3.20}$$

when [S  Q] is a full-rank solution of

$$[S \quad Q] \begin{bmatrix} E \\ V_f \end{bmatrix} = 0 \tag{3.21}$$

It is also possible to obtain direct rather than recursive expression for the W's and at the same time to expose the relationship between the inward and outward processes that we will use in Section V. Using the expression (3.1) for the outward process $z_o$ and (2.4) we can write

$$\begin{bmatrix} -A^k & E^k & 0 & 0 \\ 0 & -A^{\ell-k} & E^{\ell-k} & 0 \\ 0 & 0 & -A^{N-\ell} & E^{N-\ell} \\ V_i & 0 & 0 & V_f \end{bmatrix} \begin{bmatrix} x(0) \\ x(k) \\ x(\ell) \\ x(N) \end{bmatrix} = \begin{bmatrix} z_o(0,k) \\ z_o(k,\ell) \\ z_o(\ell,n) \\ v \end{bmatrix} \tag{3.22}$$

As we did earlier, we construct a full-rank matrix

$[T_i(k,\ell),\ T_f(k,\ell),\ P(k,\ell)]$ so that

$$[T_i(k,\ell),\ T_f(k,\ell),\ P(k,\ell)] \begin{bmatrix} -A^k & 0 \\ 0 & E^{N-\ell} \\ V_i & V_f \end{bmatrix} = 0 \tag{3.23}$$

If we then multiply (3.22) by

$$\Omega\ (k,\ell) = \begin{bmatrix} 0 & I & 0 & 0 \\ T_i(k,\ell) & 0 & T_f(k,\ell) & P(k,\ell) \end{bmatrix} \tag{3.24}$$

we obtain

$$\begin{bmatrix} -A^{\ell-k} & E^{\ell-k} \\ T_i(k,\ell)E^k & -T_f(k,\ell)A^{N-\ell} \end{bmatrix} \begin{bmatrix} x(k) \\ x(\ell) \end{bmatrix} = \begin{bmatrix} z_o(k,\ell) \\ T_i(k,\ell)z_o(0,k)+T_f(k,\ell)z_o(\ell,n)+P(k,\ell)v \end{bmatrix} \tag{3.25}$$

Equation (3.25) is essentially the result of eliminating all variables in (2.4) other than $x(k)$ and $x(\ell)$ by propagating outward to summarize all inputs between $k$ and $\ell$ and inward to summarize the effect of the boundary condition and inputs from 0 to $k$ and $\ell$ to $N$. Consequently we can identify the second block of equations as specifying an unnormalized version of the inward process. Therefore letting

$$\Lambda(k,\ell) = [T_i(k,\ell)E^{\ell} - T_f(k,\ell)A^{N-k}]^{-1} \tag{3.26}$$

we have

$$W_i(k,\ell) = \Lambda(k,\ell)T_i(k,\ell)E^k \tag{3.27}$$

$$W_f(k,\ell) = -\Lambda(k,\ell)T_f(k,\ell)A^{N-\ell} \tag{3.28}$$

and

$$z_i(k,\ell) = \Lambda(k,\ell)[T_i(k,\ell)z_o(0,k) + T_f(k,\ell)z_o(\ell,N) + P(k,\ell)v] \tag{3.29}$$

In the case of standard linear systems reachability corresponds to the ability to drive the state of the system to an arbitrary value by appropriate choice of the input sequence. It is well known that if such a system is reachable it is possible to reach an arbitrary state value by proper choice of the n previous input values, where n is the dimension of the system. In the case of a TPBVDS, however, there is a distinction between the concept of reachability by choosing the inputs in an n-point neighborhood and the concept of reachability by choosing the inputs in the whole domain of definition (i.e. [0, N]). The first concept we shall refer to as strong reachability and the second concept as weak reachability. These concepts correspond, respectively, to Krener's reachability on and reachability off which he in turn defines in terms of the outward and inward processes, respectively. We shall do the same in the next two sections in which we also analyze the corresponding observability concepts.

## IV. Strong Reachability and Observablity

We begin with an examination of reachability, and for this we need the following

Definition 4.1: The system (2.1), (2.2) is strongly reachable on [K,L] if the map

$$\{u(k)\,|\,k\in[K,L]\} \rightarrow z_o(K,L)$$

is onto. The system is strongly reachable if it is strongly reachable on some interval.

From (3.2) we can write

$$z_o(K,L) = R_s(L-K)\begin{bmatrix} u(K) \\ \vdots \\ u(L-1) \end{bmatrix} \tag{4.1}$$

where

$$R_s(j) = [A^{j-1}B \,|\, EA^{j-2}B \,|\, \ldots \,|\, E^{j-1}B] \tag{4.2}$$

In anticipation of the following result, define the strong reachability matrix

$$R_s = R_s(n) \tag{4.3}$$

and strongly reachable subspace

$$\mathcal{R}_s = \mathrm{Im}(R_s) \tag{4.4}$$

Theorem 4.1: The following statements are equivalent:

(a) The system (2.1), (2.2) is strongly reachable.

(b) The strong reachability matrix $R_s$ has full rank.

(c) The matrix $[sE - tA \,|\, B]$ has full rank for all $(s,t) \neq (0,0)$

(d) The state x and any point $k \in [n, N-n]$ can be made to assume any desired value by proper choice of inputs $u(j)$, $j \in [k-n, k+n-1]$, and this can be accomplished for any choices of $V_i$ and $V_f$ for which (2.1), (2.2) is well-posed.

Before proving this result, let us make several comments. Note first that condition (c) is one of the reachability conditions found in the descriptor literature [7,26]. By introducing the standard form of a regular pencil we are able to obtain a condition, namely that (4.2) is of full rank for j = n, that is far simpler than those presented previously. Note also that as for standard linear systems, condition (6) tells us that a system is strongly reachable if and only if it is strongly reachable over intervals of length n. On the other hand, in condition (d) we require that x(k) can be driven to an arbitrary value by applying appropriate inputs over the 2n-point symmetric neighborhood of k. In fact, one only needs an n-point neighborhood of k, but the extent of this interval before and after k depends on the matrices E, A, and B (i.e. on the causal/anticausal structure of (2.1)). Condition (d) simply uses the union of all such n-point intervals and therefore is approprate for <u>all</u> TPBVDS. Finally, note that strong reachability does not depend on the boundary matrices $V_i$ and $V_f$ (as long as (2.1), (2.2) is well-posed). This can be seen directly from the definition of $z_o(k,\ell)$ or from condition (b).

Proof: The equivalence of (a) and (b) follows immediately from the generalized Cayley-Hamilton theorem (statement (3) of Proposition 2.1)). As an alternate proof, note that

$$\mathrm{Im}[R_s(k+1)] = ER_s(k) + AR_s(k) \tag{4.5}$$

so that $\mathrm{Im}[R_s(k+2)] = \mathrm{Im}[R_s(k+1)]$ if $\mathrm{Im}[R_s(k+1)] = \mathrm{Im}[R_s(k)]$. Also, thanks to (2.6)

$$\mathrm{Im}[R_s(k)] \subseteq \mathrm{Im}[R_s(k+1)] \tag{4.6}$$

Simple dimension counting then shows that

Finally, consider the equivalence of statements (b) and (d). Because of the linearity of the system, we can assume that $v = 0$ and $u(j) = 0$ for $j\epsilon[0,k-n-1]$ and $j\epsilon[k+n,N]$. In this case (2.16), (2.17), and (3.2) allows us to write

$$x(k) = A^k[A-E^{N-k}(V_iA+V_fE)E^k]\Gamma^{-1}A^{N-k-n}z_o(k,k+n) \\ + E^{N-k}[E-A^k(V_iA+V_fE)A^{N-k}]\Gamma^{-1}E^{k-n}z_o(k-n,k) \quad (4.12)$$

Let $\xi$ be an arbitrary vector and choose inputs $u(j)$, $j\epsilon[k-n,k-1]$ so that $z_o(k-n,k) = E^n\xi$ and $u(j)$, $j\epsilon[k,k+n-1]$ so that $z_o(k,k+n) = -A^n\xi$. With these choices which can be found since $R_s$ has full rank, (4.12) reduces to

$$x(k) = \xi \quad (4.13)$$

This shows that (a) implies (d). To show the reverse implication, we make the following choice for $V_i$ and $V_f$:

$$V_i = \Lambda^{-1}E \quad (4.14a)$$

$$V_f = \gamma\Lambda^{-1}A \quad (4.14b)$$

where

$$\Lambda = E^{N+1} + \gamma A^{N+1} \quad (4.15)$$

and $\gamma$ is any number that makes $\Lambda$ invertible. Note that (2.1), (2.2) with this choice for $V_i$ and $V_f$ is in normalized form. Let us take $v = 0$ and $u(j) = 0$ for $j\epsilon[0,k-n-1]$ and $j\epsilon[k+n,N]$. Then in this case (2.16), (2.17) reduces to

$$x(k) = \Delta\Big[A^{n-1}E^{N-n-1}Bu(k-n) + A^{n-2}E^{N-n}Bu(k-n-1) + \dots$$
$$+ E^{N}Bu(k-1) + \gamma A^{N}Bu(k) + \gamma A^{N-1}EBu(k+1) + \dots$$
$$+ \gamma E^{n-1}A^{N-n+1}Bu(k+n-1)\Big] \tag{4.16}$$

The range of the mapping defined in (4.16) is

$$\Delta\left[E^{N-n-1}\mathcal{R}_s + A^{N-n+1}\mathcal{R}_s\right]$$

Assuming that (d) is true, this must also be all of $\mathbb{R}^n$. Consequently we conclude that $\mathcal{R}_s = \mathbb{R}^n$ for this choice of $V_i, V_f$. Thanks, then, to statement (c) of the theorem, we see that $\mathcal{R}_s = \mathbb{R}^n$ for any $V_i, V_f$ for which the TPBVDS is well-posed, so that statement (a) also must hold.

We next wish to consider the dual concept of strong observability. To do this we proceed in a manner analogous to that for casual linear systems. Specifically, for such systems observability corresponds to being able to reconstruct the state at some point in time, given present and future observations, when all future inputs are zero. The counterpart to this in our context is the following.

<u>Definition 4.2</u>: The system (2.1) - (2.3) is <u>strongly observable on</u> [K,L] if the map

$$z_i(K,L) \rightarrow \{y(k)\,|\,k\epsilon[K,L]\} \tag{4.17}$$

defined by (3.11), (3.12) with $u \equiv 0$ is one-to-one. The system is <u>strongly observable</u> if it is strongly observable on [K,L] for all K,L such that $L-K \geq n-1$.

Since (3.11), (3.12) is in normalized form, we can adapt the Green's function solution (2.16) to obtain an explicit expression for the mapping defined in (4.17). Specifically

$$\begin{bmatrix} y(K) \\ y(K+1) \\ \vdots \\ y(L) \end{bmatrix} = O_s(L-K)z_i(K,L) \tag{4.18}$$

where

$$O_s(j) = \begin{bmatrix} CE^j \\ CAE^{j-1} \\ \vdots \\ CA^j \end{bmatrix} \tag{4.19}$$

In analogy with our reachability results, we define the strong observability matrix

$$O_s = O_s(n-1) \tag{4.20}$$

and the strongly unobservable subspace

$$\mathcal{O}_s = \ker(O_s) \tag{4.21}$$

Theorem 4.2: The following statements are equivalent.

(a) The system (2.1) - (2.3) is strongly observable.

(b) The strong observability matrix $O_s$ has full rank.

(c) The matrix

$$\begin{bmatrix} sE - tA \\ C \end{bmatrix}$$

has full rank for all $(s,t) \neq (0,0)$.

(d) The state x at any point $k \in [n, N-n]$ can be uniquely determined from the outputs $y(j)$, $j \in [k-n, k+n-1]$ and $u(j)$, $j \in [k-n, k+n-2]$. This can be accomplished for any choice of $V_i$ and $V_f$ for which (2.1), (2.2) is well-posed.

The proof of this theorem is analogous to that for Theorem 4.1 and therefore is omitted. Also, one can make similar comments concerning this result. For example, thanks to the generalized Cayley-Hamilton theorem, statement (b) is considerably simpler than expressions that have appeared previously. Also, strong observability depends only on E, A, and C and not on the particular choice of boundary matrices $V_i$ and $V_f$.

## V. Weak Reachability and Observability

As Krener noted, in contrast to strong reachability and observability, the concepts of weak reachability and observability depend intimately on the particular choice of boundary matrices, as the structure of these matrices can increase reachability and observability beyond that which might be apparent from an examination of system dynamics alone. The examination of these weaker concepts for TPBVDS is somewhat more complicated than in Krener's case because of the possible singularity of E and A.

<u>Definition 5.1</u>: The system (2.1), (2.2) is <u>weakly reachable off</u> [K,L] if the map $F_{KL}$ defined in (3.8), with $v \equiv 0$, is onto. The <u>weakly reachable subspace</u> $\mathcal{R}_w(K,L)$ is the range of this map. The system is <u>weakly reachable</u> if it is weakly reachable off [K,L] (i.e., if $\mathcal{R}_w(K,L) = \mathbb{R}^n$) for all $K,L \in [n,N-n]$.

Note that the weak reachability condition is a natural counterpart to the causal reachability definition in which we require that the state can be driven to an arbitrary value from zero initial condition. Also, note the use of the wording "reachable <u>off</u>", emphasizing the fact that the inputs used in this case are confined to the exterior of the interval [K,L].

An important property of a causal system is that the dimension of reachable space does not change, and in fact the reachable space itself is time-invariant. The following theorem shows that the first of these statements is also true for TPBVDS's. Example 5.1 later in this section shows that the second is not.

<u>Theorem 5.1</u>: The dimension of $\mathcal{R}_w(K,L)$ is constant for $K,L \in [n,N-n]$.

<u>Proof</u>: Let K,L be any points in [n,N-n]. From (3.29) (with v set to 0) we see that

$$\mathcal{R}_w(K,L) = \Lambda(K,L)[T_i(K,L)\mathcal{R}_s + T_f(K,L)\mathcal{R}_s] \tag{5.1}$$

Now assume that $K-1 \in [n,N-n]$ as well. We would like to show that

$$\dim \mathcal{R}_w(K-1,L) = \dim \mathcal{R}_w(K,L) \qquad (5.2)$$

To do this, we first must find $T_i(K-1,L)$ and $T_f(K-1,L)$. In fact, what we show is that a possible set of choices for $T_i$, $T_f$, and P is

$$T_i(K-1,L) = T_i(K,L)\tilde{A} \qquad (5.3a)$$

$$T_f(K-1,L) = T_f(K,L) \qquad (5.3b)$$

$$P(K-1,L) = P(K,L) \qquad (5.3c)$$

where $\tilde{A}$ has the same eigenstructure as A except that the zero eigenvalue in A has been replaced by 1 in $\tilde{A}$. Without loss of generality[3] we can assume that A is in the Jordan form

$$A = \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} \qquad (5.4)$$

where J is invertible and N is nilpotent. In this case

$$\tilde{A} = \begin{bmatrix} J & 0 \\ 0 & N+I \end{bmatrix} \qquad (5.5)$$

For (5.3) to be a valid choice, two conditions must be satisfied. First $[T_i(K-1,L), T_f(K-1,L), P(K-1,L)]$ must be of full rank. This is obviously the case since $[T_i(K,L), T_f(K,L), P(K,L)]$ is, and $\tilde{A}$ is invertible. Secondly we must

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[3]Since similarity transformations have no effect on the dimension of the reachability spaces.

show that (3.23) is satisfied with $k = K-1$ and $\ell = L$, i.e., we must verify

$$-T_i(K,L)\tilde{A}A^{K-1} + P(K,L)V_i = 0 \tag{5.6}$$

when we know that

$$-T_i(K,L)A^K + P(K,L)V_i = 0 \tag{5.7}$$

However, since $K-1 \geq n$, $N^{K-1} = 0$, so that $\tilde{A}A^{K-1} = A^K$.

Consequently, we can write

$$\mathcal{R}_w(K-1,L) = \Lambda(K-1,L)[T_i(K,L)\tilde{A}\mathcal{R}_s + T_f(K,L)\mathcal{R}_s] \tag{5.8}$$

(Note that (4.25) may not be valid if $K-1 < n$, since $\mathcal{R}_s(K-1)$ may be smaller than $\mathcal{R}_s$.) Comparing (5.1) and (5.8) and using the fact that the $\Lambda(k,l)$ are all invertible, we see that (5.3) will hold if we can show that

$$\tilde{A}\mathcal{R}_s = \mathcal{R}_s \tag{5.9}$$

Note first that $A\mathcal{R}_s \subseteq \mathcal{R}_s$, so that (5.9) is clearly true if A is invertible. If A is singular, note that $\alpha$ cannot be zero in $\alpha E + \beta A = I$, so that $\mathcal{R}_s$ is given by (4.8). Then assuming that A and $\tilde{A}$ are as in (5.4) and (5.5) and using the fact that J is invertible, we see that (5.9) will hold if we can show that

$$\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\mathcal{R}_s \subseteq \mathcal{R}_s \tag{5.10}$$

If we partition B compatably with (5.4) we see that

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{5.11}$$

$$\mathcal{R}_s = \mathrm{Im}\begin{bmatrix} B_1 & JB_1 & \cdots\cdots\cdots\cdots\cdots & J^{n-1}B_1 \\ B_2 & NB_2 & \cdots N^{\mu-1}B_2 \; 0 \cdots & 0 \end{bmatrix} \tag{5.12}$$

where $\mu$ is the nilpotency degree of N. Let J be $n_1 x n_1$ and N be $n_2 x n_2$ (so that $n_1+n_2 = n$ and $\mu \leq n_2$). Suppose that $[\xi_1', \xi_2']' \epsilon \mathcal{R}_s$; we wish to show that $[0, \xi_2']' \epsilon \mathcal{R}_s$. However, if $[\xi_1', \xi_2']' \epsilon \mathcal{R}_s$, there exist inputs $u_i, i = 0, \ldots, \mu - 1$ so that

$$\xi_2 = \sum_{i=0}^{\mu-1} N^i B_2 u_i \tag{5.13}$$

We then wish to show that we can augment this sequence with $u_i, i = \mu, \ldots, n$ so that

$$\sum_{i=0}^{n-1} J^i B_1 u_i = 0 \tag{5.14}$$

i.e., so that

$$\sum_{i=\mu}^{n-1} J^{i-\mu} B_1 u_i = -J^{-\mu}\Big(\sum_{i=0}^{\mu-1} J^i B_1 u_i\Big) \tag{5.15}$$

The right-hand side of (5.15) is in the reachable space of $(J, B_1)$. Furthermore, since $n-1-\mu \geq n_1-1$, the left-hand side of (5.15) can be driven to any point in the reachable space of $(J,B_1)$.

So far we have shown that $\mathcal{R}_w(K-1,L)$ has the same dimension as $\mathcal{R}_w(K,L)$ as long as $K-1 \geq n$. In a similar manner we can show that $\mathcal{R}_w(K,L+1)$ has the same dimension as well, as long as $L+1 \leq N-n$. This then completes the proof of the theorem.

Note that one immediate consequence of Theorem 5.1 is the following

Corollary: The system (2.1) - (2.2) is weakly reachable if it is weakly reachable off some $[K,L]$ with $K,L \in [n,N-n]$.

Hence, in order to test for weak reachability we need only examine the reachability space $\mathcal{R}_w(k,k)$ of $z_i(k,k) = x(k)$ for any $k \in [n,N-n]$. Note further that $\mathcal{R}_w(k,k)$ is the range space for the map from $\{u(0),\ldots,u(N-1)\}$ to $x(k)$ (with the boundary value set to zero); i.e., weak reachability corresponds to being able to drive $x(k)$ to an arbitrary value using the entire interval of the controls. Thanks to statement (d) of Theorem 4.1, we see that weak reachability is indeed weaker than strong reachability which corresponds to being able to drive $x(k)$ to an arbitrary value using only inputs within n time steps of k.

While (5.1) provides in principal a method for computing weakly reachable subspaces, it involves a significant amount of computation in order to determine $\Lambda(K,L)$, $T_i(K,L)$, and $T_f(K,L)$. As the next theorem shows, there is an easier method for computing $\mathcal{R}_w(k,k)$.

Theorem 5.2: Let $k\epsilon$ [n,N-n]. Then

$$\mathcal{R}_w(k,k) = \mathrm{Im}[A^kE^{N-k}(V_iA + V_fE)R_s \mid R_s]$$
$$= \mathrm{Im}[A^kE^{N-k}V_iR_s \mid A^kE^{N-k}V_fR_s \mid R_s] \tag{5.16}$$

Proof: From (2.16), (2.17) (with $\omega = 1$ for simplicity) we see that

$$\mathcal{R}_w(k,k) = \mathrm{Im}[A^k(A-E^{N-k}(V_iA+V_fE)E^k)R(N-k) \mid E^{N-k}(E-A^k(V_iA+V_fE)A^{N-k})R(k)] \tag{5.17}$$

That is, if $w\epsilon\mathcal{R}_w(k,k)$, then there exist $x,y\epsilon\mathcal{R}_s$ so that

$$w = A^k[A-E^{N-k}(V_iA+V_fE)E^k]x + E^{N-k}[E-A^k(V_iA+V_fE)A^{N-k}]y$$

$$= (A^{k+1}x + E^{N-k+1}y) - A^kE^{N-k}(V_iA+V_fE)[E^kx+A^{N-k}y] \tag{5.18}$$

Since $\mathcal{R}_s$ is E- and A- invariant, we see that

$$\mathcal{R}_w(k,k) \subseteq \mathrm{Im}[A^kE^{N-k}(V_iA+V_fE)R_s \mid R_s] \tag{5.19}$$

The first equality in (5.16) will be proved then if we can show that any w in the range of $[A^kE^{N-k}(V_iA+V_fE)R_s \mid R_s]$ is in $\mathcal{R}_w(k,k)$. Clearly any such w can be written as

$$w = s - A^kE^{N-k}(V_iA+V_fE)t \tag{5.20}$$

with $s,t\in\mathscr{R}_s$. Comparing this to (5.18) we see that we will be finished if we can show that there exists $x,y\in\mathscr{R}_s$ so that

$$\begin{bmatrix} A^{k+1} & E^{N-k+1} \\ E^k & A^{N-k} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} \tag{5.21}$$

The matrix on the left-hand side of (5.21) is invertible, and solving (5.21) we obtain

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \Gamma^{-1}A^{N-k} & -\Gamma^{-1}E^{N-k+1} \\ -\Gamma^{-1}E^{k} & \Gamma^{-1}A^{k+1} \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \tag{5.22}$$

where $\Gamma$ is defined in (2.9) (with $\omega = 1$). Since $\mathscr{R}_s$ is E- and A- invariant, it is also $\Gamma^{-1}$ invariant, so that x and y are in $\mathscr{R}_s$.

Finally we need to verify the second equality in (5.16). Since $\mathscr{R}_s$ is E- and A- invariant and $V_iE^N+V_fA^N = I$, we see that

$$\mathrm{Im}[(V_iA+V_fE)R_s \,|\, R_s] \subseteq \mathrm{Im}[V_iR_s \,|\, V_fR_s] \tag{5.23}$$

On the other hand,

$$\begin{aligned} \mathrm{Im}[V_fR_s \,|\, R_s] &= \mathrm{Im}[V_f(E^{N+1}-A^{N+1})R_s \,|\, R_s] \\ &\subseteq \mathrm{Im}[(V_iA+V_fE)E^NR_s \,|\, (V_iE^N+V_fA^N)AR_s \,|\, R_s] \\ &\subseteq \mathrm{Im}[(V_iA+V_fE)R_s \,|\, R_s] \end{aligned} \tag{5.24}$$

Similarly

$$\mathrm{Im}[V_iR_s \,|\, R_s] \subseteq \mathrm{Im}[(V_iA+V_fE)R_s \,|\, R_s] \tag{5.25}$$

Combining (5.23), (5.24), (5.25) we see that

$$\mathrm{Im}[(V_iA+V_fE)R_s \mid R_s] = \mathrm{Im}[(V_iR_s \mid V_fR_s] \quad (5.26)$$

Finally

$$\begin{aligned}\mathrm{Im}[A^kE^{N-k}(V_iA+V_fE)R_s \mid R_s] &= A^kE^{N-k}\mathrm{Im}[(V_iA+V_fE)R_s \mid R_s] + \mathcal{R}_s \\ &= A^kE^{N-k}\mathrm{Im}[V_iR_s \mid V_fR_s] + \mathcal{R}_s \\ &= \mathrm{Im}[A^kE^{N-k}V_iR_s \mid A^kE^{N-k}V_fR_s \mid R_s] \quad (5.27)\end{aligned}$$

Note from (5.16) that $\mathcal{R}_s \subseteq \mathcal{R}_w(k,k)$ for $k \in [n,N-n]$, consitent with our earlier statement that weak reachability is indeed a weaker condition.

Example 5.1: Consider the system (2.1), (2.2) with

$$E = I, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (5.28)$$

$$V_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad V_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (5.29)$$

This system is in normalized form for all N. Since $E = I$, strong reachability reduces to the usual notion of reachability for causal systems. Clearly then $\mathcal{R}_s$ is spanned by the vector $[1,0,0]^T$. From (5.16) we find that $\mathcal{R}_w(k,k)$ is spanned by $[1,0,0]^T$ and $[0,0,1]^T$ for k even and by $[1,0,0]^T$ and $[0,1,0]^T$ for k odd. This example illustrates the mechanism through which some states may be weakly but not strongly reachable. It also demonstrates another fact peculiar to boundary-value systems: while the dimension of $\mathcal{R}_w(k,k)$ remains constant for $k \in [n,N-n]$, this subspace is not dynamically-invariant. In particular, while the dynamics (5.28) allow the input to influence only the first component of x(k), the boundary matrices (5.30) couple the first and third components, allowing indirect control of the third. The A-matrix then produces the oscillatory behavior in $\mathcal{R}_w(k,k)$.

Theorem 5.2 provides us with a computable weak reachability condition: we check to see if either of the matrices in (5.16) is full rank. The following result provides a simpler result of this type as no powers of E or A must be computed.

<u>Theorem 5.3</u>: The system (2.1),(2.2) is weakly observable if and only if either of the matrices

$$[EA(V_iA+V_fE)R_s \,\vdots\, R_s] \tag{5.30a}$$

or

$$[EAV_iR_s \,\vdots\, EAV_fR_s \,\vdots\, R_s] \tag{5.30b}$$

has full rank.

<u>Proof</u>: We begin by showing that for any subspace $\mathcal{D}$ of $\mathbb{R}^n$

$$E\mathcal{D} + \mathcal{R}_s = \mathbb{R}^n \leftrightarrow E^2\mathcal{D} + \mathcal{R}_s = \mathbb{R}^n \tag{5.31}$$

Let $\mathcal{F}$ be a subspace so that

$$\mathcal{F} \oplus \mathcal{R}_s = E\mathcal{D} + \mathcal{R}_s \tag{5.32}$$

Then

$$E^2\mathcal{D} + \mathcal{R}_s = E(E\mathcal{D} + \mathcal{R}_s) + \mathcal{R}_s = E(\mathcal{F} \oplus \mathcal{R}_s) + \mathcal{R}_s$$
$$= E\mathcal{F} \oplus \mathcal{R}_s \tag{5.33}$$

Dimension counting then shows that the right-to-left implication in (5.31) is true. Suppose that $E\mathscr{D} + \mathscr{R}_s = \mathbb{R}^n$. Then

$$E^2\mathscr{D} + \mathscr{R}_s = E(E\mathscr{D} + \mathscr{R}_s) + \mathscr{R}_s = E(\mathbb{R}^n) + \mathscr{R}_s \supseteq E\mathscr{D} + \mathscr{R}_s = \mathbb{R}^n \tag{5.34}$$

Note that by iterating (5.31) we see that if $E^k\mathscr{D} + \mathscr{R}_s = \mathbb{R}^n$ for some $k > 0$, it equals $\mathbb{R}^n$ for all $k > 0$. A similar statement can be made with E replaced by A, and combining these we have that $E^kA^\ell\mathscr{D} + \mathscr{R}_s = \mathbb{R}^n$ for some pair $k,\ell > 0$ if and only if $EA\mathscr{D} + \mathscr{R}_s = \mathbb{R}^n$. The theorem then follows from the application of this result with $\mathscr{D} = \mathrm{Im}\{(V_iA+V_fE)R_s\}$.

Now let us briefly present the corresponding concept of and results on weak observability.

<u>Definition 5.2</u>: The system (2.1)-(2.3) is <u>weakly observable</u> off [K,L] if the map from $z_o[K,L]$ to $\{y(j)\,|\,j\in[0,K]\cup[L,N]\}$, defined by (2.3) and the four-point boundary-value problem (3.5)-(3.7) with $v = 0$, $u \equiv 0$, is one-to-one. The <u>weakly unobservable subspace</u> $\mathcal{O}_w(K,L)$ is the kernel of this map. The system is <u>weakly observable</u> if it is weakly observable off [K,L] (i.e., if $\mathcal{O}_w(K,L) = \{0\}$) for all $K,L\in[n-1,N-n+1]$.

<u>Theorem 5.4</u>: The dimension of $\mathcal{O}_w(K,L)$ is constant for $K,L\in[n-1,N-n+1]$.

<u>Corollary</u>: The system (2.1)-(2.3) is weakly observable if it is weakly

observable off some [K,L].

A consequence of this last result is that in order to test for weak observability we need only examine the unobservability space $\mathcal{O}_w(k,k+1)$ of $z_o(k,k+1) = Bu(k)$. Furthermore, note that $\mathcal{O}_w(k,k+1)$ is the kernel of the mapping from Bu(k) to the <u>full</u> sequence of measurements y(0),...,y(N) (with v set to zero). This is weaker than strong observability which involves the use of outputs restricted to lie within n time steps of k.

<u>Theorem 5.5</u>: Let $k \in [n,N-n]$. Then

$$\mathcal{O}_w(k,k) = \ker \begin{bmatrix} \mathcal{O}_s \\ \mathcal{O}_s(V_i A + V_f E) A^{N-k-1} E^k \end{bmatrix}$$

$$= \ker \begin{bmatrix} \mathcal{O}_s \\ \mathcal{O}_s V_i A^{N-k-1} E^k \\ \mathcal{O}_s V_f A^{N-k-1} E^k \end{bmatrix} \qquad (5.35)$$

Note that $\ker\mathcal{O}_w(k,k) \subseteq \ker\mathcal{O}_s$, demonstrating again that weak observability is a weaker condition.

<u>Theorem 5.6</u>: The system (2.1)-(2.3) is weakly observable if and only if either of the matrices

$$\begin{bmatrix} O_s \\ O_s(V_i A + V_f E) AE \end{bmatrix} \qquad (5.36a)$$

$$\begin{bmatrix} O_s \\ O_s V_i AE \\ O_s V_f AE \end{bmatrix} \tag{5.36b}$$

has full rank.

## VI. Efficient Solution of TPBVDS's

Unlike causal systems, the solution of a TPBVDS cannot be computed using a simple recursion since the solution x(k) depends on inputs over the entire interval. There are, however, several efficient methods for solution which we describe in this section.

### 6.1 The Two-Filter Solution

In his study Krener derived a solution by solving his continuous-time linear system assuming a zero initial condition and then correcting for the actual boundary conditions. Since E and A may both be singular for a TPBVDS, the analogous procedure, first described in [22], is somewhat more complex as we must identify which parts of the system can be solved in the forward and backward directions.

From Kronecker's canonical form for a regular pencil [17] we can find nonsingular matrices T and F so that[4]

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[4]The decomposition in [17] splits the pencil zE-A into forward dynamics corresponding to a pencil of the form $zI-\tilde{A}_1$ and backward dynamics

$$FET^{-1} = \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} \qquad (6.1a)$$

$$FAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \qquad (6.1b)$$

so that all of the eigenvalues of $A_1$ and $A_2$ have magnitudes no larger than 1. Define

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = Tx(k) \qquad (6.2)$$

Then we obtain

$$x_1(k+1) = A_1x_1(k) + B_1u(k) \qquad (6.3a)$$

$$x_2(k) = A_2x_2(k+1) - B_2u(k) \qquad (6.3b)$$

where

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = FB \qquad (6.4)$$

Note that (6.3a), (6.3b) are asymptotically stable recursions if $|zE-A|$ has no zeros on the unit circle. Finally, given the transformation (6.2), the boundary condition (2.2) takes the form

$$[V_{1i} \vdots V_{2i}]\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + [V_{1f} \vdots V_{2f}]\begin{bmatrix} x_1(N) \\ x_2(N) \end{bmatrix} = v \qquad (6.5)$$

---

corresponding to $z^{-1}-\tilde{A}_2$ where $\tilde{A}_2$ is nilpotent. The only difference in (6.1), (6.2) is that the unstable forward modes of $\tilde{A}_1$ have been shifted into the backward dynamics $A_2$.

$$[V_{1i} \vdots V_{2i}] = V_i T^{-1} \quad , \quad [V_{1f} \vdots V_{2f}] = V_f T^{-1} \tag{6.6}$$

Employing the forward/backward representation (6.3) of the dynamics, a general solution to (2.1), (2.2) is derived as follows. Let $x_1^0(k)$ denote the solution to (6.3a) with zero initial condition, and let $x_2^0(k)$ denote the solution of (6.3b) with zero final condition. Then

$$x_1(k) = A_1^k x(0) + x_1^0(k) \tag{6.7a}$$

$$x_2(k) = A_2^{N-k} x_2(N) + x_2^0(k) \tag{6.7b}$$

Substituting (6.7) into (6.5) and solving for $x_1(0)$ and $x_2(K)$ yields

$$\begin{bmatrix} x_1(0) \\ x_2(N) \end{bmatrix} = H^{-1}\{v - V_{1f}x_1^0(N) - V_{2i}x_2^0(0)\} \tag{6.8}$$

where

$$H = [V_{1i} + V_{1N}A_1^N \vdots V_{2i}A_2^N + V_{2f}] = V_i T^{-1}(FET^{-1})^N + V_f T^{-1}(FAT^{-1})^N \tag{6.9}$$

Finally, substituting (6.8) into (6.7) yields

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} A_1^k & 0 \\ 0 & A_2^{N-k} \end{bmatrix} H^{-1} \left[v - V_{1f}x_1^0(N) - V_{2i}x_2^0(0)\right] + \begin{bmatrix} x_1^0(k) \\ x_2^0(k) \end{bmatrix} \tag{6.10}$$

The solution in the original basis can then be obtained by inverting (6.2).

Note that the transformed matrices in (6.1), (6.2) commute and are in fact in a form close to our normalized form (see discussion in the next section). However, the full importance of transforming the system into normalized form, and in particular its implication for a generalized Cayley-Hamilton theorem and the resulting form of reachability and observability results, has not been previously recognized. Also, the algorithm just described provides an equivalent well-posedness condition, namely the invertibility of H in (6.9).

## 6.2 A Parallel Outward-Inward Solution

A second efficient algorithm can be constructed by noting that the solution x can be recovered from the outward process $z_o$ and the inward process $z_i$. For simplicity, let us assume that N is odd and that E and A commute (as they would if (2.1), (2.2) is in normalized form). It is then possible to specify a recursive algorithm for the computation of $z_o(j,N-j)$ for $j = 0,\ldots,(N-1)/2$, starting from the initial condition at the center of the interval (with $j = (N-1)/2$):

$$z_o((N-1)/2,\ (N+1)/2) = Bu((N-1)/2) \tag{6.11}$$

and propagating symmetrically outward from the center:

$$z_o(j-1,N-j+1) = EAz_o(j,N-j) + A^{N-2j+1}Bu(j-1) + E^{N-2j+1}Bu(N-j) \tag{6.12}$$

Similarly we can compute $z_i(j,N-j)$ recursively inward from the initial condition

$$z_i(0,N) = v \qquad (6.13)$$

using a recursive procedure based on that outlined in Section III (see (3.13) - (3.21)).

The solution x can then be computed as

$$\begin{bmatrix} x(j) \\ x(N-j) \end{bmatrix} = \begin{bmatrix} -A^{N-2j} & E^{N-2j} \\ W_i(j,N-j) & W_f(j,N-j) \end{bmatrix}^{-1} \begin{bmatrix} z_o(j,N-j) \\ z_i(j-N-j) \end{bmatrix} \qquad (6.14)$$

where the inverse on the right-hand side of (6.14) is guaranteed to exist thanks to the well-posedness of (2.1), (2.2).

## 6.3 A Serial Outward-Inward Solution

As a first step in this algorithm we compute $z_o(j,N-j)$ outward from the interval center as in (6.11), (6.12). We then use these values, together with the boundary condition v, to solve for x(j) and x(N-j) recursively as we propogate back toward the interval center. To begin, note that

$$\begin{bmatrix} x(0) \\ x(N) \end{bmatrix} = \begin{bmatrix} -A^{N} & E^{N} \\ V^i & V^f \end{bmatrix}^{-1} \begin{bmatrix} z_o(0,N) \\ v \end{bmatrix} \qquad (6.15)$$

where the inverse indicated on the right-hand side of (6.15) is again guaranteed to exist thanks to well-posedness. To continue with the inward recursion, note that from (3.1)

$$-A^{N-2j}x(j) + E^{N-2j}x(N-j) = z_o(j,N-j) \tag{6.16}$$

while from (2.1)

$$\delta_j Ex(j) + Ax(N-j) = \delta_j Ax(j-1) + Ex(N-j+1) + \delta_j Bu(j-1) - Bu(N-j) \tag{6.17}$$

for any $j \in [1,(N-1)/2]$ and any scaler $\delta_j$. We then have the recursion

$$\begin{bmatrix} x(j) \\ x(N-j) \end{bmatrix} = \begin{bmatrix} -A^{N-2j} & E^{N-2j} \\ \delta_j E & A \end{bmatrix}^{-1} \begin{bmatrix} z_o(j,N-j) \\ \delta_j Ax(j-1)+Ex(N-j+1)+\delta_j Bu(j-1)-Bu(N-j) \end{bmatrix} \tag{6.18}$$

where $\delta_j$ is chosen so that the inverse on the right-hand side of (6.18) exists (for example, if $|zE-A|$ has no roots on the unit circle, $\delta_j$ can be taken equal to 1).

## VII. Conclusion

In this paper we have analysed some of the system-theoretic properties of TPBVDS's. As in Krener's analysis of continuous-time, non-descriptor, boundary-value systems, there are actually two distinct concepts for reachability and for observability of TPBVDS's, and in this paper we have investigated each of these. In addition, we have described three methods for the efficient solution of TPBVDS's, one based on a variation on Kronecker's form for a regular pencil and two on the inward/outward recursions and processes that play such an important role in the analysis of these systems.

An important step in our analysis is the introduction of a standard form for regular pencils. This form permits us to obtain a simple form for a generalized Cayley-Hamilton theorem which in turn leads to simpler reachability and observability results than have appeared previously in the literature. It is worth noting that this generalized Cayley-Hamilton theorem and the resulting reachability and observability results continue to hold if E and A take the form

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \quad , \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \tag{7.1}$$

where $E_i$, $A_i$ are in standard form, i.e., $\alpha_i E_i + \beta_i A_i = I$, but where $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ need not be the same. An example of such a form is the variation of Kronecker's form given in (6.1), (6.2).

There are a variety of extensions and complements to the results presented in this paper. Many of these involve the examination of stationary TPBVDS's, i.e., models as in (6.1), (6.2) but for which the Green's function $G(k,\ell)$ in (2.17) depends only on $k-\ell$. As we will describe in a subsequent paper, the analysis of stationary TPBVDS's can be significantly simplified and extended. For example, the description and recursive computation of the inward process $z_i(k,\ell)$ is far simpler in the stationary case. In addition, for such systems the weak reachability and unobservability spaces are time-invariant (i.e., they don't rotate). As a simple example, consider the class of cyclic processes, i.e., processes for which $V_i = -V_f = I$ (so that $x(0) = x(N)$ if $v = 0$). Not only is such a process stationary but from (5.16), (5.35) we see that in this case weak and strong reachability and observability concepts coincide.

There are also a number of other system-theoretic concepts that can be developed in detail for a stationary TPBVDS's. For example, there exists a minimal realization theory for such systems analogous to that described by Krener. In addition, it is possible to develop a concept of stability for such systems, reflecting the effect that the boundary conditions have on the process near the center of the interval as the boundaries recede. Not only is such a concept useful in determining the numerical well-posedness of algorithms such as those described in Section VI, but it also provides the basis for analyzing stochastic TPBVDS's and the properties of optimal estimators for such processes [22].

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