Multi-Item Single-Vendor-Single-Buyer Problem with Consideration of Transportation Quantity Discount

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Abstract — this paper deals with the problem of shipping multiple commodities from a single vendor to a single buyer. Each commodity is assumed to be constantly consumed at the buyer, and periodically replenished from the vendor. Furthermore, these replenishments are restricted to happen at discrete time instants, e.g., a certain time of the day or a certain day of the week. At any such time instant, transportation cost depends on the shipment quantity according to certain discount scheme. Specifically, we consider two transportation quantity discount schemes: LTL (less-than-truckload) incremental discount and TL (truckload) discount. For each case, we develop MIP (mixed integer programming) mathematical model whose objective is to make an integrated replenishment and transportation decision such that the total system cost is minimized. We also derive optimal solution properties and give numerical studies to investigate the problem.

Index Terms — *Vendor-Buyer Problem, Transportation Quantity Discount, LTL Incremental Discount, TL Discount*

I. PROBLEM CONTEXT

In this paper, we study the multi-item single-vendor-single-buyer problem (a.k.a. single-warehouse-single-retailer problem), in which transportation quantity discount is provided by external carrier. The problem of interest is partially motivated by the growth of emerging 3PL (third-party logistics) industry and application of VMI (vendor-managed inventory) systems.

Consider the following example. Super-Vendor, a consumer goods supplier, provides one of its clients Super-Mart some dozens of commodities from 50-cent diapers to 50-dollar cosmetics. Super-Mart adopts VMI system, that is, Super-Vendor has the liberty of making replenishment decisions for Super-Mart rather than filling replenishment orders from it. A 3PL company, Super-Logistics, transports the commodities. And according to the transportation contract, trucks from Super-Logistics arrive at 8am in the morning if shipment is scheduled in that day and no other arrival time of a day is available. We consider two quantity discount schemes as follows.

LTL incremental discount transportation: When the shipment quantity is less than the vehicle capacity, quantity discount is applied only to the additional shipment quantities beyond the predetermined breakpoint. In this situation, the logistics manager of Super-Vendor faces a basic tradeoff: infrequent replenishments raise inventory holding costs, and lower transportation costs, while the contrary happens with frequent replenishments.

TL discount transportation: When the shipment quantity is larger than the vehicle capacity, TL (truckload) discount transportation is used. In TL discount scheme, Super-Vendor pays LTL transportation cost until it has paid for the cost of a full truckload, at which point, there is no charge for the remaining quantities shipped in that truckload.

Thus, we obtain two different single-vendor-single-buyer problems, which can be meaningful and interesting.

The remainder of this paper is organized as follows. Next section presents a review of the relevant literature. The description of the problem under consideration is given in §III. Centralized models for the LTL incremental discount transportation and TL discount transportation are respectively presented in §IV and §V. Finally, we give a few concluding remarks in §VI.

II. RELEVANT LITERATURE

We note that the literature on VMI system or transportation quantity discount schemes is abundant. In the interest of brevity, our literature review in this section mainly focuses on the single-vendor- single-buyer problem.

The Single-vendor-single-buyer problem was firstly introduced by Goyal (1976) which studied an integrated inventory problem of shipping a single commodity from a single supplier to a single customer. Goyal showed that system cost savings can be achieved if the supplier and customer cooperate to determine the economic joint inventory policy.

Monahan (1984) developed a model from a vendor's perspective for establishing an optimal price discount schedule with the premise that the vendor's order processing cost is larger than the buyer's fixed order cost. Banerjee (1986a) extended and generalized these results to account for the situations where the vendor is a manufacturer. He demonstrated the equivalence between approaches suggested by Monahan (1984) and Banerjee (1986b) for lot size modification accompanied by a price discount in order to increase vendor profits. Goyal (1988) illustrated that manufacturing a batch which is made up of an integral number of equal shipments generally produces a lower cost solution. The related literature up to 1989 was well summarized in the review paper of Goyal and Gupta (1989).

Lu (1995) developed a heuristics algorithm to the single-vendor-single-buyer problem with the assumption of a production batch providing an integral number of equal shipments. And Goyal (1995) used the same numerical example from Lu (1995) to investigate an alternative policy involved successive shipments within a production batch increasing by a constant factor. This policy is based on a much earlier idea set out by Goyal (1977) to solve a very similar problem in a slightly different setting. In Hill (1997, 1999), different types of policy were considered with the assumption of successive shipments to the buyer within a single production batch.

Hoque and Goyal (2000) assumed the vendor's replenishment/production rate is finite. Their model incorporates a capacity constraint limiting the replenishment quantities of the buyer. Toptal et al. (2003) generalized the single-vendor-single-buyer problem to simultaneously consider truck capacity constraints and inbound/outbound transportation costs. They provided both exact solution procedures and heuristics algorithms.

A special single-vendor-single-buyer problem called single-link problem was introduced by Speranza and Ukovich (1994) to consider shipping multiple commodities from an origin to a destination with consideration of discrete shipment frequencies. They assumed the FTL (full truckload) transportation. Their model determines the number of trucks to be used and allocates different commodities to trucks. The branch-and-bound algorithm of Speranza and Ukovich (1996) was used for the solution of this single-link problem.

An extension to the single-link problem was studied by Bertazzi et al. (1997) with considering one origin and multiple destinations. They presented different heuristics by solving a single link problem for each of the given destinations first, and then on improving the solution through local search techniques. Bertazzi et al. (2000) proposed an improved branch-and-bound algorithm for the single-link problem. Bertazzi and Speranza (2002) gave a framework for the identification of optimal continuous and discrete shipping strategies for the single link problem.

In reviewing the previous work in vendor-buyer problem, we found that, all the papers overlook practical transportation quantity discount consideration when modeling the problem. Therefore, analyzing the impact of transportation quantity discount on vendor-buyer problem is one of the main contributions of this paper.

III. Problem Description

The following Table 1 displays the problem notations used in this section, along with the measure units for the considered quantities.

TABLE 1. PROBLEM NOTATIONS

k, K:	Index and set of commodities, ($k \in K$).
n, N:	Index and set of discrete time instants, ($n \in N$).
d_k:	Demand rate of commodity k at the buyer (quantity per time instant).
v_k:	Unit volume of commodity k (volume).
h_k:	System holding cost rate of commodity k ($ per quantity per time instant).
t, T:	Discrete value and set of possible replenishment/shipment periods, ($t \in T$).
δ_t^n :	Binary coefficient about if shipment with period t happens at time instant n.
x_k^t :	Fraction of commodity k replenished in period of t, $0 \le x_k^t \le 1$.
t_k:	Replenishment/shipment period of commodity k (when SF policy applies).
z_n:	Shipment quantity under *Time Instant Consolidation Policy* (volume).
z_t:	Shipment quantity under *Frequency Consolidation Policy* (volume).
$F_1(z)$:	Transportation cost function of LTL incremental discount scheme
$F_2(z)$:	Transportation cost function of TL discount scheme
$\lvert N\rvert$:	Cycle of total shipment quantity pattern

A. Commodity Assumptions

We use k to denote the index of commodities and K to denote the set of commodities. At the buyer, commodities are assumed to be continuously consumed at deterministic and constant rates. And no backorder is allowed. Let d_k denote the demand rate for commodity k. Each commodity k is characterized with a unit volume v_k. And all these commodities are assumed to have the same density. Thus, we ignore the shipment class category issue, in which commodities are charged according to different cost schemes based on their density values.

B. Discrete Periodic Replenishment Policy

We assume that each commodity is periodically replenished. Furthermore, the minimum replenishment (in this paper, the terms "shipment" and "replenishment" are used interchangeably) interval is assumed to be an integral value, and all possible replenishment periods must be a multiplier of this value. This assumption was justified by Hall (1985), Muckstadt and Roundy(1993), and Speranza and Ukovich (1994) that the traditional EOQ-type formulas may produce an impractical shipment policy to implement such as shipping a commodity every 1.4142 days.

We use n to denote the possible discrete shipment time instant, and t to denote the value of shipment period. We also assume that there is a given set for t: $t \in T$ .We assume the minimum interval between shipments $t_{\min}$ is 1. Thus we have n = 0, 1, 2… Now, Let us consider two possible policies of deciding how commodities are replenished:

Multiple frequency (MF) policy: each commodity can be partially replenished in different frequencies. Let variable x_k^t denote the fraction of commodity k replenished in period t. We have $0 \le x_k^t \le 1$ and

$$\sum_{t \in T} x_k^t = 1 . \tag{3.1}$$

Single frequency (SF) policy: each commodity k must be replenished in a single frequency. Therefore, x_k^t is a binary variable. Under the single frequency policy, commodity k can only be assigned one shipment period denote by t_k :

$$t_k = \sum_{t \in T} x_k^t t . \tag{3.2}$$

In principle, MF policy outperforms SF policy due to more flexibility allowed. We will show in section 4 that these two policies are equivalent for the centralized model of LTL incremental discount transportation problem.

C. Transportation Costs

First, we have an assumption that all the commodities are shipped at time instant 0 and no shipment staggering is allowed. Under this assumption, shipment with period t happens at the time instants $0, t, 2t, 3t, ...$

We use z to denote the shipment quantity, and $F(z)$ to denote the transportation cost function. Let C_{trans} denote the average transportation cost. This cost considered is defined on an infinite time horizon. However it is obviously that we only need to consider a finite time horizon $|N|$, which is

$$|N| = lcm\{t : \exists k \text{ with } x_k^t > 0\} . \tag{3.3}$$

In this paper, such horizon is assumed to start at time instant 0 and end at time instant $|N|$-1. This time instant set [0, 1, 2, …, $|N|$-1] is denoted by N.

In terms of calculation of shipment quantity z, we consider two consolidation policies as follows:

Time instant consolidation policy: all the commodities shipped at the same time instant are consolidated into one shipment. We use z_n to denote this total shipment quantity scheduled at time instant n. Furthermore, we use a binary coefficient δ_t^n to denote if the shipment with period t happens at time instant n. Thus we have

$$z_n = \sum_{t \in T} \sum_{k \in K} t d_k v_k x_k^t \delta_t^n, \qquad \forall n \in N \tag{3.4}$$

$$C_{trans}^n = \frac{1}{|N|} \sum_{n \in N} F(z_n) \tag{3.5}$$

Frequency consolidation policy: only the commodities shipped in the same frequency can be consolidated. We use z_t to denote the shipment quantity with shipment period t.

$$z_t = \sum_{t \in T} \sum_{k \in K} t d_k v_k x_k^t, \qquad \forall t \in T \tag{3.6}$$

$$C_{trans}^t = \sum_{t \in T} \frac{F(z_t)}{t} \tag{3.7}$$

D. Holding Costs

We assume that the system holding cost is proportional to the total inventory carried in the system. Let h_k denote the holding cost rate for commodity k in the system. Then we have the expression of $C_{holding}$

$$C_{holding} = \sum_{k \in K} \sum_{t \in T} h_k d_k t x_k^t \tag{3.8}$$

E. Summary

In this section, we describe a multi-item single-vendor-single-buyer problem that differs from the previous work by incorporating transportation quantity discount. This single-vendor-single-buyer problem models many practical situations such as component manufacturer and assembly plant, central warehouse and local retailer, and so on. Moreover, it may contribute to the analysis of complex supply chain networks when decomposition approach is applied and each decomposed subproblem can be optimized independently as a single-vendor-single-buyer problem as considered in this paper.

IV. LTL Incremental Discount Transportation

In this section, we study a centralized model for the LTL incremental discount transportation problem. As discussed in §I, such model is meaningful in the situation with two assumptions: (1) Under certain strategic alliance (e.g. VMI), vendor and buyer cooperate to minimize the system-wide cost. (2) The shipment quantity is less than the capacity of a vehicle. The objective of this centralized model is to make optimal replenishment and transportation decisions such that the total system cost is minimized.

A. LTL Incremental Discount Transportation Cost

In the LTL incremental discount cost scheme, quantity discount is applied only to the additional shipment quantities beyond the predetermined breakpoint. As described presented in Balakrishnan and Graves (1989), this cost structure can be modeled as a piece-wise linear and concave function $F_1(z)$ as depicted in Figure 4.1.

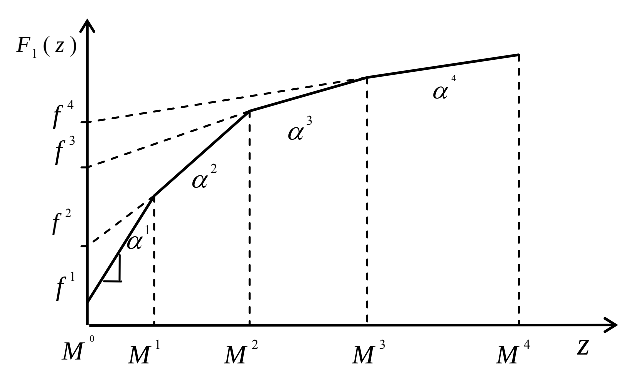

Figure 4.1.Incremental discount cost function $F(z)$

Let r be the index of different slopes of the cost function, and R be the set of r $(r \in R)$. Let M^{r-1}, M^r denote the lower and upper limits, respectively, on the rth interval of shipment quantities. Here $M^0 = 0$ and M^R can be set to the possible maximum shipment quantity. Let f^r and α^r denote the fixed and variable shipment cost associated with the rth interval. We can express the piece-wise linear concave cost function as:

$$F_1(z) = f^r + \alpha^r z \text{ , given } z \in \left(M^{r-1}, M^r\right] \tag{4.1}$$

We assume that time instant consolidation policy is used. The average system cost TC_1^n consists of the average transportation cost C_{trans}^n and average holding cost $C_{holding}$:

$$TC_1^n = \frac{1}{|N|}\sum_{n\in N} F_1(z_n) + \sum_{t\in T}\sum_{k\in K} td_k h_k x_k^t \tag{4.2}$$

B. *Mathematical Model*

In this section, we develop a mixed integer programming model. We use a binary decision variable y_n^r to denote if shipment quantity z_n falls in the range of $\left(M^{r-1}, M^r\right]$ at time instant n. And the decision variable z_n^r is equal to shipment quantity z_n if z_n falls in the range of $\left(M^{r-1}, M^r\right]$. These decision variables are:

y_n^r = 1, if shipment quantity $z_n \in \left(M^{r-1}, M^r\right]$ at time instant n, = 0, otherwise.

z_n^r $= z_n$, if shipment quantity $z_n \in \left(M^{r-1}, M^r\right]$ at time instant n, = 0, otherwise.

A *MIP* (*mixed integer programming*) problem can be formulated as follows.

Problem Φ_1

$$\text{Min } \sum_{k\in K}\sum_{t\in T}(h_k td_k x_k^t) + \frac{1}{|N|}\sum_{n\in N}\sum_{r\in R}(f^r y_n^r + \alpha^r z_n^r) \tag{4.3}$$

$$\text{s.t. } \sum_{t\in T} x_k^t = 1 \text{ ,} \qquad \forall k \in K \tag{4.4}$$

$$\sum_{k\in K}\sum_{t\in T}(td_k v_k x_k^t \delta_t^n) = \sum_{r\in R} z_n^r \text{ , } \forall n \in N \tag{4.5}$$

$$z_n^r \le M^r y_n^r \text{ ,} \qquad \forall n \in N,\ r \in R \tag{4.6}$$

$$z_n^r \ge M^{r-1} y_n^r \text{ ,} \qquad \forall n \in N,\ r \in R \tag{4.7}$$

$$\sum_{r\in R} y_n^r \le 1 \text{ ,} \qquad \forall n \in N \tag{4.8}$$

$$0 \le x_k^t \le 1 \text{ ,} \qquad \forall t \in T,\ k \in K \tag{4.9}$$

$$y_n^r \in \{0,1\} \text{ ,} \qquad \forall n \in N,\ r \in R \tag{4.10}$$

$$z_n^r \ge 0 \text{ ,} \qquad \forall n \in N,\ r \in R \tag{4.11}$$

The objective function (4.3) presents the average system cost TC_1^n. Constraints (4.4) ensure that each commodity is completely assigned shipment periods, and constraints (4.5) set the total shipment quantity at time instant n. Constraints (4.6)- (4.8) make sure that if cost index r is used at time instant n, then the shipment quantity at time instant n must fall in its associated interval $(M^{r-1}, M^r]$. Finally constraints (4.9) indicate that at most one cost range can be selected at each time instant. Constraints (4.9) specify that *multiple frequency policy* is used.

C. *Optimal Solution Properties*

We show two interesting properties for the optimal solution of problem Φ_1. Let us first introduce the following notations. Under the *multiple frequency policy*, we use $t_k^{\min}$ and $t_k^{\max}$ to denote the minimum and maximum of the shipment periods assigned to commodity k.

$$t_k^{\min} = \min\left(t : x_k^t > 0\right), \qquad \forall k \in K \tag{4.12}$$

$$t_k^{\max} = \max\left(t : x_k^t > 0\right), \qquad \forall k \in K \tag{4.13}$$

*LEMMA 1***:** Let the commodities be indexed in a nondecreasing order of the ratio h_k/v_k such that $(h_1/v_1) \le (h_2/v_2) \le ... \le (h_K/v_K)$. Then in the optimal solution of problem Φ_1, we have the following relationship.

$$t_1^{\max} \ge t_1^{\min} \ge t_2^{\max} \ge t_2^{\min} ... \ge t_K^{\max} \ge t_K^{\min} \tag{4.14}$$

PROOF: See the Appendix A.1

Since the proof does not need any particular shipment cost structure assumption, Lemma 1 is true for a general centralized model in which the transportation cost only depends on the shipment quantity and the inventory cost is proportional to the total inventory carried in the system.

LEMMA 2: In problem Φ_1, *multiple frequency policy* and *single frequency policy* are equivalent. Then, it is optimal to replenish each commodity with a single period. Furthermore, for all the commodities of the same ratio of h/v, it is optimal to replenish them with the same periods. That is,

$$t_i = t_j, \quad \text{given } \frac{h_i}{v_i} = \frac{h_j}{v_j} \qquad \forall i, j \in K \tag{4.17}$$

PROOF: See the Appendix A.2

The proof needs the assumptions of concave transportation cost structure and no shipment capacity restriction. Lemma 2 makes sense since any one unit volume of such commodities contributes the same to transportation and inventory costs.

D. *Numerical Example*

In this section, we present a numerical example that illustrates the problem Φ_1 . The main purpose is to show how our model actually works. We also consider a simple shipment strategy in which all the commodities must be shipped with the same frequency. We refer this strategy as *Unified-T* policy. We investigate the situations in which our model outperforms the simple *Unified-T* model.

In our example, 12 commodities are shipped. The vehicle will arrive at the vendor every Monday morning, that is, the basic discrete shipment period is one week. The possible shipment period set T is [1, 2, 3, 4, 6, 8, 12 (weeks)]. Consequently, we only need to consider a planning horizon of 24 weeks. We assume demand rate d_k is 30 (quantity per week) and unit volume v_k is 1 (volume per unit commodity) for every commodity.

For the LTL incremental discount cost structure considered. There are four cost ranges which are [500, 500, 1000, 2000]. The fixed cost of each shipment is 1000 dollars and unit shipment cost rates at four cost ranges are [10, 8, 7, 6 (dollars)]. The holding cost rates (h_k) of commodities are from 0.05 dollar per unit per week to 5 dollars per unit per week. (see talble 2).

TABLE 2 NUMERICAL EXAMPLE OF INCREMENTAL DISCOUNT CASE

		holding cost rate h_k , replenishment period T_k and system cost													*UnifiedT*	Saving
#	k	1	2	3	4	5	6	7	8	9	10	11	12	Cost	Cost	
1	h_k	0.05	0.05	0.85	0.85	1.65	1.65	2.45	2.45	3.25	3.25	4.05	4.05	126784	128112	1.04%
	T_k	12	12	1	1	1	1	1	1	1	1	1	1		1	
2	h_k	0.05	0.05	0.05	1.25	1.25	1.25	2.45	2.45	2.45	3.65	3.65	3.65	123472	125088	1.29%
	T_k	12	12	12	1	1	1	1	1	1	1	1	1		2	
3	h_k	0.05	0.05	0.05	0.05	0.05	0.05	2.45	2.45	2.45	2.45	2.45	2.45	110000	114720	4.12%
	T_k	12	12	12	12	12	12	2	2	2	2	2	2		2	
4	h_k	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	2.45	2.45	2.45	2.45	99008	106512	7.05%
	T_k	12	12	12	12	12	12	12	12	2	2	2	2		3	
5	h_k	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	4.25	4.25	4.25	4.25	109376	118176	7.45%
	T_k	12	12	12	12	12	12	12	12	2	2	2	2		2	
6	h_k	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	5	5	5	5	112256	122496	8.36%
	T_k	12	12	12	12	12	12	12	12	1	1	1	1		2	

4.5. Summary

In this section, we studied the centralized model whose objective is to minimize the sum of LTL incremental discount transportation costs and holding costs. This model accounts for the specificities of a practical situation where vendor-buyer strategic alliance exists and shipment quantity is less than the vehicle capacity. We first introduced the cost structure of the LTL incremental discount transportation, and developed a *MIP* formulation for the problem considered. Then we showed two interesting optimal solution properties of the problem. Finally, we gave a numerical example to show how our model actually works.

V. TRUCKLOAD DISCOUNT TRANSPORTATION

In last section, we discuss the single-vendor-single buyer problem with consideration of LTL incremental discount transportation mode, which is widely used when shipment quantity is smaller compared with the capacity of a transportation vehicle (e.g. truck, container etc). In this section, we assume multiple vehicles are needed to deliver large shipment quantity. This situation was studied in Speranza and Ukovich (1994) where a so-called single-link model was introduced to consider FTL (full-truckload) transportation mode. They showed that system cost savings can be obtained through making replenishment and transportation decisions simultaneously. However, the assumption of exclusive FTL transportation option in their single-link model may not be practical in some situations because sending an almost empty truck costs the same as a full one in FTL transportation mode.

In this section, we investigate the TL (TruckLoad) discount transportation mode which can model two practical situations as follows.

(1) With TL discount cost structure, carrier can give incentives to shippers to practice less-than-truckload shipment so that inventory cost is decreasing. Consequently, the carrier can also gain profits by charging in higher transportation cost.

(2) Both FTL carrier and LTL carrier are available in the market. The shipper uses FTL carrier to ship the quantities of truckloads. While for the delivery of the leftover quantities, the shipper chooses FTL carrier or LTL carrier based on the cost charged.

In the numerical case analysis, our model results in more system cost savings than the single-link model does.

A. *TruckLoad Discount Transportation Cost*

We are now introducing the TL discount transportation cost structure. We use P to denote the truckload capacity in volume. And a predetermined number P' ($0 < P' < P$) divides the truckload P into two segments: $(0, P']$ and $(P', P]$. Transportation cost charged for any vehicle depends on shipment quantity z carried on the vehicle:

(1). When $z \in (0, P']$: if the shipment quantity z falls into the first segment $(0, P']$, transportation cost consists of a fixed and a variable component. The fixed cost, denoted by c_0, is incurred independent of the shipment quantity z as long as it is not zero. The variable cost, denoted by α, is incurred on a per-unit-volume basis. We also refer to c_0 as the setup cost and α as the proportional shipment cost.

(2). When $z \in (P', P]$: if the shipment quantity z falls into the second segment $(P', P]$, transportation cost is a constant value c independent of the shipment quantity z. We also refer to c as full-truckload cost.

Furthermore, the TL discount transportation function $F_2(z)$ is continuous over the range of $(0, P]$, that is to say, we have the equation relationship of $c_0 + \alpha P' = c$. The general TL discount transportation cost function $F_2(z)$ can be described as follows:

(1) if $(\eta - 1)P < z \le (\eta - 1)P + P'$

$$F_2(z) = (\eta - 1)c + c_0 + \alpha\left[z - (\eta - 1)P\right]$$

(2) if $(\eta - 1)P + P' < z \le \eta P$

$$F_2(z) = \eta c \tag{5.1}$$

Where η denotes the number of trucks used to carry shipment quantity z. The cost function $F_2(z)$ can be modeled as a piece-wise linear cost structure as below.

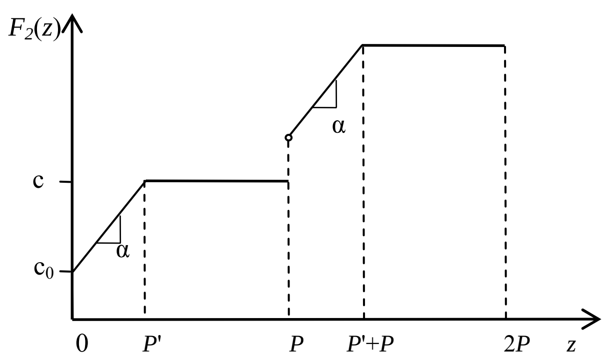

Figure 5.1. Truckload discount transportation cost

B. *Mathematical Model*

In this section, we develop a mixed integer programming model. We assume frequency consolidation policy is used. We use binary decision variable y_t to denote if the leftover quantity of z_t falls in the segment of $(0, P']$. The decision variable z_t^1 is equal to the leftover quantity of z_t if $z_t \in \left((\eta_t - 1)P, (\eta_t - 1)P + P'\right]$, and the decision variable z_t^2 is equal to the leftover quantity of z_t if $z_t \in \left((\eta_T - 1)P + P', \eta_t P\right]$. These decision variables are:

$y_t \;=\; 1$, if $z_t \in \left((\eta_t - 1)P, (\eta_t - 1)P + P'\right]$.

$\;\;\;\;= 0$, if $z_t \in \left((\eta_T - 1)P + P', \eta_t P\right]$.

$z_t^1 = z_t - (\eta_t - 1)P$, if $z_t \in \left((\eta_t - 1)P, (\eta_t - 1)P + P'\right]$

$\;\;\;\;= 0$, otherwise

$z_t^2 = z_t - (\eta_t - 1)P$, if $z_t \in \left((\eta_T - 1)P + P', \eta_t P\right]$

$\;\;\;\;= 0$, otherwise

According to the decision variables given as above, the TL discount transportation function $F_2(z_t)$ can be formulated as:

$$F_2(z_t) = \begin{cases} (\eta_t - 1)c + c_0 + \alpha z_t^1 & \text{if } y_t = 1 \\ \eta_t \, c & \text{if } y_t = 0 \end{cases} \qquad \eta = 1, 2, 3\ldots \tag{5.2}$$

We also have the relationship $c = c_0 + \alpha P'$. Thus (5.2) can be reformulated as:

$$F_2(z_t) = \begin{cases} \eta_t c - \alpha\left(P' - z_t^1\right) & \text{if } y_t = 1 \\ \eta_t c & \text{if } y_t = 0 \end{cases} \qquad \eta = 1, 2, 3\ldots \tag{5.3}$$

Furthermore, we can use the binary variable y_t to derive a general expression:

$$F_2(z_t) = \eta_t c - \alpha(P' y_t - z_t^1) \tag{5.4}$$

A mixed integer programming problem can be formulated as follows.

Problem Φ_2

$$\text{Min} \sum_{k \in K} \sum_{t \in T} \left(h_k t d_k x_k^t\right) + \sum_{t \in T} \frac{1}{t}\left[\eta_t c - \alpha\left(P' y_t - z_t^1\right)\right] \tag{5.5}$$

$$\text{s.t.} \quad \sum_{t \in T} x_k^t = 1 \;, \qquad \forall k \in K \tag{5.6}$$

$$\sum_{k \in K} t v_k d_k x_k^t = (\eta_t - 1)P + z_t^1 + z_t^2 \;, \quad \forall t \in T \tag{5.7}$$

$$0 \le z_t^1 \le P' y_t \;, \qquad \forall t \in T \tag{5.8}$$

$$P'(1 - y_t) \le z_t^2 \le P(1 - y_t) \;, \quad \forall t \in T \tag{5.9}$$

$$0 \le x_k^t \le 1 \;, \qquad \forall t \in T, k \in K \tag{5.10}$$

$$y_t \in \{0, 1\} \;, \qquad \forall t \in T \tag{5.11}$$

$$z_t^1, z_t^2 \ge 0 \qquad \forall t \in T \tag{5.12}$$

$$\eta_t \;\text{ integer} \qquad \forall t \in T \tag{5.13}$$

The objective function (5.5) expresses the minimization of average system cost: the first term presents the average system holding cost, and the second term presents the average system transportation cost. Constraints (5.6) ensure that, for each commodity, the whole quantity is shipped and assigned to different shipment periods. The quantity relationships between variables x_k^t, η_t, z_t^1 and

z_t^2 are defined in constraints (5.7). Cosntraints (5.8) and (5.9) specify that if shipment with period *t* is full-truckload charged, the binary variable y_T must be 0; otherwise y_T must be 1.

C. Numerical Example

In this section, we consider a problem of shipping 5 commodities. The unit volume v_k for each commodity is 1 (volume per unit commodity) and the demand rate d_k is the same for all commodities. The holding cost rates h_k of each commodity is [1, 1.2, 1.5, 1.8, 2]. The vehicle capacity *P* is 100 (volume unit) and the transportation cost of full truckload *c* is 300 (dollar). The setup cost c_0 is 40 and variable cost α is 4. The possible replenishment periods are [1, 2, 3, 4, 5, 6, 7, 8]

$d_k = 22$ TL Discount Model

X_j^t	*t*=1	*t*=2	*t*=3	*t*=4	*t*=5	*t*=6	......
k=1	.747	.253	0	0	0	0	......
k=2	1	0	0	0	0	0	......
k=3	1	0	0	0	0	0	......
k=4	1	0	0	0	0	0	......
k=5	1	0	0	0	0	0	......
η_t	1	1	0	0	0	0	......

Total Cost 516.08

$d_k = 22$ Single-Link Model

X_j^t	*t*=1	*t*=2	*t*=3	*t*=4	*t*=5	*t*=6	......
k=1	.747	0	0	0	.253	0	......
k=2	1	0	0	0	0	0	......
k=3	1	0	0	0	0	0	......
k=4	1	0	0	0	0	0	......
k=5	1	0	0	0	0	0	......
η_t	1	0	0	0	1	0	......

Total Cost 546.08

$d_k = 27$ TL Discount Model

X_j^t	*t*=1	*t*=2	*t*=3	*t*=4	*t*=5	*t*=6	......
k=1	1	0	0	0	0	0	......
k=2	1	0	0	0	0	0	......
k=3	1	0	0	0	0	0	......
k=4	1	0	0	0	0	0	......
k=5	1	0	0	0	0	0	......
η_t	2	0	0	0	0	0	......

Total Cost 664.28

$d_k = 27$ Single-Link Model

X_j^t	*t*=1	*t*=2	*t*=3	*t*=4	*t*=5	*t*=6	......
k=1	.28	.72	0	0	0	0	......
k=2	1	0	0	0	0	0	......
k=3	1	0	0	0	0	0	......
k=4	1	0	0	0	0	0	......
k=5	1	0	0	0	0	0	......
η_t	1	1	0	0	0	0	......

Total Cost 749.17

$d_k = 31$ TL Discount Model

X_j^t	*t*=1	*t*=2	*t*=3	*t*=4	*t*=5	*t*=6	......
k=1	.014	.896	.09	0	0	0	......
k=2	1	0	0	0	0	0	......
k=3	1	0	0	0	0	0	......
k=4	1	0	0	0	0	0	......
k=5	1	0	0	0	0	0	......
η_t	1	1	1	0	0	0	......

Total Cost 749.17

$d_k = 31$ Single-Link Model

X_j^t	*t*=1	*t*=2	*t*=3	*t*=4	*t*=5	......	*t*=8
k=1	.014	.896	0	0	0	......	.09
k=2	1	0	0	0	0	......	0
k=3	1	0	0	0	0	......	0
k=4	1	0	0	0	0	......	0
k=5	1	0	0	0	0	......	0
η_t	1	1	0	0	0	......	1

Total Cost 778.34

D. Summary

In this section, we considered the TL discount transportation mode. This model can be seemed as an extension of the single-link model introduced by Speranza and Ukovich (1994). In the numerical example, we investigated both TL discount model and single-link model. We found that TL discount model will lead to more system cost savings and more frequent replenishments.

VI. Conclusion

This paper investigates the single-vendor-single- buyer problem with incorporation of transportation quantity discount. We study consider two discount schemes: LTL incremental discount and TL discount. MIP models are developed and numerical examples are carried out. The future wok can be developing centralized models for the problems considered in this paper.

Appendices

A.1. Proof of Lemma 1

To prove Lemma 1, it is sufficient to prove the following statement: for any two commodities *i* and *j*, if we have the relationship of $(h_i/v_i) \le (h_j/v_j)$, then commodity *i* should not be replenished more frequent than commodity *j*, that is

$$\text{If } (h_i/v_i) \le (h_j/v_j) \text{, then } t_i^{\min} \ge t_j^{\max} \quad \forall i, j \in K \qquad \text{(A.1)}$$

This can be proved by contradiction. Suppose there exists an optimal solution X_k^t where we have $t_i^{\min} < t_j^{\max}$. Let $\delta = \min\left(v_i d_i x_i^{t_i^{\min}}, v_j d_j x_j^{t_j^{\max}}\right)$. We construct a new feasible solution X'^t_k identical to X_k^t except for the shipment periods of commodity *j* and *i*. Consider $\delta/(v_i d_i)$ of commodity *i* with shipment period $t_i^{\min}$ and $\delta/(v_j d_j)$ of commodity *j* with shipment period $t_j^{\max}$ in the original solution X_k^t. We change their shipment periods to each other's to get the new solution X'^t_k. Then the quantities shipped at $t_i^{\min}$ and $t_j^{\max}$ remain the same, so does the average system transportation cost.

Therefore, the cost difference between solutions X'^t_k and X_k^t is only attributable to the average holding cost for δ of commodity *i* and δ of commodity *j*:

$$\begin{aligned}
&TC(X'^t_k) - TC(X_k^t) \\
&= \left[\left(\frac{\delta}{v_i d_i}\right) d_i t_j^{\max} h_i + \left(\frac{\delta}{v_j d_j}\right) d_j t_i^{\min} h_j\right] \\
&\quad - \left[\left(\frac{\delta}{v_i d_i}\right) d_i t_i^{\min} h_i + \left(\frac{\delta}{v_j d_j}\right) d_j t_j^{\max} h_j\right] \\
&= \left(\frac{\delta}{v_i}\right)\left(t_j^{\max} - t_i^{\min}\right) h_i + \left(\frac{\delta}{v_j}\right)\left(t_i^{\min} - t_j^{\max}\right) h_j \\
&= -\delta\left(\frac{h_j}{v_j} - \frac{h_i}{v_i}\right)\left(t_j^{\max} - t_i^{\min}\right) \le 0 \qquad \text{(A.2)}
\end{aligned}$$

The first term in A.2 presents the average holding cost for $\delta/(v_i d_i)$ of commodity *i* and $\delta/(v_j d_j)$ of commodity *j* in the new solution X'^t_k, and the second term presents the counterpart in the original solution X_k^t. The inequality is due to the assumptions of $t_i^{\min} < t_j^{\max}$ and $(h_i/v_i) \le (h_j/v_j)$. Thus the original solution X_k^t does not outperform the new solution X'^t_k. Therefore, A.1 is proved. Consequently, Lemma 1 is true.

A.2. Proof of Lemma 2

To prove Lemma 2, it is sufficient to prove the following statement: under *single frequency policy*, for any two commodities *i* and *j* with the relationship of $(h_i/v_i) = (h_j/v_j)$, it is optimal to let them shipped in the same frequency, that is

$$\text{If } (h_i/v_i) = (h_j/v_j), \text{ then } t_i = t_j \quad \forall i, j \in K \qquad \text{(A.3)}$$

This can be proved by contradiction as follows. Suppose there is a unique optimal solution X_k^t, in which $t_i \ne t_j$. Let us define a notation N_k to denote the set of time instants at which commodity k is shipped. Thus commodity *i* and *j* have the different shipment time instant sets: N_i and N_j. Suppose that, at time instant *n*, the total shipment quantity $z_n \in (M^{r-1}, M^r]$ and let $f_n = f^r$, $\alpha_n = \alpha^r$. Thus the corresponding shipment cost at time instant *n* can be described as $F_1(z_n) = f_n + \alpha_n z_n$. The average system cost associated with solution X_k^t is:

$$\begin{aligned}
TC(X_k^t) &= \sum_{k\in K} h_k d_k t_k + \frac{1}{|N|}\sum_{n\in N} F_1(z_n) \\
&= \sum_{k\in K} h_k d_k t_k + \frac{1}{|N|}\sum_{n\in N}\left[f_n + \alpha_n z_n\right]
\end{aligned} \qquad \text{(A.4)}$$

Let's construct two new feasible solutions X'^t_k and X''^t_k identical to X_k^t except for the shipment period of commodity *i* and *j*:

$$X'^t_k:\ t'_j = t_i, \text{ and } t'_k = t_k \qquad \forall k \ne j$$
(A.5)

$$X''^t_k:\ t''_i = t_j, \text{ and } t''_k = t_k \qquad \forall k \ne i$$
(A.6)

Note that, we need the assumption of no transportation capacity restriction to make sure that these two new solutions are feasible. The average system cost associated with the new solution X'^t_k is:

$$\begin{aligned}
&TC(X'^t_k) \\
&= \sum_{k\in K} h_k d_k t'_k + \frac{1}{|N|}\sum_{n\in N} F_1(z'_n) \\
&= h_j d_j t_i + \sum_{k\in K\setminus\{j\}} h_k d_k t_k + \frac{1}{|N|}\Big\{\sum_{n\in N\setminus(N_i\cup N_j)} F_1(z_n) \\
&\quad + \sum_{n\in N_i\setminus N_j} F_1(z_n + v_j d_j t_i) + \sum_{n\in N_j\setminus N_i} F_1(z_n - v_j d_j t_j) \\
&\quad + \sum_{n\in N_i\cap N_j} F_1(z_n + v_j d_j(t_i - t_j))\Big\} \\
&\le h_j d_j t_i + \sum_{k\in K\setminus\{j\}} h_k d_k t_k + \frac{1}{|N|}\Big\{\sum_{n\in N\setminus(N_i\cup N_j)} F_1(z_n) \\
&\quad + \sum_{n\in N_i\setminus N_j}[F_1(z_n) + \alpha_n v_j d_j t_i] + \sum_{n\in N_j\setminus N_i}[F_1(z_n) + \alpha_n(-v_j d_j t_j)] \\
&\quad + \sum_{n\in N_i\cap N_j}[F_1(z_n) + \alpha_n v_j d_j(t_i - t_j)]\Big\} \\
&\le h_j d_j(t_i - t_j) + \sum_{k\in K} h_k d_k t_k + \frac{1}{|N|}\Big\{\sum_{n\in N} F_1(z_n) \\
&\quad + \sum_{n\in N_i}\alpha_n v_j d_j t_i - \sum_{n\in N_j}\alpha_n v_j d_j t_j\Big\} \qquad \text{(A.7)}
\end{aligned}$$

The above inequality holds because of the concavity and monotonicity of the incremental discount shipment cost function $F_1(z_n)$: $F_1(z_n + \delta) \le F_1(z_n) + \alpha_n\delta$. Then the difference in system cost between solution X'^t_k and X''^t_k is:

$$\begin{aligned}
\Delta' &= TC(X'^t_k) - TC(X_k^t) \\
&\le h_j d_j(t_i - t_j) + \frac{1}{|N|}\left(\sum_{n\in N_i}\alpha_n v_j d_j t_i - \sum_{n\in N_j}\alpha_n v_j d_j t_j\right) \\
&\le d_j v_j\left[\frac{h_j}{v_j}(t_i - t_j) + \frac{1}{|N|}\left(\sum_{n\in N_i}\alpha_n t_i - \sum_{n\in N_j}\alpha_n t_j\right)\right] \qquad \text{(A.8)}
\end{aligned}$$

Now we consider the other new solution $X^{"t}_{k}$. By a similar derivation as A.7, we can show the difference in average system cost between solutions $X^{'t}_{k}$ and $X^{"t}_{k}$ is:

$$\begin{aligned}\Delta^{"} &= TC(X^{"t}_{k}) - TC(X^{t}_{k}) \\ &\le h_i d_i (t_j - t_i) + \frac{1}{|N|}\left(\sum_{n\in N_j} \alpha_n v_i d_i t_j - \sum_{n\in N_i} \alpha_n v_i d_i t_i\right) \\ &\le -d_i v_i \left[\frac{h_i}{v_i}(t_i - t_j) + \frac{1}{|N|}\left(\sum_{n\in N_i} \alpha_n t_i - \sum_{n\in N_j} \alpha_n t_j\right)\right] = -\frac{d_i v_i}{d_j v_j}\Delta^{'} \quad \text{(A.9)}\end{aligned}$$

The last equality in A.9 holds because of the assumption $(h_i / v_i) = (h_j / v_j)$. Thus, we have $\min(\Delta^{'}, \Delta^{"}) = \min(\Delta^{'}, -\frac{d_i v_i}{d_j v_j}\Delta^{'}) \le 0$, which means that the origin solution X^{t}_{k} does not outperform both of the new solutions $X^{'t}_{k}$ and $X^{"t}_{k}$. And this contradicts the initial assumption that X^{t}_{k} is the unique optimal solution. Therefore, statement A.3 is true. To treat commodity *i* and *j* as the different amount of the same commodity, we can prove Lemma 2.

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